You Need a Lemma

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The Goal

The Goal: Demystifying Definitions

- To understand them. This often means unifying concepts in various places, or drawing analogies between them. Consider all the different examples of a Free Thingamabop
- To use them. Often the way we *use* things is often not how we've constructed them. Consider the construction of the integral or the construction of the tensor product in linear algebra.

But to do all this, you need a lemma.

What's a Category?

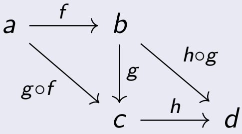
The Cold Hard Definition

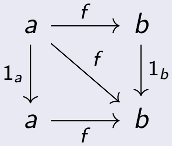
Definition

A category C is a collection of objects $\mathrm{Ob}(C)$ equipped with a collection of arrows $\mathrm{Hom}(a, b)$ for any two objects $a, b \in \mathrm{Ob}(C)$ together with a composition law:

$$\circ : \mathrm{Hom}(b, c) \times \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, c)$$

And identities $1_a \in \mathrm{Hom}(a, a)$ for each $a \in \mathrm{Ob}(C)$ together with the laws:





Ok ok, Some Examples???

- The Category of all sets as our objects along with functions as our arrows. We denote this by **Set**
- The Category of all vector spaces over $\mathbb{R}$, called $\mathbf{Vect}_{\mathbb{R}}$, with linear maps between them
- The Delooping of a group $\mathbf{B}G$, which has a single object called $\star$ and so that $\mathrm{Hom}(\star, \star)$ is the group G with the inherited composition law.

What's a Presheaf?

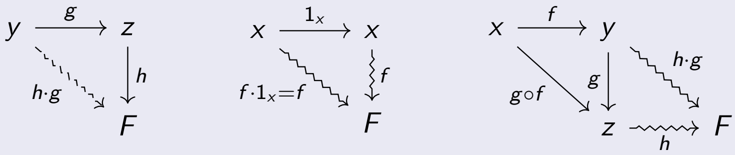
What about a Presheaf?

The Idea

Let's consider a category C and think of *generalized codomains* for the objects of C, that is an F with a sensible definition of an arrow $x \to F$ for $x \in \mathrm{Ob}(C)$

Definition

Conider a category C. A presheaf F on C asociates a set of arrows $F(x)$ to every $x \in \mathrm{Ob}(C)$. We also have a composition law $\cdot$ which allows you to compose diagrams of this form:



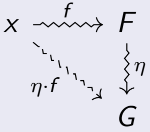
Some Nice Examples

- These are *generalized* codomains! So plain codomains $x \in \mathrm{Ob}(C)$ give rise to a presheaf $\mathcal{Y}(x) := \mathrm{Hom}(-, x)$, which associates to each $a \in \mathrm{Ob}(C)$ the set $\mathrm{Hom}(a, x)$ and which has the same composition law as C.
- A group action G on a set X is a way of combining any $g \in G$ with some $x \in X$ to form $g \cdot x$ which is compatible with the group operation!!! But this is just a presheaf on $\mathbf{B}G$
- Dually, we can think of generalized domains. With this in mind, given any two vector space V, W we can form a copresheaf which takes a vector space X and returns all the multilinear maps from $V \times W$ to X.

Natural Transformation

Definition

Let F and G be presheafs on a category C. We call $\eta : F \to G$ a natural transformation from F to G provided that it for every $f : x \to f$ it allows us to fill in diagrams of this form in a coherent way:



Concretely, this means η induces a map $\eta_x : F(x) \to G(x)$, and these maps fit together so that this composition law is compatible with the one in C and the ones defined by F and G

The Yoneda Lemma

The Yoneda Lemma: The Meat

Lemma

For any presheaf F on C and any object $x \in \mathrm{Ob}(C)$, we have a natural isomorphism:

$$\mathrm{Nat}(\mathcal{Y}(-), F) \cong F$$

In other words, the natural transformations from $\mathcal{Y}(x)$ to F are exactly the "arrows" from x to F for any $x \in \mathrm{Ob}(C)$.

Proof.

An exercise in chasing diagrams and being a bit clever. I won't spoil it for you ♡

Representables

We now make a key definition

Definition

Given a presheaf F on C and an object $x \in \text{Ob}(C)$, we say that x is a represents F provided that there is a natural isomorphism between $\mathcal{Y}(x)$ and F. In this case we say that F is representable

The Payoff

Corollary

For any two objects $x, y \in \text{Ob}(C)$ *we have that:*

$$\text{Nat}(\mathcal{Y}(x), \mathcal{Y}(y)) \cong [\mathcal{Y}(y)](x) = \text{Hom}(x, y)$$

By the Yoneda lemma and its dual, this bijection is moreover a natural isomorphism. Therefore, if two objects induce the same presheaf, they must be isomorphic, and naturally so.

The Idea

But wait!!! This means that we can define things by how they act when you map into them, or dually, by how you map out of them!!! ♡

A Motivating Example: Tensor Product

The Traditional Construction

Suppose we have two vector spaces V, W, then $V \otimes W$ is the vector space given by taking $V \times W$ as the basis for the free vector space $F(V \times W)$, and then quotienting out by the smallest equivalence relation satisfying:

- $(v, w) + (v', w) \sim (v + v', w)$ and $(v, w) + (v, w') \sim (v, w + w')$
- $c(v, w) \sim (cv, w) \sim (v, cw)$.

So then $V \otimes W = F(V \times W)/\sim$, and we denote $v \otimes w$ as the equivalence class of (v, w) under $\sim$.

The Copresheaf

Fix two vector spaces V, W. Consider the copresheaf on $\mathbf{Vect}_{\mathbb{R}}$ defined by:

$$F(X) = \{f \mid f : V \times W \to X \text{ is multilinear}\}$$

Then F is representable, and its representative is $V \otimes W$. If we then consider the identity on $V \otimes W$ must correspond to some multilinear map $V \times W \to V \otimes W$, we recover the familiar bilinear map:

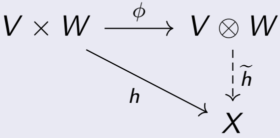
$$\phi : V \times W \to V \otimes W$$
$$(v, w) \mapsto v \otimes w$$

But what does that mean???

We can use naturality in the Yoneda Lemma along which the map ϕ to make the following definition for the tensor product

Definition

For any two vector spaces V, W, there is a vector space $V \otimes W$ and a bilinear map $\phi : V \times W \to V \otimes W$ which is unique up to a canonical isomorphism such that given any multilinear map $h : V \times W \to X$ there must be a unique linear map $\widetilde{h} : V \otimes W \to X$ such that:



commutes.

Computing Dimension

Let V have dimension n and let W have dimension m. Now consider a linear function $f : V \otimes W \to \mathbb{R}$. Then this is just a multilinear function $\widetilde{f} : V \times W \to \mathbb{R}$ and we can break this up as:

$$\widetilde{f}(v, w) = \sum_{i=1}^{n} a_i \widetilde{f}(v_i, w) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \widetilde{f}(v_i, w_j)$$

The scalars $\widetilde{f}(v_i, w_j)$ uniquely determine $\widetilde{f}$, and so they uniquely determine f. There are nm such scalars, and so $V \otimes W$ has dimension nm.

Free Constructions

Concrete Categories

The Idea

Many categories have the same arrows as those in **Set**, except that they satisfy additional properties. This motivates the following definitions.

Definition

We call a category C concrete provided that for every object x in C there is some set $U(x)$, and furthermore for any two objects x, y in C we must have $\text{Hom}_C(x, y) \subseteq \text{Hom}_{\textbf{Set}}(U(x), U(y))$, and the composition of C is merely the composition of **Set**.

Free Constructions

Definition

Consider some concrete category C. Now consider a set a. Then $F(a)$ in C is called a free object for a provided that there is the following natural isomorphism for any object y in C:

$$\mathrm{Hom}_C(F(a), y) \cong \mathrm{Hom}_{\mathbf{Set}}(a, U(y))$$

In other words, maps out of $F(a)$ consist of *all* functions out of a. Of course by the Yoneda lemma this uniquely characterizes $F(a)$ up to canonical isomorphism

The Examples

Fix a set a, we will exhibit a free object $F(a)$ in each category

- The Free Monoid is given by the set of all finite words in the alphabet a. The multiplication in this monoid is concatenation
- The Free Group is given by taking the free monoid and then adjoining inverses and using the group laws to simplify words
- The Free Abelian Group then allows us to commute elements when simplifying words
- The Free Topological space on a set a is the discrete topology on a
- The Free Vector Space on a set a is given by taking the elements of a as a basis for your vector space.

What to Take Away?

The Take-Away

The Yoneda lemma is a sort of meta-mathematical result that allows us to examine mathematical objects by looking at how they interact with other mathematical objects. Furthermore, we can even define mathematical objects using this idea, so long as we actually construct them at some point.

References and Credits

- My primary reference for this talk was *Category Theory in Context* by Emily Riehl
- I want to give credit to Sarah Rovner-Frydman, one of my close friends who introduced me to the idea of presheaves as generalized codomains. You can find her @sarah_zrf on Twitter.
- Finally, I want to thank the DRP and my mentor Lukas.

Questions?