1 Introduction

Many physical phenomena involve interactions between small and large scales. Sometimes these
interactions are hiding in plain sight. For example, when bread is produced at an industrial scale, large
machines with rotating paddles are used to mix the different ingredients together. Even though the paddles
move smoothly at a very large scale, the mixture soon becomes homogeneous at small scales. Another
example comes from metallurgy: although steel is a mixture of iron and carbon, it has different properties
than either of its constituents. The small-scale structure of the iron and carbon atoms increases the strength
and fracture toughness of the large-scale material. These examples show that the interactions between small
and large scales can go either way: large scales can affect small ones or vice versa.

My research concerns the quantitative properties of physical systems which exhibit these small and
large-scale interactions. On one hand, I am interested in the mixing properties of fluids, which involves the
creation of highly oscillatory structure at small scales. On the other hand, I am interested in homogenization,
which studies the large-scale properties of solutions to equations whose coefficients have some regular
structure, either periodic or random. In both the large and small scales, quantitative information about rates of
convergence can reveal qualitative properties about the underlying models, as in Theorem 5 below.

Below, I outline three projects based on my current work [13–16], each related to studying quantitative
rates at small and large scales. First, I plan to study optimal passive scalar mixing rates in domains with a
no-slip or partial-slip condition at the boundary. Second, I plan to study the rate of periodic homogenization
for non-convex Hamilton-Jacobi equations. Third, I plan to study the relationship between homogenization of
the G equation and the allowed compressibility of the random environment. Along the way, there are related
problems and steps which I outline below.

2 Passive scalar mixing by shear flows

**Background.** Given a time-dependent vector field \( b : [0, 1] \times (\mathbb{R}^2 / \mathbb{T}^2) \rightarrow \mathbb{R}^2 \) on the flat torus and some mean-zero initial data \( u_0 : \mathbb{R}^2 / \mathbb{T}^2 \rightarrow \mathbb{R} \), consider the solution \( u : [0, 1] \times (\mathbb{R}^2 / \mathbb{T}^2) \rightarrow \mathbb{R} \) to the associated transport equation

\[
\begin{cases}
D_t u(t, x) + b(t, x) \cdot D_x u(t, x) = 0 & \text{if } t > 0, \; x \in \mathbb{R}^2 / \mathbb{T}^2 \\
 u(0, x) = u_0(x) & \text{if } x \in \mathbb{R}^2 / \mathbb{T}^2.
\end{cases}
\]

Assume that \( b \) is divergence-free so that the associated flow preserves Lebesgue measure. Given some bound
on \( b \), how small can \( u(1, \cdot) \) be, in a suitable weak sense?

We define the geometric mixing scale by

\[
\text{mix}(b) := \sup \left\{ \text{radius}(B) \mid \frac{1}{|B|} \int_B u(1, x) \, dx > 1 \right\},
\]

where the supremum is over all balls \( B \subseteq \mathbb{R}^2 / \mathbb{T}^2 \) and the initial data \( u_0(x_1, x_2) := 2^k 1_{x_1 \leq 1} - 2^k 1_{x_1 > 1} \). Under this definition, Bressan [7] conjectured that the optimal mixing rate is exponential in \( \|D_x b\|_{L^1} \), i.e.

\[
| \log \text{mix}(b) | \leq C \|D_x b\|_{L^1}, \tag{1}
\]

for some constant \( C > 0 \). There are many examples of vector fields \( b \) which achieve this rate [1, 7], but after
nearly two decades there is no known upper bound for \( | \log \text{mix}(b) | \) in terms of \( \|D_x b\|_{L^1} \).

Under some modified assumptions, there has been substantial progress toward an inequality resembling (1).
Crippa–De Lellis [17] proved a version of (1) where \( L^1 \) is replaced with \( L^p \) for any \( p > 1 \), and the constant
\( C > 0 \) depends on \( p \). This was a huge step, but there are qualitative differences between vector fields \( b \in W^{1,1} \)
and \( b \in W^{1,p} \) which make the \( p = 1 \) case more difficult. For instance, any discontinuous shear can be
approximated to arbitrary precision by a vector field bounded in \( W^{1,1} \), but not by vector fields in \( W^{1,p} \).
Conversely, we look for particularly regular vector fields $b$ which still mix at the exponential rate. Blumenthal–Coti Zelati–Gvalani [6] recently proved a decades-old conjecture of Pierrehumbert [24] that the vector field

$$b(t, x) := \begin{cases} 
  (\sin(2\pi x_2 + \omega_1), 0) & \text{if } [\tau] \text{ is odd} \\
  (0, \sin(2\pi x_1 + \omega_1)) & \text{if } [\tau] \text{ is even}
\end{cases}$$

mixes at the exponential rate almost surely, where $\omega_1, \omega_2, \ldots$ are chosen i.i.d. from $[0, 2\pi]$. This example is so regular that it satisfies $|\log \text{mix}(b)| \geq c\|b\|_{C^k}$ for a constant $c = c(k) > 0$ for all $k \in \mathbb{N}$.

**Current research.** I recently proved a version of Bressan’s conjecture (1) [14] under the additional assumption that $b$ is a shear flow at each time, (that is, $b(t, x)$ is parallel to $b(t, y)$ for each $t \in [0, 1]$ and $x, y \in \mathbb{R}^2/\mathbb{Z}^2$) answering a question of Hadžić–Seeger–Smart–Street [19].

![Fig. 1: The flow under the vector field in (3) with random durations](image)

I also adapted ideas of Blumenthal–Coti Zelati–Gvalani [6] to show that, almost surely, an even simpler vector field mixes exponentially. Instead of randomizing phases, we only randomize durations; the example is

$$b(t, x) := \begin{cases} 
  \tau_{[\tau]}(\sin(2\pi x_2), 0) & \text{if } [\tau] \text{ is odd} \\
  \tau_{[\tau]}(0, \sin(2\pi x_1)) & \text{if } [\tau] \text{ is even}
\end{cases}$$

where $\tau_1, \tau_2, \ldots$ are chosen independently and uniformly from $[0, T]$ for $T > 0$ sufficiently large. This vector field alternates between just two distinct shears. To prove that the mixing rate for $b$ is exponential, one must contend with the existence of fixed points $\{0, \frac{1}{2}\}^2$ of the flow. After removing these fixed points, one works on a noncompact space where drift conditions ensuring mixing become more delicate.

**Future research.** There is a clear candidate for a next step toward Bressan’s conjecture (1): a version where $b$ is built up from both shears and certain kinds of vortices, as in Hadžić–Seeger–Smart–Street [19]. The only known proof for shears relies on a notion of energy which is explicitly not multiscale, while the only known proof for vortices relies on a multiscale energy. A proof which unifies these different perspectives should reveal further understanding of the general case.

On the other hand, I am especially interested in investigating the more physically relevant setting where boundary conditions complicate the mixing, in the following question.

**Question 1.** What is the optimal rate of mixing in a domain $\Omega \subset \mathbb{R}^d$ with no-slip or partial-slip boundary conditions on the vector field $b$? Is the optimal rate achieved for a wide class of random vector fields?

Since, in the no-slip setting, the boundary is effectively a codimension one fixed point, I expect that my techniques for handling the fixed points in [14] will prove useful here.
3 Periodic homogenization of convex Hamilton-Jacobi equations

**Background.** Let the Hamiltonian \( H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be continuous, \( \mathbb{Z}^d \)-periodic in the first variable, \( x \), and coercive in the second variable, \( p \). Assume that the coercivity is uniform in \( x \); that is,

\[
\liminf_{|p| \to \infty} \inf_{x \in \mathbb{R}^d} H(x, p) = +\infty.
\]

We are interested in studying, as \( \varepsilon \to 0^+ \), the solution to the initial-value problem

\[
\begin{align*}
D_t u^\varepsilon(t, x) + H\left(\frac{x}{\varepsilon}, D_x u^\varepsilon(t, x)\right) &= 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\
u^\varepsilon(0, x) &= u_0(x) & \text{in } \mathbb{R}^d.
\end{align*}
\tag{4}
\]

In a celebrated work, Lions–Papanicolaou–Varadhan [23] proved that \( u^\varepsilon \to \overline{u} \) locally uniformly as \( \varepsilon \to 0^+ \), where \( \overline{u} : \mathbb{R}^d \to \mathbb{R} \) is the solution to the effective problem

\[
\begin{align*}
D_t \overline{u}(t, x) + \overline{H}(D_x \overline{u}(t, x)) &= 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\
\overline{u}(0, x) &= u_0(x) & \text{in } \mathbb{R}^d,
\end{align*}
\tag{5}
\]

where the effective Hamiltonian \( \overline{H} : \mathbb{R}^d \to \mathbb{R} \) is determined uniquely by \( H \).

From a formal two-scale asymptotic expansion, it was expected that the rate of convergence \( \|u^\varepsilon - u\|_{L^\infty} \) should be \( O(\varepsilon) \), but the result of Lions–Papanicolaou–Varadhan [23] was purely qualitative.

The first quantitative version of the result of Lions–Papanicolaou–Varadhan [23] was given by Capuzzo-Dolcetta–Ishii [10], who proved a rate of \( O(\varepsilon^{1/3}) \), assuming only that the initial data \( u_0 \) is Lipschitz and the Hamiltonian \( H \) is locally Lipschitz. Nearly twenty years later, assuming also that \( H \) is convex in \( p \), Mitake–Tran–Yu [22] improved the rate from \( O(\varepsilon^{1/3}) \) to \( O(\varepsilon^{1/2}) \), using techniques from weak KAM theory. They also improved the rate to the optimal \( O(\varepsilon) \) in \( d = 2 \) dimensions, under the additional assumption that the Hamiltonian \( H \) is positively homogeneous of degree \( k \) for some \( k \geq 1 \).

Improving the rate further in dimensions \( d \geq 3 \) and extending the result in dimension \( d = 2 \) to cover inhomogeneous Hamiltonians was a major open problem in the area.

**Current research.** Under the same convexity assumption, I proved [15] the following theorem, which improved the rate of convergence to the near-optimal \( O(\varepsilon \log \varepsilon) \) in all dimensions.

**Theorem 2** (Theorem 1 in [15]). If \( H \) is convex in \( p \) and \( u_0 \) is Lipschitz, then there is a constant \( C = C(H, \text{Lip}(u_0)) > 0 \) such that, for all \( t > 0 \) and \( x \in \mathbb{R}^d \),

\[
|u^\varepsilon(t, x) - \overline{u}(t, x)| \leq C \varepsilon \log(C + t\varepsilon^{-1}).
\]

I also found a simple proof of the optimal \( O(\varepsilon) \) rate of convergence in \( d = 2 \) dimensions, without any assumption of homogeneity on \( H \). The main point of the proof was a shift in perspective; viewing the problem from the standpoint of optimal control theory, one can apply a result of Alexander [2], involving the convergence of subadditive functions in the context of first-passage percolation.

Further, Tran–Yu [25] pointed out that by replacing a lemma in [15] with a lemma of Burago [9], one achieves the optimal \( O(\varepsilon) \) rate in all dimensions.

**Future research.** Although the question of a quantitative rate for convex Hamiltonians is completely settled, the corresponding question for nonconvex Hamiltonians is wide open. Indeed, a counterexample of Ziliotto [26] in the random setting shows that convexity can play an important role in homogenization. I suspect that the optimal rate for nonconvex Hamiltonians is slower than \( O(\varepsilon) \); the construction of a Hamiltonian which admits a slower rate would yield insight into the large-scale geometry of solutions to the corresponding Hamilton-Jacobi equations.
**Question 3.** Is there a periodic, coercive Hamiltonian $H: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $\|u^\varepsilon - u\|_{L^\infty} \geq c \varepsilon^{1-\alpha}$ for some constants $c, \alpha > 0$? If so, how large can $\alpha$ be?

I expect that the main insight of [15], use of the optimal control interpretation, will continue to provide insight in the non-convex case. Since the representation formula becomes more complicated, I aim to prove a result similar to that of Alexander [2], but in the context of two-player games, which characterize general (nonconvex) Hamilton-Jacobi equations.

4  **Stochastic homogenization of the G equation**

**Background.** The G equation is a Hamilton-Jacobi equation which models combustion of a flammable gas subject to advection. The equation at scale $\varepsilon > 0$ is given by

$$
\begin{aligned}
D_t u^\varepsilon(t,x) + |D_x u^\varepsilon(t,x)| - V(\varepsilon^{-1} x) \cdot D_x u^\varepsilon(t,x) &= 0 & \text{ in } \mathbb{R}_{>0} \times \mathbb{R}^d \\
u^\varepsilon(0,x) &= u_0(x) & \text{ in } \mathbb{R}^d.
\end{aligned}
$$

(6)

The vector field $V: \mathbb{R}^d \to \mathbb{R}$ models the wind velocity, and a level set of $u^\varepsilon$ models the flame front. Since the Hamiltonian $H(x,p) = |p| - V(x) \cdot p$ has a particularly simple form, the G equation is an ideal first candidate for studying the homogenization of Hamilton-Jacobi equations which lack coercivity. Indeed, if the wind speed $|V(x)|$ is greater than $1$, then the Hamiltonian is not coercive. Physically, this means that if a flame started at a point $x$ where $|V(x)| > 1$, then, even after a small time $t > 0$, there exist points arbitrarily close to $x$ (in the up-wind direction) where the flame has not reached.

The lack of coercivity, which corresponds to controllability in the optimal control setting, is the main obstacle to homogenization. Even for $\mathbb{Z}^d$-periodic $V$, it is possible that homogenization may not occur. For example, if the flame starts at the origin and $V(x) = -Cx$ for all $|x| \leq \frac{1}{C}$ for some constant $C > 0$, the flame will never escape the ball of radius $1/C$ centered at the origin.

The first proof of homogenization, for $\mathbb{Z}^d$-periodic $V$, is due to Cardaliaguet–Nolen–Souganidis [11], under the assumption that $\|\text{div } V\|_{L^\infty} \leq \varepsilon$ where $\varepsilon = \varepsilon(d) > 0$ is a constant. Since $\|\text{div } V\|_{L^\infty}$ bounds the compressibility of the gas, this assumption rules out the non-homogenization behavior described above.

In the random setting, Cardaliaguet–Souganidis [12] proved homogenization, under the assumption that $V$ is stationary ergodic and divergence-free. Under the very general assumption that $V$ is stationary ergodic, the homogenization result of [12] was purely qualitative.

**Current research.** I proved [13, 16] the following quantitative rate of homogenization under the assumption that $V$ has finite range of dependence, a continuous analogue of the i.i.d. assumption.

**Theorem 4 (Theorem 1 in [13]).** Let $V: \mathbb{R}^d \to \mathbb{R}^d$ be a random Lipschitz vector field which has unit range of dependence and is $\mathbb{Z}^d$-translation invariant. Then there is a function $\bar{H}: \mathbb{R}^d \to \mathbb{R}$, which is positively homogeneous of degree one and coercive, and a constant $C = C(d) > 0$ such that, if

$$|\text{div } V| \leq C^{-1} (\|V\|_{C^{0,1}} + 1)^{-C}
$$

almost surely, then there is a random variable $T_0$, with

$$\mathbb{E}[\exp(C^{-1} (\|V\|_{C^{0,1}} + 1)^{-C} \log^{3/2} T_0)] \leq C,$
$$

such that

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| \leq C(\|V\|_{C^{0,1}} + 1)^C(T\varepsilon)^{1/2} \log^2(\varepsilon^{-1}T)
$$

(7)

for all $T \geq \varepsilon T_0$ and $t, |x| \leq T$, where $u^\varepsilon$ is the solution to the G equation (6) and $\bar{u}$ is the solution of the effective equation (5).
There are two main novel features of this theorem. First, we allow for the more general situation where the compressibility $|\text{div} V|$ is small but nonzero. This contrasts with the divergence-free assumption in [12], which was necessary due to the lack of a quantitative decorrelation assumption on $V$. Second, we achieve a quantitative rate, which may be necessary for some applications. For example, in [16] I use the quantitative Theorem 4 to deduce the following stability result for the effective Hamiltonians.

**Theorem 5** (Theorem 2 in [16]). Let $\mathbb{P}, \mathbb{P}^*$ be probability measures on $C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ which satisfy the same assumptions as in Theorem 4. Let $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}^*$ be the effective Hamiltonians for $\mathbb{P}^*$ and $\mathbb{P}$ respectively. Assume that $\pi(\mathbb{P}, \mathbb{P}^*) \leq \varepsilon$, where $\pi$ denotes the Lévy-Prokhorov metric. Then

$$|\overline{\mathcal{H}}(p) - \overline{\mathcal{H}}^*(p)| \leq |p| \varepsilon^{1/3} \log^3 \varepsilon^{-1}.$$ 

In particular, the effective Hamiltonian $\overline{\mathcal{H}}$ is a weakly-star continuous function of the law $\mathbb{P}$ of the environment, where the topology on $\overline{\mathcal{H}}$ is given by uniform convergence on compact sets.

**Future research.** Although the assumptions of Theorem 4 allow for some small compressibility of the environment, the exact amount of compressibility allowed seems to be far from optimal. In the periodic setting, Cardaliaguet–Nolen–Souganidis [11] manage to assume a bound for compressibility which depends only on the dimension; their bound has a beautiful relationship with the isoperimetric inequality for periodic sets. I expect that the bound on compressibility can be relaxed by studying connections with the isoperimetric inequality for sets that have finite range of dependence in a certain sense.

It should also be possible to further refine the quantitative rate. Specifically, methods from first-passage percolation [5] which show that the variance of first-passage time is sublinear suggest that a rate of $O\left(\sqrt{\varepsilon / \log \varepsilon}\right)$ may be achievable.

**References**


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