GAMMA FUNCTIONS, MONODROMY AND APÉRY CONSTANTS

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Introduction

In an important paper [4], Golyshev and Zagier introduce what we will refer to as an *Apéry series* $\kappa(s)$ associated to an ordinary differential operator L. Assuming L has a regular singularity at $t = 0$ with a unique local exponent ρ, and $t = c$ is a nearby conifold point, $\kappa(s) = \sum_{n\geq 0} \kappa_n s^n \in \mathbb{C}[\![s]\!]$ is a power series which describes the variation around c of an inhomogeneous Frobenius solution $\Phi = \sum_{n\geq 0} \phi_n(t)s^n$ to L near $t = 0$ (see Definition 17 and Definition 18 below). Golyshev and Zagier show in certain cases that the κ_n are periods, and they raise the question quite generally how to describe the κ_n motivically.

The purpose of this work is to develop the theory (first suggested to us by Golyshev) of *motivic Mellin transforms* or *motivic gamma functions*. Our main result (Theorem 19) relates the motivic Mellin transform associated to a solution of L with the corresponding Apéry series. It follows from this that the κ's are always periods when L is a Picard–Fuchs differential operator, (Remark 22).

Finally, in Section 4 we relate the κ's to periods of limiting mixed Hodge structures.

1. Motivic Γ-functions

Let C be a compete, smooth algebraic curve over $\mathbb{C}$, and let $S \subset C$ be a non-empty, finite set of points. Let M be an algebraic connection on $U := C - S$. The de Rham cohomology of the connection, $H^*_{DR}(U, M)$ is the cohomology of the 2-term complex of modules (placed in degrees $0, 1$) over $\Gamma(U, \mathcal{O}_U)$

$$M \xrightarrow{\nabla_M} M \otimes \Omega^1_U.$$

Recall by definition a *solution* for M is a horizontal section of the dual connection $\mu \in M_{an}^{\vee,\nabla^\vee=0}$. Whereas M and $H^*_{DR}(U, M)$ are purely algebraic in nature, interesting horizontal sections are usually multi-valued and only defined locally analytically, so we consider the analytic connection $M^\vee_{an}$ on U_{an}. By coupling solutions to suitable topological chains in C_{an}, one defines rapid decay homology groups $H_{*,rd}(U_{an}, M^\vee_{an})$, [1], and there is a period pairing which is perfect pairing of finite dimensional vector spaces

$$H^1_{DR}(U, M) \times H_{1,rd}(U, M^\vee) \to \mathbb{C}.$$

This construction is valid even when M has irregular singular points. It can be used, for example, to construct the classical Bessel and confluent hypergeometric differential equations. In this paper, we will consider only the motivic case, where M has regular

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singular points. In this case, one can ignore the rapid decay condition for homology and work with the standard topological homology of the local system of solutions. We write $\mathcal{M}^\vee := M_{an}^{\vee,\nabla^\vee=0}$ for the local system. Note that this local system can often be defined over a subfield $K \subset \mathbb{C}$). Homology can be computed over K, e.g., by fixing a basepoint $p \in U_{an}$ and interpreting $\mathcal{M}^\vee$ as a representation of $\pi_1(U_{an}, p)$ on $\mathcal{M}_p^\vee$. Let $\widetilde{U}_{an} \to U_{an}$ be the universal cover, and let $C_*(\widetilde{U}_{an}, K)$ be the complex of topological chains on the universal cover. Homology is then defined by coupling the chains to the representation

$$H_*(U_{an}, \mathcal{M}^\vee) := H_*(C_*(\widetilde{U}_{an}, K) \otimes_{K[\pi_1(U,p)]} \mathcal{M}_p^\vee).$$

Concretely, in degree 1, the period pairing can be represented as follows. For us, Ω_U^1 will always be a free, rank 1 module. We fix $\omega \in \Omega_U^1$ a generator. A de Rham 1-cocycle c lifts to an element $m \otimes \omega \in M \otimes \Omega_U^1$. An homology class $\mu \in H_1(U, \mathcal{M}^\vee)$ (to simplify notation, we no longer write the subscript $_{an}$) can be represented by a finite sum $\sum_j \sigma_j \otimes \varepsilon_j$ where $\sigma_j \in \pi_1(U,p)$, $\varepsilon_j \in \mathcal{M}_p^\vee$ and $\sum_j \sigma_j^{-1} \varepsilon_j = \sum_j \varepsilon_j$. The latter condition means that μ is a 1-cycle (and not just a 1-chain). The resulting period is

$$\langle c, \mu \rangle = \sum_j \int_{\sigma_j^{-1}} \langle m, \varepsilon_j \rangle \omega. \tag{1}$$

Remark 1. *The inverse sign in σ_j^{-1} arises because the inhomogeneous bar complex $B_*[\pi_1]$ for the group $\pi_1 = \pi_1(U,p)$ is a complex of left π_1-modules. B_n is a free $\mathbb{Z}[\pi_1]$-module on symbols $[g_1, \dots, g_n]$ with $g_i \in \pi_1$. To compute homology with values in a left π_1-module V, we view $B_*[\pi_1]$ as a complex of right π_1-modules with $g \in \pi_1$ acting by left multiplication by g^{-1}. The homology $H_*(\pi_1, V)$ is identified with the homology of the complex $B_*[\pi_1] \otimes_{\mathbb{Z}[\pi_1]} V$. We will use the following formulas for the differentials in low degrees in computations throughout the paper:*

$$\partial([g_1] \otimes v) = g_1^{-1} v - v \tag{2}$$

and

$$\partial([g_1, g_2] \otimes v) = [g_2] \otimes g_1^{-1} v - [g_1 g_2] \otimes v + [g_1] \otimes v. \tag{3}$$

For 1-chains we will often omit $[*]$ and write $[g] \otimes v$ simply as $g \otimes v$. Using (2) one can easily see that the period pairing (1) vanishes when $c = \nabla_M(m')$ is a de Rham coboundary:

$$\langle \nabla_M(m'), \mu \rangle = \sum_j \int_{\sigma_j^{-1}} d\langle m', \varepsilon_j \rangle = \langle m', \sum_j (\sigma_j^{-1} \varepsilon_j - \varepsilon_j) \rangle = 0.$$

The reader may also check that (1) vanishes when μ is a boundary given by (3).

Example 2. *Let $f : X \to U$ be a smooth, proper map of algebraic varieties. Let $M := H_{DR}^n(X/U)$ be the relative de Rham cohomology, endowed with the algebraic Gauß–Manin connection ∇. Here, again, we are totally in the realm of algebraic geometry so if, for example, f, X, U are all defined over a subfield $k \subset \mathbb{C}$, then our connection M will be defined over k as well. In the Gauß–Manin setup, solutions typically arise from continuously varying closed chains on the fibres. Since the homology of the fibres is a* *local system defined over $\mathbb{Q}$, we can think of $\mathcal{M}_{an}^\vee$ as having a Betti structure and take $K = \mathbb{Q}$.*

Example 3. *The category of connections on U has a tensor product, so we can add interest to our study by coupling e.g. a Gauß–Manin connection M as in the previous example to one of a number of standard connections on U. The effect of tensoring connections is to multiply solutions appearing in the period integral.*

Three examples are
(i)(Mellin transform). Fix $t \in \mathcal{O}_U^\times$ a unit on U and take the connection on $\mathcal{O}_U$ given by $\nabla_{Mellin}(1) := sdt/t$. Somewhat abusively, this connection is denoted t^s. It has t^s as solution. The period integrals for $M \otimes t^s$ are of the shape $\int_\sigma \langle m, \varepsilon\rangle t^s\omega$. Here s is a parameter, so in fact our periods become functions of s.
(ii) (Fourier transform). Let $t \in \mathcal{O}_U$ and define a connection on $\mathcal{O}_U$ by $\nabla_{Fourier}(1) = sdt$. Again s is a parameter. The solution is e^{st}.
(iii)(Kummer). Take $t \in \mathcal{O}_U^\times$, and let K_t be the rank 2 connection with solutions given by $1, \log t$. As an exercise, the reader can write out the connections $Sym^n(K_t)$ and describe the integrands involved in calculating periods for $M \otimes Sym^n(K_t)$.

Definition 4. *A Γ-function is the function of s associated to a period of the Mellin transform of a regular singular point connection M on U. If M is a Gauß–Manin connection, then respective gamma functions are called* motivic.

Let us associate explicit gamma functions with homology classes in $H_1(U, \mathcal{M}^\vee \otimes t^s)$. For that we fix a basepoint p such that $t(p) \neq 0$ and consider the representation of $\pi_1(U, p)$ on the stalk

$$(\mathcal{M}^\vee \otimes t^s)_p = \mathcal{M}_p^\vee \otimes_K K[e^{\pm 2\pi i s}], \tag{4}$$

where the group acts on the second component through the monodromy of t^s. That is, the homotopy group acts through its quotient $\pi_1(\mathbb{G}_{m,t}, p) \cong \mathbb{Z}$ and the generator acts as multiplication by $e^{2\pi i s}$.

Definition 4'. *Fix $m \otimes \omega \in M \otimes \Omega_U^1$. A homology class $\xi \in H_1(U, \mathcal{M}^\vee \otimes t^s)$ can be represented by a 1-cycle*

$$\xi \sim \sum_j \sigma_j \otimes \varepsilon_j \otimes e^{2\pi i s n_j},$$

where the sum is finite, σ_j are loops based at p, $\varepsilon_j \in \mathcal{M}_p^\vee$ are solutions in a neighbourhood of p and $n_j \in \mathbb{Z}$. The respective gamma function is given by

$$\Gamma_\xi(s) = \sum_j e^{2\pi i s n_j} \int_{\sigma_j^{-1}} \langle m, \varepsilon_j\rangle t^s \omega. \tag{5}$$

Here we also assume that a branch of t^s at the base point p is fixed. It is thus the same branch in every integral in the right-hand sum, while the coefficient $e^{2\pi i s n_j}$ accounts for the possibility of choosing different branches.

Lemma 5. *Expression in the right-hand side of* (5) *depends only on the homological class of ξ in $H_1(U, \mathcal{M}^\vee \otimes t^s)$.*

Proof. According to (3), boundaries of 2-chains are generated over $K[e^{\pm 2\pi i s}]$ by expressions of the form

$$\partial([\sigma_1, \sigma_2] \otimes \varepsilon \otimes 1) = [\sigma_2] \otimes \sigma_1^{-1}(\varepsilon \otimes 1) - [\sigma_1 \sigma_2] \otimes \varepsilon \otimes 1 + [\sigma_1] \otimes \varepsilon \otimes 1 .$$

Vanishing of (5) on such expressions is the composition formula. Namely, to integrate $\langle m, \varepsilon \rangle t^s \omega$ over $(\sigma_1\sigma_2)^{-1} = \sigma_2^{-1}\sigma_1^{-1}$ we first integrate it over σ_1^{-1} and then integrate $[\sigma_1^{-1}](\langle m, \varepsilon\rangle t^s\omega) = \langle m, [\sigma_1^{-1}](\varepsilon \otimes 1)\rangle t^s \omega$ over σ_2^{-1}. If $[\sigma_1^{-1}]t^s = e^{2\pi i s n(\sigma_1^{-1})} t^s$ we can write this as

$$\int_{(\sigma_1\sigma_2)^{-1}} \langle m, \varepsilon\rangle t^s \omega = \int_{\sigma_1^{-1}} \langle m, \varepsilon \rangle t^s \omega + e^{2\pi i s n(\sigma_1^{-1})} \int_{\sigma_2^{-1}} \langle m, [\sigma_1^{-1}]\varepsilon\rangle t^s \omega .$$

□

Note that the stalk (4) is a free module over $K[e^{\pm 2\pi i s}]$ of rank $\dim \mathcal{M}_p^\vee = \mathrm{rank}(M)$. The action of $\pi_1(U, p)$ commutes with the $K[e^{\pm 2\pi i s}]$-module structure and therefore $H_1(U, \mathcal{M}^\vee \otimes t^s)$ is a $K[e^{\pm 2\pi i s}]$-module. It is clear that evaluation

$$\xi \mapsto \Gamma_\xi(s)$$

in (5) is additive and commutes with multiplication by $e^{\pm 2\pi i s}$. Therefore we obtain a $K[e^{\pm 2\pi i s}]$-module of gamma functions. As a module, this is a quotient of $H_1(U, \mathcal{M}^\vee \otimes t^s)$ by classes whose gamma functions vanish. It follows that all gamma functions are $K[e^{\pm 2\pi i s}]$-linear combinations of a finite number of them:

Proposition 6. *The $K[e^{\pm 2\pi i s}]$-module of gamma functions is finitely generated.*

Proof. Since U is an affine curve then $\pi_1 = \pi_1(U, p)$ is a free group. The chain complex of the universal cover C_* will be a chain complex of finitely generated $\mathbb{Z}[\pi_1]$-modules. Since the stalk representation $N = \mathcal{M}_p^\vee \otimes_K K[e^{\pm 2\pi i s}]$ is a finitely generated module over a Noetherian ring $R = K[e^{\pm 2\pi i s}]$, then $C_* \otimes_{\mathbb{Z}[\pi_1]} N$ is a complex of finitely generated R-modules, so it has finitely generated homology. In particular, $H_1(U, \mathcal{M}^\vee \otimes t^s)$ is finitely generated. □

Example 7. *The double cover $f : \mathbb{P}^1_y \to \mathbb{P}^1_t$ given by $t = 1 - y^2$ is ramified at $t = 1, \infty$. We also remove the point $t = 0$, getting*

$$C^\circ := \mathbb{P}^1_y \setminus \{1, -1, 0, \infty\} \xrightarrow{f^\circ} U := \mathbb{P}^1 \setminus \{0, 1, \infty\} .$$

We have $f^\circ_* \mathcal{O}_{C^\circ} = \mathcal{O}_U \oplus \mathcal{O}_U[y]$. The line bundle $M := \mathcal{O}_U[y]$ carries a connection with $\nabla_{d/dt}[y] = -\frac{1}{2(1-t)}[y]$. Choose a point $p \in U$ and a horizontal section of the dual bundle $\varepsilon \in \mathcal{M}_p^\vee$. Let σ_0 and σ_1 be based at p loops around 0 and 1 respectively. In $\mathcal{M}_p^\vee \otimes \mathbb{Q}[e^{\pm 2\pi i s}]$ we have $[\sigma_0](\varepsilon \otimes 1) = \varepsilon \otimes e^{2\pi i s}$ and $[\sigma_1](\varepsilon \otimes 1) = -\varepsilon \otimes 1$. The loop $\sigma = \sigma_0^{-1}\sigma_1\sigma_0\sigma_1$ fixes $\varepsilon \otimes 1$, hence the element

$$\xi = \sigma_0^{-1}\sigma_1\sigma_0\sigma_1 \otimes (\varepsilon \otimes 1)$$

is a 1-cycle. The respective gamma function is

$$\Gamma_\xi(s) = \int_{\sigma_1^{-1}\sigma_0^{-1}\sigma_1^{-1}\sigma_0} t^s \langle [y], \varepsilon \rangle \omega .$$

The 1-cycle condition here converts into the fact that σ is a closed path along which $t^s\langle [y], \varepsilon\rangle$ is single-valued, thus the above integral is well defined.

To be more specific, we choose $\omega = \frac{dt}{t}$ *and notice that, since* ε *is horizontal, the pairing* $\langle [y], \varepsilon\rangle$ *is a solution to the differential operator* $(1-t)\frac{d}{dt} - \frac{1}{2}$ *hence is a constant multiple of* $(1-t)^{-1/2}$. *Possibly rescaling* ε, *we obtain the following beta integral:*

$$\Gamma_\xi(s) = \int_{\sigma_1^{-1}\sigma_0^{-1}\sigma_1^{-1}\sigma_0} t^{s-1}(1-t)^{-1/2}dt = \int_1^0 + e^{2\pi i s}\int_0^1 - e^{2\pi i s}\int_1^0 - \int_0^1$$
$$= 2(e^{2\pi i s} - 1)\int_0^1 t^{s-1}(1-t)^{-1/2}dt = 2(e^{2\pi i s} - 1)\frac{\Gamma(s)\Gamma(1/2)}{\Gamma(s+1/2)}\,.$$

This quotient is a motivic gamma function. Note that it is an entire function of s. *This is a feature of all our motivic gamma functions, as one can easily see from their definition* (5).

Remark 8. *The reader will have noticed that the classical* Γ*-function is not a motivic* Γ*-function. The connection in this case is* $\nabla(1) = dt$ *which has an irregular singular point at* $t = \infty$. *The notion of period can be extended to the irregular case using some form of "rapid decay" homology (* [1], [3]*). The classical path of integration from* 0 *to* ∞ *is not allowed because the Mellin connection* $\nabla_{Mellin}(1) = sdt/t$ *has a singular point at* $t = 0$. *However, if we replace* $[0, \infty]$ *with a "keyhole" path starting at* ∞, *following the positive real axis to* $+\varepsilon$, *looping counterclockwise about* 0, *and then going back to* $+\infty$, *the resulting "period",* $(e^{2\pi i s} - 1)\Gamma(s)$ *suggests a natural generalization of motivic gammas to the irregular case. Notice that again the period is an entire function of* s.

It is a general fact that Mellin transforms satisfy difference equations (see [9]). Remember that our choice is to fix $m \otimes \omega$, in which case we obtain a finitely generated $K[e^{\pm 2\pi i s}]$-module of gamma functions indexed by $\xi \in H_1(U, \mathcal{M}^\vee \otimes t^s)$. All of them satisfy the same difference equation which can be found as follows. Our periods depend on m and ω only through the function-linear tensor $m \otimes_{\mathcal{O}_U} \omega$. We will assume that $\omega = f dt/t$ for $f \in \mathcal{O}_U$. In this way, replacing m by fm, we can take $\omega = dt/t$. Put

$$m \otimes \omega = m \otimes dt/t,$$

in Definition 4'. Let $r = \operatorname{rank}(M)$ and consider the derivation $D = t\, d/dt \in T_C$. Then there exist $q_0, \ldots, q_r \in \mathcal{O}_U$ such that the differential operator

$$L = q_0\, D^r + q_1\, D^{r-1} + \ldots + q_r$$

annihilates m. Here and throughout the paper we shall adopt the convention that D acts on M via $\nabla_M(D)$; for example, $Lm = 0$ means that

$$\sum_{j=0}^{r} q_{r-j} \nabla_M(D)^j\, m = 0.$$

Observe that for a solution $\varepsilon \in \mathcal{M}^\vee$ the analytic function $\phi = \langle m, \varepsilon\rangle \in \mathcal{O}^{an}$ satisfies the differential equation $L\phi = 0$.

Proposition 9. *Assume that $q_j = q_j(t) \in \mathbb{C}[t]$ for $0 \le j \le r$ and rearrange terms in the differential operator*

$$L = \sum_{j=0}^{r} q_{r-j}(t) D^j = p_0(D) + t\, p_1(D) + \ldots + t^a\, p_a(D) \tag{6}$$

with polynomials $p_0, \ldots, p_a \in \mathbb{C}[D]$ of degree at most r. Then for every homological class ξ the gamma function (5) *satisfies the difference equation*

$$\sum_{j=0}^{a} p_j(-s-j)\Gamma_\xi(s+j) = 0\,. \tag{7}$$

Proof. For any $m'' \in M$ let us denote

$$\Gamma_\xi(m'', s) = \sum_j e^{2\pi i s n_j} \int_{\sigma_j^{-1}} \langle m'', \varepsilon_j \rangle t^s dt/t,$$

which is just the gamma function (5) corresponding to $m'' \otimes dt/t \in M \otimes \Omega^1_U$. Since ξ is a 1-cycle, the condition

$$\partial \xi = \sum_j e^{2\pi i s n_j} (\sigma_j^{-1} - 1)(\varepsilon_j \otimes 1) = 0$$

together with the fundamental theorem of calculus imply

$$\sum_j e^{2\pi i s n_j} \int_{\sigma_j^{-1}} D(\langle m'', \varepsilon_j \rangle t^s) dt/t = 0.$$

Expanding out, using the fact that ε_j is a horizontal section of $M^\vee$, we get

$$-s\Gamma_\xi(m'', s) = \sum_j e^{2\pi i s n_j} \int_{\sigma_j^{-1}} \langle \nabla_D m'', \varepsilon_j \rangle t^s dt/t = \Gamma_\xi(\nabla_D m'', s).$$

One also has trivially

$$\Gamma_\xi(t\, m'', s) = \Gamma_\xi(m'', s+1).$$

Since $\sum_j q_j \nabla_D^j m = 0$, using formula (6) we get

$$0 = \Gamma_\xi(\sum_j q_j(t) \nabla_D^j m, s) = \sum_{j=0}^{r} \Gamma_\xi(p_j(\nabla_D) m, s+j) = \sum_j p_j(-s-j)\Gamma_\xi(m, s+j). \tag{8}$$

$\square$

Quite generally, the coefficients $q_j \in \mathcal{O}_U$ of the differential operator L can be expressed as analytic functions of t which need not be polynomials (as in Proposition 9) and hence gamma functions may satisfy difference equations of infinite length. We will not assume that $q_j(t)$ are polynomials in the rest of this paper.

2. Monodromy and existence of gamma functions

An important class of motivic Γ functions arises when ξ lifts to a class in $H_1(V, \mathcal{M}^\vee \otimes t^s)$, where $V \subset U$ is open in the $\mathbb{C}$-topology and $\pi_1(V)$ is a free group on 2 generators.

The setting is as in the first section: we are given an algebraic connection M on an open curve U; there is a chosen unit $t \in O_U^\times$. Somewhat ambiguously, we will use t as a local coordinate. Let $c \in \mathbb{P}^1 \setminus \{0, \infty\}$ be such that $t = c$ is a singular value, that is the value of t at a singular point of M in $S = C \setminus U$. The local system of solutions is denoted by $\mathcal{M}^\vee = (M_{an}^\vee)^{\nabla^\vee = 0}$. This local system is defined over a field $K \subseteq \mathbb{C}$.

Proposition 10. *Let $d = \dim_K Image(\sigma_c - 1|\mathcal{M}^\vee)$ be the rank of the variation of the local monodromy of $\mathcal{M}^\vee$ around $t = c$. Let $V \subset U_{an}$ be an open neighbourhood of a path between $t = 0$ and $t = c$. We assume $V = V_0 \cup V_c$ where V_0 and V_c are punctured disks centered at 0 and c respectively, and $V_0 \cap V_c$ is contractible. The $K[e^{\pm 2\pi i s}]$-module of gamma functions*

$$\Gamma_\xi(s), \qquad \xi \in H_1(V, \mathcal{M}^\vee \otimes t^s)$$

is generated by at most d elements.

Proof. The following lemma will be used.

Lemma 11. *Write $\mathbb{Z} \cong u^{\mathbb{Z}}$ for the free abelian group on one generator, written multiplicatively. Let A be an abelian group, and suppose we are given $\phi : A \to A$ an automorphism. We view A as a $u^{\mathbb{Z}}$-module, with u acting as ϕ. Then*

$$H_1(u^{\mathbb{Z}}, A) \cong A^{\phi = id}.$$

Proof of lemma. We compute in the bar complex using (2) and (3):

$$\partial([u^j] \otimes a) = u^{-j}a - a,$$
$$\partial([u^j, u^k] \otimes a) = [u^k] \otimes u^{-j}a - [u^{j+k}] \otimes a + [u^j] \otimes a.$$

Clearly, any 1-chain $\sum_i [g_i] \otimes a_i$ is equivalent modulo boundaries of 2-chains as above to a 1-chain of the form $\sum_i [u^{\pm 1}] \otimes b_i$. Similarly, taking $j = k = 0$ shows 1-chains $[1] \otimes a$ are coboundaries, and $[u^{-1}] \otimes a \sim -[u] \otimes ua$. In this way, any 1-chain is equivalent to a 1-chain of the form $\sum_i [u] \otimes a_i = [u] \otimes (\sum_i a_i)$. We have

$$\partial([u] \otimes \sum_i a_i) = (u^{-1} - 1)(\sum a_i) \in A.$$

The lemma follows. ☐

Returning to the proof of the proposition, consider the long exact sequence for the relative homology with coefficients in $\mathcal{L} = \mathcal{M}^\vee \otimes t^s$:

$$\dots \to H_1(V_c; \mathcal{L}) \xrightarrow{Cor} H_1(V; \mathcal{L}) \to H_1(V, V_c; \mathcal{L}) \xrightarrow{b} H_0(V_c; \mathcal{L}) \to \dots \tag{9}$$

Fix the base point $p \in V_0 \cap V_c$. Since t^s has no monodromy around c we have $H_1(V_c; \mathcal{L}) = \mathcal{L}_p^{\sigma_c = id} = \mathcal{M}_p^{\vee, \sigma_c = id} \otimes_K K[e^{\pm 2\pi i s}]$. It follows that gamma functions of classes in the image of the corestriction map Cor vanish. Indeed, over $K[e^{\pm 2\pi i s}]$ such classes are generated by $\xi = \sigma_c \otimes \varepsilon \otimes 1$ where $\varepsilon \in \mathcal{M}_p^{\vee, \sigma_c = id}$ is σ_c-invariant and $\Gamma_\xi(s) = \oint_c t^s \langle m, \varepsilon \rangle \omega = 0$. From exactness, the cokernel of Cor is isomorphic to the kernel of the connecting map b so the $K[e^{\pm 2\pi i s}]$-module of Γ's is a subquotient of $Ker(b)$.

We remark that $K[e^{\pm 2\pi is}]$ is a principal ideal domain. Quite generally, if E is a module generated by d elements over a principal ideal domain, and F is a module which is a subquotient of E, then F is generated by $\leq d$ elements. It will therefore suffice to show $Ker(b)$ is contained in a $K[e^{2\pi is}]$-module generated by at most d elements.

We have a diagram

$$\begin{tikzcd} 0 \arrow[r] & H_1(V_0,\mathcal{L}) \arrow[r] \arrow[d] & H_1(V_0,p;\mathcal{L}) \arrow[r] \arrow[d, "\text{excision}", "\cong"'] & H_0(p,\mathcal{L}) \arrow[r] \arrow[d, "\alpha"] & H_0(V_0,\mathcal{L}) \arrow[r] & 0 \\ & H_1(V,\mathcal{L}) \arrow[r] & H_1(V,V_c;\mathcal{L}) \arrow[r, "b"'] & H_0(V_c,\mathcal{L}) & & \end{tikzcd} \tag{10}$$

The top line is identified with the exact sequence

$$0 \to \mathcal{L}_p^{\sigma_0=id} \to \mathcal{L}_p \xrightarrow{\sigma_0-1} \mathcal{L}_p \to \mathcal{L}_p/(\sigma_0-1)\mathcal{L}_p \to 0.$$

(To see this, one can, for example, think of H_1 as being given by 1-chains coupled to sections of $\mathcal{L}$. Chains with boundary at p yield relative homology classes.) The map $\alpha : \mathcal{L}_p \to \mathcal{L}_p/(\sigma_c-1)\mathcal{L}_p$ is the evident one induced from the inclusion $p \in V_c$. A diagram chase identifies $Ker(b)$ with the kernel of the composition

$$\mathcal{L}_p \xrightarrow{\sigma_0-1} \mathcal{L}_p \to \mathcal{L}_p/(\sigma_c-1)\mathcal{L}_p$$

which is $(\sigma_0-1)\mathcal{L}_p \cap (\sigma_c-1)\mathcal{L}_p \subset (\sigma_c-1)\mathcal{L}_p$. (Note that σ_0-1 is injective.)

Write $I_c := \text{Image}(\sigma_c - 1 : \mathcal{M}_p^\vee \to \mathcal{M}_p^\vee)$. By assumption I_c is a vector space of dimension d over K, and we have

$$(\sigma_c-1)\mathcal{L}_p = I_c \otimes_K K[e^{\pm 2\pi is}],$$

a $K[e^{\pm 2\pi is}]$-module of rank d. As remarked above, this implies that the module of Γ's has rank $\leq d$. □

Let us focus on the case when Proposition 10 guarantees there is a unique generator of the module of gamma functions. We make the following

Assumption 12. *The variation of the local monodromy of $\mathcal{M}^\vee$ around the point $t=c$ has one dimensional image.*

We fix a base point $t=p$ and a non-zero solution $\delta \in \mathcal{M}_p^\vee$ spanning the image $(\sigma_c-1)\mathcal{M}_p^\vee$.

A conifold point of a family of algebraic varieties provides an example of a situation when Assumption 12 is satisfied, in which case M is self-dual and one can take δ to be the vanishing cycle at c. By the Picard–Lefschetz theorem the variation of the monodromy around c satisfies

$$(\sigma_c-1)\varepsilon = \pm\langle\varepsilon,\delta\rangle\delta \tag{11}$$

for any section $\varepsilon \in \mathcal{M}$. When fibres of the family have even dimension we have $\langle\delta,\delta\rangle = \pm 2$, σ_c is semisimple on $\mathcal{M}$ with $\sigma_c^2 = 1$ and $\sigma_c\delta = -\delta$. In the case of odd dimensional fibres one has $\langle\delta,\delta\rangle = 0$ and $\sigma_c\delta = \delta$.

Lemma 13. *Let hypotheses on M be as in Assumption 12. Let $\varepsilon \in \mathcal{M}_p^\vee$ be such that $(\sigma_c - 1)\varepsilon = \delta$. For every relation*

$$(\sum_m \lambda_m \sigma_0^{-m})\delta = 0 \quad (\text{ a finite sum with } \lambda_m \in K) \tag{12}$$

the element

$$\xi = \sum_m \lambda_m \sigma_0^m \otimes \delta \otimes e^{2\pi i m s} + \sigma_c^{-1} \otimes \varepsilon \otimes \sum_m \lambda_m e^{2\pi i m s} \tag{13}$$

is a 1-cycle with coefficients in $\mathcal{M}_p^\vee \otimes_K K[e^{\pm 2\pi i s}]$. The resulting map to the homology of the local system $\mathcal{L} = \mathcal{M}^\vee \otimes t^s$

$$Ann_{K[\sigma_0^{\pm 1}]}(\delta) \to H_1(V, \mathcal{L}) \tag{14}$$

is a homomorphism of $K[T^{\pm 1}]$-modules, where T acts via multiplication by σ_0^{-1} on relations (12) and by multiplication by $e^{2\pi i s}$ on homology. The group $H_1(V, \mathcal{L})$ is spanned by the images of $H_1(V_c, \mathcal{L})$ and (14).

Proof. To check that ξ is a cycle, we compute

$$\begin{aligned}
\partial \xi &= \sum_m \lambda_m (\sigma_0^{-m} - 1)(\delta \otimes e^{2\pi i m s}) + (\sigma_c - 1)\varepsilon \otimes \sum_m \lambda_m e^{2\pi i m s} \\
&= \sum_m \lambda_m \sigma_0^{-m}(\delta \otimes e^{2\pi i m s}) = (\sum_m \lambda_m \sigma_0^{-m}\delta) \otimes 1 = 0.
\end{aligned}$$

The $K[T^{\pm 1}]$-structure is straightforward and left for the reader.

It remains to check that the map (14) is surjective modulo the image of $Cor : H_1(V_c, \mathcal{L}) \to H_1(V, \mathcal{L})$. Every 1-cycle for $\langle \sigma_0, \sigma_c \rangle$ can be written modulo boundaries in the form

$$\sigma_0^{-1} \otimes \Big(\sum_n \psi_n \otimes e^{2\pi i n s} \Big) + \sigma_c^{-1} \otimes \Big(\sum_m \gamma_m \otimes e^{2\pi i m s} \Big). \tag{15}$$

Here $\psi_n, \gamma_m \in \mathcal{M}_p^\vee$. (The point is that modifying by a boundary can remove any words in the σ_c and σ_0. See (3).) We write each $\gamma_m = \lambda_m \varepsilon + \gamma_m^{inv}$ with $\gamma_m^{inv} \in (\mathcal{M}_p^\vee)^{\sigma_c = id}$. The chain (15) is a sum of

$$\sigma_c^{-1} \otimes \Big(\sum_m \gamma_m^{inv} \otimes e^{2\pi i m s} \Big)$$

which is itself a 1-cycle on the subgroup $\langle \sigma_c \rangle$, and

$$\tilde{\xi} := \sigma_0^{-1} \otimes \Big(\sum_{n=n_{min}}^{n_{max}} \psi_n \otimes e^{2\pi i n s} \Big) + \sigma_c^{-1} \otimes \varepsilon \otimes \sum_m \lambda_m e^{2\pi i m s}. \tag{16}$$

The 1-cycle condition yields

$$0 = \partial \tilde{\xi} = \sum_n (\sigma_0 \psi_{n-1} - \psi_n + \lambda_n \delta) \otimes e^{2\pi i n s}.$$

(Recall the action of σ_0 includes multiplication by $e^{2\pi i s}$, (4).) This equation can be solved recursively

$$\begin{gathered}\lambda_n = 0,\ n < n_{min}\\ \psi_{n_{min}} = \lambda_{n_{min}}\delta\\ \psi_{n_{min}+1} = \lambda_{n_{min}}\sigma_0\delta + \lambda_{n_{min}+1}\,\delta\\ \vdots\\ 0 = \psi_{n_{max}+1} = \textstyle\sum_{j=0}^{n_{max}-n_{min}+1}(\lambda_{n_{min}+j}\,\sigma_0^{(n_{max}-n_{min}+1)-j})\delta\end{gathered}$$

It follows that $\Big(\sum_m \lambda_m\sigma_0^{-m}\Big)\delta = 0$. In fact, (16) is homologous to (13). To check this, we can assume $\lambda_m = 0$ for $m \le 0$. For $v \in \mathcal{L}_p$ we have by (3)

$$\sigma_0^m \otimes v \sim -\sigma_0^{-1} \otimes (\sigma_0^{-1} + \cdots + \sigma_0^{-m})v,$$

and therefore

$$\sum_m \lambda_m\sigma_0^m \otimes \delta \otimes e^{2\pi i m s} \sim \sigma_0^{-1} \otimes \Psi,$$

with

$$\begin{aligned}\Psi &= -\sum_m \lambda_m(\sigma_0^{-1} + \cdots + \sigma_0^{-m})(\delta \otimes e^{2\pi i m s})\\ &= -\sum_m \lambda_m \sum_{j=1}^{m} \sigma_0^{-j}\delta \otimes e^{2\pi i(m-j)s} = \sum_n \Big(-\sum_{m>n}\lambda_m\sigma_0^{n-m}\delta\Big) \otimes e^{2\pi i n s}\\ &= \sum_n \Big(\sum_{m\le n}\lambda_m\sigma_0^{n-m}\delta\Big) \otimes e^{2\pi n i s} = \sum_n \psi_n \otimes e^{2\pi i n s}.\end{aligned}$$

□

Let us assume that the solution $\langle m, \delta\rangle$ can be integrated between $t = 0$ and $t = c$. (For example, this condition is satisfied when both singularities are regular.) In this case the above lemma can be used to evaluate gamma functions as follows. For a polynomial $P(T) = \sum_m \lambda_m T^m$ such that $P(\sigma_0^{-1})\delta = 0$, the gamma function corresponding to the homology class (13) is given by

$$\begin{aligned}\Gamma_\xi(s) &= \sum_m \lambda_m e^{2\pi i m s}\int_{\sigma_0^{-m}} \langle m, \delta\rangle t^s\omega + P(e^{2\pi i s})\int_{\sigma_c}\langle m, \varepsilon\rangle t^s\omega\\ &= \sum_m \lambda_m e^{2\pi i m s}\int_0^p \langle m, (e^{-2\pi i m s}\sigma_0^{-m} - 1)\delta\rangle t^s\omega + P(e^{2\pi i s})\int_c^p \langle m, (\sigma_c - 1)\varepsilon\rangle t^s\omega\\ &= \int_0^p \langle m, P(\sigma_0^{-1})\delta\rangle - P(e^{2\pi i s})\int_0^c \langle m, \delta\rangle t^s\omega\\ &= -P(e^{2\pi i s})\int_0^c \langle m, \delta\rangle t^s\omega.\end{aligned}$$

Example 14 (polylogarithm). *For an integer $n \ge 1$, the nth polylogarithm is a multivalued holomorphic function, one of whose branches in the open unit circle $|t| < 1$ is given* *by the convergent series* $Li_n(t) = \sum_{k=1}^{\infty} k^{-n}t^k$. *One can easily see that this function is annihilated by the differential operator*

$$L = \left((1-t)t\frac{d}{dt} - 1\right)\left(t\frac{d}{dt}\right)^n.$$

The operator L *has regular singularities and the local system of its solutions on* $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ *is spanned by* $Li_n(t)$ *and* $\log^k(t)$ *for* $0 \leq k \leq n-1$. *The local monodromy at* $c = 1$ *satisfies Assumption 12 with*

$$\delta(t) := (\sigma_1 - 1)Li_n(t) = -2\pi i \frac{\log^{n-1}(t)}{(n-1)!},$$

see e.g. [5, Proposition 2.2]. *The annihilator of* $\delta(t)$ *in* $\mathbb{C}[\sigma_0^{\pm 1}]$ *is generated by* $(\sigma_0 - 1)^n$ *and hence the* $\mathbb{C}[e^{\pm 2\pi i s}]$*-module of gamma functions for the* n*th polylogarithm is generated by*

$$(1 - e^{2\pi i s})^n \int_0^1 \delta(t) t^{s-1} dt = 2\pi i \left(\frac{e^{2\pi i s} - 1}{s}\right)^n.$$

Note that for connections on $U = \mathbb{P}^1 \setminus \{0, c, \infty\}$ Proposition 10 applies globally, that is with $V = U$. It follows that there is a unique gamma function attached to every hypergeometric connection, of which Example 14 is a degenerate case. We will return to hypergeometric connections in Example 20.

As above, we consider gamma functions restricted to the neighbourhood $V \subset U_{an}$ of a path between $t = 0$ and $t = c$. Under Assumption 12 there is a unique generator of the module of gamma functions in Proposition 10. In what follows we will relate this gamma function to the monodromy of Frobenius deformations introduced in [4] by Golyshev and Zagier. We shall work under the following

Assumption 15. $t = 0$ *is a regular singular point for* M *and there is a unique eigenvector for the local monodromy operator* σ_0 *on* $\mathcal{M}^\vee$.

The unique eigenvalue of σ_0 *is denoted by* $\lambda \in \mathbb{C}^\times$. *We also denote* $K' = K(\lambda)$.

Remark 16. *One can check that if Assumptions 12 and 15 hold for* M *then they also hold for the dual connection* $M^\vee$. *The unique eigenvalue of* σ_0 *on* $M^\vee$ *is given by* λ^{-1}.

The following is a slight generalization of the object introduced in [4]:

Definition 17. *With the notation* $D = t\frac{d}{dt}$, *let* $L = \sum_{j=0}^{r} q_j(t) D^{r-j}$ *be an ordinary differential operator of order* r *which has a regular singularity at* $t = 0$ *with a unique local exponent equal to* ρ.

Assuming that $q_0(0) \neq 0$, *a Frobenius deformation for* L *at* $t = 0$ *is a formal series* $\Phi = \sum_{n \geq 0} \phi_n(t) s^n$ *whose coefficients* $\phi_n(t)$ *are analytic functions in a neighborhood of the base point* $p \neq 0$ *such that* Φ *satisfies the unhomogeneous differential equation in the variable* t

$$L\Phi = s^r t^{s+\rho} \tag{17}$$

and transforms under the local monodromy at $t = 0$ *as*

$$\sigma_0 \Phi = e^{2\pi i (s+\rho)} \Phi. \tag{18}$$

Existence of Frobenius deformations was shown in [4]. Later we shall also check their existence and show that, as soon as the branch of of t^s is fixed, the Frobenius deformation Φ is unique (Proposition 30 in Section 3.) Let us denote by $Sol(L)$ the local system of solutions to L and let $Sol_p(L)$ be the stalk of this local system at $t = p$, which is the r-dimensional $\mathbb{C}$-vector space of solutions near p. Note that the first r coefficients of the Frobenius deformation $\phi_0(t), \ldots, \phi_{r-1}(t)$ give a basis in $Sol_p(L)$ called *the Frobenius basis* in the classical theory of differential equations.

Proposition-Definition 18. *Suppose there is a non-zero solution $\psi \in Sol_p(L)$ such that $(\sigma_c - 1)Sol_p(L) = \mathbb{C}\psi$. The monodromy of a Frobenius deformation around $t = c$ is then given by*

$$(\sigma_c - 1)\Phi = \kappa(s)\,\psi \tag{19}$$

for some formal series $\kappa(s) = \sum_{n\ge 0} \kappa_n s^n \in \mathbb{C}[\![s]\!]$. We will call $\kappa(s)$ an Apéry series for L; its coefficients κ_n will be called Apéry constants.

We will prove this proposition in Section 3. The Apéry series is defined up to multiplication by an integral power of $e^{2\pi i s}$ (this ambiguity comes from the choice of branch of t^s in Definition 17) and a constant in $\mathbb{C}^\times$ (due to the choice of ψ.) Note that, since $\phi_0, \ldots, \phi_{r-1}$ is a basis in $Sol_p(L)$, at least one of the first N Apéry constants $\kappa_0, \ldots, \kappa_{r-1}$ is non-zero. We can now state our result. The equality in (20) below is to be understood modulo the ambiguity in the definition of $\kappa^\vee$ discussed above and a similar ambiguity in the definition of ξ_0.

Theorem 19. *Let M be a connection of rank r satisfying Assumptions 12 and 15. We choose a generator $m \in M$ and consider gamma functions for $m \otimes dt/t$ as in Definition 4'. Let $L = \sum_{j=0}^r q_j(t) D^{r-j}$ with $D = t\frac{d}{dt}$ and $q_0(0) \neq 0$ be a differential operator annihilating m and denote by $\rho \in \mathbb{C}$ the unique local exponent of L at $t = 0$ (it satisfies* $\exp(2\pi i \rho) = $ *λ). Let $V \subset U^{an}$ be a neighborhood of a path between $t = 0$ and $t = c$ as in Proposition 10. Then the $K'[e^{\pm 2\pi i s}]$-module of gamma functions*

$$\Gamma_\xi(s), \xi \in H_1(V, \mathcal{M}^\vee \otimes t^s) \otimes_K K'$$

has rank 1 and its generators satisfy $\Gamma_{\xi_0}(-\rho) \neq 0$. Moreover, we have

$$\Gamma_{\xi_0}(s - \rho) = \Big(\frac{e^{2\pi i s} - 1}{s}\Big)^{r-\nu}\ \frac{\kappa^\vee(s)}{s^\nu}, \tag{20}$$

where $\kappa^\vee(s)$ is an Apéry series for the adjoint *differential operator*

$$L^\vee = (-D)^r q_0(t) + (-D)^{r-1} q_1(t) + \ldots + q_r(t)$$

and $0 \le \nu < r$ is the order of vanishing of $\kappa^\vee(s)$ at $s = 0$.

We will prove this theorem in Section 3. One can easily check that $L^\vee$ has a unique local exponent at $t = 0$ which is equal to $-\rho$. In Section 3 we will show that $L^\vee$ corresponds to the dual connection $M^\vee$ (see also [7, §2.9]).

Let us finish this section with a few examples of Apéry series. In Example 7 the connection is self-dual and we can take the differential operator to be $L = D - t(D + 1/2)$.

For this operator the local exponent at $t = 0$ is $\rho = 0$, $L^\vee = -L$ and Theorem 19 implies that

$$\kappa(s) = s\frac{\Gamma(s)\Gamma(\frac{1}{2})}{\Gamma(s+\frac{1}{2})} = 2^{2s}\frac{\Gamma(1+s)^2}{\Gamma(1+2s)}$$
$$= \exp\Big((2\log 2)s + \sum_{j=2}^{\infty}\frac{\zeta(j)}{j}(2-2^j)(-s)^j\Big)$$

is an Apéry series for L.

Example 20 (hypergeometric connections)**.** *A hypergeometric connection of rank $N \geq 1$ is a connection on $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ such that its solutions are annihilated by the differential operator*

$$L = \prod_{j=1}^{r}(D+\beta_j) - t\prod_{j=1}^{r}(D+\alpha_j). \tag{21}$$

Here $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r \in \mathbb{C}$ are fixed parameters, and the global monodromy representation is irreducible if and only if $\alpha_i - \beta_j \notin \mathbb{Z}$ for any pair of indices i, j. Under this assumption the variation of the local monodromy around $t = 1$ has 1-dimensional image. By Proposition 10 there is a unique generator of the $\mathbb{C}[e^{\pm 2\pi i s}]$-module of gamma functions $\Gamma_\xi(s)$, $\xi \in H_1(U, Sol(L)\otimes t^s)$. It follows from the functional equation (Proposition 9) that every $\Gamma_\xi(s)$ is a multiple of $\prod_{j=1}^{r}\Gamma(\alpha_j - s)/\prod_{j=1}^{r}\Gamma(1+\beta_j - s)$ by a function invariant under the translation $s \mapsto s+1$.

Let us consider the case $\beta_1 = \ldots = \beta_r = 0$. The local monodromy around $t = 0$ is maximally unipotent and the unique local exponent is $\rho = 0$. One can check that the Frobenius deformation for L is given by $\Phi(s,t) = \sum_{k\geq 1} A_k(s)t^{k+s}$ with $A_k(s) = A_{k+s}/A_s$ and

$$A_x = \frac{\prod_{j=1}^{r}\Gamma(\alpha_j + x)}{\Gamma(1+x)^r}.$$

Note that $\Phi(0,t) = \sum_{k=0}^{\infty} A_k(0)t^k$ is the unique solution to L analytic at $t = 0$, and this solution is not analytic at $t = 1$. (Indeed, if there was a solution preserved by both σ_0 and σ_1, then the global monodromy representation would be reducible when $r > 1$ or trivial when $r = 1$.) It follows that Apéry series $\kappa(s)$ do not vanish at $s = 0$. Such a series normalized as $\kappa(0) = 1$ can be found from the condition that $\Phi(s,t) - \kappa(s)\Phi(0,t)$ is invariant under σ_1. Following the arguments in [4, §2.2] *we find that*

$$\kappa(s) = \lim_{k\to\infty}\frac{A_k(s)}{A_k(0)} = \lim_{k\to\infty}\frac{A_{k+s}A_0}{A_k A_s} = \frac{A_0}{A_s} = \frac{\Gamma(1+s)^r\prod_{j=1}^{r}\Gamma(\alpha_j)}{\prod_{j=1}^{r}\Gamma(\alpha_j+s)}.$$

One can now use Theorem 19 to obtain the unique generator of the $\mathbb{C}[e^{\pm 2\pi i s}]$-module of gamma functions attached to the hypergeometric differential operator (21) *with maximally unipotent monodromy around $t = 0$. In the case $\rho = 0$ (that is, all $\beta_j = 0$) we have*

$(-1)^r L^\vee = D^r - t\prod_{j=1}^r(D+1-\alpha_j)$ and the module of gamma functions is generated by

$$\Gamma_{\xi_0}(s) = \left(\frac{e^{2\pi is}-1}{2\pi is}\right)^r \kappa^\vee(s) = \left(\frac{e^{2\pi is}-1}{2\pi is}\right)^r \frac{\Gamma(1+s)^r\prod_{j=1}^r\Gamma(1-\alpha_j)}{\prod_{j=1}^r\Gamma(1-\alpha_j+s)}$$
$$= \frac{\prod_{j=1}^r(e^{2\pi i\alpha_j}-e^{2\pi is})\Gamma(\alpha_j-s)}{\Gamma(1-s)^r\prod_{j=1}^r(e^{2\pi i\alpha_j}-1)\Gamma(\alpha_j)}.$$

The reader may check this is an entire function satisfying the difference equation corresponding to L (see Proposition 9).

Example 21 (Apéry family)**.** *The equation $1 - tf(x_1,x_2,x_3) = 0$ with*

$$f(x) = \frac{(x_1-1)(x_2-1)(x_3-1)(1-x_1-x_2+x_1x_2-x_1x_2x_3)}{x_1x_2x_3}$$

defines a family X/U of K3 surfaces of Picard rank 19 over

$$U = \mathbb{P}^1_t \setminus \{0, 17\pm 12\sqrt{2}, \infty\}.$$

The variation $M = H^2(X/U)/NS$ is of rank 3 and there is a class of differential 2-forms $m \in M$ annihilated by the differential operator

$$L = D^3 - t(34D^3+51D^2+27D+5) + t^2(D+1)^3, \tag{22}$$

see [8]*. The local monodromy of L around $t=0$ is maximally unipotent with the local exponent $\rho = 0$. The closest singularity $c = 17-12\sqrt{2}$ is a conifold point. In* [4] *Golyshev and Zagier computed the Apéry constants along with the first higher one for the direct path joining $t=0$ and $t=c$:*

$$\kappa_0 = 1, \quad \kappa_1 = 0, \quad \kappa_2 = -\frac{\pi^2}{3} = -2\zeta(2), \quad \kappa_3 = \frac{17}{6}\zeta(3).$$

(In loc.cit. *the authors use the term* Frobenius limits *for κ_n.) Note that the objective of their paper is the computation of κ_3 for the 17 similar families of K3 surfaces, in order to verify the Gamma Conjecture in mirror symmetry. Golyshev and Zagier also evaluate a few more of the higher constants experimentally (see* [4, page 46]*):*

$$\begin{aligned}
\kappa_4 &= \frac{\pi^4}{45} = \frac{4}{5}\zeta(2)^2 = 2\zeta(4)\\
\kappa_5 &= \frac{7}{5}\zeta(5) - \frac{17}{3}\zeta(2)\zeta(3)\\
&\dots\\
\kappa_{11} &= \frac{2}{3}\zeta(3,5,3) + \text{ a $\mathbb{Q}$-linear combination of products of}\\
&\qquad\qquad\qquad\qquad \text{zeta values of total weight } 11
\end{aligned}$$

Remarkably, κ_{11} is the first one involving multiple zeta values along with ordinary zeta values. David Broadhurst was able to find similar experimental expressions in terms of MZVs for the Apéry constants κ_n in this example up to $n = 15$, [2].

The fact that κ_3 for the differential operator (22) is a rational multiple of $\zeta(3)$ is essentially related to the proof of irrationality of this number given in 1978 by Roger Apéry. The above example greatly motivated our study of the series $\kappa(s)$, that is why we named its coefficients *Apéry constants.*

Remark 22. *Let us remark that even if we start from a Gauss–Manin connection M with a conifold point, as in Examples 7 and 21, the definition of the higher Apéry constants (κ_n with $n \geq rank(M)$) involves extensions of M by powers of the Kummer connection. We don't know whether these extensions are geometric, and from this point of view the fact that the higher Apéry constants are periods is surprising. On the other hand, Theorem 19 implies that all Apéry constants are periods and gives an expression for them in terms of explicit iterated integrals.*

3. Computation of the gamma function associated to a conifold point

In this section we prove Theorem 19. We start with some preparations. Let M be an algebraic connection of rank r on an open subset $U \subseteq \mathbb{G}_m$ and $\mathcal{O} = \mathcal{O}_U$ be the ring of functions of U. The ring of differential operators $\mathcal{D} = \mathcal{D}_U$ is generated over $\mathcal{O}$ by the derivation $D = t\frac{d}{dt}$. We fix a generator $m \in M$ and a differential operator

$$L = q_0(t)D^r + q_1(t)D^{r-1} + \ldots + q_r(t), \quad q_i \in \mathcal{O}$$

such that $Lm = 0$. Possibly after passing to a smaller open set U, we assume that $q_0 \in \mathcal{O}^\times$ is a unit. It follows that $M \cong \mathcal{D}/\mathcal{D}L$ is a free $\mathcal{O}$-module of rank r with the basis $m, Dm, \ldots, D^{r-1}m$.

The dual connection on $M^\vee = Hom_{\mathcal{O}}(M, \mathcal{O})$ is determined by the identity

$$\langle \xi, D\eta\rangle + \langle D\xi, \eta\rangle = D\langle \xi, \eta\rangle, \qquad \xi \in M, \eta \in M^\vee. \tag{23}$$

Let $e_0, \ldots, e_{r-1} \in M^\vee$ be the basis dual to $m, Dm, \ldots, D^{r-1}m \in M$, that is we have $\langle D^j m, e_i\rangle = \delta_{i,j}$. The adjoint differential operator is defined as

$$L^\vee = (-D)^r q_0(t) + (-D)^{r-1} q_1(t) + \ldots + q_r(t).$$

In the next series of lemmas we check that $M^\vee \cong \mathcal{D}/\mathcal{D}L^\vee$ and give an explicit form of pairing on horizontal sections.

Lemma 23. *Let $u \in \mathcal{O}^\times$ be a unit. Then we have isomorphisms of left $\mathcal{D}$-modules*

$$\mathcal{D}/\mathcal{D}L \cong \mathcal{D}/\mathcal{D}uL \cong \mathcal{D}/\mathcal{D}Lu.$$

Proof. The two left ideals $\mathcal{D}uL$ and $\mathcal{D}L$ coincide, so the lefthand identity is clear. On the other hand, right multiplication by u is an isomorphism of left $\mathcal{D}$-modules $\mathcal{D}/\mathcal{D}L \cong \mathcal{D}/\mathcal{D}Lu$. □

Scholie 24. *In our case, $q_0(t) \in \mathcal{O}^\times$ is a unit. Using the lemma, the left $\mathcal{D}$-module structure on $\mathcal{D}/\mathcal{D}L$ and $\mathcal{D}/\mathcal{D}L^\vee$ is reduced to the case $q_0 = 1$. We will assume this throughout.*

It will be convenient to define

$$\bar{e}_i := e_{r-1-i}; \quad 0 \leq i \leq r-1.$$

Using (23) one can easily check that the $\mathcal{D}$-module structure on $M^\vee$ is given by

$$D\bar{e}_i + \bar{e}_{i+1} = q_{i+1}\bar{e}_0,\ i \neq r-1; \quad D\bar{e}_{r-1} = q_r\bar{e}_0. \tag{24}$$

Lemma 25. *We have $L^\vee\bar{e}_0 = 0$. Thus, the map $1 \mapsto \bar{e}_0$ yields an isomorphism $\mathcal{D}/\mathcal{D}L^\vee \cong M^\vee$.*

Proof. We compute

$$L^\vee\bar{e}_0 = \sum_{i=0}^{r}(-D)^i q_{r-i}\bar{e}_0 = \sum_{i=0}^{r-1}(-D)^i(D\bar{e}_{r-i-1} + \bar{e}_{r-i}) + (-D)^r\bar{e}_0 = 0.$$

□

Lemma 26. *Recall $\{e_i\} \subset M^\vee$ is the dual basis to $\{D^i m\} \subset M$. Let $\{\rho_i\} \subset M$ be dual to $\{D^i\bar{e}_0\} \subset M^\vee$. With this notation, define $\eta : \mathcal{O} \to M^\vee$ (resp. $\xi : \mathcal{O} \to M$) by*

$$\eta(\phi) = \sum_{i=0}^{r-1}(D^i\phi)e_i; \quad \xi(\psi) = \sum_{i=0}^{r-1}(D^i\psi)\rho_i. \tag{25}$$

(We can also write

$$\begin{aligned} &\eta(\phi) : M \to \mathcal{O}; \quad \eta(\phi)(D^i m) = D^i\phi, \quad 0 \le i \le r-1 \\ &\xi(\psi) : M^\vee \to \mathcal{O}; \quad \xi(\psi)(D^i\bar{e}_0) = D^i\psi, \quad 0 \le i \le r-1.) \end{aligned} \tag{26}$$

We define a bracket $\{*,*\} : \mathcal{O} \otimes_{\mathbb{C}} \mathcal{O} \to \mathcal{O}$ by composing the tensor product

$$\xi \otimes \eta : \mathcal{O} \otimes_{\mathbb{C}} \mathcal{O} \to M \otimes_{\mathcal{O}} M^\vee;$$
$$(\xi \otimes \eta)(\psi \otimes \phi) = \sum_{j,i}(D^j\psi)(D^i\phi)\rho_j \otimes e_i$$

with the duality map $\langle *,* \rangle : M \otimes_{\mathcal{O}} M^\vee \to \mathcal{O}$:

$$\{\psi,\phi\} = \sum_{j,i}(D^j\psi)(D^i\phi)\langle\rho_j, e_i\rangle = \sum_{j,k}(D^j\psi)(D^{N-k-1}\phi)\langle\rho_j,\bar{e}_k\rangle. \tag{27}$$

Then

$$\{\psi,\phi\} = \sum_{\substack{h,\nu,i \ge 0 \\ h+\nu+i = r-1}} (-D)^h(q_\nu\psi)D^i\phi. \tag{28}$$

.

Proof. As a consequence of (24) we get

$$\bar{e}_k = (-D)^k\bar{e}_0 + (-D)^{k-1}(q_1\bar{e}_0) + \cdots - D(q_{k-1}\bar{e}_0) + q_k\bar{e}_0.$$

It follows that

$$\langle\rho_j,\bar{e}_k\rangle = \begin{cases} 0 & j > k \\ (-1)^k & j = k \\ \sum_{\nu=1}^{k-j}(-1)^{k-\nu}\binom{k-\nu}{j}D^{k-\nu-j}q_\nu & j < k. \end{cases} \tag{29}$$

What we must show, then, is that the coefficient of $D^j(\psi)D^{r-k-1}(\phi)$ when we expand the right-hand side of (28) coincides with (29). Expanding (28), the coefficient at $D^j(\psi)D^i(\phi)$ is

$$\sum_{h+\nu+i=r-1}(-1)^h\binom{h}{j}D^{h-j}q_\nu,$$

which is equal to (29) when $i = r-1-k$. $\square$

Lemma 27. *With notation as above, we have*

$$D\left(\eta(\phi)\right) = (L\phi)\bar{e_0};\quad D\left(\xi(\psi)\right) = (L^\vee\psi)m.$$

Proof. This follows directly from (24) and (25). Note that replacing $\bar{e}_i$ in (24) by e_{r-1-i} yields the identity $De_j + e_{j-1} = q_{r-j}e_{r-1}$ with the convention that $e_{-1} = 0$. $\square$

Lemma 28. *With notation as above, we have*

$$\langle\xi(\psi), D\left(\eta(\phi)\right)\rangle = \psi(L\phi);\quad \langle D\left(\xi(\psi)\right), \eta(\phi)\rangle = (L^\vee\psi)\phi.$$

Proof. For example, we have from (29)

$$\langle\xi(\psi), D\left(\eta(\phi)\right)\rangle = (L\phi)\langle\sum_{i=0}^{r-1}(D^i\psi)\rho_i, \bar{e}_0\rangle = L(\phi)\psi.$$

The other identity is proved similarly. $\square$

The above lemmas give an identification of $\mathcal{M}^\vee = M_{an}^{\vee,\nabla^\vee=0}$ and $\mathcal{M} = M_{an}^{\nabla=0}$ (after $\otimes_K\mathbb{C}$) with the local systems of solutions $Sol(L)$ and $Sol(L^\vee)$ respectively. Moreover, the bracket

$$\begin{aligned}\{*,*\} &: \mathcal{O}^{an}\otimes_{\mathbb{C}}\mathcal{O}^{an}\to\mathcal{O}^{an}\\ \{\psi,\phi\} &= \sum_{\substack{h,\nu,i\geq 0\\ h+\nu+i=r-1}}(-D)^h(q_\nu\psi)D^i\phi\end{aligned}\tag{30}$$

satisfies

$$D\{\psi,\phi\} = \psi(L\phi) + (L^\vee\psi)\phi\,.\tag{31}$$

In particular, this bracket is constant on $Sol(L^\vee)\otimes_{\mathbb{C}} Sol(L)$ and coincides with the duality pairing $\langle *,*\rangle$ on $\mathcal{M}\otimes_K\mathcal{M}^\vee$. In the next lemma we use (31) to perform integration by parts, which shows the basic relation between gamma functions and Frobenius deformations.

Lemma 29. *Let $\xi\in H_1(U,\mathcal{M}^\vee\otimes t^s)$. We identify this homology group with $H_1(\pi_1(U,p),\mathcal{M}_p^\vee\otimes_K K[e^{\pm 2\pi is}])$ and choose a 1-cycle presentation*

$$\xi\sim\sum_j\sigma_j\otimes\varepsilon_j\otimes e^{2\pi isn_j}$$

with $\sigma_j\in\pi_1(U,p), \varepsilon_j\in\mathcal{M}_p^\vee, n_j\in\mathbb{Z}$. Let $\Phi = \sum_{n\geq 0}\phi_n(t)s^n\in\mathcal{O}^{an}[\![s]\!]$ be a solution to

$$L^\vee\Phi = s^rt^{s+\alpha},\qquad \alpha\in\mathbb{C}$$

near $t = p$. Then

$$s^N \Gamma_\xi(s+\alpha) = \sum_j e^{2\pi i(s+\alpha)n_j} (\sigma_j^{-1} - 1)\{\Phi, \psi_j\},$$

where the gamma function is (5) with $\omega = dt/t$ and $m \in M$ satisfying $Lm = 0$, $\{*,*\}$ is the bracket (30) and $\psi_j = \langle m, \varepsilon_j \rangle \in Sol_p(L)$ for each j.

Proof. Using (5) and (31) we have

$$\begin{aligned} s^r \Gamma_\xi(s+\alpha) &= s^r \sum_j e^{2\pi i(s+\alpha)n_j} \int_{\sigma_j^{-1}} t^{s+\alpha-1} \psi_j(t) dt = \sum_j e^{2\pi i(s+\alpha)n_j} \int_{\sigma_j^{-1}} (L^\vee \Phi) \psi_j \frac{dt}{t} \\ &= \sum_j e^{2\pi i(s+\alpha)n_j} \int_{\sigma_j^{-1}} D\{\Phi, \psi_j\} \frac{dt}{t} = \sum_j e^{2\pi i(s+\alpha)n_j} (\sigma_j^{-1} - 1)\{\Phi, \psi_j\}. \end{aligned}$$

□

To prove Theorem 19 we shall use Lemma 29 for Φ being a Frobenius deformation for $L^\vee$ (see Definition 17). Let us now demonstrate existence and uniqueness of Frobenius deformations and Apéry series.

Proposition 30. *Under the assumptions of Definition 17, there exists a unique Frobenius deformation $\Phi = \sum_{n \ge 0} \phi_n(t) s^n$. It can be written as $\Phi = t^{s+\rho} \Phi^{an}$, where $\Phi^{an} = \sum_{n \ge 0} \phi_n^{an}(t) s^n$ is a formal series whose coefficients $\phi_n^{an}(t)$ are uniquely determined analytic at $t = 0$ functions taking values*

$$\phi_0^{an}(0) = q_0(0)^{-1}, \qquad \phi_n^{an}(0) = 0 \text{ for all } n > 0.$$

Proof. It is sufficient to give a proof in the case $\rho = 0$. Indeed, Φ is a Frobenius deformation for L if and only if $t^{-\rho}\Phi$ is a Frobenius deformation for $L' = t^{-\rho} L t^\rho = \sum_j q_j(t)(D+\rho)^{r-j}$, which is a differential operator with the local exponent 0 at $t = 0$.

Case $\rho = 0$. Let $L = \sum_{j=0}^r q_{r-j}(t) D^j$ be a differential operator with $q_0(0) \neq 0$ and $q_j(0) = 0$ for $j > 0$. Condition (18) is equivalent to the series $\Phi^{an} := t^{-s}\Phi = \sum_{n \ge 0} \phi_n^{an}(t) s^n$ being σ_0-invariant. Therefore we shall look for sequences of meromorphic at $t = 0$ functions $\{\phi_n^{an}(t)\}$ such that the unhomogeneous differential equation (17) is satisfied. One can write

$$L\Phi = t^s \sum_{j=0}^r \frac{s^j}{j!} L^{(j)} \Phi^{an} \text{ with } L^{(j)} := \frac{\partial^j L}{\partial D^j}.$$

Therefore Φ is a Frobenius deformation if and only if $\sum_{j=0}^r \dfrac{s^j}{j!} L^{(j)} \Phi^{an} = s^r$, which is equivalent to the infinite system of equations

$$L\phi_n^{an} + L^{(1)}\phi_{n-1}^{an} + \frac{1}{2!} L^{(2)} \phi_{n-2}^{an} + \ldots + \frac{1}{r!} L^{(r)} \phi_{n-r}^{an} = \delta_{n,r} \tag{32}$$

for all $n \ge 0$. We will first show that there is a unique solution to the system (32) in formal Laurent series $\phi_n^{an} \in \mathbb{C}[\![t]\!][t^{-1}]$, $n \ge 0$.

Let us write $L = \sum_{i\geq 0} t^i p_i(D)$ with $p_i \in \mathbb{C}[D]$ of degree at most r and $p_0(D) = \alpha D^r$, where $\alpha = q_0(0) \neq 0$. Equation $L(\sum a_n t^n) = \sum b_n t^n$ is then equivalent to the recurrence relation

$$\alpha n^r a_n + p_1(n-1)a_{n-1} + p_2(n-2)a_{n-2} + \ldots = b_n. \tag{33}$$

One can easily see that there is a unique (up to multiplication by a constant in $\mathbb{C}^\times$) non-zero Laurent series solution $\phi_0^{an}(t) = \sum_n a_n t^n$ to $L\phi_0^{an} = 0$, and this solution has $\phi_0^{an}(0) \neq 0$. Secondly, we observe that $L(\mathbb{C}[\![t]\!]) \subset t\mathbb{C}[\![t]\!]$ and, more generally, we have

$$L^{(j)}(\mathbb{C}[\![t]\!]) \subset t\mathbb{C}[\![t]\!], \qquad 0 \leq j < r.$$

The third observation we can make from formula (33) is that the map

$$L : t\mathbb{C}[\![t]\!] \to t\mathbb{C}[\![t]\!]$$

is invertible. With these three observations we can now solve (32) as follows. We start with $n = 0$ and make an arbitrary choice in the normalization of ϕ_0^{an}. For $1 \leq n < r$ equation (32) has shape $L\phi_n^{an} = b$ with $b \in t\mathbb{C}[\![t]\!]$, which has a unique solution in $t\mathbb{C}[\![t]\!]$ to which we can add an arbitrary constant multiple of $\phi_0^{an}(t)$. So far we make an arbitrary choice of $\phi_n^{an}(0)$ for each $0 \leq n < r$. When $n = r$ we have $\frac{1}{r!}L^{(r)}\phi_0^{an} = q_0(t)\phi_0^{an}(t) \in q_0(0)\phi_0^{an}(0) + t\mathbb{C}[\![t]\!]$, and hence (32) can be solved for $\phi_r^{an} \in \mathbb{C}[\![t]\!]$ if and only if $\phi_0^{an}(0) = q_0(0)^{-1}$. Again, there is a unique solution in $t\mathbb{C}[\![t]\!]$ to which one can add an arbitrary constant multiple of $\phi_0^{an}(t)$. We make an arbitrary choice for $\phi_r^{an}(0)$ and continue. For $n > r$ we have $\frac{1}{r!}L^{(r)}\phi_{n-r}^{an} = q_0(t)\phi_{n-r}^{an}(t) \in q_0(0)\phi_{n-r}^{an}(0) + t\mathbb{C}[\![t]\!]$ and (32) can be solved for $\phi_n^{an} \in \mathbb{C}[\![t]\!]$ if and only if $\phi_{n-r}^{an}(0) = 0$. By induction in n we conclude that there is a unique sequence of formal Laurent series $\{\phi_n^{an}, n \geq 0\}$ satisfying the system of equations (32) and these series satisfy

$$\phi_0^{an} \in q_0(0)^{-1} + t\mathbb{C}[\![t]\!], \quad \phi_n^{an} \in t\mathbb{C}[\![t]\!] \text{ for } n \geq 1.$$

Finally, we observe that for $k \geq 0$ series $\phi_n(t) = \sum_{j=0}^{n} \frac{\log(t)^j}{j!}\phi_{n-j}^{an}(t) \in \mathbb{C}[\![t]\!][\log t]$ with $0 \leq n \leq N-1+k$ are solutions to the differential equation $D^k L\phi_n = 0$. It then follows from the classical Cauchy theorem in the theory of ordinary differential operators that each $\phi_n^{an}(t)$ is an actual analytic function (i.e. convergent series) in a neighborhood of $t = 0$. □

Proof of Proposition 18. Coefficients of the Frobenius deformation $\Phi = \sum_n \phi_n(t)s^n$ satisfy differential equations $(D-\rho)^k L\phi_n = 0$ when $n < r + k$. Therefore it is enough to show that if Assumption 12 holds for $M = \mathcal{D}/\mathcal{D}L$, then it also holds for $M' = \mathcal{D}/\mathcal{D}(D+\alpha)L$ with any $\alpha \in \mathbb{C}$. Indeed, iterating this statement with $\alpha = -\rho$ we obtain that $(\sigma_c - 1)Sol_p((D-\rho)^k L) = \mathbb{C}\psi$ for all $k \geq 0$.

Suppose that Assumption 12 holds for $M = \mathcal{D}/\mathcal{D}L$. Then it also holds for $M^\vee \cong \mathcal{D}/\mathcal{D}L^\vee$ (see Remark 16). If we show that the same assumption holds for $M'' = \mathcal{D}/\mathcal{D}L^\vee(D-\alpha)$, then it also holds for $(M'')^\vee \cong \mathcal{D}/\mathcal{D}(D+\alpha)L = M'$. To show that Assumption 12 holds for M'', let $\phi \in Ker(\sigma_c - 1)$ be a solution of $L^\vee$. Then any ϕ' such that $(D-\alpha)\phi' = \phi$ can be recovered (up to adding a constant multiple of t^α) as $\phi' = t^\alpha \int t^{-\alpha-1}\phi(t)dt$. Since ϕ is an analytic function at $t = c$, we see that ϕ' will be also analytic at $t = c$. It follows that the space of invariant under σ_c solutions of $L^\vee(D-\alpha)$

is at least one dimension bigger than the same space for $L^\vee$, and therefore the image of $\sigma_c - 1$ stays of dimension 1. □

Proof of Theorem 19. Recall that for an order r differential operator $L = \sum_{j=0}^r q_j(t)D^{r-j}$ with a regular singularity at $t = 0$ the respective local exponents are solutions to the indicial equation, which is the algebraic equation $\sum_{j=0}^r (q_j/q_0)(0)X^{r-j} = 0$. Since m generates M as a $\mathcal{D}$-module, Assumption 15 implies that the indicial equation for L reads as $(X - \rho)^r = 0$ for a unique $\rho \in \mathbb{C}$ satisfying $\exp(2\pi i\rho) = \lambda$. It is then clear that the indicial equation for the adjoint operator $L^\vee$ is given by $(X + \rho)^r = 0$. By Proposition 30 there exists a Frobenius deformation at $t = 0$ for $L^\vee$. Let $\Phi = \sum_n \phi_n(t)s^n$ be such Frobenius deformation, where $\phi_n(t)$ are analytic functions near our base point p. This series satisfies

$$L^\vee\Phi = s^r\, t^{s-\rho} \tag{34}$$

and

$$\sigma_0\Phi = e^{2\pi i(s-\rho)}\Phi. \tag{35}$$

Possibly passing to a smaller open subset of our curve, we can assume that $q_0 \in \mathcal{O}^\times$ and hence $M \cong \mathcal{D}/\mathcal{D}L$. Assumption 12 for M implies the same assumption for $M^\vee$, which by Lemma 25 is isomorphic to $\mathcal{D}/\mathcal{D}L^\vee$. Therefore the conditions of Proposition-Definition 18 are satisfied and we have

$$(\sigma_c - 1)\Phi = \kappa^\vee(s)\,\psi, \tag{36}$$

where $\psi \in Sol_p(L^\vee)$ is a solution spanning the one dimensional subspace $(\sigma_c - 1)Sol_p(L^\vee)$.

Consider homology with coefficients in the local system $\mathcal{L} = \mathcal{M}^\vee \otimes t^s$. It is clear from the proof of Proposition 10 that gamma functions $\Gamma_\xi(s)$ vanish when $\xi \in H_1(V, \mathcal{L})$ comes from $H_1(V_c, \mathcal{L})$. By Lemma 13, the quotient of $H_1(V, \mathcal{L})$ by the image of the corestriction map is a $K[e^{\pm 2\pi i s}]$-module of rank 1 isomorphic to the $K[\sigma_0^{\pm 1}]$-module of relations $Ann_{K[\sigma_0^{\pm 1}]}(\delta)$. For a polynomial $P(T) = \sum_m \lambda_m T^m$ such that $P(\sigma_0^{-1})\delta = 0$ the respective homology class is represented by the 1-cycle (13). Let us evaluate $\Gamma_\xi(s - \rho)$ using Lemma 29. We denote $\psi_\delta = \langle m, \delta\rangle, \psi_\varepsilon = \langle m, \varepsilon\rangle \in Sol_p(L)$ and use properties (35), (36) of the Frobenius deformation Φ:

$$\begin{aligned}
s^r\Gamma_\xi(s-\rho) &= \sum_m \lambda_m e^{2\pi i m(s-\rho)}(\sigma_0^{-m} - 1)\{\Phi, \psi_\delta\} + P(e^{2\pi i(s-\rho)})(\sigma_c - 1)\{\Phi, \psi_\varepsilon\} \\
&= \sum_m \lambda_m e^{2\pi i m(s-\rho)}\big(\{e^{-2\pi i m(s-\rho)}\Phi, \sigma_0^{-m}\psi_\delta\} - \{\Phi, \psi_\delta\}\big) \\
&\qquad + P(e^{2\pi i(s-\rho)})\big(\{\sigma_c\Phi, \sigma_c\psi_\varepsilon\} - \{\Phi, \psi_\varepsilon\}\big) \\
&= \{\Phi, P(\sigma_0^{-1})\psi_\delta\} - P(e^{2\pi i(s-\rho)})\{\Phi, \psi_\delta\} \\
&\qquad + P(e^{2\pi i(s-\rho)})\big(\{\Phi + \kappa^\vee(s)\psi, \psi_\varepsilon + \psi_\delta\} - \{\Phi, \psi_\varepsilon\}\big) \\
&= P(e^{2\pi i(s-\rho)})\kappa^\vee(s)\{\psi, \psi_\varepsilon + \psi_\delta\}.
\end{aligned}$$

Note that for $\phi \in Sol_p(L)$ one has $\{\psi, \phi\} = 0$ if and only if ϕ is σ_c-invariant. One can easily check that

$$(\sigma_c - 1)(\varepsilon + \delta) = \delta + (\sigma_c - 1)\delta = \sigma_c\delta \neq 0,$$

and hence $\{\psi, \psi_\varepsilon + \psi_\delta\} \in \mathbb{C}^\times$. We can now renormalize ψ so that $\{\psi, \psi_\varepsilon + \psi_\delta\} = 1$.

Now let us extend coefficients to $K' = K(\lambda)$ and take the polynomial P to be of the minimal possible degree. Denote this minimal degree by $d > 0$. Since $(\sigma_0 - \lambda)^r$ vanishes on $Sol_p(L)$, a minimal polynomial $P(T)$ should divide $(\lambda T - 1)^r$. Therefore $d \le r$ and with the 1-cycle ξ given by (13) with $P(T) = (\lambda T - 1)^d$ the respective Γ_ξ will be a generator of the module of gamma functions over $K'[e^{\pm 2\pi i s}]$. We then have $P(e^{2\pi i(s-\rho)}) = (e^{2\pi i s} - 1)^d$ and

$$s^r\, \Gamma_{\xi_0}(s - \rho) = (e^{2\pi i s} - 1)^d \kappa^\vee(s). \tag{37}$$

It remains to show that $\Gamma_{\xi_0}(-\rho) \neq 0$ or, equivalently, that the order of vanishing of $\kappa^\vee(s)$ at $s = 0$ is given by $\nu = r - d$. Since the gamma function is holomorphic at $-\rho$, formula (37) shows that $d + \nu \ge r$. On the other hand, by the definition of ν we have that $\phi_0, \dots, \phi_{\nu-1} \in Sol_p(L^\vee)$ are σ_c-invariant and therefore $\{\phi_j, \psi_\delta\} = 0$ for $0 \le j \le \nu - 1$. Since σ_0 preserves the span $A = \sum_{j=0}^{\nu-1} \mathbb{C}\phi_j \subset Sol_p(L^\vee)$, it follows that A is orthogonal to the d-dimensional space $\mathbb{C}[\sigma_0]\psi_\delta \subset Sol_p(L)$. Therefore we have $d + \nu \le r$. The two estimates that we have imply $d + \nu = r$, which completes the proof of the theorem. □

4. The limiting MHS and Apéry constants

We are given a Zariski open $U \subset \mathbb{G}_m \subset \mathbb{P}^1_t$ and a regular singular point connection $\nabla : M \to M \otimes \Omega^1_U$ defined over a subfield $K \subseteq \mathbb{C}$. We assume that (M, ∇) is a direct summand in the Gauß–Manin connection on $H^w_{DR}(X/U)$ for some projective, smooth $X \to U$. In this case M carries a variation of pure Hodge structures of weight w.

We further assume that M satisfies Assumptions 12 and 15 and there is a fixed element in the lowest Hodge submodule $m \in F^w M$ and a differential operator $L = D^r + \sum_{j=1}^r q_j(t) D^{r-j}$ with $D = t\frac{d}{dt}$ and some $q_j \in \mathcal{O}_U$ such that $Lm = 0$ and $M \cong \oplus_{j=0}^{r-1} \mathcal{O}_U\, D^j m$. Let $\kappa(s) = \sum_{k \ge 0} \kappa_k s^k$ be an Apéry series for the differential operator L (as in Definition 18) defined using a Betti-rational solution $\psi = \langle m, \delta\rangle$, $\delta \in \mathcal{M}^\vee(\mathbb{Q})$. In this section we will show that certain expressions in the Apéry constants $\kappa_0, \dots, \kappa_{r-1}$ and $2\pi i$ are periods of the limiting mixed Hodge structure at $0 \in \mathbb{P}^1 \setminus U$. In order to include the higher Apéry constants $\kappa_r, \kappa_{r+1}, \dots$ in the picture, we start with building a variation of mixed Hodge structure on extensions of M by powers of the Kummer connection.

We are interested in the analytic structure of M in a punctured disk Δ^* about $0 \in \mathbb{P}^1 - U$. The following is classic.

Theorem 31. *The evident functor is an equivalence of categories between the category of analytic connections on Δ^* meromorphic at 0 and having at worst regular singular points there and the category of all analytic connections on Δ^*.*

Proof. See for example [10], théorème (1.1), p. 24. □

Of course, the category of analytic connections is equivalent to the category of local systems, so, for example, M_{Δ^*} is determined by its monodromy at 0. Our hypotheses will be as in Assumption 15. To simplify the discussion, we will further suppose that the monodromy σ_0 is maximally unipotent (that is, $\lambda = 1$ in Assumption 15), so for a suitable basis the monodromy for the fibre at a base point p for the local system of solutions $\mathcal{M}_p^\vee$ is given by $\sigma_0 = \exp(N)$ where N is the nilpotent matrix of size r given by

$$\begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ & \dots & & \end{pmatrix}. \tag{38}$$

We write $\mathcal{O}$ for the ring of analytic functions on Δ^* which are meromorphic at 0. For an integer $n \geq 0$, define a free $\mathcal{O}$-module with basis $\{e_i\}$

$$\mathbb{E}_n := \mathcal{O}e_{-n-1} \oplus \mathcal{O}e_{-n} \oplus \cdots \mathcal{O}e_0 \oplus \mathcal{O}e_1 \oplus \cdots \mathcal{O}e_{r-1}.$$

We define a connection on $\mathbb{E}_n$

$$\nabla e_i := e_{i-1}\frac{dt}{t}; \quad \nabla(e_{-n-1}) = 0.$$

Let $\mathbb{F}_n \subset \mathbb{E}_n$ be the submodule spanned by $e_{-1}, \dots, e_{-n-1}$. The connection restricts to a connection on $\mathbb{F}_n$, so there is an induced connection on $\mathbb{E}_n/\mathbb{F}_n$. (Note $\mathbb{F}_0 = \mathcal{O}e_{-1}$ with trivial connection.)

Definition 32. *The Kummer connection $K_t = \mathcal{O}\varepsilon \oplus \mathcal{O}\eta$ is the rank 2 connection with $\nabla(\varepsilon) = \eta\frac{dt}{t}$ and $\nabla(\eta) = 0$.*

Lemma 33. *We have an isomorphism of connections*

$$\mathbb{F}_n \cong Sym^n K_t.$$

(Here $Sym^0 K_t = \mathcal{O}$ with the trivial connection.)

Proof. The identification is given by

$$e_{-1} = \varepsilon^n/n!,\ e_{-2} = \varepsilon^{n-1}\eta/(n-1)!, \dots, e_{-n-1} = \eta^n.$$

□

The dual connection on the free $\mathcal{O}$-module $\mathbb{E}_n^\vee$ with basis $e_i^\vee,\ -n-1 \leq i \leq r-1$ is given by

$$\nabla^\vee e_i^\vee = -e_{i+1}^\vee \frac{dt}{t}.$$

Consider horizontal sections of $\mathbb{E}_n^\vee$, (that is, solutions for $\mathbb{E}_n$) given by

$$\varepsilon_k(t) = \frac{\log(t)^k}{k!}e_{r-1}^\vee + \frac{\log(t)^{k-1}}{(k-1)!}e_{r-2}^\vee + \cdots + \log(t)e_{r-k}^\vee + e_{r-k-1}^\vee \tag{39}$$

for $k = 0, \dots, n+r$. Note that in the basis $\rho_k := (2\pi i)^{-k}\varepsilon_k$ operator $N = \log(\sigma_0)$ acts by the nilpotent matrix (38) of size $n+r+1$.

Lemma 34. *Under Assumption 15 with $\lambda = 1$, we get an exact sequence of meromorphic connections with regular singular points at the origin*

$$0 \to Sym^n K_t \to \mathbb{E}_n \to M_{\Delta^*} \to 0.$$

Proof. The assumption guarantees that the monodromy for $\mathbb{E}_n/\mathbb{F}_n$ coincides with that of M_{Δ^*}, so one can apply Theorem 31. □

We now proceed to define a variation of Hodge structure on $\mathbb{E}_n$. Using the ring of differential operators $\mathcal{D} = \mathcal{O}_U[D]$ we have Assume now that M is a pure variation of Hodge structure on U, with Hodge filtration $0 \subset F^pM \subset F^{p-1}M \subset \cdots \subset F^0M = M$. We give ourselves $m \in F^pM$ satisfying a differential equation $Lm = 0$ with $L \in \mathcal{D} = \mathbb{C}[t, t^{-1}, D]$ and $D = td/dt$. We assume further that

$$M \cong \mathcal{D}/\mathcal{D}L; \tag{40}$$

the isomorphism being such that $m \mapsto 1$. Multiplying L on the left by a power of the unit t (see Lemma 23), we can assume that the unique local exponent of L at $t = 0$ is $\rho = 0$. In this case the monodromy of solutions of the differential operator $D^{n+1}L$ around $t = 0$ is again maximally unipotent (this follows from Proposition 30) and hence it is the same as the monodromy of the local system of horizontal sections of $\mathcal{E}_n^\vee = \mathbb{E}_n^{\vee,\nabla^\vee=0}$. We can again apply Theorem 31 and build a commutative diagram of connections

$$\begin{array}{ccccccccc} 0 & \to & (\mathcal{D}/\mathcal{D}D^{n+1})_{\Delta^*} & \xrightarrow[R_L]{} & (\mathcal{D}/\mathcal{D}D^{n+1}L)_{\Delta^*} & \to & (\mathcal{D}/\mathcal{D}L)_{\Delta^*} & \to & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \to & \mathrm{Sym}^n K_t & \to & \mathbb{E}_n & \to & M_{\Delta^*} & \to & 0. \end{array} \tag{41}$$

(Here R_L means right multiplication by L.) The classical Frobenius solutions for $D^{n+1}L$ are given by

$$\phi_k = \sum_{j=0}^{k} \frac{\log(t)^j}{j!} \phi_{k-j}^{an}(t), \quad 0 \le k \le n + r,$$

where $\phi_j^{an} \in \mathcal{O}$ are uniquely defined analytic functions satisfying the condition

$$\phi_0^{an}(0) = 1; \quad \phi_j^{an}(0) = 0,\ j > 0. \tag{42}$$

These solutions were constructed in the proof of Proposition 30; they satisfy

$$\begin{aligned} &L(\phi_0) = \cdots = L(\phi_{r-1}) = 0, \\ &L\phi_{r+j} = \frac{\log(t)^j}{j!}; \quad 0 \le j \le n. \end{aligned} \tag{43}$$

Consider the element

$$\phi^{an} := \sum_{j=0}^{n+r} \phi_j^{an}(t) e_{r-1-j} \in \mathbb{E}_n. \tag{44}$$

Applying the solutions ε_k defined in (39), we observe that

$$\phi_k = \langle \phi^{an}, \varepsilon_k \rangle \tag{45}$$

for each $0 \le k \le n + r$. It then follows that $\phi^{an} \in \mathbb{E}_n$ satisfies $D^{n+1}L\phi^{an} = 0$. In particular, one can assume that the vertical isomorphism in the diagram (41) is given by $1 \in \mathcal{D} \mapsto \phi^{an} \in \mathbb{E}_n$ and that in the bottom row ϕ^{an} maps to the element $m \in M_{\Delta^*} \cong \mathbb{E}_n/\mathrm{Sym}^n K_t$.

We can use ϕ^{an} to define a variation of mixed Hodge structure on $\mathbb{E}_n$ as follows. Since $m, Dm, D^2m, \ldots, D^{r-1}m$ form an $\mathcal{O}$-basis for M_{Δ^*}, mapping $D^im \mapsto D^i\phi^{an}$ defines an

$\mathcal{O}$-module splitting $s : M_{\Delta^*} \to \mathbb{E}_n$, i.e.

$$\mathbb{E}_n \cong \mathrm{Sym}^n K_t \oplus s(M_{\Delta^*}). \tag{46}$$

One knows that K_t is the connection underlying a variation of mixed Tate Hodge structure (Kummer variation). Its nth symmetric power is the variation on $\mathrm{Sym}^n K_t$ where the Hodge filtration is given by

$$F^0 \subset F^{-1} \subset \ldots \subset F^{-n} = \mathrm{Sym}^n K_t, \quad F^{-k} = \oplus_{j=0}^{k} \mathcal{O}\varepsilon^{n-j}\eta^j$$

and the weight filtration is given by

$$\begin{aligned} &W_{-2n} = W_{1-2n} \subset W_{2-2n} = W_{3-2n} \subset \ldots \subset W_0 = \mathrm{Sym}^n K_t, \\ &W_{-2k} = \oplus_{j=k}^{n} \mathcal{O}\varepsilon^{n-j}\eta^j. \end{aligned}$$

The Hodge filtration is opposite to the weight filtration in the sense that $F^{-k} \oplus W_{-2(k+1)} = \mathrm{Sym}^n K_t$, from which one can easily compute that $gr^W_{-2k}\mathrm{Sym}^n K_t \cong \mathbb{Q}(k)$. The corresponding weight-graded Hodge variation is thus given by $\mathbb{Q}(n) \oplus \ldots \oplus \mathbb{Q}(1) \oplus \mathbb{Q}(0)$. For our purposes we will need the twist $(\mathrm{Sym}^n K_t)(1)$, which is the variation on the same connection where the respective filtrations are shifted as $F^i((\mathrm{Sym}^n K_t)(1)) = F^{i+1}(\mathrm{Sym}^n K_t)$ and $W_i(\mathrm{Sym}^n K_t)(1)) = W_{i+2}(\mathrm{Sym}^n K_t)$.

We use the splitting (46) to define a Hodge filtration on $\mathbb{E}_n$ as follows

$$\begin{aligned} F^i\mathbb{E}_n &= 0 \oplus s(F^i M_{\Delta^*}); \quad i \geq 0 \\ F^i\mathbb{E}_n &= F^i((\mathrm{Sym}^n K_t)(1)) \oplus M_{\Delta^*}; \quad i < 0. \end{aligned} \tag{47}$$

Proposition 35. *The Hodge filtration* (47) *on* $\mathbb{E}_n$ *satisfies Griffiths transversality:*

$$\nabla(F^a\mathbb{E}_n) \subset F^{a-1}\mathbb{E}_n \otimes \Omega^1_{\Delta^*}.$$

The following is a curious consequence of the maximal unipotency of the local monodromy of M at $t = 0$ (Assumption 15 with $\lambda = 1$). It will be useful in our proof of Proposition 35.

Lemma 36. *Suppose M is a connection on $U \subset \mathbb{G}_m$ carriying a polarized variation of pure HS and such that the monodromy of the local system of its flat sections around $t = 0$ is maximally unipotent. Let $p \in \Delta^* \subset U$ and consider the pure Hodge structure on the fiber M_p. Then for some $a < b$ the Hodge graded is given by*

$$gr_F M_p = \bigoplus_{a \leq i \leq b} F^i M_p / F^{i+1} M_p$$

with each $F^i M_p / F^{i+1} M_p \cong \mathbb{C}$.

Proof. The point of Schmid's construction of the limiting mixed Hodge structure for M at $0 \in \Delta$ is that the Hodge filtration passes to a limit as $p \to 0$, so it suffices to prove the assertion for the Hodge graded of the limiting mixed Hodge structure M_{lim}. But M_{lim} has a logarithm of monodromy operator $N : M_{lim} \to M_{lim}(-1)$. The monodromy weight filtration $L_* M_{lim}$ has a very simple form

$$\begin{aligned} M_{lim} = L_r M_{lim} = L_{r-1} M_{lim}; \quad L_{r-2} M_{lim} = L_{r-3} M_{lim} = N M_{lim} \\ \cdots L_{-r+1} M_{lim} = N^{r-1} M_{lim}; \quad L_{-r} M_{lim} = (0). \end{aligned} \tag{48}$$

Finally, $N^iM_{lim}/N^{i+1}M_{lim}$ is a one dimensional vector space. For $i \geq 0$, the map $N : gr_N^i M \to gr_N^{i+1} M(-1)$ is a surjective map of Hodge structures of dimensions ≤ 1. For a Hodge structure H we have $F^i(H(-1)) = F^{i-1}H$, so there are two possibilities. Either $gr_N^{i+1}M = (0)$, or $gr_N^{i+1}M = \mathbb{Q}(k)$ for some k, in which case $gr_N^i M = \mathbb{Q}(k-1)$. Since $\mathbb{C}(k)$ has Hodge filtration degree $-k$, the lemma follows. □

Proof of Proposition 35. By Lemma 36, the Hodge filtration on M is $0 = F^rM \subset F^{r-1}M \subset F^{r-2}M \subset \cdots \subset F^0M = M$ with F^i/F^{i+1} rank 1. Since we assumed that $m \in M$ is chosen in the smallest part of the Hodge filtration, it follows that the weight is given by $w = r-1$ and $F^kM = \sum_{j=0}^{r-1-k} \mathcal{O}_U D^j m$.

Recall the $\mathcal{O}$-splitting $s : M_{\Delta^*} \hookrightarrow \mathbb{E}_n$ is defined by $D^i m \mapsto D^i\phi^{an}, 0 \leq i \leq r-1$. It follows that $F^i\mathbb{E}_n = \sum_{j=0}^{r-1-i} \mathcal{O}D^j\phi^{an}$ for $i \geq 0$, which immediately implies the statement of the proposition for all $a > 0$. To deal with $a \leq 0$ we will need the following observation.

Lemma 37. *We have $D^jL\phi^{an} = e_{-1-j},\ 0 \leq j \leq n.$*

Proof of lemma. Since $Lm = 0$, we can write $L\phi^{an} = B_1e_{-1} + \cdots + B_{n+1}e_{-n-1}$ with $B_i \in \mathcal{O}$. From (43) we get

$$\begin{aligned}\frac{(\log t)^{k-r}}{(k-r)!} &= L\phi_k = \langle L\phi^{an}, \varepsilon_k\rangle \\ &= \langle B_1e_{-1} + \cdots + B_{r+n}e_{-n-1}, \varepsilon_k\rangle = \sum_{j=r}^{k} \frac{\log(t)^{k-j}}{(k-j)!} B_{j-r+1}.\end{aligned}$$

Since the $B_i \in \mathcal{O}$ cannot involve powers of $\log(t)$, we conclude that $B_1 = 1$ and $B_j = 0$ for $j > 1$. Thus, $L\phi^{an} = e_{-1}$ and hence $D^jL\phi^{an} = e_{-1-j}$. □

Note that $F^{-j}\Big((\mathrm{Sym}^n K_t)(1)\Big) = \mathcal{O}e_{-1} \oplus \cdots \oplus \mathcal{O}e_{-j}$. Since $L\phi^{an} = e_{-1}$ we see that $D^r\phi^{an} \in F^{-1}\mathbb{E}_n$ (remember we shrank Δ^* so that the leading coefficient of L is a unit in $\mathcal{O}$.) The case $a = 0$ follows from $D\left(F^0\mathbb{E}_n\right) \subset \sum_{j=0}^{r} \mathcal{O}D^j\phi^{an} = F^{-1}\mathbb{E}_n$. Finally, $(\mathrm{Sym}^n K_t)(1)$ satisfies Griffiths transversality, and the proposition follows for all a. □

It is clear from the proof of Proposition 35 that the filtration (47) is actually given by

$$F^i\mathbb{E}_n = \sum_{j=0}^{r-1-i} \mathcal{O}\, D^j\phi^{an}, \quad -n-1 \leq k \leq r-1. \tag{49}$$

With the Hodge filtration (49) and the weight filtration defined as

$$\begin{aligned}&W_i\mathbb{E}_n = W_i((\mathrm{Sym}^n K_t)(1)) \ \text{ for } \ i < r-1,\\ &W_{r-1}\mathbb{E}_n = \mathbb{E}_n,\end{aligned} \tag{50}$$

it is clear that our variation of mixed Hodge structure on $\mathbb{E}_n$ is an extension of the pure variation of weight $w = r-1$ on M_{Δ^*} by the mixed Tate variation $(\mathrm{Sym}^n K_t)(1)$.

As it stands, our variation of HS $\mathbb{E}_n$ is only a variation of $\mathbb{C}$-HS. We shall now lift it to the variation of $\mathbb{Q}$-HS. Recall that the $\mathbb{Q}$-structure on solutions $\mathcal{M}_p^\vee \subset (\mathcal{E}_n^\vee)_p$ is given by rational Betti homology classes. The subspace $\mathcal{M}_p^\vee(\mathbb{Q})$ is clearly preserved by the monodromy. Since we assume that the action of σ_0 is maximally unipotent, we have the nilpotent operator $N = \log(\sigma_0)$ acting on $\mathcal{M}_p$ and preserving the $\mathbb{Q}$-structure. Quite generally, associated to a nilpotent operator on a finite dimensional vector space there is a filtration L_* (Jacobson filtration). In our case N is maximally nilpotent and the Jacobson filtration is given by $L_k = N^{\lfloor \frac{r-k}{2} \rfloor}(\mathcal{M}_p)$ for $1-r \le k \le r-1$, (48). As M carries a pure HS of weight $w = r-1$, then basic result of Schmid and Deligne implies that there exists the limiting mixed HS $M_{\lim}$ with $M_{\lim}(\mathbb{Q}) = \mathcal{M}_p(\mathbb{Q})$ and the weight filtration given by

$$W_k M_{\lim} = L_{k-w} = N^{r-\lceil \frac{k+1}{2} \rceil}(\mathcal{M}_p), \quad 0 \le k \le 2(r-1).$$

It is clear in our case that $N^{r-1-j}(\mathcal{M}_p^\vee(\mathbb{C})) = \mathbb{C}\varepsilon_0 + \ldots + \mathbb{C}\varepsilon_j$. It appears that, in the situation when the Apéry constants κ_j are defined and $\kappa_0 \neq 0$, these numbers can be used to describe the $\mathbb{Q}$-subspaces of the Jacobson filtration. Suppose that $\delta \in \mathcal{M}_p^\vee$ in Assumption 12 is chosen to be rational, $\delta \in (\sigma_c - 1)\mathcal{M}_p^\vee(\mathbb{Q})$, and consider the Apéry constants defined by

$$(\sigma_c - 1)\varepsilon_j = \kappa_j \delta, \tag{51}$$

where ε_j are the horizontal sections (39) yielding the classical Frobenius solutions, (45). As a connection, we have a natural globalization of $\mathbb{E}_n$, namely $\mathbb{E}_n = \mathcal{D}/\mathcal{D}D^{n+1}L$, (41). Thus, it makes sense also to talk about the variation around c for horizontal sections of $\mathbb{E}_n$. Proposition 18 shows that the variation at c for $\mathbb{E}_n$ also has rank 1. The dual connection also has variation of rank 1 (see Remark 16) and hence formula (51) with $j \ge r$ defines the higher Apéry constants. Recall that solutions $\varepsilon_0, \ldots, \varepsilon_{n+r}$ form a $\mathbb{C}$-basis for the multivalued horizontal sections $(\mathcal{E}_n^\vee)_p$; it will be convenient to drop the base point p from our notation and denote their $\mathbb{C}$-span by $\mathcal{E}_n^\vee = \mathbb{C}\varepsilon_0 + \ldots + \mathbb{C}\varepsilon_{n+r}$.

Proposition 38. *Suppose that δ in Assumption 12 for M is chosen in the rational subspace, $\delta \in \mathcal{M}_p^\vee(\mathbb{Q})$, and the Apéry constants $\kappa_0, \kappa_1, \ldots$ are defined by* (51). *Assume further that $\sigma_c(\varepsilon_0) \neq \varepsilon_0$. Then there exists a unique $\mathbb{Q}$-structure $\mathcal{E}_n^\vee(\mathbb{Q})$ on $\mathcal{E}_n^\vee$ such that*
(i) $\kappa_0^{-1}\varepsilon_0 \in \mathcal{E}_n^\vee(\mathbb{Q})$ and the filtration $fil_j\mathcal{E}_n^\vee := \mathbb{C}\varepsilon_0 + \mathbb{C}\varepsilon_1 + \cdots + \mathbb{C}\varepsilon_j$ is defined over $\mathbb{Q}$;
(ii) $\mathcal{E}_n^\vee(\mathbb{Q})$ is stable under both loops σ_0, σ_c around $t = 0$ and $t = c$ respectively.
A $\mathbb{Q}$-basis for $\mathcal{E}_n^\vee(\mathbb{Q})$ is then given by

$$\eta_k := (2\pi i)^{-k} \sum_{j=0}^{k} \alpha_j \varepsilon_{k-j}, \quad 0 \le k \le n+r, \tag{52}$$

where α_j are the coefficients of the series

$$\begin{aligned} \sum_{j=0}^{\infty} \alpha_j s^j &:= 1 \Big/ \left(\sum_{j=0}^{\infty} \kappa_j s^j \right) \\ &= \frac{1}{\kappa_0} + \frac{-\kappa_1}{\kappa_0^2}\, s + \frac{-\kappa_2\kappa_0 + \kappa_1^2}{\kappa_0^3} s^2 + \ldots \end{aligned} \tag{53}$$

(Strictly speaking, in our current setting we only use (53) *modulo $O(s^{n+r+1})$).*

Proof. The η_k defined in (52) are linearly independent over $\mathbb{C}$ (because the ε_k are.) It is clear that $\mathcal{E}_n^\vee(\mathbb{Q}) := \sum_{k=0}^{n+r} \mathbb{Q}\eta_k$ satisfies (i). Let us check it also satisfies (ii). One can easily check that $N = \log(\sigma_0)$ acts as $N\eta_k = \eta_{k-1}$, and hence $\sigma_0 = \exp(N)$ preserves $\mathcal{E}_n^\vee(\mathbb{Q})$. As for the action of σ_c, we observe that

$$(\sigma_c - 1)\eta_k = (2\pi i)^{-k} \sum_{j=0}^{k} \alpha_j (\sigma_c - 1)\varepsilon_{k-j} = (2\pi i)^{-k} \left(\sum_{j=0}^{k} \alpha_j \kappa_{k-j} \right) \delta = \begin{cases} 0, & k \geq 0, \\ \delta, & k = 0. \end{cases}$$

It remains to show that $\delta \in \mathcal{E}_n^\vee(\mathbb{Q})$. We write $\delta = \sum_{k=0}^{r-1} \mu_k \eta_k$ with $\mu_k \in \mathbb{C}$. Since $\delta \in \mathcal{M}_p^\vee(\mathbb{Q})$ and this space is preserved by the monodromy operators, it follows from $(\sigma_c - 1)\delta = \mu_0 \delta$ that $\mu_0 \in \mathbb{Q}$. To access other coefficients, note that $(\sigma_c - 1)N^k \delta = \mu_k \delta$ implies that $\mu_k \in \mathbb{Q}$ for all k.

To show uniqueness, denote $V_k = \langle \eta_0, \ldots, \eta_k \rangle_{\mathbb{Q}}$ and suppose we have another $\mathbb{Q}$-structure with these properties, say $W_k = \langle \nu_0, \ldots, \nu_k \rangle_{\mathbb{Q}}$ with $W_k \otimes \mathbb{C} = V_k \otimes \mathbb{C}$ for $k \geq 0$. It is clear from (i) that $V_0 = W_0$. Suppose $k > 0$ and assume inductively that $W_{k-1} = V_{k-1}$. Due to (ii) we must have that $\delta = (\sigma_c - 1)\eta_0 \in W_{r+n}$, and hence for any $w \in W_k$ one has $(\sigma_c - 1)w \in \mathbb{Q}\delta$. It follows that $W_k = W_0 \oplus W_k^0$ where $W_k^0 := \ker(\sigma_c - 1 : W_k \to W_{r+n})$. Since $(\sigma_0 - 1)(W_0) = (0)$, it follows that $W_k^0 \subset W_k$ defines a splitting of $\sigma_0 - 1 : W_k \twoheadrightarrow W_{k-1} = V_{k-1}$. We have $V_k^0 \otimes \mathbb{C} = W_k^0 \otimes \mathbb{C} \subset \langle \varepsilon_0, \ldots, \varepsilon_k \rangle$ and with this identification, the two splittings coincide. Thus $V_k^0 = W_k^0$ so $V_k = W_k$. ☐

Note that $fil_{r-1}\mathcal{E}_n^\vee(\mathbb{Q}) = \mathcal{M}_p^\vee(\mathbb{Q})$. Indeed, $\delta = (\sigma_c - 1)(\kappa_0^{-1}\varepsilon_0) \in \mathcal{E}_n^\vee(\mathbb{Q})$ and under the condition $\sigma_c \varepsilon_0 \neq \varepsilon_0$ the space of Betti cycles $\mathcal{M}_p^\vee(\mathbb{Q})$ is generated by the images of δ under σ_0 (see the last paragraph in the proof of Theorem 19.) Since $fil_j \mathcal{E}_n^\vee = N^{n+r-j}(\mathcal{E}_n^\vee)$, Proposition 38 defines a unique lift of the Betti structure on $\mathcal{M}^\vee$ to a $\mathbb{Q}$-structure on $\mathcal{E}_n^\vee$ for which the Jacobson filtration for $N = \log(\sigma_0)$ is defined over $\mathbb{Q}$.

The final objective in this section is to link the Apéry constants κ_j to periods of a limiting mixed Hodge structure. As we mentioned earlier, for a pure variation of Hodge structure M of weight w the limiting mixed Hodge structure M_{lim} can be identified with the fiber $\mathcal{M}_p$ with the weight filtration given by the Jacobson filtration for $N = \log(\sigma_0)$ shifted by w. More generally, for a variation of mixed Hodge structure H with the weight filtration W_*H, a *monodromy weight filtration* on the fiber $\mathcal{H}_p$ is a filtration $\mathcal{W}_*\mathcal{H}_p$ such that $N(\mathcal{W}_j) \subset \mathcal{W}_{j-2}$ and such that for each k the filtration induced by $\mathcal{W}_*$ on $gr_k^W \mathcal{H}_p$ is the Jacobson filtration defined by N on the pure HS $gr_k^W \mathcal{H}_p$ and then shifted by $-k$. There is at most one monodromy weight filtration satisfying these conditions, but it can happen that no such monodromy weight filtration exists. As earlier, we denote the fiber $(\mathcal{E}_n)_p$ simply by $\mathcal{E}_n$.

Proposition 39. *Let L_* be the Jacobson filtration for $N = \log(\sigma_0)$ on $\mathcal{E}_n$. Then $\mathcal{W}_* = L_*[n + 2 - r]$ is the monodromy weight filtration on $\mathcal{E}_n$.*

Proof. One can check that $N^k(\mathcal{E}_n) = (\mathbb{C}\varepsilon_0^\vee + \ldots + \mathbb{C}\varepsilon_{k-1}^\vee)^\perp = \mathbb{C}e_{-n-1} + \ldots + \mathbb{C}e_{r-1-k}$, which yields

$$\begin{aligned} \mathcal{W}_k \mathcal{E}_n &= L_{k+n+2-r}\mathcal{E}_n = N^{\lfloor \frac{n+r+1-(k+n+2-r)}{2} \rfloor}(\mathcal{E}_n) = N^{r-1-\lfloor \frac{k}{2} \rfloor}(\mathcal{E}_n) \\ &= \mathbb{C}e_{-n-1} + \ldots + \mathbb{C}e_{\lfloor \frac{k}{2} \rfloor}. \end{aligned}$$

The weight filtration $W_*\mathbb{E}_n$ is given by (50). One can check that for $k < 0$ we have $W_k \mathcal{E}_n = \mathcal{W}_k \mathcal{E}_n$ and hence the filtration induced by $\mathcal{W}_*$ on each $gr_{-2k}^W \mathcal{E}_n$ is zero in degrees

$< -2k$ and everything in degrees $\geq -2k$. This is precisely the Jacobson filtration for N on this rank one subquotient shifted by $2k$. It remains to check the rank r graded piece $gr^W_{r-1}\mathcal{E}_n \cong \mathcal{M}$. There the induced filtration is given by

$$\mathcal{W}_k gr^W_{r-1}\mathcal{E}_n = \mathbb{C}\overline{e}_0 + \ldots + \mathbb{C}\overline{e}_{\lfloor \frac{k}{2} \rfloor} = N^{r-1-\lfloor \frac{k}{2} \rfloor}\left(gr^W_{r-1}\mathcal{E}_n\right),$$

which is again the Jacobson filtration shifted by $1-r$. □

Proposition 40. *The limiting mixed Hodge structure $(\mathbb{E}_n)_{lim}$ exists. If α_k are the coefficients of the inverse Apéry series* (53)*, then numbers $\alpha_k(2\pi i)^{-h}$ with $0 \leq k \leq h \leq n+r$ are periods of $(\mathbb{E}_n)_{lim}$.*

The following observation will be a key step in the proof of Proposition 40.

Lemma 41. *Let $K \subset \mathbb{C}$ be a field and $L = \sum_{j=0}^{m} q_j(t)D^{m-j}$, $q_j \in K(t)$ be a differential operator such that the monodromy of its solutions around $t=0$ is maximally unipotent. Let $V = Sol_p(L)$ be the space of solutions near a base point $t=p$. We further assume that $q_0(0) \neq 0$ and $q_j(0) = 0$ when $j > 0$, so that the local exponent of L at $t=0$ equals* 0*. Fix some branch of* $\log(t)$ *near p and consider the classical Frobenius basis in V:*

$$\phi_k(t) = \sum_{j=0}^{k} \frac{\log(t)^j}{j!}\phi^{an}_{k-j}(t), \qquad k = 0, \ldots, m-1,$$

where $\phi^{an}_0, \ldots, \phi^{an}_{m-1} \in K[\![t]\!]$ are uniquely determined (i.e. independent of the choice of $\log(t)$*) analytic near $t=0$ functions satisfying $\phi^{an}_j(0) = \delta_{j,0}$.*

(i) Consider some Zarisky open set $U \subset \mathbb{G}_m$ and the ring of differential operators $\mathcal{D} = \mathcal{O}_U[D]$. Consider a connection on U given by $H = \mathcal{D}/\mathcal{D}L \cong \sum_{j=0}^{m-1}\mathcal{O}_U D^j\omega$, where $\omega \in H$ is the image of $1 \in \mathcal{D}$. Consider a filtration on H given by

$$F^k H = \sum_{j=0}^{m-1-k} \mathcal{O}_U D^j \omega, \quad 0 \leq k \leq m-1. \tag{54}$$

Then the limiting filtration in the sense of Schmid ([12, (6.15)]*) exists and is given on the dual space $V^\vee$ by*

$$F^k_\infty\, V^\vee = \mathbb{C}\phi^\vee_0 + \ldots + \mathbb{C}\phi^\vee_{m-1-k}.$$

Moreover, limits of algebraic classes yield $F^k_\infty(K) := K\phi^\vee_0 + \ldots + K\phi^\vee_{m-1-k}$.

(ii) Consider the nilpotent operator $N = \log(\sigma_0)$ and let the respective Jacobson filtration on $V^\vee$ shifted by $m-1$; this filtration is given by

$$\mathcal{W}_* V^\vee := N^{m-1-\lfloor \frac{k}{2} \rfloor}(V^\vee). \tag{55}$$

Then $\mathbb{H} = (V^\vee, \mathcal{W}_*, F^*_\infty)$ is a mixed Hodge structure with $gr^{\mathcal{W}}\mathbb{H} = \oplus_{k=0}^{m-1}\mathbb{C}(-k)$.

Proof. (i) We will apply Schmid's limiting procedure as in [12, §6] to the filtration (54) and describe the limit as a filtration on the dual space $V^\vee$. For this we shall restrict our connection to a small punctured disk $\Delta^* = \Delta \setminus \{0\} \subset U_{an}$. Since $q_0(0) \neq 0$, connection $H_{an} = \sum_{j=0}^{m-1-k}\mathcal{O}_{\Delta^*}D^j\omega$ extends to a connection on Δ simply by $\widetilde{H}_{an} = \sum_{j=0}^{m-1-k}\mathcal{O}_\Delta D^j\omega$ (this extension is a logarithmic connection, it has a simple pole at $t=0$). To deal with the monodromy, we pullback solutions to the universal covering of Δ^*. On a subset of the upper halfpalne $G \subset \mathbb{C}$ we take $e : G \to \Delta^*$, $e(z) = \exp(2\pi i z)$ and identify the space of solutions with

$$\mathbb{V} := \left\{ u(z) = \sum_{j=0}^{m-1} u_j(e(z)) z^j \mid u_j \in \mathcal{O}_\Delta^{an}, (e^*L)u = 0 \right\} \cong V.$$

For each $z \in G$ there is a map $\pi_z : H_{an} \to \mathbb{V}^\vee$. In concerete terms, for an element $v = \sum_j v_j(t) D^j \omega \in H_{an}$ the pairing of $\pi_z(v)$ with a solution $u \in \mathbb{V}$ is given by $\sum_j v_j(e(z))(D^j u)(z) \in \mathbb{C}$. Indeed, if we identify u with $\langle \omega, \varepsilon \rangle$ for a horizontal section $\varepsilon \in \mathcal{H}^\vee$, then $\langle D^j \omega, \varepsilon \rangle = D^j \langle \omega, \varepsilon \rangle = D^j u$ and since u is represented by a function of z, then $D = e(z) \frac{d}{de(z)} = \frac{1}{2\pi i} \frac{d}{dz}$. We now restrict our filtration to $\widetilde{\mathcal{F}}^k \widetilde{H}_{an} := \sum_{j=0}^{m-1-k} \mathcal{O}_\Delta^{an} D^j \omega$ and define $\mathcal{F}_z^k \, \mathbb{V}^\vee := \pi_z \left(\widetilde{\mathcal{F}}^k \widetilde{H}_{an} \right)$. The limiting filtration is defined in [1, (6.15)] by

$$\mathcal{F}_\infty^k \, \mathbb{V}^\vee := \lim_{Im(z) \to +\infty} \exp(-zN) \mathcal{F}_z^k \, \mathbb{V}^\vee. \tag{56}$$

Using as a basis in $\mathbb{V}$ the Frobenius solutions

$$\phi_k(z) = \sum_{j=0}^{k} \frac{(2\pi i z)^j}{j!} \phi_{k-j}^{an}(e(z)), \quad 0 \le k \le m-1,$$

we need to compute the limits

$$\lim_{Im(z) \to +\infty} \exp(-zN) \pi_z(D^s \omega)(\phi_k) = \lim_{Im(z) \to +\infty} \exp(-zN)(D^s \phi_k)(z). \tag{57}$$

Let us see how to apply $\exp(-zN)$ to a function of the shape $u = \sum_{j \ge 0} u_j(e(z)) z^j$. Recall that σ_0 acts by the shift $(\sigma_0 u)(z) = u(z+1)$. We claim that $N = \log(\sigma_0) = \sum_{h \ge 1} (-1)^{h-1} (\sigma_0 - 1)^h / h$ acts by $(Nu)(z) = \sum_{j \ge 1} u_j(e(z)) j z^{j-1}$. Indeed, since σ_0 is trivial on each $u_j(e(z))$, we clearly have $Nu = \sum_{j \ge 0} u_j(e(z)) N z^j$, and to check that $N z^j = j z^{j-1}$ we observe that on the space of polynomials the shift σ_0 is equal to $\exp(\frac{d}{dz})$ due to the Taylor formula. Finally,

$$\begin{aligned}
(\exp(-zN)u)(z) &= \sum_{h \ge 0} \frac{(-z)^h}{h!} (N^h u)(z) = \sum_{j \ge 0} u_j(e(z)) \sum_{h \ge 0} \frac{(-z)^h}{h!} \left(\frac{d}{dz} \right)^h z^j \\
&= \sum_{j \ge 0} u_j(e(z)) \left(\sum_{h \ge 0} (-1)^h \binom{j}{h} \right) z^j = u_0(e(z)).
\end{aligned}$$

We see that applying $\exp(-zN)$ simply kills the part with the monodromy. Hence to compute the limit in (57) we need to extract the analytic part of $D^s \phi_k$ and evaluate it at $t = 0$. Namely, since

$$D^s \phi_k = \sum_{j=0}^{k} \sum_{h=0}^{s} \binom{s}{h} D^h \left(\frac{(2\pi i z)^j}{j!} \right) (D^{s-h} \phi_{k-j}^{an})(e(z))$$

the analytic part (that is, we gather the terms with z^0) consists of the summands where $h = j$. Further, if $h < s$ or $j < k$ then $(D^{s-h} \phi_{k-j}^{an})(0) = 0$. Hence the limit (57) vanishes unless $k = s$, and in the latter case it is equal $\phi_0^{an}(0) = 1$. We conclude that

$$\lim_{Im(z)\to+\infty} \exp(-zN)\pi_z\left(D^s\omega\right) = \phi_s^\vee$$

and, more generally,

$$\lim_{Im(z)\to+\infty} \exp(-zN)\pi_z\left(\sum_s v_s(t)D^s\omega\right) = \sum_s v_s(0)\phi_s^\vee.$$

Part (i) follows immediately from this formula.

For (ii) we note that $N^a(V^\vee) = Span_{\mathbb{C}}(\phi_a^\vee, \ldots, \phi_{m-1}^\vee)$ and hence filtrations $F_\infty^* H$ and $\mathcal{W}_*$ are opposite in the sense that $V^\vee = \mathcal{W}_{2k} \oplus F_\infty^{k+1}$ for any k. It follows that the filtration induced by F_∞^* on $gr_{2k}^{\mathcal{W}} V^\vee$ is zero in degrees $> k$ and everything in degree k. □

Remark 42. *Suppose that L in Lemma 41 is a Picard–Fuchs differential operator. More precisely, we assume that for a smooth projective family of algebraic varieties $f : X \to U$ there is a class in the smallest Hodge part $\omega \in F^{m-1}H_{dR}^{m-1}(X/U)$ annihilated by the differential operator L. Then $H = \mathcal{D}/\mathcal{D}L$ carries a polarized variation of Hodge structure of pure weight $m-1$, and using Griffiths' transversality along with Lemma 36 we conclude that* (54) *is the Hodge filtration. The limiting MHS is constructed in* [12, Theorem (6.16)]*, and this is precisely $\mathbb{H}$ from (ii) in Lemma 41.*

The K-structure from part (i) of our Lemma is the de Rham structure on $\mathbb{H}$. To see this, we change notation in order to appeal to the work of Steenbrink ([13]*, as corrected in* [6]*). Steenbrink considers a geometric situation where H is the DR-cohomology of a projective family $f : \mathcal{X} \to S$, where $\mathcal{X}$ is smooth and S is a smooth, affine curve. t is a parameter on S and f is smooth away from $t = 0$. We assume $Y := f^{-1}(0)$ is a reduced normal crossings divisor. The link with our standard notation $f : X \to U$ is $X = \mathcal{X} - Y$; $U = S - \{t = 0\}$.*

Define

$$\omega_S^* = \Omega_S^*(\log\{0\}); \quad \omega_{\mathcal{X}}^* = \Omega_{\mathcal{X}}^*(\log Y); \quad \omega_Y^* = \Omega_{\mathcal{X}/S}^*(\log Y) \otimes_{\mathcal{O}_S} K.$$

Steenbrink's basic result identifies $\omega_Y^* \otimes_K \mathbb{C}$ with the nearby cycle complex $R\Psi(\mathbb{C})$ for $Y \subset \mathcal{X}$. This identification depends on the choice of t and also of $\log t$. Given n, Steenbrink's result enables one to put a mixed Hodge structure on $H^n(Y, \omega_Y^*)$ which is then identified with the limiting MHS H_{lim}^n as defined by Deligne and Schmid. The fact that ω_Y^* is defined algebro-geometrically automatically endows H_{lim}^n with a K-structure (DR structure) which can be used to define periods.

We introduce a variable denoted $\log t$ and consider the complexes

$$\omega_S[\log t]; \quad \omega_{\mathcal{X}}[\log t]; \quad \omega_{\mathcal{X}/S}[\log t].$$

Sections e.g. of $\omega_S^*[\log t]$ are polynomials in $\log t$ with coefficients which are sections of ω^*. The differential is extended from ω by setting $d\log t = dt/t$. Note that $dt/t = 0$ in $\omega_{\mathcal{X}/S}$.

Let $i : Y \hookrightarrow \mathcal{X}$ be the inclusion, and write i^{-1} for the sheaf-theoretic restriction functor from sheaves on $\mathcal{X}$ to sheaves on Y. (Note $i^{-1} \neq i^*$, the pullback in the category *of sheaves of $\mathcal{O}$-modules.) Steenbrink's basic result is that the composition*

$$i^{-1}(\omega_{\mathcal{X}}[\log t]) \xrightarrow{\log t \mapsto \log t} i^{-1}(\omega_{\mathcal{X}/S}[\log t]) \xrightarrow{\log t \mapsto 0} i^{-1}\omega_{\mathcal{X}/S} \to \omega_Y \tag{58}$$

is a quasi-isomorphism.

Consider the diagram

$$\begin{array}{ccccccccc}
0 & \to & i^{-1}(\omega^{\cdot -1}_{\mathcal{X}/S} \otimes \omega^1_S[\log t]) & \to & i^{-1}(\omega_{\mathcal{X}}[\log t]) & \xrightarrow{\alpha} & i^{-1}(\omega_{\mathcal{X}/S}[\log t]) & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \ker\beta & \to & i^{-1}(\omega_{\mathcal{X}}[\log t]) & \xrightarrow{\beta} & i^{-1}(\omega_{\mathcal{X}/S}) & \to & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \to & i^{-1}(\omega^{\cdot -1}_{\mathcal{X}/S} \otimes \omega^1_S) & \to & i^{-1}(\omega^*_{\mathcal{X}}) & \to & i^{-1}(\omega^*_{\mathcal{X}/S}) & \to & 0. \\
 & & & & & & \downarrow & & \\
 & & & & & & \omega_Y & &
\end{array} \tag{59}$$

A piece of the long-exact sequence of cohomology sheaves on Y associated to the top line reads (for any p)

$$0 \to \mathcal{H}^p(i^{-1}(\omega_{\mathcal{X}}[\log t])) \xrightarrow{\mathcal{H}^p(\alpha)} \mathcal{H}^p(i^{-1}(\omega_{\mathcal{X}/S}[\log t])) \xrightarrow{\nabla_{GM}} \mathcal{H}^p(i^{-1}(\omega^{\cdot}_{\mathcal{X}/S} \otimes \omega^1_S[\log t])) \tag{60}$$

The boundary map coincides with the Gauß–Manin connection as indicated. Also, the result of Steenbrink cited above implies that $\mathcal{H}^p(\alpha)$ is injective, both on the sheaf level and for global cohomology groups. Thus, (60) *identifies*

$$H^p(Y, i^{-1}(\omega^*_{\mathcal{X}}[\log t]) \cong H^p(Y, i^{-1}(\omega_{\mathcal{X}/S}[\log t]))^{\nabla_{GM}=0} \tag{61}$$

We know by Frobenius that we have a full set of horizontal sections defined over $K[\![t]\!][\log t]$, so we conclude

$$H^p(Y, i^{-1}(\omega^*_{\mathcal{X}}[\log t]) \cong \{\textit{Horizontal sections of the GM connection on } H^p(Y, i^{-1}\omega^*_{\mathcal{X}/S})\} \cong H^p(Y, \omega^*_Y). \tag{62}$$

The assignment $\log t \to 0$ in (58) *coincides with the vanishing of $z^j, j > 0$ in the computation of* (57)*. The K-structure from $H^p(Y, \omega_Y)$ is Steenbrink's DR-structure. It matches the K-structure on $\mathbb{H}$, and if one expresses period functions in the classical Frobenius basis, the coefficients are periods of the limiting Hodge structure.*

Example 43. *The period function of the Legendre family of elliptic curves*

$$\phi(t) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}} = \pi \sum_{n=0}^\infty \binom{2n}{n}^2 \left(\frac{t}{16}\right)^n = \pi \cdot {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1|t)$$

is a Betti-rational solution to the hypergeometric differential operator $L = D^2 - t(D+\frac{1}{2})^2$. *Then* π *here is the period the limiting MHS because the hypergeometric function* $\phi_0(t) = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1|t)$ *is the Frobenius solution analytic at* $t = 0$.

Lemma 41 can be also applied to variations of mixed Hodge structure. Namely, if the Hodge filtration is a shift of F^*H and if the same shift of $\mathcal{W}_*$ yields the monodromy weight filtration, then the respective shift of $\mathbb{H}$ is the limiting mixed Hodge structure.

Example 44. $L = D^{n+1}$ *corresponds to the symmetric power of the Kummer variation* $Sym^n K_t$. *One can check that the Hodge filtration is given by the shift* $F^*[n] = F^{*+n}$ *of the filtration* (54) *and that the shift* $\mathcal{W}_*[n] = \mathcal{W}_{*+2n}$ *of the filtration* (55) *is the monodromy weight filtration. It follows that the limiting Hodge structure is* $(Sym^n K_t)_{lim} = \mathbb{H}[n]$. *The Frobenius basis is given by* $\phi_k(t) = \frac{\log(t)^k}{k!}$ *and the rational structure is spanned by* $\left(\frac{\log(t)}{2\pi i}\right)^k = (2\pi i)^{-k} k!\, \phi_k(t)$. *Hence* $(2\pi i)^{-k}$, $0 \le k \le n$ *are periods of* $(Sym^n K_t)_{lim}$.

Proof of Proposition 40. We apply Lemma 41 for the operator $D^{n+1}L$ where L is an operator of order r satisfying the usual assumptions used throughout this section. Here $m = n + r + 1$ and as connections we haev $H = \mathcal{D}/\mathcal{D}D^{n-1}L \cong \mathbb{E}_n$, see (41). By (49) the Hodge filtration $F^*\mathbb{E}_n$ is the shift of F^*H in (54) by $n+1$. By Proposition 39 the monodromy weight filtration $\mathcal{W}_*\mathcal{E}_n$ is the shift of $\mathcal{W}_*H$ by $n+1$. We conclude that the limiting MHS for our variation $\mathbb{E}_n$ exists and is given by $(\mathbb{E}_n)_{lim} = \mathbb{H}[n+1]$.

The rational structure on $\mathcal{E}_n^\vee$ was defined in Proposition 38. Since $\eta_k \in \mathcal{E}_n^\vee(\mathbb{Q})$ and

$$\langle \phi^{an}, \eta_k \rangle = (2\pi i)^{-k}(\alpha_k \phi_0 + \alpha_{k-1}\phi_1 + \ldots + \alpha_0 \phi_k),$$

then $\alpha_k (2\pi i)^{-h}$ with $0 \le k \le h \le n + r$ are periods of $(\mathbb{E}_n)_{lim}$. □

We showed that numbers α_k from Proposition 38 divided by certain powers of $2\pi i$ are periods of the limiting MHS associated to the extension

$$0 \to \mathrm{Sym}^n(K_t)(1) \to \mathbb{E}_n \to M_{\Delta^*} \to 0. \tag{63}$$

Though we assume M is motivic, i.e. M is the Gauß–Manin connection for a family of varieties over $\mathbb{P}^1$ as in the beginning of this section, it is not clear that the extension (63) is motivic. Indeed, we do not expect it to be so in general. To better understand this question, we consider briefly some calculations inspired by work of Kerr [8, Section 5.3]. Kerr considers the example of Apéry which is a pencil of K3-surfaces defined by $1 - t\, f(x_1, x_2, x_3) = 0$ with

$$f = \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(1 - x_1 - x_2 + x_1x_2 - x_1x_2x_3)}{x_1x_2x_3},$$

as in our Example 21. He shows that the Milnor symbol $\{x_1, x_2, x_3\}$ defines classes in motivic cohomology $H^3_{mot}(X_\lambda, \mathbb{Q}(3))$ where $\lambda = 1/t$ and X_λ is a suitable compactification of the divisor $f(x_1, x_2, x_3) = \lambda$ in $\mathbb{G}_m^3$. Associated to such a motivic class, one has the Beilinson regulator

$$reg(\{x_1, x_2, x_3\}) \in \mathrm{Ext}^1_{MHS}(H^2(X_\lambda, \mathbb{Q}(0)), \mathbb{Q}(1)).$$

We speculate that, replacing t with λ in (63), the extension $reg\{x_1, x_2, x_3\}$ coincides with (63) for $n = 0$. If so, this will say in particular that (63) for $n = 0$ is in this case motivic. (NB. “speculation”$<<$ “conjecture” $<<$ “theorem”)

We may try to go further and consider the Apéry example for $n = 1$. The extension (63) becomes

$$0 \to K_\lambda(1) \to \mathbb{E}_1 \to H^2(X_\lambda, \mathbb{Q}) \to 0$$

This extension lies in

$$\operatorname{Ext}^1_{MHS}(\mathbb{Q}(0), H^2(X_\lambda) \otimes K_\lambda(3)) \cong \operatorname{Ext}^1_{MHS}(\mathbb{Q}(0), H^3(X_\lambda \times (\mathbb{G}_m, \{1, \lambda\}), \mathbb{Q}(4))) \tag{64}$$

Formally, we would expect such a class to arise as the Beilinson regulator of a relative motivic class in $H^4_{mot}(X_\lambda \times (\mathbb{G}_m, \{1, \lambda\}), \mathbb{Q}(4))$.

Actually, it is more precise to look at the whole family, allowing λ to vary. To this end, consider the pair

$$\Big(\mathbb{G}_m \times \mathbb{G}_m, (\mathbb{G}_m \times \{1\}) \cup \Delta_{\mathbb{G}_m}\Big)$$

where $\Delta_{\mathbb{G}_m}$ is the diagonal. We view this as a family over $\mathbb{G}_m$ via $pr_1 : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$, and we want a class in

$$H^4_{mot}(X \times_{\mathbb{G}_m} (\mathbb{G}_m \times \mathbb{G}_m, (\mathbb{G}_m \times \{1\}) \cup \Delta_{\mathbb{G}_m}), \mathbb{Q}(4))$$

Let u be the coordinate in the righthand $\mathbb{G}_m$ factor. We consider the Milnor symbol $\{x_1, x_2, x_3, u\}$. Informally speaking, to define a relative motivic class, we need to trivialize this symbol along the diagonal $u = \lambda = f(x_1, x_2, x_3)$. (A convenient and rigorous treatment of relative motivic cohomology can be given using higher cycle complexes, but here our intention is merely to suggest a way forward. We do not attempt to give details.) Informally, one trivializes this symbol by invoking the Steinberg relations, viz.

$$\begin{aligned}
&\{x_1, x_2, x_3, u\} = \{x_1, x_2, x_3, f(x_1, x_2, x_3)\} = \\
&\{x_1, x_2, x_3, \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(1 - x_1 - x_2 + x_1x_2 - x_1x_2x_3)}{x_1x_2x_3}\} \\
&= \{x_1, x_2, x_3, 1 - x_1 - x_2 + x_1x_2 - x_1x_2x_3\} = \\
&\{x_1, x_2, x_3, (1 - x_1)(1 - x_2)\Big(1 - \frac{x_1x_2x_3}{(1 - x_1)(1 - x_2)}\Big)\} = \\
&\{x_1, x_2, x_3, \Big(1 - \frac{x_1x_2x_3}{(1 - x_1)(1 - x_2)}\Big)\} = \\
&\{x_1, x_2, \Big(\frac{(1 - x_1)(1 - x_2)}{x_1x_2}\Big), \Big(1 - \frac{x_1x_2x_3}{(1 - x_1)(1 - x_2)}\Big)\} = 1
\end{aligned} \tag{65}$$

(On the last line we use the identity $\{x, 1 - ax\} = \{a^{-1}, 1 - ax\}$.)

Intuitively, at least, the above argument can be used in the Apéry example to construct our extension motivically for $n = 1$. We may hope to apply a similar aggument for $n > 1$, working with

$$X \times_{\mathbb{G}_m} \mathcal{G}^n; \mathcal{G} = (\mathbb{G}_m \times \mathbb{G}_m, \mathbb{G}_m \times 1 \cup \Delta_{\mathbb{G}_m}).$$

Here $\mathcal{G}^n = \mathcal{G} \times_{\mathbb{G}_m} \cdots \times_{\mathbb{G}_m} \mathcal{G}$ where the structure maps are again $pr_1 : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$. The symbol becomes $\{x_1, x_2, x_3, u_1, \ldots, u_n\}$. The first order trivializations along diagonals are as above, but there are now higher order compatibilities on multiple diagonals that are not understood.

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