A padic Riemann–Hilbert functor for $\mathbb{F}_p$-coefs
(joint with Lurie)

Last time: $\mathbb{C}/\mathbb{Q}_p$ complete + alg closed field, $k = \mathbb{Q}_c/m$
$R = p$-complete $\mathbb{Q}_c$-algebra

$\Rightarrow R_{\text{pf}} \subseteq \text{CAlg}(\frac{D^0}{\text{comp}}(R))$

a) $R_{\text{pf}} \otimes_k k \simeq (R \otimes_k k)_{\text{perf}}$

b) If $R/\mathbb{Q}_c$ is formally smooth, then $\ker(i^!)$

Today: Build a Riemann–Hilbert functor using $R_{\text{pf}}$ as a coherent counterpart of $\mathbb{Z}_p$ x
Setup: \( C/\Omega_p \) as above.

\( X/\Omega_c \) any scheme

\[ X_0 = X \otimes_{\Omega_c} \Omega_c/p \] scheme \( (\Omega_c/p)^a \)

Recall: Recall that the max ideal \( m \subset \Omega_c \) satisfies

\[ m \not\subset \Omega_c \]

\[ \Rightarrow \text{restriction of scalars} \quad D(\Omega_c/m) \rightarrow D(\Omega_c) \]

is fully faithful.

\[ \therefore \quad D(\Omega_c)^a := D(\Omega_c)/D(\Omega_c/m) \rightarrow \text{almost derived category} \]

Likewise: \( D_{qc}(X_0)^a \) = almost quasi-coherent derived category of \( X_0 \)

\[ \left( = D_{qc}(X_0)/\{ M \in D_{qc}(X_0) \mid M \text{ almost 2-exact over } \Omega_c \} \right) \]

Have a semi-orthogonal decomposition.

\[ D_{qc}(X_0) \rightarrow D_{qc}(X_0) \rightarrow D_{qc}(X_0)^a \]

\( \uparrow \) Should be \( X_0 \otimes_{\Omega_c} \)
Classically, we have

\[ D(C_r, F_p) \xrightarrow{i^*} D(C, F_p) \xrightarrow{j^*} D(C_c, F_p) \]

**Thm:** $J$ a colimit preserving functor

\[ RH: D(C, F_p) \longrightarrow D_{qc} (C_0) \]

with the fill features:

a) Normalization: $RH(C_{F, x}) = \emptyset_x, \text{perf} l_p$

b) Proper pushforwards: If $f: Y \rightarrow X$ is proper,

\[ RH \circ Rf_x \cong Rf_x \circ RH \]

Now assume $X/\mathcal{O}_C$ is fin. presented + flat

Then $RH$ is fully faithful and satisfies:

c) Almost coherent: If $F \in D^b_{\text{coh}} (X; F_p)$, then

$RH(F)$ is almost coherent $/\mathcal{O}_0$, $\mathfrak{e} F \in \mathcal{E}_M$, $\exists$ a coherent $N_F \in D^b_{\text{coh}} (\mathcal{O}_0) + N_F \rightarrow RH(F)$ with $\text{Cone killed by } \mathfrak{e}$
d) Duality: On the subcategory $D^b_{\text{cons}}(X_{/\mathbb{F}_p})$

\[ \overset{\text{Nil ! - ext}}{D(X/\mathbb{F}_p)} \]

\[ \overset{a}{\text{RH} \cdot D \text{ Verdier} \leq D_{\text{Guth}} \cdot \text{RH}} \]

e) Reverse: $\overset{q}{\text{RH}} \left( p \overset{D^{\leq 0}(X_{/\mathbb{F}_p})}{\leq D^{\leq 0}(\mathcal{O}_X)} \right) \]

Rmk: 1) Special fibres: If $p = 0$ on $X$, get a
covariant version of RH for schemes of char $p$

2) Almost coherence: inspired by Zazapoulos' thesis

\[ \overset{\text{ext}}{\mathfrak{O}_C/p^n} \text{ is almost fg over } \mathcal{O}_C \]

\[ \overset{n \geq 0}{\prod \overset{\text{f.g.}}{\mathfrak{O}_C/p^n}} \]

3) RH is lax symmetric monoidal, and thus gives

\[ \overset{\sim}{\text{RH}}: D(X/\mathbb{F}_p) \rightarrow D_{\text{qc}}(X_0, \mathcal{O}_X, \text{perf}_p) \]

This is symmetric monoidal and compatible

with pullbacks.
1) A version with mod $p^n$ coeff of (3)

Take a limit, invert $p$, to get:

$$D(x_c, O_p) \rightarrow D_{x_c} \left( x \left[ \frac{1}{p} \right], O \left[ x \frac{1}{p} \right] \right)$$

$$\parallel x/\mathcal{O}_c \text{ smooth}$$

$$D_{x_c} \left( x \left[ \frac{1}{p} \right], \mathcal{O} \left[ x \frac{1}{p} \right] \right)$$

Sketch of construction:

- define by hand (using (a) + (b))
  for very large $\mathcal{O}_c$-schemes

- rest by arc descent

Application

Then $DS: R \text{ noeth ring, } X/R \text{ proper scheme}$

Then $\pi: Y \rightarrow X$ s.t.

$$\text{RP} \left( x, O_x(p) \right) \rightarrow \text{RP} \left( y, O_y(p) \right) \text{ factors over}$$
Remark: 1) This was previously known:
   - $R$ has char $p$
   - dim $(R) = 1$; in Beilinson's proof of
     $\text{Gal}_R$ (geom input of p-adic P.I.)

2) This implies:

   $R$ noether $\mathbb{Z}_p$-scheme $p \in \text{Rad}(R)$

Assume $R$ is a splitter (is any finite inj map

$R \to S$ has an $R$-linear splitting)

Then $R$ has "rational singularities"

Example: $R = \mathbb{Z}_p \left[ x, y \right] / (x^3 + y^3 + p^3)$ a 2 dim normal local domain

$X = \mathbb{B}_1 (\text{Spec } (R)) \xrightarrow{\pi} \text{Spec } (R)$

\[ E = V(x^3 + y^3 + z^3) \]

$\mathbb{P}^2_{\mathbb{F}_p}$ \to $\mathbb{P}^2_{\mathbb{F}_p}$
Can show $H^i(x, O_x) \cong H^i(E, O_E) \cong F_{\mathfrak{p}} \neq 0$

**Exercise**: Prove Thm for this example

**Proof of Thm**:

May assume $X, R$ integral, $X \to \text{Spec}(R)$ dominant

Let $Y \to X$ be an absolute integral closure of $X$

By limit arguments, need to show

\((\ast) \quad R^i(Y, O_Y/\mathfrak{p}) \cong H^i(Y, O_Y)/\mathfrak{p}\)

Have a diagonal

\[\begin{array}{ccc}
Y & \to & X \\
\uparrow f^* & & \downarrow f \\
\text{Spec}(R^t) & \to & \text{Spec}(R)
\end{array}\]

Abs integral closure of $R$

Both $\hat{Y}$ and $\hat{R}$ are perfectoid (easy exercise)

By proper pushforward compact of $R^t$, we have
\[ Rf_+^{*} 0_{v/p} = Rf_+^{*} 0_{v,nid/p} = RH \left( Rf_+^{*} F_{p} \right) \]

\[ \therefore \text{(x)} \text{ is implied by} \]

\[ (x^+) : \quad Rf_+^{*} F_{p} \cong F_{p} \]

This follows from

Lemma: \text{Y integral normal scheme with alg closed function field, } A = \text{ab group}. \text{ Then}

\[ Rf^{*} (Y_{et}, A) \subseteq A \]

\[ Rf_{1} : \quad Y_{et} = Y_{zar} \]

\[ \text{Groth's thm } Rf^{*} (\text{irred space}, A) = A \]