LECTURES ON ALGEBRO-GEOMETRIC CHERN-WEIL AND CHEEGER-CHERN-SIMONS THEORY FOR VECTOR BUNDLES

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Abstract. An algebraic theory of differential characters [16], [17] is outlined. A Riemann-Roch theorem for regular flat bundles, using a weak form of these classes, is described. Full details of the proof are available in a manuscript on the web [6].

1. Connections and Characteristic Classes

1.1. **Introduction.** The purpose of these two lectures will be to describe some work on the general topic of algebro-geometric Chern-Weil and Cheeger-Chern-Simons theory for characteristic classes of vector bundles. The trick is to cling to algebraic methods (K-theory, the Zariski topology, algebraic differential forms,...) so the resulting theory gives information about algebraic cycles, while at the same time drawing inspiration from the beautiful calculus of differential forms and connections lying behind the differential geometry.

We start with an algebraic variety X which will always be taken smooth and quasi-projective over an algebraically closed field k of characteristic 0. Let E be a vector bundle on X. We suppose given an algebraic connection on E, that is a k-linear map

$$\nabla : E \to E \otimes \Omega^1_{X/k} \tag{1.1.1}$$

satisfying the Leibniz rule

$$\nabla(f \cdot e) = f\nabla(e) + e \otimes df \tag{1.1.2}$$

for f a function and e a section of E. Note that in the $\mathcal{C}^\infty$ category a partition of unity argument can be used to give any bundle E a connection. Algebraically, if e.g. X is projective, most bundles E will not admit a connection and our ship would seem to have sunk before leaving the harbor. In fact, we may dodge that bullet as follows.

Proposition 1.1.1. [1] *There exists a class*

$$At(E) \in H^1(X, rmHom(E,E) \otimes \Omega^1_X)$$

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the vanishing of which is necessary and sufficient for E to admit a connection.

Proof. As the difference of two connections is $\mathcal{O}_X$-linear (see (1.1.1) and (1.1.2)), connections on E form a torsor under $rmHom(E, E\otimes\Omega^1_X)$. For more detail, one can argue as follows. Let $I\subset\mathcal{O}_{X\times X}$ be the ideal of the diagonal. Recall

$$\Omega^1_X \cong I/I^2; \qquad dx\mapsto x\otimes 1 - 1\otimes x \mod I^2$$

Write $P := \mathcal{O}_{X\times X}/I^2$, so we have an exact sequence

$$0\to\Omega^1_X\to P\to\mathcal{O}_X\to 0 \tag{1.1.3}$$

Note P has two $\mathcal{O}_X$-module structures pulling back via the two projections $X\times X\to X$. Tensor (1.1.3) with E via the left $\mathcal{O}_X$-structure:

$$0\to E\otimes_{\mathcal{O}_X}\Omega^1_X\to E\otimes_{\mathcal{O}_X} P \overset{\overset{s}{\leftarrow}}{\to} E\to 0 \tag{1.1.4}$$

and view this as a sequence of $\mathcal{O}_X$-modules via the *right* $\mathcal{O}_X$-module structure on P. The middle term $E\otimes_{\mathcal{O}_X} P$ is the vector bundle of 1-jets of sections of E. The evident splitting $s(e) = e\otimes 1$ is not an $\mathcal{O}_X$ splitting. In fact, it is straightforward to check that there is a $1-1$ correspondence [11]

(1.1.5)

$$\{\mathcal{O}_X\text{-linear splittings } t \text{ of } (1.1.4)\}\leftrightarrow\{\text{connections } \nabla : E\to E\otimes\Omega^1_X\}$$

given by $t\mapsto\nabla := s-t$. The proposition follows by considering the sequence obtained by $rmHom(E, (1.1.4))$

(1.1.6)

$$0\to rmHom(E, E\otimes\Omega^1_X)\to rmHom(E, E\otimes P)\to Hom(E,E)\to 0$$

and defining $\mathrm{conn}(E) = \partial(Id_E)\in H^1(X, Hom(E, E\otimes\Omega^1_X))$. □

Corollary 1.1.2. *If X is affine, any vector bundle E admits a connection.*

Proposition 1.1.3 (Jouanolou). *Let X be quasi-projective. There exists $\pi:\tilde{X}\to X$ with the following properties:*
1. $\tilde{X}$ is an affine variety.
2. Locally over X, $\tilde{X}\cong X\times\mathbb{A}^N$ for some N.

Proof. Embed $X\hookrightarrow Y$ with Y projective. Blowing up $Y-X$, we may assume $Y-X$ is a Cartier divisor. Since the complement of a Cartier divisor in an affine scheme is affine, we reduce in this way to the case X projective. By pulling back, it suffices to do the case $X=\mathbb{P}^N$. Let X' be the dual projective space, and let $I\subset X'\times X$ be the incidence correspondence. Define $\tilde{X} := X'\times X - I$. □

The crucial point is that because of condition 2,

$$\pi^* : CH^*(X) \cong CH^*(\tilde{X}),$$

i.e. the two varieties have the same motive. Thus, we may pull back our bundle to $\tilde{X}$ and assume E admits a connection.

In what follows we will consider two sorts of questions. First, for an arbitrary vector bundle, we develop Chern-Weil and Cheeger-Chern-Simons theory for an arbitrary connection on a pullback to an affine as in proposition 1.1.3. Second, when E comes with an integrable connection, we discuss Riemann-Roch theorems and results about chern classes in the Chow group.

1.2. **Differential Characters.** We start with the $\mathcal{C}^\infty$ case (compare [33]). Let M be a $\mathcal{C}^\infty$-manifold and let E be a $\mathcal{C}^\infty$-vector bundle on M. Write

$$\begin{gathered} S.(M,A) := \mathcal{C}^\infty \text{ singular chains on } M \text{ with values in the abelian group } A \\ Z_k(M,A) \subset S_k(M,A) := \text{closed chains} \\ S^{\cdot}(M,A) := \mathrm{Hom}(S.(M,\mathbb{Z}),A) \\ \mathcal{A}^{\cdot}(M) := \mathcal{C}^\infty \ \mathbb{C}\text{-valued forms on } M \\ \mathcal{A}^{\cdot}(M)_0 := \{\omega \in \mathcal{A}^{\cdot}(M) \mid d\omega = 0 \text{ and } \int_\gamma \omega \in \mathbb{Z},\ \forall \gamma \in Z.(M,\mathbb{Z})\} \\ I : \mathcal{A}^{\cdot}(M) \to S^{\cdot}(M,\mathbb{C}) \text{ integration map} \end{gathered} \tag{1.2.1}$$

The group of differential characters of degree k, $\widehat{H}^k(M,\mathbb{C}/\mathbb{Z})$, is defined by the fibre product

$$\begin{array}{ccc} \widehat{H}^k(M,\mathbb{C}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(Z_k(M,\mathbb{Z}),\mathbb{C}/\mathbb{Z}) \\ \downarrow & & \downarrow \chi \mapsto \chi \circ \partial \\ \mathcal{A}^{k+1} \to S^{k+1}(M,\mathbb{C}) & \xrightarrow{\text{restrict}} & S^{k+1}(M,\mathbb{C}/\mathbb{Z}) \end{array} \tag{1.2.2}$$

i.e.

$$\widehat{H}^k(M,\mathbb{C}/\mathbb{Z}) = \{(a,\omega) \in \mathrm{Hom}(Z_k(M,\mathbb{Z}),\mathbb{C}/\mathbb{Z}) \times \mathcal{A}^{k+1} \mid a \circ \partial = \text{restrict} \circ I(\omega)\}. \tag{1.2.3}$$

Proposition 1.2.1. *One has an exact sequence*

$$0 \longrightarrow H^k(M,\mathbb{C}/\mathbb{Z}) \longrightarrow \widehat{H}^k(M,\mathbb{C}/\mathbb{Z}) \xrightarrow{(a,\omega)\mapsto\omega} \mathcal{A}_0^{k+1} \longrightarrow 0$$

Proof. Straightforward. □

Now let ∇ be a $\mathcal{C}^\infty$-connection on E, and let $N = \mathrm{rank}(E)$. Let

$$F = \nabla^2 \in \mathrm{Hom}(E, E \otimes \mathcal{A}^2(M)) \tag{1.2.4}$$

be the curvature of ∇. Let $P \in \mathrm{Sym}^p(gl_N^*)$ be a polynomial which is invariant under the adjoint action. The central point of Chern-Weil theory is that $P(F) \in A^{2p}(M)$ is closed and represents the characteristic de Rham class in cohomology associated to E and P. Write D for a diagonal matrix with diagonal entries $x_1, \ldots, x_N$. If $P(D)$ is the p-th elementary symmetric function (resp. $P(D) = \sum_i \frac{x_i^p}{p!}$), $P(F)$ represents the p-th chern class $c_p(E)$ (resp. the component of the chern character $ch(E)$ in cohomological degree $2p$). Cheeger-Chern-Simons theory refines this:

Proposition 1.2.2. *With notation as above, assume further that the characteristic class defined by P is integral. Then there exist canonically defined classes $\hat{c}(P, E, \nabla) \in \widehat{H}^{2p-1}(M, \mathbb{C}/\mathbb{Z})$ lifting $P(F) \in \mathcal{A}^{2p}(M)_0$. These classes are functorial for pullbacks and additive for exact sequences.*

Proof. Omitted. □

Now we return to the case X, E algebraic and ∇ an algebraic connection. Suppose for a moment E is a line bundle. Define

$$AD^1(X) := \mathbb{H}^1(X_{\mathrm{zar}}, \mathcal{O}_X^* \xrightarrow{d\log} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \to \ldots) \tag{1.2.5}$$
$$\widehat{AD}^1(X) := \mathbb{H}^1(X_{\mathrm{zar}}, \mathcal{O}_X^* \xrightarrow{d\log} \Omega_X^1)$$

(Here $\mathcal{O}_X^*$ is the Zariski sheaf of invertible functions, and AD stands for "algebraic differential character".)

Proposition 1.2.3. *1. $\widehat{AD}^1(X)$ is isomorphic to the group $Pic^\nabla(X)$ of isomorphism classes of line bundles with connections on X.*
2. $AD^1(X)$ is isomorphic to the group $Pic^{0,\nabla}(X)$ of isomorphism classes of line bundles with integrable connections on X.
3. Let $\Gamma(X, \Omega_X^2)_0$ denote the group of global closed algebraic 2-forms ω on X such that $[\omega] \in H^2_{DR}(X)$ is the class of an algebraic divisor. Then there is an exact sequence

$$0 \to AD^1(X) \to \widehat{AD}^1(X) \to \Gamma(X, \Omega_X^2)_0 \to 0.$$

4. Assume $k = \mathbb{C}$ and X is projective. Then

$$AD^1(X) \cong H^1(X_{rman}, \mathbb{C}/\mathbb{Z}2\pi i).$$

Proof. Let (L, ∇) be a line bundle with connection on X. With respect to an open cover $\{U_i\}$ trivializing L, we have transition functions

$$a_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$$

and 1-forms $\omega_i = \nabla(1_i) \in \Gamma(U_i, \Omega^1_X)$. The data $\{a_{ij}, \omega_i\}$ represent a class in $\widehat{AD}^1(X)$. The assertions of the proposition are straightforward from this. $\square$

The idea to define the higher AD-groups [15], [17] is to work with the higher Milnor K-sheaves

$$\mathcal{K}_n := \mathrm{Image}\Big((\mathcal{O}_X^*)^{\otimes n} \to K_n^{\mathrm{Milnor}}(k(X))\Big) \tag{1.2.6}$$

These sheaves are less abstract than the Quillen K-sheaves. In [30] they are shown to satisfy many of the same properties. In particular, one has a Gersten resolution, and

$$H^n(X, \mathcal{K}_n) \cong CH^n(X). \tag{1.2.7}$$

There is a $d\log$ map

$$d\log : \mathcal{K}_n \to \Omega^n_X; \quad d\log\{x_1, \ldots, x_n\} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}. \tag{1.2.8}$$

We define

$$\begin{aligned} AD^n(X) &:= \mathbb{H}^n(X, \mathcal{K}_n \overset{d\log}{\to} \Omega^n_X \to \Omega^{n+1} \to \ldots) \\ \widehat{AD}^n(X) &:= \mathbb{H}^n(X, \mathcal{K}_n \overset{d\log}{\to} \Omega^n_X \to \Omega^{n+1}_X \to \ldots \to \Omega^{2n-1}_X) \end{aligned} \tag{1.2.9}$$

Write

$$\begin{aligned} &\Gamma(X, \Omega^{2n}_X)_0 = \\ &\ker\Big(\Gamma(X, \Omega^{2n}_{X,\mathrm{closed}}) \to \mathbb{H}^n(X, \Omega^n \to \Omega^{n+1} \to \ldots)\Big/ d\log H^n(X, \mathcal{K}_n)\Big) \\ &= \Big\{\omega \in \Gamma(X, \Omega^{2n}_{X,\mathrm{closed}}) \;\Big|\; [\omega] \in \mathbb{H}^{2n}(X, \Omega^{\geq n}_X) \\ &\qquad \text{is the class of an algebraic cycle}\Big\} \end{aligned} \tag{1.2.10}$$

Proposition 1.2.4. 1. *There is an exact sequence*

$$0 \to AD^n(X) \to \widehat{AD}^n(X) \to \Gamma(X, \Omega^{2n}_X)_0 \to 0$$

2. *Let (E, ∇) be a rank N algebraic bundle with connection on X, and let P be an invariant polynomial on gl_N as above. Then one has defined functorial characteristic classes*

$$\hat{c}(P, E, \nabla) \in \widehat{AD}^n(X) \mapsto P[F] \in \Gamma(X, \Omega^{2n}_X)_0.$$

3. *When $k = \mathbb{C}$, there is a canonical map of exact sequences from the above sequence to the classical differential character sequence from proposition 1.2.1. This map carries the algebraic classes from 2 to the classes defined in proposition 1.2.2.*

Proof. For full details the reader is referred to [17]. Let us sketch one of the constructions of the characteristic classes which uses a "Weil algebra" construction due to Beilinson and Kazhdan [2]. First, we push out the sequence (1.1.6) along the trace on the left to get an exact sequence (defining $\Omega^1_{X,E}$)

$$0 \to \Omega^1_X \to \Omega^1_{X,E} \to End(E) \to 0 \tag{1.2.11}$$

We consider the composed differential $\mathcal{O}_X \xrightarrow{\delta} \Omega^1_{X,E}$ and build a differential graded algebra $\Omega^*_{X,E}$ subject only to the relation that $\delta(f) = df$ for $f \in \mathcal{O}_X$. Even the algebra structure here is tricky; $\Omega^*_{X,E}$ is not simply an exterior algebra on $\Omega^1_{X,E}$. One has as graded algebras

$$\bigwedge \Omega^1_{X,E} \otimes \mathrm{Sym}(End(E)) \cong \Omega^*_{X,E} \tag{1.2.12}$$

where $End(E)$ has graded degree 2. (To see this, think about the case X=point, $\Omega^1_X = (0)$. The universal DGA structure in this case makes

$$\Omega^*_{X,E} = \text{ext. algebra on complex in deg. } [1,2]\ End(E) \xrightarrow{=} End(E).)$$

One defines a Hodge filtration

(1.2.13)

$$F^p\Omega^*_{X,E} := \mathrm{Image}\Big(\oplus_{a+b\geq p} \wedge^a \Omega^1_{X,E} \otimes \mathrm{Sym}^b(End(E))\Big) \to \Omega^*_{X,E}.$$

With respect to the Hodge filtrations, the natural map

$$(\Omega^*_X, \Omega^{\geq p}) \to (\Omega^*_{X,E}, F^p\Omega^*_{X,E}) \tag{1.2.14}$$

is a filtered quasi-isomorphism.

A connection ∇ on E is equivalent to a splitting of (1.2.11) and hence also (1.2.14):

$$\begin{array}{ccccccc} \mathcal{O}_X & \to & \Omega^1_{X,E} & \to & \wedge^2\Omega^1_{X,E} & \oplus\ End(E) & \to \ \dots \\ \| & & \downarrow \nabla & & \downarrow & \swarrow \nabla^2 & \\ \mathcal{O}_X & \to & \Omega^1_X & \to & \Omega^2_X & & \dots \end{array} \tag{1.2.15}$$

The curvature ∇^2 is the obstruction to compatibility of this splitting with the Hodge filtration.

Let $N = \mathrm{rank}(E)$. Invariant polynomials in $\mathrm{Sym}^p(End(E))$ define a subcomplex with trivial differential

$$S := (\mathrm{Sym}(k_X^{\oplus N}[-2]))^{\mathcal{S}_N} \to \Omega^*_{X,E} \tag{1.2.16}$$

Note S^{2p} (viewed as a complex placed in degree $2p$) is generated by invariant polynomials of degree p and maps to $F^p\Omega^*_{X,E}$ (viewed as a complex beginning in degree p). A universal argument on BGL_N shows that characteristic classes for E associated to invariant polynomials of degree p can be defined in

$$\mathbb{H}^{2p}\Big(X, \mathrm{Cone}(\mathcal{K}_p[-p] \oplus S^{2p} \to F^p\Omega^*_{X,E})[-1]\Big) \tag{1.2.17}$$

Finally, the splitting (1.2.15) induced by a connection carries $S^{2p} \to F^{2p}\Omega^*_X$, so we get characteristic classes

$$\begin{aligned}\hat{c}(P, E, \nabla) \in \mathbb{H}^{2p}\Big(X, \mathrm{Cone}(\mathcal{K}_p[-p] \oplus F^{2p}\Omega^*_X \to F^p\Omega^*_X)[-1]\Big)\\ \cong \widehat{AD}^p(X).\end{aligned} \tag{1.2.18}$$

If ∇ is integrable, the map on S^{2p} is zero and the classes fall in

$$AD^p(X) = \mathbb{H}^{2p}\Big(X, \mathrm{Cone}(\mathcal{K}_p[-p] \to F^p\Omega^*_X)[-1]\Big). \tag{1.2.19}$$

$\square$

1.3. **Cheeger-Chern-Simons Classes.** This is all a bit abstract, but there are much more down to earth variants on the differential character construction which can be studied locally. Examining the complexes (1.2.9) defining the $\widehat{AD}$ groups, one sees there are vertical maps and commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & AD^1(X) & \longrightarrow & \widehat{AD}^1(X) & \longrightarrow & \Gamma(X, \Omega^2_X)_0 & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \Gamma(X, \Omega^1_{X,cl}/d\log(\mathcal{O}^*_X)) & \longrightarrow & \Gamma(X, \Omega^1_X/d\log(\mathcal{O}^*_X)) & \xrightarrow[d]{} & \Gamma(X, \Omega^2_X)_0 & \longrightarrow & 0 \end{array} \tag{1.3.1}$$

We write $\mathcal{H}^n := \Omega^n_{X,cl}/d\Omega^{n-1}_X$. For $p > 1$, the diagram corresponding to (1.3.1) becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & AD^p(X) & \longrightarrow & \widehat{AD}^p(X) & \longrightarrow & \Gamma(X, \Omega^{2p}_X)_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(X, \mathcal{H}^{2p-1}) & \longrightarrow & \Gamma(X, \Omega^{2p-1}_X/d\Omega^{2p-2}_X) & \xrightarrow[d]{} & \Gamma(X, \Omega^{2p}_{X,cl}) & & \end{array} \tag{1.3.2}$$

Note when $p > 1$ the vertical arrows are no longer isomorphisms. We will write

$$H^{2p}_{CS}(X) := \begin{cases} \Gamma(X, \Omega^1_X/d\log\mathcal{O}^*_X) & p = 1 \\ \Gamma(X, \Omega^{2p-1}_X/d\Omega^{2p-2}_X) & p \geq 2. \end{cases} \tag{1.3.3}$$

For an invariant polynomial P of degree p, there are classes

$$w(P, E, \nabla) \in H^{2p}_{CS}(X) \tag{1.3.4}$$

deduced from the $\widehat{AD}$ classes via the vertical maps in (1.3.1) and (1.3.2).

Proposition 1.3.1. *Let $\eta = \mathrm{Spec}\,(k(X)) \xrightarrow{j} X$ be the generic point of X. Then the pullback maps $j^* : H^{2p}_{CS}(X) \to H^{2p}_{CS}(\eta)$ are injective.*

Proof. When $p = 1$ this amounts to the assertion that a logarithmic 1-form $\frac{df}{f}$ is regular along a divisor D if and only if D is neither a zero nor a pole of f. For $p > 1$ it follows from the Gersten style resolution [8]

$$0 \to \mathcal{H}^n \to j_*(H^n_{DR}(\eta)) \to \dots \tag{1.3.5}$$

where $\mathcal{H}^n = \Omega^n_{\mathrm{cl}}/d\Omega^{n-1}$. □

Thus, as sections of a sheaf, the $w(P, E, \nabla)$ may be calculated locally. Further, by the above proposition, they are determined by their value at the generic point. One of the main results in [5] is that these classes can be calculated locally using the connection matrix as in Chern-Simons theory.

Proposition 1.3.2. *Let $U \subset X$ be non-empty and open such that $E|U \cong \mathcal{O}_U^{\oplus N}$. Let A be the corresponding connection matrix of 1-forms on U. For an invariant polynomial P of degree $p \geq 2$ define*

$$TP(A) := p\int_0^1 P(A \wedge F(tA)^{p-1})dt; \quad F(tA) := tdA - t^2 A \wedge A.$$

then

$$dTP(A) = P(F(A)),$$

and

$$TP(A)|_U \text{ represents } w(P, E|_U, \nabla) \in \Gamma(U, \Omega^{2p-1}/d\Omega^{2p-2}) = H^{2p}_{CS}(U).$$

Proof. The first assertion is the basic result of Chern-Simons theory. The reader is referred to theorem (4.0.1) in [5] for the proof of the second assertion. □

Example 1.3.3. *Suppose E is a rank 2 bundle with structure group SL_2, and ∇ is a connection which is written on some open U via its connection matrix*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

after the chhoice of a basis. Then for the invariant polynomial $P(M) = tr(M^2)$ the Chern-Simons class $w(P, E, \nabla)$ is represented by

$$2\alpha d\alpha + \beta d\gamma + \gamma d\beta - 4\alpha\beta\gamma \in \Gamma(U, \Omega^3).$$

If ∇ is flat, that is $\nabla^2 = 0$, then

$$d\alpha = \beta\gamma, d\gamma = 2\gamma\alpha, d\beta = 2\alpha\beta$$

and thus $w(P,E,\nabla)$ is represented by

$$2\alpha d\alpha \in \Gamma(U,\mathcal{H}^3) \subset H^4_{CS}(U).$$

Important question: Do there exist examples with X projective where this class is not trivial, i.e. where $\alpha d\alpha$ is not an exact form in $\Omega^3_{k(X)}$?

Example 1.3.4. *Let $\pi : Y \to X$ be a finite étale morphism, and define $E = \pi_*\mathcal{O}_Y$ with the natural (Gauß-Manin) connection ∇. Then the class associated to $P(M) = tr(M)$ is calculated locally as follows. We may suppose $X = \mathrm{Spec}\,(R)$, $Y = \mathrm{Spec}\,(R[t]/(f(t))$, where f is a monic polynomial. Write $disc(f)$ for the discriminant of f. Then*

$$w(trace,\pi_*\mathcal{O}_Y,\nabla) = [\frac{1}{2}d\log(disc(f))] \in \Gamma(\mathrm{Spec}\,(R),\Omega^1_{R,cl}/d\log(\mathcal{O}^*)).$$

This class is trivial if and only if the discriminant is a square in R. Said another way, if we factor $f = \prod(t-r_i)$, then the discriminant is $\prod_{i\neq j}(r_i - r_j)$ and the class is represented by $d\log(\prod_{i<j}(r_i-r_j))$.

Finally, we want to explain the relation between these classes and the Griffiths group of algebraic cycles. For simplicity we will stick to the case of codimension 2. Analogous results are available in higher codimensions but the appropriate equivalence relation on cycles is defined via a spectral sequence and the business becomes more technical.

Let X be smooth and projective over $\mathbb{C}$. For an abelian group A, write $\mathcal{H}^n(A)$ for the Zariski sheaf associated to the presheaf $U \mapsto H^n(U_{\mathrm{an}},A)$. Thus, the $\mathcal{H}^n$ above is $\mathcal{H}^n(\mathbb{C})$. One has a spectral sequence of Leray type associated to the "continuous map" $X_{\mathrm{an}} \to X_{\mathrm{zar}}$

$$E_2^{p,q} = H^p(X_{\mathrm{zar}},\mathcal{H}^q(A)) \Rightarrow H^{p+q}(X_{\mathrm{an}},A) \tag{1.3.6}$$

The analysis of this spectral sequence carried out in [8] shows that one gets an exact sequence ($\mathbb{Z}(2) := \mathbb{Z}\cdot(2\pi i)^2$.)

$$\begin{aligned} H^3(X_{\mathrm{an}},\mathbb{Z}(2)) \xrightarrow{a} \Gamma(X,\mathcal{H}^3(\mathbb{Z}(2))) \xrightarrow{d_2} H^2(X_{\mathrm{zar}},\mathcal{H}^2(\mathbb{Z}(2))) \\ \xrightarrow{b} H^4(X_{\mathrm{an}},\mathbb{Z}(2)) \end{aligned} \tag{1.3.7}$$

Further, $H^2(X_{\mathrm{zar}},\mathcal{H}^2(\mathbb{Z}))$ is identified with the group of codimension 2 algebraic cycles modulo algebraic equivalence, so $\mathrm{Griff}^2(X) := \ker(b)$ is the Griffiths group of codimension 2 algebraic cycles homologous to zero modulo algebraic equivalence. Write $\tilde{H}^3(X,\mathbb{Z}(2)) := \mathrm{image}(a)$, so we get an exact sequence

$$0 \to \tilde{H}^3(X,\mathbb{Z}(2)) \to \Gamma(X,\mathcal{H}^3(\mathbb{Z}(2))) \to \mathrm{Griff}^2(X) \to 0. \tag{1.3.8}$$

this can be interpreted as an exact sequence of (possibly infinite dimensional) mixed Hodge structures, where the Griffiths group is given a trivial mixed Hodge structure (direct limit of Hodge structures of the form $\oplus\,\mathbb{Z}(0) + \text{torsion}$).

Now let (E, ∇) be a vector bundle of rank ≥ 2 with a flat connection on X. We write

$$w_p(E, \nabla) = w(P, E, \nabla)$$

where P is the invariant polynomial associated to the p-th elementary symmetric function. Thus w_2 is associated to

$$P(M) = \frac{1}{2}(-\mathrm{tr}(M^2) + \mathrm{tr}(M))^2)$$

Let $z = c_2(E)$ be the second Chern class, which we interpret as an element in $\mathrm{Griff}^2(X) \otimes \mathbb{Q}$. Tensoring (1.3.8) with $\mathbb{Q}$ and pulling back to the fibre over $\mathbb{Q} \cdot z$ yields an exact sequence of finite dimensional $\mathbb{Q}$-mixed Hodge structures, which we write

$$0 \to \tilde{H}^3(X_{an}, \mathbb{Q}(2)) \to \mathcal{E}_z \to \mathbb{Q}(0) \to 0 \tag{1.3.9}$$

The Hodge structure on the left is pure of weight -1, so the sequence splits canonically when tensored with $\mathbb{R}$. That is, there exists a unique $w \in F^0\mathcal{E}_z \otimes \mathbb{R}$ lifting z.

Proposition 1.3.5. *Under the inclusion $\mathcal{E}_z \otimes \mathbb{R} \subset \Gamma(X, \mathcal{H}^3(\mathbb{C}))$, the above element w lifting z is identified with $w_2(E, \nabla)$. In particular, $w_2(E, \nabla) = 0 \iff c_2(E) = 0$ in $\mathrm{Griff}^2(X) \otimes \mathbb{Q}$.*

Proof. This is theorem (5.6.2) in [5]. ∎

2. Riemann-Roch

2.1. **Introduction.** In this lecture we will explain a sort of Riemann-Roch theorem for bundles with flat connections having regular singular points. A preprint with details of these results is on the algebraic geometry server [6]. Such results extend to the irregular rank one case [7], but we don't address this subject in those notes. We just say that then, the top Chern class of differential forms, which in our theory plays the role of the Todd class, then depends on the connection.

A bundle with a flat connection is an example of a holonomic $\mathcal{D}$-module, and very probably the correct Riemann-Roch theorem should be formulated in that context (possibly including also the hypothesis of regular singular points.) One has various Riemann-Roch theorems for D-modules ([27], [23]) but in the context of flat connections these give little more than the Euler characteristic of the underlying coherent sheaf. On a projective variety they depend only on the rank of the local system. A model for a more precise theorem has been proven in the analytic category by Bismut and Lott [3]. A Riemann-Roch theorem with values in the AD-classes explained in the previous lecture could be expected to yield an algebraic proof of the Bismut-Lott result. For technical reasons, however, the algebraic result we want to explain is currently limited to the Chern-Simons classes in $H^*_{CS}(X)$.

Definition 2.1.1. *Let $f : X \to S$ be a flat morphism of smooth varieties. Let $Y \subset X$ and $T \subset S$ be normal crossings divisors, and assume that set-theoretically $f^{-1}(T) \subset Y$. The data $\{f : X \to S, Y \subset X, T \subset S\}$ is said to be a relative normal crossings if*

$$\Omega^1_{X/S}(\log(Y)) := \Omega^1_X(\log Y)/f^*\Omega^1_S(\log T) \tag{2.1.1}$$

is locally free of rank $d = \dim(X/S)$.

Let Z_i denote the components of Y not lying in $f^{-1}(T)$. One has residue maps

$$\mathrm{res}_{Z_i} : \Omega^1_{X/S}(\log(Y)) \to \mathcal{O}_{Z_i} \tag{2.1.2}$$

which may be viewed as "partial trivializations" of $\Omega^1_{X/S}(\log(Y))$. Using ideas of T. Saito [32] one can define relative top chern classes

$$\begin{aligned} c_d(\Omega^1_{X/S}(\log(Y)), \{\mathrm{res}_{Z_i}\}) \in CH^d(X, Z_\bullet) :=& \\ \mathbb{H}^d(X, \mathcal{K}_{d,X} \to \mathcal{K}_{d,Z^{(1)}} \to \mathcal{K}_{d,Z^{(2)}} \to \dots&) \end{aligned} \tag{2.1.3}$$

Here as before $\mathcal{K}_d$ denotes the Milnor K-sheaf, and $Z^{(i)}$ is the i-fold intersections in $Z = \bigcup Z_i$.

We continue to assume $\{f : X \to S, Y \subset X, T \subset S\}$ is a relative normal crossing. Let E be a locally free sheaf on X, and let

$$\nabla : E \to E \otimes \Omega^1_X(\log(Y))$$

be a flat connection with logarithmic poles along Y. As in the first lecture, such a bundle with connection supports CS-classes with log poles. For the Riemann-Roch theorem we will need only the "Newton classes". Let P be the invariant polynomial of degree p which maps the diagonal matrix with x_i entries on the diagonal to $\sum x_i^p$. One defines

$$(2.1.4)\quad Nw_p(E,\nabla) = w(P,E,\nabla) \in H^{2p}_{CS}(X(\log Y)) := \begin{cases} \Gamma(X, \Omega^1_X(\log Y)/d\log(j_*\mathcal{O}^*_{X-Y})) & p=1 \\ \Gamma(X, \Omega^{2p-1}_X(\log Y)/d\Omega^{2p-1}_X(\log Y)) & p\geq 2 \end{cases}$$

Assume further that $f : X \to S$ is projective. The partial trivialization built into the class (2.1.3) compensates for the log poles in (2.1.4), so one has a well-defined product-pushforward

$$(2.1.5)\qquad f_*\Big(Nw_p(E,\nabla)\cdot c_d(\Omega^1_{X/S}(\log Y), \mathrm{res}_{Z_i})\Big) \in H^{2p}_{CS}(S(\log T)).$$

We can use the classical Gauß-Manin construction to push forward E as follows (here again it would be much more convenient to work with D-modules). Consider the diagram of complexes (with differentials defined using the flat connection ∇)

$$(2.1.6)\quad 0 \to E\otimes \Omega^*_{X/S}(\log Y)[-1]\otimes\Omega^1_S(\log T) \to E\otimes\Omega^*_X(\log Y) \to E\otimes\Omega^*_{X/S}(\log Y)\to 0$$

The boundary map puts S-connections on the cohomology sheaves

$$(2.1.7)\quad \nabla_{GM} : \mathbb{R}^i f_{*DR}(E) := \mathbb{R}^i f_*(E\otimes\Omega^*_{X/S}(\log Y)) \to \mathbb{R}^i f_{*DR}(E)\otimes_{\mathcal{O}_S}\Omega^1_S(\log T).$$

these connections are again integrable, and one can try to relate their CS-classes to the CS-classes of E on X. (Warning: because of log poles, the cohomology sheaves are not necessarily locally free on S.)

One further notation: write $\widetilde{E} := E - \mathrm{rk}(E)\mathcal{O}_X$. One extends relative de Rham cohomology and characteristic classes to such virtual bundles with connection by additivity.

Theorem 2.1.2 (Riemann-Roch for flat bundles). *With notation as above ($d = \dim(X/S)$) we have*

$$w_p(\mathbb{R}^* f_{*DR}(\widetilde{E}), \nabla_{GM}) = (-1)^d f_*\Big(c_d(\Omega^1_{X/S}(\log(Y)), \{res_{Z_i}\})\cdot w_p(E,\nabla)\Big).$$

Remark 2.1.3. *1. Notice this Riemann-Roch has a very simple shape.* $w_p(\mathbb{R}^* f_{*DR}(\widetilde{E}), \nabla_{GM})$ *is determined by* $w_p(E, \nabla)$*. This should be contrasted with Riemann-Roch for coherent sheaves, where the inhomogeneous nature of the Todd class* $Td(T_{X/S})$ *results in a mixing of degress of characteristic classes. Essentially, for flat bundles one can think one is taking the chern character of* $E \otimes \Omega^*_{X/S}$*. The homogeneous result then results from the identity*

$$(-1)^{\dim(X/S)} c_d(\Omega^1_{X/S}) = Td(T_{X/S}) \cdot \sum_i (-1)^i ch(\Omega^i_{X/S}). \tag{2.1.8}$$

2. When $p = 1$*, this result fixes the isomorphism class of the local system determinant of cohomology*

$$(\det(\mathbb{R}^* f_{*DR}(\widetilde{E})), \nabla_{GM})$$

For $p \geq 2$ *the* CS*-classes are rather mysterious. Using non-flat connections (see 4.), one can give examples where they are non-zero, but the situation is far from being understood.*
3. This Riemann-Roch theorem only gives information for virtual bundles of rank 0*. It would be interesting to have a "Noether's theorem" calculating the* $Nw_p(\mathbb{R}^* f_{*DR}(\mathcal{O}_X)$*. Suppose, e.g.,* S *is a curve, and the morphism* f *degenerates over a divisor* $T \subset S$*. The class* Nw_1 *represents the determinant* det *of de Rham cohomology, which is a rank* 1 *local system on* $S - T$*. One can show (e.g. by looking at the representation on cohomology with* $\mathbb{Z}$*-coefficients) that* $\det^2$ *is trivial, so*

$$\det := \det \mathbb{R} f_*(\mathbb{C}) \in H^1(S - T, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial} H^0(T, \mathbb{Z}/2\mathbb{Z}). \tag{2.1.9}$$

One question would be for $t \in T$ *to compute the* t*-component of* $\partial(\det)$ *in* $\mathbb{Z}/2\mathbb{Z}$ *in terms of characteristic classes supported on the singular fibre* $f^{-1}(t)$*.*
4. The Riemann-Roch theorem proved in [6] *is somewhat more precise in that the curvature* ∇^2 *is assumed to be basic, i.e. in* $End(E) \otimes f^*\Omega^2_S$*, rather than actually zero.*

2.2. **Sketch of Proof.** We want to give some indication of the basic ideas of the proof of theorem 2.1.2.

2.2.1. *Step 1.* Since the CS-classes are determined at the generic point, one reduces to the case $S = \operatorname{Spec}(F)$ is the spectrum of a function field. By a Lefschetz pencil argument, one reduces further to the case $X = \mathbb{P}^1_F$, $E = \oplus \mathcal{O}_{\mathbb{P}^1}(m_i)$, $m_i \geq 0$. The Nw_p take values in a $\mathbb{Q}$-vector space (except for Nw_1, which requires a special argument) so we can replace S by a finite cover and assume the connection has logarithmic poles along a divisor $D = \{a_1, \ldots, a_\delta\} \subset \mathbb{P}^1(F)$. One shows further that one is free to augment D by adding points where the connection does not have log poles. In particular, we may assume $\infty \in D$ but ∇ does not have a pole at ∞.

2.2.2. *Step 2.* We may assume $E = \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$. In the original version of the paper, this reduction contained a serious error, which was pointed out to us by O. Gabber. Thus, for the sake of credibility, I will give the argument here in full. Identify $E = \oplus \mathcal{O}(m_i \cdot \infty)$, and define

$$E' = \bigoplus_{m_i=0} \mathcal{O}_P \oplus \bigoplus_{m_i>0} \mathcal{O}((m_i - 1)\cdot\infty) \subset E$$

Lemma 2.2.1. *With notation as above, $E' \subset E$ is stable under ∇.*

proof of lemma. The assertion is invariant under an extension of F, so we may assume $D = \{a_1, \ldots, a_\delta, \infty\}$ with all $a_\nu \in P(F)$. Let 1_j, $1 \le j \le r$ be the evident basis of E on $\mathbb{A}^1 = P - \{\infty\}$, and let z be the standard parameter on P. An element $\gamma \in \Gamma(P, \mathcal{O}(n) \otimes \Omega^1_{\mathbb{P}^1_F}(\log D))$ for $n \ge 0$ can be uniquely written in the form

$$\sum_{\nu=1}^{\delta} A_\nu d\log(z - a_\nu) + \sum_{i=0}^{n} z^i \eta_i + \sum_{j=1}^{n} C_j (z - a_1)^j d\log(z - a_1),$$

with $A_\nu, C_j \in F$ and $\eta_i \in \Omega^1_F$. Since $\nabla(z^{m_j} 1_j) \in \Gamma(E \otimes \Omega^1_{\mathbb{P}^1_F}(\log D))$, we may write

$$(2.2.1) \quad \nabla(1_j) = \sum 1_k \otimes \Big[\sum_{i=0}^{m_k - m_j} \eta_j^{ki} (z - a_1)^\ell + \sum A_j^{k\nu} d\log(z - a_\nu) + \sum_{\ell=1}^{m_k - m_j} C_j^{k\ell} (z - a_1)^\ell d\log(z - a_1) \Big]$$

If $m_j > m_k$, the sums over i and ℓ on the right are not there. If $m_j = m_k$, the sum over ℓ is absent. With respect to (2.2.1) we have the following facts:

$$(2.2.2) \quad C_j^{k, m_k - m_j} = 0$$

For $m_j \ge m_k$,

$$(2.2.3) \quad \sum_\nu A_j^{k\nu} d\log(z - a_\nu) \in \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1_F}(\log D)((m_k - m_j)\cdot\infty)).$$

(2.2.4)

$$\sum A_j^{k\nu} d\log(z - a_\nu) \in \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1_F/F}(\log D)((m_k - m_j - 1)\cdot\infty)).$$

To check (2.2.2) we may suppose $m_k > m_j$. The composition

$$\begin{aligned}(2.2.5)\quad \mathcal{O}(m_j) \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega^1_{\mathbb{P}^1_F}(\log D) \to E \otimes \Omega^1_{\mathbb{P}^1_F/F}(\log D)\\ \xrightarrow{\mathrm{res}_\infty} E|_\infty \twoheadrightarrow \mathcal{O}(m_k)|_\infty = \mathcal{O}(m_k \cdot \infty)/\mathcal{O}((m_k-1)\cdot\infty)\end{aligned}$$

maps

$$(z-a_1)^{m_j} 1_j \mapsto C_j^{k,m_k-m_j}(z-a_1)^{m_k} \pmod{\mathcal{O}((m_k-1)\cdot\infty)}$$

By assumption, the connection has zero residue at infinity, so this is zero. The inclusion (2.2.3) follows because

$$\begin{aligned}&\nabla((z-a_1)^{m_j}1_j) =\\ &\sum (z-a_1)^{m_k} 1_k \otimes \Big[\delta_{m_j,m_k}\cdot \eta_j^{k0} + (z-a_1)^{m_j-m_k}\sum A_j^{k\nu} d\log(z-a_\nu)\Big]\\ &\qquad\qquad + m_j(z-a_1)^{m_j}1_j \otimes d\log(z-a_1)\end{aligned}$$

is assumed to extend across infinity. finally, (2.2.4) holds because of the vanishing of the residue (2.2.5). In the case $m_j \geq m_k$ the residue map is

$$1_j \mapsto \Big((z-a_1)^{m_j-m_k}\sum A_j^{k\nu} d\log(z-a_\nu)\Big)\Big|_\infty.$$

Now view (2.2.1) as defining the connection matrix $B = (b_j^k)$ for ∇ on $P - \{\infty\}$. The above assertions can be summarized as follows. For $m_k > m_j$

$$b_j^k \in \Gamma\Big(\mathbb{P}^1, f^*\Omega^1_F((m_k-m_j)\cdot\infty) + \Omega^1_{\mathbb{P}^1_F}(\log D)((m_k-m_j-1)\cdot\infty)\Big)$$

and for $\eta \in \Omega^1_F$ and $m_k \leq m_j$,

$$b_j^k \in \Gamma\Big(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1_F}(\log D)((m_k-m_j)\cdot\infty)\Big)$$

$$b_j^k \in \Gamma\Big(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1_F/F}(\log D)((m_k-m_j-1)\cdot\infty)\Big)$$

$$\eta\wedge b_j^k \in \Gamma\Big(\mathbb{P}^1, \Omega^1_F\otimes\Omega^1_{\mathbb{P}^1_F/F}(\log D)((m_k-m_j-1)\cdot\infty)\Big).$$

It follows that whenever $m_i \leq m_k$ we have

$$b_i^j b_j^k \in \Gamma(\mathbb{P}^1, \Omega^1_F\otimes\Omega^1_{\mathbb{P}^1_F/F}(\log D)(m_k-m_i-1)\cdot\infty)).$$

Vanishing of C_i^{k,m_k-m_i} and the trivial curvature condition

$$db_i^k = \sum_j b_i^j b_j^k$$

implies

$$db_i^k = (m_k-m_i)\eta_i^{k,m_k-m_i}(z-a_1)^{m_k-m_i} d\log(z-a_1) + \epsilon$$

for $\epsilon \in \Gamma\Big(\mathbb{P}^1, \Omega^2_{\mathbb{P}^1_F}(\log(D))((m_k - m_i - 1)\infty))\Big)$. It follows for $m_k > m_i$ that $\eta_i^{k, m_k - m_i} = 0$ as claimed.

It follows now from (2.2.1) that ∇ stabilizes $E' \subset E$, proving the lemma. ☐

The lemma implies reduction to the case $E = \oplus \mathcal{O}_{\mathbb{P}^1}$ by induction on $\max\{m_i\}$.

2.2.3. *Step 3.* We continue to assume $E = \oplus \mathcal{O}_P$ with $P := \mathbb{P}^1_F$ and connection given by

$$\nabla(1_j) = \sum_{k\nu} A_j^{k\nu} 1_k \otimes d\log(z - a_\nu) + \sum_k 1_k \otimes \eta_j^k. \tag{2.2.6}$$

we define an F-linear splitting σ of the natural reduction from absolute to relative E-valued 1-forms

$$\begin{aligned} \sigma : \Gamma(P, \Omega^1_{P/F}(\log D)) &\to \Gamma(P, \Omega^1_{P/k}(\log D)); \\ \sigma(1_k \otimes (z - a_1)^\nu dz) &= 1_k \otimes (z - a_1)^\nu d(z - a_1), \\ \sigma(1_k \otimes d\log(z - a_\nu)) &= 1_k \otimes d\log(z - a_\nu) \end{aligned} \tag{2.2.7}$$

Now consider the diagram

(2.2.8)

$$\begin{array}{ccccc} & & \Gamma(E) & = & \Gamma(E) \\ & & \downarrow \nabla_1 & & \downarrow \nabla_{P/F} \\ \Gamma(E) \otimes \Omega^1_F & \to & \Gamma(E \otimes \Omega^1_P(\log D)) & \overset{\sigma}{\leftrightarrows} & \Gamma(E \otimes \Omega^1_{P/F}(\log D)) \\ \downarrow \nabla_{P/F} \otimes 1 & & \downarrow \nabla_2 & & \\ \Gamma(E \otimes \Omega^1_{P/F}(\log D)) \otimes \Omega^1_F & = & \Gamma(E \otimes \Omega^2_P(\log D)/\Omega^2_F) & & \end{array}$$

Here ∇_1 and ∇_2 are the absolute connection maps. Define

$$\Phi := \nabla_1 - \sigma\nabla_{P/F}; \quad \Psi = -\nabla_2\sigma. \tag{2.2.9}$$

The diagram

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\Phi} & \Gamma(E) \otimes \Omega^1_F \\ {\scriptstyle \nabla_{P/F}} \big\downarrow & & \big\downarrow {\scriptstyle \nabla_{P/F} \otimes 1} \\ \Gamma(E \otimes \Omega^1_{P/F}(\log D)) & \xrightarrow{\Psi} & \Gamma(E \otimes \Omega^1_{P/F}(\log D)) \otimes \Omega^1_F \end{array} \tag{2.2.10}$$

represents $(\sum_{i=0}^{i=2} (-1)^i (R^i f_*(E \otimes \Omega^*_{P/F}, \nabla_{P/F}), GM^i(\nabla))$ in the Grothendieck group $\mathcal{K}(F)$ of F-vector spaces with connection. We see from (2.2.6) that

$$\Phi(1_j) = (\nabla - \sigma\nabla_{P/F})(1_j) = \sum_k 1_k \otimes \eta_j^k. \tag{2.2.11}$$

Also

$$(2.2.12)\quad \Psi(1_j \otimes d\log(z-a_\nu) = -\nabla(1_j)\wedge d\log(z-a_\nu) =$$
$$= \Big(-\sum_{k\tau} A_j^{k\tau} 1_k \otimes d\log(z-a_\tau) - 1_k \otimes \eta_j^k\Big) \wedge d\log(z-a_\nu) =$$
$$= -\sum_{k\tau} A_j^{k\tau} 1_k \otimes \Big(d\log(z-a_\tau) - d\log(z-a_\nu)\Big) \otimes d\log(a_\nu - a_\tau) +$$
$$+ \sum_k 1_k \otimes d\log(z-a_\nu)\otimes \eta_j^k.$$

Define

$$(2.2.13)\qquad B_{j\nu}^{k\tau} = \begin{cases} -A_j^{k\tau} d\log(a_\nu - a_\tau) & \tau \neq \nu \\ \eta_j^k + \sum_{\theta\neq\nu} A_j^{k\theta} d\log(a_\nu - a_\theta) & \tau = \nu \end{cases}.$$

Then

$$(2.2.14)\qquad \Psi(1_j \otimes d\log(z-a_\nu)) = \sum_{k\tau} 1_k \otimes d\log(z-a_\tau)\wedge B_{j\nu}^{k\tau}.$$

The left hand side of the Riemann-Roch formula for E (in this case there is no need to subtract off $\mathcal{O}^{\oplus \mathrm{rk}(E)}$) is given by

$$(2.2.15)\quad Nw_n((\sum_{i=0}^{i=2}(-1)^i(R^i f_*(E\otimes \Omega^*_{P/F}, \nabla_{P/F}), GM^i(\nabla)))$$
$$= Nw_n(\Phi) - Nw_n(\Psi).$$

2.2.4. *Step 4.* We next make some observations about $Nw_n(\Psi)$. Define B_ν^τ (resp. B) to be the $N\times N$ matrix (resp. $\delta\times\delta$ matrix of $N\times N$ matrices)

$$(2.2.16)\qquad B_\nu^\tau := (B_{j\nu}^{k\tau})_{1\le j,k\le N} \quad (\text{resp. } B = (B_\nu^\tau)_{1\le \nu,\tau\le n}).$$

Lemma 2.2.2. *Let $M(B) = B^{r_1}(dB)^{r_2}\cdots B^{r_{2s-1}}(dB)^{r_{2s}}$ be some (non-commuting) monomial in B and dB. Then*

$$\mathrm{Tr}\ (M(B)) = \sum_{\tau=1}^n \mathrm{Tr}\ (M(B_\tau^\tau)).$$

Proof. Write as above

$$M(B)_\nu^\tau := (M(B)_{j\nu}^{k\tau})_{1\le j,k\le N} \quad (\text{resp. } M(B) = (M(B)_\nu^\tau)_{1\le\nu,\tau\le n}).$$

Then $\mathrm{Trace}(M(B)) = \sum_\tau \mathrm{Trace}(M(B)_\tau^\tau)$. Now

$$M(B)_\tau^\tau = \sum_{\tau_1,\ldots,\tau_{r_{2s}-1}} B_\tau^{\tau_1} B_{\tau_1}^{\tau_2}\cdots B_{\tau_{r_1}-1}^{\tau_{r_1}} dB_{\tau_{r_1}}^{\tau_{r_1}+1}\cdots dB_{\tau_{r_{2s}-1}}^\tau.$$

For $\nu \neq \tau$ we can write $B_\nu^\tau = C_\nu^\tau d\log(a_\nu - a_\tau)$. Possibly introducing some signs, the $d\log$ terms can be pulled to the right. Suppose, among $\{\tau_1, \tau_2, \ldots, \tau_{r_{2s}-1}\}$ we have $\tau_{j_1}, \ldots, \tau_{j_a} \neq \tau$ and all the other $\tau_k = \tau$. Then that particular summand on the right multiplies

$$d\log(a_\tau - a_{\tau_{j_1}}) \wedge \cdots \wedge d\log(a_{\tau_{j_a}} - a_\tau) = 0.$$

(Note $x_1 + \cdots + x_{a+1} = 0 \Rightarrow dx_1 \wedge \ldots \wedge dx_{a+1} = 0$.) Thus, one one term on the right is non-zero, and

$$M(B)_\tau^\tau = M(B_\tau^\tau),$$

proving the lemma. □

Since $Nw_n(\Psi)$ is a sum of terms Tr $(M(B))$ as in the lemma, we conclude

$$Nw_n(\Psi) = \sum_{\tau=1}^{\delta} Nw_n(\Psi_\tau), \tag{2.2.17}$$

where Ψ_τ is the connection on $F^{\oplus N}$ given (with notation as above) by

$$1_j \mapsto \sum_{k=1}^{N} 1_k \otimes (\eta_j^k + \sum_{\theta\neq\tau} A_j^{k\theta} d\log(a_\tau - a_\theta))$$

The connection matrix for Ψ_τ is thus

$$\Phi + \sum_{\theta\neq\tau} A^\theta d\log(a_\tau - a_\theta) \tag{2.2.18}$$

where $\Phi = (\eta_j^k)$ and $A^\theta = (A_j^{k\theta})$.

2.2.5. *Step 5.* We now consider the right hand side of the Riemann-Roch formula, which in our case takes the form

$$-Nw_n(E, \nabla) \cdot c_1(\Omega^1_{P/F}(\log D), \mathrm{res}_D)$$

Since $\Omega^1_{P/F}(\log D)$ has rank 1, the relative chern class can be computed in a standard way to be the divisor of any meromorphic section ω of the bundle such that ω is regular along D and $\mathrm{res}_D(\omega) = 1$. We shall assume that $0 \notin D$. (This is easy to arrange by applying an automorphism to P.) We take for our meromorphic section

$$\omega := \Big(\sum_{\tau=1}^{\delta} \frac{1}{z - a_\tau} - \frac{\delta + 1}{z}\Big) dz. \tag{2.2.19}$$

Clearing denominators

(2.2.20)
$$\omega = \frac{F(z)}{z\prod(z-a_\tau)}dz; \quad F(z) = \sum_{\tau=1}^{\delta} a_\tau \prod_{\theta\neq\tau}(z-a_\theta) - \prod_{\tau=1}^{\delta}(z-a_\tau).$$

Writing formally $F = \prod_{i=1}^{\delta}(z-\beta_i)$, we get
$$c_1(\Omega^1_{P/F}(\log D), \mathrm{res}) = (\omega) = \sum(\beta_i) - (0).$$

2.2.6. *Step 6.* We shall need to compute

$$-\sum_{i=1}^{\delta} Nw_n(E,\nabla)|_{z=\beta_i} + Nw_n(E,\nabla)|_{z=0} \tag{2.2.21}$$

and compare the answer to $Nw_p(\Phi) - Nw_p(\Psi)$ (cf. (2.2.11), (2.2.13), (2.2.15), (2.2.17), (2.2.18)).

Proposition 2.2.3. *Suppose given* $\{j_1,\dots,j_r\} \subset \{1,\dots,\delta\}$. *Then*

$$d\log(z-a_{j_1}) \wedge \cdots \wedge d\log(z-a_{j_r})|(\omega) :=$$
$$\sum_{i=1}^{\delta} d\log(\beta_i - a_{j_1}) \wedge \cdots \wedge d\log(\beta_i - a_{j_r}) - d\log(a_{j_1}) \wedge \cdots \wedge d\log(a_{j_r}) =$$
$$= \sum_{t\neq j_1,\dots,j_r} d\log(a_t - a_{j_1}) \wedge \cdots \wedge d\log(a_t - a_{j_r}).$$

Proof. To simplify notation we consider the case $j_k = k,\ 1 \le k \le r$.

Lemma 2.2.4.

$$\sum_{i=1}^{\delta} d\log(\beta_i - a_1) \wedge \cdots \wedge d\log(\beta_i - a_r) =$$
$$= \sum_{j=1}^{r}(-1)^{j-1} d\log(F(a_j)) \wedge d\log(a_j - a_1) \wedge \cdots \wedge \widehat{d\log(a_j - a_j)} \wedge \cdots$$
$$\cdots \wedge d\log(a_j - a_r).$$

proof of lemma 2.2.4. We can argue universally. Consider the rings

$$A = \mathbb{Q}[z_1,\dots,z_\delta,t_1\dots,t_r] \subset B = A[x]/(x^\delta + \sum_{i=1}^{\delta} z_i x^{\delta-i}) \cong$$
$$\cong \mathbb{Q}[z_1,\dots,z_{\delta-1},t_1\dots,t_r,X].$$

Note that both A and B are polynomial rings over $\mathbb{Q}$. Let $L \subset M$ be their quotient fields, and consider the symbol

$$S = \{X - t_1, \dots, X - t_r\} \in K_r(M).$$

We will compute the norm, $N(S) \in K_r(L)$. Note the tame symbol is given by

$$\mathrm{tame}(S) = \sum_{k=1}^{r} (-1)^{k-1} \{t_k - t_1, \dots, \widehat{t_k - t_k}, \dots, t_k - t_r\}|_{X=t_k}$$

Let $\pi : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$. We have a map on divisors

$$\pi_*(X - t_i = 0) = (F(t_i) = 0)$$

with degree 1, so

$$\begin{aligned}
&\mathrm{tame}(N(S)) = N(\mathrm{tame}(S)) = \\
&\quad = \sum_{k=1}^{r} (-1)^{k-1} \{t_k - t_1, \dots, \widehat{t_k - t_k}, \dots, t_k - t_r\}|_{F(t_k)=0} = \\
&\quad = \mathrm{tame}\Big(\sum_{k=1}^{r} (-1)^{k-1} \{F(t_k), t_k - t_1, \dots, \widehat{t_k - t_k}, \dots, t_k - t_r\} \Big).
\end{aligned}$$

(The last equality holds because $F(t_j)/F(t_k) = 1$ on the divisor $t_j = t_k$.). Since L is purely transcendental over $\mathbb{Q}$, this determines $N(S)$ upto constant symbols, which can be ignored because we want to apply $d\log$. Specializing the z_i to the coefficients of our F and the $t_i \mapsto a_i$ and applying $d\log$, we deduce the lemma. □

Lemma 2.2.5.

$$\begin{aligned}
&d\log(b_1) \wedge \cdots \wedge d\log(b_r) = \\
&\quad = \sum_{k=1}^{\delta} (-1)^{k-1} d\log(b_k) \wedge d\log(b_k - b_1) \wedge \cdots \wedge d\log\widehat{(b_k - b_k)} \wedge \cdots \\
&\qquad\qquad \cdots \wedge d\log(b_k - b_r).
\end{aligned}$$

proof of lemma 2.2.5. As above, we argue universally and prove the corresponding identity for symbols. For this it suffices to compare the images under the tame symbol. At the divisor $b_j - b_k = 0$ for $j < k$ we need

$$\begin{aligned}
0 = {}& (-1)^{j+k} \{b_k, b_k - b_1, \dots, \widehat{b_k - b_j}, \dots, \widehat{b_k - b_k}, \dots, b_k - b_r\}|_{b_k = b_j} + \\
&+ (-1)^{j+k-1} \{b_j, b_j - b_1, \dots, \widehat{b_j - b_k}, \dots, \widehat{b_j - b_j}, \dots, b_j - b_r\}|_{b_k = b_j},
\end{aligned}$$

which is clear. Finally at the divisor $b_k = 0$ we need

$$(-1)^{k-1}\{-b_1, \ldots, \widehat{-b_k}, \ldots, -b_r\} = (-1)^{k-1}\{b_1, \ldots, \hat{b}_k, \ldots, b_r\} + \epsilon$$

where ϵ dies under $d\log$. Again this is clear. $\square$

Returning to the proof of proposition 2.2.3, we apply lemmas 2.2.4 and 2.2.5 (with $b_j = a_j$) to conclude

$$\begin{aligned}
&d\log(z - a_{j_1}) \wedge \cdots \wedge d\log(z - a_{j_r})|_{(\omega)} = \\
&\quad = \sum_{s=1}^{r} (-1)^{s-1} d\log\Big(\prod_{k \notin \{j_1, \ldots, j_r\}} (a_{j_s} - a_k)\Big) \wedge d\log(a_{j_s} - a_{j_1}) \wedge \cdots \\
&\qquad\qquad \cdots \wedge d\log\widehat{(a_{j_s} - a_{j_s})} \wedge \cdots \wedge d\log(a_{j_s} - a_{j_r}) \\
&= \sum_{\substack{k \neq j_1, \ldots, j_r \\ s=1}}^{s=r} (-1)^{s-1} d\log(a_{j_s} - a_k) \wedge d\log(a_{j_s} - a_{j_1}) \wedge \cdots \wedge d\log\widehat{(a_{j_s} - a_{j_s})} \wedge \cdots \\
&\qquad\qquad \cdots \wedge d\log(a_{j_s} - a_{j_r}).
\end{aligned}$$

Finally we apply lemma 2.2.4 again to this last expression, taking $b_s = a_{j_s} - a_k$, to get the assertion of the proposition:

$$\begin{aligned}
&d\log(z - a_{j_1}) \wedge \cdots \wedge d\log(z - a_{j_r})|_{(\omega)} = \\
&\qquad = \sum_{k \neq j_1, \ldots, j_r} d\log(a_{j_1} - a_k) \wedge \cdots \wedge d\log(a_{j_r} - a_k).
\end{aligned}$$

$\square$

Proposition 2.2.6. *With notation as above, the Riemann-Roch formula holds for* (E, ∇).

Proof. The computation mentioned in (2.2.21) can be done as follows. Let ρ_ν be closed 1-forms. For $J = \{j_1 < \ldots < j_r\} \subset \{1, \ldots, \delta\}$ define $\rho_J = \rho_{j_1} \wedge \cdots \wedge \rho_{j_r}$. Write

(2.2.22)

$$Nw_n(\mathcal{O}_P^{\oplus N}, \sum_{\nu=1}^{\delta} A^\nu \rho_\nu + \Phi) = \sum_{J \subset \{1, \ldots, \delta\}} P_J(A^\nu, dA^\nu, \Phi, d\Phi)\rho_J + Nw_n(\mathcal{O}_P^{\oplus N}, \Phi)$$

Here A^ν (resp. Φ) are matrices with coefficients in F (resp. Ω^1_F), and the P_J are independent of the ρ_j. Then, using proposition 2.2.3, we get

$$(2.2.23) \quad -Nw_n(\sum_{\nu=1}^{\delta} A^\nu\rho_\nu + \Phi)|_{(\omega)} =$$

$$= -\sum_{\substack{J\subset\{1,\ldots,\delta\}\\ r=|J|\geq 1}} P_J(A^\nu, dA^\nu, \Phi, d\Phi)\sum_{k\notin J} d\log(a_{j_1}-a_k)\wedge\cdots\wedge d\log(a_{j_r}-a_k)+$$

$$+(1-\delta)Nw_n(F^{\oplus N}, \Phi).$$

On the other hand, if we fix $\tau \leq \delta$ and take $\rho_\nu = d\log(a_\tau - a_\nu)$ for $\nu \neq \tau$ and $\rho_\tau = 0$ we find

(2.2.24)

$$Nw_n(\mathbb{R}f_*(E\otimes\Omega^*_{P/F}(\log D))) = Nw_n(F^N, \Phi) - \sum_{\tau=1}^{\delta} Nw_n(F^N, \Psi_\tau) =$$

$$= -\sum_{\tau=1}^{\delta}\sum_{\substack{J\subset\{1,\ldots,\delta\}\\ \tau\notin J}} P_J(A^\nu, dA^\nu, \Phi, d\Phi)d\log(a_{j_1}-a_\tau)\wedge\cdots\wedge d\log(a_{j_r}-a_\tau)+$$

$$+(1-\delta)Nw_n(\Phi).$$

The right hand sides of (2.2.23) and (2.2.24) coincide, proving the proposition. $\square$

This completes the sketch of the proof of Riemann-Roch for bundles with regular connections. For more details, the reader is referred to [6].

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