A FORMULA FOR GAUSS-MANIN DETERMINANTS

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ABSTRACT. We give an explicit formula for the determinant of the Gauß-Manin connection for an irregular connection on a Zariski open set of the projective line $\mathbb{P}^1_K$ over a function field K over a field k of characteristic zero.

I hasten to write down in verse what I saw then,
For the scene lost to sight can't be revived again.

Su Shi

1. INTRODUCTION

Let K be a function field over a field k of characteristic 0, and let $j : U \subset \mathbb{P}^1_K$ be a Zariski open set of the projective line. We consider a flat connection (E, ∇) on U. The de Rham cohomology groups $H^i_{DR}(U/K, \nabla_{/K})$ carry a K/k connection, the Gauß-Manin connection, and taking the alternate tensor of the determinant connections

$$\otimes(\det H^i_{DR}(U/K, \nabla_{/K}), \text{Gauß} - \text{Manin})^{(-1)^i},$$

one defines the Gauß-Manin determinant connection, denoted by

$$\det H_{DR}(U/K, \nabla_{/K}).$$

This invariant is living in the group of isomorphism classes of K-lines endowed with a connection, which is the abelian group

$$\Omega^1_K / d\log K^\times.$$

The aim of this article is to give an explicit formula for it (see theorem 1.3 for a vague formulation, and 2.8 for a precise one) under a genericity assumption on (E, ∇). Special examples are contained in [4].

We comment briefly on the meaning and interest in such a formula. There is a deep analogy between connections on curves over function

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fields in characteristic 0 and ℓ-adic sheaves $\mathcal{E}_\ell$ on curves U over finite fields $\mathbb{F}_q$. Irregular singular points for the connection correspond to wild ramification at ∞ for the ℓ-adic sheaf, and the Gauß-Manin determinant connection corresponds to the global epsilon factor

$$\epsilon(\mathcal{E}_\ell) := \det(-f \mid \det R\Gamma(U, \mathcal{E}_\ell)).$$

Our hope is that the local formula we obtain for higher rank irregular connections will suggest local formulas for ϵ-factors extending the abelian Tate theory.

The Gauß-Manin construction is fairly standard and we do not recall it in detail. By way of example, we cite two classical formulas (Gauß hypergeometric and Bessel functions, respectively) :

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z) = \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a}du$$
$$J_n(z) = \frac{1}{2\pi i}\int_{S_0} u^{-n}\exp\ \frac{z}{2}\Big(u+\frac{1}{u}\Big)\frac{du}{u} \qquad (S_0 = \text{circle about } 0).$$

In both cases, the integrand is a product of a solution of a rather simple degree 1 differential equation in u, the solution being either

$$u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a} \quad \text{or} \quad u^{-n}\exp\ \frac{z}{2}\Big(u+\frac{1}{u}\Big),$$

with an algebraic 1-form (du or $\frac{du}{u}$). The integral is taken over a chain in the u-plane. The resulting functions $F(a,b;c;z)$ and $J_n(z)$ satisfy Gauß-Manin equations, which are much more interesting degree 2 equations in z.

It is not our purpose to go further into the classical theory, but, to understand the role of the determinant, we remark that in each of the above cases, there is a second path and a second algebraic 1-form such that the two integrals, say $f_1(z)$ and $f_2(z)$, satisfy the same second order equation. The Wronskian determinant

$$\begin{vmatrix} f_1 & f_2 \\ \frac{df_1}{dz} & \frac{df_2}{dz} \end{vmatrix}$$

satisfies the degree 1 equation given by the determinant of Gauß-Manin. It seems to us to be possible using the theory of Stokes structures to formulate a theory of period integrals for irregular connections in such a way that the Gauß-Manin determinant connection has as solution the determinant of the period matrix. We hope to return to this in a future paper.

Let X be a complete, smooth curve over K. For purposes of this article, we define the group of relative algebraic differential characters

$$AD^2(X/K) := \mathbb{H}^2(X, \mathcal{K}_2 \xrightarrow{d\log} \Omega^2_X/(\mathcal{O}_X \otimes \Omega^2_K)) \tag{1.1}$$

(The notation here differs from [7], as one has factored out 2-forms coming from the base and in particular truncated the differential forms of degree 3). The transfer

$$f_* : AD^2(X/K) \to AD^1(K) = \Omega^1_K/d\log K^\times \tag{1.2}$$

maps the group of relative algebraic differential characters of degree 2 on X to the group of algebraic characters of degree 1 on K, which is the group of connections on K. Indeed this is an isomorphism (lemma 2.7). But to write connections on K as coming from differential characters on X allows to single out two types of classes, global decomposable classes and local classes, which we discuss in the sequel.

Let $\mathcal{D} = \sum m_i x_i$ be an effective divisor on X, and let $D = \sum x_i$ be the corresponding reduced divisor. We define a sheaf of meromorphic 1-forms

$$\Omega^1_X(\mathcal{D} - D) \subset \Omega^1_X\{\mathcal{D}\} \subset \Omega^1_X(\mathcal{D}) \tag{1.3}$$

as follows. If z is a local parameter at a point x of multiplicity m in $\mathcal{D}$, a 1-form is a section of $\Omega^1_X\{\mathcal{D}\}$ if it can be written in local coordinates in the form

$$\frac{f dz}{z^m} + \frac{\eta}{z^{m-1}} \tag{1.4}$$

where $f \in \mathcal{O}_{X,x}$ and $\eta \in \mathcal{O}_{X,x} \otimes \Omega^1_K$ are regular at x. We define $\Omega^p_X\{\mathcal{D}\} = \Omega^1_X\{\mathcal{D}\} \wedge \Omega^{p-1}_X \subset \Omega^p_X(\mathcal{D})$. There is an exact sequence

$$0 \to \mathcal{O}_X(\mathcal{D} - D) \otimes \Omega^1_K \to \Omega^1_X\{\mathcal{D}\} \to \omega_{X/K}(\mathcal{D}) \to 0 \tag{1.5}$$

where we write $\omega_{X/K}$ for the sheaf of relative 1-forms.

The graded algebra $\oplus_n \wedge^n (\Omega^1_X\{\mathcal{D}\})$ is closed under exterior d. Further, writing $\mathcal{I} = \mathcal{O}(-\mathcal{D}) \subset \mathcal{O}_X$ for the ideal sheaf, we have

$$d\log((1+\mathcal{I})^\times) \wedge \Omega^p_X\{\mathcal{D}\} \subset \Omega^{p+1}_X. \tag{1.6}$$

Let E be a vector bundle on X, and let

$$\nabla : E \to E \otimes \Omega^1_X\{\mathcal{D}\} \tag{1.7}$$

be an absolute connection (i.e. parameters from K are also differentiated).

Definition 1.1. The connection ∇ is *vertical*, if the curvature

$$\nabla^2 : E \to E(*D) \otimes \Omega^2_K.$$

We assume our connection ∇ is vertical. (Of course the most important case is that of a flat connection $\nabla^2 = 0$.) We write $\nabla_{/K} : E \to \omega_{X/K}(\mathcal{D})$ for the corresponding relative connection. Viewing $\mathcal{D}$ as a nilpotent subscheme of X, it is easy to check that ∇ induces a function linear "polar part" map

$$(1.8) \qquad \nabla_{\mathcal{D}} : E|_{\mathcal{D}} \to E \otimes \left(\Omega^1_X\{\mathcal{D}\}/\Omega^1_X\right) \qquad \text{(Absolute)}$$
$$\nabla_{\mathcal{D}/K} : E|_{\mathcal{D}} \to E \otimes \omega_{\mathcal{D}/K} = E \otimes (\omega_{X/K}(\mathcal{D})/\omega_{X/K}) \quad \text{(Relative)}$$

Definition 1.2. The connection (1.7) is said to be *admissible* if the relative polar parts map $\nabla_{\mathcal{D}/K} : E|_{\mathcal{D}} \to E \otimes \omega_{\mathcal{D}/K}$ is an isomorphism in a singular point of multiplicity ≥ 2, and if in a singular point of multiplicity 1, Deligne's condition [6] that the eigenvalues of the residue do not belong to $\{0, 1, 2 \dots\}$ is fulfilled.

Notice that the notion of admissibility depends on the extension of E to all of X. Although the definition makes reference only to the relative connection, admissibility depends also on the absolute connection, which is required to take values in $\Omega^1_X\{\mathcal{D}\} \subset \Omega^1_X(\mathcal{D})$. It is a local formal property. The motivation for this definition comes from a struture theorem ([8], [12], [9], compare also with [10], p.124) asserting that locally formally, after ramification of the curve, a flat connection (E, ∇) becomes a direct sum of summands $L \otimes \Lambda$, where Λ is a higher rank flat connection with logarithmic singularities, and L is either a rank 1 trivial connection or a rank 1 flat connection with multiplicity ≥ 2 (strictly speaking, this is proven in [9] only over $\mathbb{C}$, outside of a Baire set). Since rank 1 vertical connections are admissible ([3], lemma 3.1), we see that up to ramification, any flat connection is locally of sum of admissible and logarithmic connections.

For a rank 1 vertical connection we may view

$$(1.9) \qquad \nabla_{\mathcal{D}/K} : E|_{\mathcal{D}} \cong E|_{\mathcal{D}} \otimes \omega_{X/K}(\mathcal{D})|_{\mathcal{D}}$$

as defining a trivialization of $\omega_{X/K}(\mathcal{D})|_{\mathcal{D}}$, i.e. a class $(\omega_{X/K}(\mathcal{D}), \nabla_{\mathcal{D}/K}) \in \mathrm{Pic}(X, \mathcal{D})$. There is a cohomological pairing $(j : X - D \hookrightarrow X)$

$$(1.10) \qquad \{\ ,\ \} : \mathrm{Pic}(X, \mathcal{D}) \otimes \mathbb{H}^1(X, j_*\mathcal{O}^\times_{X-D} \to \Omega^1_X\{\mathcal{D}\})$$
$$\to AD^2(X) = \mathbb{H}^2(X, \mathcal{K}_2 \xrightarrow{d\log} \Omega^2_X)$$

and in the rank 1 case (see main theorem of [3])

$$(1.11) \qquad \det H_{DR}(U/K, \nabla_{/K}) = -f_*\{(\omega_{X/K}(\mathcal{D}), \nabla_{\mathcal{D}/K}), (E, \nabla)\}$$
$$\in \Omega^1_K / d\log K^\times \otimes \mathbb{Q}.$$

In higher rank, however, we have examples of connections (E, ∇) with $(\det(E), \det(\nabla))$ trivial but non-trivial Gauß-Manin determinant connection (see [4], remark 3.3, equation 3.27, and remark 2.9).

Still, the choice of a meromorphic section $s \in \omega_{X/K}(\mathcal{D}) \otimes K(X)$ which generates the sheaf at the points of $\mathcal{D}$ defines a rigidification $c_1(\omega_{X/K}(\mathcal{D}), s) \in \mathrm{Pic}(X, \mathcal{D})$, and allows to define a class

$$\{c_1(\omega_{X/K}(\mathcal{D}), s), \det(E, \nabla)\} \in AD^2(X).$$

We refer to it as the global factor (see (2.5)).

In each singularity, we define local factors (see proposition 2.4) which play the rôle of the local epsilon factors defined to express the global epsilon of an ℓ-adic sheaf. If $A_i = g_i \frac{dz}{z^m} + \frac{\eta_i}{z^{m-1}}$ is the local equation of ∇ in a local basis around the point $a_i \in D$, then one defines $\mathrm{Tr} dg_i g_i^{-1} A_i \in AD^2(X/K)$. We also define a 2-torsion local factor $\frac{1}{2} d \log(\det g_i(a_i))) \in AD^2(X/K)$ (see definition 2.5).

The main theorem of this article says

Theorem 1.3. *Let (E, ∇) be an admissible connection on $\mathbb{P}^1_K$ having at least one point of multiplicity ≥ 2. Then*

$$\det H_{DR}(U, \nabla_{/K}) = -f_*\{c_1(\omega_{X/K}(\mathcal{D}), s), \det(E, \nabla)\} +$$
$$\sum_i \mathrm{res}_{a_i} \mathrm{Tr}\left(dg_i g_i^{-1} A_i\right) + \frac{1}{2} \sum_i m_i d \log(\det(g_i(a_i)))$$
$$\in \Omega^1_K / d \log(K^\times).$$

(See theorem 2.8 for a slightly more precise formulation).

The cases rank $(E) = 1$, resp. ∇ with logarithmic poles, were considered in the earlier articles [3], resp. [2], except that for the rank 1 case, our results did not include torsion. In those two cases, there is a well defined class $\gamma(E, \nabla) \in AD^2(X)$ such that $f_*\gamma(E, \nabla) = \det H_{DR}(U/K, \nabla_{/K})$. In the higher rank, non-logarithmic case considered here, one still has the global factor in $AD^2(X)$, but the local factors are well defined in $AD^2(X/K)$ only (see proposition 2.4).

It would be of great interest to find some variant of this formula which applied to epsilon factors for ℓ-adic sheaves.

We now discuss the proof of the main theorem. Let K be a field in characteristic 0. A classical theorem by Euler ([11], III, 6, lemma 2) asserts that if $g \in K[t]$ is a polynomial, and $h \in L = K[u]/gK[u]$, then the trace of multiplication by h, viewed as a K-linear map from L to itself, is computed by

$$-\mathrm{res}_{u=\infty} dg g^{-1} h.$$

We define a generalization of this in the non-commutative situation as follows. Let V be a finite dimensional K-vectorspace, and $g =$

$\sum_{i=0} g_i u^i \in \mathrm{End}(V)[u]$ be a polynomial with coefficients in the endomorphisms of E, such that the leading coefficient $g_m \in \mathrm{Aut}(V)$ is invertible. The invertibility of g_m allows us to write an element of $W = V[u]/gV[u]$ as the class of an element $\sum_{r=0}^{m-1} v_i u^i$, where $v_i \in V$, yielding a splitting $\sigma : W \to V[u]$ of the natural projection $p : V[u] \to W$. For any $h \in \mathrm{End}(V)[u]$, $\phi(h) := p \circ h \circ \sigma : W \to W$ will have a trace. Proposition 5.1 says

$$\mathrm{Tr}_{V[u]/gV[u]}(\phi(h)) = -\mathrm{Tr}_V \mathrm{res}_{u=\infty}(dgg^{-1}h).$$

The second point is to relate the trace of this linear operator with the trace of a differential operator. The crucial case to understand is that of a connection on a trivial bundle $E = V \otimes_K \mathcal{O}_{\mathbb{P}^1}$ on $\mathbb{P}^1_K$. Such a connection is given by a matrix which has the shape

$$\sum_{i=1}^{N} \sum_{r=1}^{m_i} \frac{g_r^{(i)} d(t-a_i)}{(t-a_i)^r} + \eta = g + \eta, \tag{1.12}$$

where $g_r^{(i)} \in \mathrm{End}(V)$ and $\eta \in \mathrm{End}(V) \otimes \Omega^1_K \otimes \mathcal{O}_{\mathbb{P}^1_K}(*D)$. We write g also for the corresponding matrix of relative forms, so $\nabla_{/K} = d + g$. We fix a certain finite-dimensional vector subspace $\sigma : H \hookrightarrow H^0(\mathbb{P}^1_K, V \otimes_K \omega(*D))$ such that composition with the natural projections give isomorphisms

$$\begin{aligned} H \xrightarrow{\sigma} H^0(\mathbb{P}^1_K, V \otimes_K \omega(*D)) &\xrightarrow{p_\nabla} H^1_{DR}(U/K, \nabla_{/K}) \\ &= H^0(\mathbb{P}^1_K, V \otimes_K \omega(*D))/\mathrm{Im}\nabla_{/K} \end{aligned} \tag{1.13}$$

$$H \xrightarrow{\sigma} H^0(\mathbb{P}^1_K, V \otimes_K \omega(*D)) \xrightarrow{p_g} H^0(\mathbb{P}^1_K, V \otimes_K \omega(*D))/\mathrm{Im}\ g$$

The operators

$$\eta_\nabla := (p_\nabla \circ \sigma)^{-1} p_\nabla \circ \eta \circ \sigma : H \to H \otimes \Omega^1_K$$
$$\eta_\gamma := (p_g \circ \sigma)^{-1} p_g \circ \eta \circ \sigma : H \to H \otimes \Omega^1_K$$

will be referred to as the (Gauß-Manin) de Rham operator and Higgs operator respectively. The traces of these operators play a central role in the Gauß-Manin determinant, and the remarkable fact is that for an admissible, vertical connection on a trivial bundle on $\mathbb{P}^1_K$ one finds

$$\mathrm{Tr}(\eta_\nabla - \eta_\gamma) \equiv \frac{1}{2} \sum_i m_i d\log(\det(g^{(i)}_{m_i})) \mod d\log(K^\times). \tag{1.14}$$

(Here the $g^{(i)}_{m_i}$ are as in (1.12).) This result is theorem 3.6. It is reminiscent of Hitchin's comparison of de Rham and Higgs twisted cohomologies on projective manifolds, and of Kontsevich's theorem comparing de Rham and Higgs cohomology of df, where f is a regular function on a manifold. Algebraically, if

$$(\frac{g_m}{z^m}+\frac{g_{m-1}}{z^{m-1}}+\dots)dz+\frac{\eta_{m-1}}{z^{m-1}}+\frac{\eta_{m-2}}{z^{m-2}}+\dots,$$

represents the polar part of our admissible, vertical connection at a point $z=0$, the essential result (proposition 3.8) is that

$$\mathrm{Tr}\sum_{s=0}^{m-1} g_m^{-1}[g_{m-s},\eta_s]$$

is identically vanishing. We must confess that, even after performing the computation, we don't really understand its meaning.

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2. Admissible Connections

Let K be a function field over a field k of characteristic 0, $f: X \to \operatorname{Spec} K$ be a smooth projective curve, $j: U \subset X$ a non-trivial Zariski open set such that the closed points of $D = X \setminus U$ are K rational points, and (E,∇) a global connection of rank r on X which is regular on U. Barring express mention to the contrary, we shall always assume ∇ to be vertical (definition 1.1). We shall need a small generalization of the notion of admissibility introduced in definition 1.2.

Definition 2.1. The connection (E,∇) is *pseudo-logarithmic* at $x \in X \setminus U$ if the local equation of ∇ in some basis of E has the shape

$$A = g\frac{dz}{z}+\frac{\eta}{z},$$

where z is a local parameter around x, $\eta \in M(r\times r, \Omega^1_K \otimes \mathcal{O}_X)$, $g \in GL(r,\mathcal{O}_x)$. The connection (E,∇) is *pseudo-admissible* if it is admissible in singularities with multiplicities $m \geq 2$ and pseudo-logarithmic in points with multiplicities $m=1$.

We will use a very special simple shape of pseudo-logarithmic singularities, which we single out in the following definition.

Definition 2.2. A *special pseudo-logarithmic point* of a connection (E,∇) is pseudo-logarithmic, and there is local basis

$$(e_\nu) = ((e_1,\dots,e_s),(e_{s+1},\dots,e_r))$$

with respect to which the block-matrix of the connection has the shape

$$\begin{pmatrix} A + m\frac{dz}{z} & zB \\ \frac{C}{z} & D + n\frac{dz}{z} \end{pmatrix},$$

where the connection matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has no poles and $m, n \in k$.

A connection is special pseudo-logarithmic if it is admissible, and special pseudo-logarithmic in pseudo-logarithmic points.

Working with pseudo-admissible connections will enable us to reduce the Gauß-Manin determinant computation for a general (E, ∇) on $\mathbb{P}^1$ to the case where $E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. We use the following:

Theorem 2.3. *[Compare with* [2]*, lemma 4.2 and reduction 4.1] Let (E, ∇) be an admissible connection on $\mathbb{P}^1_K$ having a singularity of multiplicity ≥ 2. Then there are finitely many points $p_i \in U(K)$, such that if $\lambda : V = U \setminus \{p_i\} \to \mathbb{P}^1_K$ denotes the open embedding, then $(E|_V, \nabla|_V)$ extends to a special pseudo-admissible connection $(\oplus_1^r \mathcal{O}_{\mathbb{P}^1_K}, \nabla)$ on $\mathbb{P}^1_K$ such that*

$$(\oplus_1^r \mathcal{O}_{\mathbb{P}^1_K}) \xrightarrow{\nabla_{/K}} \omega(\mathcal{D} + \sum_i p_i) \otimes (\oplus_1^r \mathcal{O}_{\mathbb{P}^1_K})$$

$$\to (\lambda_* E_V \xrightarrow{\nabla_{/K}} \lambda_*(\omega \otimes E_V))$$

is a quasiisomorphism.

Proof. Without loss of generality, we may assume that ∞ is a smooth point of the connection. Let x be a point of multiplicity ≥ 2, and let z be a local coordinate at x. Note the effect of twisting (i.e. replacing E by $E(Nx)$) is to replace the local connection matrix A at x with respect to a basis e_i with $A - N\frac{dz}{z}I$ for the basis $\frac{e_i}{z^N}$. In particular, x will remain an admissible singularity for the new connection. After such a twist, we may assume $E = \oplus_{i=1}^r \mathcal{O}(n_i)$, with $0 \leq n_1 \leq n_2 \leq \ldots$. We argue by induction on $n_r - n_1$. If $n_r - n_1 = 0$, we replace E by $E(-n_1 x)$ and argue as above.

Assume $n_r - n_1 > 0$. Let $E' = \oplus_{i=1}^{r-1} \mathcal{O}(n_i) \oplus \mathcal{O}(n_r - 1)$, and embed E' in E via $\mathcal{O}((n_r - 1)\infty) \to \mathcal{O}(n_r \infty)$. If z is a local parameter at ∞, and e_ν is a local basis of $\mathcal{O}(n_\nu)$ at ∞, then $((e_\nu, \mu \leq r-1), ze_r)$ is a local basis of E', and if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1}$$

is the local block matrix of the connection ∇ in the basis $((e_1, \dots, e_{r-1}), e_r)$, then

$$\begin{pmatrix} A & zB \\ \frac{C}{z} & D + \frac{dz}{z} \end{pmatrix} \tag{2.2}$$

is the local block matrix of the connection in the basis $((e_1, \dots, e_{r-1}), ze_r)$. If C has a local expansion $C = C_0 + C_1 z + \dots$, then the polar part of this connection is

$$\begin{pmatrix} 0 & 0 \\ \frac{C_0}{z} & \frac{dz}{z} \end{pmatrix}. \tag{2.3}$$

Thus replacing now E' by $E'' = E'(2\infty) \cong \oplus_{i=1}^{r-1}\mathcal{O}(n_i + 2) \oplus \mathcal{O}(n_r + 1)$, the local equation of the connection at ∞ becomes

$$\begin{pmatrix} A - 2\frac{dz}{z} & 0 \\ \frac{C}{z} & D - \frac{dz}{z} \end{pmatrix}, \tag{2.4}$$

and therefore, is pseudo-admissible. On the other hand, $n_r - n_1$ has decreased. We conclude by induction. $\square$

Next we describe the class $\gamma(E, \nabla) \in AD^2(X/K) := \mathbb{H}^2(X, \mathcal{K}_2 \xrightarrow{d\log} \Omega^2_X/\mathcal{O}_X \otimes \Omega^2_K)$ from theorem 2.8. If we choose a meromorphic section s of $\omega_{X/K}(\mathcal{D})$ which generates this sheaf in a neighborhood of $\mathcal{D}$, we may view s as defining a trivialization of $\omega_{X/K}(\mathcal{D})|_{\mathcal{D}}$, i.e. a class $(\omega(\mathcal{D}), s) \in \mathrm{Pic}(X, \mathcal{D})$. As in (1.10), we may consider the product

$$\{(\omega(\mathcal{D}), s), (\det(E), \det(\nabla))\} \in AD^2(X). \tag{2.5}$$

We refer to this class as the global factor.

Fix a basis e_i for E in a neighborhood of $\mathcal{D}$. the choice of e_i determines a local connection matrix A, so $\nabla = d + A$. Let $\mathcal{O}_{X,\mathcal{D}}$ denote the semi-local ring of functions regular at all points of $\mathcal{D}$. The choice of s determines $g \in GL_r(\mathcal{O}_{X,\mathcal{D}})$ such that the relative connection $\nabla_{/K} = d + gs$. Note the hypothesis of pseudo-admissibility insures that g is invertible.

The basic local invariant we consider is

$$\mathrm{Tr}(dgg^{-1}A) \in H^0\Big(X, \Omega^2_X\{\mathcal{D}\}\Big/\big(\Omega^2_X + \mathcal{O}_X(\mathcal{D} - D) \otimes \Omega^2_K\big)\Big). \tag{2.6}$$

The boundary map from the exact sequence

$$\begin{aligned} 0 \to \Omega^2_X/\mathcal{O}_X \otimes \Omega^2_K \to \Omega^2_X\{\mathcal{D}\}\Big/\Big(\mathcal{O}_X(\mathcal{D} - D) \otimes \Omega^2_K\Big) \\ \to \Omega^2_X\{\mathcal{D}\}\Big/\Big(\Omega^2_X + \mathcal{O}_X(\mathcal{D} - D) \otimes \Omega^2_K\Big) \to 0 \end{aligned} \tag{2.7}$$

together with the evident map $H^1(X, \Omega^2_X/\mathcal{O}_X \otimes \Omega^2_K) \to AD^2(X/K)$ enables us to define an element, which we denote by abuse of notation

$$\mathrm{Tr}(dgg^{-1}A) \in AD^2(X/K). \tag{2.8}$$

Proposition 2.4. *Let (E, ∇) be a pseudo-admissible vertical connection on X. The element $\mathrm{Tr}(dgg^{-1}A)$ (2.8) is independent of the choice of local bases around $\mathcal{D}$. The element*

$$-\{(\omega(\mathcal{D}), s), (\det(E), \det(\nabla))\} + \mathrm{Tr}(dgg^{-1}A) \in AD^2(X/K) \tag{2.9}$$

is independent both of the choice of local bases and the trivializing meromorphic section s of $\omega(\mathcal{D})$.

Proof. We show first that $\mathrm{Tr}(dgg^{-1}A)$ is independent of the local bases. We work locally around a point x which is a singular point of the connection with multiplicity m. We assume first that $s = \frac{dz}{z^m}$ for a local coordinate, and that the connection matrix is

$$A = \frac{gdz}{z^m} + \frac{\eta}{z^m}.$$

with $g \in GL_r(\mathcal{O})$ and $\eta \in M_r(\mathcal{O} \otimes \Omega^1_K)$. (This includes both the admissible and pseudolog cases.) We take a gauge transformation of the form $A \mapsto \phi A \phi^{-1} + d\phi\phi^{-1}$ with $\phi \in GL_r(\mathcal{O})$. This amounts to

$$g \mapsto \phi g \phi^{-1} + z^m \frac{d\phi}{dz}\phi^{-1};\ \eta \mapsto \phi\eta\phi^{-1} + z^m d_K\phi\phi^{-1}.$$

Here $d = d_z + d_K$. We claim

$$\mathrm{Tr}(dgg^{-1}A) \in \Omega^2_X(\mathcal{D}) \Big/ \Big(\mathcal{O}(\mathcal{D}) \otimes \Omega^2_K + \Omega^2_X\Big)$$

is invariant. We compute (writing $T(A) := \mathrm{Tr}(dgg^{-1}A)$, and computing modulo Ω^2_X)

$$
\begin{aligned}
&(2.10)\quad T(\phi A\phi^{-1} + d\phi\phi^{-1}) = \\
&\mathrm{Tr}\Big(d(\phi g\phi^{-1} + z^m\frac{d\phi}{dz}\phi^{-1})(\phi g\phi^{-1} + z^m\frac{d\phi}{dz}\phi^{-1})^{-1}(\phi A\phi^{-1} + d\phi\phi^{-1})\Big) \equiv \\
&\qquad \mathrm{Tr}\Big(d(\phi g\phi^{-1} + z^m\frac{d\phi}{dz}\phi^{-1})\phi g^{-1}\phi^{-1}\phi A\phi^{-1}\Big) \equiv \\
&\qquad \mathrm{Tr}\Big((d\phi\phi^{-1} + \phi dgg^{-1}\phi^{-1} - \phi g\phi^{-1}d\phi\phi^{-1}\phi g^{-1}\phi^{-1} \\
&\qquad\qquad + mz^{m-1}\frac{d\phi}{dz}g^{-1}\phi^{-1}dz)\phi A\phi^{-1}\Big) \equiv \\
&\mathrm{Tr}\Big(\phi^{-1}d\phi A + dgg^{-1}A - g\phi^{-1}d\phi g^{-1}A + mz^{m-1}\phi^{-1}\frac{d\phi}{dz}g^{-1}dz\wedge A\Big) \equiv \\
&\quad \mathrm{Tr}\Big(dgg^{-1}A + \phi^{-1}d\phi(A - g^{-1}Ag) + mz^{m-1}\phi^{-1}\frac{d\phi}{dz}g^{-1}dz\wedge A\Big) \equiv \\
&\qquad\qquad \mathrm{Tr}\Big(dgg^{-1}A + \phi^{-1}\frac{d\phi}{dz}g^{-1}dz(gA - Ag + mz^{m-1}A)\Big)
\end{aligned}
$$

(Note that $gA - Ag$ has entries in $\Omega^1_K \otimes K(X)$, justifying replacing $d\phi$ by $\frac{d\phi}{dz}dz$.) To show invariance, it will suffice to show

$$dz \wedge (gA - Ag + mz^{m-1}A) \equiv 0 \mod \Omega^2_X.$$

This expression can be written

$$(*) = \frac{dz}{z^m} \wedge [g, \eta] + m\frac{dz}{z} \wedge \eta.$$

Verticality gives

$$dg \wedge \frac{dz}{z^m} + m\eta \wedge \frac{dz}{z^{m+1}} + \frac{d_z\eta}{z^m} = \frac{dz}{z^{2m}}[g, \eta]$$

Multiplying through by z^m, the expression $(*)$ above becomes

$$(*) \equiv m\frac{dz}{z} \wedge \eta + m\eta \wedge \frac{dz}{z} \equiv 0 \mod \Omega^2_X.$$

It remains to show independence of s. Let $s' = fs$ where f is meromorphic on X and invertible on D. Consider a diagram

(2.11)

$$
\begin{array}{ccccc}
H^0(\mathcal{O}^\times_{\mathcal{D}}) & \xrightarrow{\partial} & H^1((1+\mathcal{I}_{\mathcal{D}})^\times) = \mathrm{Pic}(X, \mathcal{D}) & \longrightarrow & H^1(\mathcal{O}^\times_X) = \mathrm{Pic}(X) \\
\times & & \times & & \times \\
\Omega^1_X\{\mathcal{D}\}/\Big(\Omega^1_X + d\log j_*\mathcal{O}^\times_{X-D}\Big) & \longleftarrow & \mathbb{H}^1(j_*\mathcal{O}^\times_{X-D} \to \Omega^1_X\{\mathcal{D}\}) & \longleftarrow & \mathbb{H}^1(\mathcal{O}^\times_X \to \Omega^1_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2_X\{\mathcal{D}\}\Big/\Big(\Omega^2_X + d\log j_*\mathcal{K}_{2,\mathcal{O}_{X-D}}\Big) & \longrightarrow & AD^2(X/K) & = & AD^2(X/K)
\end{array}
$$

The classes of the rigidified bundles $(\omega(\mathcal{D}), s)$ and $(\omega(\mathcal{D}), fs)$ differ by $\partial(\tilde{f})$, where $\tilde{f} \in H^0(\mathcal{O}_{\mathcal{D}})$ is the image of f. It follows that (with notation as above)

$$\{(\omega(\mathcal{D}), fs)(\omega(\mathcal{D}), s)^{-1}, (\det(E), \det(\nabla))\} = \mathrm{Tr}(\frac{d\tilde{f}}{\tilde{f}}A) \tag{2.12}$$

Replacing s with fs in (2.9), this gives the desired invariance. □

To complete the construction of the class $\gamma(E, \nabla) \in AD^2(X/K)$ we need one more invariant. Let $x \in D$ and assume $m \geq 2$, where m is the multiplicity of x in $\mathcal{D}$. Let z be a local coordinate at x. Write the local relative connection $\nabla_{/K} = d + g\frac{dz}{z^m} = \frac{g_m dz}{z^m} + \frac{g_{m-1}dz}{z^{m-1}} + \ldots$ with $g_m \in GL_r(K)$.

Definition 2.5. If the multiplicity is ≥ 2, the invariant

$$\tau_x(E, \nabla) := \frac{m}{2} d\log(\det(g_m)) \in \frac{1}{2} d\log(K^\times)/d\log(K^\times)$$

is associated to the rank 1 quadratic form

$$\det g_m^m \in K^\times/(K^\times)^2 \cong H^1(K, \mathbb{Z}/2\mathbb{Z}).$$

In a point of multiplicity 1, we set $\tau_x(E, \nabla) = 1$.

A change of local gauge replaces g with $hgh^{-1} + z^m \frac{dh}{dz} h^{-1}$ with $h \in GL_r(K[[z]])$. It follows that $\det(g_m)$ is invariant under gauge transformation of $m \geq 2$. On the other hand, replacing z with $z' = uz$ leads to $g'_m = u^{m-1} g_m$, whence $\det(g'_m) = u^{r(m-1)} \det(g_m)$ and

$$\frac{m}{2} d\log(\det(g'_m)) = \frac{m}{2} d\log(\det(g_m)) + \frac{rm(m-1)}{2} d\log(u),$$

so the definition is independent of the choice of z.

For $x \in X(K)$, we define a map

$$\rho_x : K^\times/K^{\times 2} \to AD^2(X/K) \tag{2.13}$$

as follows. One has a map of exact sequence of complexes

(2.14)

$$\begin{array}{ccccc}
\mathcal{K}_{2X} & \longrightarrow & j_{x*}\mathcal{K}_{2,X-\{x\}} & \longrightarrow & i_{x*}K^\times \\
\downarrow & & \downarrow & & \downarrow a \\
\Omega^2_X/\mathcal{O}_X \otimes \Omega^2_K & \longrightarrow & \Omega^2_X\{x\}/\mathcal{O}_X \otimes \Omega^2_K & \longrightarrow & \Omega^2_X\{x\}/\Big(\mathcal{O}_X \otimes \Omega^2_K + \Omega^2_X\Big).
\end{array}$$

Write z for a local coordinate and let a be as in (2.14). The mapping

$$K^\times \otimes \mathbb{Q}/\mathbb{Z} \to \mathrm{coker}(a); \quad \kappa \otimes \frac{1}{n} \mapsto \frac{1}{n}\frac{dz}{z} \wedge \frac{d\kappa}{\kappa} \tag{2.15}$$

is well-defined. We compose this with the boundary map from (2.14) to define ρ_x.

Definition 2.6. Let (E, ∇) be a pseudo-admissible connection as above. Then

$$\begin{aligned} (2.16)\quad \gamma(E,\nabla) = -\{(\omega(\mathcal{D}), s), (\det(E), \det(\nabla))\} \\ + \operatorname{Tr}(dgg^{-1}A) + \sum_{x\in\mathcal{D}, m_x\geq 2} \rho_x(\tau_x(E,\nabla)) \in AD^2(X/K). \end{aligned}$$

We continue to assume $f : X \to \operatorname{Spec}(K)$ is a smooth, projective curve. The transfer map $f_* : AD^2(X/K) \to \Omega^1_K/d\log(K^\times)$ is defined as follows. We remark that $H^2(X, \mathcal{K}_2) = (0)$ and $\Omega^2_X/\mathcal{O}_X \otimes \Omega^2_K \cong \omega_{X/K} \otimes \Omega^1_K$, and we define f_* from the diagram

$$(2.17)\quad \begin{array}{ccccccc} H^1(X,\mathcal{K}_2) & \longrightarrow & H^1(X,\omega_{X/K})\otimes\Omega^1_K & \longrightarrow & AD^2(X/K) & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \mathrm{Tr}} & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle f_*} & & \\ K^\times & \longrightarrow & \Omega^1_K & \longrightarrow & \Omega^1_K/d\log(K^\times) & \longrightarrow & 0 \end{array}$$

The check that with $\gamma(E,\nabla)$ defined as in (2.16), $f_*\gamma(E,\nabla)$ has the form as in theorem 1.3 is straightforward and will be omitted. Since the trace map $\operatorname{Tr} : H^1(X,\mathcal{K}_2) \to K^\times$, which is simply defined on the generators $\oplus_x\lambda_x \in \oplus_{x\in X^{(1)}} K(x)^\times$ by $\operatorname{Tr}_{[K(x):K]} d\log\lambda_x$, is surjective, we obtain the

Lemma 2.7. *The transfer map*

$$f_* : AD^2(X/K) \to AD^1(K) = \Omega^1_K/d\log K^\times$$

is an isomorphism.

Now we are in the position to give a slightly more precise formulation of our main theorem.

Theorem 2.8. *Let (E,∇) be a special pseudo-admissible connection on $\mathbb{P}^1_K$, smooth over $\emptyset \neq U \subset \mathbb{P}^1_K$, such that the singularities $\mathbb{P}^1_K \backslash U = D$ of ∇ consist of K-rational points. Then, with the notation of definition 2.6, one has*

$$\det H_{DR}(U/K, \nabla_{/K}) = f_*\gamma(E,\nabla) \in AD^1(K) = \Omega^1_K/d\log K^\times.$$

Finally, we take a moment to point out some simple consequences of theorem 2.8.

Remark 2.9. (i). Let (E,∇) be an admissible connection on $\mathbb{P}^1_K$, and let $(E^\vee, \nabla^\vee)$ be the dual connection. then

$$(2.18)\quad \det H_{DR}(E,\nabla) \cong \det H_{DR}(E^\vee,\nabla^\vee)^{(-1)}.$$

Indeed, replacing E with $E^\vee$ replaces g with $-{}^tg$ and A with $-{}^tA$. Verticality and admissibility imply that $[g, A]$ has no pole, so

$$\begin{aligned}\mathrm{Tr}(d(-{}^tg)(-{}^tg)^{-1}(-{}^tA)) &= -\mathrm{Tr}({}^t(g^{-1}dg){}^tA) = -\mathrm{Tr}(g^{-1}dgA)\\ &= -\mathrm{Tr}(dgAg^{-1}) = -\mathrm{Tr}(dgg^{-1}A).\end{aligned}$$

(ii). In [4], remark 3.3, equation 3.27, we give an example of an admissible connection on $\mathbb{P}^1_K$ with trivial determinant, for which the determinant of the Gauß-Manin connection is not torsion. More generally, given the main theorem of [3], if (L, ∇_L) and (M, ∇_M) are two connections, say on $\mathbb{P}^1_K$, with the same singularities $\mathcal{D}$, which are generic enough so that the singularites of $(L \otimes M, \nabla_L \otimes \nabla_M)$ are exactly $\mathcal{D}$ as well, then det A, with

$$A = \begin{pmatrix} L \otimes M & 0 & 0 \\ 0 & L^{-1} & 0 \\ 0 & 0 & M^{-1} \end{pmatrix} \tag{2.19}$$

is trivial, while the main theorem of [3] says that the Gauß-Manin determinant is computed by

$$\begin{aligned} &c_1(\omega(\mathcal{D}), \Gamma_L + \Gamma_M) \cdot c_1(\nabla_L + \nabla_M) \\ &-c_1(\omega(\mathcal{D}), \Gamma_L) \cdot c_1(\nabla_L) - c_1(\omega(\mathcal{D}), \Gamma_M) \cdot c_1(\nabla_M), \end{aligned} \tag{2.20}$$

where Γ_L and Γ_M are the principal parts of ∇_L and ∇_M. For generic L and M, this won't vanish.

3. Higgs and de Rham Traces

In this section we introduce the concepts of Higgs and de Rham operators associated to a vertical pseudo-admissible connection on the trivial bundle on $\mathbb{P}^1_K$, and analyse the difference between the traces of those operators.

It will be convenient to write $E = V \otimes \mathcal{O}$ with $V = \Gamma(\mathbb{P}^1, E)$. Let (e_μ), $\mu = 1, \ldots, r$ be a basis of V. The connection ∇ will be a vertical pseudo-admissible connection on E with poles on $D = \sum_{i=1}^N (a_i)$ where for simplicity we take $a_i \in \mathbb{P}^1(K)$. We assume $\infty \notin D$. The relative connection is given by

$$\nabla_{/K} e_\mu = \sum_{i=1}^N \sum_{r=1}^{m_i} \frac{g_r^{(i)}(e_\mu)dt}{(t-a_i)^r}. \tag{3.1}$$

Here the $g_r^{(i)}$ are matrices with entries in K. The admissibility condition implies that $g_{m_i}^{(i)}$ is invertible over $\mathcal{O}_{\mathbb{P}^1_K, a_i}$. Regularity of the connection at infinity means

$$\sum_{i=1}^{N} g_1^{(i)} = 0. \tag{3.2}$$

Definition 3.1. Let $M_i = m_i - 1$ if $m_i \geq 2$, and $M_i = m_i = 1$ else.

The absolute connection has the following equation

$$\nabla e_\mu = \sum_{i=1}^{N} \sum_{r=1}^{m_i} \frac{g_r^{(i)}(e_\mu) d(t-a_i)}{(t-a_i)^r} + \sum_{i=1}^{N} \sum_{r=1}^{M_i} \frac{\eta_r^{(i)}(e_\mu)}{(t-a_i)^r} + \eta_0(e_\mu), \tag{3.3}$$

where the η are matrices with entries in Ω^1_K.

Definition 3.2. We define γ_K by $\nabla_{/K} = d + \gamma_K$, thus concretely

$$\gamma_K = \sum_{i=1}^{N} \sum_{r=1}^{m_i} \frac{g_r^{(i)} dt}{(t-a_i)^r}.$$

We define η by

$$\eta = \sum_{i=1}^{N} \sum_{r=1}^{M_i} \frac{\eta_r^{(i)}}{(t-a_i)^r} + \eta_0,$$

and γ by

$$\gamma = \sum_{i=1}^{N} \sum_{r=1}^{m_i} \frac{g_r^{(i)} d(t-a_i)}{(t-a_i)^r},$$

so that $\nabla = d + \gamma + \eta$.

We have natural identifications

$$\Gamma(E(*D)) = V[\frac{1}{t-a_1}, \ldots, \frac{1}{t-a_N}] \tag{3.4}$$

$$\Gamma(E \otimes \omega(*D)) \subsetneq V[\frac{1}{t-a_1}, \ldots, \frac{1}{t-a_N}] dt \tag{3.5}$$

where the strict inclusion in (3.5) comes from the requirement of no poles at infinity.

The following lemma will be useful in the sequel.

Lemma 3.3. *For integers $r, s \geq 1$ one has a formal identity*

$$\frac{1}{(t-a)^r (t-b)^s} = \sum_{p=1}^{r} \frac{A_p(a,b)}{(t-a)^p} + \sum_{q=1}^{s} \frac{B_q(a,b)}{(t-b)^q}.$$

One has

$$A_r = (a-b)^{-s}; \quad B_s = (-1)^r (a-b)^{-r} = (b-a)^{-r}.$$

The partial fraction expansion of $\frac{1}{(t-b)^s(t-a)}$ *begins*

$$\frac{1}{(t-b)^s(t-a)} = \frac{(a-b)^{-s}}{t-a} - \frac{(a-b)^{-s}}{t-b} + \dots$$

In particular, we have

$$\begin{aligned}(3.6)\quad & \frac{1}{(t-b)^s}\Big(\frac{d(a-b)\wedge dt}{t-a} - \frac{d(a_N-b)\wedge dt}{t-a_N}\Big) \\ &= (a-b)^{-s}d(a-b)\wedge dt\Big(\frac{1}{t-a}-\frac{1}{t-a_N}\Big) \\ &+ \Big((a_N-b)^{-s}d(a_N-b)\wedge dt - (a-b)^{-s}d(a-b)\wedge dt\Big) \\ &\times\Big(\frac{1}{t-b}-\frac{1}{t-a_N}\Big) + \text{ terms involving } \frac{1}{(t-b)^r} \text{ for } r\geq 2.\end{aligned}$$

Proof. We have

$$\begin{aligned}(3.7)\quad & \frac{1}{(t-b)^s(t-a)^r} \\ &= \frac{1}{(s-1)!(r-1)!}\Big(\frac{d}{db}\Big)^{s-1}\Big(\frac{d}{da}\Big)^{r-1}\Big(\frac{(a-b)^{-1}}{t-a} - \frac{(a-b)^{-1}}{t-b}\Big).\end{aligned}$$

The formulas in the lemma follow easily from this. □

Definition 3.4. We denote by $H \overset{\sigma}{\hookrightarrow} \Gamma(E\otimes\omega(*D))$ be the K-vector subspace with basis

$$(3.8)\qquad \frac{e_\mu dt}{(t-a_i)^r},\ \begin{cases} 2\leq r\leq m_i & 1\leq i\leq N-1 \\ 2\leq r\leq m_N-1 & i=N\end{cases},$$

$$e_\mu dt\Big(\frac{1}{t-a_i}-\frac{1}{t-a_N}\Big),\ i<N.$$

There are two splittings of σ,

$$\pi_\gamma : \Gamma(E\otimes\omega(*D)) \to \Gamma(E\otimes\omega(*D))/\gamma_K\Gamma(E(*D)) \cong H$$
$$\pi_\nabla : \Gamma(E\otimes\omega(*D)) \to \Gamma(E\otimes\omega(*D))/\nabla_{/K}\Gamma(E(*D)) \cong H$$

There is a multiplication map

$$\eta : \Gamma(E\otimes\omega(*D)) \to \Gamma(E\otimes\omega(*D))\otimes\Omega^1_K.$$

We define now two K-linear operators.

Definition 3.5. The composite map

$$\eta_\gamma := \pi_\gamma\circ\eta\circ\sigma : H\to H\otimes\Omega^1_K$$

(resp.

$$\eta_\nabla := \pi_\nabla\circ\eta\circ\sigma : H\to H\otimes\Omega^1_K)$$

will be called the *Higgs* (resp. *de Rham*) operator. Similarly, if h is one of the terms $\frac{\eta_r^{(i)}}{(t-a_i)^r}$ appearing in the definition of η, we denote by h_γ and h_∇ the corresponding Higgs and de Rham operators.

The rest of this section is devoted to the comparison of the trace of those two K-linear operators. We will show

Theorem 3.6. *Let (E, ∇) be a pseudo-admissible connection on the trivial bundle $E \cong \oplus_1^r \mathcal{O}_{\mathbb{P}^1_K}$ on $\mathbb{P}^1_K$ having at least one singularity of order ≥ 2. Then*

$$\mathrm{Tr}(\eta_\gamma - \eta_\nabla) \equiv \frac{1}{2} \sum_{m_i \geq 2} m_i d \log(\det(g_{m_i}^{(i)})) \mod d\log K^\times.$$

We assume henceforth that $m_N \geq 2$.

Suppose first a_i is a pseudo-logarithmic point for the connection. We write $h = \frac{\eta^{(i)}}{t-a_i}$, and we compute $\mathrm{Tr}(h_\nabla) - \mathrm{Tr}(h_\gamma)$.

The notation $x = y + (H)$ will mean x and y differ by an element in H. The pattern is then we take x in the basis of H. We write

$$h(x) = \gamma_K y + (H) = \nabla_{/K} y' + (H). \tag{3.9}$$

Then

$$z := h_\nabla(x) - h_\gamma(x) = -dy' + \gamma_K(y - y'). \tag{3.10}$$

Of course, if $h(x) \in H$ then $y = y' = z = 0$. Also, we are only interested in the trace, so if the expansion of z in the basis of H does not involve x, we can ignore it. Suppose e.g. $x = \frac{e_\mu dt}{(t-a_j)^r}$, $j \neq i$. Then $h(x) = \frac{(*)dt}{(t-a_i)(t-a_j)^r} \in H$ (the condition to lie in H amounts to a bound on the pole order together with no pole of the differential form at infinity.) Thus such elements x contribute 0. Similarly, if $j \neq i$ then

$$\frac{(*)dt}{(t-a_i)(t-a_j)} - \frac{(*)dt}{(t-a_i)(t-a_N)} \in H \tag{3.11}$$

so

$$x = e_\mu dt \Big(\frac{1}{t-a_j} - \frac{1}{t-a_N} \Big) \tag{3.12}$$

contributes 0.

It remains to consider $x = e_\mu dt \Big(\frac{1}{t-a_i} - \frac{1}{t-a_N} \Big)$. We have

$$h(x) = \frac{\eta_1^{(i)}(e_\mu)dt}{(t-a_i)^2} + (H). \tag{3.13}$$

So we can take

$$y = \frac{(g_1^{(i)})^{-1}\eta_1^{(i)}(e_\mu)dt}{t-a_i}; \quad y' = \frac{(g_1^{(i)}-I)^{-1}\eta_1^{(i)}(e_\mu)dt}{t-a_i}. \tag{3.14}$$

Write $\gamma_K = \frac{g_1^{(i)}dt}{t-a_i} + \gamma_K'$. Then

$$z = \gamma_K'(y-y'). \tag{3.15}$$

Since we are interested in the trace, we need only consider the coefficient of $e_\mu dt\Big(\frac{1}{t-a_i} - \frac{1}{t-a_N}\Big)$ in the expansion of z in the basis of H. This coefficient is the coefficient of e_μ in

(3.16)
$$\gamma_K'|_{t=a_i}(y-y') = \sum_{\substack{j\neq i\\ r}} (a_i-a_j)^{-r} g_r^{(j)}\Big((g_1^{(i)}-I)^{-1} - (g_1^{(i)})^{-1}\Big)\eta_1^{(i)}(e_\mu).$$

Summing over μ yields finally

(3.17)
$$\mathrm{Tr}(h_\nabla - h_\gamma) = \mathrm{Tr}\Big(\sum_{\substack{j\neq i\\ r}} (a_i-a_j)^{-r} g_r^{(j)}\Big((g_1^{(i)}-I)^{-1} - (g_1^{(i)})^{-1}\Big)\eta_1^{(i)}\Big).$$

Next we consider i with $m_i \geq 2$. Take first

$$x = \frac{e_\mu dt}{(t-a_i)^r}; \ \begin{cases} 2\leq r\leq m_i & 1\leq i\leq N-1 \\ 2\leq r\leq m_N-1 & i=N \end{cases}.$$

Since we have already handled the $h = \frac{\eta^{(i)}}{t-a_i}$ in pseudo-logarithmic points, we introduce the notation

$$\eta' = \eta - \sum_{\mathrm{pseudo-log}} \frac{\eta^{(i)}}{t-a_i}, \tag{3.18}$$

$$\gamma_K' = \gamma_K - \sum_{\mathrm{pseudo-log}} \frac{g_i^{(i)}}{(t-a_i)}dt.$$

Take

$$y_0 = (g_{m_i}^{(i)})^{-1}\eta_{m_i-1}^{(i)}(e_\mu)/(t-a_i)^{r-1}. \tag{3.19}$$

It follows from lemma 3.3 that $\eta' x = \gamma_K' y_0 +$ lower order terms. Here "lower order terms" means terms with denominators $(t-a_j)^s$ where $s\leq m_j,\ j\neq i$ and $s\leq m_i+r-2$ for $j=i$. Now continue in this way, replacing y_0 by

$$y = y_0 + \sum_{s=0}^{r-2} \frac{v_s}{(t-a_i)^s}. \tag{3.20}$$

We may write

(3.21)

$$\eta'(e_\mu)dt/(t-a_i)^r = \gamma'_K y + \sum_{j=1}^{N-1}\sum_{u=1}^{m_j} \frac{w_{j,u}dt}{(t-a_j)^u} + \sum_{u=1}^{m_N-1} \frac{w_{N,u}dt}{(t-a_N)^u}.$$

From equations (3.19) and (3.20) we may also write

$$\begin{aligned}\frac{\eta'(e_\mu)dt}{(t-a_i)^r} &= (d+\gamma'_K)y + \sum_{j=1}^{N-1}\sum_{u=1}^{m_j} \frac{w_{j,u}dt}{(t-a_j)^u} + \sum_{u=1}^{m_N-1} \frac{w_{N,u}dt}{(t-a_N)^u} \\ &+ (r-1)(g^{(i)}_{m_i})^{-1}\eta^{(i)}_{m_i-1}(e_\mu)dt/(t-a)^r + \sum_{s=1}^{r-2} \frac{s v_s dt}{(t-a_i)^{s+1}}.\end{aligned} \tag{3.22}$$

For $r=1$ we may write for suitable $v, w_{j,u} \in \Gamma(E)$

(3.23)

$$\eta'(e_\mu)dt\Big(\frac{1}{t-a_i} - \frac{1}{t-a_N}\Big) = \gamma'_K v + \sum_{j=1}^{N-1}\sum_{u=1}^{m_j} \frac{w_{j,u}dt}{(t-a_j)^u} + \sum_{u=1}^{m_N-1} \frac{w_{N,u}dt}{(t-a_N)^u}.$$

Since $dv = 0$ in (3.23) we conclude from equations (3.17), (3.22) and (3.23) that

$$\begin{aligned}\mathrm{Tr}(\eta_\nabla - \eta_\gamma) &= \sum_{i=1}^{N-1}\sum_{r=2}^{m_i}(r-1)\mathrm{Tr}\Big((g^{(i)}_{m_i})^{-1}\eta^{(i)}_{m_i-1}\Big) \\ &+ \sum_{r=2}^{m_N-1}(r-1)\mathrm{Tr}\Big((g^{(N)}_{m_N})^{-1}\eta^{(N)}_{m_N-1}\Big) \\ &+ \sum_{\substack{i \\ m_i=1}} \mathrm{Tr}\Big(\sum_{\substack{j\neq i \\ r}}(a_i-a_j)^{-r}g^{(j)}_r\Big((g^{(i)}_1 - I)^{-1} - (g^{(i)}_1)^{-1}\Big)\eta^{(i)}_1\Big).\end{aligned} \tag{3.24}$$

(Notice that replacing γ_K by γ'_K in (3.21), (3.22) and (3.23) will not affect the trace calculation.) We now use the verticality condition $dA =$

$A \wedge A \bmod \Omega^2_K \otimes K(X)$.

$$(3.25)\quad 0 = \sum_{i=1}^{N}\sum_{r=1}^{m_i} \frac{dg_r^{(i)}dt}{(t-a_i)^r} + \sum_{i=1}^{N}\sum_{s=1}^{M_i} \frac{s\eta_s^{(i)}dt}{(t-a_i)^{s+1}} + \sum_{i,j=1}^{N}\sum_{r,s=1}^{m_i,M_j}[g_r^{(i)},\eta_s^{(j)}]\frac{dt}{(t-a_i)^r(t-a_j)^s} + \sum_{i=1}^{N}\sum_{r=1}^{m_i}[g_r^{(i)},\eta_0]\frac{dt}{(t-a_i)^r}.$$

Dropping terms with poles at $t=a_i$ of degree $> M_i+1$, we find

$$(3.26)\quad 0 = \sum_{i=1}^{N}\sum_{r=1}^{m_i} \frac{dg_r^{(i)}dt}{(t-a_i)^r} + \sum_{i=1}^{N}\sum_{s=1}^{M_i} \frac{s\eta_s^{(i)}dt}{(t-a_i)^{s+1}} + \sum_{i,j=1;\ i\neq j}^{N}\sum_{r,s=1}^{m_i,M_j}[g_r^{(i)},\eta_s^{(j)}]\frac{dt}{(t-a_i)^r(t-a_j)^s} + \sum_{i=1}^{N}\sum_{r+s\le M_i+1}[g_r^{(i)},\eta_s^{(i)}]\frac{dt}{(t-a_i)^{r+s}} + \sum_{i=1}^{N}\sum_{r=1}^{m_i}[g_r^{(i)},\eta_0]\frac{dt}{(t-a_i)^r}.$$

In an admissible point, we calculate as before, multiplying through by

$$(g_{m_i}^{(i)})^{-1}(t-a_i)^{m_i}/dt$$

and set $t=a_i$ to get

$$(3.27)\quad 0 = (g_{m_i}^{(i)})^{-1}dg_{m_i}^{(i)} + (m_i-1)(g_{m_i}^{(i)})^{-1}\eta_{m_i-1}^{(i)} + \sum_{j\neq i}\sum_{s=1}^{M_j}(g_{m_i}^{(i)})^{-1}[g_{m_i}^{(i)},\eta_s^{(j)}]/(a_i-a_j)^s + \sum_{r+s=m_i}(g_{m_i}^{(i)})^{-1}[g_r^{(i)},\eta_s^{(i)}] + (g_{m_i}^{(i)})^{-1}[g_{m_i}^{(i)},\eta_0].$$

Taking traces gives

$$(3.28)\quad 0 = d\log(\det(g_{m_i}^{(i)})) + (m_i-1)\mathrm{Tr}((g_{m_i}^{(i)})^{-1}\eta_{m_i-1}^{(i)}) + \sum_{r+s=m_i}\mathrm{Tr}\Big((g_{m_i}^{(i)})^{-1}[g_r^{(i)},\eta_s^{(i)}]\Big).$$

On the other hand, in a pseudo-logarithmic point, we multiply through by $(t-a_i)^2/dt$ and then set $t=a_i$ getting

$$(3.29)\qquad \eta_1^{(i)} = [\eta_1^{(i)}, g_1^{(i)}].$$

Now discard those terms, multiply by $(t-a_i)/dt$ and set $t=a_i$. One gets

(3.30)
$$0 = dg_1^{(i)} + \sum_{j\neq i}\sum_{s=1}^{M_j}(a_i-a_j)^{-s}\Big([g_1^{(i)},\eta_s^{(j)}]+[g_s^{(j)},\eta_1^{(i)}]\Big)+[g_1^{(i)},\eta_0].$$

Formula (3.29) gives

$$\eta_1^{(i)}(g_1^{(i)}-I) = g_1^{(i)}\eta_1^{(i)}, \tag{3.31}$$

whence, assuming the indicated matrices invertible, one has

$$\eta_1^{(i)}(g_1^{(i)}-I)^{-1} = (g_1^{(i)})^{-1}\eta_1^{(i)}. \tag{3.32}$$

Using $\mathrm{Tr}(a[b,c])=0$ if $[a,b]=0$, multiplying equation (3.30) on the left by $(g_1^{(i)})^{-1}$ (resp. by $(g_1^{(i)}-I)^{-1}$) and taking traces yields

$$\begin{gathered}\mathrm{Tr}\Big(\sum_{j\neq i}\sum_{s=1}^{M_j}(a_i-a_j)^{-s}(g_1^{(i)})^{-1}[g_s^{(j)},\eta_1^{(i)}]\Big)\in d\log K^\times,\\ \mathrm{Tr}\Big(\sum_{j\neq i}\sum_{s=1}^{M_j}(a_i-a_j)^{-s}(g_1^{(i)}-I)^{-1}[g_s^{(j)},\eta_1^{(i)}]\Big)\in d\log K^\times.\end{gathered} \tag{3.33}$$

We now get

$$\begin{aligned}&\sum_{\substack{i\\ m_i=1}}\mathrm{Tr}\Big(\sum_{\substack{j\neq i\\ r}}(a_i-a_j)^{-r}g_r^{(j)}\Big((g_1^{(i)}-I)^{-1}-(g_1^{(i)})^{-1}\Big)\eta_1^{(i)}\Big)\\ &=\sum_{\substack{i\\ m_i=1}}\mathrm{Tr}\Big(\sum_{\substack{j\neq i\\ r}}(a_i-a_j)^{-r}\Big((g_1^{(i)}-I)^{-1}-(g_1^{(i)})^{-1}\Big)\eta_1^{(i)}g_r^{(j)}\Big)\\ &\equiv\sum_{\substack{i\\ m_i=1}}\mathrm{Tr}\left(\sum_{\substack{j\neq i\\ r}}(a_i-a_j)^{-r}\Big(\eta_1^{(i)}(g_1^{(i)}-I)^{-1}-(g_1^{(i)})^{-1}\eta_1^{(i)}\Big)g_r^{(j)}\right)\\ &\equiv 0 \mod (d\log K^\times).\end{aligned} \tag{3.34}$$

Now one can compare (3.24), (3.28), and (3.34) and deduce:

Proposition 3.7. *With notation as in definition 3.5 above, and assuming that a_N has multiplicty $m_N \geq 2$, one has*

$$\begin{aligned}\mathrm{Tr}(\eta_\nabla - \eta_\gamma) \equiv &- \sum_{i=1}^{N-1} \frac{m_i}{2}\Big(d\log(\det(g^{(i)}_{m_i})) + \mathrm{Tr}\sum_{r+s=m_i} (g^{(i)}_{m_i})^{-1}[g^{(i)}_r, \eta^{(i)}_s]\Big)\\ &- \frac{m_N-2}{2}\Big(d\log(\det(g^{(N)}_{m_N})) + \mathrm{Tr}\sum_{r+s=m_N} (g^{(N)}_{m_N})^{-1}[g^{(N)}_r, \eta^{(N)}_s]\Big)\\ &\hspace{8cm}\mod d\log K^\times.\end{aligned}$$

To complete the proof of theorem 3.6, we must show

Proposition 3.8. *Let (E, ∇) be a vertical admissible connection, with local equation*

$$\frac{g_m dz}{z^m} + \frac{g_{m-1}dz}{z^{m-1}} + \ldots + \frac{\eta_{m-1}}{z^{m-1}} + \frac{\eta_{m-2}}{z^{m-2}} + \ldots$$

in an admissible point. Then

$$\Phi = \mathrm{Tr}(g_m^{-1}\sum_{s=0}^{m-1}[g_{m-s}, \eta_s]) = 0$$

Proof. The connection ∇ is vertical. Vanishing for curvature terms involving z^{-p} for $p \geq m+1$ implies

$$\begin{gathered}[g_m, \eta_\ell] + [g_{m-1}, \eta_{\ell+1}] + \ldots [g_{\ell+1}, \eta_{m-1}] = 0\\ \text{for } \ell = 1, \ldots, m-1.\end{gathered} \tag{3.35}$$

The tactic is to eliminate first η_1 from Φ, then η_2 etc. One easily verifies matrix relations

$$\begin{gathered}[a^{-1}, b] = -a^{-1}[a, b]a^{-1}\\ \mathrm{Tr}(g^{-1}[a, b]) = -\mathrm{Tr}(a[g^{-1}, b]) = \mathrm{Tr}(ag^{-1}[g, b]g^{-1})\\ \mathrm{Tr}(ag^{-1}[b, \eta]g^{-1}) = \mathrm{Tr}(ag^{-1}b\eta g^{-1}) - \mathrm{Tr}(bg^{-1}ag^{-1}\eta).\end{gathered} \tag{3.36}$$

In particular,

$$\mathrm{Tr}(ag^{-1}[b, \eta]g^{-1}) = \mathrm{Tr}(ag^{-1}bg^{-1}[g, \eta]g^{-1}) \text{ if } \; ag^{-1}b = bg^{-1}a. \tag{3.37}$$

Write

$$\Phi = \mathrm{Tr} g_m^{-1}[g_{m-1}, \eta_1] + \mathrm{Tr}\sum_{s=2}^{m-1} g_m^{-1}[g_{m-s}, \eta_s]. \tag{3.38}$$

This yields

$$\Phi = -\mathrm{Tr} g_m^{-1} g_{m-1} g_m^{-1}[g_{m-1}, \eta_2] + \mathrm{Tr} g_m^{-1}[g_{m-2}, \eta_2] \tag{3.39}$$
$$-\mathrm{Tr} \sum_{s=3}^{m-1} g_m^{-1} g_{m-1} g_m^{-1}[g_{m-s+1}, \eta_s] + \mathrm{Tr} \sum_{s=3}^{m-1} g_m^{-1}[g_{m-s}, \eta_s].$$

Applying again now the relations (3.36) yields

$$\Phi = \mathrm{Tr}(-g_m^{-1} g_{m-1} g_m^{-1} g_{m-1} g_m^{-1} + g_m^{-1} g_{m-2} g_m^{-1})[g_m, \eta_2] \tag{3.40}$$
$$-\mathrm{Tr} \sum_{s=3}^{m-1} g_m^{-1} g_{m-1} g_m^{-1}[g_{m-s+1}, \eta_s] + \mathrm{Tr} \sum_{s=3}^{m-1} g_m^{-1}[g_{m-s}, \eta_s].$$

Assume inductively that for some $t \geq 2$, one can write Φ as follows:

$$\Phi = \mathrm{Tr}(\sum_{a=1}^{t} (-1)^{a-1} \sum_{\tau_1+\ldots+\tau_a=t} g_m^{-1} g_{m-\tau_1} \cdots g_m^{-1} g_{m-\tau_a} g_m^{-1})[g_m, \eta_t] \tag{3.41}$$
$$+\mathrm{Tr} \sum_{s=t+1}^{m-1} \sum_{\ell=0}^{t-1} (\sum_{a=0}^{\ell} (-1)^a \sum_{\tau_1+\ldots+\tau_a=\ell} g_m^{-1} g_{m-\tau_1} \cdots g_m^{-1} g_{m-\tau_a} g_m^{-1})[g_{m-s+\ell}, \eta_s].$$

Applying (3.35) to the first line, and isolating the terms in η_{t+1} and in η_s, $s \geq (t+2)$, one obtains

$$\Phi = F(t+1) + \tag{3.42}$$
$$\mathrm{Tr} \sum_{s=t+2}^{m-1} \sum_{\ell=0}^{t} (\sum_{a=0}^{\ell} (-1)^a \sum_{\tau_1+\ldots+\tau_a=\ell} g_m^{-1} g_{m-\tau_1} \cdots g_m^{-1} g_{m-\tau_a} g_m^{-1})[g_{m-s+\ell}, \eta_s],$$

with

(3.43)

$$F(t+1) = \mathrm{Tr}(\sum_{a=1}^{t} (-1)^a \sum_{\tau_1+\ldots\tau_a=t} g_m^{-1} g_{m-\tau_1} \cdots g_m^{-1} g_{m-\tau_a} g_m^{-1})[g_{m-1}, \eta_{t+1}]$$
$$+\mathrm{Tr} \sum_{\ell=0}^{t-1} (\sum_{a=0}^{\ell} (-1)^a \sum_{\tau_1+\ldots+\tau_a=\ell} g_m^{-1} g_{m-\tau_1} \cdots g_m^{-1} g_{m-\tau_a} g_m^{-1})[g_{m-t-1+\ell}, \eta_{t+1}].$$

It remains to arrange $F(t+1)$. To this aim, write

$$F(t+1) = \sum_{\ell=0}^{t} \sum_{a=0}^{\ell} \sum_{\tau_1+\ldots+\tau_a=\ell} \mathrm{Tr} \tag{3.44}$$
$$((-1)^a g_{m-\tau_1} g_m^{-1} \cdots g_{m-t-1+\ell} \eta_{t+1} g_m^{-1} - (-1)^a g_{m-t-1+\ell} g_m^{-1} \cdots g_{m-\tau_a} g_m^{-1} \eta_{t+1}).$$

Now we group those terms differently. To a tuple $(\tau_1, \ldots, \tau_a)$, with $\tau_1 + \ldots + \tau_a = \ell$, we associate the tuple $(\tau_1', \ldots, \tau_a')$ with $\tau_1' + \ldots \tau_a' + \tau_1 =$

$t+1$, $\tau'_a = t+1-\ell$, and otherwise $\tau_i = \tau'_{i-1}$ for $i \geq 2$. Using the first relation of (3.35) again, this gives for those 2 terms together

$$\mathrm{Tr}(-1)^a g_m^{-1} g_{m-\tau_1} g_m^{-1} \cdots g_{m-t-1+\ell} g_m^{-1} [g_m, \eta_{t+1}]. \tag{3.45}$$

This shows that the relation (3.41) is true, with t replaced by $t+1$. As the last equation of (3.35) for $\ell = m-1$ is $[g_m, \eta_{m-1}] = 0$, one obtains by induction that Φ vanishes on the variety defined by (3.35). $\square$

Remark 3.9. Writing $a_i = g_i$ and $b_i = \eta_{i-1}$, the above proposition can be restated as follows. Suppose $a(t) = a_m t^m + \ldots + a_1 t$ and $b(t) = b_m t^m + \ldots + b_1 t$ are polynomials with matrix coefficients satisfying $a(0) = b(0) = 0$ and a_m invertible. Assume $[a(t), b(t)] = c_m t^m +$ lower order terms. Then $\mathrm{Tr}(a_m^{-1} c_m) = 0$.

4. The Gauss-Manin Determinant: Step 1

In this section we begin the computation of the Gauß-Manin determinant appearing in the main theorem 2.8.

We keep the same notations as in section 3. In particular, E is a trivial bundle on $\mathbb{P}^1_K$ with basis e_μ, having at least one point of multiplicity ≥ 2, $D = \{a_1, \ldots, a_N\}$, and $H \hookrightarrow \Gamma(E \otimes \omega(*D))$ is the K-subspace with basis defined in 3.4. We continue to write $\nabla = d + \gamma + \eta$ and $\nabla_{/K} = d + \gamma_K$ as in definition 3.2.

The Gauß-Manin connection is computed from the diagram

(4.1)
$$\begin{tikzcd}
 & \Gamma(E(*D)) \arrow[r, equal] \arrow[d, "\nabla"] & \Gamma(E(*D)) \arrow[d, "\nabla_{/K}"] \\
\Gamma(E(*D)) \otimes \Omega^1_K \arrow[r] \arrow[d, "\nabla_{/K} \otimes 1"] & \Gamma(E \otimes \Omega^1(*D)) \arrow[r, shift right] \arrow[d, "\nabla"] & \Gamma(E \otimes \omega(*D)) \arrow[l, shift right, "s"'] \\
\Gamma(E \otimes \omega(*D)) \otimes \Omega^1_K \arrow[r, "\cong", "\iota"'] & \Gamma(E \otimes \Omega^2(*D)/F^2) &
\end{tikzcd}$$

Here in the central column Ω refers to the Kähler differentials $\Omega_{\mathbb{P}^1_K/k}$, and $F^2 := \mathcal{O}_{\mathbb{P}^1}(*D) \otimes \Omega^2_K \subset \Omega^2(*D)$. We are interested in the induced map from $H^1_{DR} = H^1_{DR/K}(\mathbb{P}^1 - D, (E, \nabla))$, which is the cokernel of the right hand column, to $H^1_{DR} \otimes_K \Omega^1_K$, which is the cokernel of the left hand column. Let $p : \Gamma(E \otimes \omega(*D)) \to H^1_{DR}$ be the projection. By construction, $H \hookrightarrow \Gamma(E \otimes \omega(*D))$ splits p. Let $q : H^1_{DR} \cong H$ denote the splitting. The section s is given on H by

$$(4.2)\quad s\Big(\frac{e_\mu dt}{(t-a_i)^r}\Big)=\frac{e_\mu d(t-a_i)}{(t-a_i)^r};$$
$$s\Big(e_\mu dt\big(\frac{1}{t-a_i}-\frac{1}{t-a_N}\big)\Big)=\frac{e_\mu d(t-a_i)}{t-a_i}-\frac{e_\mu d(t-a_N)}{t-a_N}.$$

Lemma 4.1. *The Gauß-Manin determinant is the trace of the map*

$$(q\otimes 1)\circ(p\otimes 1)\circ\iota^{-1}\circ\nabla\circ s: H\to H.$$

Proof. Straightforward. □

Explicitly, this map is obtained by applying the projection $(q\otimes 1)\circ(p\otimes 1)$ to the right hand side in

$$(4.3)\quad \frac{e_\mu dt}{(t-a_i)^r}\mapsto\sum_{j=1}^N\sum_{s=1}^{m_j}\frac{g_s^{(j)}(e_\mu)d(a_i-a_j)\wedge dt}{(t-a_j)^s(t-a_i)^r}+\sum_j\sum_{s=1}^{M_j}\frac{\eta_s^{(j)}(e_\mu)\wedge dt}{(t-a_j)^s(t-a_i)^r}$$

$$(4.4)\quad e_\mu dt\Big(\frac{1}{t-a_i}-\frac{1}{t-a_N}\Big)\mapsto$$
$$\sum_{j=1}^N\sum_{s=1}^{m_j}g_s^{(j)}(e_\mu)\Big(\frac{d(a_i-a_j)\wedge dt}{(t-a_j)^s(t-a_i)}-\frac{d(a_N-a_j)\wedge dt}{(t-a_j)^s(t-a_N)}\Big)$$
$$+\sum_j\sum_{s=1}^{M_j}\eta_s^{(j)}(e_\mu)\wedge dt\Big(\frac{1}{(t-a_j)^s(t-a_i)}-\frac{1}{(t-a_j)^s(t-a_N)}\Big).$$

We leave aside for the moment the terms in the trace involving η and focus on the trace of the map which we rewrite using lemma 3.3 as

$$(4.5)\quad \frac{e_\mu dt}{(t-a_i)^r}\mapsto\sum_{j=1}^N\sum_{s=1}^{m_j}g_s^{(j)}(e_\mu)d(a_i-a_j)\wedge dt\Big(\frac{(a_i-a_j)^{-s}}{(t-a_i)^r}+\dots\Big)$$

$$
\begin{aligned}
(4.6)\quad & e_\mu dt\Big(\frac{1}{t-a_i}-\frac{1}{t-a_N}\Big)\mapsto \\
& \sum_{j=1}^{N}\sum_{s=1}^{m_j} g_s^{(j)}(e_\mu)\Bigg[(a_i-a_j)^{-s}d(a_i-a_j)\wedge dt\Big(\frac{1}{t-a_i}-\frac{1}{t-a_N}\Big) \\
& +\Big((a_N-a_j)^{-s}d(a_N-a_j)\wedge dt-(a_i-a_j)^{-s}d(a_i-a_j)\wedge dt\Big) \\
& \qquad\qquad\times\Big(\frac{1}{t-a_j}-\frac{1}{t-a_N}\Big)+\dots\Bigg].
\end{aligned}
$$

Terms are to be dropped if some factor becomes 0. The terms represented by ellipses $(\dots)$ do not enter into the trace calculation. Also all terms lie in the K-span of the basis (3.19). Let Ψ denote the resulting endomorphism of H. The contribution to $\mathrm{Tr}(\Psi)$ from (4.5) is

$$
\begin{aligned}
(4.7)\quad & \sum_{i=1}^{N}(m_i-1)\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{s=1}^{m_j}\mathrm{Tr}(g_s^{(j)})(a_i-a_j)^{-s}d(a_i-a_j)\wedge dt \\
& \qquad -\sum_{j=1}^{N-1}\sum_{s=1}^{m_j}\mathrm{Tr}(g_s^{(j)})(a_N-a_j)^{-s}d(a_N-a_j)\wedge dt.
\end{aligned}
$$

The second sum arises because $\frac{e_\mu dt}{(t-a_N)^{m_N}}$ is not a basis element for H. The contribution from (4.6) is

$$
\begin{aligned}
(4.8)\quad & \sum_{i=1}^{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{s=1}^{m_j}\mathrm{Tr}(g_s^{(j)})(a_i-a_j)^{-s}d(a_i-a_j)\wedge dt \\
& \qquad +\sum_{j=1}^{N-1}\sum_{s=1}^{m_j}\mathrm{Tr}(g_s^{(j)})(a_N-a_j)^{-s}d(a_N-a_j)\wedge dt.
\end{aligned}
$$

In total, this gives

$$
\begin{aligned}
(4.9)\quad & \sum_{i=1}^{N}m_i\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{s=1}^{m_j}\mathrm{Tr}(g_s^{(j)})(a_i-a_j)^{-s}d(a_i-a_j)\wedge dt \\
& \qquad -\sum_{j=1}^{N-1}\sum_{s=1}^{m_j}\mathrm{Tr}(g_s^{(j)})(a_N-a_j)^{-s}d(a_N-a_j)\wedge dt.
\end{aligned}
$$

In fact, the above analysis of $\mathrm{Tr}(\Psi)$ omits some terms. On the right in (4.3) taking $j = N$ and $s = m_N$ gives a term

$$\frac{g^{(N)}_{m_N}(e_\mu)d(a_i - a_N) \wedge dt}{(t - a_N)^{m_N}(t - a_i)^r}.$$

Expanding this by lemma 3.3 yields a term (for $2 \leq r \leq m_i$)

$$\begin{aligned}(4.10)\quad & \frac{g^{(N)}_{m_N}(e_\mu)(a_N - a_i)^{-r}d(a_i - a_N) \wedge dt}{(t - a_N)^{m_N}} \\ & \qquad \equiv -\sum_{j,s}{}' \frac{g^{(j)}_s(e_\mu)(a_N - a_i)^{-r}d(a_i - a_N) \wedge dt}{(t - a_j)^s}.\end{aligned}$$

Here $\equiv$ means equivalent in $H^1_{DR} \otimes \Omega^1_K$. The prime in the sum means omit the pair $j = N,\ s = m_N$.

Similarly, from (4.4) we get a term

$$\begin{aligned}(4.11)\quad & \frac{g^{(N)}_{m_N}(e_\mu)(a_N - a_i)^{-1}d(a_i - a_N) \wedge dt}{(t - a_N)^{m_n}} \\ & \qquad \equiv -\sum_{j,s}{}' \frac{g^{(j)}_s(e_\mu)(a_N - a_i)^{-1}d(a_i - a_N) \wedge dt}{(t - a_j)^s}.\end{aligned}$$

Of course, in (4.10) and (4.11) the contribution to the trace comes from $j = i$. These precisely cancel the second double sum on the right in (4.11). Thus, one gets

$$(4.12)\qquad \mathrm{Tr}(\Psi) = \sum_{i=1}^{N} m_i \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{s=1}^{m_j} \mathrm{Tr}(g^{(j)}_s)(a_i - a_j)^{-s}d(a_i - a_j) \wedge dt.$$

Finally, comparing (4.3), (4.4), and definition 3.5, we have

Proposition 4.2. *With notation as above (definition 3.5 and equation* (4.12)*), the Gauß-Manin trace on* H^1_{DR} *is given by*

$$\begin{aligned}&\mathrm{Tr}_{GM}(H^1_{DR}) = \mathrm{Tr}(\Psi) + \mathrm{Tr}(\eta_\nabla) \\ &\overset{\text{thm.3.6}}{\equiv} \mathrm{Tr}(\Psi) + \mathrm{Tr}(\eta_\gamma) + \frac{1}{2}\sum_{i;\ m_i\geq 2} m_i d\log(\det(g^{(i)}_{m_i})) \mod d\log K^\times.\end{aligned}$$

5. The Higgs Trace

The purpose of this section and of the next one is to rewrite the Higgs trace $\mathrm{Tr}(\eta_\gamma)$ as a sum of terms which are in some sense local, associated to the singularities a_i of the connection. In the next section we will compute these local traces.

It will be convenient to write

$$\mathcal{V} := \Gamma(E(*D)); \quad \mathcal{W} := \Gamma(E\otimes\omega(*D)) \subset \mathcal{V}dt; \tag{5.1}$$

$$\gamma_K = \sum_{j=1}^{N}\sum_{r=1}^{m_j} \frac{g_r^{(j)}dt}{(t-a_j)^r} : \mathcal{V} \to \mathcal{W}.$$

(See equations (3.5) and definition 3.2).

We identify $H \hookrightarrow \mathcal{W}$ with basis (3.8). As in section 3, there is a splitting $H \cong \mathcal{W}/\gamma_K\mathcal{V}$. We write

$$\eta = \eta_0 + \sum_{j=1}^{N}\sum_{s=1}^{M_j} \frac{\eta_s^{(j)}}{(t-a_j)^s} = \eta_0 + \sum_j \eta^{(j)}. \tag{5.2}$$

The $\eta_s^{(j)}$ are matrices with entries in Ω^1_K. We view these objects as linear maps $H \to H \otimes_K \Omega^1_K$:

$$H \hookrightarrow \mathcal{W} \xrightarrow{\eta^{(i)}} \mathcal{W} \otimes \Omega^1_K \twoheadrightarrow H \otimes \Omega^1_K$$

Now fix an i. It will be convenient to put a_i at ∞, so we set

$$u := \frac{1}{t-a_i}. \tag{5.3}$$

Define

$$\begin{aligned} R := \Gamma(\mathbb{P}^1 - D, \mathcal{O}) &= K[\frac{1}{t-a_1}, \ldots, \frac{1}{t-a_N}] \\ &= K\Big[\frac{u}{1-(a_1-a_i)u}, \ldots, u, \ldots, \frac{u}{1-(a_N-a_i)u}\Big]. \end{aligned} \tag{5.4}$$

In the u coordinates,

$$\gamma_K = \sum g_s^{(j)}\Big(\frac{u}{1-(a_j-a_i)u}\Big)^s. \tag{5.5}$$

The basis of H is

$$\begin{aligned} &e_\mu dt\Big(\frac{u}{1-(a_j-a_i)u}\Big)^r; \quad 2 \le r \le m_j \text{ (resp. } 2 \le r \le m_N - 1), \\ &e_\mu dt\Big(\frac{u}{1-(a_j-a_i)u} - \frac{u}{1-(a_N-a_i)u}\Big) \\ &= e_\mu dt \frac{(a_j-a_N)u^2}{(1-(a_j-a_i)u)(1-(a_N-a_i)u)}; \quad j \ne N. \end{aligned} \tag{5.6}$$

Define

$$
\begin{aligned}
\theta &:= \prod_{j\neq i}(1-(a_j-a_i)u)^{m_j}, \\
\theta_1 &:= \theta/(1-(a_N-a_i)u), \\
g &= \theta\cdot\gamma_K = \sum_{j,s} g_s^{(j)}u^s.
\end{aligned}
\tag{5.7}
$$

Note that θ is a unit in R. We can write

$$
\begin{aligned}
g &= \prod_{j\neq i}(a_i-a_j)^{m_j} g_{m_i}^{(i)}u^m + \text{lower order terms} \\
&= u^2 + \text{higher order terms},
\end{aligned}
\tag{5.8}
$$

where $m=\sum m_j$. Let $V=\oplus_\mu Ke_\mu = \Gamma(\mathbb{P}^1, E)$ and write

$$H' := Vu^2dt\oplus\ldots\oplus Vu^{m-1}dt \subset V[u]dt. \tag{5.9}$$

As a consequence of (5.9) we have

$$\theta_1 H = H'. \tag{5.10}$$

We are interested in the trace of $\eta^{(i)} = \sum_{s=1}^{M_i}\eta_s^{(i)}u^s$. Consider the diagram

(5.11)

$$
\begin{tikzcd}
H \arrow[r, "\subset"] \arrow[d, "\cong"] & \mathcal{W} \arrow[r, "\eta^{(i)}"] \arrow[d, "\theta_1"', "\cong"] & \mathcal{W}\otimes\Omega_K^1 \arrow[r] & \mathcal{W}/\gamma_K\mathcal{V}\otimes\Omega_K^1 & H\otimes\Omega_K^1 \arrow[l, "\cong"'] \\
H' \arrow[r, "\subset"] \arrow[d, equal] & \mathcal{W} \arrow[r, "\eta^{(i)}"] & \mathcal{W}\otimes\Omega_K^1 \arrow[r] \arrow[u, "\theta_1^{-1}"', "\cong"] & \mathcal{W}/\gamma_K\mathcal{V}\otimes\Omega_K^1 \arrow[u, "\theta_1^{-1}"', "\cong"] & H'\otimes\Omega_K^1 \arrow[l, "\cong"'] \arrow[u, "\cong"] \\
H' \arrow[r, "\subset"] & u^2V[u]dt \arrow[r, "\eta^{(i)}"] \arrow[u, "\cup"] & u^2V[u]dt\otimes\Omega_K^1 \arrow[r] \arrow[u, "\cup"] & \frac{u^2V[u]}{gV[u]}dt\otimes\Omega_K^1 \arrow[u, "\cong"', "a"] & H'\otimes\Omega_K^1 \arrow[l, "\cong"'] \arrow[u, equal]
\end{tikzcd}
$$

Proposition 5.1. *The following maps have the same trace*

$$
\begin{gathered}
\eta^{(i)} : H \to H\otimes\Omega_K^1 \\
\eta^{(i)} : H' \to H'\otimes\Omega_K^1 \\
\eta^{(i)} : \frac{V[u]}{gV[u]}dt \to \frac{V[u]}{gV[u]}dt\otimes\Omega_K^1.
\end{gathered}
$$

Here the first two maps are given by horizontal rows in (5.11). The third is given by embedding

$$\frac{V[u]}{gV[u]}dt \hookrightarrow V[u]dt$$

via the basis $V \oplus Vu \oplus \ldots \oplus Vu^{m-1}$ *and then proceeding as in* (5.11).

Proof. The first two traces are equal by the diagram. For the third, note that since $g = u^2 +$ higher, it follows that one has an exact sequence compatible with the endomorphism multiplication by u

$$0 \to H' \to V[u]/gV[u]dt \to V[u]/u^2V[u]dt \to 0.$$

Since $\eta^{(i)}$ has no constant term in u, it acts nilpotently on the right. □

6. The Higgs trace: local calculation

In this section we give a formula for the trace of $\eta^{(i)}$ as in proposition 5.1, involving residues. As already mentioned in the introduction, the method here is reminiscent of the classical residue calculation for the trace of an element in a field extension.

To simplify, we write h in place of $\eta^{(i)}$. We also suppress the Ω^1_K and treat $h(u)$ as a polynomial with matrix coefficients. The case of matrices with coefficients in Ω^1_K follows immediately by applying arbitrary derivations $\Omega^1_K \to K$ to the entries.

To avoid confusion we write

$$\phi(h) : V[u]/gV[u] \hookrightarrow V[u] \xrightarrow{h\cdot} V[u] \twoheadrightarrow V[u]/gV[u], \tag{6.1}$$

where $V[u]/gV[u] \hookrightarrow V[u]$ is defined as in proposition 5.1 via the invertibility of the leading coefficient of g

$$V \oplus Vu \oplus \ldots \oplus Vu^{m-1} \hookrightarrow V[u]. \tag{6.2}$$

By (4.9) and admissibility, the leading coefficient of $g(u)$ is invertible so $g^{-1} \in \mathrm{End}(V)((u^{-1}))$. Write $dg = \frac{dg}{du}du$ and let

$$\mathrm{res}_{u=\infty} : \mathrm{End}(V)((u^{-1})) \to \mathrm{End}(V)$$

be the evident extension of the residue map.

One has

Proposition 6.1. *The notations being as above, one has*

$$\mathrm{Tr}_{V[u]/gV[u]}(\phi(h)) = -\mathrm{Tr}_V \mathrm{res}_{u=\infty}(dg g^{-1} h).$$

Proof. Write $g = a_0 u^m + a_1 u^{m-1} + \ldots + a_m$. Note that neither side of the identity changes if we replace g by $g a_0^{-1}$ so we may assume $a_0 = 1$.

Also, by linearity, we may assume $h = cu^p$. The matrix for the action of u on $V[u]/gV[u]$, the entries of which are themselves matrices, is

$$M = \begin{pmatrix} 0 & 0 & \ldots & -a_n \\ 1 & 0 & \ldots & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & -a_1 \end{pmatrix}$$

The matrix for u^p is M^p. We write $\mathrm{Tr}M^p \in \mathrm{End}(V)$ for the naive trace, i.e. the sum of the diagonal elements. E.g. $\mathrm{Tr}M = -a_1 \in \mathrm{End}(V)$. Then

$$\mathrm{Tr}(\phi(cu^p)) = \mathrm{Tr}_V(\mathrm{Tr}(M^p)c).$$

Also

$$\mathrm{Tr}_V\mathrm{res}_{u=\infty}(dgg^{-1}cu^p) = \mathrm{Tr}_V(\mathrm{res}\ (dgg^{-1}u^p)\cdot c).$$

(The residue is computed in the ring $\mathrm{End}(V)((u^{-1}))$. Since u is in the center of this ring, we may move c past u^p under the residue.) It will therefore suffice to show

$$\mathrm{Tr}(M^p) = -\mathrm{res}_{u=\infty}(dgg^{-1}u^p). \tag{6.3}$$

Let $z = u^{-1}$ and write $g = u^m G(z)$ with $G(z) = I + a_1 z + \ldots + a_m z^m$. The assertion becomes

$$dGG^{-1} = (\mathrm{Tr}(M) + \mathrm{Tr}(M^2)z + \ldots)dz. \tag{6.4}$$

Lemma 6.2. *Let $X = (x_{ij})$ be an $m \times m$ matrix. Then*

$$\mathrm{Tr}(XM^p) = \sum_{q=0}^{p}(-1)^q \sum_{\substack{1\le m_1,\ldots,m_q\le m\\ 1\le i,i_1\le m\\ \sum m_k = p+i-i_1\\ m_1\ge m-i_1+1}} x_{i,i_1}a_{m_1}\ldots a_{m_q}. \tag{6.5}$$

In particular, taking $X = I$ it follows that

$$\mathrm{Tr}(M^p) = \sum_{q=1}^{p}(-1)^q \sum_{\substack{1\le m_1,\ldots,m_q\le m\\ 1\le i\le m\\ \sum m_k = p\\ m_1\ge m-i+1}} a_{m_1}\ldots a_{m_q}. \tag{6.6}$$

proof of lemma. Write $M = (M_{ij})_{1\le i,j\le m}$. We have $M_{i+1,i} = 1$, $M_{i,m} = -a_{m+1-i}$, and $M_{ij} = 0$ otherwise. Thus

(6.7)

$$\mathrm{Tr}(XM^p) = \sum_{i,i_1,\ldots,i_p} x_{i,i_1}M_{i_1,i_2}\cdots M_{i_p,i} = \sum_{i,i_1,q,j_1,\ldots,j_q} x_{i,i_1}M_{j_1,m}\cdots M_{j_q,m}$$

$$= \sum_{i,i_1,q,j_1,\ldots,j_q}(-1)^q x_{i,i_1}a_{m-j_1+1}\cdots a_{m-j_q+1}$$

The conditions on the tuples $\{i, i_1, q, j_1, \ldots, j_q\}$ over which the right hand sums are taken become

$$0 \le q \le p;\ j_1 \le i_1; \tag{6.8}$$

$$(i_1 - j_1) + 1 + (m - j_2) + 1 + \ldots + (m - j_q) + 1 + (m - i) = p$$

Replacing j_k by $m_k := m - j_k + 1$, these become

$$(6.9) \qquad 0 \le q \le p;\ m_1 \ge m - i_1 + 1;\ \sum m_k = p + i - i_1,$$

proving the lemma. □

Write $c_p = \mathrm{Tr}(M^p)$. We must show

$$(6.10) \quad -(c_1 + c_2 z + c_3 z^2 + \dots)(1 + a_1 z + \dots + a_m z^m) = a_1 + 2a_2 z + \dots + m a_m z^{m-1}.$$

This amounts to

$$(6.11) \qquad c_p + c_{p-1}a_1 + \dots + c_1 a_{p-1} = \begin{cases} -pa_p & p < m \\ 0 & \text{else} \end{cases}$$

Suppose first $p < m$. With reference to (6.6), one can isolate the terms in c_p ending in a_k for $1 \le k \le p-1$ and write

$$(6.12) \qquad c_p = c_{p,1}a_1 + c_{p,2}a_2 + \dots + c_{p,p-1}a_{p-1} + R_p.$$

Here

$$(6.13) \qquad c_{p,k} = -\sum_{r=1}^{p-k} (-1)^r \sum_{\substack{1 \le m_1, \dots, m_r \le m \\ 1 \le i \le m \\ \sum m_k = p-k \\ m_1 \ge m-i+1}} a_{m_1} \dots a_{m_r} = -c_{p-k}.$$

(Notice that since each $m_j \ge 1$, terms with $r > p - k$ are impossible. Also, since $k < p$, necessarily $r \ge 1$.) The remainder R_p is given by the terms a_p in c_p. In the sum for c_p these terms arise when $q = 1$ and $p \ge m - i + 1$, i.e. $m - p + 1 \le i \le m$. There are p such terms:

$$(6.14) \qquad R_p = -a_p.$$

Finally, in (6.11) we consider the terms with $p \ge m$. Writing $t_k = \mathrm{Tr}(XM^k)$ and replacing X with M^j for some j, it suffices to show

$$(6.15) \qquad t_0 a_m + \dots + t_{m-1}a_1 + t_m = 0.$$

We start with

$$(6.16) \qquad t_m = \sum_{q=1}^{p} (-1)^q \sum_{\substack{1 \le m_1, \dots, m_q \le m \\ 1 \le i, i_1 \le m \\ \sum m_k = n+i-i_1 \\ m_1 \ge n-i_1+1}} x_{i,i_1} a_{m_1} \dots a_{m_q}.$$

Note $q = 0$ is not possible because $\sum m_k = m + i - i_1 \ge 1$. Again, by grouping together the terms ending with a_k we get

$$t_m = -t_{m-1}a_1 - \dots - t_0 a_m,$$

which is the desired equation. □

Corollary 6.3. *We have with notation as in definition 3.5*

$$\mathrm{Tr}(\eta_\nabla) \equiv (m-2)\mathrm{Tr}(\eta_0) - \sum_i \mathrm{res}_{\ t=a_i} \mathrm{Tr}(dgg^{-1}\eta^{(i)}) + \frac{1}{2}\sum_{i;\ m_i\geq 2} d\log(\det(g^{(i)}_{m_i})) \quad \mathrm{mod}\ d\log(K^\times).$$

Proof. By theorem 3.6, we may replace η_∇ with η_γ. By proposition 4.2 $\mathrm{Tr}(\eta_\gamma - \eta_0)$ is the sum of the $\mathrm{Tr}(\eta^{(i)})$ on $V[u]/gV[u]dt$. By proposition 5.1 this is the same as $-\sum_i \mathrm{Tr}_V \mathrm{res}_{\ t=a_i}(dgg^{-1}\eta^{(i)})$. Note the factor $m-2$ on the right is because as a matrix η_0 acts on $V = \Gamma(\mathbb{P}^1, E)$ while the trace one wants is the action on $H \cong V^{\oplus m-2}$. □

7. The proof of the main Theorem

In this section we deduce the main theorem 2.8 from the equality of the Higgs and de Rham traces (theorem 3.6) and from the shape of the Higgs trace (proposition 6.1) via residues.

We start with an admissible connection (E, ∇) on $\mathbb{P}^1_K$. Let $V \subset U$ be a Zariski open subset, with complement Z. By localization, one obtains

$$\det H_{DR}(U, \nabla_{/K}) = \det H_{DR}(V, \nabla_{/K}) + \det \nabla|_Z. \tag{7.1}$$

On the other hand, at a special pseudo-logarithmic point the local factor $\mathrm{Tr}(dgg^{-1}A) = 0$. Indeed, writing the connection as $g_1\frac{dz}{z} + g_0 dz + \ldots + \frac{\eta_1}{z} + \frac{\eta_0}{+}\ldots$, the local factor is $\mathrm{Tr} g_0 g_1^{-1}\eta_1$. The special pseudo-logarithmic points have, in the notations of the definition 2.2, a local matrix of the shape

$$\begin{pmatrix} A + m\frac{dz}{z} & zB \\ \frac{C}{z} & D + n\frac{dz}{z} \end{pmatrix}. \tag{7.2}$$

Thus in particular

$$\eta_1 = \begin{pmatrix} 0 & 0 \\ \gamma_0 & 0 \end{pmatrix}, \tag{7.3}$$

where $C = cdz + \gamma_0 + \gamma_1 z + \ldots$, while g_0 and g_1 are both of the shape

$$\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}. \tag{7.4}$$

Thus $\mathrm{Tr} g_0 g_1^{-1}\eta_1 = 0$.

Thus the difference of the right hand side of the theorem 2.8 for U and V is the difference of the global factors, which is $\det \nabla|_Z$, as one sees taking a trivializing section which is good for V. Thus by theorem 2.3, we may assume that $E = \oplus_1^r \mathcal{O}_{\mathbb{P}^1_K}$.

Let $G = \gamma_K \cdot \prod_j (t-a_j)^{m_j}$. Note $G = u^{-m} g(u)$ with u as in (5.3). Write the absolute connection as $\nabla = d + A$ with $A = \gamma + \eta$ as in definition 3.2.

Proposition 7.1. *We have*

$$\sum_i \mathrm{Tr}\ \mathrm{res}_{t=a_i} dGG^{-1}A = -\sum_{\substack{i,j,r\\ j\neq i}} \mathrm{Tr}(g_r^{(i)}) m_j (a_j - a_i)^{-r} d(a_j - a_i) + \sum_i \mathrm{Tr}\ \mathrm{res}_{t=a_i}(dGG^{-1}\eta^{(i)}). \tag{7.5}$$

Proof. Define absolute forms s, $s(i)$ and $\eta(i)$:

$$s = \frac{dt}{\prod_j (t-a_j)^{m_j}}; \quad s(i) = \frac{d(t-a_i)}{\prod_j (t-a_j)^{m_j}}; \quad \nabla = d + G\cdot s(i) + \eta(i). \tag{7.6}$$

The local term at $t = a_i$ is

$$\mathrm{Tr}\ \mathrm{res}_{t=a_i} dGG^{-1}A = \mathrm{Tr}\ \mathrm{res}_{t=a_i} dG\cdot s(i) + \mathrm{Tr}\ \mathrm{res}_{t=a_i} dGG^{-1}\eta(i).$$

Applying trace to $dA = A\wedge A$ yields

$$\mathrm{Tr}(dG\cdot s(i) + G\cdot ds(i) + d\eta(i)) = 0.$$

(Note here that $s(i)$ is not closed as an absolute form!) Also, modulo Ω^2_K, we have res $\mathrm{Tr}(d\eta(i)) = 0$ because the residue of an exact form vanishes. The local term thus becomes

$$\mathrm{Tr}\ \mathrm{res}_{t=a_i} dGG^{-1}A = -\mathrm{Tr}\ \mathrm{res}_{t=a_i} G\cdot ds(i) + \mathrm{Tr}\ \mathrm{res}_{t=a_i} dGG^{-1}\eta(i). \tag{7.7}$$

We have

$$G\cdot ds(i) = \tag{7.8}$$

$$\Big(\prod_j (t-a_j)^{m_j} \sum_{r,k} \frac{g_r^{(k)}}{(t-a_k)^r}\Big) \frac{\sum_j m_j \prod_{k\neq j}(t-a_k) d(t-a_j)\wedge d(t-a_i)}{\prod_j (t-a_j)^{m_j+1}}$$

$$= \Big(\sum_{r,k} \frac{g_r^{(k)}}{(t-a_k)^r}\Big) \sum_j m_j \frac{d(a_i - a_j)\wedge dt}{t-a_j}.$$

Thus

$$\mathrm{res}_{t=a_i}\ \mathrm{Tr}(G\cdot ds(i)) = \mathrm{res}_{t=a_i}\Big(\sum_{\substack{j,k,r\\ j\neq i}} \frac{\mathrm{Tr}(g_r^{(k)}) m_j d(a_i-a_j)\wedge dt}{(t-a_k)^r (t-a_j)}\Big). \tag{7.9}$$

Expanding

$$\frac{1}{(t-a_j)} = -(a_j-a_i)^{-1}\sum_{n=0}^{\infty}\Big(\frac{t-a_i}{a_j-a_i}\Big)^n$$

and substituting on the right in (7.9)

$$\text{(7.10)} \qquad \mathrm{res}_{t=a_i}\ \mathrm{Tr}(G\cdot ds(i)) = \sum_{\substack{j,r\\ j\neq i}} \mathrm{Tr}(g_r^{(i)})m_j(a_j-a_i)^{-r}d(a_j-a_i)$$

Thus, (7.9) becomes after summing over i

$$\text{(7.11)} \quad \sum_i \mathrm{Tr}\ \mathrm{res}_{t=a_i} dGG^{-1}A = -\sum_{\substack{i,j,r\\ j\neq i}} \mathrm{Tr}(g_r^{(i)})m_j(a_j-a_i)^{-r}d(a_j-a_i) + \sum_i \mathrm{Tr}\ \mathrm{res}_{t=a_i}(dGG^{-1}\eta(i)).$$

It remains to compare $\eta^{(i)}$ and $\eta(i)$. Recall $\gamma_K = G\cdot s$. We have

$$\text{(7.12)} \qquad \gamma = \gamma_K - \sum_{j,r}\frac{g_r^{(j)}da_j}{(t-a_j)^r}$$

$$\text{(7.13)} \qquad G\cdot s(i) = \gamma_K - \sum_{j,r}\frac{g_r^{(j)}da_i}{(t-a_j)^r}$$

$$\text{(7.14)} \qquad \nabla = d + G\cdot s(i) + \eta(i) = d + \gamma + \eta_0 + \sum_j \eta^{(j)}.$$

From these equations it follows that

$$\eta(i) - \eta^{(i)} = \sum_{j\neq i}\frac{k_r^{(j)}}{(t-a_j)^r}.$$

Since this difference has no pole at a_i, and since $G(a_i)$ is invertible, we find

$$\mathrm{res}_{t=a_i}(dGG^{-1}\eta(i)) = \mathrm{res}_{t=a_i}(dGG^{-1}\eta^{(i)})$$

Making this substitution in (7.11) proves the proposition. □

Our calculations to this point have been on H^1_{DR} which introduces a minus sign in the final formula. (Note, admissibility forces $H^0_{DR} = (0)$.)

We find therefore

$$
\begin{aligned}
(7.15)\quad -\operatorname{Tr}_{GM}(H^*_{DR}) &= \operatorname{Tr}(\Psi) + \operatorname{Tr}(\eta_\nabla) \quad \text{(proposition 4.2)}\\
&\equiv \operatorname{Tr}(\Psi) + (m-2)\operatorname{Tr}(\eta_0) - \sum_i \operatorname{Tr}\ \operatorname{res}_{t=a_i}(dgg^{-1}\eta^{(i)})\\
&+ \frac{1}{2}\sum_{i;\ m_i\geq 2} d\log(\det(g^{(i)}_{m_i})) \mod d\log K^\times \quad \text{(corollary 6.3)}\\
&\equiv \operatorname{Tr}(\Psi) + (m-2)\operatorname{Tr}(\eta_0) - \sum_i \operatorname{Tr}\ \operatorname{res}_{t=a_i}(dGG^{-1}\eta^{(i)})\\
&+ \frac{1}{2}\sum_{i;\ m_i\geq 2} d\log(\det(g^{(i)}_{m_i})) \mod d\log K^\times\\
&\equiv (m-2)\operatorname{Tr}(\eta_0) - \sum_i \operatorname{Tr}\ \operatorname{res}_{t=a_i}(dGG^{-1}A)\\
&+ \frac{1}{2}\sum_{i;\ m_i\geq 2} d\log(\det(g^{(i)}_{m_i})) \mod d\log K^\times \quad \text{(proposition 7.1).}
\end{aligned}
$$

Note here we can replace g by $G = u^{-m}g$ because $\eta^{(i)}$ only involves terms of degrees ≥ 1 in u, so res $(\frac{du}{u}\eta^{(i)}) = 0$.

View $s = \frac{dt}{\prod_j (t-a_j)^{m_j}}$ as a relative form, i.e. as a meromorphic section of $\omega(\sum m_j(a_j))$. As such, it has a zero of order $m-2$ at infinity, $(s) = (m-2)\infty$. Note $\det(E,\nabla)|_\infty = (K, \operatorname{Tr}(\eta_0))$. Since the relative connection is given by $\nabla_{/K} = G\cdot s$, the desired formula is

$$
\begin{aligned}
(7.16)\quad \operatorname{Tr}_{GM}(H^*_{DR}) \equiv -\det(E,\nabla)|_{(s)} &+ \sum_i \operatorname{res}_{t=a_i}\operatorname{Tr}(dGG^{-1}A)\\
&+ \frac{1}{2}\sum_{i;\ m_i\geq 2} d\log(\det(g^{(i)}_{m_i})) \mod d\log K^\times.
\end{aligned}
$$

Finally, comparing (7.15) and (7.16) we see that theorem 2.8 holds for pseudo-admissible connections on bundles on $\mathbb{P}^1$.

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