Representations of reductive Lie groups

Lectures delivered by Joe Harris
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Contents

Introduction

Joe Harris taught a course (Math 224) on representations of reductive Lie groups at Harvard in Spring 2013. These are my "live-TEXed" notes from the course.

Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are my fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe.

Please email corrections to `amathew@college.harvard.edu`.

Lecture 1
1/28

This is Math 224. The topic of this course is not "representation theory of reductive Lie groups," but simply "representation theory," or simply the study of Lie groups.

Today's lecture is basically going to be a mishmash. The first part of the lecture will describe the course mechanics, and the prerequisites. The second part of the lecture will talk about the philosophy: how we're going to approach the subject, why it's important, and what our attitude is going to be. In the third part of the lecture, we'll get started— hopefully!

§1 Mechanics

Our course assistant: Francesco Cavvazani. He will grade the homeworks and hold weekly recitations, time TBD. We will have weekly homework assignments; the first one is due a week from today and is already up on the course webpage and is on the reverse of the syllabus. In general, the homework will be weekly and will be timed in conjunction with the section. We'll get the scheduling straight between Wednesday and Friday.

The text is *Representation Theory,* by Fulton and Harris. That's the only textbook. We're not going to do the first part of the book, which is representation theory of finite groups. It's a good way to get warmed up, to get a sense of the goals of representation theory and the basic constructions. I am going to be assuming that you have some familiarity with representations of finite groups. If you want to get a sense of what that entails, look at the first two or three chapters of this book. It should be something that you might have seen before. In any case, I would urge you to review it.

Let me mention also — by way of algebra — that we will be working with a lot of multilinear algebra.

Here are the prerequisites:

- Algebra: representations of finite groups (ch. 1-2 of F+H or Part I of Serre's *Linear representations of finite groups*).
- Algebra: Multilinear algebra. By this, I mean simply the construction of various objects built up of vector spaces via multilinear constructions: tensor products, symmetric powers, exterior powers, etc. This material is in F+H, appendix B, parts I-II.
- Manifolds: What a C^∞ manifold is. I'll say what this is in just a few minutes. This is one of these things where it's helpful where you had prior experience. (No need for metrics, curvature, etc.) In particular, what a tangent space is. Again, there are many different ways of viewing this: some fairly abstract, some fairly concrete. It's useful to have the ability to pass back and forth between these descriptions.
- Since we don't offer any courses here on several complex variables: we are going to be talking about Lie groups in real and complex settings. A **real Lie group**

is an object that is both a group and a differentiable manifold. You can carry out the same constructions in the category of complex manifolds. To even define a complex manifold, technically, you need to know what a holomorphic function in several complex variables is. What I'm telling you is that you **don't.** In the structure of the course, we're going to be passing fairly soon from Lie groups to Lie algebras — they're much easier to study. One of the great triumphs of the subject is that you can study Lie groups via Lie algebras. So we're not going to need too much about several complex variables.

The **grading** of the course is going to be based largely on the weekly homeworks. We have the option of having a final exam as well. I'd like your opinion on that. (**Laughter.**) Let me try to convince you that it's not such an unreasonable thing to do. If it's not too ominous or threatening – and believe me, it won't be — it's not a bad way to organize the thoughts of the semester and to review the material. Those of you who are seniors, in particular, will be wandering around aimlessly in the fog, and a final exam is a good way to focus your energy. I can see this is hopeless, but who would like to have a final exam? (**Nays.**) There will be no final exam and problem sets will determine the grade.

§2 Philosophy

The first thing, I have to put in a plug for the subject as a whole. The theory of Lie groups is, I think, without question the most central topic in mathematics. It's hard to think of a subject that *doesn't* involve Lie groups and Lie algebras. It's very much a part of differential geometry, in the complex side of algebraic geometry, mathematical physics, and even number theory. Now number theory may seem like a far stretch from Lie groups, but not entirely. If you study fancy number theory, you know that things like groups like GL_n over local fields arise very often. When we talk about GL_n over local fields, we use a lot of the ideas of classical Lie groups. So it's really one of the most central, one of the most topics, in modern mathematics. (Someone objects: **logic** is the most foundational.)

The other thing about representation theory is that it's a **huge success story.** So much of the time, we try to solve problems, and they turn out to be tough. And we wind up at a point where we are dealing with problems beyond our means. We understand how to solve quadratic polynomials, so we think, let's do cubics. And now we're talking about elliptic curves and it turns out to be tougher. Then you introduce invariants that let you describe the solutions you want to find. Pretty soon you get to the point that even a major breakthrough in the subject has little to do with the problem you want to solve. The great thing about representation theory is that it's not like that: we're going to ask whether we can classify Lie groups and classify their representations. The answer is **basically yes.**

We're going to identify a class of Lie groups (or Lie algebras) that are sort of atomic: the **simple** or **semisimple** Lie algebras. We are going to develop the coarse classification and decide to focus on these. Once we focus on those, we can try to classify the simple Lie groups and the representations. By the end of the semester, we'll have a complete list of all the simple Lie algebras and their representations, and we'll construct them explicitly. So the basic problem of classification is **solved**, and we will solve it.

There's one more thing I wanted to say: this is something true in a lot of subjects. This is a difference in point of view in **abstract versus embedded**. Let me say a few words about this. There's a fundamental shift in viewpoint that took place in many fields almost simultaneously at the beginning of the twentieth century. All the names — Lie was a nineteenth century mathematician — but manifolds were defined only in the twentieth century. Lie was studying subsets of GL_n closed under composition and inversion. There was no such thing as an abstract group.

Nowadays, you define a group as a set with a binary operation which satisfies a bunch of axioms. It's not a priori realized as a subgroup of a fixed group. This is analogous to what happened in topology or differential geometry. In the nineteenth century, when foundational work in manifolds was done, they didn't have a manifold: a manifold to them was a subset of $\mathbb{R}^n$ that was defined by smooth functions with independent differentials. The point was, it was imbedded in $\mathbb{R}^n$. There wasn't the notion of an abstract manifold. In algebraic geometry, a variety was a subset of affine or projective space. In the 1920s Weil and others came up with the notion of an abstract variety, which lives independently of an imbedding. In every case, this broke up a subject that was intractable into two tractable pieces:

- Describe all abstract objects.
- Given an abstract object, give its representations as a subobject of a concrete thing.

That decomposition has been very successful in every field. The one thing that's different in representation theory is that you have one additional fact present here, which we'll come to in due course: the existence of the **adjoint representation.** A group comes with a representation for free. If we understand the structure of representations, we can try to understand the groups themselves—that's how we'll understand the classification of simple Lie groups.

§3 Basic definitions

Let's talk about the basic objects we will be dealing with. Our objects of interest are **primarily Lie groups.** We are going to use Lie algebras as a tool to understand them. Lie algebras are much simpler but encapsulate a tremendous amount of the structure of the groups they come from.

1.1 Definition. A **smooth (C^∞) manifold** is a set X with additional structure:

- A topological space structure on X.
- An **atlas**, i.e. a collection of open sets $\{U_\alpha\}$ covering X and homeomorphisms

  $$\phi_\alpha : U_\alpha \simeq \Delta \subset \mathbb{R}^n$$

  where Δ is a disk.

- These homeomorphisms differ by smooth maps from the disk to itself where defined. In other words, we require that
$$\phi_\alpha \circ \phi_\beta^{-1}$$
is smooth where defined.

(Strictly speaking, we should say — an equivalence class of atlases.)

In other words, we know how to identify small open subsets of X with open subsets of $\mathbb{R}^n$. Given this condition, we can talk about C^∞ functions on X: what we mean is, given
$$f : X \to \mathbb{R},$$
we want $f|_{U_\alpha} \circ \phi_\alpha^{-1}$ to be smooth as a function $\Delta \to \mathbb{R}$. (By the compatibility condition, it doesn't depend on the choice of atlas.) Moreover, you can talk about C^∞ maps $X \to Y$ between smooth manifolds.

1.2 Definition. A map $X \to Y$ between smooth manifolds is **smooth** if when you compose it with a coordinate system, it is smooth.

The other half of the definition of a Lie group is the group part.

1.3 Definition. A **group** is a set X with the structure of two maps
$$m : X \times X \to X, \quad i : X \to X,$$
and an element $e \in X$ satisfying the usual group axioms.

1.4 Definition. A **Lie group** is simply a set endowed with both of these structures in a compatible way. It's a set G which is both a smooth manifold and a group, and the two structures are compatible in the only sense: in the sense that the maps
$$m : G \times G \to G, \quad i : G \to G$$
are smooth. In other words, we want the group law to respect the manifold structure on G.

There are a couple of associated definitions.

1.5 Definition. A map $f : G \to H$ between Lie groups is a **Lie group homomorphism** exactly if it is:

- A homomorphism of groups.
- A smooth map.

In other words, it's a map that respects both of the structures.

When we talk, in general, about maps between Lie groups, I'm not going to use this language: I'm just going to talk about *maps.* We're never going to consider maps between Lie groups that aren't of this form.

There is one issue here: the notion of a Lie subgroup. We have to be a little cautious on that score. You would be completely within your rights to assume that a Lie subgroup is the image of an injection between Lie groups; that's not quite right.

1.6 Definition. A **Lie subgroup** is defined as follows. If G is a lie group, then $H \subset G$ is a Lie subgroup if H is simultaneously:

- A **closed** submanifold.
- A subgroup.

A closed submanifold means that the topology inherited by H by the inclusion is the topology of H itself.

There's one standard example to keep in mind. Take a plane $\mathbb{R}^2$ (which is a Lie group with respect to addition), mod out by a discrete lattice like $\mathbb{Z}^2$. We get a torus $\mathbb{T}^2$. Here in $\mathbb{R}^2$, the Lie subgroups are easy to describe: they're lines through the origin. Lines are subgroups and submanifolds. But the image in the torus is not necessarily a Lie subgroup. If the slope is irrational, then the image in the torus is going to be this horrible thing that goes around and around and every point in the torus is going to be in the closure of the irrational slope line. So we have an inclusion, but the induced topology goes awry. There are maps

$$\mathbb{R} \to \mathbb{T}^2$$

as above, which are **immersed subgroups**, but not necessarily closed submanifolds.

§4 Examples

Having defined Lie groups, let's talk about examples. I'd like to give you a roster of all the players we're going to be dealing with. When we do an actual classification of simple Lie groups, it will turn out that I will have described for you **all** the simple Lie groups, with exactly **five exceptions.** So it's a nearly complete list.

1.7 Example. $\mathrm{GL}_n(\mathbb{R})$ is the set of invertible n-by-n matrices. This is an open subset of $\mathbb{R}^{n^2}$ (where the determinant doesn't vanish), so it's automatically a manifold. It gets the manifold structure from $\mathbb{R}^{n^2}$. The group structure is matrix multiplication.

The only thing that has to be checked — and this does have to be checked — is that the group law is C^∞. So in other words, if I have two matrices $(a_{ij}), (b_{ij})$, the entries of the product should be smooth functions of the factors. But that's easy: it's just a bilinear form in the a's and b's. It's less obvious that the inversion map is smooth. There's one word you should say here: **cofactors.** You can write the inverse as the matrix of cofactors divided by the determinant. The cofactors are polynomials, hence smooth; ditto for the determinant.

I want to be able to talk about actions of groups on vector spaces, not just $\mathbb{R}^n$. When I was a kid, a long time ago, I didn't understand the fuss about finite-dimensional vector spaces: why not call them $\mathbb{R}^n$? Then I learned about representations and vector bundles, and I learned that there is different data here. I don't just want to think about $\mathrm{GL}_n(\mathbb{R})$. If V is an n-dimensional real vector space, then we introduce the Lie group

$$\mathrm{GL}(V) = \mathrm{Aut}(V),$$

and of course, that's isomorphic to $\mathrm{GL}_n(\mathbb{R})$ after choosing a basis. I want to be able to talk about these things without necessarily choosing a basis, though. When we introduce more Lie groups, we're going to describe them as subgroups of matrix groups and as automorphism groups of objects.

This lets us make the following basic definition:

1.8 Definition. A **representation** of a Lie group G on a vector space V is a Lie group homomorphism

$$G \to \mathrm{GL}(V).$$

We want to say it this way, and not in terms of matrices, so that we can do things like form tensor products of representations.

1.9 Example. $\mathrm{SL}_n(\mathbb{R})$ consists of n-by-n matrices A such that $\det(A) = 1$. This is visibly a subgroup of $\mathrm{GL}_n(\mathbb{R})$. You do have to check something here: that it's a submanifold. This is a good way to check that it's a submanifold. How would you check that it's a submanifold? If you think of

$$\det : M_n(\mathbb{R}) \to \mathbb{R}$$

as a function on the space $M_n(\mathbb{R})$ of n-by-n matrices, then we're looking at the level set where $\det = 1$. We just have to show that the differential is nonzero at that point. Because you have the group structure, it's *enough to check it at the identity.* If you write out the differential of the determinant at the identity, it follows that it's nowhere zero on $\mathrm{SL}_n(\mathbb{R})$, hence it's a submanifold. The compatibility issue follows from the corresponding fact on $\mathrm{GL}_n(\mathbb{R})$.

You can also view $\mathrm{SL}_n(\mathbb{R})$ in terms of abstract vector spaces. Let V be a vector space. Then $\mathrm{SL}(V)$ is the subgroup of $\mathrm{GL}(V)$ consisting of automorphisms *which preserve a volume form*, i.e. a nonzero element in $\wedge^{\dim V} V^*$. A volume form is an alternating map

$$\phi : \prod_{\dim V} V \to \mathbb{R}$$

which is not identically zero. This is a useful way to think of it: most of the Lie groups we define are going to be subgroups of $\mathrm{Aut}(V)$ preserving some additional structure. Here that structure is a volume form.

Lecture 2
1/30

Today, I want to do two things:

- Examples of Lie groups. Just to give you some idea who the players are in this game — the ones that keep coming up in mathematics.
- To get started on the analysis: in particular, the notion of **isogeny.**

§1 Examples

The basic idea is that there are two ways of describing a Lie group, in general.

- As a subgroup of automorphisms of a vector space defined by preserving some structure.
- As a subgroup of GL_n by specifying matrices.

I want to do both.

We may specify a Lie group as a group of automorphisms of V, an n-dimensional $\mathbb{R}$-vector space, preserving some additional structure on V.

2.1 Example. $\mathrm{SL}_n(\mathbb{R})$ consists of automorphisms of $\mathbb{R}^n$ preserving a volume form.

Here's a more elementary example.

2.2 Example. Given the vector space V, a **flag** $\mathcal{V}$ in V is a nested sequence of subspaces

$$V_1 \subsetneq \cdots \subsetneq V_k \subsetneq V.$$

This is a type of structure that an automorphism could preserve. We define:

2.3 Definition. $B(\mathcal{V})$ is the group of automorphisms of V which preserve the flag $\mathcal{V}$. In other words, this consists of automorphisms $\phi : V \to V$ such that

$$\phi(V_i) = V_i, \quad \forall i.$$

2.4 Example. In the previous example/definition, suppose you have a **full flag**: one subspace of each dimension. Then $k = n - 1$ and $\dim V_i = i$. Choose a basis for the vector space

$$e_1, \ldots, e_n \in V$$

such that V_i is spanned by $\{e_1, \ldots, e_i\}$. The group $B(\mathcal{V})$, in terms of this basis, is **the group of upper-triangular matrices.** That's pretty much the picture in general. If the whole flag consists of just one subspace $V_1 \subsetneq V$, then the automorphisms preserving that subspace are the **block upper-triangular matrices.** In other words, they look like

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

where each $*$ is a block.

Remark. The B here is for **Borel,** who first studied the role that these play in other Lie groups.

2.5 Example. Given a flag $\mathcal{V}$ in V, we can look at the automorphisms which preserve the flag and act as the identity on successive quotients.

2.6 Definition. $N(\mathcal{V})$ is the set of automorphisms $\phi : V \to V$ such that

- $\phi(V_i) \subset V_i$.
- $(\phi - 1)(V_i) \subset V_{i-1}$, i.e. $\phi|_{V_i/V_{i-1}} = 1$.

Here the "N" stands for "nilpotent:" $\phi - 1$ is nilpotent.

In the case of a full flag, this is upper triangular with ones on the diagonal. In general, it consists of block upper triangular matrices with identities on the diagonal.

§2 More examples

Another case consists of automorphisms that preserve some element of the *tensor algebra.* This is how most of the classical groups are defined. The standard example:

2.7 Example. Suppose Q is a **symmetric bilinear form** on the vector space V, which is a map

$$Q : V \times V \to \mathbb{R}$$

which is linear in each factor, and symmetric. I.e., $Q \in \mathrm{Sym}^2(V^*)$. We assume for the most part that Q is **nondegenerate**, i.e. if we think of Q as defining a map

$$V \to V^*,$$

then that map is an isomorphism.

2.8 Definition. We set $O(V, Q)$ to be the set of automorphisms of V preserving Q. In other words, we are looking at automorphisms $\phi : V \to V$ such that

$$Q(\phi v, \phi w) = Q(v, w), \quad \forall v, w \in V.$$

In other words, an automorphism of V induces an automorphism of $\mathrm{Sym}^2(V^*)$ and we want this automorphism to preserve Q.

In matrix language (if we take $V = \mathbb{R}^n$), we can write Q as the bilinear form given by a symmetric n-by-n matrix. We can write

$$Q(v, w) = v^t M w,$$

for M a symmetric $n \times n$-matrix. Of course, if you do that, and write out what this condition states, then $O(V, Q)$ consists of *matrices* $A \in \mathrm{GL}_n(\mathbb{R})$ such that

$$A^t M A = M. \tag{1}$$

That's true for any symmetric bilinear form. If you take Q to be the standard inner product on $\mathbb{R}^n$, then M is the identity, then we get the **orthogonal group** $O(n)$, which is the space of (invertible) matrices A with

$$A^t A = I.$$

Given any nondegenerate symmetric bilinear form on V, we can find a basis such that M is given by block form

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

which is to say that

$$(v, w) = \sum_{i=1}^{k} v_i w_i - \sum_{k+1}^{l} v_i w_i$$

in coordinates. The form is indefinite, and we say that the group that we get is $O(k, l)$.

Remark. This condition does not imply that the map has determinant one. If n is odd, for example, $-I$ has determinant -1 and has this condition.

2.9 Definition. We define the **special orthogonal group** $SO(n)$ to be the subgroup preserving both the inner product and the determinant. In other words, A satisfying (1) and $\det A = 1$.

2.10 Example. We can instead take Q to be a *skew-symmetric* bilinear form, $Q \in \wedge^2 V^*$. If Q is nondegenerate, then $\dim V$ is even and we can always write Q via the block antidiagonal matrix
$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

2.11 Definition. The group of automorphisms preserving the nondegenerate skew-symmetric form Q is called the **symplectic group** and is denoted Sp_{2n}.

§3 General remarks

It seems like I've made some choices along the way. What if I don't assume that this thing is nondegenerate? To say that Q is degenerate is to say that the induced map $V \to V^*$ has a kernel, i.e. there is a nonzero subspace of vectors in V which pair to zero against every other vector in V. Any automorphism preserving Q preserves the kernel. The upshot is that you can describe the degenerate case in terms of the nondegenerate case.

Why symmetric or skew-symmetric? If Q is any bilinear form, I can still make the same definition. But any bilinear form decomposes into a symmetric and anti-symmetric form. And if an automorphism preserves a bilinear form, it preserves the symmetric and anti-symmetric part. So we get intersections of orthogonal and symplectic groups in GL_n.

2.12 Example. Given a random bilinear form (not symmetric or anti-symmetric), the automorphism group of it is the identity — maybe with some roots of unity. Prove this.

Why should we not look at higher symmetric powers? Why not skew-symmetric trilinear forms and automorphisms that preserve that? Or take any Schur functor, apply it to V^*, fix an element and see what we get by looking at symmetries that preserve it. It's *almost always trivial.* If you have a vector space of dimension eight or more, and a general skew-symmetric trilinear form, then there are no automorphisms of it. It's not hard to see via a dimension count: just count the dimension of the automorphism group GL_n and count how many distinct trilinear forms. You'll see that there are a lot more trilinear forms (dimension $\binom{n}{3}$) than $n^2 = \dim \mathrm{GL}_n(\mathbb{R})$.

In general, if I take higher degree tensors, I *get a trivial group.*

2.13 Example. If $\dim V = 7$ and $\alpha \in \wedge^3 V^*$ is general, then the group of automorphisms $\phi : V \to V$ preserving α is a nontrivial group, called G_2. When we classify the simple Lie groups, we'll see that all of them are the classical groups that are on the board right now, except for *five exceptions:* this is one of the exceptions. There's a smattering of other constructions that yield the remaining five simple Lie groups. (The other exceptional Lie groups are defined using things like the octonions.) There's a whole chapter devoted to G_2 in the book.

I'd suggest that you just read about the *unitary group.*

The last variation on all this is to work with a *complex* vector space, and we get subgroups of $\mathrm{GL}_n(\mathbb{C})$: this is a complex Lie group, but you can also think of it as a real Lie group. We have that $\mathrm{GL}_n(\mathbb{C})$ is an open submanifold of $\mathbb{C}^{n^2}$. The one thing to bear in mind via the $SO(n,q)$ is that the classification of symmetric bilinear forms via *signature* doesn't apply over the complex numbers. There are no positive definite symmetric bilinear forms over $\mathbb{C}$. There's no sign. In fact, $Q(v,w) = -Q(iv,iw)$ so there's no such thing as a positive definite form. There's only one orthogonal group over the complex numbers.

Remark. The **quaternions** also provide a source of examples, and are important in defining the compact simple Lie groups.

However, as we'll see on Friday or next week, passing through the Lie algebra simplifies many things, and equates many different Lie groups. The need for the compact versions of the symplectic group and so forth becomes less.

§4 Neighborhoods of e generate

There's one thing I want to do explicitly before we go any further. There's a basic fact about Lie groups that actually appears at the beginning of ch. 8, it's exercise 8.1. Reading through the text now, it strikes me that it's such a fundamental idea that we should prove it now.

2.14 Proposition. *If G is a connected Lie group, and $U \subset G$ is any open subset containing the identity $e \in G$, then U generates G.*

So if I have a Lie group map, if I know the map on *any* neighborhood of the identity, then that determines the map entirely. It suggests that we can understand maps between Lie groups by understanding what they do locally — and that's much simpler to describe. (Then we don't have to care about the global topology of the Lie group.)

Proof. Without loss of generality, replace U by $U \cap U^{-1}$ which contains a neighborhood of e. So assume that

$$U = U^{-1}.$$

Say $H \subset G$ is the subgroup generated by U. I claim that $H = G$. Let's say that we have an element $g \in \partial H$, i.e.

$$g \in \overline{H} \setminus H.$$

Look at the translate gU, which is a neighborhood of the point g. This means that gU must intersect H nontrivially. Let's say

$$h \in gU \cap H.$$

Now we're done, basically. Since $h \in H$, we can write $h = u_1 \dots u_k$ for the $u_i \in U$. We can also write it as $gu, u \in U$. Thus

$$u_1 \dots u_k = gu,$$

which implies that $g \in H$. ▲

The point is that we can look at maps of Lie groups by looking at the *local structure* near the identity.

§5 Isogenies and covering spaces

Let me state and prove a couple of theorems. I want to talk about a class of maps between Lie groups which are called *isogenies.* Let H be any Lie group. I want to consider a *discrete subgroup* of H, contained in the center $Z(H)$. Let

$$\Gamma \subset Z(H) \subset H$$

be a *discrete subgroup.* In this case, we can form the quotient $G = H/\Gamma$. A priori, this is just a group, but the claim is that we can give G the structure of a Lie group in a unique way, so that the quotient map

$$H \to G \simeq H/\Gamma$$

is a Lie group map (i.e., respects the smooth structure). In fact, that's obvious. All I want to say is that a small enough neighborhood of the identity in H maps isomorphically to a neighborhood of the identity in G, and we give G a differentiable structure by transfer from H. So there's a natural way of giving the quotient a Lie group.

Remark. $H \to H/\Gamma = G$ is a *covering space map* and the fibers are the cosets of Γ.

The other side of this is a less obvious theorem, which will require a little more work. It states that I can go in the other direction.

Let G be a Lie group. Let H be any connected topological space and consider a map

$$H \to G$$

which is a covering space map. (For instance, the covering space $S^1 \overset{z \mapsto z^n}{\to} S^1$.) Since H is a covering space of a manifold, it's also a manifold.

2.15 Proposition. *H can be given the uniquely structure of a Lie group once an identity e' in H is picked lying above $e \in G$, such that $H \to G$ is a Lie group homomorphism.*

In particular, any connected covering space of a Lie group naturally acquires the structure of a Lie group in its own right, if we specify which point is going to be the identity.

Proof. This is a good example of how everything is determined by what goes on in a neighborhood of the identity. If I want to define a group law in some neighborhood of the identity in H, I use the covering space projection and the group law on G. It's not clear that I can do this on all of H consistently. The product is unambiguous close to the identity in H (because near e', the projection to G is a local isomorphism). Let me give you a brief indication of how you prove this.

To prove the result, I'm going to do it in a special case and deduce it from it. Let's suppose $H \to G$ is the **universal** covering space, or equivalently H is simply connected: $\pi_1(H) = *$. How do I see in this case that I can necessarily lift? The example is the covering space

$$\mathbb{R} \to S^1.$$

We consider the group law on G (we can do the same thing for the inverse map), a map

$$G \times G \to G$$

and try to lift it to a map $H \times H \to H$. I claim that there's a unique way to do that with the identity. That's because $H \times H \to G \times G$ is a covering space map. So we have to find a lifting

$$\begin{tikzcd} H \times H \arrow[r, dashed] \arrow[d] \arrow[dr] & H \arrow[d] \\ G \times G \arrow[r] & G \end{tikzcd}$$

and the lifting comes from the universal property of the universal covering space. Namely, $H \times H \to G$ lifts to the cover $H \to G$ once an identity is chosen. The issue is, why does it satisfy the group axioms — for instance, why is it associative? This is what the group law has to be. We're going to do this on Friday. ▲

Lecture 3
2/1

§1 Recap

I want to finish the proposition from last time. This is what we were in the middle of saying in the previous class:

- Let H be any Lie group, $\Gamma \subset Z(H)$ a discrete subgroup. Then $G = H/\Gamma$ (the quotient group) has the structure of a Lie group such that

  $$H \to G$$

  is a morphism of Lie groups (and is a covering space!). The idea is that if I have any point in the quotient, it corresponds to a *coset* of the subgroup Γ. I can describe coordinate charts in H/Γ by taking small coordinate charts upstairs in G.

- If G is any Lie group and $H \xrightarrow{\pi} G$ a connected covering space, then H has the structure of a Lie group and π is a morphism of Lie groups. This is what we didn't finish proving (we'll do it below).

I wanted to go through this because it illustrates something that we do all the time. Here is an exercise which is a simple case.

3.1 Exercise. Let G be a connected Lie group and let $\Gamma \subset G$ be discrete and normal. Then $\Gamma \subset Z(G)$. In particular, Γ itself is abelian. This is very straightforward to prove: we want to say that $h \in \Gamma$ commutes with every g. We'd like to prove that

$$ghg^{-1} = h.$$

Draw an arc $\gamma = \gamma(t)$ in the manifold G which starts out at 1 and winds up at g. Consider what happens when we apply conjugation by this element to h. We have another arc

$$\widetilde{\gamma} : t \mapsto \gamma(t) h \gamma(t)^{-1}.$$

We consider the conjugates of h by elements along this arc. This $\widetilde{\gamma}$ is again a continuous arc, and the image at time $t = 0$ is just h. At the end, we get ghg^{-1}. Finally, since since Γ is *normal*, the image of this arc is contained in Γ throughout. So I have a continuous map from an interval to a discrete space and it follows that $\widetilde{\gamma}$ is constant.

Again, you use the fact that the conjugate varies *continuously* along the arc and takes values in a discrete space.

Remark. Alternatively, use the fact that the continuous action of a connected group acting on a discrete set is constant.

The above exercise shows that, in the proposition at the end of last time, we could have just assumed that Γ was *normal.*

Proof of the second part. Let G be a Lie group and $\pi : H \to G$ be a covering map. To start with assume that H is the *universal* cover so H is simply connected.

Choose $e' \in \pi^{-1}(e) \subset H$ and we'd like to *lift* the group structure to H, with e' the identity. In other words, we have a diagram

$$\begin{tikzcd} H \times H \arrow[r, dashed, "m_H"] \arrow[d] & H \arrow[d] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

and we'd like to lift the multiplication on G to produce a dotted arrow which will be the multiplication on H. Similarly, we'd like to produce a diagram

$$\begin{tikzcd} H \arrow[r, dashed] \arrow[d] & H \arrow[d] \\ G \arrow[r, "\iota_G"] & G \end{tikzcd} .$$

Using the theory of covering spaces, we get unique liftings once we've picked a basepoint: that's e'. The point is now to check that we actually have a group law on H. For example, if you want to check that m_H is associative, then we have to show

$$m_H(a, m_H(b,c)) = m_H(m_H(a,b),c),$$

or more colloquially

$$a(bc) = (ab)c.$$

We can check that $a(bc)\left((ab)c\right)^{-1} = 1$ always by drawing arcs from the identity to each of a, b, c. This product at all times (as the arcs go from the identity to a, b, c) starts from the identity and is a continuous arc in H lie entirely in $\pi^{-1}(e)$—that's because the group law is true in G. Since $\pi^{-1}(e)$ is discrete, we get that everything is constant and equal to the identity. (**Not included:** this argument works for any connected cover, not necessarily the universal one.) ▲

§2 Isogeny

The reason that I am introducing this material is that it is the *first* of our reductions. We are trying to study representations of Lie groups, or more generally morphisms between Lie groups. For any two *connected* Lie groups G, H, we say that G, H are **isogeneous** if there exists a Lie group map

$$G \to H$$

which is also a covering space map (i.e., with discrete kernel). Isogeny is *not* an equivalence relation per se, but it *generates* an equivalence relation, which is called **isogeny**. In each isogeny class, there exists a *unique* initial member of the class: the universal covering space. We want to look at all the Lie groups isogeneous to a given one, and what I'm saying is that they're all quotients of the universal cover by discrete subgroups of the center.

3.2 Example. Let's denote by H the simply connected member of an isogeny class. If $Z(H)$ is discrete, then there also exists a final object (in the isogeny class) which is $H/Z(H)$. This is the "bottom" member of the isogeny class. This is called the **adjoint form.** Meanwhile, H is called the **simply connected form** (of an isogeny class).

In most of this semester, we are going to focus on the simply connected form: this is the one with the most representations. Given a representation of anything else in that form, compose it with the universal cover map to get a representation of the simply connected form. There are two problems here:

- First, describe representations of the simply connected form.
- Describe which representations descend to a given (non-simply connected) member of the isogeny class.

Remark. If you have a disconnected Lie group, then break up the problem of describing its representation into two parts. The connected component at the identity is a *normal* subgroup whose quotient is discrete (and often finite). Hence, for most of the semester, we will work with representations of connected — even simply connected — and adopt this simplifying assumption. When we solve this problem with respect to this simplifying assumption, we'll solve the general problem later. We'll be able to say *exactly* which representations descend to a given form in the isogeny class.

Remark. Algebraic geometry does this as much as any other field. At first, when people started studying polynomials, they started with polynomials over the *real numbers.* Eventually in the nineteenth century people realized that it's much easier to work with algebraic curves or varieties over $\mathbb{C}$ than over $\mathbb{R}$. So they made a deal: it's not what we set out to do, but let's describe polynomials over $\mathbb{C}$ and once we finish that we'll go back and describe real polynomials. Of course, that's a good example of a broken promise. But really — it's much easier to classify algebraic varieties over $\mathbb{C}$ and the starting point to understand real varieties is to understand $\mathbb{C}$ varieties. But unfortunately, most people have abandoned that promise.

§3 The adjoint representation

The construction that will enable us to do this is the theory of *Lie algebras.* Today, I'd like to describe the overall framework, and we'll go back and prove all the assertions on Monday.

The **problem**, in general, is to classify all morphisms between Lie groups G, H. (That subsumes the problem of representation theory, which is where $H = \mathrm{GL}(V)$.) I am going to assume, for now, that G is *connected* and *simply connected.* The starting point is the remark I made last time. If I want to understand a map

$$\rho : G \to H$$

of Lie groups, I can look at the identities e_G, e_H and any small neighborhoods of the origin. A small neighborhood of e_G maps to a small neighborhood of e_H. It's *enough* to know what the map is doing on that open set. I can *zoom in* and look at the map under a closer and closer magnifying class. What I see on that small neighborhood determines the entire map. That's because for a connected Lie group, any neighborhood of the origin generates the whole group.

Now I'm going to do something that induces a certain amount of vertigo. Let's zoom in on these shrinking open sets in G. What I want to say is that:

3.3 Proposition (First basic principle)**.** *Any Lie group map $\rho : G \to H$ is determined by its differential $d\rho_G$ at e_G.*

It's enough to know the map just to *first order* at the origin and that determines everything. That's not obvious. We will prove it on Monday. Let's just take that as given for now. It's a crucial observation: that maps are determined by their *linearizations* from

$$d\rho_e : T_eG \to T_eH.$$

We thus have an inclusion

$$\mathrm{Map}(G, H) \subset \mathrm{Hom}_{\mathrm{Vect}}(T_eG, T_eH).$$

This is pretty striking. The set of all maps turns out to be bounded by this finite-dimensional vector space. Now all we have to do is ask one simple question: **which linear maps** arise in this way, as differentials of Lie group maps? Somehow, the condition that the map is a Lie group homomorphism should be reflected in the differential. At the end of the day, we'll get a complete answer to this question.

Recall that to say that ρ is a Lie group map means that

$$\rho(gh) = \rho(g)\rho(h), \tag{2}$$

and I want to say this in a more confusing way. Let $m_g : G \to G$ be the multiplication by g map. Similarly, for $h \in H$, let $m_h : H \to H$ be multiplication by h on H. To say that ρ is a Lie group map simply says that we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "m_g"] & H \arrow[d, "m_{\rho(g)}"] \\ G \arrow[r, "\rho"] & H \end{tikzcd} .$$

It's hard to say something about the differential at the identity via these diagrams because none of these maps have *fixed points*! We'd like something that involves just the tangent space at the identity.

Instead of looking at multiplication by g, look at conjugation by g. For $g \in G$, define

$$\phi_g : G \to G, \quad g' \mapsto gg'g^{-1}.$$

(Do the same for H.)

If you have a group in general, if you know what the conjugate of any element by any other element is, does that determine the group structure? No — abelian groups. So this is *less* information than knowing m_g. But we do know that if ρ is a morphism of Lie groups, then we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\psi_g"] & H \arrow[d, "\psi_{\rho(g)}"] \\ G \arrow[r, "\rho"] & H \end{tikzcd} .$$

Of course, ψ_g fixes the identity, so it has a differential

$$d\psi_g : T_eG \to T_eG,$$

and similarly we can differentiate conjugation at the identity in H,

$$d\psi_h : T_eH \to T_eH.$$

So for each element in the group, we get an action on the tangent space at the identity. We get a map, of groups,

$$G \stackrel{\mathrm{Ad}}{\to} \mathrm{Aut}(T_eG) = \mathrm{GL}(T_eG).$$

3.4 Corollary. *The vector space T_eG has a natural representation of G on it.*

This is important because it's a representation for G that we get for free.

3.5 Definition. The representation of G on T_eG given as above is called the **adjoint representation.**

Granted, this could be a trivial representation, e.g. if G is abelian. So it may not be a faithful representation. But it is canonically associated to the group G. In the cases that occupy us for most of the semester — in the case where the simply connected group has discrete center — then the image of the adjoint representation is the adjoint form (in fact, its kernel includes the center).

Remark. We have redundant notations: for a vector space V, $\mathrm{GL}(V)$ and $\mathrm{Aut}(V)$ mean the same thing.

§4 Differentiating the adjoint representation

Again, what is true is that a Lie group map has to respect the adjoint representation. If $\rho : G \to H$ is a Lie group map, then the induced map on tangent spaces has to respect this. But we're not quite there yet. What I have is still something that associates to every element of the *group* an automorphism of the tangent space. It still explicitly invokes the *group.* I want some structure that involves only the tangent space. So how do I get that?

I have a map

$$G \stackrel{\mathrm{Ad}}{\to} \mathrm{Aut}(T_eG)$$

and I want something that only sees the tangent space. So take the differential of Ad at the identity! I get a map

$$\mathrm{ad} : T_eG \to \mathrm{End}(T_eG) = \mathrm{Hom}(T_eG, T_eG).$$

Note that $\mathrm{Aut}(T_eG) \subset \mathrm{Hom}(T_eG, T_eG)$ as an open subset, which determines the tangent space.

Now, we can think of Ad as a trilinear thing: we can take a transpose

$$\boxed{\mathrm{ad} : T_eG \times T_eG \to T_eG, \quad (X, Y) \mapsto [X, Y]} \tag{3}$$

which is a bilinear "bracket" operation. This is a binary operation on this vector space T_eG. The key fact is that this has to be preserved under any morphism of Lie groups. It seems like we've been throwing information right and left. But the point is:

3.6 Proposition. *Any Lie group map $\rho : G \to H$, then its differential $d\rho_e : T_eG \to T_eH$ respects this binary operation. In other words, there is a commutative diagram:*

$$\begin{array}{ccc} T_eG \times T_eG & \xrightarrow{d\rho \times d\rho} & T_eH \times T_eH \\ \downarrow{\scriptstyle [\cdot,\cdot]_G} & & \downarrow{\scriptstyle [\cdot,\cdot]_H} \\ T_eG & \xrightarrow{d\rho} & T_eH \end{array}.$$

This bracket operation must be preserved. A *necessary* condition that a map $T_eG \to T_eH$ arise as the differential of a Lie group map is that it preserves this bracket structure. The second basic fact is that it is sufficient:

3.7 Proposition (Second basic principle)**.** *Suppose G is connected and simply connected. Then any map $\phi : T_eG \to T_eH$ is a differential of a Lie group map if and only if it respects this bracket operation, i.e. if $[\phi(x), \phi(y)] = \phi([x, y])$.*

We have, in some sense, answered the question. The first basic principle was that a Lie group map was determined by its infinitesimal structure. The second basic principle determines when a map on tangent spaces comes from a Lie group map. Thus, in the case when G is simply connected, we can — in principle — understand maps of Lie groups out of G. We'll need to understand this bracket operation for that.

Lecture 4
2/4

§1 The basic setup

Today's lecture is going to be a turning point: we are going to get to the definition of a Lie algebra and answer the question from last time. We were looking at maps between Lie groups, and we observed naively that (if the source is connected) a map of topological groups $G \to H$ (with G connected) is determined locally, by its value in a neighborhood of the identity e. But it was better for a Lie group: it was determined "micro-locally," in terms of the map tangent space.

From last time:

- If G is a connected Lie group, then any Lie group map $\rho : G \to H$ is determined by its differential $\phi = (d\rho)_e : T_eG \to T_eH$. We therefore have an inclusion
$$\mathrm{Hom}(G, H) \hookrightarrow \mathrm{Hom}(T_eG, T_eH).$$
There's a finite amount of data that determines the set of Lie group maps.

- We wondered — and this is the key question — which linear maps $\phi : T_eG \to T_eH$ arise as the differentials of homomorphisms. We also had an adjoint representation
$$\mathrm{Ad} : G \to \mathrm{Aut}(T_eG), \quad g \mapsto (d\psi_g)_e.$$

In more detail, for any $g \in G$, we had a map

$$\psi_g : G \to G, \quad \psi_g(x) = gxg^{-1}$$

given by conjugation by g, which is an automorphism of the Lie group G. The differential $d\psi_g$ gives an action of g on T_eG. We thus get a representation of G on its tangent space T_eG, which comes for free.

Given this, we can take the derivative again, and what we get is a map

$$\mathrm{ad} : T_eG \to T_{\mathrm{id}}\mathrm{Aut}(T_eG) = \mathrm{End}(T_eG) = \mathrm{Hom}(T_eG, T_eG).$$

What we wind up is a map from the tangent space T_eG to the vector space of homomorphisms from T_eG to itself. That's a trilinear object. We can take its "transpose" and we arrive at a bilinear map

$$T_eG \times T_eG \to T_eG.$$

It's defined simply by sending

$$(X, Y) \mapsto \mathrm{ad}(X)(Y).$$

This is a piece of structure on *the vector space itself,* making no reference to the group. That structure, we observe, has to be preserved by any map between the tangent space of Lie groups that arises from a homomorphism of Lie groups.

4.1 Definition. We write

$$[X, Y] \stackrel{\mathrm{def}}{=} \mathrm{Ad}(X)(Y), \quad X, Y \in T_eG. \tag{4}$$

This is called the **Lie bracket** of X, Y.

The basic statement that we made last time was:

4.2 Proposition. *Suppose G is (connected and) simply connected. Then any linear map $\phi : T_eG \to T_eH$ is the differential of a Lie group homomorphism if and only if it preserves the Lie bracket. In other words, if*

$$[\phi X, \phi Y] = [X, Y]. \tag{5}$$

I owe you a proof of both of these fundamental statements: that ρ is determined by its differential and conversely the characterization of derivatives of homomorphisms. On Wednesday, we'll introduce a construction that will allow us to prove these statements; for now we'll go on.

§2 Describing the bracket

At this point, you might be a little bit skeptical. We have this operation, which comes from differentiating the group law twice. It sounds pretty airy and abstract. The key point is that this is something that can be made very explicit. We can describe the bracket in all cases *explicitly.*

4.3 Example. We start with the basic example:

$$G = \mathrm{GL}(V) = \mathrm{Aut}(V),$$

where $T_eG = \mathrm{End}(V)$. We want to know, in this setting, what the bracket is. Here's one thing that's worth bearing in mind; it's a little difficult to visualize explicitly.

We know that if $g \in G$, then we have an automorphism

$$\psi_g : G \to G$$

given by conjugation, which extends to a linear map on all of $\mathrm{End}(V)$, sending $h \mapsto ghg^{-1}$. That's a perfectly well-defined map on not just invertible matrices, but all matrices. So ψ_g is a linear map from $\mathrm{End}(V) \to \mathrm{End}(V)$ and it is its own differential. The differential of ψ_g at I is exactly

$$h \mapsto ghg^{-1}, \quad h \in \mathrm{End}(V).$$

We get a description of the adjoint representation Ad:

4.4 Proposition. *$\psi_g = d\psi_g$ is the map $\mathrm{End}(V) \to \mathrm{End}(V)$ sending an arbitrary map $X \in \mathrm{End}(V)$ to gXg^{-1}.*

This was pretty easy. Now we've got to differentiate this in g, which is a little trickier since it's no longer linear. To describe the ad adjoint representation, we start with $X \in \mathrm{End}(V)$, we choose an arc in G with that tangent vector.

So choose a smooth map

$$\gamma : (-\epsilon, \epsilon) \to G, \quad \gamma(0) = 1, \dot{\gamma}(0) = X.$$

I want to take the corresponding family of maps from $\mathrm{End}(V) \to \mathrm{End}(V)$ and differentiate it. In other words,

$$\mathrm{ad}(X)(Y) = \frac{d}{dt}|_{t=0}\gamma(t)Y\gamma(t)^{-1}. \tag{6}$$

That's not so bad, since this is a matrix product. I can apply the product rule. This is

$$\mathrm{ad}(X)(Y) = \gamma'(0)Y\gamma(0)^{-1} + \gamma(0)Y(\gamma^{-1})'(0). \tag{7}$$

Here I'm talking about the derivative of the arc γ^{-1} at time zero. How do you differentiate the inverse of an arc? If it was a real-valued function, this is the quotient rule. What is it in the world of matrices? If I have an arc in the space of invertible matrices, whose derivative I know, I get a second arc by taking the pointwise inverse. I should be able to take the derivative and do it. I know that

$$\gamma(t)\gamma(t)^{-1} = 1,$$

so that

$$\gamma'(0)\gamma^{-1}(0) + \gamma(0)(\gamma^{-1})'(0) = 0. \tag{8}$$

This gives

$$(\gamma^{-1})'(0) = -\gamma^{-1}(0)\gamma'(0)\gamma^{-1}(0). \tag{9}$$

In our example, we know that $\gamma(0) = 1$, so we get from (7)

$$\mathrm{ad}(X)(Y) = XY - YX. \tag{10}$$

You're probably pretty mad: why didn't I just define the Lie bracket as the commutator to begin with? Let me come back to this. But we've seen that the bracket operation on the tangent space of $\mathrm{GL}(V)$ is just the commutator. Given this, it makes everything much more concrete. It also applies to any *subgroup* of $GL(V)$. If I have a subgroup $H \subset \mathrm{GL}(V)$, I can carry out the exact same analysis, and I find again the bracket on $T_eH \subset \mathrm{End}(V)$ is the commutator.

There are some other consequences of this formula for $\mathrm{GL}(V)$. We thus get:

4.5 Proposition. *The Lie bracket (for subgroups for* $\mathrm{GL}(V)$*, at least) is skew-symmetric,*

$$[X, Y] = -[Y, X]. \tag{11}$$

This isn't at all obvious from the initial definition. Another consequence, which will use once but never again in this course, is the **Jacobi identity:**

4.6 Proposition. *For a subgroup of* $\mathrm{GL}(V)$*,*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{12}$$

If you just write this out in terms of the commutators, this is what you get. This leads to the following definition:

4.7 Definition. A **Lie algebra** is a vector space $\mathfrak{g}$ with a skew-symmetric bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

satisfying the Jacobi identity (12).

It follows that **any Lie group has an associated Lie algebra** given by the tangent space at the identity.

Thus we can rephrase the previous result in the language of Lie algebras:

4.8 Proposition. *The space of maps* $G \to H$ *where* G, H *are Lie groups and* G *is simply connected, then*

$$\mathrm{Hom}(G, H) = \mathrm{Hom}_{Lie\ alg}(\mathfrak{g}, \mathfrak{h}),$$

for $\mathfrak{g}, \mathfrak{h}$ *the associated Lie algebras.*

§3 Some general remarks

This is the beginning of the shift. It's going to take us a couple of days to prove the assertions on the board, but this is the fundamental shift in our understanding of Lie groups. We can express everything we want to know about Lie groups in terms of the corresponding Lie algebras, which are much simpler. Lie algebras don't have this topology, and they're much easier to deal with. Given a nonlinear problem, we linearize it, and describing maps between Lie groups is a very nonlinear problem. Now we've magically converted it to a problem about maps between vector spaces and we know which maps between vector spaces work.

Again, we haven't proved it. We've seen that a map of Lie groups has to induce a map of Lie algebras. We're going to prove that this characterizes maps between Lie groups over the next week. The main thing, in some sense, that this process of passage of Lie groups to Lie algebras does is to give us a much simpler object to deal with. There are other things that are equally important, in practice.

- In the definition above, I didn't specify that the vector space be finite-dimensional. There are infinite-dimensional Lie algebras which do arise: for instance, the Lie algebras of all global vector fields on a smooth manifolds (with the Lie bracket). We can extend the notion of Lie algebra to infinite-dimensional ones which brings in many other examples and applications.

- We can talk about a Lie algebra over any field. We can talk about a Lie algebra over a finite field. It doesn't make sense if you think of it as something arising from Lie groups but the definition makes perfect sense, and people do it. The remarkable thing is that the correspondence between Lie groups and Lie algebras persists in this much broader context. We're not going to talk about Lie algebras over fields other than $\mathbb{R}$ or $\mathbb{C}$.

- However, note that a Lie algebra over $\mathbb{R}$ leads to a complex Lie algebra by tensoring with $\mathbb{C}$. There's no analogous assertion at the level of Lie groups. In fact, the first step in transforming the problem into a solvable one is to pass from a Lie group to its Lie algebra. The second step is to *complexify,* because everything is simpler over the complex numbers. We're going to classify representations over complex Lie algebras before going back to $\mathbb{R}$. (The whole classification is much simpler over $\mathbb{C}$.)

Note however that I did promise that once we described representations of an arbitrary simply connected Lie group, we will be able to describe representations of an arbitrary Lie group.

§4 Lie brackets and commutators

Let's go back to the description of the Lie bracket as a *commutator.* Why not just define the bracket $[X, Y]$ to be the commutator? The answer is simply that XY is *not defined.* When we imbed $G \hookrightarrow \mathrm{GL}(V)$, its tangent space T_eG is imbedded in $\mathrm{End}(V)$. The space of endomorphisms of V is an algebra in that we can take the product of any two elements. But, that product doesn't respect T_eG: if we start with two elements of T_eG and take their product in $\mathrm{End}(V)$, that *depends on the particular representation.*

To restate: given an inclusion $G \hookrightarrow \mathrm{GL}(V)$, we get an inclusion

$$T_eG \hookrightarrow \mathrm{End}(V)$$

and given $X, Y \in T_eG$, the product XY need not even be an element of T_eG (or rather in the image). Composing two elements in the image doesn't preserve the image. Even when the composition is in the image, it generally depends on the choice of representation. The product of two elements of a Lie algebra is **not well defined.**

In the end, we have to define the Lie bracket in a way that does not involve a particular representation, and then we can use the representation to try to describe it explicitly. That's why the definition is what it is.

4.9 Exercise. Come up with an example where the "product" (in $\mathrm{End}(V)$) of two elements of a Lie algebra isn't in the Lie algebra.

§5 Some more terminology

The word "algebra" in this context means two different things:

- Up till now, the word "algebra" means essentially a ring: you have an addition and a multiplication law. Like the algebra of endomorphisms of a vector space:

you can add maps and compose them. But now we're using the word "algebra" in a different sense: **Lie algebras are not algebras,** which is a bit of a problem, linguistically.

- When we write down $\mathrm{End}(V)$, it's both an algebra and a Lie algebra.
- "Lie" is not an adjective! "Lie algebra" is a unitary expression that means something different from "algebra" satisfying a bunch of properties.

So, a bit of notation: when we write $\mathrm{End}(V)$, we'll mean it as an algebra in the usual sense. When we want to think of it as a Lie algebra, we will write

$$\mathfrak{gl}(V),$$

but sometimes this might not happen. You're own your own.

§6 Representations of Lie algebras

There's one more point to set down here.

4.10 Definition. If $\mathfrak{g}$ is a Lie algebra, a **representation** of $\mathfrak{g}$ on a vector space V is a Lie algebra map

$$\mathfrak{g} \to \mathfrak{gl}(V).$$

That means that $\mathfrak{g}$ is *acting* on V: for each $x \in \mathfrak{g}$, we get a multiplication

$$V \to V, \quad v \mapsto xv.$$

When we talk about action of groups, we think of automorphisms: here we're associating to each $x \in \mathfrak{g}$ an *endomorphism* (which could be zero!). The one thing it does have to satisfy is the commutator relation

$$[x, y]v = x(yv) - y(xv), \quad x, y \in \mathfrak{g}, v \in V. \tag{13}$$

When we talk about representations of Lie algebras, some of the most basic constructions for finite groups will work for Lie algebras as well. For example, we can form tensor product representations. Given a Lie group G acting on two vector spaces V, W, we get a representation of G on $V \otimes W$ by

$$g(v \otimes w) = gv \otimes gw, \quad g \in G. \tag{14}$$

That is, starting with maps $G \to \mathrm{GL}(V), G \to GL(W)$ (representations on V, W respectively), we got a representation

$$G \to GL(V \otimes W).$$

What about Lie algebras? In each case, we had an associated Lie algebra and representations of them, and we want to know how they relate to one another. Suppose we have Lie algebra maps

$$\phi_V : \mathfrak{g} \to \mathfrak{gl}(V), \phi_W : \mathfrak{g} \to \mathfrak{gl}(W), \quad \phi_{V\otimes W} : \mathfrak{g} \to \mathfrak{gl}(V \otimes W),$$

which we got from differentiating the above Lie group maps. How do these relate to one another? They do not satisfy (14). Draw an arc γ through the identity G and $\gamma'(0) = X \in \mathfrak{g}$. By definition, the action of X comes from taking the action of $\gamma_t \in G$ and differentiating at $t = 0$. Then

$$\gamma_t(v \otimes w) = \gamma_t(v) \otimes \gamma_t(w).$$

Then differentiating at zero, we get

$$X(v \otimes w) = \frac{d}{dt}(\gamma_t(v \otimes w)) = Xv \otimes w + v \otimes Xw. \tag{15}$$

This is what the tensor product representation of Lie algebras looks like.

4.11 Proposition. *Given representations of a Lie algebra $\mathfrak{g}$ on V, W, we get a representation on $V \otimes W$ via*

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw. \tag{16}$$

4.12 Example (Some other examples)**.** If a Lie group G acts on V, it acts on $\mathrm{Sym}^2(V)$ given by

$$g(v^2) = (gv)^2,$$

and extending by linearity. (The squares span $\mathrm{Sym}^2(V)$.) We are in characteristic not 2.

We can do the same for Lie algebras. Given a representation of $\mathfrak{g}$ on V, it acts on $\mathrm{Sym}^2(V)$ via

$$X(v^2) = 2vX(v). \tag{17}$$

Lecture 5
2/6

§1 Recap

From last time, recall:

5.1 Definition. A **Lie algebra** is a vector space $\mathfrak{g}$ with a skew-symmetric bilinear form (**Lie bracket**)

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity (12).

Last time, we got to the end of the sequence of reductions and were led to the definition of a *Lie algebra.* Again, that's the definition, and the basic property — which we hopefully will prove today — is that:

- Every Lie group G has a Lie algebra $\mathfrak{g} = T_eG$ together with the differential of the adjoint map, which defines the Lie bracket.

- We have a bijection between *isogeny classes* of Lie groups and (finite-dimensional) Lie algebras over $\mathbb{R}$. In particular, every Lie algebra has associated to it a Lie group, which we can take to be simply connected.

The basic thing to prove is that if G is connected and simply connected and H is any Lie group, then we have a bijection

$$\mathrm{Hom}_{\mathrm{LG}}(G, H) \simeq \mathrm{Hom}_{\mathrm{LA}}(\mathfrak{g}, \mathfrak{h}).$$

Once we've proved this, we will switch from studying the Lie group to studying the Lie algebra. We're going to study Lie algebras and morphisms between them, with the ultimate goal of understanding the theory of Lie groups. (Conversely, we will show that any Lie algebra over $\mathbb{R}$ comes from a Lie group.)

There are a few remarks to make about this definition, which are worth making as they involve blanket assumptions:

- The definition doesn't specify whether $\mathfrak{g}$ is finite-dimensional. As I said last time, there are lots of examples of Lie algebras which occur "in nature" which are infinite-dimensional, e.g. the vector space of vector fields on a manifold.

- However, we're going to be concerned in this class entirely with finite-dimensional ones. So, we adopt a convention: we will assume that **a Lie algebra is finite-dimensional.**

- The definition also doesn't specify the ground field, and the definition makes sense over any field. You can do a lot of interesting things with Lie algebras over other fields, even ones that don't correspond to geometric objects. We won't do that. We will **make the assumption that the ground field is $\mathbb{R}$ or $\mathbb{C}$,** and **by default we will work over $\mathbb{R}$.**

- Most of the work we will do will be with complex Lie algebras. Working over $\mathbb{C}$ is a simplifying assumption. If your goal is to classify Lie algebras (or Lie groups) over $\mathbb{R}$, you'd want to start with $\mathbb{C}$ anyway.

- Let me also put down the following fact, and challenge you to think about how to prove something like this. It'll become clearer in the next chapter. In fact, any Lie algebra **can be imbedded in $\mathfrak{gl}_n(\mathbb{R})$ for some n.** It's not that hard: every Lie algebra acts on itself by the adjoint action on itself. Any Lie algebra has an *adjoint representation* that sends any element x to $\mathrm{Ad}x = [x, \cdot]$. That's not necessarily an imbedding, or a faithful representation. But, it only fails to be an imbedding because the Lie algebra may have a *center*: elements that commute with everything else. We can deal with those. (We haven't defined these terms, but in the next chapter, we will get to work and do that.)

  Once we have a Lie algebra imbedded in $\mathfrak{gl}_n(\mathbb{R})$, we will be able to say that every Lie algebra **arises from a Lie group.**

Remark. Given the previous remarks, it's natural to ask whether every Lie group is imbeddable in $\mathrm{GL}_n(\mathbb{R})$ for some n. The answer is **no**; we'll get to this point relatively soon. Not every Lie group has a faithful representation. The simplest example I know is the universal cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of $\mathrm{SL}_2(\mathbb{R})$ (on the homework, you showed that $\pi_1(\mathrm{SL}_2(\mathbb{R})) = \mathbb{Z}$). We'll see this by analyzing representations of the Lie algebra: every representation of the Lie algebra gives a representation of the Lie group. We'll see that every representation of $\widetilde{SL}_2(\mathbb{R})$ has a kernel.

Remark (Wickelgren)**.** Consider the group G of unipotent upper-triangular 3-by-3 matrices

$$\begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}$$

whose center consists of matrices

$$Z(G) = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and if we take the subgroup of $Z(G)$ consisting of matrices with $b \in \mathbb{Z}$, it follows that $\mathbb{Z} \subset G$ as a central subgroup. If we take $G/\mathbb{Z}$, then this has no finite-dimensional representations.

Let me just make a couple of more definitions and then we can get to prove these assertions.

5.2 Definition. A **morphism of Lie algebras** is a map $\phi : \mathfrak{g} \to \mathfrak{h}$ that preserves the brackets, i.e. such that

$$\phi([x, y]) = [\phi(x), \phi(y)]. \tag{18}$$

A **representation** of a Lie algebra $\mathfrak{g}$ on a vector space V is simply a morphism

$$\mathfrak{g} \to \mathfrak{gl}(V).$$

This is the same as an action of $\mathfrak{g}$ on V such that for all $x, y \in \mathfrak{g}$, then

$$x(y(v)) - y(x(v)) = [x, y]v, \quad v \in V. \tag{19}$$

The main point here is that a representation is the same thing as an action on a vector space. As we saw last time, we can carry out multilinear operations on representations of a Lie algebra $\mathfrak{g}$ in such a way that it is compatible with corresponding representations of Lie groups. In other words, if V, W are representations of a Lie group G, yielding representations of $\mathfrak{g} = \mathrm{Lie}(G)$, then to get the representation of $\mathfrak{g}$ on $V \otimes W$, we differentiate the action of G on $V \otimes W$. We get

$$g(v \otimes w) = g(v) \otimes g(w), \quad g \in G \tag{20}$$

so that

$$X(v \otimes w) = X(v) \otimes w + v \otimes X(w), \quad X \in \mathfrak{g}. \tag{21}$$

Let's now do the case of the dual representation. A bit of notation: given $\phi : V \to W$ between vector spaces, we write $\phi^t : W^* \to V^*$ to be the transpose. Given a representation $\rho_V : G \to \mathrm{GL}(V)$ of a Lie group G on a vector space V, we define a representation on V^* via

$$\rho_{V^*}(g) = \rho_V(g^{-1})^t. \tag{22}$$

On the level of Lie algebras, we get

$$\phi_{V^*}(X) = -\phi_V(X)^t. \tag{23}$$

5.3 Exercise. Check that this works.

§2 The exponential map

We need a construction for the proofs of these theorems. For most of the course, we won't need this construction, as we will be working with Lie algebras, but it will come back later. I want to construct the **exponential map**

$$\exp : \mathfrak{g} \to G$$

from the Lie algebra $\mathfrak{g}$ of a Lie group G.

Suppose $X \in \mathfrak{g} = T_eG$. We can define a *vector field* on G by taking this one tangent vector and translating it around. A vector field on a manifold associates to each point in the manifold a tangent vector at that point. I want to do this by saying that $v = v_X$ is such that

$$v_X(g) = d(m_g)_e(X). \tag{24}$$

In other words, we use left translation by g to translate the vector X at e to g.

Remark. Given a map $\alpha : M \to N$ sending $p \mapsto q$, then we get a differential

$$d\alpha = (d\alpha)_p : T_pM \to T_qN.$$

This is also denoted α_*, or α' (especially if $M = \mathbb{R}$).

A bit of checking shows that this is a smooth vector field. I'm now going to invoke a classical theorem from differential geometry. It's not a big theorem, but it runs:

5.4 Theorem. *If M is any manifold, and $p \in M$, and v a vector field on M, then there exists a unique germ*[1] *of an arc $\gamma : (-\epsilon, \epsilon) \to M$ such that:*

- $\gamma(0) = p$.
- $\gamma'(t) = v(\gamma(t))$ *for each t.*

In other words, if I have a vector field on a manifold, then I can integrate it. I can find an arc which passes through a given point and whose tangent vector at every point comes from the vector field.

Now, I want to combine these two ideas. If I have a Lie group G and I choose a tangent vector $X \in T_eG = \mathfrak{g}$, then let v_X be the corresponding vector field. Let $\gamma : (-\epsilon, \epsilon) \to G$ be the corresponding integral curve with $\gamma(0) = e$. I claim two things:

[1] Two such things are equivalent if and only if they agree on an open subset of 0.

- You can extend this to the entire real line $\gamma : \mathbb{R} \to \mathbb{R}$.
- What you get is actually a homomorphism of Lie groups.

I'm going to do the second part first. We're going to show that γ is a group homomorphism *where defined.* For s, t small, we want

$$\gamma(s+t) = \gamma(s)\gamma(t). \tag{25}$$

The reason is pretty clear. If I fix s, and set $\alpha(t) = \gamma(s+t)$ and set $\beta(t) = \gamma(s)\gamma(t)$, then the derivatives of these arcs are exactly the same. We have $\alpha'(t) = v(\gamma(s+t))$ while $\beta'(t) = (dm_{\gamma(s)})\gamma'(t)$. Since v is a left-invariant vector field, these are the same. So we have two arcs with the same derivative at every point (or rather, satisfy the same differential equation). Since we're in characteristic zero, this means that they are the same arc. That in turn tells me that I can extend the arc forever by adding images of small neighborhoods of the origin here. It continues to be an integral curve for this vector field and becomes a homomorphism.

What I'm saying here is that given a tangent vector at the origin, I can "integrate" it to get a Lie group homomorphism

$$\mathbb{R} \to G.$$

We've proved:

5.5 Proposition. *For all $X \in \mathfrak{g}$, we get a unique Lie group morphism $\phi_X : \mathbb{R} \to G$ with $\phi'_X(0) = X$.*

A couple of remarks:

- We have this map ϕ_X for every $X \in \mathfrak{g}$, which is a homomorphism.
- ϕ_X need not be injective! For one thing, if I started with $X = 0$, I'd get the constant map $\mathbb{R} \to G$ at e.
- If it has a kernel (which is not all of $\mathbb{R}$), it has a discrete kernel, isomorphic to $\mathbb{Z}$, and we get a circle $S^1 \hookrightarrow G$. (E.g., for instance, if $G = S^1$, this is what happens.)

Lecture 6
2/8

Alright, let's get started. Today we want to wrap up the discussion from this past week. The crux of this matter is to prove the two key assertions we made:

- (G connected): Any Lie group map $\rho : G \to H$ is determined by its differential $d\rho : \mathfrak{g} \to \mathfrak{h}$ at the identity. Again, it was relatively elementary to see that it was determined by its values in any open neighborhood of the identity, but we're saying something stronger: just knowing the first derivatives at the origin determines the map.

- (G connected and simply connected): A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a differential of a Lie group map $\rho : G \to H$ if and only if ρ is a map of Lie algebras. This entirely characterizes linear maps that arise as differentials of Lie group maps.

Modulo the distinction between a group and its simply connected form, the study of morphisms of Lie groups is reduced to the study of Lie algebras. Starting on Monday, we will undertake in earnest the study of Lie algebras. Ultimately our goal is to apply that back to understand Lie groups and morphisms between them. That's our program.

§1 The exponential map

Last time, we had a basic construction. Given a Lie group G with Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$, then we defined a vector field $v = v_X$ which was left-invariant and with $v(e) = X$: that is, we translated X around by the group elements. In other words,

$$v_X(g) = dm_g(X), \quad \text{multiplication by } g \quad , m_g : G \to G.$$

We integrated v to arrive at an integral curve $\gamma : (-\epsilon, \epsilon) \to G$ with $\gamma(0) = 1$ and

$$\gamma'(t) = v(\gamma(t)), \quad \forall t \in (-\epsilon, \epsilon),$$

and, because of the left-invariance of the vector field v_X, γ turns out to extend to all of $\mathbb{R}$ **to be a homomorphism**, satisfying $\gamma(s+t) = \gamma(s)\gamma(t)$.

Proof. (Recap of the proof last time.) Let's show that

$$\gamma(s+t) = \gamma(s)\gamma(t) \tag{26}$$

where defined. Fix s, and set $\alpha(t) = \gamma(s)\gamma(t)$. Set $\beta(t) = \gamma(s+t)$. We want to show that

$$\alpha \equiv \beta,$$

where they are defined. We'll show that α, β are *both integral curves* for the vector field $v = v_X$. Since $\alpha(0) = \beta(0)$, then they're the same throughout. Now

$$\beta'(t) = \gamma'(s+t) = v(\gamma(s+t)) = v(\beta(t)).$$

Likewise,

$$\alpha'(t) = m_{\gamma(s)}\gamma'(t) = m_{\gamma(s)}v(\gamma(t)) = v(\gamma(s)\gamma(t)) = v(\alpha(t))$$

by left-invariance of the vector field v. This proves that γ is a homomorphism where defined, which in turn means that we can extend it to all of $\mathbb{R}$ by translating. We conclude that γ extends to a **Lie group homomorphism**

$$\phi = \phi_X : \mathbb{R} \to G, \quad \dot{\phi}(0) = X.$$

▲

Conversely, given any Lie group morphism $\mathbb{R} \to G$, it arises in this way under the above construction, using the same uniqueness result for integrating vector fields. By uniqueness, it's **natural.** That is, if $\rho : G \to H$ is any Lie group map, then if I take $X \in \mathfrak{g}$, we have a commutative diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\phi_X"] \arrow[dr, "\phi_{d\rho(X)}"'] & G \arrow[d, "\rho"] \\ & H \end{tikzcd} .$$

This follows from the uniqueness of ϕ_X (notation as in the previous proof).

6.1 Definition. A map $\phi : \mathbb{R} \to G$ of Lie groups is called a **1-parameter subgroup**: it's a subgroup of G which is one-dimensional.

A priori, this map ϕ is just an immersion (injective differential everywhere); it may or may not be one-to-one. Here are the three possibilities.

- It has a kernel, which is a discrete subgroup of $\mathbb{R}$, which after rescaling is the integers.
- ϕ is injective and closed, so $\mathbb{R}$ is a closed subgroup of G.
- ϕ is injective but only an immersion; it isn't a closed map.

6.2 Example. Take $G = S^1 \times S^1$. The Lie algebra is $\mathfrak{g} = \mathbb{R}^2$ with trivial bracket. Given $X \in \mathfrak{g}$, integrating amounts to drawing a line on the torus. If I think of the torus as a plane mod a lattice, then I draw the line on the plane and project to the torus. If the line has rational slope, then it repeats itself and has a kernel: it defines a subgroup $S^1 \subset S^1 \times S^1$. If the line has irrational slope, then it defines an immersed (but not imbedded) submanifold $\mathbb{R} \subset S^1 \times S^1$. Observe that possibilities are dense in the Lie algebra.

Nonetheless, the claim is that I can fit all of these maps together and get a single map which will at least locally behave well.

For each $X \in \mathfrak{g}$, we've defined a map $\phi = \phi_X : \mathbb{R} \to G$. Consider all vectors in the Lie algebra and define them simultaneously.

6.3 Definition. The **exponential map** $\exp : \mathfrak{g} \to G$ by setting $\exp(x) = \phi_X(1)$.

Remark. Observe that $\phi_{\lambda X}(t) = \phi_X(\lambda t)$. In particular, if I look at the exponential map and restrict it to any line through the origin in the Lie algebra, I get a one-parameter subgroup of the Lie group.

The exponential map is a smooth map $\mathfrak{g} \to G$, as we'll see below. In fact, $\exp : \mathfrak{g} \to G$ is the unique smooth map such that:

- $(d\exp)_0 = \mathrm{id}$.
- exp restricted to any line is a one-parameter subgroup (i.e., it's a homomorphism).

So we actually get a *diffeomorphism* between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of the identity in the group G. I don't even want to think about what it does globally. But at least in a neighborhood of zero, it's well-defined.

The uniqueness of the exponential map gives:

6.4 Proposition. *The exponential map is natural. Given a Lie group map $\rho : G \to H$, there's a commutative diagram:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\exp"] \arrow[d, "d\rho_e"] & G \arrow[d, "\rho"] \\ \mathfrak{h} \arrow[r, "\exp"] & H \end{tikzcd} .$$

This proves the first principle. We've already observed that any Lie group map is determined in a neighborhood of the identity. Now, we've said that if you know the differential at the identity, you know the Lie group map in a neighborhood of the identity, and hence everywhere.

I still owe you a proof that exp is a smooth map. How do we show that? I keep talking about how it's good to work in a coordinate-free manner, but now I'm going to work in GL_n and do it explicitly. Whatever I deduce from this will apply to any subgroup of GL_n.

6.5 Example. Explicitly, let's suppose $G = \mathrm{GL}(V)$ for a vector space V. The corresponding Lie algebra is $\mathfrak{gl}(V)$, which is the algebra of endomorphisms of a vector space. In this case, I can write out the exponential map completely explicitly, and you'll see why it's called that: given an endomorphism X of V, we get

$$\exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \dots ; \tag{27}$$

this converges and gives a well-defined automorphism of V. It is an automorphism, because $\exp(-X)\exp(X) = \mathrm{id}$. Moreover, by multiplying power series we find that

$$\exp((s+t)X) = \exp(sX)\exp(tX).$$

This is evidently a smooth map which satisfies the conditions that characterize the exponential map, so this must be it. The same must be true for any subgroup of $\mathrm{GL}(V)$.[2]

§2 The Baker-Campbell-Hausdorff formula

Let G be a Lie group with $\mathfrak{g}$ the Lie algebra.

The point of the previous analysis is that this smooth map

$$\exp : \mathfrak{g} \to G$$

is a diffeomorphism from a neighborhood of the origin to the neighborhood of the identity. Now I can ask: this map is a homomorphism *when restricted to a line.* It

[2]Though as we stated last time, there are Lie groups which don't imbed in $\mathrm{GL}(V)$.

isn't a homomorphism in general. I'd like to transport the group law in G over to $\mathfrak{g}$ and see what it looks like. If I have two small vectors v, w in the Lie algebra $\mathfrak{g}$, I can exponentiate them to get elements of the group, and I can *compose* them and then take the inverse $\exp^{-1}$: what is it the image of? Can I say, in terms of the Lie algebra structure, what the Lie group structure looks like locally?

6.6 Definition. We define the **logarithm** map

$$\log : U \to \mathfrak{g}$$

on a small neighborhood $U \subset G$ of the identity as the inverse to the exponential map.

6.7 Example. If the group imbeds in $\mathrm{GL}(V)$, then the logarithm map is

$$\log(g) = (g - I) - \frac{(g-I)^2}{2} + \frac{(g-I)^3}{3} \pm \dots .$$

Note that the individual terms in this power series (which converges when g is close to the identity) are not intrinsically defined but the sum does.

I'd like to describe the group law on the group ported over to the Lie algebra via the logarithm or the exponential map.

6.8 Definition. For $X, Y \in \mathfrak{g}$ small enough, define the auxiliary operation

$$X * Y = \log(\exp(X)\exp(Y)). \tag{28}$$

This makes sense if X, Y are close enough to zero and the logarithm is defined. This is the operation we're going to study. Now, we want to write it out. This is the crucial formula, which you will use once and never again, so I don't know how much to emphasize this.

What is $X * Y$, for a subgroup of $\mathrm{GL}(V)$? It is

$$X * Y = \log\left(\left(1 + X + \frac{X^2}{2} + \dots\right)\left(1 + Y + \frac{Y^2}{2} + \dots\right)\right)$$
$$\log\left(1 + (X+Y) + \frac{X^2}{2} + XY + \frac{Y^2}{2} + \dots\right).$$

Now expand this out in a power series:

$$\left(1 + (X+Y) + \frac{X^2}{2} + XY + \frac{Y^2}{2} + \dots\right) - \frac{1}{2}\left(1 + (X+Y) + \frac{X^2}{2} + XY + \frac{Y^2}{2} + \dots\right)^2 \pm \dots .$$

The quadratic terms in this expansion are

$$X + Y + \frac{X^2}{2} + XY + \frac{Y^2}{2} - \frac{(X+Y)^2}{2} = X + Y + \frac{[X,Y]}{2}.$$

Then:

6.9 Theorem. *We have:*

$$X * Y = X + Y + \frac{[X,Y]}{2} + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) + \dots. \tag{29}$$

What this is good for is one thing: we have taken the group law and translated it back to the Lie algebra. The $*$ operation is what corresponds, via the exponential map, to the group law, and the key fact is that this is expressible purely in terms of the Lie bracket. If you know the Lie algebra structure, you can express the Lie group structure in these terms. The exponential map is generally not a homomorphism: if X, Y don't commute, there are additional terms in the above formula. But these correction terms are expressed purely in terms of the Lie algebra structure. So once I know the Lie algebra structure, I can create the Lie group structure locally. That is, essentially, the proof of the second principle.

Remark. This hasn't been completely proved: we haven't shown that the higher terms are also expressible in terms of Lie brackets.

6.10 Corollary. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace. The subgroup of G generated by $\exp(\mathfrak{h})$ with Lie algebra $\mathfrak{h}$ is an immersed subLie group of G with tangent space $T_eH = \mathfrak{h}$ if and only if $\mathfrak{h}$ is a Lie subalgebra.*

In general, if I take two vectors $x, y \in \mathfrak{h}$ and take $\exp(x)\exp(y)$, the product is not necessarily going to be the exponential of a linear combination of x, y. The exponential map applied to a two-dimensional subspace will generally not be a local Lie subgroup. But if we started with a Lie subalgebra, then we can appeal to the Baker-Campbell-Hasudorff formula. We get a one-to-one correspondence between:

- Immersed, connected subgroups $H \subset G$.
- Lie subalgebras $\mathfrak{h} \subset G$.

That's the first step in the correspondence between Lie groups and Lie algebras. Think about this over the weekend. We're going to start building a dictionary between phenomena we're interested in via the group and via the Lie subalgebra. We're saying that immersed subgroups correspond to subalgebras. What do normal subgroups correspond to? We are going to carry over the standard definitions of group theory to Lie algebras.

Proof of the second principle. Given a map $\phi : \mathfrak{g} \to \mathfrak{h}$, let $\mathfrak{j} \subset \mathfrak{g} \oplus \mathfrak{h}$ be the graph. This is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$ if and only if ϕ is a Lie algebra map. In this case, there exists a connected Lie group $J \subset G \times H$, an immersed subgroup, with Lie algebra $\mathfrak{j}$. That means that composition of this map

$$J \to G \times H \stackrel{p_1}{\to} G$$

is an isomorphism on Lie algebras: it's therefore a covering space map. (That's because $\mathfrak{j} \to \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}$ is an isomorphism.) As G is simply connected, it follows that $J \simeq G$ and J is the graph of a Lie group map $G \to H$ with the desired properties. ▲

Lecture 7
2/11

Last time, on Friday, we talked about the passage from Lie groups to Lie algebras. We saw that there is a **one-to-one correspondence** between Lie algebras and connected, simply connected Lie groups. We can describe morphisms of Lie groups entirely in terms of Lie algebras. So we will work mostly with Lie algebras, with the idea that this is eventually going to apply to Lie groups. Still, what we need to do know is to create a dictionary between the basic properties of Lie groups and Lie algebras. Mostly, this is a matter of translation.

§1 The dictionary

First, we should define the notion of an **abelian** Lie algebra. Given an abelian Lie group, the associated Lie algebra has all Lie brackets zero.

7.1 Definition. A Lie algebra is **abelian** if $[\cdot,\cdot] \equiv 0$: if all brackets are zero. The **center** $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the collection of $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$.

Given a connected Lie group G with Lie algebra $\mathfrak{g}$, connected (possibly immersed) subgroups $H \subset G$ correspond to subalgebras $\mathfrak{h} \subset \mathfrak{g}$. What corresponds to the notion of normlaity? Given $H \subset G$ which is normal, then that means that

$$gHg^{-1} = H, \quad g \in G.$$

If we want to translate this back into the world of Lie algebras, say we have an arc $\gamma : I \to G$ with $\gamma(0) = e$ and $\gamma'(0) = X \in \mathfrak{g}$, then to say that

$$\gamma(t)H\gamma(t)^{-1} = H, \quad \forall t$$

implies after differentiating that $\mathfrak{h}$ is stable under the adjoint representation: that is,

$$\mathrm{Ad}(\gamma_t)\mathfrak{h} \subset \mathfrak{h}$$

under the adjoint representation of G on $\mathfrak{g}$. Differentiating with respect to t, we find that

$$\mathrm{ad}(X) : \mathfrak{h} \to \mathfrak{h},$$

or that if $Y \in \mathfrak{h}$, then $[X, Y] \in \mathfrak{h}$ (for any $X \in \mathfrak{g}$). This leads to the following definition, which is the Lie algebra-theoretic analog of the notion of a normal subgroup:

7.2 Definition. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is **an ideal** if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Remark. It's natural to ask whether I can form quotients of Lie algebras in the same way that I can form quotient groups. Given a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, then we can form the quotient $\mathfrak{g}/\mathfrak{h}$ as a Lie algebra *if and only if* $\mathfrak{h}$ *is an ideal.*

In analogy with group theory, we have:

7.3 Definition. We say that $\mathfrak{g}$ is **simple** if $\mathfrak{g}$ has no nontrivial ideals (that is, no ideals other than 0 and $\mathfrak{g}$).

We're going to justify focusing on the simple Lie algebras in classification. Over the next week, we'll describe various types of Lie algebras, and we'll see that a Lie algebra can be broken into a relatively easily understood piece (the maximal solvable Lie algebra) and the quotient, which will be a direct sum of simple Lie algebras.

§2 Nilpotent, solvable, and semisimple Lie algebras

I want to introduce three more important classes of Lie algebras. To do that, it will be useful to introduce some auxiliary constructions which we'll use this week and then never see again. We're going to define **solvable** and **nilpotent** Lie algebras.

7.4 Definition. Given a Lie algebra $\mathfrak{g}$, we define two sequences of ideals in $\mathfrak{g}$.

- The **lower central series** is defined via $D_0\mathfrak{g} = \mathfrak{g}$ and inductively,

$$D_i\mathfrak{g} = [\mathfrak{g}, D_{i-1}\mathfrak{g}]. \tag{30}$$

  In particular, $D_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. This is a descending filtration

$$D_0\mathfrak{g} \supset D_1\mathfrak{g} \supset D_2\mathfrak{g} \supset \dots .$$

  These are evidently ideals.

- The **derived series** is a descending sequence of algebras with $D^0\mathfrak{g} = \mathfrak{g}$ and

$$D^i\mathfrak{g} = [D^{i-1}\mathfrak{g}, D^{i-1}\mathfrak{g}]. \tag{31}$$

  These are ideals, and you can prove it using the Jacobi identity.

Now we make a couple of definitions:

7.5 Definition. We say that a Lie algebra $\mathfrak{g}$ is **nilpotent** if the lower central series eventually terminates at zero: that is, if $D_k\mathfrak{g} = 0$ for $k \gg 0$.

7.6 Definition. We say that $\mathfrak{g}$ is **solvable** if the derived series eventually terminates at zero: that is, if $D^k\mathfrak{g} = 0$ for $k \gg 0$.

Of course, nilpotent is a stronger condition than solvable: the terms in the lower central series contain the analogous terms in the derived series.

7.7 Definition. We say that $\mathfrak{g}$ is **semisimple** if $\mathfrak{g}$ has no nonzero solvable ideals.

As we will see shortly, an equivalent definition to semisimple is that $\mathfrak{g}$ be a direct sum of simple Lie algebras. If we can classify simple Lie algebras, we can classify semisimple ones.

Remark. A subalgebra of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable).

There are key examples of nilpotent and solvable Lie algebras.

7.8 Example. The Lie algebra $\mathfrak{n}$ of strictly upper-triangular matrices sitting inside $\mathfrak{gl}_n$ is nilpotent. (This is the Lie algebra of the Lie group of upper-triangular unipotent matrices.) Any Lie subalgebra of $\mathfrak{n}$ is nilpotent. In fact, we're going to show that the converse is true: every nilpotent Lie algebra can be realized as a subalgebra of $\mathfrak{n}$.

To see that $\mathfrak{n}$ is nilpotent, observe that the commutator of strictly upper-triangular matrices has zeros on the main diagonal and on the diagonal adjacent to it. Repeating, we get more and more zeros.

7.9 Example. There is a similar example for solvable Lie algebras. This is the Lie algebra $\mathfrak{b}$ of all upper-triangular matrices, viewed as a subalgebra of $\mathfrak{gl}_n$. This is no longer nilpotent, but it is solvable. When you take the commutator, the first time, you get $\mathfrak{n} \subset \mathfrak{b}$. But on the other hand, $[\mathfrak{b}, \mathfrak{n}] = \mathfrak{n}$ so $\mathfrak{b}$ is not nilpotent. Analogously, every solvable Lie algebra imbeds inside some $\mathfrak{b}$.

Remark. Question: how does this question justify saying that we can ignore solvable Lie algebras? It's not true. The idea is to focus on irreducible representations. The result we're going to prove is that any representation of a solvable Lie algebra, we can choose a basis so that the representation consists of upper-triangular matrices. This means that there is always a one-dimensional subrepresentation. So the only *irreducible* representations of a solvable Lie algebra are one-dimensional. That's our justification for focusing on the simple case.

Now, I want to say something about solvable Lie algebras that does *not* have an analog of nilpotent ones; it is much more the right characterization of them.

Let's make the following observation.

7.10 Proposition. *Given a Lie algebra $\mathfrak{g}$, it is solvable if and only if there exists a sequence of subalgebras*

$$0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_k \subset \mathfrak{g}$$

such that:

- $\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ *is an ideal in* $\mathfrak{g}_{i+1}$ *(not necessarily an ideal in* $\mathfrak{g}$*).*
- *The quotient* $\mathfrak{g}_{i+1}/\mathfrak{g}_i$ *is abelian.*

Proof. If $\mathfrak{g}$ is solvable, the derived series is such a sequence of subalgebras. Conversely, if you have such a sequence of subalgebras, then the elements of the derived series will be carrying the filtration further and further back and we get solvability. ▲

As a consequence, we get:

7.11 Corollary. *If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.*

7.12 Corollary. *Given a pair of solvable Lie ideals $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, the span $\mathfrak{a}+\mathfrak{b}$ is a solvable Lie ideal of $\mathfrak{g}$.*

Proof. In fact, this follows from the previous corollary, and from the isomorphism

$$(\mathfrak{a}+\mathfrak{b})/\mathfrak{b} \simeq \mathfrak{a}/(\mathfrak{a}\cap\mathfrak{b}),$$

which is a solvable Lie algebra. ▲

Now we've come to a definition which doesn't seem to have a definition in the analog of group theory.

7.13 Corollary. *Given a Lie algebra $\mathfrak{g}$, there exists a unique maximal solvable ideal in $\mathfrak{g}$.*

7.14 Definition. The **radical** $\mathrm{rad}(\mathfrak{g})$ of $\mathfrak{g}$ is the maximal solvable Lie ideal.

We have an exact sequence

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \to 0,$$

and in fact $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. In fact, if it had any nonzero solvable ideals, we could take the preimage in $\mathfrak{g}$ to get a strictly larger solvable ideal in $\mathfrak{g}$. Therefore, any Lie algebra can be thought of as having two component parts: a solvable part and a semisimple part. It's not a direct sum, but it is in some sense a decomposition.

If we want to understand irreducible representations of $\mathfrak{g}$, we can hope to do so by understanding the irreducible representations of each piece. As we'll see, those of a solvable Lie algebra are easy to describe.

§3 Engel's and Lie's theorems

I'm going to prove two theorems that give the characterization of solvable and nilpotent Lie algebras.

7.15 Theorem (Engel)**.** *Let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. Suppose that for all $x \in \mathfrak{g}$, $\phi(x)$ is a nilpotent endomorphism of V. Then there exists a nontrivial vector $v \in V$ such that all of $\mathfrak{g}$ kills v.*

The hypothesis is that everything in $\mathfrak{g}$ acts nilpotently, so everything in $\mathfrak{g}$ has a kernel: we can get a common kernel.

7.16 Corollary. *Suppose $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a Lie subalgebra which consists of nilpotent endomorphisms of V. Then there is a basis for V such that $\mathfrak{g}$ can be simultaneously strictly-upper-triangularized: that is, $\mathfrak{g} \subset \mathfrak{n}$.*

Proof of the corollary. If I know that there's a single vector $v \in V$ killed by $\mathfrak{g}$, I observe that $\mathfrak{g}$ acts on V/v and then keep applying the same logic over and over. ▲

Proof of Engel's theorem. One preliminary remark: if $X \in \mathfrak{gl}(V)$ is an endomorphism of V which is nilpotent, then that means that there's a diminishing sequence of subspaces

$$V \supsetneq V_1 \supsetneq \cdots \supset V_k = 0,$$

such that the action of X carries each subspace in the sequence to the next smaller one. **In particular,** $\mathrm{ad}X \in \mathfrak{gl}(\mathfrak{gl}(V))$ **(that is,** $\mathrm{ad}X$ **is viewed as an endomorphism of** $\mathrm{gl}(V)$**) is nilpotent.** In fact, $(\mathrm{ad}X)^m Y \in \mathfrak{gl}(V)$ carries V_i into V_{i+k-m} — let's come back to this next time.

Given this fact, let's prove Engel's theorem by **induction on** $\dim \mathfrak{g}$.

7.17 Lemma. $\mathfrak{g}$ *contains an ideal* $\mathfrak{h}$ *of codimension one.*

Then we're going to do the natural thing and look at the action of this subalgebra $\mathfrak{h}$.

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be any maximal proper subalgebra. Then the claim is that $\mathfrak{h}$ is an ideal and of codimension one. Look at the adjoint action of $\mathfrak{h}$ on the vector space $\mathfrak{g}/\mathfrak{h}$. Every element of $\mathfrak{h}$ acts nilpotently. By the inductive hypothesis, we can apply Engel's theorem to $\mathfrak{h}$ acting on $\mathfrak{g}/\mathfrak{h}$ and find a vector $Y \in \mathfrak{g} \setminus \mathfrak{h}$ such that

$$[Y, \mathfrak{h}] \subset \mathfrak{h}.$$

Consider the span of $Y + \mathfrak{h}$. This is itself a subalgebra, and since $\mathfrak{h} \subset \mathfrak{g}$ is maximal, it follows that $Y + \mathfrak{h} = \mathfrak{g}$. Moreover, $[Y, \mathfrak{h}] \subset \mathfrak{h}$ implies that $\mathfrak{h}$ is an ideal. ▲

We'll finish the proof of Engel's theorem on Wednesday. ▲

Lecture 8
2/13

In this course, we are going to focus almost exclusively on the *classical* Lie algebras, in particular the semisimple Lie algebras, and I want to explain why that's a reasonable restriction to make. Today, we're going to finish with chapter 9 of Fulton and Harris. On Friday, we will do chapter 10, which is a bit of a digression, in the sense that it is simply a matter of trying to get more familiar with concrete examples of Lie algebras by describing all Lie algebras of dimensions $1, 2, 3$. It's good to develop your familiarity but it is skippable. So why don't you take a look, between now and Friday, and I'll ask you how much time to spend on chapter 10. Next week, we'll hit chapter 11, which is where the course starts. Chapter 11 is where we start *answering* the questions that we've set up. So that's our plan.

§1 Engel's theorem

Last time, we had **Engel's theorem:**

8.1 Theorem (Engel's theorem)**.** *If* $\mathfrak{g} \subset \mathfrak{gl}(V)$*, and* $X : V \to V$ *is nilpotent for each* $X \in \mathfrak{g}$*, then there exists* $v \in V \setminus \{0\}$ *such that* $Xv = 0$ *for all* $X \in \mathfrak{g}$*. In other words, the kernels all have a common intersection.*

For the purposes of comparison, let me put up the version of this for solvable Lie algebras.

8.2 Theorem (Lie's theorem)**.** *Let k be algebraically closed of characteristic zero. If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is any solvable Lie algebra, then there exists $v \in V \setminus \{0\}$ which is a simultaneous eigenvector for every $X \in \mathfrak{g}$. That is, for each $X \in \mathfrak{g}$, we have*

$$Xv = \lambda(X)v,$$

for some linear map $\lambda : \mathfrak{g} \to k$.

Engel's theorem states, under these hypotheses, that there exists a *basis* for the vector space of V such that in terms of the corresponding identification of $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n$, the subalgebra $\mathfrak{g}$ is identified with a subalgebra of the *upper-triangular matrices.* The argument here is straightforward: find v killed by every element of $\mathfrak{g}$, make that the first basis vector, and then look at the action on the quotient of $V/\{v\}$ and repeat.

Here, the conclusion of Lie's theorem is — analogously — that there exists a basis $v_1, \dots, v_n \in V$ such that $\mathfrak{g}$ becomes contained (in this basis) in the upper-triangular matrices. Again, the reasoning is analogous, by passing to quotients repeatedly.

Notice one difference between the two theorems. The second theorem depends only on the Lie algebra: it states that *any* representation of a solvable Lie algebra can be represented via upper-triangular matrices. The hypothesis of Engel's theorem assumes that every element in $\mathfrak{g}$ is a nilpotent endomorphism. It is *not* true that any representation of a nilpotent Lie algebra can be represented via strictly upper-triangular matrices (e.g. a diagonal representation).

Proof of Engel's theorem. This starts with a preliminary remark, which needs to be fixed (thanks to Omar Antolin-Camerana). Assume every element $X \in \mathfrak{g}$ is nilpotent (as an element of $\mathrm{End}(V)$). The claim is that the adjoint action

$$\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$$

is nilpotent for all $X \in \mathfrak{g}$. The proof, which kind of got sloppy last time, is actually very simple, as Omar pointed out: we're saying that $X : V \to V$ is nilpotent, so that $X^k = 0$ on V (for some k). Now

$$\mathrm{Ad}(X)^m Y = [X, [X, \dots, [X, Y] \dots].$$

If you expanded that out as a product of X's and Y's, each term of the expansion would consist of a string of X's with one Y. If $m > 2k$, there has to be a consecutive string of k X's somewhere and the product has to be zero. That's going to be a useful point as we go on.

Now, **we're going to assume inductively that Engel's theorem is true for Lie algebras of smaller dimension.**

The next point in this proof is:

8.3 Lemma. *Under these hypotheses, there exists $\mathfrak{h} \subset \mathfrak{g}$ of codimension one.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal proper subalgebra. We'll show that $\mathfrak{h}$ is codimension one and an ideal. In this case, $\mathfrak{h}$ acts on both $\mathfrak{g}$ on $\mathfrak{h}$ via the adjoint action, so we get an induced action of $\mathfrak{h}$ on the quotient $\mathfrak{g}/\mathfrak{h}$. The action of $\mathfrak{h}$ is nilpotent on this quotient space (cf. preliminary remark above: in fact it's even nilpotent on $\mathfrak{g}$), which—by the inductive hypothesis—means that there is a nonzero vector $\overline{Y} \in \mathfrak{g}/\mathfrak{h}$ such that $\overline{Y}$ is killed by the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$.

Now we just choose $Y \in \mathfrak{g}$ living over $\overline{Y} \in \mathfrak{g}/\mathfrak{h}$. I can now say that

$$\mathrm{ad}(X)(Y) \in \mathfrak{h}, \quad \text{i.e. } [X, Y] \in \mathfrak{h} \quad \text{for } X \in \mathfrak{h}.$$

That states that if I take the span of $\mathfrak{h}, Y$, then this is a *subalgebra* of $\mathfrak{g}$. That implies, of course, that it's equal to all of $\mathfrak{g}$, since $\mathfrak{h}$ was maximal. Thus $\mathfrak{h}$ is codimension one and it's also clear that $\mathfrak{h}$ is an ideal since

$$[\mathfrak{h}, Y] \subset \mathfrak{h}.$$

▲

Now let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be as in Engel's theorem, and choose an ideal $\mathfrak{h} \subset \mathfrak{g}$ of codimension one and let $Y \in \mathfrak{g} \setminus \mathfrak{h}$. By the inductive hypothesis, there exists $v \in V \setminus \{0\}$ killed by $\mathfrak{h}$. That's not quite right. But here's what we do: consider the *subspace of all such vectors.* Set

$$W = \{v \in V : \mathfrak{h}v = 0\}.$$

We know that $W \neq 0$ (by the inductive hypothesis), but let's look at all such vectors.

Our last claim is:

8.4 Lemma. *The subspace W is stable under $\mathfrak{g}$: that is, the action of Y preserves W.*

Everything in $\mathfrak{h}$ kills W, so the two claims are equivalent. Once we know that, we know that Y is nilpotent, so it is nilpotent on W, so it must have a kernel in W: that means that there is a nonzero vector w in W killed by Y. That means that $\mathfrak{g}w = 0$ and w is the vector we want to prove Engel's theorem.

Proof. This last claim is the reason we're doing this proof: it's a basic calculation that we're going to see over and over, and it's going to come up in Lie's theorem as well. The proof of the claim is to suppose $w \in W$ is any vector. We want to show that $Yw \in W$. How can I tell that? The space W is defined as the set of vectors killed by everything in $\mathfrak{h}$. If I want to know that a given element is in W, I just have to see that it's killed by everything in $\mathfrak{h}$.

In other words, to show $Yw \in W$, I have to show that for each $X \in \mathfrak{h}$, we have

$$X(Yw) = 0, \quad X \in \mathfrak{h}. \tag{32}$$

But

$$X(Yw) = Y(X(w)) + [X, Y]w, \tag{33}$$

and $X(w) = 0$, so the first term drops out. On the other hand, $[X, Y] \in \mathfrak{h}$ (Because $\mathfrak{h}$ is an ideal), so $[X, Y]w = 0$ as well. So we conclude that (32) holds. This means that $Yw \in W$ and this is what we wanted to show. ▲

This calculation, which seems pretty mindless, is what's going to unlock a lot of what we're going to do. Observe that w is an eigenvector for all of $\mathfrak{h}$, and we're saying that if some element of the Lie algebra outside $\mathfrak{h}$ acts on it, we still get an eigenvector for all of $\mathfrak{h}$, again with eigenvalue zero. We'll say this again, but we've now proved Engel's theorem. ▲

§2 Lie's theorem

The argument of Lie's theorem is going to go along very similar lines. Instead of looking for elements which are eigenvectors of $\mathfrak{g}$ with eigenvalue zero (i.e., elements killed by $\mathfrak{g}$), we are going to look simply for eigenvectors. The first point, in particular, is that there a codimension one ideal again.

8.5 Lemma. *If $\mathfrak{g}$ is solvable, then there exists $\mathfrak{h} \subset \mathfrak{g}$ of codimension one which is an ideal.*

Proof. This is *easier* than the corresponding lemma in Engel's theorem, because in Engel's theorem we didn't assume anything about the Lie algebra to begin with, just about the elements as endomorphisms. We know that

$$[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g},$$

because $\mathfrak{g}$ is solvable. We observe that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian and therefore I can find an ideal in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ by taking any *codimension one* vector space $\overline{\mathfrak{h}}$. Take $\mathfrak{h}$ to be the preimage of $\overline{\mathfrak{h}}$. ▲

It's easier, again, because we're assuming something about the Lie algebra. Now comes the part which requires a little more work. Let $\mathfrak{g}$ be solvable, $\mathfrak{h} \subset \mathfrak{g}$ an ideal of codimension one. Let $\mathfrak{g}$ act on V.

Assume inductively that Lie's theorem is true for solvable Lie algebras of smaller dimension. Therefore, since $\mathfrak{h}$ acts on V, we can find a common eigenvector $v \in V \setminus \{0\}$ for all of $\mathfrak{h}$. In other words,

$$Xv = \lambda(X)v, \quad X \in \mathfrak{h},$$

where $\lambda(X)$ is the eigenvalue: then

$$\lambda : \mathfrak{h} \to k$$

is a linear functional.

8.6 Definition. In general, when we talk about **eigenvectors** for the action of a Lie algebra, we'll mean that it's an eigenvector for every element of the Lie algebra, and the associated **eigenvalue** is a *linear functional* on the Lie algebra.

We want to say that we can choose v such that some $Y \in \mathfrak{g} \setminus \mathfrak{h}$ has v has an eigenvector. To do this, fix λ as above, and define

$$W = W_\lambda = \{v \in V : Xv = \lambda(X)v, \quad X \in \mathfrak{h}\} \tag{34}$$

where, again, λ is a *fixed* linear functional on $\mathfrak{h}$. For *some* choice of λ, this is nonzero. The main lemma is:

8.7 Lemma. *Choose $Y \in \mathfrak{g} \setminus \mathfrak{h}$ (so that Y and $\mathfrak{h}$ span the Lie algebra). Then $Y(W) \subset W$. In particular, $\mathfrak{g}W \subset W$.*

Given this, Y has an eigenvector in W, and that's all we need to know to prove Lie's theorem, and we're done. Again, we can't take any vector in W, but a specific one: we find that one by looking at the space of all eigenvectors of $\mathfrak{h}$ with eigenvalue λ. Remember that *we are working with an algebraically closed field of characteristic zero, e.g.* $\mathbb{C}$.

In general, this is what we're going to do: given a representation of a Lie algebra, we're going to restrict to a subalgebra, break up the action of that subalgebra into eigenspaces, and look at how the rest of the Lie algebra acts on the decomposition.

Let's prove the lemma:

Proof. Let $w \in W \setminus \{0\}$. We have to show that $Yw \in W$. Since W is characterized by how $\mathfrak{h}$ acts on it, we have to show simply that if $X \in \mathfrak{h}$, then

$$X(Y(w)) = \lambda(X)Yw. \tag{35}$$

We do exactly the same thing now: we have

$$X(Y(w)) = Y(Xw) + [X,Y]w.$$

The first term is easy to calculate because $w \in W$: it is $Y(\lambda(X)w) = \lambda(X)Yw$. That's what we want. The second term is a little more of a problem: by definition it is $\lambda([X,Y])w$ since $[X,Y] \in \mathfrak{h}$. We get

$$X(Y(w)) = \lambda(X)Yw + \lambda([X,Y])w.$$

In particular, *we have to show that* $\lambda([X,Y]) = 0$*.* This is a potential problem. How can we do that? This is going to involve another trick.

8.8 Lemma. $\lambda([X,Y]) = 0$ *for* $X \in \mathfrak{h}, Y \in \mathfrak{g} \setminus \mathfrak{h}$*.*

A priori, λ is just a linear functional on $\mathfrak{h}$: the claim is that if $W_\lambda \neq 0$, then λ vanishes on commutators. Let's see why that is.

Proof. Fix $w \in W\setminus\{0\}$. Consider the subspace U which is the span of $\{w, Yw, Y^2w, \dots\}$. This is a finite-dimensional vector space and there is thus a basis of U of the form

$$w, Yw, Y^2w, \dots, Y^nw$$

for some n.

How does $X \in \mathfrak{h}$ act on U? That's pretty easy: we have

$$X(w) = \lambda(X)w$$

to start with. Next,

$$X(Yw) = \lambda(X)Yw + \text{some multiple of } w.$$

Similarly,

$$X(Y^2w) = \lambda(X)Y^2w + \text{a linear combination of } w \text{ and } Yw\ .$$

And so forth. In particular, X carries U into itself (thus U is *stable under all of* $\mathfrak{g}$). In terms of this basis, it acts as an *upper-triangular matrix.* The diagonal entries are all equal to $\lambda(X)$.

Now, we're home. This states that the *trace* of the action of X on U is simply $n\lambda(X)$ (where $n = \dim U$), because X is represented by an upper-triangular matrix. If X is a commutator, then the trace must be zero. If $X \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$, then the action of X on U is the commutator of two endomorphisms of U and therefore has trace equal to zero, so $\lambda(X) = 0$. This proves the claim. ▲

▲

The thing to take away from this, apart from the statement itself, is simply this basic technique: if you want to understand a representation of a Lie algebra, find a subalgebra such that you can decompose the restriction of this representation into eigenspaces. And then look at the action of the rest of the Lie algebra on the decomposition.

Lecture 9
2/15

(No class next Monday; the homework is due on Wednesday.) I thought some more about the issue of how much time to spend on chapter 10, and I think the answer is **not much:** first of all it's something you can read on your own (and it's something I encourage you to do, as you'll get a sense of how many Lie algebras are out there in low dimensions). I'm going to say some general things today and then we're going to jump into the end of chapter 10, where we discuss the last case.

Today, I want to finish the reduction of the problem of classifying representations to the problem of classifying representations of **simple Lie algebras.** What's the simple Lie algebra? Let's find it: we'll do that today. That corresponds to the last section of chapter 10 and we'll go from there. Starting Wednesday, we are going to analyze representations of actual Lie algebras. We'll start with chapter 11, representations of $\mathfrak{sl}_2(\mathbb{C})$ (or of the Lie group $SL_2(\mathbb{C})$).

§1 The radical

First, I want to give a proof of a basic proposition that justifies our restriction to semisimple Lie algebras. Recall that for any Lie algebra $\mathfrak{g}$, we have an exact sequence

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \to 0$$

where the **radical** $\mathrm{rad}(\mathfrak{g})$ is the maximal solvable subalgebra and

$$\mathfrak{g}_{ss} \stackrel{\text{def}}{=} \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$$

is semisimple. This isn't a split sequence, but we can think of it as a type of decomposition in some sense. I want to claim that if we can describe irreducible representations of $\mathfrak{g}_{ss}$, then we can describe irreducible representations of $\mathfrak{g}$.

9.1 Proposition. *If V is an irreducible representation of $\mathfrak{g}$, then $V \simeq V_0 \otimes L$ where:*

- *V_0 is a representation of V_{ss} (i.e., the radical of $\mathfrak{g}$ acts trivially).*
- *L is one-dimensional.*

I haven't said this explicitly, but it's pretty clear: one dimensional representations of Lie algebras are straightforward. A one-dimensional representation of a given Lie algebra $\mathfrak{g}$ is the same as a linear functional

$$\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to k,$$

because each element of $\mathfrak{g}$ must act by a scalar, and each element of $[\mathfrak{g}, \mathfrak{g}]$ must act by zero. This proposition thus suggests that to understand irreducible representations of a general Lie algebra, we may restrict to semisimple ones.

Remark (Warning)**.** There is an error in the first printing of Fulton and Harris.

Proof. The proof is based on the idea we introduced last time. Recall that if $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, and if V is a representation of $\mathfrak{g}$, then we can introduce for each $\lambda \in \mathfrak{h}^*$, the eigenspace

$$V_\lambda = \{v \in V : Xv = \lambda(X)v, \quad \text{for all } X \in \mathfrak{h}\}.$$

The calculation we made last time is that **the entire Lie algebra preserves V_λ.**

Now, let $\mathfrak{h} = \mathrm{rad}(\mathfrak{g})$. By Lie's theorem, proved last time, there is an eigenvector for the action of $\mathfrak{h}$ on V. For some $\lambda \in \mathfrak{h}^*$, we have $V_\lambda \neq 0$. Since V is irreducible and since V_λ is a subrepresentation (by last time), we have

$$V = V_\lambda,$$

so that $\mathfrak{h}$ acts on the *entire* representation as an eigenspace. In particular, $\lambda|_{\mathrm{rad}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]} \equiv 0$. So we **extend** the function $\lambda : \mathfrak{h} \to k$ (where $k = \mathbb{C}$ here) to a function

$$\widetilde{\lambda} : \mathfrak{g} \to \mathbb{C}$$

such that $\widetilde{\lambda}|_{[\mathfrak{g},\mathfrak{g}]} \equiv 0$. Let L be the corresponding one-dimensional representation of $\mathfrak{g}$. Then we're done, as $\mathfrak{h}$ acts trivially on $V_0 \stackrel{\text{def}}{=} V \otimes L^*$ and V_0 becomes a representation of $\mathfrak{g}_{ss}$. ▲

Remark. This is the philosophical basis for the focus on representations of semisimple Lie algebras. Now, when we're dealing with say finite groups, the first thing that we say is that we can take any representation and break it up as a direct sum of irreducibles. That's **not the case** for representations of Lie groups and Lie algebras in general, so the idea that it is enough to study irreducible representations is somewhat suspect, but I'm still going to take this as justification for focusing on the semisimple (and ultimately the simple) case.

§2 Jordan decomposition

The next topic I want to talk about has to do with this notion of irreducibility. Recall that if G is a *finite group*, and $\rho : G \to \mathrm{GL}(V)$ is a representation (over $\mathbb{C}$), there are two things that are absolutely fundamental to the analysis of representations:

- We have complete reducibility: if $W \subset V$ is a subrepresentation, it has a *complementary* G-invariant subspace.
- For all $g \in G$, the corresponding endomorphism of V is *diagonalizable* because it has finite order: any finite-order automorphism of a complex vector space is diagonalizable. But we're going to miss this when it's gone.

Neither of these is true for representations of Lie groups or Lie algebras in general. That's going to hurt us, but let me talk about what is true. Here is what is true for representations of Lie algebras (or Lie groups).

- **Reducibility fails.** A natural example: look at the Lie algebra of endomorphisms of a 2-dimensional vector space that look like

  $$\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$

  which, for a basis $\{e_1, e_2\}$, send $e_1 \mapsto 0$ or $e_2 \mapsto te_1$. The subspace spanned by e_1 is invariant and has no complement. The corresponding group under exponentiation is the group of *shears*

  $$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

  which preserve the x-axis but which shift lines parallel to the x-axis by varying tilts.

  The **good news** is that reducibility holds for a *semisimple* Lie algebra. This is part of the payoff for restricting our attention to semisimple Lie algebras. There's a discussion in the proof of how you prove it. There is a purely algebraic proof going back to the basic definitions, which is kind of a slog and I'm not going to do it. There is another (and the original) proof, which is an interesting idea called the **unitary trick.** I'm just going to say a couple of words about this and then move on.

  How do we prove the complete reducibility of the finite group representation theory? Introduce a G-invariant hermitian inner product and take orthogonal complements. The orthogonal complement of a G-invariant subspace is still G-invariant. How do you know that there's a G-invariant hermitian inner product? Take *any* hermitian inner product and average over the group.

  We can't do that with a Lie group, and generally there won't be invariant inner products. It is true if we have a **compact group.** If we want to average a family of hermitian inner products over a compact group, we can do this: we have to integrate the translates of the inner product under the group action. We can't do that with an arbitrary group as the volume will be infinite. In other words, complete reducibility holds for **compact Lie groups.** If we're trying to analyze representations of $SL_2(\mathbb{C})$, that doesn't help us.

But here's the trick: representations of a group like $SL_2(\mathbb{C})$ are classified by their representations of their Lie algebra. A given complex Lie algebra may be the complexification of a lot of different real Lie algebras (there may be different ones). For instance, for $\mathfrak{sl}_2(\mathbb{C})$, it is the complexification of $\mathfrak{sl}_2(\mathbb{R})$, but it is *also* the complexification of $\mathfrak{su}_2$. Now $\mathfrak{su}_2$ is the Lie algebra of a compact group. If you're looking at representations of $\mathfrak{sl}_2(\mathbb{C})$, you can work with $\mathfrak{su}_2$, and then for the compact group SU_2. So if I know reducibility for SU_2 (which I do because it's compact), then I can deduce it for $SL_2(\mathbb{C})$ going via the Lie algebra. The argument is written out in somewhat excruciating detail in the book, but what makes this work is that if I have a **complex semisimple Lie algebra,** then it has a **real form** which is the Lie algebra of a **compact Lie group.** This was Weyl's original proof of reducibility.

- **Diagonalizability also fails in general.** For example, if I have the Lie algebra $\mathbb{R}$, that acts by multiplication by scalars on a one-dimensional vector space, so everything is diagonalizable. But in the previous representation via $\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$, things are not diagonalizable. I could take the same Lie algebra and have it act on a vector space and not have the elements be diagonalizable.

  Recall that if I have any endomorphism $A \in \mathrm{End}(V)$, then I can write it as a sum

  $$A = A_{ss} + A_n$$

  of a diagonalizable and a nilpotent part. You can say more: this is the content of the theorem on the Jordan canonical form. If I choose a basis for which A is in Jordan canonical form, I can take the diagonal and off-diagonal part of A. In fact, the two parts commute with each other, and A_{ss} and A_n are expressible as polynomials in A. Given an endomorphism of a vector space, we can thus talk about its diagonalizable and nilpotent parts.

  Now, in a Lie algebra $\mathfrak{g}$, the same element can act diagonalizably in one representation, nilpotently in another representation, and in neither way in a third. Moreover, the nilpotent and semisimple parts might not be in the image of the representation. But there is a remarkable fact that holds for semisimple Lie algebras. I can't tell you that everything acts diagonalizably or nilpotently, but I can tell you something I won't prove now:

  9.2 Theorem. *If $\mathfrak{g}$ is semisimple, then every $X \in \mathfrak{g}$ has a decomposition*

  $$X = X_{ss} + X_n \in \mathfrak{g} \tag{36}$$

  with $[X_{ss}, X_n] = 0$, such that under any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$,

  1. *$\rho(X)_{ss} = \rho(X_{ss})$.*
  2. *$\rho(X)_n = \rho(X_n)$.*

In particular, the decomposition into semisimple and nilpotent parts is independent of the representation.

The theorem also states that if $\mathfrak{g}$ acts on some vector space V, and take the endomorphism of V induced by some $X \in \mathfrak{g}$, then the nilpotent and semisimple parts of that endomorphism are also in the image of the representation.

Remark. The word "semisimple" is overused here. I would really like to call X_{ss} the "diagonalizable part of X." The two uses of "semisimple" on the blackboard are not entirely consistent.

Remark. Justin Campbell points out that there is a nice and comprehensible homological proof of complete reducibility in Weibel's book on homological algebra. The **unitary trick** relies on the fact that a complex semisimple Lie algebra arises as the complexification of a real Lie algebra that comes from a compact group, which we'll prove at the end of the semester.

We're trying to think about Lie groups and Lie algebras in conjunction with one another. It is possible to derive all these results without the use of Lie algebras (or without the use of Lie groups). There is a book by J. F. Adams that does not acknowledge the existence of a Lie algebra, and derives all the results about representations by using the unitary trick a lot. But at this point, it's almost a perverse exercise. You can also study Lie algebras without acknowledging that Lie groups exist and derive everything we're going to derive.

§3 An example: $\mathfrak{sl}_2$

Let's get started. We should *find* a simple Lie algebra.[3] What does that mean? A Lie algebra $\mathfrak{g}$ is a vector space together with a Lie bracket $[\cdot,\cdot] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity. You can try to order them in terms of complexity either by looking at $\dim \mathfrak{g}$ or by looking at the rank of the commutator. If the rank is zero, you have an abelian subalgebra, and if the rank is maximal, then every element is a commutator.

Let's ask a question: **What is the smallest-dimensional simple Lie algebra?**

This can't happen in dimension one because $\dim \bigwedge^2 \mathfrak{g} < \dim \mathfrak{g}$ if $\dim \mathfrak{g} < 3$. Let's say $\dim \mathfrak{g} = 3$, so we must have an *isomorphism*

$$[\cdot,\cdot] : \bigwedge^2 \mathfrak{g} \simeq \mathfrak{g}.$$

For all $h \in \mathfrak{g}$, then the map

$$\mathrm{Ad}(H) : \mathfrak{g} \to \mathfrak{g}$$

must have rank two and its kernel must be the multiples of H itself.

The next thing (and this won't take too long), is to make the claim that *there exists an $H \in \mathfrak{g}$ such that* $\mathrm{ad}(H)$ *has an eigenvector with nonzero eigenvalue.*

[3] We're not counting one-dimensional algebras. Recall that a simple Lie algebra is a nonabelian Lie algebra with no nontrivial ideals.

Proof. Start with any $X \in \mathfrak{g} \setminus \{0\}$. Look at $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$. Is this nilpotent or not? To say that it has an eigenvector with nonzero eigenvalue is to say that it's not nilpotent. So if it's not nilpotent, we can take $H = X$. Suppose $\mathrm{ad}(X)$ is nilpotent.

If $\mathrm{ad}(X)$ (which has rank two) is nilpotent, then the kernels of successive powers are growing. This means that there exists $Y \in \ker \mathrm{ad}(X)^2 \setminus \ker \mathrm{ad}(X)$. In other words,

$$\mathrm{Ad}(X)Y \in \ker \mathrm{ad}(X) = \mathbb{C}\{X\},$$

so that

$$[X, Y] = \alpha X, \quad \alpha \in \mathbb{C},$$

and since $Y \notin \ker \mathrm{ad}(X)$, we find that $\alpha \neq 0$. Hence, we can take $H = Y$. ▲

Start with such an H, and let X be an eigenvector for $\mathrm{ad}(H)$. We can write

$$[H, X] = \alpha X. \tag{37}$$

We can scale H to make α anything we want. Think about it: $\mathrm{ad}(H)$ is a map $\mathfrak{g} \to \mathfrak{g}$ which has a one-dimensional kernel. It has one eigenvector with eigenvalue α. Because it's a commutator (*everything* in $\mathfrak{g}$ is a commutator), it has *trace zero.* So it must have an eigenvalue $-\alpha$ as well and we let Y by an eigenvector for $-\alpha$,

$$[H, Y] = -\alpha Y. \tag{38}$$

Now we're almost there. We have a basis $\{H, X, Y\}$ (this is necessarily a basis), and we know two out of the three brackets. All that's left is to say something about $[X, Y]$. But that's also easy. I'm not going to go through this, because we have two minutes left, but by Jacobi,

$$[H, [X, Y]] = 0, \tag{39}$$

using (37) and (38). This shows that $[X, Y]$ is a **multiple of H.** Rescaling, we can assume that

$$[X, Y] = H. \tag{40}$$

We thus have a complete description of the Lie algebra: we have the relations:

$$[H, X] = \alpha X \tag{41}$$

$$[H, Y] = -\alpha Y \tag{42}$$

$$[X, Y] = H. \tag{43}$$

This is $\mathfrak{sl}_2(\mathbb{C})$. The three elements are such that

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{44}$$

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{45}$$

$$Y = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \tag{46}$$

and these satisfy the above relations with $\alpha = 2$. On Wednesday, we will understand all the representations of $\mathfrak{sl}_2(\mathbb{C})$.

Lecture 10
2/20

§1 $\mathfrak{sl}_2(\mathbb{C})$

Today, we're going to begin the heart of the course. To wrap up what we've done so far:

- Largely, we've been setting the page. We're trying to frame the problem in a way that focuses on the core issues.
- We've gone from the initial definition of a **Lie group** (which is fairly natural) to the simpler definition of a **Lie algebra**, which still encodes a lot of information about the group (i.e., it gets the group up to isogeny).
- We can **complexify** the Lie algebra to get a complex Lie algebra. A complex Lie algebra is actually a simpler object than a real Lie algebra. (However, for a given complex Lie algebra, there may be different real Lie algebras that realize it.)
- The last step was to focus, among complex Lie algebras, on the **simple ones:** the wonderfully named **simple complex Lie algebras.** This is going to be our focus for the next month or so.

For those of you who are doing the homework, let's start fresh. If you have homeworks that are still due, I would suggest that you forget about them and do the current ones. Homework 4 (the latest) should not be too onerous and should be a chance for those of you who are behind.

Today, we are going to talk about the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. One thing which I didn't say explicitly, but which is in the last section of the text in ch. 10: for a given complex Lie algebra there may be several real Lie algebras with that complexification.

10.1 Example. There are exactly two real Lie algebras that complexify to $\mathfrak{sl}_2(\mathbb{C})$: namely, the obvious $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2$ (the Lie algebra of the special unitary group).

Every time we shift in this scheme of things, we simplify in the sense that many objects that reduce to one object. There's a chart in the textbook of Lie groups whose complexified Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$.

Recall:

10.2 Definition. $\mathfrak{sl}_2(\mathbb{C})$ is the vector space of traceless 2×2-matrices.

Everything in the next month is going to be over $\mathbb{C}$. Last time, we saw that there was a basis

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{47}$$

The adjoint action was easy to describe; we had

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{48}$$

§2 Irreducible representations

Let V be an irreducible representation of $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$. Let's start with an idea that will be useful in other contexts. We want to invoke the preservation of Jordan decomposition. That states that under *any* representation of a simple Lie algebra (which this), we can decompose an endomorphism coming from the representation into semisimple and nilpotent parts; those parts are in the image of the representation.

In particular, since H is diagonalizable in the faithful representation on $\mathbb{C}^2$, it acts faithfully in *any* representation. We get a decomposition of V into eigenspaces of H, i.e.

$$V = \bigoplus_\alpha V_\alpha,$$

where $V_\alpha = \{v \in V : Hv = \alpha v\}$. A priori, the α that appear can be any complex numbers. If you want to draw a picture, we just have a bunch of complex numbers, with a subspace attached to each of them. Now, of course, we ask what X, Y do to each of these subspaces.

Question. What does X do to these subspaces V_α?

Start with a given $v \in V_\alpha$. Where does X send v? A priori, the decomposition of $V = \bigoplus V_\alpha$ is obtained by looking at the action of H. That means, we need to figure out what H does to Xv to figure out where Xv lands. Is Xv again an eigenvector of H? If so, with what eigenvalue?

So we need to look at $H(Xv)$. Here is the same calculation that we've seen before:

$$H(Xv) = X(Hv) + [H, X]v = X(\alpha v) + 2Xv = (\alpha + 2)Xv, \tag{49}$$

because $v \in V_\alpha$. We find that if $v \in V_\alpha$ (i.e., v is an eigenvector with eigenvalue α), then Xv will again be an eigenvector, with eigenvalue $\alpha+2$. That answers the question.

10.3 Proposition. *X carries the subspace V_α into $V_{\alpha+2}$. Similarly, Y carries V_α into $V_{\alpha-2}$.*

We find that X, Y shift the eigenvalues over by two, horizontally in the complex plane. That's the crucial thing. By the way, what if $V_{\alpha+2} = 0$? That just says that $X|_{V_\alpha} \equiv 0$.

In particular, for any α, if we consider the complex numbers congruent to α mod $2\mathbb{Z}$, and form

$$W = \bigoplus_{n \in \mathbb{Z}} V_{\alpha+2n}.$$

Then, since X, Y shift eigenvalues by two (to the left or the right), it follows that $W \subset V$ is *invariant* under $\mathfrak{sl}_2$. Hence, since we assumed irreducibility, we have $W = V$ (if $V_\alpha = 0$).

10.4 Corollary. *The eigenvalues of an irreducible representation of $\mathfrak{sl}_2$ form an unbroken string of complex numbers separated by twos.*

This a priori arbitrary configuration of eigenvalues in facts has to consist of an unbroken string of complex numbers differing from one another by twos.

Now comes the last part of the puzzle. Let's look at this string of eigenvectors. The natural thing is to start at one end of the string and apply X or Y to that. Start with an eigenspace V_α at the *end* of the string, in the right-handed sense (so $\Re(\alpha)$ is maximized). This implies that

$$Xv = 0, \quad v \in V_\alpha. \tag{50}$$

because $Xv \in V_{\alpha+2} = 0$. The key claim is:

10.5 Proposition. *Choose $v \in V_\alpha \setminus \{0\}$ where the eigenvector α is such that $\Re(\alpha)$ is maximized. The entire (irreducible) representation V has as basis the vectors $v, Yv, Y^2v, \dots,$ (that is, these vectors are linearly independent as long as they're nonzero).*

The linear independence of these vectors is clear because they are eigenvectors of H with different eigenvalues. The claim is that the nonzero vectors in this sequence form a basis. The string $\alpha, \alpha-2, \alpha-4, \dots$ of eigenvalues keeps going: Y keeps moving the eigenvalues to the left. Part of the consequence of this result is that all the eigenspaces are one-dimensional.

This one-dimensionality, by the way, is one thing — as we look at more complicated Lie algebras — which is special to $\mathfrak{sl}_2$.

Proof. The proof will tell us more than the statement itself. We've seen that $v, Yv, \dots,$ forms a basis as long as it is nonzero. Let W be the subspace spanned by $v, Yv, Y^2v, \dots$: the claim is that W is $\mathfrak{sl}_2$-invariant. Since V is irreducible, it follows $W = V$.

Why is W invariant? It is Y-invariant. Every element Y^iv is an eigenvector for H, so H acts acts on the basis of W by scalar multiplication. Finally, we have to show that $XW \subset W$.

- To start with, $Xv = 0$: we started with a vector killed by v.
- What about $X(Yv)$? We have

  $$X(Yv) = Y(Xv) + [X, Y]v = Hv = \alpha v.$$

  So X kills v and sends Yv to αv. (A priori, we knew from the above that $X(Yv)$ would be an eigenvector for H with eigenvalue H, but we're saying more.)
- Let's do $X(Y^2v)$. We have

  $$X(Y^2v) = YXYv + [X, Y]Yv = \alpha Yv + (\alpha - 2)Yv.$$

- The same reasoning is going to work forever. X is going to shift each of these basis vectors Y^kv to a multiple of the previous one. In particular, X preserves W (which is the claim).
- More precisely, the claim (proved inductively) is that

  $$X(Y^kv) = (\alpha + (\alpha - 2) + \cdots + (\alpha - 2k + 2))\, Y^{k-1}v. \tag{51}$$

The proof is now complete.

▲

The proof gives us a bit more. At some point, $Y^k v = 0$: we're in a finite-dimensional vector space. In fact, Y is nilpotent. We can choose Y such that $Y^k v = 0, Y^{k-1} v \neq 0$. Then the coefficient

$$(\alpha + (\alpha - 2) + \cdots + (\alpha - 2k + 2)) = 0,$$

which gives us a strong constraint on α.

Anyway, the proof gives us a decomposition of V into a direct sum of *one-dimensional* eigenspaces from α on down. Since H is a commutator, the sum of all the eigenvalues that appear in this decomposition has to be zero. This means that the diagram of eigenvalues has to live on the real values, and it shows that the *maximal* eigenvalue has to be an integer.

10.6 Proposition. *The maximal eigenvalue $\alpha \in \mathbb{Z}_{\geq 0}$ and the eigenvalues are symmetric around* 0*. In other words,*

$$V = V_\alpha \oplus V_{\alpha-2} \oplus \cdots \oplus V_{-\alpha}.$$

The eigenvalues that arise form a string of integers, separated by two and symmetric about the origin.

We now have the question of existence and uniqueness. But we've understood them. We have written down a basis for a putative irreducible representation with a given highest eigenvalue and written down how X, Y, H act, and they satisfy the commutation relations.

10.7 Proposition. *There is a unique irreducible representation W_n of dimension $n+1$ of $\mathfrak{sl}_2$ with highest eigenvector n for each $n \in \mathbb{Z}_{\geq 0}$. The eigenvalues are $n, n-2, \ldots, -n$ and the eigenspaces are one-dimensional.*

As above, Y shifts everything over to the left; X shifts everything over to the right; and H multiplies things in the graded pieces by suitable scalars.

10.8 Example. If $n = 0$, we get the trivial 1-dimensional representation (everything acts by 0).

10.9 Example. If $n = 1$, we get the **standard** representation on $\mathbb{C}^2$. If we realize $\mathfrak{sl}_2$ as a space of 2×2-matrices, it acts on a 2-dimensional vector space. Then H is the diagonal matrix with eigenvalues ± 1, X moves everything to the right, and Y moves everything to the left.

10.10 Example. If $n = 2$, we get the **adjoint representation**. This follows from the commutation relations $[H, X] = 2X, [H, H] = 0, [H, Y] = -2Y$: the eigenvalues of H acting on $\mathfrak{sl}_2$ are X, H, Y, and X has the highest eigenvalue.

In some sense, we have already constructed all the irreducible representations of $\mathfrak{sl}_2$, but there is a more intrinsic way of doing it.

10.11 Example. If $V \simeq \mathbb{C}^2$ is the standard representation, then the irreducible representation V_n is $\mathrm{Sym}^n V$. For example, let's do Sym^2.

If $V = \langle a, b \rangle$ has a basis a, b, then $\mathrm{Sym}^2 V$ has a basis a^2, ab, b^2. Let's say the representation is the standard one:

$$Ha = a, \quad Hb = -b \tag{52}$$

$$Xa = 0, \quad Xb = a \tag{53}$$

$$Ya = b, \quad Yb = 0. \tag{54}$$

Then in $\mathrm{Sym}^2 V$,

$$H(ab) = (Ha)b + a(Hb) = 0,$$

and likewise

$$H(a^2) = 2aH(a) = 2a^2$$

and

$$H(b^2) = -2b^2.$$

This is the description of the adjoint representation.

Here's a fact which is true for $\mathfrak{sl}_2$ but not in general.

10.12 Corollary. *All irreducible representations are isomorphic to their dual.*

In fact, there is a unique irreducible representation of each dimension! The dualization flips the eigenvalues around zero, which preserves the string of eigenvalues.

10.13 Corollary. *If V is any representation of $\mathfrak{sl}_2$, then $V \simeq V^*$.*

Proof. It's true for irreducibles, and any representation is a direct sum of irreducibles. ▲

This corollary won't be true for general simple Lie algebras.

10.14 Corollary. *Every irreducible representation has either 0 or 1 as an eigenspace. Therefore, V is irreducible if and only if*

$$\dim V_0 + \dim V_1 = 1,$$

and in general, $\dim V_0 + \dim V_1$ is the number of irreducible factors.

Given an *arbitrary* representation of $\mathfrak{sl}_2$, if I tell you the dimensions of the eigenspaces, then that will tell you which irreducibles appear in the decomposition. To indicate a representation, we can draw a sequence of dots, and draw circles around the dots to indicate eigenspaces (with multiplicities). The multiplicities of these eigenvalues is going to form a symmetric string of positive numbers, increasing to the left of the origin and decreasing to the right of the origin. Given the diagram with the dimensions of the eigenspaces, then I can tell you what the decomposition into irreducibles is.

The promise was, if I understand representations of $\mathfrak{sl}_2$, then I could understand representations of the Lie groups. You should go back and check this. But I want to go back to the Lie algebras and play a game, basically. A sort of test of our understanding of these representations is: if I carry out a multilinear operation on these irreducible representations, can I identify the decomposition of the resulting one? Let me phrase this as a question which we're going to spend a little while on Friday.

Question. Which irreducibles appear in the representation $W_3 \otimes W_4$?

Lecture 11
2/22

§1 Recap

Let's recall where we are. I want to do two things today:

- I want to finish the discussion of representations of $\mathfrak{sl}_2$ by talking about **plethysm.**
- We're going to start on a crucial example, which is $\mathfrak{sl}_3$. In case you're worried that we are going to slog through the Lie algebras one after another, these are the two crucial cases. Once we've done them, you will have seen the paradigm that will work to analyze the representations of every simple Lie algebra.

Recall from last time:

11.1 Proposition. *The irreducible representations of $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ are exactly the symmetric powers of the standard representation $\mathbb{C}^2$.*[4]

We wrote

$$W_n = \mathrm{Sym}^n V.$$

What's more important is to recall how we arrived at this. We would start with an irreducible representation V, and we would look at the action of the diagonal element $H \in \mathfrak{sl}_2$ and decompose by eigenspaces (by the theorem on the Jordan decomposition, H will act diagonalizably on every representation). So we would get a decomposition

$$V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha, \quad V_\alpha = \{v \in V : Hv = \alpha v\}$$

and we represented these subspaces by points in the complex plane corresponding to α. We saw, however, that the eigenvalues were actually **integers** and the eigenvalues were given by

$$n, n-2, \ldots, -n$$

and each eigenspace has dimension one. The picture of the Lie algebra action on this decomposition is very simple. H carries each of these one-dimensional subspaces by eigenvalues, by assumption. Now X moves each of the eigenspaces by two to the right: this was the fundamental calculation. Given an eigenvector $v \in V_\alpha$ for H, we had $Xv = V_{\alpha+2}$. By the same token, $YV_\alpha \subset V_{\alpha-2}$.

You can see directly that this is a description of the nth symmetric power. Take the monomials in two basis elements of $\mathbb{C}^2$ and check that they have the right eigenvalues. The point is that **any irreducible representation is of this form.**

Given an irreducible representation, we know that the eigenvalues form an **unbroken string of complex numbers differing by two,** and take the eigenvector for the eigenvalue on the far right. We take a vector $v \in V$ which is **killed by** X and starting with that, generated the entire representation. So we got a pretty complete description of representations of $\mathfrak{sl}_2$.

I promised you at the end of the discussion, I was going to describe representations of the groups we started with. Let's give an example of how it plays out.

[4] This is just the realization of $\mathfrak{sl}_2$ as a subspace of 2-by-2 matrices.

11.2 Example. There are **exactly two** groups with Lie algebra $\mathfrak{sl}_2$. They are $SL_2(\mathbb{C})$ and—since $SL_2(\mathbb{C})$ is simply connected, it sits at the top of the identity—the quotient $SL_2(\mathbb{C})/\{\pm I\} = PSL_2(\mathbb{C})$. That's because the center of $SL_2(\mathbb{C})$ is ± 1. I didn't do this in class, but I think it's worthwhile if you're trying to think back to Lie groups, it's worth looking at the diagram at the end of chapter 10. The diagram describes all real Lie groups whose complexified Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$.

Now, incidentally, $PSL_2(\mathbb{C}) = SO_3(\mathbb{C})$ is also the automorphism group of a complex vector space preserving a nonzero symmetric bilinear form (and having determinant one).

The representations of $SL_2(\mathbb{C})$ are exactly the representations of $\mathfrak{sl}_2(\mathbb{C})$ (by simple connectivity), so the representations are exactly the $W_n = \mathrm{Sym}^n\mathbb{C}$. To figure out the representations of $PSL_2(\mathbb{C})$, we have to look at when the center acts trivially on the representations. The center acts trivially on W_n if and only if n is even. This is sort of the picture — representations will be parametrized by some lattice in a cone and representations of the adjoint form will be parametrized by some sublattice.

§2 Plethysm

In representation theory, we take the representations we know and apply constructions from multilinear algebra: symmetric powers, tensor products, and so forth. We'd like to describe how they decompose into irreducibles. It's all over the homework this week. In this case, since we know the representations in terms of eigenspaces and eigenvalues, we can answer the questions completely mechanically. There are some interesting things we can see along the way, for instance the isomorphism asserted earlier

$$PSL_2(\mathbb{C}) \simeq SO_3(\mathbb{C}).$$

11.3 Example. Start with the **representation** $V = \mathbb{C}^2$ and consider $V \otimes V$. Now V has eigenvalues (of H) consisting of ± 1. The eigenvalues on the tensor product are going to be the four pairwise sums and so the eigenvalues on $V \otimes V$ are $-2, 0, 0, 2$. In terms of the diagrams we've been drawing, we'd have dots at $-2, 0, 2$ and with 0 circled twice to indicate the double eigenvalue. It's pretty clear that there's only one way to decompose this into irreducibles. We get

$$V \otimes V \simeq \mathrm{Sym}^2(V) \oplus \mathbb{C}$$

where $\mathbb{C}$ is the trivial representation. Of course, you knew this already: the tensor square decomposes into Sym^2 and $\bigwedge^2$. And since $\mathfrak{sl}_2$ acts tracelessly, the wedge square of the standard representation is trivial.

11.4 Example. Let's take $\mathrm{Sym}^2(V)\otimes\mathrm{Sym}^2(V)$. The eigenvalues of this tensor product will be the pairwise sums of the eigenvalues of $\mathrm{Sym}^2 V$, which are $\{-2, 0, 2\}$. If I take all pairwise sums of these, I'm going to get

$$\{-4, -2, -2, 0, 0, 0, 2, 2, 4\}.$$

This tells us what the decomposition into irreducibles. Namely,

$$\mathrm{Sym}^2(V) \otimes \mathrm{Sym}^2(V) \simeq \mathrm{Sym}^4(V) \oplus \mathrm{Sym}^2(V) \oplus \mathbb{C}.$$

If you wanted to know more than this abstractly, if you wanted to actually see the decomposition, then you also know what to do. Start with an eigenvector for $\mathrm{Sym}^2(V)\otimes \mathrm{Sym}^2(V)$ which is extremal: so X kills it. The argument we discussed last time is that the representation generated by this extremal eigenvector gives the copy of Sym^4. Then we peel that off and keep going. So, we can do this both abstractly and concretely.

11.5 Example. I said a moment ago that tensor square is naturally a direct sum of the exterior square and its symmetric square, i.e.

$$\mathrm{Sym}^2(V) \otimes \mathrm{Sym}^2(V) = \mathrm{Sym}^2(\mathrm{Sym}^2(V)) \oplus \bigwedge^2(\mathrm{Sym}^2(V)). \tag{55}$$

If I look at the eigenvalues on $\mathrm{Sym}^2(\mathrm{Sym}^2(V))$, the eigenvalues are the *pairwise unordered* sums of $\mathrm{Sym}^2(V)$, i.e. the pairwise unordered sums of $\{-2, 0, 2\}$. So we easily get

$$\mathrm{Sym}^2(\mathrm{Sym}^2(V)) \simeq \mathrm{Sym}^4(V) \oplus \mathbb{C}.$$

Similarly, we get

$$\bigwedge^2 \mathrm{Sym}^2(V) \simeq \mathrm{Sym}^2(V),$$

which is not surprising, because $\bigwedge^2$ of a three-dimensional thing is isomorphic to its dual, and representations of $\mathfrak{sl}_2$ are self-dual.

Let's look at homogeneous polynomials of degree two whose arguments are homogeneous polynomials of degree 2. I could substitute and get a polynomial of degree four: that's the factor of $\mathrm{Sym}^4(V)$ in $\mathrm{Sym}^2\mathrm{Sym}^2(V)$. But the trivial factor that occurs is interesting. There's an element of $\mathrm{Sym}^2\mathrm{Sym}^2(V)$ which is fixed under the action of the group $SL_2(\mathbb{C})$. That's a symmetric bilinear form preserved under the action of the group.

If I look at the action of $SL_2(\mathbb{C})$ on $\mathrm{Sym}^2(V) \otimes \mathrm{Sym}^2(V)$, it acts preserving a symmetric bilinear form. That induces the isomorphism

$$\mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{so}_3(\mathbb{C}).$$

There's a long discussion in the book about geometric plethysm, which is near to my heart. I would urge you to take a look at the discussion in the book, which is old-style classical algebraic geometry (like Veronese surfaces, rational normal curves, and so on).

§3 $\mathfrak{sl}_3$

The Lie algebra $\mathfrak{sl}_2$ is central because it is the simplest example of a simple Lie algebra. Aside from that, one of the ways we're going to describe more complicated Lie algebras like $\mathfrak{sl}_3$ and so forth is as a span of copies of $\mathfrak{sl}_2$.

But there's a lot that you don't see in $\mathfrak{sl}_2$: it's in some sense too simple. On the other hand, when we get to $\mathfrak{sl}_3$, we will see all the wrinkles that will appear in general. Once we've done $\mathfrak{sl}_3$, there will be **no more surprises.** We will see the pattern in general.

When we did $\mathfrak{sl}_2$, we wrote down three elements which formed a basis. We just went on from there. If I want to extend a similar analysis for $\mathfrak{sl}_2$, I have to ask what was special about H, X, Y.

- The first answer is that H is **diagonalizable**, in any representation by the preservation of Jordan canonical form.
- What made X, Y work? The basic idea is that if I have an eigenvector $v \in V$ such that $Hv = \alpha v$ (for some representation V of $\mathfrak{sl}_2$), then

  $$H(Xv) = X(Hv) + [H, X]v = \alpha Xv + 2Xv = (\alpha + 2)Xv, \tag{56}$$

  so we get that Xv is an eigenvector of H of eigenvalue $\alpha + 2$. This calculation worked because $[H, X]$ was proportional for X. In other words, X **is an eigenvector for the adjoint action of H.**

That's going to be our plan. We are going introduce the analog of H for $\mathfrak{sl}_3$. We're going to decompose the rest of the Lie algebra looking at the action of H. We'll start carrying this out today, and should finish on Monday.

What plays the role of H? This is the big change. In $\mathfrak{sl}_3$, we don't have just one diagonalizable element up to scalars: we have a two-dimensional space. **That is the fundamental change.** The role of the single element H in $\mathfrak{sl}_2$ is going to be played by a vector space of matrices (again, the traceless diagonal ones).

11.6 Definition. We write $\mathfrak{h} \subset \mathfrak{sl}_3$ for the subspace of diagonal matrices,

$$\mathfrak{h} = \left\{ \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix} \right\}.$$

Let's make one crucial observation now. If I have *commuting* diagonalizable endomorphisms of a vector space, then they are *simultaneously diagonalizable.* (This is a special case of the calculation in (56).) If I have a diagonalizable endomorphism H of a vector space V and I have another endomorphism that commutes with H, then it preserves the decomposition of V into eigenvectors. *That's* what I want to apply here.

Remember I said about a week ago—we wanted to generalize the notion of eigenvector and eigenvalue. Given a vector space of endomorphisms, an **eigenvector** for that vector space of endomorphisms is a vector which is carried into a multiple of itself under any element of that vector space; what multiple it is will be a linear functional on that vector space.

11.7 Definition. Let V be a representation of $\mathfrak{sl}_3$. Given $\alpha \in \mathfrak{h}^*$, we set

$$V_\alpha = \{v \in V : Hv = \alpha(H)v, \quad \text{for all } H \in \mathfrak{h}\}.$$

We call this the **eigenspace** for the action of $\mathfrak{h}$ with **eigenvalue** α.

The main thing here is that eigenvalues are no longer numbers; they're linear functionals on $\mathfrak{h}$. Just by way of notation, since we're going to be working with $\mathfrak{h}^*$, we should introduce some generators for it.

11.8 Definition. We can imbed $\mathfrak{h} \subset \mathbb{C}^3$: it is the subspace of triples of complex numbers with sum zero. Therefore, $\mathfrak{h}^*$ is the complex span of three linear functions L_1, L_2, L_3 with the relation $L_1 + L_2 + L_3 = 0$. It comes to us naturally as a quotient.

For any representation V, I can write

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha,$$

and this is the eigenspace decomposition for the action of $\mathfrak{h}$.

Now, that is, for the time being at least, what takes the role of H and the eigenspace decomposition. The next step is to figure out what plays the role of X, Y. Again, the observation is that X, Y were eigenvectors for the action of H. So, consider the *adjoint* action of $\mathfrak{h}$ on all of $\mathfrak{sl}_3(\mathbb{C})$. In this way, I get a decomposition as a vector space

$$\mathfrak{sl}_3(\mathbb{C}) = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha.$$

We're going to choose as our analogs of X, Y, elements of these eigenspaces.

11.9 Example. One eigenspace I know: $\mathfrak{h}$, since it's abelian, acts by zero on itself. So I can write the above decomposition in a more refined form

$$\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha.$$

How do I go about finding eigenvectors for this action? It's pretty easy. The eigenvectors for $\mathfrak{h}$ acting on $\mathfrak{sl}_3$ itself are what you think. Let E_{ij} be the matrix with (i, j)-entry 1 and all other entries zero. There are six of these for $i \neq j$. The claim is that the E_{ij} span exactly the eigenspaces that appear here. That makes sense, by counting dimensions. It's easy to figure out the eigenvalues. The eigenvalue of $\mathfrak{h}$ on E_{ij} is $L_i - L_j$. Those are the six eigenvectors in the Lie algebra that are going to play the role of X, Y.

How do we draw these? When the eigenvalues were scalars, we drew them as points on a real line. Now that they belong to a two-dimensional vector space, we draw them as points in the plane.

We'd like to retain some symmetry here and not make too many choices: when we have a vector space $\mathbb{C}\langle L_1, L_2, L_3 \rangle / (L_1 + L_2 + L_3)$, the natural thing is to draw it on a hexagonal lattice. We can put L_1, L_2, L_3 as the third roots of unity in the plane.

We now have:

- A decomposition of $\mathfrak{sl}_3$ into a two-dimensional eigenspace $\mathfrak{h}$ (the dot at 0) and one-dimensional subspaces (of $\mathfrak{h}$) corresponding to a hexagon in the plane.

- The fundamental calculation goes through as before. An element $X \in \mathfrak{g}_\alpha$ will carry $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{\alpha+\beta}$ under the adjoint action. The adjoint action by an eigenvector just shifts the eigenspaces of $\mathfrak{h}$. That is,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}.$$

- This doesn't quite describe the Lie algebra, though. But the structure turns out to be implicit in the above diagram. This is going to be the fundamental diagram we're going to be working with.

Lecture 12
2/25

§1 Recap on $\mathfrak{sl}_3$

Today, we're going to see what representations of $\mathfrak{sl}_3$ look like. We won't finish the story today, because some aspects of the story that were immediate for $\mathfrak{sl}_2$ that were immediate—existence and uniqueness—will take a little longer. Let's recall where we were.

To describe the structure of $\mathfrak{sl}_3$, we introduce the *subalgebra* $\mathfrak{h}$ consisting of the diagonal (trace-free) matrices, so

$$\mathfrak{h} = \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix}, \quad a_1 + a_2 + a_3 = 0, \tag{57}$$

and we observe that $\dim_{\mathbb{C}} \mathfrak{h} = 2$. The action will also take place in the dual vector space $\mathfrak{h}^*$, which is spanned by the linear functions L_1, L_2, L_3 which pick out a_1, a_2, a_3 in a matrix. We have $L_1 + L_2 + L_3 = 0 \in \mathfrak{h}^*$ and this is the only relation. So

$$\mathfrak{h}^* = \mathbb{C}\langle L_1, L_2, L_3\rangle / (L_1 + L_2 + L_3).$$

We're going to take the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$, and *decompose* it by the action of $\mathfrak{h}$. Because $\mathfrak{h}$ acts diagonally in the standard representation, it acts diagonally in any representation, e.g. the adjoint one. The action of $\mathfrak{h}$ on $\mathfrak{g}$ gives an eigenspace decomposition.

The zero eigenspace is $\mathfrak{h} \subset \mathfrak{g}$ itself. Then we have other eigenspaces $\mathfrak{g}_\alpha, \alpha \in \mathfrak{h}^* \setminus \{0\}$ where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{h}\}, \tag{58}$$

and we have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha. \tag{59}$$

In fact, α runs over the pairs $L_i - L_j$, for $1 \leq i \neq j \leq 3$, and

$$\mathfrak{g}_{L_i - L_j} = \mathbb{C}E_{ij}, \tag{60}$$

where E_{ij} is the matrix with 1 in the (i, j)th entry and zero elsewhere. This sort of decomposition is going to work for any simple Lie algebra.

The idea, again, is to find this abelian subalgebra, look at how it acts on the whole Lie algebra by the adjoint action, and break it up into eigenspaces. This describes the action of $\mathfrak{h}$ on $\mathfrak{g}$. We want to know the action of $\mathfrak{g}$ on $\mathfrak{g}$, though—the structure of the Lie algebra.

However, we can say how the Lie bracket acts with respect to the decomposition:

12.1 Proposition. *If* $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$, *then* $\mathrm{ad}(X)(Y) = [X, Y] \in \mathfrak{g}_{\alpha+\beta}$, *i.e.*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}. \tag{61}$$

Proof. We need to see how $\mathfrak{h}$ acts on the commutator $[X,Y]$; we have

$$\begin{aligned}\mathrm{ad}(H)(\mathrm{ad}(X)Y) &= \mathrm{ad}(X)(\mathrm{ad}(H)Y) + \mathrm{ad}[X,Y](Y)\\ &= \mathrm{ad}(X)(\beta(H)Y) + \alpha(H)\mathrm{Ad}(X)(Y)\\ &= (\alpha+\beta)(H)[X,Y].\end{aligned}$$

This is the standard calculation. ▲

(There is a picture that can be drawn by placing the L_i on a hexagonal lattice... To be added.)

Notice that if I start with a one-dimensional eigenspace $\mathfrak{g}_\alpha$, and look at $\mathfrak{g}_{-\alpha}$ (the opposite eigenspace), and then I throw in their commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$, then we get a subalgebra

$$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

It is a subalgebra since $\mathfrak{h}$ acts diagonally on $\mathfrak{g}_{\pm\alpha}$.

In fact, what we have in here is a *copy of* $\mathfrak{sl}_2$. For every pair of opposite eigenvalues, we get a copy of $\mathfrak{sl}_2$. One of the techniques we're going to be using is to restrict to these subalgebras.

12.2 Definition. We write $\mathfrak{s}_\alpha$ for the subalgebra $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, which is isomorphic to $\mathfrak{sl}_2$.

Concretely, taking $\alpha = L_i - L_j$, we can write

$$\mathfrak{s}_\alpha = \mathbb{C}E_{ij} \oplus \mathbb{C}E_{ji} \oplus \mathbb{C}H_{ij} \tag{62}$$

where H_{ij} is the diagonal matrix with 1 in the ith place and -1 in the jth place. This is more or less the structure of $\mathfrak{sl}_3$, although I haven't told you how to take brackets fully. But, we have all the relevant information we need to analyze representations.

§2 Irreducible representations of $\mathfrak{sl}_3$

Let V be an irreducible representation of $\mathfrak{g} = \mathfrak{sl}_3$. Let's look at the decomposition of V into eigenspaces of $\mathfrak{h}$; we write

$$V = \bigoplus_\alpha V_\alpha, \quad \alpha \in \mathfrak{h}^*, \tag{63}$$

where

$$V_\alpha = \{v \in V : Hv = \alpha(H)v, \quad \text{for all } H \in \mathfrak{h}\}. \tag{64}$$

The crucial observation is again the same:

12.3 Proposition. *If $v \in V_\beta, X \in \mathfrak{g}_\alpha$, then*

$$Xv \in V_{\alpha+\beta}. \tag{65}$$

Proof. For any $H \in \mathfrak{h}$, we have

$$H(Xv) = X(Hv) + [H,X]v = X(\beta(H)v) + \alpha(H)Xv = (\alpha+\beta)(H).(Xv).$$

So Xv is an eigenvector for $\mathfrak{h}$ with eigenvalue $\alpha+\beta$. ▲

In other words, $\mathfrak{g}_\alpha$ **carries the eigenspace** V_β **into** $V_{\alpha+\beta}$**.** This is very much like the picture for $\mathfrak{sl}_2$. If I have an irreducible representation, all the eigenvalues that appear all differ from each other by linear combinations of the eigenvalues $L_i - L_j$ that appear in the decomposition of $\mathfrak{g}$ into $\mathfrak{g}_\alpha$. In other words, $\{\alpha : V_\alpha \neq 0\}$ **is contained in a translate of the hexagonal lattice generated by the** $L_i - L_j$**.** If I have any irreducible representation and take its corresponding eigenvalues, they will all differ from each other by integral linear combinations of the $L_i - L_j$. Otherwise, we would be able to find a proper subrepresentation: given an irreducible representation, I have to be able to get from any eigenspace to any other eigenspace.

At this point, we need some language. Just because it's getting a bit complicated, here's the language we will use:

12.4 Definition.

- For an arbitrary representation V, the $\alpha \in \mathfrak{h}^*$ such that $V_\alpha \neq 0$ are called the **weights.**
- The corresponding subspaces V_α are called the **weight spaces**.
- A vector in V_α is called a **weight vector** with weight α.

The weights of a given representation are contained in a translate of this lattice. We should distinguish special weights, which are the weights of the adjoint representation. These are the weights $\{L_i - L_j\}$ for $\mathfrak{sl}_3$.

12.5 Definition. The **roots**, denoted R, are the weights of the adjoint representation. Again, we can talk about **root spaces** and **root vectors**.

The lattice spanned by the roots is called the **root lattice.** We will denote it by Λ_R.

Now we can get to work. This assertion amounts to the analog for $\mathfrak{sl}_3$ of the assertion we made for $\mathfrak{sl}_2$: the eigenvalues for $H \in \mathfrak{sl}_2$ formed an unbroken string of scalars differing in succession by twos (for an irreducible representation). We'll come to the unbroken chain assertion in time, but we have seen that the eigenvalues of an irreducible representation of $\mathfrak{sl}_3$ live in a translate of the root lattice.

What did we do next for $\mathfrak{sl}_2$? We started with an extreme eigenvalue. For $\mathfrak{sl}_2$, we said that the weights form a finite string of numbers. We said, let's go to one *end* of that string. We went to a vector with maximal eigenvalue for H, and we saw that it was killed by X. Then we saw that you could get the whole representation by repeatedly applying Y. That gave us everything we need. The key was to start with the far right (or far left) of the representation. That raises a question: what are we going to do in this case? We're not on a one-dimensional vector space anymore and the eigenvectors now form a string differing by elements of a lattice.

We're going to just ignore this, and go to what we call an *extreme* eigenvalue, and *extreme* with respect to a seemingly random linear functional. The procedure involves an arbitrary choice. Going back to our representation V, we're going to choose a linear functional on $\mathfrak{h}^*$ which we're going to take as an arbitrary measure of size.

Procedure. Choose a linear functional ℓ on $\mathfrak{h}^*$ which is irrational with respect to Λ_R. For a given V, let α_0 be the weight of V with maximal real part on ℓ.

In other words, we're looking at the diagram, choosing an irrational direction, and saying "that way is up." Then we're picking the highest eigenvalue with respect to that notion of "up." Given α_0 extremal as above, choose $v \in V_{\alpha_0}$. What we have now is a vector v such that

$$Hv = \alpha_0(H)v, \quad \forall H \in \mathfrak{h},$$

and it's killed by half of the root space.

Let's actually go ahead and simply make a choice.

12.6 Definition. We set $\ell(aL_1 + bL_2 + cL_3) = a_1a + a_2b + a_3c$ where $a_1 > a_2 > a_3$ and $a_1 + a_2 + a_3 = 0$.

Then we get:

12.7 Proposition. *If $X \in \mathfrak{g}_\alpha$ such that $\ell(\alpha) > 0$, then $XV_{\alpha_0} = 0$.*

For example, that means

$$\mathfrak{g}_{L_1-L_2}v = \mathfrak{g}_{L_1-L_3}v = \mathfrak{g}_{L_2-L_3}v = 0.$$

In effect, we've separated the roots into *two halves:* the **positive roots** and the **negative roots** (depending on what ℓ does). For an arbitrary representation, go to the eigenvalue that's furthest out in terms of this linear functional. If I apply any of these *positive* root vectors, I get zero since we started at an extremal eigenspace. So, we can find an eigenvector for $\mathfrak{h}$ that is killed by half of the root vectors (by the positive root vectors).

(The choice of ℓ was in some sense not really necessary: the really important point was to decide on the positive and negative roots.)

12.8 Proposition. *Given such a vector $v \in V_{\alpha_0}$, then the images of that one vector under successive applications of the negative root spaces (in the example, $E_{2,1}, E_{3,2}, E_{3,1}$) span V.*

If I go out to my highest (or extremal) weight vector, then if I go out in the direction of any of the positive root spaces, there's nothing there and I get zero. Now I just want to do the obvious thing and start applying the negative root spaces and see what I generate.

Proof. We haven't proved this result yet. We have to show that the images of v under successive applications of the negative root spaces is a subrepresentation of $\mathfrak{g}$. We'll do this on Wednesday. ▲

Let's write down some consequences.

- All the weights in V are obtained from α_0 after adding *negative* roots (or $\mathbb{Z}_{\geq 0}$-combinations of negative roots). In particular, all the weights lie inside **a third of a plane** obtained by starting at v_0 and applying $E_{2,1}$ and $E_{3,2}$. That's the counterpart for $\mathfrak{sl}_3$ for the claim in $\mathfrak{sl}_2$: we start with a highest weight vector and then keep applying Y's to it.

- If I look at the *edges* of this third of a plane, e.g. by taking α_0 and adding $L_2 - L_1$ successively, then all the eigenspaces have dimension one. There's only way to get from α_0 to one of these eigenspaces along these lines. It's not true that the weight spaces interior to this one-third of a plane are necessarily one dimensional. There might be multiple ways to get from α_0 to another weight space. This is something that never happens for $\mathfrak{sl}_2$.

- The next thing I want to do is the following. Remember that I said that in analyzing representations of $\mathfrak{sl}_3$, we would use these copies of $\mathfrak{sl}_2$ sitting inside. Here for example, if I just look at the weight spaces of these representation that live along this edge $(\alpha_0 - n(L_2 - L_3))$, then that forms a subrepresentation of $\mathfrak{s}_{L_2-L_3}$ (but not a subrepresentation of $\mathfrak{sl}_3$!). That means we're going see an unbroken string of one-dimensional eigenspaces which is going to have a certain symmetry.

Lecture 13
2/27

§1 Continuation of $\mathfrak{sl}_3$

I'd like to recall where we are, and today hopefully I can get to the statement of the main theorem, where we promise to describe all representations of $\mathfrak{sl}_3 = \mathfrak{sl}_3(\mathbb{C})$.

Recall:

- We have a decomposition
$$\mathfrak{sl}_3 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha \tag{66}$$
where $\mathfrak{h}$ is the diagonal matrices and there are six nonzero $\mathfrak{g}_\alpha$, each one-dimensional.

- Here α is always $L_i - L_j, 1 \leq i \neq j \leq 3$.

- The natural way to draw them (the $L_i - L_j$) is as a hexagonal figure.

- $\mathfrak{g}_{L_i - L_j} = \mathbb{C}E_{ij}$ where E_{ij} is 1 in the (i,j)th entry and zero elsewhere.

- If we look at two opposite root spaces, we can take their direct sum and their commutator and get a copy of $\mathfrak{sl}_2$ in $\mathfrak{sl}_3$. We're going to use what we know about these three copies of $\mathfrak{sl}_2$ to get information.

- We define
$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]. \tag{67}$$
If $\alpha = L_i - L_j$, the commutator part is $\mathbb{C}H_{ij}$ where H_{ij} is the diagonal matrix with a 1 on the ith diagonal entry and a -1 in the jth diagonal entry. In other words,
$$\mathfrak{s}_{L_i - L_j} = \mathbb{C}\left\langle E_{ji}, E_{ij}, H_{ij} \right\rangle. \tag{68}$$

- This is called the **root space decomposition.** The roots describe a large part of the structure of the Lie algebra. We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and we know the adjoint action of $\mathfrak{h}$ on each $\mathfrak{g}_\alpha$. A priori, we don't get all the structure of the Lie algebra, but in fact this diagram does tell you everything you need to know to reconstruct the Lie algebra. (In fact, with five exceptions, the simple Lie algebras are all ones we're familiar with; the exceptional ones are going to be constructed from their root system. Look in the book for $\mathfrak{g}_2$, which is the simplest case.)

Now we want to analyze a general irreducible representation. Let V be an irreducible representation of $\mathfrak{sl}_3$. The basic idea here is, we have the weight space decomposition

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha, \tag{69}$$

where the V_α are *eigenspaces* for the action of $\mathfrak{h}$. We'd like to know if we can go on, given this decomposition, and say how the other parts of the Lie algebra act on these subspaces V_α. Again, we saw that

$$XV_\beta \subset V_{\alpha+\beta}, \quad X \in \mathfrak{g}_\alpha. \tag{70}$$

In other words, we're going to see that the weights of an irreducible representation will differ from one another by linear combinations of these roots. That's because if I start with any eigenvector and start applying all these different operators, we have to get the whole thing (by irreducibility). So, all the weights of an irreducible representation will differ from one another by elements of the **root lattice,** which is the lattice generated by the root vectors.

In the case of $\mathfrak{sl}_2$, the next thing we did was to go to the extremal eigenvalue (either go to the far right or far left). We want to do the same thing here. We're going to take this diagram of roots and we're going to draw a random irrational line, and instead of the "farthest right" or "farthest left" we're going to look at the weight farthest from this line. That's going to be called the **highest weight** for the representation. The idea is, if I go to the weight space farthest away from this line, it's necessarily going to get killed by these three vectors $L_i - L_j, i < j$ (for a certain choice of linear functional). We're splitting the roots into **positive and negative roots.** When we got to an extremal weight, by definition, we can't go any further, and consequently that vector is going to be killed by the positive roots $L_i - L_j, i < j$.

We find:

13.1 Proposition. *Given the irreducible representation V, there exists $v \in V_{\alpha_0} \setminus \{0\}$ for some α_0 with*

$$\mathfrak{g}_\alpha v = 0, \quad \alpha \in R^+ \tag{71}$$

where R^+ denotes the set of positive roots.

13.2 Definition. Such a vector v is called a **highest weight vector.**

13.3 Proposition. *Under these circumstances, we can generate the whole representation V by taking a highest weight vector v and taking the images of v under the successive applications of the three negative root spaces.*

This means that the weights that occur are going to lie in a third of a plane, which is what you get when you take the lines through the highest weight in the directions $L_2 - L_1$ and $L_3 - L_2$. In other words, you start with v, and apply E_{21}, E_{32}, E_{31} successively, and that's going to move us along two rays starting from α_0.

(**Pictures should be added here.**)

A consequence is:

13.4 Corollary. *The subspace V_{α_0} (the highest weight subspace) is one-dimensional.*

In fact, we can say any the same thing of any weight that occurs along any of the two edges of the third of a plane. If there's only way to get from the highest weight to another weight by applying negative roots, then that means the analogous subspaces are one-dimensional. I.e., $\alpha_0 + n(L_2 - L_1), \alpha_0 + n(L_3 - L_2)$.

13.5 Corollary. *V has a unique highest weight vector (up to scalars).*

This is once we've chosen a decomposition of the roots into positive and negative roots.

Proof of Proposition 13.3. Let w_n be any word, any string of length $\leq n$, in E_{21} or E_{32}. We let W_n be the subspace of V spanned by all $w_n(v)$. WE write $W = \bigcup_n W_n$.

The claim is:

$$E_{12} W_n \subset W_{n-1}, \quad E_{23} W_n \subset W_{n-1}. \tag{72}$$

This means that W is stable under $\mathfrak{g}$. It's clearly preserved under $\mathfrak{h}$, and it's clearly carried into itself by the negative root spaces. But the claim is that it is carried into itself by the positive roots. I don't have to prove it for E_{31} since it's in the commutator of positive root spaces.

The proof is pretty much the same as it always is, and it's by induction. Observe that $W_0 = \mathbb{C}\langle v \rangle$ and $W_{-1} = 0$. In this case, E_{12}, E_{23} kill v so they go from W_0 to $W_{-1} = 0$, so that's the base case. Let's suppose $u \in W_n$, which means

$$u = E_{21} w_{n-1} v, \quad \text{or} \quad u = E_{32} w_{n-1} v.$$

Suppose we're in the first case. Then

$$E_{12} u = E_{12} E_{21} w_{n-1} v = E_{21} E_{12} w_{n-1} v + [E_{12}, E_{21}] w_{n-1} v, \tag{73}$$

where $[E_{12}, E_{21}] \in \mathfrak{h}$ and $w_{n-1} v$ is an eigenvector for $\mathfrak{h}$. It's easy to see from this that $E_{12} u \in W_{n-1}$ by the inductive hypothesis.

Another possibility is that $u = E_{32} w_{n-1} v$. Then

$$E_{12} u = E_{12} E_{32} w_{n-1} v = E_{32} E_{12} w_{n-1} v + [E_{12}, E_{32}] w_{n-1}, \tag{74}$$

where the second term is zero as $[E_{12}, E_{32}] = 0$. Using induction, we get that this is in W_{n-1}. (The other cases are similar.) ▲

This is the same as what happened for $\mathfrak{sl}_2$: we started with a highest weight vector, kept applying Y, and got the whole thing. We had to check, though, that after applying Y, the result was invariant under X.

§2 Irreducible representations

Let V be an irreducible representation. We're starting with a highest weight vector $v \in V$. Calling it a highest weight vector means that it's killed by half the root spaces. Then, applying the negative root spaces to v gets the whole representation V.

Now, we have these copies of $\mathfrak{sl}_2$ sitting inside $\mathfrak{sl}_3$. Let's start at the highest weight vector and apply E_{21}. We get a string of one-dimensional weight spaces. That is a subrepresentation of the subalgebra $\mathfrak{s}_{L_2-L_1}$. What do we know about representations of $\mathfrak{sl}_2$? Their representations are symmetric about zero and form unbroken strings differing by two.

To put it more systematically: let $\mathfrak{s}_{L_1-L_2} = \mathbb{C}\langle E_{12}, E_{21}, H_{2,1}\rangle$; then

$$\bigoplus V_{\alpha_0+n(L_2-L_1)}$$

is a subrerpesentation of $\mathfrak{s}_{L_1-L_2}$ and the eigenspaces are symmetric about the line orthogonal to $L_1 - L_2$. Because we know that any representation of $\mathfrak{sl}_2$ has integral eigenvalues, then all the weights that occur here lie in the white lattice of points in $\mathfrak{h}^*$ with integral values on the $H_{i,j}$. That is, there is a symmetry around the "orange" line that kills $H_{1,2}$.

Now, find an element along this line $\alpha_0 + n(L_2 - L_1)$ which is killed by E_{21}; then you get to the end of a string like that. Then you get to a vector which is again killed by half the roots, but for a different half. So you keep going around and get a hexagonal diagram. This is the picture of the weights that we know about so far. They're obtained by initial highest weight by reflecting around the lines which are the kernels of the elements $H_{i,j}$. Conversely, all the points interior to the hexagon are going to occur as weights of the representation.

13.6 Proposition. *If V has a highest weight vector with weight α_0, then the weights of V are exactly the weights $\alpha \in \mathfrak{h}^*$ congruent to α_0 modulo the root lattice Λ_R and located inside the hexagon with vertices obtained from α_0 by reflecting α_0 in the lines $\ell(H_{ij}) = 0$.*

This hexagon is not a regular hexagon. In fact, it could have been a triangle. Note, however, that the choice of the highest weight was determined only up to six: the highest weight would have to be one of the vertices of the hexagon.

There are a lot of things we don't have yet: for instance, we don't have an existence or uniqueness theorem for representations given a highest weight. The theorem is:

13.7 Theorem. *For all pairs $a, b \in \mathbb{N}^2$, there exists a unique irreducible representation of $\mathfrak{sl}_3$ with highest weight $aL_1 - bL_3$.*

We'll actually see how to construct them via multilinear algebra.

Lecture 14
3/1

§1 $\mathfrak{sl}_3$

Today, we want to finish the discussion of the Lie algebra $\mathfrak{sl}_3$ and of the corresponding Lie group. Starting on Monday, we're going to discuss the general picture, for representations of any semisimple Lie algebra. (This is chapter 14, which we'll start next week.) On the one hand, $\mathfrak{sl}_3$ is just one example; on the other hand, it introduces a lot of the ideas, constructions, and techniques we'll use in general. So in fact, 90% of Monday's lecture will be stuff that you could guess already based on the examples $\mathfrak{sl}_3, \mathfrak{sl}_2$. Once we've done about the general paradigm, we're going to return to examples. We're going to talk about the special linear group or algebra, the orthogonal and symplectic groups (and algebras) and see how this plays out.

Very briefly, let me recall where we are. We drew a picture of the vector space $\mathfrak{h}^*$, the dual to the space of diagonal traceless matrices, and given any representation we could identify the **weights** as points on a certain white lattice, the **weight lattice** (see below). We saw that the set of weights that occurred forms a **hexagon**.

Namely, we take the **roots** of the Lie algebra $\mathfrak{sl}_3$, which are the weights of the adjoint representation. We saw that those are $L_i - L_j, i \neq j$. There's one issue, which is just notational: the roots are the eigenvalues of $\mathfrak{h}$ on itself, and we're not including 0 in the root diagram. Again, we get a decomposition

$$\mathfrak{sl}_3 = \mathfrak{h} \oplus \bigoplus_{i,j} \mathfrak{g}_{L_i - L_j},$$

where $\mathfrak{g}_{L_i-L_j} = \mathbb{C}\langle E_{ij}\rangle$. Having said this much, we tried to mimic the case of $\mathfrak{sl}_2$ for $\mathfrak{sl}_3$. We chose an **ordering** of the weights. We took an irrational linear functional and took the weight that was furthest out.

Remark. What is the intrinsic characterization of the weight lattice? For any root $\alpha \in R$, we have a one-dimensional root space $\mathfrak{g}_\alpha$ and its "opposite" $\mathfrak{g}_{-\alpha}$ and we have the copy of $\mathfrak{sl}_2$ given by

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

In the case of $\mathfrak{sl}_2$, there's a one-dimensional abelian subalgebra which acts on two root spaces. We have a canonical generator in $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha]$, which is the element which has eigenvalues ± 2 in the adjoint representation. Let H_α be that element (we could take $H_\alpha = H_{i,j}$ in case $\alpha = L_i - L_j$) and H_α acts on $\mathfrak{g}_{\pm\alpha}$ with eigenvalues ± 2; that determines H_α. The **weight lattice** is the lattice of all elements of $\mathfrak{h}^*$ which take integral values on H_α. Given any representation V of $\mathfrak{g}$, all the eigenvalues of H_α on V must be integers (by the analysis of $\mathfrak{sl}_2$).

When choosing an **ordering** of the roots, that's a decomposition

$$R = R^+ \sqcup R^-$$

into **positive** and **negative** roots (given by splitting along a hyperplane). There are six possible choices we could make for this, but we're going to make the choice

$$R^+ = \{L_i - L_j, i < j\}, \quad R^- = \{L_i - L_j, i > j\}.$$

If I have *any* representation V, then V contains a (nonzero) vector v with two properties:

- v is an eigenvector for $\mathfrak{h}$, so $v \in V_{\alpha_0}$ for some α_0.
- v is killed by all the positive root spaces. That is, $\mathfrak{g}_\alpha v = 0$ for $\alpha \in R^+$.

In other words, we choose a linear functional that separates the positive roots from the negative roots. Then we go to the weight of V which has the highest value under this linear functional. If we increase this weight by any positive root, we're no longer a weight, and that's how we get the claim. These are called **highest weight vectors** for the representation.

In fact, we saw:

14.1 Proposition. *If V is irreducible, then V has a unique such vector (up to scalars).*

That is, if we go to a representation and choose a highest weight vector v, then we can take the subspace generated under v by applying the *negative* root spaces and that's a subrepresentation.[5] If we started with something irreducible, the subrepresentation must be the whole thing.

In general, given any representation V, in this way we get a bijection between:

- Irreducible subrepresentations of V.
- Highest weight vectors in V (mod scalars).

Let me just remind you, finally, of a theorem we're going to prove today after doing some examples. It's exactly what's missing so far: an existence and uniqueness result. Given the symmetry in the weight diagram, we note that the highest weight has to be in a certain cone: it has to be of the form $aL_1 - bL_3$ for $a, b \in \mathbb{N}^2$. (Note that **zero is a natural number.**)

14.2 Theorem. *Given $a, b \in \mathbb{N}^2$, there is a unique irreducible representation $\Gamma_{a,b}$ of highest weight $aL_1 - bL_3$.*

These are all the irreducible representations, since all irreducible representations have a highest weight vector. We thus have a classification of all irreducible representations of $\mathfrak{sl}_3$. In the case of $\mathfrak{sl}_2$, we had a lattice: all the elements of $\mathfrak{h}^*$ which were integral on H. We saw that the weights of any representation were symmetric about reflection the origin, and we associated to any representation a highest weight. For $\mathfrak{sl}_3$, we have a two-dimensional $\mathfrak{h}^*$, a lattice, and the claim is that there is a highest weight vector in a sector. We can classify irreducible representations up to isomorphism by lattice points in that closed wedge.

[5] In fact, an irreducible one. The dimension of the highest weight space is one-dimensional: so if the representation were reducible, only one of the summands could contain the highest weight vector and that would be the whole representation.

§2 Examples

Beyond the existence and uniqueness, we'd like to be able to construct the representations. For $\mathfrak{sl}_2$, we could take the symmetric powers of the standard representation. It's going to be a little trickier for $\mathfrak{sl}_3$, and you can start thinking where these are going to come from.

Let's start with the simplest of all representations, in some sense, which is the standard representation. There are too many letters, but let me remind you of the standard representation.

14.3 Example. Consider $V \simeq \mathbb{C}^3 \simeq \mathbb{C}\langle e_1, e_2, e_3\rangle$ where

$$\begin{cases} E_{ij}e_j = e_i \\ E_{ij}e_k = 0 \quad k \neq j \end{cases}.$$

We find that the matrix

$$\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix}$$

acts with eigenvalues a_1, a_2, a_3 on e_1, e_2, e_3. These are the linear functionals L_1, L_2, L_3. We find that e_i is an eigenvector for $\mathfrak{h}$ with eigenvalue L_i. The weight diagram consists of $\{L_1, L_2, L_3\}$ and we get a triangle in the weight space. (The hexagonal picture is sort of degenerate now.) In fact,

$$V = \Gamma_{1,0}.$$

14.4 Example. On V^*, that's also $\mathbb{C}^3$, and give it the dual basis $\langle e_1^*, e_2^*, e_3^*\rangle$. By the definitions, we find that

$$\begin{cases} E_{ij}e_i = -e_j \\ E_{ij}e_k = 0 \qquad k \neq i \end{cases},$$

and the matrix

$$\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix}$$

sends e_i^* to $-a_i e_i^*$. The weights are exactly the $-L_i, 1 \leq i \leq 3$. In fact,

$$V^* = \Gamma_{0,1}.$$

Now you can take tensor products of V, V^* to build up new representations. Note that $V \otimes V$ automatically decomposes into $\mathrm{Sym}^2 V$ and $\bigwedge^2 V$.

14.5 Example. Let's look at $\bigwedge^2 V$. That's the representation whose weights are the *pairwise sums* of *distinct weights* of the original representation. That means that the weights of $\bigwedge^2 V$ are the $L_i + L_j, i < j$. (There are three of them.) But of course, we can write that as $L_k, k \in [1, 3]$, and these are exactly the weights of the dual representation. It follows from the theorem that

$$\bigwedge^2 V \simeq V^*,$$

but we can see that directly. You have a natural pairing

$$V \times \bigwedge^2 V \to \bigwedge^3 V \simeq \mathbb{C}$$

which gives the duality.

14.6 Example. What about $\mathrm{Sym}^2 V$? Now we have weights $L_i + L_j, i \leq j$. I get $2L_1, 2L_2, 2L_3$ and then I also get the other intermediate weights $L_i + L_j, i < j$. This is irreducible. You want to say that it's not a superposition of two weight diagrams. If you start with the irreducible representation with highest weight $2L_1$, you can argue that all the weights in $\mathrm{Sym}^2 V$ must occur there. So $\mathrm{Sym}^2 V$ occurs a copy of $\Gamma_{2,0}$ but in fact

$$\mathrm{Sym}^2 V \simeq \Gamma_{2,0},$$

and you see that from the weight diagram. This diagram is not a union of other allowable diagrams.

The same logic applies, and tells us:

14.7 Proposition. *The symmetric powers* $\mathrm{Sym}^n V$ *and* $\mathrm{Sym}^n V^*$ *are irreducible and are the* $\Gamma_{n,0}, \Gamma_{0,n}$.

In fact, $\mathrm{Sym}^n V$ has a basis given by the monomials $\{e_{i_1} e_{i_2} \dots e_{i_n}\}$ with weights $L_{i_1} + \cdots + L_{i_n}$ and these are all distinct weights. So we find that all the weights occur with multiplicity one. The weights of $\mathrm{Sym}^n V$ form a giant triangle. The vertices correspond to the standard diagram of monomials in three variables. Now since $\mathrm{Sym}^n V$ contains a highest weight vector e_1^n with highest weight nL_1, it contains $\Gamma_{n,0}$. Now last time we saw that the irreducible representation $\Gamma_{n,0}$ contains all the lattice points in the triangle spanned nL_1, nL_2, nL_3. Looking at the multiplicities gives the result.

Remark. If V, W are any representations of $\mathfrak{sl}_3$ with highest weight vectors $v \in V_\alpha, w \in W_\beta$, then $v \otimes w \in V \otimes W$ is again a highest weight vector with weight $\alpha + \beta$. That's visible. When $X \in \mathfrak{g}$ acts on $V \otimes W$, it acts by applying X to each factor then summing. Therefore, if I took $\Gamma_{a,b} \otimes \Gamma_{c,d}$, that *contains* a highest weight vector with weight $(a+c)L_1 - (b+d)L_3$ and therefore

$$\Gamma_{a+c,b+d} \subset \Gamma_{a,b} \otimes \Gamma_{c,d},$$

and that in particular shows existence. We find that

$$\Gamma_{a,b} \subset \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*.$$

Let's look at the first example of this.

14.8 Example. What is $V \otimes V^*$? This should give us the irreducible representation with highest weight $L_1 - L_3$. You already know that representation: it's the *adjoint* representation, which is irreducible as we have a simple Lie algebra. $L_1 - L_3$ is a root. So

$$\Gamma_{1,1} \simeq \mathfrak{sl}_3.$$

If we look at $V \otimes V^*$, the weights are $L_i - L_j, i \neq j$, each taken once; it also has the weight 0 taken three times. We find that

$$V \otimes V^* \simeq \Gamma_{1,1} \oplus \Gamma_{0,0}$$

and that corresponds to the contraction (trace) map $V \otimes V^* \to \mathbb{C}$. The kernel is $\Gamma_{1,1}$, the traceless 3-by-3 matrices.

Let's do one more interesting case, which is going to spill out.

14.9 Example. Let's try to construct something we haven't already seen. Let's look at $\mathrm{Sym}^2 V \otimes V^*$. This is the first such tensor product that we're not going to be able to construct naively. This is going to have a highest weight vector of highest weight $2L_1 - L_3$ corresponding to $e_1^2 e_3^*$. It's in the interior of the cone and when we reflect it around, we're going to get a hexagon.

So $\Gamma_{2,1} \subset \mathrm{Sym}^2 V \otimes V^*$ but we want to identify the rest of it. One observation is that all the weights on the boundary in $\mathrm{Sym}^2 V \otimes V^*$ occur with multiplicity one (this requires a check). We know that the same is true for the figure of $\Gamma_{2,1}$. In the interior, though, it's a different story. There are only nine weight spaces on the boundary (or you can write it out explicitly) and we find that the interior weights of $\mathrm{Sym}^2 V \otimes V^*$ occur with multiplicity three. So we find that some multiple of the standard representation might be left over. We'll resolve this next time.

Lecture 15
3/4

Today, the plan is to finish talking about representations of $\mathfrak{sl}_3$. Wednesday is going to be a sort of big day, in that based on the examples we've seen, we're going to lay out the analysis of simple Lie algebras in general. We're going to describe the paradigm that you apply to analyze an arbitrary simple Lie algebra and its representations. We'll lay out the steps, all of which are analogous to the steps we've seen for $\mathfrak{sl}_2, \mathfrak{sl}_3$. It still remains to go ahead and carry out the analysis, and we'll start that on Friday, or on Monday—depending on how much time we spend on the Killing form. This is going to be the paradigm that guides us through the next month or so, when we'll understand the representations of the classical Lie algebras (i.e., with five exceptions, all simple Lie algebras.) I'm going to defer this week's homework assignment to Wednesday, since we didn't cover as much as planned.

Let me just state the main theorem:

15.1 Theorem. *We have a bijection between:*

- *Irreducible representations of* $\mathfrak{sl}_3$.
- *Weights that live in a certain hexagonal lattice (generated by* L_1, L_2, L_3*) in* $\mathfrak{h}^*$*) and in the cone spanned by* L_1 *and* $-L_3$*. In other words, this is the set of all elements of* $\mathfrak{h}^*$ *of the form* $aL_1 - bL_3 : a, b \in \mathbb{Z}_{\geq 0}$.

The bijection sends a representation to its highest weight.

Let me talk about where this is coming from, and then move on. We have not fully proved this yet. The proof of existence is something we basically did last time. We observed that I have two representations V, W, and highest weight vectors $v \in V, w \in W$ of weights α, β (that is, vectors which are eigenvectors for $\mathfrak{h}$ and killed by the positive root spaces), then the element $v \otimes w \in V \otimes W$ with highest weight $\alpha + \beta$. There was one more observation:

15.2 Lemma. *If V is any representation, and $v \in V$ is a highest weight vector of weight α_0, then if I start with v and start applying the negative root spaces successively, then I get an irreducible subrepresentation of V.*

I've gotten tired of saying "applying the negative root spaces successively," so let me introduce some notation now, and then I'll come back to the lemma. Recall that when we choose that linear function on $\mathfrak{h}^*$, it has the effect of breaking the roots

$$R = R^+ \sqcup R^-$$

lying on opposite signs of a line. It's natural to introduce:

15.3 Definition. We write $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$. This is called a **Borel subalgebra,** which we'll introduce when we talk about the general paradigm.

Again, $\mathfrak{sl}_3$ consists of the two-dimensional $\mathfrak{h}$ and the six root spaces: if we take the negative root spaces and add them to $\mathfrak{h}$, we get a Lie subalgebra.

Proof. So, to restate the lemma: we take the highest weight vector v, and apply $\mathfrak{b}$ over and over. It's the smallest subspace of V that contains v and is stable under application of $\mathfrak{b}$. The main point of the lemma is that the subspace is invariant under the *positive* root spaces, which was an explicit calculation. For irreducibility, we used the following argument: if $W = \mathfrak{b}v$ (repeatedly applying $\mathfrak{b}$) and

$$W = W_1 \oplus W_2,$$

then the decomposition commutes with the decomposition into weight spaces for $\mathfrak{h}$. In particular

$$W_{\alpha_0} = (W_1)_{\alpha_0} \oplus (W_2)_{\alpha_0}$$

where W_{α_0} is one-dimensional (when we apply the negative root spaces, we never get back to the original weight). So one of W_1, W_2 contains v and that one must contain all of W.

▲

Proof of uniqueness and existence. Given this lemma and these observations, we conclude that if I look at

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*,$$

where V is the **standard representation,** this contains *an* irreducible subrepresentation with highest weight $aL_1 - bL_3$. The corresponding highest weight vector up to scalars is $e_1^a \otimes (e_3^*)^b$ inside the tensor product of symmetric powers. That proves the existence right there, and gives us a recipe for constructing irreducible representations.

We just go to this tensor product, and pick the subrepresentation generated by one vector.

Uniqueness: Let V, W be irreducible representations with highest weight vectors v, w of the same weight α, we want $V \simeq W$. If that's the case, form $V \oplus W$ where $(v, w) \in V \oplus W$ is a highest weight vector of weight α. If $U = \mathfrak{b}(v, w) \subset V \oplus W$ is the subrepresentation generated by (v, w) in the direct sum, then U is again irreducible, by that lemma. What we observe is that U is irreducible, V and W are irreducible, so when we look at the projection maps

$$U \to V, \quad U \to W,$$

these are nonzero (they send (v, w) to v, w) and are hence isomorphisms. Hence, $U \simeq V \simeq W$. ▲

That shows that there exists an irreducible representation of highest weight $aL_1 - bL_3$ for any a, b.

15.4 Definition. We denote by $\Gamma_{a,b}$ the irreducible representation with highest weight $aL_1 - bL_3$.

We don't know at this point what the relation between $\Gamma_{a,b}$ and $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ where V is the standard representation.

§1 Examples

Let's look at the first nontrivial example and see what the general pattern is. We've seen that there are irreducible representations corresponding to the highest weights $aL_1 - bL_3$. The $\mathrm{Sym}^a V$ were already irreducible as were the $\mathrm{Sym}^b V^*$. The first example not of this form was the adjoint representation $\Gamma_{1,1}$.

Let's now search for $\Gamma_{2,1}$. As we just said, the place to look for it is sitting inside

$$\Gamma_{2,1} \subset \mathrm{Sym}^2 V \otimes V^*, \quad V \simeq \mathbb{C}^3.$$

Let's start by looking at the weight diagrams. We know that $\Gamma_{2,1}$ has highest weight $2L_1 - L_3$ and to generate all the weights of this representation, we start with this representation and reflect successively about the "red lines." We get this hexagon of weights and the weights of this irreducible representations are the elements of the weight lattice congruent to any of these weights modulo the root lattice and lying inside the hexagon. Also, all the dimensions of the weight spaces on the *boundary* are one. But we don't know the dimensions of the eigenspaces inside the remaining three interior weights, and that's really important. We will have a couple of formulas that will tell us this for a simple Lie algebra.

Note that we know the weights of $\mathrm{Sym}^2 V \otimes V^*$ because we know the weights of each factor and the weights add in a tensor product.

- In the case of V^*, we have $-L_1, -L_2, -L_3$,
- In the case of $\mathrm{Sym}^2 V$, we get $2L_1, L_1 + L_2, 2L_2, L_1 + L_2, 2L_3, L_2 + L_3$.

- When we add these, we get the weights $2L_i - L_j, i \neq 3$, weights $-2L_i$, and each L_i with multiplicity three. (A total of eighteen, as we should get.)
- For example the vectors with weight L_1 are $e_1^2 \otimes e_1^*, e_1e_2 \otimes e_2^*, e_1e_3 \otimes e_3^*$.

Here's what we're going to do: start with the vector $e_1^2 \otimes e_3^*$ (a highest weight vector with weight $2L_1 - L_3$) and start pushing it around by the negative weight spaces.

OK, let's recall: E_{ij} is the matrix that carries $e_j \mapsto e_i, e_k \mapsto 0$ for $k \neq j$ and acts on the dual representation via $e_i^* \mapsto e_j^*, e_k^* \mapsto 0$ for $k \neq i$. We're just going to take this highest weight vector and keep applying the negative root spaces. Consider for instance

$$E_{21}E_{32}(e_1^2 \otimes e_3^*) = E_{21}(-e_1^2 \otimes e_2^*) = -2e_1e_2 \otimes e_2^* + e_1^2 \otimes e_1^*.$$

Consider similarly

$$E_{32}E_{21}(e_1^2 \otimes e_3^*) = 2e_1e_3 \otimes e_3^* - 2e_1e_2 \otimes e_2^*.$$

This is something you should do at home, if you do it at all. What we get is the computation of the dimensions of the weight spaces of $\Gamma_{2,1}$ and what we're left with

$$\mathrm{Sym}^2 V \otimes V^* \simeq \Gamma_{2,1} \oplus V.$$

Remark. There's another way to see this. If this is a valid isomorphism, we should have a projection map. In particular, we should have a projection map $\mathrm{Sym}^2 V \otimes V^* \to V$. But we do have such a map: **contraction.** Given a decomposable tensor uv and an element v^*, we map that to $(u, v^*)v + u(v, v^*)$. That's a map of representations which is onto V and the calculation we made shows that $\Gamma_{2,1}$ is the kernel of that map.

In general, we have contraction maps and the general statement is that there are contraction maps $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* \to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$. The claim is that:

15.5 Proposition. *The kernel of this contraction map is* $\Gamma_{a,b}$.

(See the textbook.) Equivalently, we have a decomposition of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ into $\Gamma_{a,b}$ and $\mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$, so inductively we get the decomposition in general. We find:

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* = \bigoplus_{i=0}^{\min(a,b)} \Gamma_{a-i,b-i}.$$

That in turn gives us an explicit formula for the dimensions of the irreducible representations, or equivalently the multiplicities.

Lecture 16
3/6

§1 Outline

Let me say a word about this before you get started. Anytime you teach a subject, there are two different approaches: you can start with examples and generalize from there, or you can start with the theory and see how it specializes. There's a lot to be said for both approaches, but in the case of Lie algebras, something almost unique is going on: by the time we've done all the special cases, we've done the general theory (with five exceptions). That tips the balance towards this example-driven approach.

There are nine basic steps in analyzing a simple Lie algebra $\mathfrak{g}$ (over $\mathbb{C}$):

1. Identify a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which was abelian and acted diagonalizably under all representations. We would also like it to be maximal with respect to this property. This is going to be called a **Cartan subalgebra,** by definition a maximal abelian subalgebra which acts diagonalizably. (We're relying crucially here on the preservation of Jordan decomposition.)

   In every case, it'll be pretty clear what this should be.

2. As we did before, once we've got this subalgebra $\mathfrak{h}$, we have to know how $\mathfrak{h}$ acts on the rest of the Lie algebra by conjugation. We get a **Cartan decomposition**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha, \tag{75}$$

   where $\mathfrak{g}_\alpha$ is the eigenspace of $\mathfrak{h}$ with eigenvalue α (in the adjoint representation). These $\mathfrak{g}_\alpha$'s are called the **root spaces,** and the eigenvalues $\alpha \neq 0$ that occur are called **roots**. (The set of **roots** is denoted R.)

   If you happen to choose $\mathfrak{h}$ too small, at this point you'll notice: you'll get a zero eigenspace that is larger than $\mathfrak{h}$. In fact, $\mathfrak{h}$ is maximal if and only if the centralizer of $\mathfrak{h}$ is $\mathfrak{h}$ itself.

3. Let's write down some facts which we will observe in specific cases:

   - Each $\mathfrak{g}_\alpha$ is one-dimensional.
   - $R = -R$. That is, if α is a root, then so is $-\alpha$.
   - R generates a sublattice Λ in $\mathfrak{h}^*$ of rank equal to the (complex) dimension of $\mathfrak{h}^*$.
   - One really crucial consequence is the following. Here $\mathfrak{g}$ and $\mathfrak{h}$ are complex vector spaces. However, the roots R span a *real* vector space $\mathfrak{h}^*_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ whose complexification is $\mathfrak{h}^*$. All the pictures that we will draw is of this real vector space.
   - By the standard calculation, we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}. \tag{76}$$

     This gives us a beginning picture of how the Lie algebra acts on itself. We know how $\mathfrak{h}$ acts (diagonally) in terms of this decomposition and now we've seen some information of how the different pieces bracket with each other.

4. We get a beginning picture of what representations look like. If V is any representation of $\mathfrak{g}$, then we can write down a **weight space decomposition**

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha, \tag{77}$$

where $\mathfrak{h}$ acts on V_α by the eigenvalue α. Moreover, $\mathfrak{g}_\alpha$ carries V_β into $V_{\alpha+\beta}$. So we have a picture of eigenvalues as dots in $\mathfrak{h}^*_{\mathbb{R}}$.

5. Next, we need to introduce **distinguished subalgebras** (as we did for $\mathfrak{sl}_2, \mathfrak{sl}_3$). For all $\alpha \in R$, we have

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

   Remember, we got a lot of mileage in the case of $\mathfrak{sl}_3$ from looking at these small subalgebras and looking at how they acted on representations. In the case of $\mathfrak{sl}_3$ at least, these were all copies of $\mathfrak{sl}_2$ and we could use what we knew about $\mathfrak{sl}_2$.

   I want to quote some facts in general. These distinguished subalgebras $\mathfrak{s}_\alpha$ **are all copies of $\mathfrak{sl}_2$ floating around.** In order to prove this, I need to know that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is nonzero so that it is a one-dimensional subspace of $\mathfrak{h}$ and we get a three-dimensional subalgebra $\mathfrak{s}_\alpha$. Next, we need to know that the one-dimensional subspace $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ acts nontrivially on $\mathfrak{g}_{\pm\alpha}$. That we'll assume as well.

6. Given this, observe that there's a distinct element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. This one-dimensional space acts diagonally on $\mathfrak{g}_{\pm\alpha}$ and nontrivially, so there is a unique element of this subspace which acts on $\mathfrak{g}_\alpha$ with eigenvalue 2 and on $\mathfrak{g}_{-\alpha}$ with eigenvalue -2. (In other words, I want to recreate the generators in $\mathfrak{sl}_2$.)

   Let $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with this property (i.e., so that $\alpha(H_\alpha) = 2$). We thus get these **distinguished elements** $H_\alpha \in \mathfrak{h}$.

7. We want to observe that for any representation V, the eigenvalues of the distinguished elements H_α are *integers*; that's true of any representation of $\mathfrak{sl}_2$. In other words, given any representation of $\mathfrak{g}$, all the $\mathfrak{h}$-eigenvalues will be linear functionals on $\mathfrak{h}$ that are **integral on these distinguished elements.** We thus introduce the **weight lattice** $\Lambda_W \subset \mathfrak{h}^*$ which consists of

$$\Lambda_W = \{\alpha \in \mathfrak{h}^* : \alpha(H_\beta) \in \mathbb{Z}, \quad \text{for all } \beta \in R\}.$$

   In particular, the weights of any representation are in Λ_W: in particular, $R \subset \Lambda_W$ by looking at the adjoint representation. We haven't shown this yet, but **any element of Λ_W arises as weight.**

8. I want to offer a teaser here. This is something that's way ahead of us, and for the next few weeks we're going to be sticking to Lie algebras. But at the end of the day, at least part of the goal is to understand representations of Lie groups. There are several different groups associated to a given complex Lie algebra. Once we've understood representations of Lie algebras, we want to know which representations lift to Lie groups. I said that $\Lambda_R \subset \Lambda_W$ and the quotient is a finite abelian group Λ_W/Λ_R. Given the Lie algebra $\mathfrak{g}$, it has an **adjoint form** G_0 (i.e., the simply connected form modded out by the center).

   16.1 Theorem. *Then* $\Lambda_W/\Lambda_R \simeq \pi_1(G_0)$.

In other words, the structure of the Lie algebra encodes the fundamental group (or the center of the simply connected form). It's in these terms that we can identify which representations lift to which forms of the group. The forms of the Lie algebra $\mathfrak{g}$ correspond exactly to subgroups of Λ_W/Λ_R.

16.2 Theorem. *A representation V lifts to the Lie group G if and only if the weights of V lie in the corresponding subgroup of Λ_W/Λ_R.*

In particular, the representations that descend to the adjoint form are the ones whose weights live in the root lattice. This is how we will understand representations of semisimple Lie groups.

9. We need to introduce a group of symmetries. The crucial element of this analysis is to look at these distinguished subalgebras and restrict to elements of these subalgebras to get information about the weights of the representations. We've already used this with the $\mathfrak{s}_\alpha$. We know that the eigenvalues of the diagonal elements H_α must be symmetric about the origin, though. In other words, there's a lot of symmetry in the weights of any representation.

   For any *root* $\alpha \in R \subset \mathfrak{h}^*$, I get a distinguished element $H_\alpha \in \mathfrak{s}_\alpha \subset \mathfrak{g}$. I'd like to say that under representation of $\mathfrak{g}$, the eigenvalues of H_α are integers symmetric about zero. That suggests we introduce the hyperplane $\Omega_\alpha = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) = 0\}$. We have a direct sum decomposition

   $$\mathfrak{h}^* \simeq \mathbb{C}\alpha \oplus \Omega_\alpha.$$

   We let w_α be the involution on $\mathfrak{h}^*$ with eigenvalue 1 on Ω_α and -1 on $\mathbb{C}\alpha$.

10. The basic observation is that the weights of any representation are **invariant** under this involution w_α. The group generated by the involutions w_α is called the **Weyl group** and is denoted by $\mathfrak{W}$: this is a symmetry of the roots. More generally, if V is any representation, then the weights of V forms a subset of $\mathfrak{h}^*$ invariant under the action of $\mathfrak{W}$. In the case of $\mathfrak{sl}_3$, the Weyl group is the group generated in the lines orthogonal to the roots: it is S_3.

11. Let me do something which isn't logically necessary. We saw that in the case of $\mathfrak{sl}_3$, $\mathfrak{h}$ was the space of diagonal traceless matrices, and the dual was the space of linear functions L_1, L_2, L_3 modulo $L_1 + L_2 + L_3$. When drawing a picture of the vector space, it seemed natural to draw the symmetry and draw L_1, L_2, L_3 at $2\pi/3$ degrees. We were implicitly invoking an inner product on $\mathfrak{h}^*$, which is respected by the symmetries in the Weyl group. That's again a feature of the general situation. We have a **Killing form**, which is a positive-definite inner product on $\mathfrak{h}^*_{\mathbb{R}}$ invariant under the Weyl group. (It's going to be a symmetric bilinear form on the entire Lie algebra and thus its dual.) This will help us visualize things.

12. The next step is to produce an *ordering* of the roots. We need a decomposition

    $$R = R^+ \sqcup R^-,$$

into **positive** and **negative** roots, where R^+, R^- lie in *half-spaces.* So I choose a hyperplane with irrational slope in $\mathfrak{h}^*_{\mathbb{R}}$ and look at the roots to either side.

13. Next, I want to start talking about highest weight vectors and the classification by highest weights. This will start on Friday.

Lecture 17
3/8

Today, I want to recall briefly the story thus far, and get a pretty complete algorithmic process for describing a representation of an arbitrary simple Lie algebra. Some of those things are not proved in general, but we'll observe them in specific cases. Very very briefly, the steps so far for a simple Lie algebra $\mathfrak{g}$:

1. We introduced the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This is a maximal subalgebra which is both abelian and diagonalizable

2. We get the **Cartan decomposition**
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R \subset \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha.$$
It makes sense to focus on the **distinguished subalgebras**, which are the subalgebras
$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}],$$
and we stated that this was isomorphic to $\mathfrak{sl}_2$. One of the crucial things we do is to apply our analysis of $\mathfrak{sl}_2$ to any Lie algebra by restricting to distinguished subalgebras.

3. Given these isomorphisms, we get **distinguished elements** $H_\alpha \in \mathfrak{h}$: these live in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and act on $\mathfrak{g}_\alpha$ with eigenvalue 2. This determines H_α uniquely.

4. We get the **weight lattice** Λ_W**,** which is the set of linear functions $\alpha \in \mathfrak{h}^*$ such that $\alpha(H_\beta) \in \mathbb{Z}$ for each $\beta \in R$. **Every representation of $\mathfrak{g}$ has weights that in this lattice,** because the eigenvalues of H_α are integers, so if Λ_R is the root lattice, then
$$\Lambda_R \subset \Lambda_W.$$

5. Let's illustrate this for $\mathfrak{sl}_3$. We have six roots $L_i - L_j, i \neq j$ which we can draw in a hexagonal lattice with L_1, L_2, L_3 located at the cube roots of unity. The weight lattice has index two in the root lattice.

6. From restricting to the $\mathfrak{s}_\alpha$, we already saw that the weights of any representation lie in the weight lattice. But we also got a little more: the weights are invariant under certain involutions of $\mathfrak{h}^*$.

The object we're producing is the **Weyl group** $\mathfrak{W}$**,** which is generated by the involutions

$$W_\alpha = \begin{cases} 1 & \text{on } \Omega_\alpha = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) = 0\} \\ -1 & \text{on } \mathbb{C}\alpha \end{cases}.$$

7. In order to visualize this, it's sometimes helpful to introduce an inner product called the **Killing form.** This is a symmetric bilinear form on $\mathfrak{h}$ (and correspondingly on $\mathfrak{h}^*$) and it's also positive definite on the real space $\mathfrak{h}^*_{\mathbb{R}}$ spanned by the root lattice. It's invariant under the Weyl group.

8. We introduce an **ordering** of the roots, a decomposition of R into $R = R^+ \sqcup R^-$ where each is contained in a half-space in the real vector space $\mathfrak{h}^*_{\mathbb{R}}$. The ordering of the roots is a choice, but it's a negligible choice, since any two choices differ by an automorphism of the Lie algebra.

9. We get two things out of this. We get **Borel subalgebras** (this is not mentioned in the book)

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha,$$

and then what I get is a subalgebra. In fact, we know that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ here.

10. The real point of this is that, given an ordering of the roots, we have the notion of a **highest weight vector**. If I have any representation V, I say that a vector in V is a **highest weight vector** if it is an eigenvector for $\mathfrak{h}$ and is killed by the positive root spaces. (I.e., find a weight as far from the hyperplane with irrational slope as possible.) Every representation has a highest weight vector, and *an irreducible representation has a unique one* (up to scalars). The weight of this vector is called the **highest weight** of the representation.

    The word "highest" is sometimes replaced by "dominant."

11. The third observation, which we proved in the case of $\mathfrak{sl}_2, \mathfrak{sl}_3$ and is proved in general similarly: if V is any representation and $v \in V$ any highest weight vector, then the subrepresentation $\mathfrak{b}v$ obtained by taking v and taking the sub$\mathfrak{b}$-representation generated by this one vector v is an irreducible representation of $\mathfrak{g}$.

12. If V is any irreducible representation, then the set of weights of V (contained in Λ_W) is obtained by starting with the highest weight and then adding an element of Λ_R. I.e., if α_0 is the highest weight, then all the weights of V are *congruent* to α_0 mod Λ_R. Moreover, if we take the convex hull of the orbit $\mathfrak{W}\alpha_0$, and take the intersection of the coset $\alpha_0 + \Lambda_R$ with that convex hull, then the set of all weights in the representation is that set. This is a generalization of the fact that the weights of an irreducible $\mathfrak{sl}_2$-representation form an unbroken string of integers.

    As the example of $\mathfrak{sl}_2$ suggests, there is a nice picture that describes the multiplicities, but we're not there yet.

13. When we have this picture of the weights of V, we can ask ourselves where the highest weights live. The answer is that the highest weights are weights with *nonnegative* inner product (with respect to the Killing form) with all the positive roots. We want to introduce a *cone* in $\mathfrak{h}_{\mathbb{R}}^*$, which is called a **Weyl chamber.**

    We define
$$\mathcal{W} = \left\{ \alpha \in \mathfrak{h}^* : B(\alpha, \beta) \geq 0, \quad \beta \in R^+ \right\},$$
    (where B is the Killing form) or equivalently that
$$\mathcal{W} = \left\{ \alpha : \alpha(H_\beta) \geq 0, \quad \beta \in R^+ \right\}.$$
    In other words, we're saying the following: take a connected component of the complement in $\mathfrak{h}_{\mathbb{R}}^*$ of the union of these hyperplanes Ω_α and take the closure. The basic observation is that the highest weights are always in this Weyl chamber.

    Essentially, we're repackaging the basic choice we made when ordering the roots. The set of orderings of the roots is acted upon simply transitively by the Weyl group, and the Weyl group acts simply transitively on the set of the Weyl chambers.

14. The main theorem, which describes all representations, is as follows:

    17.1 Theorem (Main theorem)**.** *The association that sends an irreducible representation V to its highest weight $\alpha \in \mathfrak{h}^*$ is a bijection between the set of irreducible representations of $\mathfrak{g}$ and the intersection of the weight lattice Λ_W with the Weyl chamber $\mathcal{W}$.*

15. I want to go back again to the example of $\mathfrak{sl}_3$. When we made the statement that we could start with a highest weight vector and apply the negative root spaces successively to get a subrepresentation, what did we say? We wanted to say that the result $\mathfrak{b}v$ was invariant under the positive root spaces. In fact, you don't have to check every positive root space. For $\mathfrak{sl}_3$, there are three positive root spaces, but one is the commutator of the other two.

    In a lot of ways, instead of dealing with all the positive roots, it's sufficient to look at a subset which generates additively.

    17.2 Definition. A **simple** (or **primitive**) root is a positive root $\alpha \in R^+$ if it isn't the sum of two other positive roots.

    We don't know this yet, but we can generate the Borel subalgebra simply by taking the root spaces corresponding to the simple roots and $\mathfrak{h}$. Every positive root is a sum of simple roots.

16. This is sort of a fun exercise in euclidean geometry, but observe that the simple roots are all at angles $\geq \pi/2$ from each other, and therefore they are linearly independent. Since Λ_R spans the vector space, it follows that the **number** of simple roots is equal to $\dim \mathfrak{h}$. In general, the number of roots — although we have not seen this yet — grows quadratically with the dimension of the Cartan subalgebra, so focusing on the simple roots is a simplification.

17. The Weyl chamber is a **simplicial cone.** We can characterize it as the set of vectors that form an acute or right angle with all the simple roots (instead of all positive roots): that is, it is an intersection of exactly $\dim \mathfrak{h}$ half-spaces (which is what it means to be a simplicial cone). That's something that you don't see in 2-space.

18. That, finally, leads us to our last definition. This is going to come out of the blue, in some sense, though it's useful in terminology, and we'll see this explicitly in every case. We're going to introduce the notion of **fundamental weights.** Look at the edges of the Weyl chambers and look at the first weight lattice point on each of these edges. We said that the Weyl chamber is a simplicial cone. It has m edges, for m the dimension. If I look at $w_1, \dots, w_m$ which are the smallest elements of the weight lattice along the edges of the Weyl chamber, then they generate $\Lambda_W \cap \mathcal{W}$ as a semigroup. What does that say? It states that we can use the same notation as we used for $\mathfrak{sl}_3$. We can identify the smallest lattice vectors along the cone, called **fundamental weights,** and we can express any lattice vector as a nonnegative linear combination of them. *So we can index all irreducible representations by m-tuples of nonnegative integers.* This tells us something important: we're going to have to prove this main theorem eventually. The uniqueness proof it's easy; it's the proof we gave for $\mathfrak{sl}_3$. Existence is something we'll see case-by-case but is harder to prove in general. This is saying that, to prove existence, it's enough to exhibit m irreducible representations with highest weights the fundamental weights.

Lecture 18
3/11

§1 The Killing form

Let me say a little about the Killing form. It's a symmetric bilinear form B on the Lie algebra $\mathfrak{g}$. In other words, it is a map

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}.$$

As we'll see, what it does in terms of the Cartan decomposition is pretty elementary. What is crucial is the restriction of B to the Cartan subalgebra itself. We'll see that it is nondegenerate, and induces an isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$. We'll use the symbol B both for the bilinear form on $\mathfrak{g}$, its restriction $B|_{\mathfrak{h}}$, as well as the induced form on $\mathfrak{h}^*$ (via the isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ that B induces).

18.1 Definition. Given $X, Y \in \mathfrak{g}$, we look at the adjoint actions on $\mathfrak{g}$. We define the **Killing form**

$$B(X, Y) = \operatorname{Tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)), \tag{78}$$

where $\operatorname{ad}(X), \operatorname{ad}(Y) : \mathfrak{g} \to \mathfrak{g}$. Observe that this is symmetric by symmetry of the trace.

The picture that we have of the adjoint action of $\mathfrak{g}$ on itself tells us what to expect. We have the Cartan decomposition

$$\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

and we typically draw that as a diagram of dots. The point is, what each of these $\mathfrak{g}_\alpha$ does is to *shift* every root space over. Commuting with $\mathfrak{g}_\alpha$ carries $\mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$, and in particular it has trace zero: everything is moving somewhere else. In other words, we have

$$\mathrm{Tr}(\mathrm{ad}(X)|_{\mathfrak{g}}) = 0, \quad X \in \mathfrak{g}_\alpha, \alpha \in R.$$

The same is true if $X \in \mathfrak{h}$, because X acts by zero on $\mathfrak{h}$, and the actions on $\mathfrak{g}_{\pm\alpha}$ cancel each other out. That is, the *trace* of any X acting on itself is zero. But when we compose, we don't have to get zero.

For example, if $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha}$, then $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ could have nonzero trace. But we have

$$B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \quad \text{unless} \quad \alpha + \beta = 0, \tag{79}$$

and so therefore we have an *orthogonal* decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

On each two-dimensional summand, the inner product looks like

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It doesn't tell us much about the root spaces, but the real point is the action on $\mathfrak{h}$. If we have $X, Y \in \mathfrak{h}$, we know how they act on the Lie algebra. They kill everything in $\mathfrak{h}$. On $\mathfrak{g}_\alpha$, they act by scalars $\alpha(X), \alpha(Y)$. In other words,

$$B(X, Y) = \sum_{\alpha \in R} \alpha(X)\alpha(Y). \tag{80}$$

A couple of other observations. Again, this is something we're going to see in practice in every case: this form is nondegenerate. Another thing, it's *positive definite* on the real subspace $\mathfrak{h}_\mathbb{R}$ spanned by the distinguished elements H_α. Likewise, on $\mathfrak{h}_\mathbb{R}^*$. Just bear in mind: if we have a symmetric bilinear form on a complex vector space, it can't have a sign. If $B(x, y) > 0$, then $B(ix, iy) < 0$. But when we restrict to a real subspace, it can have a sign, and in this case it does. The key fact again is that this is *invariant* under the Weyl group. If you want to identify the Killing form, you usually only care about it up to scalars, and you just look for something invariant under the Weyl group.

We will (necessarily) have to come back to this when analyzing arbitrary semisimple Lie algebras.

§2 $\mathfrak{sl}_n$

I'd like to go through the next few chapters of the book, i.e. all the classical Lie algebras. We're going to discuss $\mathfrak{sl}_n, \mathfrak{sp}_n, \mathfrak{so}_n$. There are many things that I could spend an unlimited amount of time talking about, but I'm open to suggestions.

Again, $\mathfrak{g} = \mathfrak{sl}_n$ consists of traceless n-by-n matrices. If we're looking for an abelian subalgebra acting diagonally under some faithful (and hence any) representation, why not just take the diagonal matrices?

18.2 Definition. We choose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_n$ to be the subalgebra of (traceless) diagonal matrices: this is the collection of n-tuples $\{(a_1, \ldots, a_n) : \sum a_i = 0\}$.

We don't know that it's maximal yet, but if it's not maximal, then there will be a zero eigenspace of the action outside itself, and we'll see that this doesn't happen. What is the Cartan decomposition?

18.3 Definition. Let $V \simeq \mathbb{C}^n$ be the standard representation of $\mathbb{C}^n$.

18.4 Definition. We let $E_{ij} \in \mathfrak{sl}_n$ (for $i \neq j$) to be the endomorphism of $\mathbb{C}^n$ that carries the jth basis vector to the ith basis vector and all other basis vectors to zero. That is,

$$E_{ij}(e_j) = e_i, \quad E_{ij}(e_k) = 0, k \neq j.$$

We let H_i (which is *not* in the Lie algebra) be the matrix with

$$H_i(e_i) = e_i, \quad H_i(e_k) = 0, k \neq i.$$

By the same calculation as always, we find that E_{ij} is an eigenvector for the action of $\mathfrak{h}$. The eigenvalue is $L_i - L_j$ where the notation $\{L_i\}$ is as before.

18.5 Definition. We can think of $\mathfrak{h}^* = \mathbb{C}\langle L_1, \ldots, L_n \rangle / (L_1 + \cdots + L_n = 0)$ where L_i is the linear functional picking out the ith coordinate.

In other words, we have

$$\mathfrak{g}_{L_i - L_j} = \mathbb{C}E_{ij},$$

and we get the Cartan decomposition for $\mathfrak{sl}_n$,

$$\mathfrak{sl}_n \simeq \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}, \quad \mathbb{C}E_{ij} = \mathfrak{g}_{L_i - L_j}. \tag{81}$$

There are no zero eigenvalues for $\mathfrak{h}$ except $\mathfrak{h}$ itself, so it is in fact a Cartan subalgebra.

This collection of roots is invariant under the symmetric group S_n, and the Killing form has to be as well. If I look at $\mathfrak{h}$, then that's a representation of S_n, and there's a unique inner product invariant under S_n, which is just the standard inner product on $\mathbb{C}^n$.

How do we visualize this? Think about the case we've already done. In the case of $\mathfrak{sl}_3$, we looked at the vectors $L_1, L_2, L_3 \in \mathbb{C}^3$ and we projected into the plane with sum zero, and we wound up with three vectors which formed the vertices of an equilateral triangle. In general, we can view the L_i as forming the vertices of an $(n-1)$-simplex centered at the origin in $\mathfrak{h}^*$.

What's the next step in our general algorithm? We have to introduce the *distinguished subalgebras.* The distinguished subalgebras correspond to pairs of opposite roots, where we take the corresponding root spaces and throw in their commutators. We get

$$\mathfrak{s}_{L_i - L_j} = \mathbb{C}\left\langle E_{ij}, E_{ji}, H_i - H_j \right\rangle. \tag{82}$$

This is obviously a copy of $\mathfrak{sl}_2$: it's just the two-by-two matrices that are zero outside the ith and jth rows and columns. The distinguished element of $\mathfrak{h}$, the $H_{L_i - L_j}$ is exactly $H_i - H_j$.

What did we get out of the distinguished elements in general? First, we got an integrality condition on the eigenvalues of any representation: the eigenvalues of any representation have to have integral eigenvalues on the $H_i - H_j$. That should identify the weight lattice.

Indeed, the **weight lattice** is

$$\Lambda_W = \mathbb{Z}\left\langle L_1, L_2, \dots, L_n \right\rangle. \tag{83}$$

The **root lattice** is the subspace spanned by the pairwise differences $L_i - L_j$, i.e.

$$\Lambda_R = \mathbb{Z}\left\langle L_i - L_j, i \neq j \right\rangle \tag{84}$$

and the quotient Λ_W / Λ_R is exactly $\mathbb{Z}/n$. (This is the center of the simply connected form $SL_n(\mathbb{C})$, which consists of the roots of unity.)

The other thing that we got was a collection of symmetries. Under any representation of $\mathfrak{sl}_n$, if we restrict to $\mathfrak{s}_{L_i - L_j}$, the eigenvalues of $H_i - H_j$ are symmetric about zero. This lets us say that the weights are invariant under a bunch of reflections.

Recall that $\mathfrak{h}^* = \left\{\sum a_i L_i\right\} / (L_1 + \cdots + L_n)$. The hyperplane $\Omega_{L_i - L_j}$ orthogonal to the root $L_i - L_j$, or equivalently the annihilator of the distinguished element $H_i - H_j$, is exactly the set $\sum a_k L_k$ such that $a_i = a_j$. The reflection operator about this hyperplane is the one that simply exchanges a_i, a_j. The Weyl group simply acts by permuting coordinates, and in particular it's S_n acting by permuting coordinates.

The next step is based on choosing an ordering of the roots, and it's a choice that we have to make (although the end result won't depend on the choice).

18.6 Definition. We set R^+ to be $\{L_i - L_j, i < j\}$, as we've done in the two cases before.

In terms of this ordering of the roots, I want to observe that the **simple roots** (the roots of this form that can't be expressed as sums of roots of this form) are exactly the $L_i - L_{i+1}$. As predicted, there are exactly enough of them. More to the point, we can describe the *Weyl chambers.* We can describe the chamber that contains the vectors with positive inner product with the simple roots. The Weyl chamber is a simplicial cone, it's an intersection of $n-1$ half-spaces in $\mathbb{R}^{n-1}$. That is,

$$\mathcal{W} = \sum a_i L_i, \quad a_1 \geq a_2 \geq \cdots \geq a_n \in \mathbb{R}.$$

This is a simplicial cone: it's defined by $n-1$ linear inequalities. In particular, the next thing we want to do is to look at the edges of this cone. Those are exactly the places where all but one of the inequalities are in fact equalities. We have a bunch of equal coefficients and one jump. The *edges* of $\mathcal{W}$ are spanned by the vectors

$$\begin{gathered} L_1 \\ L_1 + L_2 \\ L_1 + L_2 + L_3 \\ L_1 + \cdots + L_{n-1} = -L_n \end{gathered}$$

Notice that the intersection of the Weyl chamber defined by these inequalities and the weight lattice (the integer linear combinations of the L_i) is the set of integral linear combinations of the L_i satisfying these inequalities, and it is generated as a *semigroup* by these vectors $L_1, L_1 + L_2, L_1 + L_2 + L_3, \dots$. This was the observation that we made before, which doesn't follow immediately, which is that when we take the Weyl chamber and intersect with the weight lattice, it's generated by the primitive vectors along the edges. These are what we called the **fundamental weights**: every weight in the Weyl chamber is a nonnegative linear combination of these.

There is exactly one irreducible representation for each element in $\mathcal{W} \cap \Lambda_W$, corresponding to the highest weight.

18.7 Definition. We will call the unique irreducible representation with highest weight $a_1 L_1 + a_2 L_2 + \cdots + a_n L_n = b_1 L_1 + b_2(L_1 + L_2) + \cdots + b_{n-1}(L_1 + \cdots + L_{n-1})$ by the name $\Gamma_{b_1,\dots,b_{n-1}}$.

In particular, the irreducible representations correspond to $(n-1)$-tuples of nonnegative integers. Let's just look at a first couple of examples, to prove the existence half of the fundamental theorem.

Now, take $n = 4$**.**

18.8 Example. The standard representation $V \simeq \mathbb{C}^4$. The weights are the L_i, which are alternate vertices of a cube (as you might draw). The highest weight is simply L_1. So, we're also writing $V = \Gamma_{1,0,0}$, but I'll still call it V.

18.9 Example. The next simplest is the dual representation V^*. The weights of the duals are the negatives of the weights of V, so the weights are $\{-L_i\}$. In other words, they correspond to the remaining weights of a cube that you might draw; the highest weight is $-L_4 = L_1 + L_2 + L_3$. That gives us $V^* = \Gamma_{0,0,1}$.

18.10 Example. What's the next thing to look? Tensor, exterior, symmetric products of these. The simplest would be $\bigwedge^2 V$. If we look at, the weights are $L_i + L_j, i < j$. Those correspond to the midpoints of the faces of the cube you might draw. This is again irreducible. You can't write this as a union of two other weight diagrams symmetric under S_4 and the highest weight is $L_1 + L_2$, so this is $\Gamma_{0,1,0}$.

Thus, we've proved the existence half! Every weight in the intersection of the weight lattice with this cone is a nonnegative linear combination of the fundamental weights, so we get highest weight representations as we want by tensoring up.

Lecture 19
3/13

§1 $\mathfrak{sl}_n$

We consider $\mathfrak{g} = \mathfrak{sl}_n$ and choose the Cartan subalgebra consisting of diagonal traceless matrices in $\mathfrak{sl}_n$. We let $\mathfrak{h}^* = \mathbb{C}\langle L_1, \dots, L_n\rangle / (L_1 + \cdots + L_n)$. The roots $R = \{L_i - L_j, i \neq j\}$ and we let the positive roots be $R^+ = \{L_i - L_j, i < j\}$. The Weyl group $\mathfrak{W}$ is S_n acting on $\mathfrak{h}^*$ by permuting the generators L_i.

19.1 Example. For instance, the reflection through the root $L_i - L_j$ interchanges L_i and L_j and gives a transposition.

The Weyl chamber is the collection

$$\mathcal{W} = \left\{ \sum a_i L_i, \quad a_1 \geq a_2 \geq \cdots \geq a_n \right\}.$$

The intersection of the Weyl chamber with the weight lattice is generated by the primitive vectors, the *fundamental weights*, which live on a cone. These are the weights

$$L_1, L_1 + L_2, L_1 + L_2 + L_3, \quad \dots.$$

Any point in the weight lattice and in the Weyl chamber is a nonnegative integral linear combination of the fundamental weights. The fundamental theorem is:

19.2 Theorem. *There is a unique irreducible representation $\Gamma_{a_1,\dots,a_{n-1}}$ (for $a_1, \dots, a_{n-1} \in \mathbb{Z}_{\geq 0}$) with highest weight $a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_n(L_1 + \cdots + L_n)$.*

Let's do some examples with $n = 4$.

19.3 Example. The standard representation $V \simeq \mathbb{C}^4$ has weights $L_1, \dots, L_4$: it has highest weight L_1 and is what we call $\Gamma_{1,0,\dots,0}$.

19.4 Example. Then we have the dual representation V^*, which we can identify with $\bigwedge^3 V$ using the pairing

$$V \otimes \bigwedge^3 V \to \mathbb{C}$$

which is $\mathfrak{sl}_4$-equivariant. We can see it in any case. What are the weights of $\bigwedge^3 V$? They are the triple sums of distinct weights of V. We can compare this with the weights of V^*, which are $\{-L_i\}$. This is therefore a $\Gamma_{0,0,1}$.

19.5 Example. We can also look at $\bigwedge^2 V$, whose highest weight is $L_1 + L_2$. The diagram of weights looks like an octahedron if you draw it. This is irreducible (no proper subconfiguration invariant under the Weyl group), and it's $\Gamma_{0,1,0}$.

At this point, we have proved the existence theorem. We have exhibited representations with highest weights $(1,0,0), (0,1,0), (0,0,1)$, and if we look at tensor products of these, we can get representations with *a* highest weight vector of any weight (a,b,c). Then, we take the subrepresentation spanned by that vector.

There's a similar description in $\mathfrak{sl}_n$ in general. For $\mathfrak{sl}_n$:

19.6 Example. $V \simeq \mathbb{C}^n$ has highest weight L_1. To get the other representations with fundamental highest weights, we look at the exterior products $\bigwedge^k \mathbb{C}^n$ whose weights are the sums of k distinct L_i's. The highest weight is $L_1 + \cdots + L_k$, and this is a $\Gamma_{0,\ldots,1,\ldots 0}$ where 1 is in the kth slot. When we get to $\bigwedge^{n-1} V \simeq V^*$, we get the $\Gamma_{0,\ldots,1}$. In general,

$$\Gamma_{a_1,\ldots,a_{n-1}} \subset \mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}\left(\bigwedge^2 V\right) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\bigwedge^{n-1} V).$$

It is generated (under the negative root spaces) by the highest weight vector $(e_1)^{a_1} \otimes (e_1 \wedge e_2)^{a_2} \otimes \ldots$. This proves the existence theorem for $\mathfrak{sl}_n$.

Let's go back to $n = 4$. We've the three fundamental representations, the exterior powers of V. It's natural to ask what happens when we take tensor and symmetric powers of these. We consider, for example:

19.7 Example. Consider $V \otimes \bigwedge^2 V$ and see how it decomposes. The weights of this are exactly the sums of the L_i and the pairwise distinct sums. There are two types of sums. There are the $2L_i + L_j, i \neq j$ (twelve of these, with multiplicity one) and also the $L_i + L_j + L_k$ for $i \neq j \neq k$ (where each of the latter occurs three times). This accounts for the twenty-four dimensions. This is *not* an irreducible representation. For example, we have a map, a surjective map

$$\phi : V \otimes \bigwedge^2 V \twoheadrightarrow \bigwedge^3 V \simeq V^*$$

and the claim is that the kernel of this map is the irreducible representation with highest weight $2L_1 + L_2$. In other words,

$$V \otimes \bigwedge^2 V \simeq \Gamma_{1,1,0} \oplus V^*.$$

The representation $\Gamma_{1,1,0} \subset V \otimes \bigwedge^2 V$ is obtained by applying $E_{2,1}, E_{3,2}, E_{4,3}$ to the highest weight vector there, which is $e_1 \otimes (e_1 \wedge e_2)$. In fact, we have to show that $\Gamma_{1,1,0}$ has weights $L_i + L_j + L_k, i \neq j \neq k$ occur with multiplicity *two*, and we can do this by writing down $\Gamma_{1,1,0} \subset V \otimes \bigwedge^2 V$.

19.8 Example. Let's look at $V \otimes \bigwedge^3 V$ (for $\mathfrak{sl}_4$) and this contains a copy of the highest weight $L_1 - L_4$. We have a map to $\bigwedge^4 V \simeq \mathbb{C}$ and the kernel of V is the representation $\Gamma_{1,0,1}$. This is exactly the adjoint representation, whose weights are the $L_i - L_j$ (including the case $i = j$). If I think of this as the kernel of $V \otimes V^* \to \mathbb{C}$, then I'm exactly looking at traceless endomorphisms of V, i.e. $\mathfrak{sl}_4$. So

$$V \otimes \bigwedge^3 V \simeq \mathbb{C} \oplus \Gamma_{1,0,1}.$$

That's something you can try to think about in general for $\mathfrak{sl}_n$: how does the adjoint representation fit in?

In general, we'd like to describe $\Gamma_{a_1,\ldots,a_n}$ as a subrepresentation of the tensor product of symmetric powers as a kernel of contraction maps. But that still isn't enough to an irreducible representation. Here's what does work (and these are things called *Schur functors*); I'd urge those of you who are interested to read about in the text (sec. 6.1 and 15.3). You may also have to look at section 4.1, 4.3.

The basic idea is very simple and straightforward. A basic idea that we've seen many times is that

$$V \otimes V \simeq \mathrm{Sym}^2 V \oplus \bigwedge^2 V.$$

All I want to ask: is there an analogous decomposition of $V \otimes V \otimes V$? By analogous, I mean that if V is a representation of anything, then the summands are likewise representations. How would you characterize this? Here's the point: if V is a representation of a group G, then $V \otimes V$ is a representation of G. But it's also a C_2-representation by switching the two factors and that commutes with the action of G. There are two irreducible representations of C_2, the trivial and the alternating one, and we break up $V \otimes V$ via the C_2-pieces.

For $V \otimes V \otimes V$, that's a representation of whatever group acted on V, but it's also a representation of S_3, and I can take this and break it up into the irreducible representations of S_3. There are three irreducible representations of S_3: the trivial one, the alternating one, and the *standard* on $\mathbb{C}^2$, so I get a decomposition

$$V^{\otimes 3} \simeq \mathrm{Sym}^3 V \oplus \bigwedge^3 V \oplus \mathbb{S}_{2,1}(V)^{\oplus 2},$$

where $\mathbb{S}_{2,1}(V)$ is a *Schur functor*. In general, these come up by decomposing tensor powers under the symmetric group action.

To construct the irreducible representations of $\mathfrak{sl}_n$, apply Schur functors to the stnadard one.

Lecture 20
3/15

§1 Geometric plethysm

The goal is to understand decompositions of irreducible representations in *geometric* terms. I'm going to be using the language of algebraic geometry to some degree. I'm going to ask lots of questions that I don't know the answer to, even though they deal with very simple objects, and I urge you to think about them if you have any interest.

The basic idea is: up to now we've been studying representations of groups and algebras, i.e. actions on vector spaces. If I have a linear automorphism of a vector space, that induces an automorphism of the projective space, and we can talk about the geometry of that action. If I look at a group like SL_n, it acts on a vector space $V = \mathbb{C}^n$, and it's transitive on nonzero vectors. Correspondingly the action on $\mathbb{P}(V)$ is transitive. If I look at other representations of SL_n, like $\mathrm{Sym}^n V$, then the action of $SL(V)$ is no longer transitive. Instead, the orbits in projective space are interesting algebro-geometric objects: they are locally closed algebraic subsets whose closures are varieties. We can try to understand the geometry of these varieties.

20.1 Example. Let's work with SL_2. Let V be the standard representation, $V \simeq \mathbb{C}^2$. We have now a language for all the irreducible representations of SL_2: the irreducible representations are exactly $\mathrm{Sym}^p V = \Gamma_n$ by the usual notation system for irreducible representations. Recall, SL_2 acts correspondingly on $\mathbb{P}(V) \simeq \mathbb{P}^1$ and correspondingly on $\mathbb{P}(\mathrm{Sym}^n V) \simeq \mathbb{P}^n$. The action is transitive on $\mathbb{P}^1$ but for $n \geq 2$, it's no longer transitive.

What does this look like? How can I distinguish one polynomial of degree two on a 2-dimensional vector space from another? There really is only one distinction you can make. Among homogeneous degree two polynomials on a two-dimensional vector space, there are the squares, and the non-squares. The products of linearly independent forms, and the squares of forms. There are thus *two* linearly independent orbits. All the squares are congruent mod SL_2, and all the products of distinct linear factors are congruent mod SL_2, because SL_2 acts 3-transitively on $\mathbb{P}^1$. What do the two orbits look like? The squares form a conic curve. The complement of that conic curve are the general quadratic polynomials.

For cubic polynomials, there will also be three orbits under the SL_2-action: the cubes of linear forms (when there is a single triple root), products of squares of linear forms with another linear form (when there is a double root and a distinct simple root), and the general polynomial (a product of three distinct factors). Again, only finitely many orbits.

For quartic polynomials, if you look at the action of SL_2 on $\mathbb{P}(\mathrm{Sym}^4 V)$, you have *continuous* families of quartics: they're distinguished by the j-function. The roots of the quartic are four points on $\mathbb{P}^1$ (defined up to conjugation), and the invariant of them is the j-function. (Once you're above dimension three, there are continuous families.)

In general, SL_2 acts on $\mathbb{P}(\mathrm{Sym}^n V) \simeq \mathbb{P}^n$ and this action preserves the locus of polynomials which are simply nth powers of linear forms.

Remark. If V is any vector space, then the *projectivization* $\mathbb{P}(V)$ of V refers to the space of one-dimensional linear subspaces of V: that is, $(V \setminus \{0\})/\mathbb{C}^*$. (There's some dispute about this; in modern algebraic geometry this would be $\mathbb{P}(V^\vee)$.) Given $v \in V \setminus \{0\}$, we let $[v]$ denote the corresponding point in projective space.

In particular, we have a map

$$\mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^n V), \quad [v] \mapsto [v^n];$$

it isn't a linear map, but it is an algebraic map and the image of this map is a curve called the **rational normal curve.** These are, in some sense, the simplest and most fundamental algebraic varieties which are not complete intersections. Beyond the conic, they're not complete intersections. In coordinates, this is the image of the map in homogeneous coordinates

$$[x, y] \mapsto [x^n : x^{n-1}y : \dots : y^n].$$

(You could object that there should be binomial coefficients here: if I raise $(xe_1+ye_2)^n$, then I'd see these, but I'm just rescaling the coordinates. In characteristic p, this is a big issue.)

In affine coordinates, this is simply the map

$$t \mapsto (1, t, t^2, \dots, t^n).$$

The final thing to say about it is that I can describe the ideal in the ring of all polynomials on $\mathbb{P}^n$: it is generated by quadratic polynomials, specifically if I call the homogeneous coordinates $z_0, \dots z_n$, the polynomnials are

$$z_i z_j - z_k z_l, \quad i+j=k+l,$$

and these cut out the ideal of this polynomial. In particular, the space of *squares* in $\mathbb{P}^2$ is a conic, cut out by a single polynomial, the **discriminant.**

20.2 Example. Let's now focus on the case of SL_2, where we already have a picture. Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$: I can think of them as quadratic polynomials on $\mathrm{Sym}^2 V$. As such, I naturally have a map

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4(V)$$

given by restricting to $\mathbb{P}^1$ to get a homogeneous quartic on $\mathbb{P}^1$. This is clearly surjective. What is the kernel? The kernel is simply the trivial representation, by counting dimensions: it's just the *ideal* of the rational normal curve in degree two, so generated by the discriminant. The kernel is spanned by the unique quadratic polynomial in $\mathbb{P}^2$ vanishing on $\mathbb{P}^1$.

Here's a question. *What is the splitting?* In other words, we have constructed a very natural and obvious exact sequence: but as a representation of SL_2, $\mathrm{Sym}^2\mathrm{Sym}^2 V$ is uniquely a direct sum of $\mathbb{C}$ and $\mathrm{Sym}^4 V$. It's more than an extension, it's a splitting in exactly one way. In particular, $\mathrm{Sym}^4 V \subset \mathrm{Sym}^2\mathrm{Sym}^2 V$. Here $\mathrm{Sym}^4 V$ is quartic polynomials on $\mathbb{P}^1$ and $\mathrm{Sym}^2\mathrm{Sym}^2 V$ is quadratic polynomials on $\mathbb{P}^2$. I can tell you the answer in one form. SL_2 acts as the automorphisms of $\mathbb{P}^2$ that preserve $\mathbb{P}^1$. I'm looking for a space of quadratics on $\mathbb{P}^2$.

Given a conic, I can take its tangent line. If I take the rational normal curve, and look at all the tangent lines, and all their squares, then the span of all those polynomials is this subspace. I want to argue on the grounds that it can't be anything else. You have a map $\mathbb{P}^1 \to \mathbb{P}^5$ sending a point to the tangent line containing it: that's a degree four map $\mathbb{P}^1 \to \mathbb{P}^5$ contained in the image of the Veronese surface, hence is a hyperplane section of that.

Evan suggests taking the map $\mathrm{Sym}^2\mathrm{Sym}^2 V \to \mathbb{C}$ via the discriminant.

20.3 Example. We want a map $\mathrm{Sym}^4 V \to \mathrm{Sym}^2(\mathrm{Sym}^2 V)$ and the claim is that the image is the hyperplane spanned by squares of tangent lines to the rational normal curve. Given a polynomial f of degree one, then there is a "natural" way to write it as a quadratic polynomial of quadratic polynomials. What is a degree four polynomial on $\mathbb{P}^1$? Up to scalars, that's just a four-tuple of points on $\mathbb{P}^1$. If I give four points on the rational normal curve, there's a natural choice of a quadratic in $\mathbb{P}^2$—-i.e., a conic curve—-that intersects the rational normal curve in four points. So I have a way of associating to a conic in the plane and four points another conic which intersects it in those four points. How is this possible?

Let's go to the case $n = 3$. In this case, I get

$$\mathbb{P}^1 \to \mathbb{P}^3$$

imbedded as a *twisted cubic*, $t \mapsto [1, t, t^2, t^3]$; this is the simplest example of a curve in algebraic geometry which is not a complete intersection. Now $\mathbb{P}^3 = \mathbb{P}(\mathrm{Sym}^3 V)$ so we get a short exact sequence

$$0 \to \mathrm{Sym}^2 V \to \mathrm{Sym}^2 \mathrm{Sym}^3 V \to \mathrm{Sym}^6 V \to 0.$$

Again, this is saying that if I have a twisted cubic curve, and I have six points on that twisted cubic, then there is a canonically associated quadric surface containing these. Second question: what is the image of the rational normal curve in $\mathbb{P}(\mathrm{Sym}^2 V) \subset \mathbb{P}(\mathrm{Sym}^2 \mathrm{Sym}^3 V)$? There are distinguished conics vanishing on it, which are the singular quadrics containing the curve...

(I was sufficiently lost here that the notes are incomplete.)

Lecture 21
3/25

§1 $\mathfrak{sp}_{2n}$

The current homework is now going to be due on next Monday instead of today. We're working on specific Lie groups and Lie algebras. Today, we want to do the **symplectic Lie group** (or Lie algebra).

21.1 Definition. Let V be a vector space. Let's start with a nondegenerate, skew-symmetric bilinear form

$$Q : V \times V \to \mathbb{C}.$$

Any bilinear form on a vector space gives a map from a vector space to its dual; it's nondegenerate if and only if this map is an isomorphism. The existence of such a form implies that the dimension of V is even.

We'll write $\dim V = 2n$. If I think of $Q \in \bigwedge^2 V^*$ (that's what it means to be skew-symmetric and bilinear), the condition of being nondegenerate is equivalent to saying that $Q \wedge \cdots \wedge Q \in \bigwedge^{2n} V^*$ is nonzero.

One consequence of that is that *any* automorphism of V that preserves Q necessarily preserves its nth wedge product, and that is a nonzero volume form. In particular, any element of the group Sp_{2n} that I'm about to define has determinant one.

21.2 Definition. We write $\mathrm{Sp}(V, Q)$ to be the set of automorphisms $A : V \to V$ such that $Q(Av, Aw) = Q(v, w)$ for all $v, w \in V$. It's the automorphisms that preserve the quadratic form.

If I take this relation and differentiate it, then I'm going to find that the Lie algebra $\mathfrak{sp}(V,Q)$ is given by the set of *endomorphisms* X of the vector space V with the property that

$$Q(Xv,w)+Q(v,Xw)=0, \quad v,w\in V. \tag{85}$$

The group is defined as the set of automorphisms that preserve the bilinear form. The Lie algebra is the subalgebra of $\mathfrak{sl}_n$ consisting of endomorphisms whose induced action of $\bigwedge^2 V^*$ *kills* Q. That's what it means to preserve in the Lie algebra sense.

We basically have a normal form here. Any such Q can be put in the following form: we can choose a basis $V\simeq\mathbb{C}^{2n}$ such that the bilinear form Q is represented by a standard skew-symmetric matrix in block-diagonal form

$$M=\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}.$$

In other words,

$$Q(v,w)=(v^t)Mw.$$

If I want to write out what condition this defines on the matrix, all I have to do is multiply. If a matrix X is written in block form

$$X=\begin{bmatrix}A & B\\ C & D\end{bmatrix},$$

then

$$\mathfrak{sp}_{2n}=\left\{X:(X^t)M+MX=0\right\}, \tag{86}$$

or in other words

$$C=C^t,\quad B=B^t,\quad A^t+D=0. \tag{87}$$

§2 Cartan decomposition

We now need to choose a maximal diagonalizable abelian subalgebra. Here we're just going to take the *diagonal elements in this Lie algebra*. If it's not maximal, we'll find out when we perform the root space decomposition.

Let's recall that E_{ij} is the matrix which is 1 in the (i,j)th entry and all other entries are zero. If you want to think of it as a linear transformation, it's the map

$$e_j\mapsto e_i,\quad e_k\mapsto 0,\forall k\neq j.$$

In other words, we take

$$\mathfrak{h}=\mathbb{C}\left\langle E_{ii}-E_{i+n,i+n}\right\rangle_{1\leq i\leq n}, \tag{88}$$

where we take $H_i=E_{ii}-E_{i+n,i+n}$ as the basic diagonal elements. For $\mathfrak{h}^*$, I'm going to choose the dual basis L_i, so that L_i is the linear functional which is value 1 on H_i and zero on the other $H_j, j\neq i$.

Now we need to describe the root spaces. If I look at the action of $\mathfrak{h}$ on the rest of the Lie algebra, what are the eigenspaces? If I have a given diagonal element, and I want to know what it's commutator with an arbitrary E_{ij}, it's precisely

$$[\sum a_rE_{rr},E_{ij}]=(a_i-a_j)E_{ij}. \tag{89}$$

This is true in any algebra of matrices.

Now let's try to pick matrices in $\mathfrak{sp}_{2n}$.

- The simplest thing to do would be to take A so that it has one nonzero entry, and B and C to be zero. So for $i \neq j \in [1, n]$, we can take
$$E_{ij} - E_{n+j,n+i}.$$
This is an eigenvector for $\mathfrak{h}$ with eigenvalue $L_i - L_j$.

- The other thing we can do is to take B to be an arbitrary nonzero symmetric matrix and everything else zero. For $i \neq j \in [1, n]$, we can take $E_{i,n+j} - E_{j,n+i}$. This is an eigenvector with eigenvalue $L_i + L_j$.

- We can do the opposite thing: take C to be nonzero and everything else to be zero. That is, we take $E_{n+i,j} + E_{n+j,i}$ and the eigenvalue is $-L_i - L_j$.

- I can also take $E_{i,n+i}$ or $E_{n+i,i}$. The eigenvalues here are $2L_i$ and $-2L_i$, respectively.

For $n = 2$, we're in the plane. The basic picture is that the roots are $L_i - L_j, i \neq j$, $L_i + L_j$, $-L_i - L_j$, and the $2L_i, -2L_i$. The roots form a square in the plane when you draw them. In the case $n = 3$, you can write them out—probably the best thing is not to draw the axes but to draw a cube whose sides have length two, centered at the origin. The midpoints of the faces are simply the $\pm L_i$. In terms of this cube, the roots are simply the midpoints of the edges, and then also the $\pm 2L_i$. What we get is the vertices of an octahedron... The vertices of the octahedron are the $\pm 2L_i$.

Remark. Also, $\mathfrak{sl}_2$ is the same as $\mathfrak{sp}_2$.

Let's get to work again. We're supposed to describe the *distinguished subalgebras* of $\mathfrak{sp}_{2n}$. If you recall, these are copies of $\mathfrak{sl}_2$ sitting inside this Lie algebra, and they're gotten by taking pairs of opposite root spaces and their commutator. That gives me a 3-dimensional subspace. For example, if I take the distinguished subalgebra associated to $L_i - L_j$, that means that I take
$$\mathbb{C}\left\langle E_{ij} - E_{n+j,n+i}, E_{ji} - E_{n+i,n+j}, H_i - H_j \right\rangle.$$
This subalgebra, which is just $\mathfrak{sl}_2$, is great: we need, though, to normalize it so that we can identify the distinguished element which acts by ± 2. What multiple of $H_i - H_j$ acts on the two preceding basis elements with eigenvalues ± 2? The answer is, it's $H_i - H_j$ itself. The distinguished element, which we typically call $H_{L_i - L_j}$, is just $H_i - H_j$.

We do the same for the rest:

- $\mathfrak{s}_{L_i+L_j}$ has distinguished element $H_i + H_j$.

- $\mathfrak{s}_{-L_i-L_j}$ has distinguished element $-H_i - H_j$. (Actually, this one is included the previous one.)

- $\mathfrak{s}_{2L_i}$ has distinguished element H_i.

- (We don't need to do $\mathfrak{s}_{-2L_i}$.)

The Killing form is pretty much as it's drawn; one thing that's in contrast to [$\mathfrak{sl}_n$ is that the roots were exactly these $L_i - L_j$ in this quotient of $\mathbb{C}\langle L_i \rangle$. They were all conjugate under the Weyl group and they all had the same length. That's not true here. There are two types of root lengths: there are the $\pm L_i \pm L_j, i \neq j$ (the **short roots**) and the **long roots** $\pm 2L_i$. There will always be **at most two** distinct lengths, in any simple Lie algebra. You'll see, when we do the odd orthogonal groups, is that the role of the long and short roots will be different.

The point of all this is, having defined and described these distinguished elements, we know that under any representation of $\mathfrak{sp}_{2n}$, the eigenvalues of these distinguished elements are always integers. So any allowable weight has to take *integer values on the H_i*. That is, the allowable weights are the integral linear combinations of the H_i. (This is going to be different in the orthogonal algebra: that's going to lead to the spin representations.) The crucial detail here is that there's no coefficient in front of the distinguished elements H_i.

21.3 Proposition. *The **weight lattice** Λ_W is simply $\mathbb{Z}\langle L_1, \dots, L_n \rangle$.*

We also see what the Weyl group is: the Weyl group is generated by reflection around the hyperplanes orthogonal to the roots. The roots are the $\pm L_i \pm L_j$.

The **Weyl group** is generated by reflections around the short roots $L_i - L_j, i \neq j$ (which exchanges L_i and L_j) and the reflection around the long roots $2L_i$ (which flips $L_i \mapsto -L_i$ and leaves the rest fixed). In general, the Weyl group takes $\{L_1, \dots, L_n\}$ into $\{\pm L_{\sigma(1)}, \dots, \pm L_{\sigma(n)}\}$. If you think of this as a group of automorphisms of $\mathfrak{h}$, it fixes the axes—as axes, as lines, and collectively (not individually). We have a map from the Weyl group $\mathcal{W}$ to S_n describing how it acts on the coordinate axes and the kernel is simply $(\mathbb{Z}/2)^n$. We have an exact sequence

$$0 \to (\mathbb{Z}/2)^n \to \mathcal{W} \to S_n \to 0,$$

and it's a so-called **wreath product.** (A semi-direct product.)

Let me just do a couple more things and then we'll get started with examples on Wednesday. We're almost at the end of our algorithm at this point. The main thing that's left at this point is to choose an ordering of the roots. So I'm going to do that. That's going to be a linear functional $\ell : \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ and I'm going to take the one that sends $\sum a_i L_i \mapsto \sum c_i a_i$ with $c_1 > c_2 > c_3 > \cdots > c_n > 0$. That's an ordering of the roots. In those terms, we can describe the positive and the negative roots.

- The positive roots are the $\pm 2L_i, L_i + L_j$ and, for $i < j$, $L_i - L_j$.
- The negative roots are the rest.
- The *primitive* positive roots are the $L_1 - L_2, L_2 - L_3, \dots, L_{n-1} - L_n$, and finally $2L_n$. (As predicted, there are n of these.)

What does this mean? It means that the **Weyl chamber** is the intersection of the half-planes corresponding to those vectors, againa simplicial cone. The Weyl chamber consists of the vectors here that form an acute angle with each of the positive roots, i.e. forms a positive angle with the primitive positive roots. So that's the locus of linear combinations of $\sum a_i L_i$ such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. That's the common intersection of n linear inequalities on this real vector space. In particular, it has n faces and n edges: the edges we get the ones where we get equality in all but one of these: that is: where $a_1 = a_2 = \cdots = a_k \geq a_{k+1} = \cdots = a_n = 0$.

The **fundamental weights** are the smallest weight vectors along each of these n edges, and they are

$$L_1, L_1 + L_2, L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_n.$$

We see that, exactly as predicted, that the intersection of the Weyl chamber with the weight lattice is just the semigroup generated by these n vectors.

Lecture 22
3/27

Today, we're going to focus on $\mathfrak{sp}_4$.

§1 Recap of $\mathfrak{sp}_{2n}$

Let's recall where we were for $\mathfrak{sp}_{2n}$ in general:

- The symplectic Lie algebra consists of block matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where B, C are symmetric and $A = -D^t$.
- $\mathfrak{h} = \mathbb{C}\langle H_i = E_{ii} - E_{n+i,n+i}\rangle$.
- The roots divide into four components:

root α	$\mathfrak{g}_\alpha$	H_α
$L_i - L_j$	$E_{ij} - E_{n+j,n+i}$	$H_i - H_j$
$L_i + L_j$	$E_{i,n+j} + E_{j,n+i}$	$H_i + H_j$
$-L_i - L_j$	$E_{n+i,j} + E_{n+j,i}$	$-H_i - H_j$
$2L_i$	$E_{i,n+i}$	H_i
$-2L_i$	$E_{n+i,i}$	$-H_i$

- The weight lattice is the standard rectilinear lattice $\mathbb{Z}\{L_i\}$. The root lattice is a sublattice of index two. This corresponds to the fact that the center of the simply connected form is $\mathbb{Z}/2 = \{\pm I\}$. If you look at the adjoint form, the fundamental group is $\mathbb{Z}/2$.
- The **positive** roots are the $2L_i, L_i + L_j$ and also $L_i - L_j$ for $i < j$.
- The **primitive positive roots** are the $L_i - L_{i+1}$ and $2L_n$.
- Once we identify a highest weight vector in an irreducible representation, we just have to take that and repeatedly apply the primitive negative root spaces to get the whole thing.

- The **Weyl chamber** is the locus of linear combinations $\sum a_i L_i$ such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$.
- We have **fundamental weights**

$$L_1, L_1 + L_2, L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_n.$$

  These are the first integral weights that occur on the edges of the Weyl chamber. Every weight lattice point inside the Weyl chamber is a nonnegative linear combination of these.

- We set some notation: for $a_1, \ldots, a_n$, we let $\Gamma_{a_1,\ldots,a_n}$ be the irreducible representation of weight $a_1 L_1 + \cdots + a_n(L_1 + \cdots + L_n)$. We'll see in just a moment that they do actually exist.
- The **Weyl group** is the group of orthogonal transformations of $\mathbb{R}^n$ that preserve the union of the coordinate axes: it's a wreath product of S_n and $(\mathbb{Z}/2)^n$.

§2 Examples

22.1 Example. Let me draw once more this standard picture: for $\mathfrak{sp}_4$, this is in the plane. There is the **standard representation** $V \simeq \mathbb{C}^4$. The standard basis vectors $e_1, \ldots, e_4$ are eigenvectors with eigenvalues $L_1, L_2, -L_1, -L_2$ for the diagonal subalgebra. This corresponds to $\Gamma_{1,0}$. The weights of any representation are symmetric under the Weyl group. In particular, for the symplectic Lie algebra, that the weights are invariant under transformation $x \mapsto -x$. Since representations are determined by their weights, we conclude that **any representation is isomorphic to its dual.** (This wasn't true for $\mathfrak{sl}_n, n > 2$.)

22.2 Example. A natural thing to start doing is to take tensor products. Let's start with $\bigwedge^2 V$. The weights here are the pairwise sums of the two distinct weights of V. We get $\pm L_i \pm L_j$ for $i \neq j$ and we also get 0 with multiplicity two. This gives us six weights. Is this irreducible? You can see that it is *not* irreducible in two different ways. Start with the highest weight vector with weight $L_1 + L_2$: that corresponds to $e_1 \wedge e_2$. Whatever it corresponds to, an irreducible representation is obtained by applying the primitive negative root spaces. The primitive negative vectors are $L_2 - L_1$ and $-2L_2$. You can only get from $L_1 + L_2$ to 0 in one way.

But also: by definition, the symplectic Lie algebra consists of endomorphisms that preserve a skew-symmetric bilinear form. We have a natural map $\bigwedge^2 V \to \mathbb{C}$ which is contraction with applying the skew-symmetric bilinear form.

In any case, I conclude that $\bigwedge^2 V$ has a one-dimensional subrepresentation. The quotient $\bigwedge^2 V/\mathbb{C}$ has weights of multiplicity one and is consequently irreducible, since otherwise 0 would have to occur twice. We now get $\Gamma_{0,1}$: the highest weight is $L_1 + L_2$.

22.3 Example. Now that we've constructed $\Gamma_{1,0}$ and $\Gamma_{0,1}$, we've proved the existence theorem for irreducible representations. In fact,

$$\Gamma_{a,b} \subset \mathrm{Sym}^a \Gamma_{1,0} \otimes \mathrm{Sym}^b \Gamma_{0,1}.$$

22.4 Example. What is $\mathrm{Sym}^2 V$? The weights, first of all, are the pairwise sums of the weights of V, but now we're allowing repeated weights. So that would be $\pm L_i \pm L_j (i \neq j)$, $\pm 2L_i$, and 0 taken twice. It's right there: these are the weights of the adjoint representation. This is also $\Gamma_{2,0}$ since the highest weight is $2L_1$. Think about this for a little bit. The adjoint representation is the symmetric square of the standard representation. For the orthogonal representation, it's $\bigwedge^2$ of the standard representation.

22.5 Example. Writing $W = \Gamma_{0,1}$, let's consider $V \otimes W$ (twenty-dimensional). The weights are going to be the pairwise sums of the weights of V and the weights of W. So we get:

- $\pm 2L_i \pm L_j$ (twelve of these).
- $\pm L_i$ (each taken three times).

This contains a weight vector for $2L_1 + L_2$. In other words, $e_1 \otimes (e_1 \wedge e_2)$. I'm claiming that if I apply the primitive negative root spaces to this, I can't get to everything else in the representation. In other words, I'm claiming that the representation is not irreducible. In fact, $V \otimes W \subset V \otimes \bigwedge^2 V$ and $V \otimes \bigwedge^2 V \to \bigwedge^3 V$. Now $\bigwedge^3 V \simeq V^* \simeq V$. So inside $V \otimes W$, there's a copy of the standard representation. In fact, the map $V \otimes \bigwedge^2 V \to \bigwedge^3 V$ is simply a natural map in the exterior algebra of V. The kernel of this can't include $V \otimes W$, and the composite map is not zero.

So we can split off a copy of V from the tensor product $V \otimes W$. Is what's left irreducible? What's left only assumes the weights $\pm L_i$ with multiplicity *two* rather than three and is sixteen-dimensional. Let's look at $\mathrm{Hom}(V \otimes W, V)$. If in fact there were two copies of V inside $V \otimes W$, then we'd have a two-dimensional space of maps $\mathrm{Hom}(V \otimes W, V)$. But we can also write this as $\mathrm{Hom}(W, V \otimes V)$. And we already know how to determine this because we know how $V \otimes V$ splits as a sum of irreducibles.

You can also do this by direct calculation. Given the time, I won't do that in full. We can check the map (which is the wedge product) $V \otimes W \to V$ is in fact irreducible, and hence isomorphic to $\Gamma_{1,1}$. By explicit calculation, we can check irreducibility.

22.6 Example. We can look at $\bigwedge^2 W$ and $\mathrm{Sym}^2 W$. Here we know the weights of W, so we get for the weights of $\mathrm{Sym}^2 W$, $\pm 2L_i \pm 2L_j$, $\pm L_i \pm L_j$, $\pm 2L_i$, and 0 with multiplicty three. It clearly contains $\Gamma_{0,2}$. Does it contain other stuff as well? We have a wedge product map

$$\bigwedge^2 V \times \bigwedge^2 V \to \bigwedge^4 V \simeq \mathbb{C},$$

which is symmetric. It's a symmetric bilinear pairing and it factors through $\mathrm{Sym}^2(\bigwedge^2 V)$ and you can restrict to $\mathrm{Sym}^2 W \to \mathbb{C}$ and that splits off a trviail summand. So

$$\mathrm{Sym}^2 W \simeq \Gamma_{0,2} \oplus \mathbb{C}.$$

(Observe that it can't have more than one trivial factor because $\mathrm{Hom}(W, W)$ is one-dimensional.)

22.7 Example. More interesting is $\bigwedge^2 W$. Here we're looking at pairwise sums of distinct sums of W. The weights of W are $\pm L_i \pm L_j$ and 0. So the weights of $\bigwedge^2 W$ are $\pm 2L_i, \pm L_i \pm L_j$ and 0 taken twice. The same trick above shows that $\mathbb{C}$ can't show up twice. But this is in fact the adjoint representation. This is the interesting observation: we have an interesting isomorphism

$$\mathrm{Sym}^2 V \simeq \bigwedge^2 W.$$

It's a reflection of something we're going to start seeing next time, when we do the next string of classical Lie algebras. There are low-dimensional coincidences among low-dimensional classical Lie algebras. This reflects

$$\mathfrak{sp}_4 \simeq \mathfrak{so}_5.$$

Now $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, $\mathfrak{so}_4 \simeq \mathfrak{sp}_4$, and $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$, and then after that there aren't any more low-dimensional coincides. We'll see how the root systems end up looking the same.

By the way, we haven't proved at this point that two simple Lie algebras with the same root system are isomorphic. Let me just do one more thing for $\mathfrak{sp}_{2n}$ in general and then we'll call it quits for symplectic Lie algebras.

22.8 Example. Let's work with $\mathfrak{sp}_{2n}$ in general. Start with the standard representation $V \simeq \mathbb{C}^{2n}$ with highest weight L_1. That suggests we can find the other irreducible representations with the fundamental weights by looking at exterior powers. Now $V \simeq \Gamma_{1,0,\dots,0}$. Then $\bigwedge^k V$ is going to contain a copy of $\Gamma_{0,\dots,1,\dots,0}$ with 1 in the kth spot. The highest weight vector is $e_1 \wedge \cdots \wedge e_k$. That proves the existence theorem in general. It's natural to ask how to actually describe the irreducible representations. There is a general construction, an analog of Weyl's construction by taking kernels of contraction maps. You can read about it in the text. There is one case where I can tell you how to identify the irreducible representations, and that is for the fundamental representations. If I look at $\bigwedge^k V$, since I have a natural element of $\bigwedge^2 V^*$, I have a contraction map

$$\bigwedge^k V \to \bigwedge^{k-2} V.$$

The claim is that the kernel of this map is the irreducible representation with highest weight $L_1 + \cdots + L_k$. You can see this by this cute trick. The time we spent analyzing representations of $\mathfrak{sl}_n$ is not wasted, because inside $\mathfrak{sp}_{2n}$, we have a subspace $\mathfrak{sl}_n$: the block-diagonal matrices $\begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix}$. So you can study the kernel as a representation of $\mathfrak{sl}_n$.

Lecture 23
3/29

Correction (courtesy Yale Fan): in Exercise 17.4 of the textbook, the correct multiplicities are 0, 1, and 4.

§1 Plans

Let's start with the **orthogonal Lie algebras**. There's a distinction between the even and odd cases, $\mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n+1}$. The even case will be notationally much simpler. The odd case will be more closely related to the symplectic Lie algebras.

One thing that's different for the orthogonal Lie algebras is that the weights of the standard representation don't generate the weight lattice. The significance of that is, you can't find all the representations of $\mathfrak{so}_n$ sitting inside the tensor algebra of the standard representation, as we have seen in examples up till now. Rather, the weights of the standard representation generate an index 2 subgroup of the weight lattice. We need to add the **spin representations,** which aren't representations of the orthogonal group but of the universal cover.

Even though the weights of the spin representation is smaller than the highest weight of the standard representation, the representation associated to it is *much bigger* (than $2n$ or $2n+1$). The spin representation has dimension of the order 2^n. Don't start looking for it among low-dimensional representations.

Also, we'll start talking about **coincidences.** The first four orthogonal Lie algebras (I'm not counting $\mathfrak{so}_4$) all coincide with Lie algebras we've seen before. For me, by far the best way to see this is by algebraic geometry. You can use the adjoint forms of these Lie algebras as automorphism groups of algebraic varieties. In those terms, it'll fall right out — the coincides we have in low dimensions.

Here is a list of coincidences:

- $\mathfrak{so}_3 \simeq \mathfrak{sl}_2$.
- $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.
- $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$.
- $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$.

We'll see why this is true, but **that's it.** After this, all the orthogonal Lie algebras will be new.

§2 $\mathfrak{so}_{2n}$

23.1 Definition. The **group** $SO(V,Q)$ (over the complex numbers) is the group of automorphisms of a complex vector space V that preserve a nondegenerate symmetric bilinear form Q on V.

Over the complex numbers, all nondegenerate symmetric bilinear forms are congruent. They're all the same. Given a nondegenerate symmetric bilinear form, we can always choose a basis in which it's the standard quadratic form given by the identity matrix. We can also choose a basis in which it's given by a block antidiagonal matrix, like the skew-symmetric form that we dealt with in the symplectic case.

Let's suppose $\dim_{\mathbb{C}} V = 2n$. Choose a basis $e_1, \dots, e_{2n}$ for V such that

$$Q(e_i, e_{n+i}) = Q(e_{n+i}, e_i) = 1, \quad Q(\text{all others}) = 0.$$

In other words, Q is given by

$$Q(x,y) = (x^t)\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} y,$$

so the matrix has the same form as in the symplectic case, except that a minus sign is missing.

The Lie algebra $\mathfrak{so}_{2n}$ is thus the Lie algebra of block matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ such that

$$A = -D^t, \quad B, C \text{ skew-symmetric.}$$

Why do we choose this quadratic form and not the diagonal form? I'm going to hold off on answering that question for a few minutes.

The rest of the analysis is going to be similar.

We're going to take $\mathfrak{h}$ to be the diagonal matrices in $\mathfrak{so}_{2n}$. In other words, $\mathfrak{h}$ is the span of the diagonal matrices $H_i = E_{ii} - E_{n+i,n+i}$. (These are the same elements as in the symplectic Lie algebra.)

We're going to write $\mathfrak{h}^* = \mathbb{C}\langle L_1, \ldots, L_n\rangle$ where the L_i are the dual basis to the E_i.

Next, we've got to describe the **roots**. We have pretty much the same roots. Now A and D satisfy the same relations as before. So we can look at block diagonal matrices (where $C = B = 0$), and there it's the same calculation we made last time.

- We take $E_{ij} - E_{n+j,n+i}$ as a root space corresponding to the root vector $L_i - L_j$.
- We can also take $E_{i,n+j} - E_{j,n+i}$, which has eigenvalue $L_i + L_j$.
- $E_{n+i,j} - E_{n+j,i}$ with eigenvalue $-L_i - L_j$.
- We don't see the remaining $2n$ roots that we saw for the symplectic case because of the skew-symmetry condition.

Next, we want to describe the distinguished elements for each of the roots. They're the same as in the symplectic case. We have

$$H_{\pm L_i \pm L_j} = \pm H_i \pm H_j. \tag{90}$$

So again, the roots are $\pm L_i \pm L_j$.

23.2 Example. For instance, when $n = 2$, we have just four roots: they form the vertices of a square. There's one thing that you can see right off the bat. This configuration of roots is contained in the union of two orthogonal lines. In general, if you look at a semisimple Lie algebra, and you find that the roots all lie in the union of complementary subspaces, that tells you that the Lie algebra is actually a direct sum. If you act on any of the roots in one subspace by something in the other, you'll get zero because there are no place to go. That gives the decomposition

$$\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2,$$

and while we'll see this from other points of view, you can see it from the root diagram.

23.3 Example. Let's consider $n = 3$. Then again the roots are $\pm L_i \pm L_j$ where the L_i are standard basis vectors. You can see this by taking a cube centered at the origin and taking the midpoints of the edges. We've seen this root diagram before, and it's the root diagram of $\mathfrak{sl}_4$, and that reflects a coincidence.

We'll prove that a (semisimple) Lie algebra is determined by its root diagram, but right now we don't know that, so we haven't proved the exceptional isomorphisms above, but it's motivation.

§3 $\mathfrak{so}_{2n+1}$

The odd orthogonal case is more complicated because we can't use the same notation. We're going to choose a basis of our $2n+1$-dimensional vector space V with quadratic form Q such that

$$Q(e_i, e_{n+i}) = 1, \quad Q(e_{2n+1}, e_{2n+1}) = 1, \quad Q(\text{all others}) = 0.$$

The matrix giving the quadratic form is

$$\begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is as close to what we can get as before. I can now write the Lie algebra $\mathfrak{so}_{2n+1}$ as the space of matrices in a similar block form:

$$\mathfrak{so}_{2n+1} = \left\{ X = \begin{bmatrix} A & B & E \\ C & D & F \\ G & H & J \end{bmatrix}, \right\} \tag{91}$$

where we have the conditions:

- B, C skew-symmetric.
- $A = -D^t$.
- $E = -H^t$, $F = -G^t, J = 0$.

The Cartan subalgebra $\mathfrak{h}$ is generated by the same elements H_i: note that we don't get any new diagonal elements from $\mathfrak{so}_{2n}$. All the root spaces we had for $\mathfrak{so}_{2n}$ are still root spaces. However, we get new root spaces. For instance, we get new root vectors

$$E_{i,2n+1} - E_{2n+1,n+i},$$

and this is an eigenvector for $\mathfrak{h}$ with root L_i. We also get new root vectors

$$E_{n+i,2n+1} - E_{2n+1,i}$$

and the root is $-L_i$. So this looks a lot like the symplectic Lie algebra, except there the roots were $\pm 2L_i$ rather than $\pm L_i$. We can also describe the distinguished elements:

$$H_{L_i} = 2H_i, \quad H_{-L_i} = -2H_i. \tag{92}$$

The difference here is that the roots here are *half* their previous twice and the distinguished elements are twice their previous size.

For example, when $n = 2$, we have the four roots $\pm L_i \pm L_j$ from before, but also the $\pm L_i$. So it's the vertices and edge-midpoints of a square. Note that if you rotate this by 45 degrees, you see the isomorphism $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$ (at least at the level of root systems).

I just want to introduce some language here. There's a distinction here that we can make between the case of $\mathfrak{sl}_n$ and the *even* orthogonal algebras $\mathfrak{so}_{2n}$. In these cases, **all roots are the same length.** However, for $\mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}$, we have exactly **two different root lengths.** This will be true for the exceptional Lie algebras as well. This distinction is significant enough in practice is that the first are called **simply laced** and the second are called **not simply laced.** Maybe you're already familiar with this from algebraic geometry, but there are a number of theorems proved for simply laced things and not in general.

§4 Weyl group and weight lattice

Let's look at the Weyl group. Once again:

- The roots of $\mathfrak{so}_{2n}$ are $\pm L_i \pm L_j$. (The distinguished elements are $\pm H_i \pm H_j$.)
- The roots of $\mathfrak{so}_{2n+1}$ are $\pm L_i \pm L_j, \pm L_i$. (The distinguished elements are $\pm H_i \pm H_j$ and $\pm 2H_i$.)

The **Weyl group** is the group of transformations generated by reflections in the hyperplanes perpendicular to the roots. In the case of $\mathfrak{so}_{2n+1}$, it is the same as for $\mathfrak{sp}_{2n}$: it's the wreath product that we saw before. (It's all transformations that preserve the union of the coordinate axes.) For $\mathfrak{so}_{2n}$, we have fewer reflections to play with: only the $\pm L_i \pm L_j$. Reflecting in $L_i - L_j$ exchanges L_i and L_j, so we get the permutations. We also get some sign changes—but we can only change an even number of signs. So again, what we see is that the Weyl group surjects on S_n (corresponding to the action on n coordinate axes) and the kernel is $(\mathbb{Z}/2)^{n-1}$ and in the kernel, we can only flip an even number of signs.

In the case of $\mathfrak{so}_{2n}$, the **weight lattice** consists of linear functionals that have integral values on all of the $\pm H_i \pm H_j$. So that includes the L_i, but it also includes $\frac{1}{2}(\sum L_i)$. That'll still have integral values on all of these. The weight lattice is

$$\mathbb{Z}\left\langle L_1, \ldots, L_n, \frac{L_1 + \cdots + L_n}{2} \right\rangle.$$

In the case of the odd orthogonal Lie algebras, the same thing happens, because of this 2 that pops up. Because we have this $2H_i$, we get the same weight lattice.

§5 Remarks

Why do we prefer this off-diagonal quadratic form to the standard one? Why not take the one given by the identity matrix? The answer has to do with the distinction of working over $\mathbb{R}$ or over $\mathbb{C}$. Over $\mathbb{C}$, any two nondegenerate symmetric bilinear forms are equivalent. Over the reals, they're different: you have the **index** of a symmetric bilinear form. The one given here has n positive and n negative eigenvalues (or $n+1$ positive and n negative).

You see the differences in the Lie groups. Ultimately, we want to understand not just complex Lie algebras, but real Lie groups. Eventually we want to ask, what are the real Lie algebras that complexify to a given one (say, $\mathfrak{so}_n$). When we take the corresponding Lie groups, we're going to get differently behaved Lie groups.

Say V is a *real* vector space with quadratic form $Q : V \times V \to \mathbb{R}$. Let $SO(V,Q)$ be the automorphisms of V preserving Q. I'll let $\mathfrak{so}(V,Q)$ be its Lie algebra. When I complexify this, I'm going to get the standard $\mathfrak{so}_n$, but over the reals, there are different Lie algebras. Let's see how they're reflected in the group.

Take $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{so}(V,Q)$ be the intersection of the usual complex subalgebra $\mathfrak{h}$ with this. Let's look at the corresponding Cartan subalgebra in the real case. What does it look like when I exponentiate? In all the cases I've looked at, when we take an abelian Lie algebra—the Cartan subalgebra in any one of these algebras—the corresponding Lie subgroup is $(\mathbb{C}^*)^n$. If I let H be the exponential of $\mathfrak{h}_{\mathbb{R}}$ inside $SO(V,Q)$, then this can take different forms. If Q is given by the identity matrix, then H is simply a compact torus. If Q is given by the off-diagonal form, then $H \simeq \mathbb{R}^n$. It's not something we're going to use logically, but when we talk about different real forms of a complex Lie algebra, then we make a distinction based on what the exponential of the Cartan subalgebra looks like (some product of (S^1)'s and $\mathbb{R}$'s). When it's S^1's, then we talk about the **compact form.** When it's a collection of copies of $\mathbb{R}$, it's called the **split form.** Over non-algebraically closed fields, you're going to see different Lie groups with the same Lie algebra after complexifications.

This is true, not just for the orthogonal Lie algebras, but for all of them. There are compact forms, split forms, and all others inbetween. If I look at $\mathfrak{sl}_{n+1}\mathbb{C}$, then I have two forms. I have $\mathfrak{sl}_n\mathbb{R}$ (the split form) and the compact form $\mathfrak{su}_n$. It's not just the Cartan subalgebra that is compact, it's the whole thing.

Lecture 24
4/1

§1 $\mathfrak{so}_n$

Once again, we're working with $\mathfrak{so}_n$. We recall that:

- The roots for $\mathfrak{so}_{2n}$ are the $\pm L_i \pm L_j$: the distinguished elements are the $\pm H_i \pm H_j$.
- For $\mathfrak{so}_{2n+1}$, the roots are $\{\pm L_i \pm L_j, \pm L_i\}$ and the distinguished elements are $\pm H_i \pm H_j$ and $\pm 2H_i$.

The weight lattice is $\Lambda_W = \mathbb{Z}\langle L_1, \dots, L_n, \alpha\rangle$ where $\alpha = \frac{1}{2}\sum L_i$. We discussed the Weyl group last time.

The **Weyl chambers are as follows:**

- For $\mathfrak{so}_{2n+1}$, the Weyl chamber is $\mathcal{W} = \{\sum a_i L_i, \quad a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}$.

- For $\mathfrak{so}_{2n}$, the Weyl chamber is twice the size: it's $\sum a_i L_i$ such that $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n|$ — in particular, a_n doesn't have to be positive.

The **edges** of $\mathcal{W}$ are as follows:

- For $\mathfrak{so}_{2n+1}$, the fundamental weights are simply $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1}$, and finally $\alpha \equiv \frac{L_1+\cdots+L_n}{2}$. They're the same as for the symplectic Lie algebra except that the last fundamental weight can be divided by two.

- For $\mathfrak{so}_{2n}$, the fundamental weights are $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-2}, \alpha \equiv \frac{L_1+\cdots+L_n}{2}, \beta \equiv \frac{L_1+\cdots+L_{n-1}-L_n}{2}$.

We've defined in particular these two fundamental weights α, β.

Remark. Remember that the L_i are the dual basis to the basis $\{H_i = E_{ii} - E_{n+i,n+i}\}$ of $\mathfrak{h}$.

Remark. The odd orthogonal algebras look a lot more like the symplectic algebras.

Remark. The weights of the standard representation are:

- For $\mathfrak{so}_{2n}$, they're $\pm L_i$ (with standard basis vectors e_i).

- For $\mathfrak{so}_{2n+1}$, they're $\pm L_i$ and 0 (the last basis vector is killed by the Cartan subalgebra).

Remark. If I have a nondegenerate bilinear form on a vector space V invariant under the action of the Lie algebra, that gives an isomorphism $V \simeq V^*$, and so

$$\mathrm{End}(V) \simeq V \otimes V \simeq \bigwedge^2 V \oplus \mathrm{Sym}^2 V,$$

and in these terms, the orthogonal Lie algebra is the skew-symmetric part (if the bilinear form is symmetric) and the symplectic Lie algebra is the symmetric part. More precisely, if the bilinear form is symmetric, then the subalgebra of $\mathfrak{gl}(V)$ preserving it is $\bigwedge^2 V$, and if the bilinear form is skew-symmetric, then the subalgebra preserving it is $\mathrm{Sym}^2 V$.

The point is, the adjoint representation for the orthogonal Lie algebra is exactly $\bigwedge^2$ of the standard representation. Its weights are just the pairwise sums of distinct elements. That lets you see the roots of these two Lie algebras: they're the pairwise sums of distinct weights of the standard representation.

Remark. If you take a *general* bilinear form, what is the subalgebra of $\mathfrak{gl}(V)$ preserving it? You'll find that it's too small to be interesting.

Remark. Why look only at bilinear forms? We could pick any element of the tensor algebra of V or V^* and look at the subalgebra of $\mathfrak{gl}(V)$ preserving it. The answer is that you don't get interesting elements that way. If you look at elements of $\bigwedge^3 V$, they don't tend to have many automorphisms.

Remark. Of course, the weights of the standard representation are all $\mathbb{Z}$-linear combinations of the $\{L_i\}$. In particular, if you take tensor products or duals of the standard representation, you don't get everything in the weight lattice. There are also **spin representations**.

But think about it: low-dimensional orthogonal subalgebras tend to coincide with other algebras already described, whose representations we've understood. So the low-dimensional spin representations should be already described in terms of what we already understand.

§2 Low-dimensional isomorphisms

I also want to show you the isomorphisms between $\mathfrak{so}_{\{3,4,5,6\}}$ and other Lie algebras via algebraic geometry.

24.1 Example. $\mathfrak{so}_3$. The roots are $\pm L_1$ and that looks like $\mathfrak{sl}_2$. Of course, it is $\mathfrak{sl}_2$: we have an isomorphism

$$\mathfrak{so}_3 \simeq \mathfrak{sl}_2.$$

From a geometric point of view, I want to give an isomorphism of the *adjoint forms* of these two Lie algebras. The adjoint form of $\mathfrak{sl}_2$ is PGL_2 and this is the group of automorphisms of the Riemann sphere, which algebraic geometers like to call $\mathbb{P}^1$ and draw as a line. You can imbed this by what's called the **Veronese imbedding** as a conic curve C in $\mathbb{P}^2$. For instance, the map $\mathbb{P}^1 \to \mathbb{P}^2$ given by monomials of degree two. The image is a conic curve: it's the zero locus of a quadratic form.

There are two things to see: all automorphisms of $\mathbb{P}^1$ induce automorphisms of $\mathbb{P}^2$ that carry the conic to itself. This gives an isomorphism

$$PGL_2 \simeq \mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{Aut}(\mathbb{P}^2, C),$$

and the automorphisms of $\mathbb{P}^2$ that preserve C are the ones which are (up to scalars) orthogonal transformations. In other words, this is $PSO_3(\mathbb{C})$.

This is the first example where we've already seen the spin representation, although we didn't know to call it that. The **standard representation** of $\mathfrak{so}_3$ is exactly the Sym^2 of the standard representation of $\mathfrak{sl}_2$. You can see this by looking at the weights. The **standard representation** of $\mathfrak{sl}_2$ is the spin representation of $\mathfrak{so}_3$. To find the spin representation, we have to take a sort of square root of the standard representation.

24.2 Example. Here there's $\mathfrak{so}_4$. The thing here is, the root diagram for $\mathfrak{so}_4$ has four roots, $\pm L_1, \pm L_2$. We saw that this is the root diagram of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$: note that the root diagram is contained in the union of two orthogonal subspaces. The adjoint representation is not irreducible: the Lie algebra is reducible.

In fact, we have an isomorphism

$$\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2.$$

Again, to identify the spin representations, we note that the adjoint representation breaks up into two irreducible representations—which are exactly the doubles of the fundamental (standard) representations of each copy of $\mathfrak{sl}_2$. These again are the symmetric squares of the spin representations.

Let me take a minute to describe this at adjoint forms. PSO_4 is automorphisms of $\mathbb{P}^3$ that preserve a *smooth quadric hypersurface* Q. A quadric hypersurface comes from taking the zero locus of a nondegenerate symmetric bilinear form. So

$$PSO_4 = \mathrm{Aut}(PSO_4, Q)^0$$

where 0 denotes the connected component at the identity. A smooth quadric surface in $\mathbb{P}^3$ has two rulings by lines: there are two families of lines on Q. These lines are there for any smooth quadric (any two smooth quadrics are isomorphic). You wouldn't see them if your picture of a quadric was a sphere in $\mathbb{R}^3$. These two families of lines correspond to an isomorphism

$$Q \simeq \mathbb{P}^1 \times \mathbb{P}^1,$$

imbedded via the **Segre map.** The automorphisms of $\mathbb{P}^3$ preserving the quadric, modulo connected components, are the automorphism of Q (we take connected components to avoid the automorphism that flips the two factors). So

$$\mathrm{Aut}(Q)^0 = \mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1) = PSL_2 \times PSL_2.$$

There's an echo of this when you talk about $\mathfrak{so}_6$.

Lecture 25
4/3

Here's the plan:

- Today: $SO_5, SO_{2n+1}, SO_6, SO_{2n}$. Friday: Clifford algebras and spin representations.
- Next week: classification (Ch. 21)! This is a long-promised result that justifies focusing on all these specific examples.
- The week after: Weyl character formula (24).
- Finally: the passage to (real) Lie groups (23, 26).

Today, we'll finish the concrete examples, modulo the fact that we have yet to exhibit all the representations of the orthogonal Lie algebras.

§1 Low-dimensional isomorphisms

Let's recall the **exceptional isomorphisms**:

- $\mathfrak{so}_3 \simeq \mathfrak{sl}_2$.
- $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$.
- $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$.
- $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$.

We've proved the first two isomorphisms. We'll see the rest of them in the future.

§2 $\mathfrak{so}_5$

Let's now look at $\mathfrak{so}_5$. This has a 2-dimensional Cartan algebra and the root diagram consists of the vertices and midpoints of the edges of a square centered at the origin. That is, the $\pm L_i, \pm L_i \pm L_j$. The Weyl chamber is spanned by L_1 and $\alpha = \frac{L_1+L_2}{2}$ and those two are the fundamental weights: the highest weights of all irreducible representations are the $\mathbb{Z}_{\geq 0}$-linear combinations of these weights.

Where are the representations? They're staring us in the face, except for the ones involving α.

25.1 Example. We look at the standard representation of $\mathfrak{so}_5$ on $\mathbb{C}^5$. I'll call this $V = V_{\mathfrak{so}_5}$. It has weights $\pm L_i, 0$. The highest weight is L_1.

25.2 Example. Next, $\bigwedge^2 V$ is the adjoint representation of $\mathfrak{so}_5$, and we can see its weights directly: they are the pairwise sums $\pm L_i \pm L_j, \pm L_i$ (and zero with multiplicity two). The highest weight is $L_1 + L_2$, so $\bigwedge^2 V$ is the irreducible representation coming from 2α.

The picture of the root diagram for $\mathfrak{sp}_4$ is analogous, except that it's rotated.

25.3 Example. Under the isomorphism $\mathfrak{sp}_4 \simeq \mathfrak{so}_5$, which has not yet been specified, the standard representation $V_{\mathfrak{so}_5}$ is obtained from $\bigwedge^2 V_{\mathfrak{sp}_4}$ by modding out by the trivial one-dimensional subspace $\mathbb{C}$. In other words, $V_{\mathfrak{sp}_4}$ is the standard representation of $\mathfrak{sp}_4$ of dimension four, we take the exterior square, and that contains a trivial summand: we mod out by that.

25.4 Example. The standard representation of $\mathfrak{sp}_4$ corresponds to the spin representation of $\mathfrak{so}_5$. In fact, we have $\bigwedge^2 V_{\mathfrak{so}_5} \simeq \mathrm{Sym}^2 V_{\mathfrak{sp}_4}$. Once again, to describe the spin representation, we have to take a symmetric square root of $\bigwedge^2 V$ (the adjoint representation).

§3 $\mathfrak{so}_{2n+1}$

In these terms, we should describe the odd orthogonal algebras in general. Recall that we have an n-dimensional Cartan subalgebra.

- The roots are $\pm L_i \pm L_j, \pm L_i$.
- The Weyl chamber is the set of linear combinations $\sum a_i L_i$, with $a_1 \geq \cdots \geq a_n \geq 0$.
- The edges of the Weyl chamber occur where $n-1$ of these n inequalities are satisfied.
- The fundamental weights are the first weight lattice vectors along these lines, namely $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1}, \frac{L_1+\cdots+L_n}{2}$.

The basic theorem is that we can get all the representations except the last one simply from the standard representation:

25.5 Theorem. *$\bigwedge^k V_{\mathfrak{so}_{2n+1}}$ is the irreducible representation with highest weight $L_1 + \cdots + L_k$ for $k = 1, \dots, n$.*

This gives us $n-1$ of the n fundamental representations: we have to find a "square root" of $\bigwedge^n V$. In other words, this theorem lets us construct exactly half the irreducible representations of $\mathfrak{so}_{2n+1}$.

Remark. In the symplectic case, every representation was self-dual (this was something that wasn't true for $\mathfrak{sl}_n, n \geq 3$). For $\mathfrak{sp}_{2n}$, the Weyl group includes -1, so all representations are isomorphic to their duals. In the case of the *odd* orthogonal algebras, -1 is also in the Weyl group and we get that all representations are self-dual. For the *even* ones, that's no longer the case in general.

§4 $\mathfrak{so}_{2n}$

For $\mathfrak{so}_6$, the root system looks like the midpoints of the edges of a reference cube centered at the origin. The Weyl chamber is exactly the cone over the triangle spanned by $L_1, L_1+L_2, L_1+L_2+L_3$, but these aren't fundamental weights: we need α. There's a second fundamental weight here, which is $\beta = \frac{L_1+L_2-L_3}{2}$. Let's look for representations.

25.6 Example. The standard representation of $\mathfrak{so}_6$ has weights $\pm L_i$; these correspond to the weights of the cube. The highest weight is L_1.

25.7 Example. $\bigwedge^2 V_{\mathfrak{so}_6}$ is the irreducible representation with weight $L_1 + L_2$. This isn't one of the fundamental representations though: it doesn't even lie on an edge.

25.8 Example. Let's look at $\bigwedge^3 V$. The weights are $\pm L_i \pm L_j \pm L_k$ and $\pm L_i$ (each taken twice). These $\pm L_i \pm L_j \pm L_k$ look like the vertices of the "reference cube." The first thing you can see is that it's not irreducible. It has a subrepresentation with highest weight $L_1 + L_2 + L_3$. In fact, $\bigwedge^3 V$ is a sum of two ten-dimensional representations. These summands are the representations with weights $L_1 + L_2 + L_3$ and $L_1 + L_2 - L_3$. This phenomenon will be true in general.

We now want to see the isomorphism $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$. Let's work with the adjoint forms PSO_6, PSL_4. Now PSL_4 is the automorphism group of $\mathbb{P}^3$. Moreover, PSL_4 acts on the Grassmannian $G(2,4)$ of 2-dimensional subspaces of $\mathbb{C}^4$. It turns out that

$$PSL_4 = \mathrm{Aut}^0(G(2,4)).$$

In general, the automorphism group of $G(k,n)$ is exactly PGL_n except when $k = n/2$. If you choose an isomorphism $V \simeq V^*$, then you get an involution of the Grassmannian taking every plane to its annihilator. Those form the second connected component.

OK, $G(2,4)$ is a projective variety. Given a 2-plane in $\mathbb{C}^4$ spanned by v, w, you can take $v \wedge w \in \bigwedge^2 \mathbb{C}^4$. Let's view this mod scalars, so this is well-defined element of $\mathbb{P}(\bigwedge^2 \mathbb{C}^4) = \mathbb{P}^5$. If I chose a different basis, I'd still get the same wedge product, up to scalars. What is the image? The image consists of all vectors $\eta \in \bigwedge^2 \mathbb{C}^4$ such that $\eta \wedge \eta = 0$. In other words,

$$\bigwedge^2 \mathbb{C}^4$$

is not a vector space, it's a vector space with a nondegenerate symmetric bilinear form, and the elements that square to zero form the Grassmannian. It follows that

$$PSL_4 = \mathrm{Aut}(G(2,4)) = \mathrm{Aut}(\mathbb{P}^5, Q) = PSO_6.$$

That gives the isomorphism

$$\mathfrak{sl}_4 \simeq \mathfrak{so}_6.$$

Finally, there's the isomorphism $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$. To get this, look at the previous isomorphism to find $PSp_4 \subset PSL_4$ that preserve a skew-symmetric bilinear form, which corresponds to automorphisms of $\mathbb{P}^5$ fixing Q and fixing a hyperplane. That's PSO_5.

Lecture 26
4/8

In this week, we're going to study the classification of complex simple Lie algebras.

To give an overview:

- It's the same thing we ever do with classification. We start with a (simple) Lie algebra $\mathfrak{g}$, introduce a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, consider the roots, and get a **root system** $R \subset \mathfrak{h}^*_{\mathbb{R}}$.
- That root system is crucial in all aspects of representation theory. We've associated to a simple Lie algebra a **root system.** We will have to prove that this data determines the Lie algebra.
- A root diagram gets associated to a **Dynkin diagram,** which is a simple object. One can classify these.

A priori, R is just a finite subset of the real euclidean (via the Killing form) vector space $\mathfrak{h}^*_{\mathbb{R}}$. (Recall that this is the real linear subspace of $\mathfrak{h}^*$ spanned by the roots.)

26.1 Definition. The pair $(\mathfrak{h}^*_{\mathbb{R}}, B)$ (for B the Killing form) is denoted E. $R \subset E$ is the **root system.**

R is a priori a finite subset of E, but it has several special properties that we'll include in the definition of a root system. Here are some of the special properties:

- R is finite and spans $\mathfrak{h}^*$ $\mathbb{C}$-linearly (so spans $\mathfrak{h}^*_{\mathbb{R}}$ $\mathbb{R}$-linearly).
- If $\alpha \in R$, then $-\alpha \in R$. If $k \in \mathbb{Z}$, then $k\alpha \notin R$ for $k \neq \{\pm 1\}$.[6]

[6] If α is a root, look at the subalgebra $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{k\alpha}$. That's a representation of $\mathfrak{s}_\alpha$. Suppose α is one of the two smallest elements in this chain. Then I see that this representation contains a copy of $\mathfrak{s}_\alpha$ itself. So I can write this as a direct sum of $\mathfrak{s}_\alpha$ and V. Now H_α acts on $\mathfrak{s}_\alpha$ with eigenvalue two, and we're saying that's the smallest eigenvalue. That means that V has no eigenvectors for H_α with eigenvectors $\pm 1, \pm 2$. That states that V is a trivial representation. In particular, there are no other root spaces further out.

- For each $\alpha \in R$, the reflection w_α about the hyperplane orthogonal to α carries R to R.
- To motivate this property, recall that for each $\alpha \in R$, we have a distinguished subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and a particular element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ which acts on $\mathfrak{g}_\alpha$ with eigenvalue two. The second key factor we encountered when we did the analysis is that the distinguished elements H_α always act with integral weight. This means that $\beta(H_\alpha) \in \mathbb{Z}$. This doesn't mean anything — H_α is some element that comes from the structure of the Lie algebra.

  But I can rewrite this as

$$\beta(H_\alpha) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

We're now going to axiomatize this.

26.2 Definition. A **root system** is a subset $R \subset E$, where E is a euclidean space, satisfying the above conditions. We say that R is **reducible** if $R \subset V \cup V^\perp$ for $V \subsetneq E$, $V \neq 0$. (These correspond to *semisimple* Lie algebras.)

It turns out that simple Lie algebras are determined by their root systems, so the problem becomes to classify root systems. The first thing is to derive some consequences of the fourth integrality condition.

Fix roots $\alpha, \beta \in R$. Let $\theta = \theta(\alpha, \beta)$ be the language between the two, so

$$\langle \alpha, \beta \rangle = \cos\theta |\alpha||\beta|.$$

I can plug this in there to get

$$\eta_{\beta,\alpha} \equiv \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2\cos\theta \frac{|\beta|}{|\alpha|}.$$

The point is, if I multiply $\eta_{\beta\alpha}\eta_{\alpha\beta}$, I get that

$$4\cos^2\theta \in \mathbb{Z}.$$

So $4\cos^2\theta \in \{0, 1, 2, 3, 4\}$. If we assume α is not simply $\pm\beta$, we get $4\cos^2\theta \in \{0, 1, 2, 3\}$. You can list the possibilities. The possibilities are,

- $\theta = \pi/2$: the two roots are perpendicular.
- $\theta = \pi/3$ or $2\pi/3$, in which case $|\alpha| = |\beta|$.
- $\theta = \pi/4$ or $3\pi/4$, in which case $|\alpha| \neq |\beta|$. In fact, the ratio of lengths is $\sqrt{2}$.
- We haven't seen this in practice, but $\theta = \pi/6$ or $5\pi/6$, and the ratio of the length is $\sqrt{3}$.

26.3 Definition. The **rank** of a root system (R, E) is the dimension $\dim E$.

(Note that any two Cartan subalgebras of a Lie algebra are conjugate under the adjoint action, although we haven't proved it.)

26.4 Example. In rank 1, it's two vectors $\alpha, -\alpha$.

26.5 Example. In rank 2, root systems are living in a plane. For vectors in $\mathbb{R}^2$, there's a notion of adjacencies. The angles between any two adjacent vectors in a rank 2 root system are equal. In fact, reflection about a vector followed by -1 preserves the two adjacent vectors. What is that angle? The angle could be any one of the four possibilities. We could get the four unit vectors (for $\mathfrak{so}_4$), we could get $\theta = \pi/3$ (for $\mathfrak{sl}_3$), $\theta = \pi/4$ (for $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$), and $\theta = \pi/6$ (where the roots alternate between long and short roots) —- which is the root system G_2.

Lecture 27
4/10

§1 Dynkin diagrams

Today, I want to introduce the **Dynkin diagram.** Let me recall what we're dealing with.

27.1 Definition. A **root system** is a collection R of nonzero vectors in euclidean space $\mathbb{E}$ (with inner product $(\cdot,\cdot)$) of some finite dimension n (called the **rank** of the root system) that satisfies the following axioms:

- R is finite and spans $\mathbb{E}$.
- For all $\alpha \in R$, then $k\alpha \in R$ if and only if $k = \pm 1$.
- The reflection w_α about the hyperplane perpendicular to α preserves R.
- For all $\alpha, \beta \in R$, the quantity

$$\eta_{\beta\alpha} \equiv \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 2\cos\theta \frac{\|\beta\|}{\|\alpha\|} \in \mathbb{Z}.$$

  Observe that w_α has the property that $w_\alpha(\beta) = \beta - \eta_{\beta\alpha}\alpha$.

We say that a root system is **irreducible** if it is not contained in the union of two orthogonal planes.

This is modeled on the root systems of Lie algebras. One of the statements we're going to prove is that *every* root system does come from a unique Lie algebra. There is a one-to-one correspondence between root systems and semisimple Lie algebras.

27.2 Example. In the rank two case, there are exactly four root systems, and they simply depend on the angle between any two adjacent vectors.

- We have $\{\pm e_1, \pm e_2\}$. (This one is reducible.)
- We have the sixth roots of unity (for $\mathfrak{sl}_3$).

- We have the vertices and midpoints of a square, for $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$.
- We have the $\mathfrak{g}_2$ root system, with angle $\pi/6$ between them.

27.3 Example. In rank three, there are only three irreducible root systems: the ones we know about. These are $\mathfrak{sl}_4 \simeq \mathfrak{so}_6$, $\mathfrak{sp}_6$, and $\mathfrak{so}_7$.

Let me tell you about Dynkin diagrams, and then we can prove some facts about root systems in general. In fact, the entire structure of the Lie algebra is encoded in the root system, but the Dynkin diagram is a more efficient representation.

27.4 Definition. Given a root system, we pass to the collection of **simple** (or **primitive**) roots.

We have a collection of roots in $\mathbb{E}$, so we choose a hyperplane H such that no root lies in H; that splits the set of roots into *positive* and *negative* roots. We get

$$R = R^+ \sqcup R^-,$$

and we can define a **primitive** or **simple** root as an element of R^+ which cannot be expressed as a sum of elements in R^+.

27.5 Example. In the case of $\mathfrak{sl}_{n+1}$, the simple roots are the elements $L_1 - L_2, \dots, L_n - L_{n+1}$ which lie in $\mathbb{C}\langle L_1, \dots, L_{n+1}\rangle / \sum L_i$.

27.6 Example. In $\mathfrak{so}_{2n+1}$, the roots live in $\mathbb{C}\langle L_1, \dots, L_n\rangle$ and the simple roots are $L_1 - L_2, \dots, L_{n-1} - L_n, L_n$.

27.7 Example. For $\mathfrak{sp}_{2n}$, we have the simple roots $L_1 - L_2, \dots, L_{n-1} - L_n, 2L_n$.

27.8 Example. For $\mathfrak{so}_{2n}$, the simple roots are $L_1 - L_2, \dots, L_{n-1} - L_n, L_{n-1} + L_n$.

It's enough to specify the configuration of simple roots, as it turns out. Since every positive root is a sum of simple roots, the simple roots span. It turns out that the simple roots are linearly independent, so there are exactly n of them. Moreover, the angle between any two simple roots cannot be acute (as we'll show). The angle between simple roots can only be $\{\pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}$.

27.9 Definition. To draw a **Dynkin diagram**, we draw one node for each simple root α. Given two nodes α, β, we ask what the angle between them is, and:

- We draw one line between α, β if the angle is $2\pi/3$; in this case the roots have the same length.
- We draw two lines if the angle is $3\pi/4$.
- We draw three lines if the angle is $5\pi/6$.
- We draw no lines if the angle is $\pi/2$.

We say that a Dynkin diagram is **irreducible** if it is connected.

If the angle between two simple roots has angle $3\pi/4, 5\pi/6$, we add an arrow pointing from the longer root to the short root (or between the corresponding nodes).

27.10 Theorem. *The only Dynkin diagrams that come from irreducible root systems are:*

- *A string of dots each connected to the next with a single line. This is A_n associated to $\mathfrak{sl}_{n+1}$.*
- *A string of dots each connected to the next with a single line, except the last is a double line with an arrow pointing to the end. This is B_n for $\mathfrak{so}_{2n+1}$.*
- *The same as B_n with the last arrow reversed, for $\mathfrak{sp}_{2n}$ (this is C_n).*
- *A string of dots each connected to the next with a single line, with a "forked tail" at the end, for $\mathfrak{so}_{2n}$ (this is D_n).*
- *Five exceptional Dynkin diagrams, called G_2, F_4, E_6, E_7, E_8.*

27.11 Example. We can see the coincidences among small Lie algebras by looking at the Dynkin diagrams. For instance, $B_2 \simeq C_2$, giving the isomorphism $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$. Similarly, D_2 has two unconnected dots, and $\mathfrak{so}_4$ is a product $\mathfrak{sl}_2 \times \mathfrak{sl}_2$.

For rank three, D_3 is isomorphic to A_3, so that $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$. These are all the coincidences.

The usual convention to avoid these coincidences is to require $n \geq 2$ for B_n, $n \geq 3$ for C_n, and $n \geq 4$ for D_n.

If I'm going to prove the main theorem above, we're going to need to prove some more properties of root systems and derive a series of consequences. After that, we need to show that we can reconstruct the entire Lie algebra from the Dynkin diagram.

§2 Returning to root systems

Let's write down some more properties of root systems that are consequences of the axioms.

- Given a pair of roots $\alpha, \beta \in R$, for $\alpha \neq \pm\beta$, I want to look at the string of elements $\beta + k\alpha$ for $k \in \mathbb{Z}$, the **α-string through** β. Let's suppose

  $$\beta - p\alpha, \ldots, \beta, \beta + \alpha, \ldots, \beta + q\alpha$$

  is the longest string around zero consisting of roots. Then this string has length at most 4: that is, $p + q \leq 3$.
- More specifically, $p + q = \eta_{\beta\alpha}$.

Proof. Apply w_α to $\beta + q\alpha$ to get the other end of the string $\beta - p\alpha$. On the other hand, that's the same thing as $\beta - \eta_{\beta\alpha}\alpha - q\alpha$. So we get

$$p - q = \eta_{\beta\alpha}.$$

Now, apply this same logic to the root at one end of the string: $\beta + q\alpha$. For that, $q = 0$, so we get that the length of the string is $\eta_{\beta+q\alpha,\alpha}$ which has absolute value at most 3. The logic here to deduce that the length is at most here is to apply this equality to the root at the end of the string. ▲

The primary consequence of this is to tell us when the sum of two roots is a root.

- If α, β are roots, and $\alpha \neq \pm\beta$, then if the angle $\theta(\alpha, \beta)$ is strictly obtuse, then $\alpha + \beta$ is a root again.
- Conversely, if the angle is strictly acute, then $\alpha - \beta$ is a root.
- If the angle is exactly $\pi/2$, then $\alpha + \beta, \alpha - \beta$ are either both roots or both non-roots.

For example, if the angle is obtuse, then that says that q is strictly positive, so $\beta + \alpha$ is a root. This is what we're going to use to build up the root system from the set of simple roots.

- If α, β are distinct simple roots, then $\alpha - \beta$ is not a root (so the angle between α, β is obtuse) and $\alpha + \beta$ is a root if and only if $\theta(\alpha, \beta) > \frac{\pi}{2}$.

Indeed, write $\beta = \alpha + (\beta - \alpha)$. Now β is a simple root so it isn't the sum of simple roots. So if $\beta - \alpha$ is a root, it's a negative root. But I can also write it the other way: $\alpha = \beta + (\alpha - \beta)$, which means that $\alpha - \beta$ can't be a positive root either. That means that $\beta - \alpha$ can't be a root at all.

That implies:

- The simple roots are linearly independent.

Indeed, if you have a collection of vectors in a euclidean space lying in a half-space with non-acute pairwise angles between them, they must be linearly independent.

We're going to use these facts to show that these are the only Dynkin diagrams arising from root systems. Next, we're going to show that a Dynkin diagram is enough to recover the entire Lie algebra. The first is a theorem in euclidean geometry. The second is a statement about Lie algebras. It's saying that we can reconstruct a Lie algebra from a very minimal amount of information.

Lecture 28
4/15

§1 Recovery

(No notes for the previous lecture.)

Today, we're going to discuss reconstructing a semisimple Lie algebra from its Dynkin diagram. The classification for simple Lie algebras has two parts. First, you introduce the Dynkin diagram. Then you have to classify all allowable Dynkin diagrams (i.e., those arising from semisimple Lie algebra), which is a matter of studying root systems via euclidean geometry. The other half is reconstruction.

There are two steps:

- Recovering the root system from the Dynkin diagram. (The Dynkin diagram is exactly the information describing the **simple** or **primitive** roots.) The idea is that if we know what the simple roots are, we can recover all the roots. Every root is a linear combination of the simple roots, so the question is which linear combinations of the roots are roots and which are not. We showed inductively last time (for which there were no notes) how to do that inductively.

  You can go back to what we said on Friday and interpret it by saying that "you get all roots by reflecting in hyperplanes Ω_α to known roots α." That is, you start with the simple roots, and reflect them about each other, and keep repeating the process. That's what you need to remember from Friday's lecture.

- We want to see how to recover the Lie algebra from the root system R.

Last time, when we discussed the Killing form, we said that it was an inner product on $\mathfrak{h}_\mathbb{R}$ which was invariant under the Weyl group. We'll need a little more:

1. Recall that we have two spaces, $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{h}^*$. The roots $R \subset \mathfrak{h}^*$. In the original $\mathfrak{h}$, we have distinguished elements H_α for each root $\alpha \in R$. Namely, for each α, we have a distinguished subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}_{\pm\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and we pick a unique element H_α to play the role of $H \in \mathfrak{sl}_2$.

2. In $\mathfrak{h}^*$, we have a hyperplane Ω_α which was the annihilator of the distinguished element H_α. Likewise, in $\mathfrak{h}$, we have the kernel of α itself.

3. The isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ given by the Killing form carries the hyperplanes to one another. It almost, but not quite, carries α into H_α: that is, it does up to a scalar factor.

   28.1 Proposition. *The isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ given by the Killing form carries Ω_α to $\ker(\alpha)$ and carries α to $\frac{2H_\alpha}{B(H_\alpha,H_\alpha)}$.*

   Proof. Remember how the Killing form is defined. Given $X, Y \in \mathfrak{g}$, we have

   $$B(X,Y) = \operatorname{Tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)),$$

   which is symmetric since the trace of a commutator is zero. Recall the following identity: for $A, B, C \in \operatorname{End}(V)$, we have

   $$\operatorname{Tr}([A,B]C) = \operatorname{Tr}(A[B,C]).$$

   This follows by writing out

   $$\operatorname{Tr}(ABC - BAC) = \operatorname{Tr}(ABC - ACB).$$

   Given this, we can write for $H \in \mathfrak{h}$,

   $$B(H_\alpha, H) = B([X_\alpha, Y_\alpha], H) = B(X_\alpha, [Y_\alpha, H]) = B(X_\alpha, \alpha(H)Y_\alpha).$$

   If $\alpha(H) = 0$, then $B(H_\alpha, H) = 0$; this is what we wanted. Moreover, let's say that $T_\alpha \in \mathfrak{h}$ is the element corresponding to $\alpha \in \mathfrak{h}^*$. By definition, that means

   $$B(T_\alpha, H) = \alpha(H).$$

That says that

$$T_\alpha = \frac{H_\alpha}{B(X_\alpha, Y_\alpha)} = \frac{2H_\alpha}{B(H_\alpha, H_\alpha)}.$$

▲

4. We know that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ where the $\mathfrak{g}_\alpha$ are one-dimensional. We need to introduce a basis for the $\mathfrak{g}_\alpha$ and then need to specify what the products are. Here's the proposal. I'm going to start with the simple roots, $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$, which come to us with the Dynkin diagram. We let $H_1, \dots, H_n \in \mathfrak{h}$ be the corresponding dual elements up to some scalar multiples still unspecified.

   Start by choosing any (nonzero) element $X_i \in \mathfrak{g}_{\alpha_i}$. Then, choose $Y_i \in \mathfrak{g}_{-\alpha_i}$ such that $[X_i, Y_i] = H_i$. That gives a total of $3n$ elements: a basis for $\mathfrak{h}$ and a basis for the simple root spaces and their opposites.

   What are we going to do now? We're going to try to describe a basis for the remaining root spaces. Namely, we're going to take brackets of these basic generators. I want to say that if I have a root space which isn't simple, I can express it as a partial sum of simple roots whose partial sums are roots, and bracketing repeatedly gives a nonzero element of the root space.

   28.2 Definition. A sequence of indices $I = i_1, \dots, i_r \in [1, n]$ is **admissible** if the partial sums of the sequence are all roots (i.e. $\alpha_{i_1} + \alpha_{i_2} + \dots \alpha_{i_k}$ is a root for all k). We let $\alpha_I = \sum_{k=1}^r \alpha_{i_k}$.

   For any $\alpha \in R^+$, we saw in the previous lecture (for which there are no notes), there is an admissible sequence I with $\alpha_I = \alpha$. In this case, if I just take the commutators in that order,

$$X_I \stackrel{\text{def}}{=} [X_{i_r}, [X_{i_{r-1}}, \dots, [X_{i_2}, X_{i_1}] \dots],$$

   is a nonzero element of the root space $\mathfrak{g}_{\alpha_I}$. We're going to take these (and the Y_I?) as a basis and now we have to specify how to multiply. Also, we have to specify when there are two different admissible sequences summing to the same element, how the two elements differ.

5. If $\alpha_I = \alpha_J$, how are X_I, X_J related?

   28.3 Lemma. *If $\alpha_I = \alpha_J$, then $X_J = qX_I$ where $q \in \mathbb{Q}$ is a function of the two sequences I, J and is determined by the Dynkin diagram.*

   Proof. Induction on r (the length of the sequence). It's trivial for $r = 1$. Since the simple roots are linearly independent, I and J are permutations of each other. If the last elements of I, J agree, then we're done by induction. In any event, the last term $k = i_r$ must appear somewhere in J, but if $k = j_r$, we're done by induction. If not, we have to figure some way to manuever this factor to the front.

   Observe that

$$X_J = q_1[X_k, [Y_k, X_J]], q = q(k, J, DD).$$

Point: we have a subalgebra $\mathfrak{s}_{\alpha_k} \simeq \mathfrak{sl}_2$ and this acts on the α_k-string, whose length is determined by the Dynkin diagram. If I tell you the length of the α_k-string, then I've told you the $\mathfrak{sl}_2$-representation.

So I can relate X_J to $[X_k, [Y_k, X_J]]$. Let s be the largest integer such that $j_s = k$. Let $K = j_1, \ldots, j_s$.

(getting lost here...) ▲

Lecture 29
4/17

§1 Summary

Last time, we were just about done with the proof of the main theorem that the Dynkin diagram of a semisimple Lie algebra determines the Lie algebra. We did everything we needed to do, so I want to summarize the result.

- The Dynkin diagram (DD) describes the simple roots, and from there we can get all the roots. That is, we get the root system $R \subset E$.
- We choose bases $H_i \in \mathfrak{h}$, $X_i \in \mathfrak{g}_{\alpha_i}, Y_i \in \mathfrak{g}_{\alpha_i}$ where the α_i are simple roots. We choose H_i to be normalized appropriately.
- For all $\alpha \in R^+$, we choose an **admissible sequence** $I = (\alpha_{i_1}, \ldots, \alpha_{i_r})$ such that $\sum \alpha_{i_k}$. "Admissible" means that all the partial sums are themselves roots.
- Write $X_I = [X_{i_r}, [X_{i_{r-1}}, [X_{i_{r-2}}, \ldots, X_{i_1}] \ldots]$. These (together with analogous Y_α and $\mathfrak{h}$ form a basis for the Lie algebra).
- If I and J are admissible with the same sum, then

  $$X_I = qX_I, \quad q = q(I, J, DD).$$

  Taking products of sequences can be done by repeatedly applying Jacobi, but that's not totally clear.
- In this way, you can write down the entire multiplication table.

This proves uniqueness, but **existence** is harder.

§2 Constructing G_2

Today, I want to talk about constructing the Lie group G_2, because that's the one exceptional group that we can describe like the classical groups. Then, I want to talk about the complex Lie groups associated to the simple Lie algebras. We're going to need a more easily manipulated formal framework of discussing representations, and we'll introduce the **representation ring** and **characters.**

So far, we've usually defined Lie groups as automorphisms of a vector space that preserve some tensor. For instance, $SL(V)$ consists of automorphisms of V that preserve a top-form $\phi \in \bigwedge^n V$ (for $n = \dim V$). Similarly, $Sp(V, Q)$ consists of automorphisms of V preserve a $Q \in \bigwedge^2 V^*$, and similarly for $SO(V, Q)$. In each case, we're looking at some element in the tensor algebra $T(V)$, and we're looking at some subgroup of $\mathrm{Aut}_{\mathbb{C}}(V)$ preserving this element. Why these three and no others? If you chose a vector space V of dimension n, then $\dim \mathrm{Aut}(V) = n^2$.

Now $\bigwedge^2 V^*$ has dimension $\binom{n}{2}$, and the group $\mathrm{Aut}(V)$ acts nearly transitively on it (consisting of the nondegenerate forms). If I pick a general element of $\bigwedge^2 V^*$, the isotropy subgroup should have dimension $n^2 - \binom{n}{2}$. If I do this with $\bigwedge^3 V^*$, I get $\binom{n}{3}$, and that grows much more quickly. In particular, $\dim \bigwedge^3 V^*$ grows much more quickly than $\dim \mathrm{Aut}(V)$ as $\dim V$ grows. The automorphism group of a general element in the tensor algebra (away from Sym^2 or $\bigwedge^2$) is trivial when $\dim V$ is large.

But let's try to do this when V is small.

Question. What is $\mathrm{Aut}(V, \phi)$ when $\phi \in \bigwedge^3 V^*$?

29.1 Example. When $\dim V = 3$, we get SL_3.

29.2 Example. When $\dim V = 4$, $\bigwedge^3 V^* \simeq V$, so we're looking at matrices of the form

$$\begin{bmatrix} 1 & * \\ 0 & A \end{bmatrix},$$

and it isn't even a semisimple group.

29.3 Example. When $\dim V = 5$, we get a "Sp_5" except that the form has a kernel. That is, an element of $\bigwedge^3 V^*$ is the same as an element of $\bigwedge^2 V$.

29.4 Example. When $\dim V = 6$, we get something interesting: we can't relate an element of $\bigwedge^3 V$ to a quadratic form. The automorphism group turns out to be $SL_3 \times SL_3$.

29.5 Example. When $\dim V = 7$, we get a 14-dimensional group, and that's G_2. You can see this worked out in chapter 22.

The remaining four exceptional Lie algebras can't be constructed this way.

§3 Groups associated to classical Lie algebras

Let's look at forms of $\mathfrak{sl}_{n+1}$.

29.6 Example. The group $SL_{n+1}(\mathbb{C})$ is connected and simply connected, so that is the simply connected form. Recall that if we have a fiber bundle $X \to M$ with connected fiber F, we have a long exact sequence of homotopy groups

$$\cdots \to \pi_2 M \to \pi_1 F \to \pi_1 X \to \pi_1 M \to *.$$

Look at the action of $SL_{n+1}(\mathbb{C})$ on $\mathbb{C}^{n+1} \backslash \{0\}$. This is a transitive action, with stabilizer H. So we get a bundle

$$H \to SL_{n+1}(\mathbb{C}) \to \mathbb{C}^{n+1} \setminus \{0\}.$$

Here the base has the homotopy type of a S^{2n+1}. Now H, homotopically, is just $SL_n(\mathbb{C})$ crossed with $\mathbb{C}^n$, so it is homotopically $SL_n(\mathbb{C})$. We get

$$\pi_1(SL_n(\mathbb{C})) \to \pi_1(SL_{n+1}(\mathbb{C})) \to \pi_1(S^{2n+1}) = *,$$

and inductively we get the simple connectivity assertion.

The center is just the scalar matrices which are roots of unity, so the set of $\exp(2\pi ik/(n+1))$ for $k = 0, 1, \ldots, n$ and is isomorphic to $\mathbb{Z}/(n+1)$. The adjoint form is $PSL_{n+1}(\mathbb{C})$, which is the quotient of the simply connected form by its center.

There are intermediate forms, for instance $SL_4/\{\pm I\}$ is isomorphic to $SO_6(\mathbb{C})$.

Which representations of $SL_{n+1}(\mathbb{C})$ (or of $\mathfrak{sl}_{n+1}(\mathbb{C})$) lift to which models?

29.7 Example.

Lecture 30
4/22

Today, we'll continue discussing forms of the simple complex Lie algebras. In the rest of the group, we'd also like to discuss which representations lift to which groups, and then we'll discuss the Weyl character formula and representation rings. Time permitting, I'd like to talk about the classification of real forms of complex Lie algebras and homogeneous spaces.

§1 Forms of the classical Lie algebras

Last time, we talked about the special linear group and its forms.

30.1 Example. Recall that in the case of $SL_n(\mathbb{C})$, the natural group was simply connected, and we proved this using fiber bundles. This was at the top of the "tower" of groups with Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. The center is $\mathbb{Z}/n$ (the scalar matrices). The forms of $\mathfrak{sl}_n(\mathbb{C})$ are the quotients of $SL_n(\mathbb{C})$ by subgroups of the center; the bottom of the "tower" is $PSL_n(\mathbb{C})$. The groups in the middle typically don't arise, but when $n = 4$, then

$$SL_4(\mathbb{C})/(\mathbb{Z}/2) \simeq SO_6(\mathbb{C}).$$

There's also the question of which representations lift to which groups. For any complex semisimple LA, we had a configuration of lattices inside $\mathfrak{h}_{\mathbb{R}}^{\vee}$. We had

$$\Lambda_R \subset \Lambda_W \subset \mathfrak{h}_{\mathbb{R}}^*.$$

I also want to consider the duals of these lattices,

$$\Lambda_W^* \subset \Lambda_R^* \subset \mathfrak{h}_{\mathbb{R}}.$$

Suppose I have a given group G. Consider the exponential map restricted to $\mathfrak{h}$; that gives a map $\exp : \mathfrak{h} \to H \subset G$ (the image is a Cartan subgroup) and there's always a kernel, which turns out to be between $2\pi i\Lambda_W^*$ and $2\pi i\Lambda_R^*$. (In each case, the group in question is isomorphic to $(\mathbb{C}^*)^n$ – this is an observed phenomenon.) In the simply connected form, the kernel is Λ_W^*, while in the adjoint form, the kernel is Λ_R^*. The point is, we have a correspondence between forms of the Lie algebra and intermediate groups between Λ_W^* and Λ_R^*. A given representation of $\mathfrak{g}$ lives to a form G if and only if the weights (which live in $\mathfrak{h}^*$) take integer values on this intermediate lattice.

30.2 Example. $Sp_{2n}(\mathbb{C})$ is simply connected. The center of $Sp_{2n}(\mathbb{C})$ is $\{\pm I\} = \mathbb{Z}/2$, so there are only two forms of $\mathfrak{sp}_{2n}(\mathbb{C})$: $Sp_{2n}(\mathbb{C})$ and $PSp_{2n}(\mathbb{C}) = Sp_{2n}(\mathbb{C})/\{\pm 1\}$.

30.3 Example. For $m \geq 3$, $\pi_1(SO_m(\mathbb{C})) = \mathbb{Z}/2$. The center of $SO_m(\mathbb{C})$ consists of scalars, and there either aren't any (if m is odd) or there are $\pm I$ (if m is even).

- For $m = 2n+1$, there are only two forms: the universal cover $Spin_{2n+1}(\mathbb{C})$ (with center $\mathbb{Z}/2$) and $SO_{2n+1}(\mathbb{C})$.
- For $m = 2n$, we have a sort of tower of groups. We have $Spin_{2n}(\mathbb{C})$, which is simply connected—by definition the universal cover. Then we have a two-fold cover $Spin_{2n}(\mathbb{C}) \to SO_{2n}(\mathbb{C})$. We get another two-sheeted cover, though $SO_{2n}(\mathbb{C}) \to PSO_{2n}(\mathbb{C})$.

  There's a further bifurcation, though. If n is odd, $Z(Spin_{2n}(\mathbb{C})) = \pi_1(PSO_{2n}(\mathbb{C})) = \mathbb{Z}/4$, meaning SO_{2n} is the unique intermediate form between the simply connected form and the adjoint form. If n is even, though, then the group is $\mathbb{Z}/2 \times \mathbb{Z}/2$ and we have two other intermediate groups. So there are either three or five forms.

Let's prove the claim made at the beginning of the example, that $Sp_{2n}(\mathbb{C})$ is simply connected. To understand the topology of these groups, you want them to act transitively on a manifold, so that you can get a fiber bundle $H \to G \to G/H$. Let's consider $Sp(V, Q)$ for V a vector space with a symplectic form Q. I want to say that this acts transitively on the manifold $M = \{(v, w) \in V \times V : Q(v, w) = 1\}$. When you prove that nondegenerate skew-symmetric bilinear forms are conjugate to one another, the standard approach is to start with any vector v in the vector space. That has self-inner-product zero, but you can find a second vector w such that $Q(v, w) = 1$. Then you take those as the first two basis vectors and choose the remaining basis vectors from the orthogonal complement of them. That argument essentially shows that M is acted upon transitively by $Sp(V, Q)$.

How to understand M? If $v \neq 0$, then there are always lots of w with $Q(v, w) = 1$. In other words, M fibers over $V \setminus \{0\}$ by sending $(v, w) \mapsto v$. Now $V \setminus \{0\}$ is topologically a sphere of dimension $4n - 1$ and so is highly connected. The fibers of this map are as follows. If I fix v, then w is constrained to lie in an affine hyperplane, which is contractible. In particular,

$$M \sim S^{4n-1}.$$

As soon as n is positive, then $\pi_1 M = \pi_2 M = 0$.

We get a fiber sequence

$$H \to G \to M,$$

where H is the stabilizer of a given element (v, w) and is therefore Sp_{2n-2} (the automorphisms of Q restricted to the orthogonal complement of (v, w)). We thus get a homotopy fiber sequence

$$Sp_{2n-2}(\mathbb{C}) \to Sp_{2n}(\mathbb{C}) \to S^{4n-1},$$

and inductively we see that Sp_{2n} is simply connected. (Note that the induction starts, since $Sp_2 \simeq SL_2$ is simply connected.)

30.4 Example. Let's now study $\pi_1(SO_m(\mathbb{C}))$, again inductively. If you're studying the real orthogonal group, you study it by acting on the unit sphere. Consider (V, Q) with $\dim V = m$ and Q a nondegenerate symmetric bilinear form. Then $SO_m(\mathbb{C})$ acts on $M = \{v : Q(v,v) = 1\}$. Here the stabilizer is exactly $SO_{m-1}(\mathbb{C})$.

Two things here will be a little more complicated. Start with $m = 3$ (since SO_2 isn't semisimple). There, we know that SO_3 is doubly covered by SL_2: we have a two-sheeted cover

$$SL_2(\mathbb{C}) \to SO_3(\mathbb{C}),$$

and we know that SL_2 is simply connected, so $\pi_1(SO_3(\mathbb{C}))$ is $\mathbb{Z}/2$.

What does the space M look like? For that, I'm going to propose choosing a particular quadratic form—they're all the same. Take $V = \mathbb{C}^m$ and $Q(z,z) = \sum z_\alpha^2$. What is the locus where this has value one? I write $z = x + iy$ out in terms of real and imaginary parts. To say that $Q(z,z) = 1$ is to say that $\sum x_\alpha^2 - \sum y_\alpha^2 = 1$ and $\sum x_i y_i = 0$. That's some subset of $\mathbb{R}^{2n}$. Let's consider a map from M to the unit sphere sending $x + iy \mapsto x/\|x\|$ and that gives a map to S^{m-1}. The fibers are simply linear subspaces. So we're good. When $m \geq 4$, this is highly connected and we get the desired homotopy groups.

Lecture 31
4/26

§1 Setup

Last time (**no notes**), we started setting up the Weyl character formula. We'll discuss it today and Monday.

Let $\mathfrak{g}$ be a semisimple Lie algebra. We associate to $\mathfrak{g}$ the **weight lattice** Λ. Choose an ordering of the roots, and a corresponding **Weyl chamber** and **fundamental weights**. The **fundamental weights** are exactly the smallest weights that occur along the edges of the Weyl chamber. In every case, we saw that the Weyl chamber is a simplicial cone (the intersection of n half-planes), so that it has n edges. If we look at the smallest lattice vectors at each edge, then they form a set of generators of Λ. We'll call the fundamental weights $\omega_1, \dots, \omega_n$, and we consider the **fundamental representations** $\Gamma_1, \dots, \Gamma_n$—these are the irreducible representations with weights ω_i.

31.1 Definition. $R(\mathfrak{g})$ is the **representation ring** of $\mathfrak{g}$: the Grothendieck group of $\mathfrak{g}$-representations (with ring structure from the tensor product). So $R(\mathfrak{g})$ is the free abelian group on the irreducible representations.

Last time, we saw that $R(\mathfrak{g})$ is simply the **polynomial ring** on the classes of $\Gamma_1, \dots, \Gamma_n$, the fundamental representations. This is just a reflection of the fact (which we didn't really explain but observed) that the fundamental weights form a basis for the lattice. Every weight in the Weyl chamber is a nonnegative linear combination of fundamental weights, so every representation appears inside a tensor product of fundamental representations.

We still want to know more, though. So we introduce a second object: the group algebra $\mathbb{Z}[\Lambda]$ of the weight lattice, which we call the **character ring**. We could write this as $\mathbb{Z}\langle e(\lambda)\rangle_{\lambda\in\Lambda}$ and where

$$e(0) = 1, \quad e(\lambda)e(\mu) = e(\lambda+\mu).$$

We have a **character map**

$$\mathrm{char} : R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]$$

This map sends a representation V to the linear combination $\sum_{\lambda\in\Lambda}(\dim V_\lambda)e(\lambda)$: that is, the character map simply keeps track of the weights of a representation. This map is an isomorphism

$$R(\mathfrak{g}) \simeq \mathbb{Z}[\Lambda]^{\mathcal{W}},$$

where $\mathcal{W}$ acts on the character ring by permuting the $e(\lambda)$.

31.2 Example. In the case of $\mathfrak{sl}_{n+1}$, the weight lattice is $\mathbb{Z}\langle L_1,\dots,L_{n+1}\rangle / \langle \sum L_i = 0\rangle$. We write $X_i = e(L_i)$, so that $X_i^{-1} = e(-L_i)$. The Weyl group acts by permuting the L_i.

We have

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1,\dots,x_{n+1}]/(\prod x_i = 1).$$

If we want to describe the subring invariant under the Weyl group, we can look at the fundamental representations. The fundamental weights are

$$L_1, L_1+L_2, \dots, L_1+\dots+L_n = -L_{n+1},$$

and the corresponding representations are the standard representation and its exterior powers:

$$V, \Lambda^2 V, \dots, \Lambda^n V \simeq V^*.$$

The weights of V are $L_1,\dots,L_{n+1}$ so the character of V is $X_1+\dots+X_{n+1}$, and so the character of $\bigwedge^k V$ is $\sigma_k(X_1,\dots,X_{n+1})$ (elementary symmetric polynomials). The conclusion is that the invariant subring is simply the polynomial ring on these symmetric polynomials.

31.3 Example. For $\mathfrak{sp}_{2n}$, the weight lattice is $\mathbb{Z}\langle L_1,\dots,L_n\rangle$. The fundamental weights are $L_1, L_1+L_2,\dots,L_1+\dots+L_n$. The fundamental representations are the standard one V, and the kernels of the maps $\bigwedge^k V \to \bigwedge^{k-2} V$ given by contraction by the skew-symmetric bilinear form.

The standard representation has weights $L_1,\dots,L_n,-L_1,\dots,-L_n$, so the character of V is

$$X_1+X_2+\dots+X_n+X_1^{-1}+\dots+X_n^{-1},$$

and the characters of $\bigwedge^k V$ are the elementary symmetric polynomials c_k in the X_i, X_i^{-1}. The characters of the fundamental representations are the differences $c_k - c_{k-2}$. So

$$R(\mathfrak{g}) = \mathbb{Z}[\Lambda]^{\mathcal{W}} = \mathbb{Z}[c_1, c_2-c_0, c_3-c_1,\dots,c_n-c_{n-2}] = \mathbb{Z}[c_1,\dots,c_n].$$

31.4 Example. Let's do $\mathfrak{so}_{2n+1}$. The weight lattice $\Lambda = \mathbb{Z}\left\langle L_1, \dots, L_n, \frac{L_1+\dots+L_n}{2}\right\rangle$. The fundamental weights are $L_1, L_1+L_2, \dots, L_1+\dots+L_{n-1}$, and then $\frac{L_1+\dots+L_n}{2}$. As before, $X_i = e(L_i)$ and $\sqrt{X_i} = e(L_i/2)$. The corresponding fundamental representations are

$$V, \bigwedge^2 V, \dots, \bigwedge^{n-1} V, \mathcal{S},$$

where $\mathcal{S}$ is the spin representation. The character of V is $X_1 + \dots + X_n + X_1^{-1} + \dots + X_n^{-1} + 1$. The character b_k of $\bigwedge^k V$ can be obtained from this. The character of $\mathcal{S}$ is

$$b = \sum X_1^{\pm 1/2} X_2^{\pm 1/2} \dots X_n^{\pm 1/2}.$$

The weights of the spin representation are the image of the highest weight under the Weyl group. The representation ring is $\mathbb{Z}[b_1, \dots, b_{n-1}, b]$.

31.5 Example. Consider $\mathfrak{so}_{2n}$. As in the odd orthogonal case, the same Λ, L_i. The Weyl group here is just a little smaller: you're allowed to permute the axes and change an even number of the axes. The fundamental weights here consist of $L_1, L_1 + L_2, \dots, L_1+\dots+L_{n-2}$, and the remaining two fundamental weights are $\frac{L_1+\dots+L_n}{2}, \frac{L_1+\dots+L_{n-1}-L_n}{2}$. The corresponding representations are $V, \bigwedge^2 V, \dots, \bigwedge^{n-2} V$, and the two spin representations $\mathcal{S}^+, \mathcal{S}^-$. The characters of $\bigwedge^k V$ are the symmetric polynomials in $X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}$. Then we have the two characters of the spin representations,

$$\sum_{\text{even}} X_1^{\pm 1/2} \dots X_n^{\pm 1/2}, \sum_{\text{odd}} X_1^{\pm 1/2} \dots X_n^{\pm 1/2},$$

where the sum is "even" or "odd" according to the parity of minus signs allowed in the exponents.

Let's now go back to the general case. There is a special weight. Take the sum $\sum_{\alpha \in R^+} \alpha$. The claim is that this is **divisible by two in the weight lattice,** and once I divide it by two, it's the smallest weight in the interior of the Weyl chamber. Let's write

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

It's also the sum of the fundamental weights. For now, let's assume these facts.

31.6 Definition. For $\lambda \in \Lambda_W$, set $A_\lambda = \sum_{w \in \mathcal{W}} (-1)^w e(w(\lambda)) \in \mathbb{Z}[\Lambda_W]$.

The Weyl group has a canonical homomorphism to $\mathbb{Z}/2$ (determinant). We're taking the vertices of the convex hull of the irreducible representation with highest weight λ and adding them up with a sign. This is *alternating* under the Weyl group. If λ is invariant under a reflection in the Weyl group, then $A_\lambda = 0$.

Let's assume the following, to be proved on Monday.

31.7 Lemma. $A_\rho = \prod_{\alpha \in R^+} (e(\alpha/2) - e(-\alpha/2)) = e(-\rho) \prod (e(\alpha) - 1)$.

31.8 Theorem (Weyl character formula)**.**

$$\operatorname{char}(\Gamma_\lambda) = \frac{A_{\rho+\lambda}}{A_\rho}. \tag{93}$$

Here Γ_λ is the irreducible representation with highest weight λ.

31.9 Example. Let's do $\mathfrak{sl}_{n+1}$. The fundamental weights are $L_1 + \cdots + L_i$. In particular, $\rho = nL_1 + (n-1)L_2 + \cdots + L_n = \frac{1}{2}\sum_{1\le i<j\le n+1}(L_i - L_j)$. Let X_i correspond to L_i. Then

$$A_\rho = \sum_{\sigma\in S_{n+1}} (-1)^\sigma \left(e(nL_{\sigma(1)} + \cdots + L_{\sigma(n)})\right) = \sum_\sigma (-1)^\sigma X^n_{\sigma(1)} \dots X_{\sigma(n)}.$$

This is a van der Monde determinant, so it's $\prod_{i<j}(X_i - X_j)$. We'll see that there's a similar phenomenon for all the groups.

Lecture 32
4/29

Let $\mathfrak{g}$ be a semisimple Lie algebra, Λ the weight lattice, $\alpha_1, \dots, \alpha_n$ the simple roots, $\omega_1, \dots, \omega_n$ the fundamental weights. We let

$$\mathrm{char} : R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^{\mathcal{W}}$$

be the character map from the representation ring of $\mathfrak{g}$. The simple roots form a very broad set, which hug the hyperplane that separates the roots, while the fundamental weights are all concentrated in the Weyl chamber.

Recall that ρ is the distinguished weight, which is $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$, and it's also $\rho = \omega_1 + \cdots + \omega_n$ (so, in particular, it is a weight). By virtue of the second expression, it's the smallest weight in the interior of the Weyl chamber (which consists of nonnegative linear combinations of $\omega_1, \dots, \omega_n$). Next, for $\lambda \in \Lambda$, we defined

$$A_\lambda = \sum_{w\in\mathcal{W}} (-1)^w e(w(\lambda)). \tag{94}$$

Observe that this is not in $(\mathbb{Z}[\Lambda])^{\mathcal{W}}$: this expression is *skew-symmetric* with respect to the Weyl group.

32.1 Theorem (Weyl character formula)**.** *If Γ_λ is the irreducible representation with highest weight λ, then*

$$\mathrm{char}\Gamma_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}. \tag{95}$$

It's not even clear that this makes sense. It's not clear that the quotient makes sense in the character ring. Note, however, that since both the numerator and the denominator are skew-invariant under the Weyl group, the quotient (if it exists) is $\mathcal{W}$-invariant.

We'll do this calculation in a larger ring, where we allow half-weights. We're going to work with the group ring of $\frac{1}{2}\Lambda$, and then I'm going to enlarge that to allow formal power series. In other words, I'm going to take

$$\mathbb{Z}[[\frac{1}{2}\Lambda]],$$

which means that we're allowing "infinite linear combinations" (but only in one direction).

32.2 Lemma. $A_\rho = \prod_{\alpha\in R^+}(e(\alpha/2) - e(-\alpha/2))$.

Proof. We're going to take $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ as the definition. So we can write the right-hand-side as

$$A \stackrel{\text{def}}{=} e(-\rho)\prod_{\alpha\in R^+}(e(\alpha) - 1).$$

We want to show that this is A_ρ.

- First, let's show that A is alternating (since A_ρ is). It's enough to do this for the reflections that come from the simple roots. If w is the reflection about the ith simple root (or about the plane perpendicular to this), then I want to say that $wA = -A$. The reflections about the simple roots generate the Weyl group.

  That will follow, in turn, from the sublemma. Remember this picture of the simple roots as the positive roots closest to the hyperplane that separates positive from negative roots. When I reflect about a root closest to the hyperplane, it's going to send that particular root to its negative, and permute the other positive roots.

  32.3 Lemma. *The reflection w about a simple root α_i carries α_i to $-\alpha_i$ and permutes the other positive roots.*

  Proof. If $\beta \in R^+$, then $\beta = \sum m_j\alpha_j$ for the $m_j \geq 0$. If I reflect about α_i, then I get

  $$w\beta = \beta - \frac{2\beta.\alpha_i}{\alpha_i.\alpha_i}\alpha_i.$$

  In particular, $w(\beta)$ is the same sum of $\sum m_j\alpha_j$ minus some multiple of α_i. If $\beta = \alpha_i$, of course, what we get is $-\alpha_i$. If β is any other positive root, there's another positive coefficient m_j ($j \neq i$), so the expression for $w(\beta)$ still has positive coefficients. But as we saw, every root is a nonnegative linear combination of simple roots or nonpositive linear coefficient. If there's one positive coefficient, then the root is positive. ▲

  As an example, we find that if w is the reflection about α_i as above, then

  $$w\rho = \rho - \alpha_i,$$

  because w basically permutes the summands and flips one sign. In particular,

  $$2\frac{(\rho, \alpha_i)}{(\alpha_i, \alpha_i)} = 1,$$

  which gives $\rho = \sum \omega_i$. (In fact, $2\frac{(\omega_j,\alpha_i)}{(\alpha_i,\alpha_i)} = \delta_{ij}$.) This is what we claimed earlier. Now, we get

  $$wA = \prod_{\alpha\in R^+}(e(w\alpha/2) - e(-w\alpha/2))$$

  and this is $-A$ by the above smaller lemma.

- A itself is a finite product, so you can multiply out, and you can see that its highest weight is ρ itself. The highest weight term appears with coefficient 1.
- Now, I want to write formally $\frac{1}{A}$, and for this purpose I want to use the last expression $A = e(\rho)\prod(1 - e(-\alpha))$. I get

  $$\frac{1}{A} = e(-\rho)\prod_{\alpha \in R^+}(1 + e(-\alpha) + e(-2\alpha) + \dots).$$

  Hence,

  $$\frac{A_\rho}{A} = A_\rho e(-\rho)\prod_{\alpha \in R^+}(1 + e(-\alpha) + e(-2\alpha) + \dots),$$

  and it follows that this expression has highest weight 0. The only term in the ratio that lives in the Weyl chamber is the constant term 0. On the other hand, it's symmetric, so it has to be 1, and we're done. The question is to find the right ring in which to make this calculation.

We can apply the same reasoning to $\frac{A_{\lambda+\rho}}{A_\rho}$, which has highest weight λ, and the same reasoning shows that it has finitely many terms in the Weyl chamber. It's invariant under the Weyl group, so it has finitely many terms. So this expression $\frac{A_{\lambda+\rho}}{A_\rho}$ lies where we'd like it to: in the ring $\mathbb{Z}[\Lambda]^{\mathcal{W}}$. So the Weyl character formula makes sense. ▲

In the case of $\mathfrak{sl}_{n+1}$, we get a very explicit expression in terms of the Schur polynomials. Originally, the Weyl character formula came from a prior knowledge of $\mathfrak{sl}_{n+1}$, and that goes back to the discussions in ch. 15 and that goes back to representations of the symmetric group. But we never did that and never introduced Schur functors.