Lifting Endo-trivial Modules

J.L. Alperin *

Endo-permutation modules arise in a number of ways in the representation theory of finite groups, in particular in connection with equivalences between blocks and as sources of simple modules. Great progress has been made recently towards a complete classification of such modules in characteristic p. The question of lifting such modules to valuation rings has been open now for some time and is now quite relevant to the classification of such modules over such rings. In this paper, we solve the problem in the key case of endo-trivial modules.

Let R be a complete discrete valuation ring of characteristic zero with maximal ideal J and residue class field k of prime characteristic p. For a fixed finite p-group P, all the kG-modules considered will be finite dimensional and all the RP-modules will be finitely generated and free as R-modules. An RG-module M (and similarly for RP-modules) is an endo-trivial module if the tensor product $M^* \otimes M$ is the direct sum of the trivial RP-module R and a free RG-module. The standard results of the theory of such modules may be found in the references [4, 5, 8].

In order to state the result of this paper, recall that an RP-module M lifts a kP-module U if M/JM is isomorphic with U.

Theorem. *Any endo-trivial kP-module lifts to an endo-trivial RP-module.*

If M is an RP-module lifting the endo-trivial kP-module U then M is also endo-trivial. For M has rank not divisible by p so $M^* \otimes M$ is the direct sum of the trivial RP-module R and another RP-module N. But then the reduction mod J of N is free, by hypothesis, so N is free as desired. Hence, to prove the theorem, we need only establish the lifting.

We denote the congruence subgroups of $\mathrm{GL}(n, R)$ by $\mathrm{GL}(n, R; m)$. Here m is a positive integer and $\mathrm{GL}(n, R; m)$ consists of the n by n matrices congruent to the identity matrix I modulo J^m so these subgroups are normal subgroups of $\mathrm{GL}(n, R)$ and form a central series of $\mathrm{GL}(n, R; 1)$ with all the successive quotients isomorphic, as modules for $\mathrm{GL}(n, k)$, to the module of all n by n matrices over k under conjugation by $\mathrm{GL}(n, k)$ (which is isomorphic as always to the tensor product of the standard module and its dual). If n is not divisible by p then this module of matrices is the direct sum of a trivial module and the module of matrices of trace zero. Similarly defining the congruence subgroups $\mathrm{SL}(n, R; m)$ of the special linear group $\mathrm{SL}(n, R)$ we get a central series of $\mathrm{SL}(n, R; 1)$ with all the successive quotients isomorphic, as modules for $\mathrm{SL}(n, k)$, to the matrices of trace zero.

If U is an endo-trivial kP-module of dimension n then the image of P in $\mathrm{GL}(n, k)$, under the corresponding representation, lies in $\mathrm{SL}(n, k)$, as P is a p-group and the multiplicative group of k has no non-identity p-elements. The question of lifting is now whether the

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extension of $\mathrm{SL}(n,R;1)$ by this image splits. However, $\mathrm{SL}(n,R;1)$ is filtered by a series of subgroups with the successive quotients each a free module for this image, by our hypothesis on U, in view of the properties of the congruence subgroups just given, so the lifting is automatic.

Notice that the argument also shows that this lifting is unique up to conjugacy in $\mathrm{SL}(n,R)$ while there may be other lifts in $\mathrm{GL}(n,R)$.

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