

DESCENT

Quick references:

- Angelo Vistoli's notes
- Etale cohomology book by Tamme
- Higher algebra by Lurie
- Akhil's paper on Galois groups of stable homotopy theories.

1. CARTESIAN FIBRATIONS

The fiber of a functor $p : \mathcal{C} \rightarrow \mathcal{D}$ at $d \in \mathcal{D}$ is the subcategory $p^{-1}(d)$ of $\mathcal{C}$ of objects lifting d and maps lifting the identity.

For $p : \mathcal{C} \rightarrow \mathcal{D}$ a morphism $c' \rightarrow c$ is p -Cartesian if

$$\begin{array}{ccc} \mathrm{Hom}(z, c') & \longrightarrow & \mathrm{Hom}(z, c) \\ \downarrow p & & \downarrow p \\ \mathrm{Hom}(pz, pc') & \longrightarrow & \mathrm{Hom}(pz, pc) \end{array}$$

is a pullback diagram for any $z \in \mathcal{C}$

A functor $p : \mathcal{C} \rightarrow \mathcal{D}$ is a Cartesian fibration if for any $d' \rightarrow d$ and c lifting d there is a morphism $c' \rightarrow c$ lifting $d' \rightarrow d$ that is p -Cartesian. The lifted morphism is unique up to a unique isomorphism, and choosing such lifts is a choice of pullbacks. So for any $f : d' \rightarrow d$ we get a pullback $f^* : p^{-1}(d) \rightarrow p^{-1}(d')$. This is unfortunately not a functor $\mathcal{D} \rightarrow \mathbf{Cat}^{op}$ because the arbitrary choices may not play well with composites. There are two ways to fix it:

- Replace $p^{-1}(d) = \mathrm{Fun}_p(\{d\}, \mathcal{C})$ with the equivalent category $\mathrm{Fun}_p(\mathcal{D}/d, \mathcal{C})$ of functors that compose with p to give the usual $\mathcal{D}/d \rightarrow \mathcal{D}$. $\mathrm{Fun}_p(\mathcal{D}/-, \mathcal{C})$ is always a functor $\mathcal{D} \rightarrow \mathbf{Cat}^{op}$ and $\mathrm{Fun}_p(\mathcal{D}/d, \mathcal{C}) \simeq p^{-1}(d)$ via the choice of pullbacks.
- View $\mathcal{C}$ as a 2-category and $p^{-1} : \mathcal{C} \rightarrow \mathbf{Cat}$ as a pseudofunctor, i.e. instead of preserving identities+composites we get natural isomorphisms $1 \implies F(1)$ and $F(f) \circ F(g) \implies F(gf)$ that make 2 diagrams commute.

We take the second approach and do the converse.

Given a pseudofunctor $F : \mathcal{C}^{op} \rightarrow \mathbf{Cat}$ the Grothendieck construction $\int F \rightarrow \mathcal{C}$ is the pullback

$$\begin{array}{ccc} \int F & \longrightarrow & \mathbf{Cat}_{*,l}^{op} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathbf{Cat}^{op} \end{array}$$

where $\mathbf{Cat}_{*,l}$ is the category of lax pointed categories i.e. has objects categories with chosen objects (A, a) and maps $(A, a) \rightarrow (B, b)$ are functors $f : A \rightarrow B$ and maps $\eta : f(a) \rightarrow b$. The right hand side vertical map is the universal Cartesian fibration.

Therefore, $\int F$ has

- objects (c, a) for $c \in \mathcal{C}$ and $a \in Fc$.
- morphisms $(c', a') \rightarrow (c, a)$ maps $f : c' \rightarrow c$ and $\eta : a' \rightarrow Ffa$ ($Ff : Fc \rightarrow Fc'$ is a functor).

$p : \int F \rightarrow \mathcal{C}$ is then a Cartesian fibration with pullbacks $f^*(c, a) = (c', Ffa)$ for $f : c' \rightarrow c$; the fibers are $p^{-1}(c) \approx Fc$ and p^{-1} is naturally isomorphic to F as pseudofunctors.

The Grothendieck construction and taking fibers give an equivalence of categories of fibrations on $\mathcal{C}$ and pseudo-presheaves of categories on $\mathcal{C}$.

2. A WORD ABOUT SITES

A Grothendieck topology on a category $\mathcal{C}$ with pullbacks is a collection of coverings $\{U_i \rightarrow U\}$ s.t.

- Any isomorphism forms a singleton covering
- If $\{U_i \rightarrow U\}$ and $\{V_{ij} \rightarrow U_i\}$ are coverings then $\{V_{ij} \rightarrow U\}$ is a covering
- If $\{U_i \rightarrow U\}$ is a covering and $V \rightarrow U$ then $\{U_i \times_U V \rightarrow U\}$ is a covering.

A category with a chosen topology is a site. In the following examples, the big site over X uses $\mathcal{C}/X$ and certain coverings, while the small site is the full subcategory of $\mathcal{C}/X$ of objects $Y \rightarrow X$ that can be included in a covering.

- The site of a topology on X uses jointly surjective open embeddings as coverings (so the small site is $Open(X)$ while the big site is $Spaces/X$).
- The étale site of spaces is $Spaces$ with jointly surjective maps that are covering maps on their images.
- The Zariski site on a scheme.
- The étale site on a scheme X uses as covers the jointly surjective étale maps.
- The fppf and fpqc sites on a scheme (see wikipedia; we will define a version of the fpqc site for affine schemes later).

If $\leq$ denotes finer than, then

$$(1) \quad \text{fpqc} \leq \text{fppf} \leq \text{étale} \leq \text{Nisnevich} \leq \text{Zariski}$$

All these topologies are subcanonical in the sense that the representable functors are all sheaves. We shall prove this in the last section.

The Čech nerve for a presheaf $\mathcal{F}$ and cover $U_i \rightarrow X$ is the cosimplicial (actually, cofacial) set

$$\coprod_i \mathcal{F}U_i \rightrightarrows \coprod_{i,j} \mathcal{F}(U_i \times_X U_j) \rightrightarrows \cdots$$

and $\mathcal{F}$ is a sheaf if the limit (equalizer of the first two arrows) is $\mathcal{F}X$. For all representable functors to be sheaves, we want

$$\coprod_i U_i \rightrightarrows \coprod_{i,j} U_i \times_X U_j \rightrightarrows \cdots$$

to have limit X (we are assuming our site is cocomplete here). If we have a singleton cover then

$$U \rightrightarrows U \times_X U \Rrightarrow \dots$$

is the cobar construction, thinking of $\times_X$ as the tensor product (this will become more concrete in a following section).

3. THE TWO GLUINGS

Let $p : \mathcal{C} \rightarrow \mathcal{D}$ be a Cartesian fibration with chosen pullbacks. For any two $x, y \in p^{-1}(d) = Fd$ we define the presheaf $\text{Hom}(x, y) : \mathcal{D}/d \rightarrow \mathbf{Set}$,

$$(2) \quad \text{Hom}(x, y)(d' \xrightarrow{f} d) = \text{Hom}_{p^{-1}(d')}(f^*x, f^*y) = \text{Hom}_{Fd'}(Ffx, Ffy)$$

(I'll let you figure out how we define it on the morphisms). These presheaves are independent of the choice of pullbacks up to natural isomorphism.

Given a family of morphisms $f_i : d_i \rightarrow d$ in $\mathcal{D}$ the category of descent data $\text{Desc}(f_i)$ has

- objects (c_i, ϕ_{ij}) where $c_i \in p^{-1}(d_i) = Fd_i$ and $\phi_{ij} : p_0^*(c_i) \rightarrow p_1^*(c_j)$ are isomorphisms in $p^{-1}(d_i \times_d d_j)$ ($p_0, p_1 : d_i \times_d d_j \rightarrow d_i, d_j$) satisfying the cocycle condition in the sense that

$$\begin{array}{ccc} p_0^*(c_i) & \xrightarrow{\phi_{ij}} & p_1^*(c_j) \\ & \searrow \phi_{ik} & \swarrow \phi_{jk} \\ & p_2^*(c_k) & \end{array}$$

commutes on $p^{-1}(d_i \times_d d_j \times_d d_k)$.

- Morphisms $(c_i, \phi_{ij}) \rightarrow (c'_i, \phi'_{ij})$ maps $c_i \rightarrow c'_i$ on $p^{-1}(d_i)$ giving commutative squares.

Different choices of pullbacks give equivalent descent categories.

Any $c \in p^{-1}(d) = Fd$ determines a canonical descent datum $(f_i^*c, 1) = (Ff_i c, 1) \in \text{Desc}(f_i)$ where 1 is the identity on $(f_i \times f_j)^*c = F(f_i \times f_j)c \in F(d_i \times_d d_j)$. This defines a functor $Fd \rightarrow \text{Desc}(f_i)$ and the objects in the essential image (i.e. those isomorphic to canonical descent data) are called effective.

A stack over a site $\mathcal{C}$ is a Cartesian fibration $p : \mathcal{S} \rightarrow \mathcal{C}$ s.t.

- The presheaf $\text{Hom}(x, y)$ on $\mathcal{C}/c$ is a sheaf for any $x, y \in p^{-1}(c) = Fc$.
- For every covering every descent datum is effective. In other words, $\text{Desc}(U_i \rightarrow c) \simeq Fc$ by the canonical map.

For a pseudofunctor $F : \mathcal{C}^{op} \rightarrow \mathbf{Cat}$, the totalization i.e. homotopy limit of its Čech nerve on a cover is the associated descent category. So while for presheaves of sets the sheaf condition translates to isomorphism with the limit of the Čech nerve, for presheaves of categories the appropriate sheaf condition is equivalence with the homotopy limit of the Čech nerve.

Some examples

- A Cartesian fibration given by a functor $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ i.e. all fibers are discrete categories (only identities) has $\mathrm{Hom}(x, y)(d \xrightarrow{f} c) = *$ if $Ffx = Ffy$ and is empty otherwise, so we have sheaves Hom 's. The descent data are $x_i \in Fd_i$ agreeing on $F(d_i \times_d d_j)$.

Thus F is a stack $\iff F$ is a sheaf. So stacks are the 2-category generalizations of sheaves.

- Let $\mathcal{S}$ be the arrow category of *Spaces* and $p : \mathcal{S} \rightarrow \mathbf{Spaces}$ be the codomain functor; $p^{-1}(X) = \mathbf{Spaces}/X$. Our chosen pullbacks are the usual presentations of pullbacks of spaces. For spaces Y, Z over X ,

$$(3) \quad \mathrm{Hom}(Y, Z)(W \rightarrow X) = \mathrm{Hom}_W(Y \times_X W, Z \times_X W)$$

is the set of continuous maps $Y' \rightarrow Z'$ where Y', Z' are the points of Y, Z projecting to the image of W in X . For open $W \subseteq X$ this is maps $f^{-1}(W) \rightarrow g^{-1}(W)$; such maps glue (if they agree on intersections) to $f^{-1}(X) \rightarrow g^{-1}(X)$ hence Hom is a sheaf.

The descent data for $X = \cup_i U_i$ are maps $Y_i \rightarrow U_i$ with homeomorphisms $p_i : Y_i \rightarrow Y_j$ satisfying the cocycle condition over $p_u^{-1}(U_i \cap U_j \cap U_k)$ for $u = i, j, k$. Thus all descent data are effective and we have a stack of spaces.

Similarly, this example for schemes in the Zariski topology gives a stack of schemes.

- Let $\mathcal{S}$ be the category of spaces with vector bundles and $p : \mathcal{S} \rightarrow \mathbf{Spaces}$ be the underlying space; $p^{-1}(X) = \mathbf{Vect}(X)$. We choose pullbacks as before and then if V, W are bundles over X

$$(4) \quad \mathrm{Hom}(V, W)(Y \xrightarrow{f} X) = \mathrm{Hom}(f^*V, f^*W)$$

More generally for a group G we can consider principal G -bundles and get a stack $BG \rightarrow \mathbf{Spaces}$.

4. FAITHFULLY FLAT DESCENT

A map of rings $A \rightarrow B$ is faithfully flat if B is a flat A -module and $B \otimes_A M = 0 \implies M = 0$. This is equivalent to B being flat over A and $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(B)$ being surjective.

We will work with a version of the fpqc site that has objects affine schemes and the coverings are flat maps $\mathrm{Spec}(B_i) \rightarrow \mathrm{Spec}(A)$ s.t. $\coprod_i \mathrm{Spec}(B_i) \rightarrow \mathrm{Spec}(A)$ is quasi-compact and surjective; quasi-compactness implies that the indices i are finitely many so we can take $B = \prod_i B_i$ and replace the cover with a single flat map $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$.

Faithfully flat descent: If $R \rightarrow S$ is faithfully flat then

$$(5) \quad R \rightarrow S \rightrightarrows S \otimes_R S$$

is an equalizer. Easy exercise! This proves that the fpqc topology is subcanonical, i.e. all representatives are sheaves. The affine case immediately implies the case of general schemes (using an appropriate definition of the fpqc site there). We note that the étale topology is also subcanonical, for the exact same reason.

Slightly more generally, if M is an R module,

$$(6) \quad M \rightarrow M \otimes_R S \rightrightarrows M \otimes_R S \otimes_R S$$

is an equalizer. This proves that if $\mathcal{F}$ is a quasi-coherent sheaf on the Zariski site of X then the flatization

$$(7) \quad \mathcal{F}(Y \xrightarrow{f} X) = (f^* \mathcal{F})(Y)$$

is a sheaf on the fpqc site over X . This allows us to consider quasi-coherent sheaves as sheaves on the flat site (or if you prefer, the étale site).

Even more generally, if we take the alternating sum of the faces of the Čech nerve, we get a complex

$$(8) \quad M \rightarrow M \otimes_R S \rightarrow M \otimes_R S \otimes_R S \rightarrow \cdots$$

that is exact. This is Grothendieck's Theorem for faithful flatness. Because the Čech complex is used to compute Čech cohomology and there is a spectral sequence from Čech to derived functor sheaf cohomology, this fact implies that the Zariski sheaf cohomology of a quasi-coherent sheaf equals the flat (or étale) sheaf cohomology of its flatization (or étalization). So in this sense, the theory of étale/flat cohomology contains the classical Zariski theory. The really interesting thing is when you mix them up, say have an SES of étale sheaves where some of them are quasi-coherent, and take étale cohomology. This will give a long exact sequence where some terms will be Zariski cohomology and the rest étale cohomology. Examples are the Artin-Schreier and Kummer exact sequences.

Finally we explain what this descent has to do with stacks. Consider the category $\mathcal{S}$ with objects $(\mathrm{Spec}(A), M)$, M an A module, with morphisms being maps of schemes and $M \otimes_A B \rightarrow N$. We have a fibering $p : \mathcal{S} \rightarrow \mathcal{A}ff$ with $p^{-1}(A) = \mathbf{Mod}_A$. For A -modules M, N ,

$$(9) \quad \mathrm{Hom}(M, N)(\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)) = \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$$

is a sheaf on the flat site $\mathcal{A}ff/\mathrm{Spec}(A)$ by faithful flat descent (exercise).

For a map $f : \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ the descent data are B -modules M with isomorphisms $\phi : M \otimes_A B \rightarrow B \otimes_A M$ of $B \otimes_A B$ modules (with the obvious factorwise action) s.t.

$$\begin{array}{ccc} M \otimes_A B \otimes_A B & \xrightarrow{\phi \otimes 1} & B \otimes_A M \otimes_A B \\ & \searrow \phi' & \swarrow 1 \otimes \phi \\ & B \otimes_A B \otimes_A M & \end{array}$$

where ϕ' fixes the second factor and acts as ϕ on the first and last factors.

A descent datum is effective if it comes from an A -module N namely $M \approx N \otimes_A B$ and $\phi : B \otimes_A M \rightarrow M \otimes_A B$ is induced from this isomorphism in the obvious way.

Grothendieck: If f is faithfully flat then all descent data in $\mathrm{Desc}(f)$ are effective. Indeed, given M, ϕ let N be the kernel of the A -map $\rho : M \rightarrow B \otimes_A M$, $\rho(m) = 1 \otimes m - \phi(m \otimes 1)$; then we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & N \otimes_A B & \longrightarrow & M \otimes_A B & \xrightarrow{\rho \otimes 1} & B \otimes_A M \otimes_A B \\
& & \downarrow & & \downarrow \phi & & \downarrow 1 \otimes \phi \\
0 & \longrightarrow & M & \longrightarrow & B \otimes_A M & \longrightarrow & B \otimes_A B \otimes_A M
\end{array}$$

(the second row is exact by faithfully flat descent and the second square is exact by the descent datum). The 5-Lemma gives that the first map is an iso as desired. So the essential image of the extension of scalars functor is the descent data.

More generally, the pseudofunctor sending scheme X over S to the category of quasi-coherent sheaves on X , is a stack on the fpqc site.

5. WHERE ARE THE SPECTRA?

Descent: If $A \rightarrow B$ is a map of E_∞ rings that is faithfully flat on π_0 and B is flat over A (i.e. $\pi_0 B$ is flat over $\pi_0 A$ and $\pi_0 B \otimes_{\pi_0 A} \pi_* A = \pi_* B$) then the adjunction $\mathbf{Mod}(A) \rightleftarrows \mathbf{Mod}(B)$ is comonadic by Barr-Beck-Lurie.

As a result,

$$(10) \quad \mathbf{Mod}(A) \simeq Tot(\mathbf{Mod}(B) \rightrightarrows \mathbf{Mod}(B \otimes_A B) \cdots)$$

This cobar construction on the right is the ∞ -category of descent data as before.

We shall now see that the faithful flatness hypothesis can be relaxed.

If X is a stable homotopy theory (presentable stable ∞ symmetric monoidal category), a commutative algebra object x admits descent if the thick $\otimes$ -ideal it generates is all of X (thick subcategory=closed under finite colimits+retracts, $\otimes$ -ideal means closed under tensoring with anything in X).

Example: An E_∞ ring spectrum E admits descent in $L_E Sp$ if by definition all spectra are E -prenilpotent (X is prenilpotent if $L_E X$ is in the thick tensor ideal generated by E). When that happens, E -localization is smashing and the Adams tower for X converges to $L_E X$. Moreover, the resulting Adams SS has a horizontal vanishing line in a finite page, hence it stabilizes.

More examples: $L_n S \rightarrow E_n$ descends i.e. E_n admits descent in $\mathbf{Mod}(L_n S) = L_n Sp$. Any faithful G -Galois extension of E_∞ spectra admits descent.

If x admits descent, $X \rightleftarrows \mathbf{Mod}(x)$ (given by tensoring with x /forgetting), is comonadic as above, hence

$$(11) \quad X \simeq Tot(\mathbf{Mod}(x) \rightrightarrows \mathbf{Mod}(x \otimes x) \cdots)$$

Moreover, the spectral sequence associated to the tower $Tot^n(x^{\otimes \bullet + 1})$ has a horizontal vanishing line at a finite page.

If $A \rightarrow B$ is faithfully flat and A_* is countable then B admits descent over A .