The Hyperbolic Plane

Definition: We define $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 |\ y > 0\}$ to be the *upper half plane*. We define on $\mathbb{H}$ a 2-tensor

$$ds^2 = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

This gives an inner product, and thus a norm $||\cdot||$, on the tangent plane at each point in $\mathbb{H}$.

Curves and lengths

If $\gamma : [a,b] \longrightarrow \mathbb{H}$ is a C^1 curve, then for each $t_0 \in [a,b]$, $(\gamma(t_0), \frac{d\gamma}{dt}(t_0))$ is a vector in tangent plane at $\gamma(t_0)$, where $\frac{d\gamma}{dt}(t_0) = \left(\frac{d\gamma_1}{dt}(t_0), \frac{d\gamma_2}{dt}(t_0)\right)$, where γ_i are the components of γ. We define the length of γ to be $l(\gamma) = \int_a^b \|\frac{d\gamma}{dt}\| dt$. It is not hard to show that the length of curve does not depend on the parametrization.

We define the distance between two points to be the infimum of the length of all the paths connecting them. This turns out to be a minimum as we will show below. The distance function can be shown to be a metric on $\mathbb{H}$. The upper half plane with the tensor ds^2 is called the *hyperbolic plane*.

Exercise 1: Let $p = (0, y_1) \in \mathbb{H}$ and $q = (0, y_2) \in \mathbb{H}$. Prove that the shortest C^1 curve connecting p and q is the straight, vertical line connecting p and q.

Definition: We can identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, that is, we identify $(x,y) \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$. For $a,b,c,d \in \mathbb{C}$ with $ad - bc \neq 0$, the map $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined as

$$f(z) = \frac{az+b}{cz+d}$$

is called a *fractional linear transformation* (or FLT). If $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$, then f preserves the upper half plane (explained later) and thus induces a map from $\mathbb{H}$ to $\mathbb{H}$. It is not hard to show that an FLT with *real coefficients*, that is $a, b, c, d \in \mathbb{R}$, can be obtained by composing the following maps:

- Translation: $T : \mathbb{H} \longrightarrow \mathbb{H}$ is defined as $T(x, y) = (x + b, y)$, where b is a constant. This corresponds to the case $a = 1, c = 0, d = 1$.
- Dilation: $D : \mathbb{H} \longrightarrow \mathbb{H}$ is defined as $D(x, y) = (ax, ay)$, where $a > 0$ is a constant. This corresponds to the case $b = 0, c = 0, d = 1$.
- Inversion: $I : \mathbb{H} \longrightarrow \mathbb{H}$ is defined as $I(x, y) = (\frac{-x}{x^2 + y^2}, \frac{y}{x^2 + y^2})$. This corresponds to the case $a = 0, b = -1, c = 1, d = 0$.

Exercise 2: Show that ds^2 is preserved under the above three maps. That is, show that it is equal to the pull-back of itself, i.e. if f is one of the above maps, then $f^*(ds^2)(p) = ds^2(f(p))$.

Exercise 3: Prove that if f and $g \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are differentiable functions, then for $\omega \in \mathcal{T}^k(\mathbb{R}^n)$, we have $(f \circ g)^*(\omega) = g^*(f^*(\omega))$. Since a FLT f is a composition of translations, dilations and inversions, and ds^2 is preserved under these maps, it follows that ds^2 is preserved under f.

Exercise 4: Let f be an FLT with real coefficients so that f induces a map $f : \mathbb{H} \longrightarrow \mathbb{H}$. Let γ be a C^1 curve in H. Then $\lambda := f \circ \gamma$ is also a C^1 curve in $\mathbb{H}$. Show that $l(\lambda) = l(\gamma)$.

Facts (which are not extremely hard to prove): FLT's take generalised circles to generalised circles (a generalised circle is a line or a circle) and preserve angles between curves. The angle between two curves at an intersection point is the angle between the two tangent vectors. Also, for any 2 triples of points, say, (u_1, u_2, u_3) and (v_1, v_2, v_3), there exists an FLT f that sends one to the other, i.e, $f(u_i) = f(v_i)$ for $i = 1, 2, 3$.

Since a generalised circle is determined by 3 points on it, it follows that given 2 generalised circles, there exists an FLT that brings one to another. An FLT with real coefficients preserves the real axis in $\mathbb{C}$. In order for it to preserve the upper half plane, we need to require that $ad - bc > 0$.

Given any two points p, $q \in \mathbb{H}$, there exists an FLT f with real coefficients such that $f(p)$, $f(q)$ are on the y–axis. This map can be constructed as follow. First, there is a circle going through p and q that is perpendicular to the x-axis, i.e. the center of this circle lies on the x-axis. Let r be one of the intersection points of the circle and the x-axis. Next, by the facts above, there is an FLT f that takes r to the origin and p, q to two other points on the y-axis (imaginary axis).

Note that the circle going through p and q must be perpendicular to the x-axis because its image under f is perpendicular to the image of the x-axis and f is angle-preserving.

Now, with this observation and Exercise 4 and Exercise 1, we see that the shortest path between two point p and $q \in \mathbb{H}$ is the generalised circle going through them that is perpendicular to the x-axis. These curves are called *geodesics*.

Areas

Let $\mathbb{M}$ be the region in $\mathbb{H}$ bounded by $x = 1$, $x = -1$ and $y = \sqrt{1 - x^2}$.

If $U \subset \mathbb{H}$, we define the area of U to be $\int_U \frac{1}{y^2}\, dA = \int_U \frac{1}{y^2}\, dxdy$ if this integral exists. (The form $\frac{1}{y^2}\, dx \wedge dy$ is the volume/area form corresponding to the "metric" ds^2.)

A triangle with vertices $A, B, C \in \mathbb{H}$ is the closed set bounded by geodesics connecting each pair of points in $\{A, B, C\}$. Up to an FLT, any triangle is contained in $\mathbb{M}$. Also, it can be shown that this volume form is preserved under FLT's as in the above exercises. Hence, the area of any triangle is bounded by the area of $\mathbb{M}$.

Exercise 5: Show that the area of any triangle in $\mathbb{H}$ is less than π by showing that

$$\lim_{h \to \infty} \int_{\mathbb{M}_h} \frac{1}{y^2}\, dA = \pi,$$

where $\mathbb{M}_h = \{(x, y) \in \mathbb{M} \mid y \leq h\}$, for $h > 1$.

In fact, this is a corollary of a much stronger theorem. The theorem of Gauss-Bonnet implies that for any triangle $\triangle$ in $\mathbb{H}$ with angles α, β, γ, we have

$$\text{Area}(\triangle) = \pi - (\alpha + \beta + \gamma).$$