I) The characteristic $p$ story

Recall: a noeth local ring $(R_m)$ is CM if it satisfies:

a) Every system of parameters is a regular sequence

b) $H^i_{m}(R) = 0$ $\forall i \leq \dim(R)$

c) Say $X = \text{Spec}(R)$. For each pt $i_x : \{x\} \subseteq \text{Spec}(R)$ we have $i_x \ast \mathcal{O}_X \in D \geq \dim(\mathcal{O}_{X,x})$

d) (If $R$ admits a dualizing complex)

$\omega_R^\ast$ is concentrated in a single deg.

Examples: regular rings, rational singularities / $\mathbb{Q}$

$F$-rational rings / $\mathbb{F}_p$, ....

Thm HH (Hochster-Huneke)
1) \((R, m)\) excellent noeth local \(1Fp\) - domain

\[R^+ = \text{absolute int closure} = \text{integral closure of } R \text{ in } \text{Frac}(R)\]

Then \(R^+\) is CM, i.e. \(H^i_{m} (R^+) = 0\) for \(i < \text{dim } (R)\)

2) Say \(k\) is field of char \(p\)

\(X/k\) proj variety, \(L\) ample \(1:b\) on \(X\)

Then \(Y\) a finite surj map \(\pi: Y \rightarrow X\)

\(\pi^*\) kills:

\[H^0(X, L^a) = 0\]

\[H^b(X, L^b) < 0\]

\[\exists X = \text{Cone } (A \subset \mathbb{P}^a) \subset \mathbb{P}^{a+1}\]

Then \(H^c(X, \Theta(-1)) \neq 0\) over any field

Remark: 1) Thm. HH (1) is false in char 0
as soon as \( \dim(R) \geq 3 \) (due to trace obstructions)

2) Thm \( HH^i(x) \Leftrightarrow R \) excellent regular \( \mathbb{F}_p \)-algebra, then \( R \to R^+ \) is faithfully flat.

Example: Say \( X \) the proj variety, \( k[X] \) char \( p \),

Consider \( H^i_c(x, \mathcal{O}_X) \) for some \( i > 0 \)

\( \cup \quad \text{Frob} \)

\( \text{p-linear algebra } \implies H^i_c(x, \mathcal{O}_X) \cong H^{i \cdot \mathfrak{f}}_c(x, \mathcal{O}_X) \text{ nilp } \oplus \)

\(( H^{i \cdot \mathfrak{f}}_c(x, \mathcal{O}_X)_{\text{Frob}} = 1 \circ \mathbb{F}_p \quad \circ \quad \mathbb{R} ) \)

Now \( H^{i \cdot \mathfrak{f}}_c(x, \mathcal{O}_X)_{\text{nilp}} \) is killed by \( \text{Frob}^n \) \( \forall n \geq 0 \)

Also, \( H^i_c(x, \mathcal{O}_X)_{\text{Frob}} = 1 \to H^i_c(x, \mathbb{F}_p) \) Artin-Schreier

Exercise: \( Y \) any qcqs scheme, \( i > 0 \) each \( \alpha \in H^i_c(Y, \mathbb{F}_p) \) is killed by a finite cover of \( Y \).
A sample application (Hochster–Roberts)

Say $S$ is smooth $G$-algebra, $G$ red. group

$G \subset S \quad R = SG \subset S$

Then, $R$ is CM

By the Reynolds operator, $R \hookrightarrow S$ is $R$-linear

By spreading out, this reduces to

($
 R \twoheadrightarrow S$ is an $R$-linearly split inclusion

of noeth. excellent $R$-algebras with $S$ regular

$\Rightarrow R$ is CM

Proof: Consider the diagram

$$
\begin{array}{ccc}
R & \rightarrow & R^+ \\
\downarrow & & \downarrow \\
S & \rightarrow & S^+ \\
\uparrow & & \\
\text{split by assumption} & & \\
\end{array}
$$

split as an $S$-module map by $\text{Thm HH for } S$

$\Rightarrow R \rightarrow R^+$ is $R$-linearly split

$\text{Thm HH for } R \Rightarrow R^+$ is (CM), so $R$ is CM
Other results one can prove similarly:

1) Direct summand cong
2) Fattings connectedness thm
3) Kollar's cong or improving local Lefschetz from SG12

Goal of this series: mixed char analogy of Thm H1H

Plan: 1) Use (perfect) prismatic cohomology to build a Riemann-Hilbert functor for $\mathbb{F}_p$-sheaves on varieties / $p$-adic fields.

2) Use (1) to "almost" accomplish Goal

3) Use (imperfect, log) prismatic coh to upgrade (2) to an honest result

II) Review of (perfect) prismatic cohomology (joint with Morra, Scholze)
Def: A perfect prism is a pair \((A,I)\) where
- \(A = \omega(S)\), \(S\) perfect \(\mathbb{F}_p\)-alg. \(\mathbb{Q}_p\)
- \(I \subset A\) ideal "primitive of deg 1", i.e.
\[ I = (d) \text{ st } d = \sum_{i \geq 0} [d_i] p^i, \text{ we have } d_1 \in S^*, \text{ } S \text{ is } \mathbb{Q}_p\text{-adically complete} \]

Ex. i) \(A = \omega(S)\) for \(S\) perfect of char \(p\)
\[ I = (p) \text{ "crystalline perfect prisms"} \]

2) Perfect \(q\)-dR prism:
\[ A = \mathbb{Z}_p \left[ q^{1/q^{\infty}} \right]^{\wedge}_{(p,q-1)} \cong \omega(F_p[q^{1/q^{\infty}}]) \]
\[ I = (\mathbb{Z}[q]), \text{ where } \mathbb{Z}[q] = \frac{q^{0-1}}{q-1} \]

Note: \(A/I = \mathbb{Z}_p[q^{1/q}] = \text{ring of integers of } \mathbb{Q}_p(\sqrt[n]{q})^\wedge \qquad q^{1/n} \rightarrow \varepsilon_n = \text{primitive } p^n\text{-th root of } 1\]

3) \(C/\mathbb{Q}_p\) complete + alg closed field

Lemma: \(\text{Ann}_p(\theta_C) = \omega(\lim_{\to \mathbb{Q}_p/\mathbb{Q}_p}) \rightarrow \theta_C \)
\[(\text{Ann}(\sigma), \ker(\sigma)) \text{ is a perfect prism}\]

\[\text{Rmk} : \{ \text{perfect prisms} \} \cong \{ \text{perfectoid rings} \}\]

\[(A, I) \mapsto A/I\]

\[\text{Thm} : \text{Fix } (A, I) \text{ perfect prism}\]

\[R = p\text{-complete } A/I\text{-algebra}\]

I a naturally defined comm. algebra object

\[\Delta_R \in D(A)\text{ comp. always } (p, I)\text{-adic completions}\]

\[\phi_{R/A} : \phi_A^* \Delta_R \to \Delta_R\text{ satisfying.}\]

c) Hodge-Tate: I an increasing mult filt on

\[\overline{\Delta}_R = \Delta_R \otimes_{A/I} A/I\text{ s.t}
\]

\[\phi_{R/A}^* (\overline{\Delta}_R) \cong \Lambda^* \frac{R}{(A/I)} \left[ -* \right]\]

b) de Rham: \[(\phi_A^* \overline{\Delta_R}) / I = \mathcal{L}\Omega^1_{R/(A/I)}\]
c) Étale: \((\Delta_R/p \left[ \frac{1}{p} \right])^\phi = 1 \cong \text{R}^\phi(\text{Spec}(R/p^\infty), \mathbb{F}_p)\)

d) Isogeny: \(R\) smooth / \((\mathbb{A}^1/\mathbb{F}_p)\), then

\[\phi_R: \phi^\ast \Delta_R \to \Delta_R\]

is an \(\Sigma\)-isogeny

Construction: For a \(p\)-complete \(\mathbb{A}^1\)-alg \(R\), set

\[\Delta_{R/\text{per}^\ast} = \left(\lim_{\leftarrow} \Delta_R \to \Delta_R \to \Delta_R \to \ldots\right)^\wedge \in \text{D}_{\text{comp}}(A)\]

\(\text{R}_{\text{per}^\ast} := (\Delta_{R/\text{per}^\ast})/I \in \text{D}_{\text{comp}}(R)\)

\[^{\text{perfectionization}}\]

Remark: i) If \(I = (p)\), then \(\text{R}_{\text{per}^\ast} \cong \text{R}_{\text{per}} = \lim_{\leftarrow} \frac{R}{I^n}

2) We always have \(\text{R}_{\text{per}^\ast} \in \text{D}^{>0}\)

If \(\text{R}_{\text{per}^\ast} \) is \(\text{no} \text{ deg} \text{0}\), then
$R_{pfld}$ is a perfectoid and $R \to R_{pfld}$ is the universal map to a perfectoid

**Example:** Say $(A, I)$ is perfect $q$-dR prism so $A/I = \mathbb{Z}_p^{qcl}$

Say $R = A/I \left[ x^{\pm 1} \right]^\wedge$. One can show:

$\phi_A^* \Delta_R = \text{"$q$-dR Thom complex of } R/A\text{"}$

$$
\phi_A^* \Delta_R = \hat{\bigoplus}_{i \in \mathbb{Z}} \left( A \cdot x^i \xrightarrow{\delta_{i,q}} A \cdot x^i \cdot d \log_q(x) \right)
$$

Using this, one shows:

$$
R_{pfld} = \hat{\bigoplus}_{i \in \mathbb{Z}^{[k]}, q_{01}} \left( A/I \cdot x^i \xrightarrow{\partial_i - 1} m A/I \cdot x^i \right)
$$

where $m \subset A/I = \mathbb{Z}_p^{qcl}$ is the max ideal $\left( \begin{array}{c} \text{note:} \\ m/m^2=0 \end{array} \right)$

In particular, $H^1(R_{pfld}) \neq 0$