The Point at ∞: What is the Degree of $\frac{1}{z}$

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The Degree of a Polynomial

Definition

The *degree* of a complex polynomial p, where p is expressed as:

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

Is the largest i so that $a_i \neq 0$. If $p = 0$ we say that the degree of p is $-\infty$. We write $\deg(p)$ to mean the degree of p.

Example

The degree of $0z^5 + 3z^4 + 2z + 6$ is

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The degree of $0z^5 + 3z^4 + 2z + 6$ is 4. Be careful of 0 terms

Why We Care: The Fundamental Theorem of Algebra

Definition

The multiplicity m of a root a of a polynomial p is defined as the unique nonnegative integer so that we can write:

$$p(z) = (z-a)^m g(z)$$

For some polynomial g where $g(a) \neq 0$.

Theorem (The Fundamental Theorem of Algebra)

Every polynomial p has a root in the complex plane. Furthermore, counting by multiplicity, every polynomial has $\deg(p)$ *roots in the complex plane. The proof is beyond the scope of this talk but it is beautiful.*

An Extension on the Fundamental Theorem, towards $\frac{1}{z}$

Corollary

Given any point w in the complex plane and polynomial p, p achieves the value w exactly $\deg(p)$ times, where we count with multiplicity.

Proof.

Consider the polynomial $q(z) = p(z) - w$. This has $\deg(q)$ roots with multiplicity by the Fundamental Theorem of Algebra. Note that $\deg(q) = \deg(p)$. □

$\frac{1}{z}$ fails the Fundamental Theorem...

- Consider that $\frac{1}{z}$ has no roots in the complex plane. That is there is no complex number z so that $\frac{1}{z} = 0$. What if we added one in? We call it ∞, with this $\frac{1}{z}$ has degree 1. And $\frac{1}{\infty} = 0$, as well as $\frac{1}{0} = \infty$.
- But wait, what about $\frac{1}{z^2}$? How do we account for multiplicity?

The Riemann Sphere $\hat{\mathbb{C}}$, the structure of $\mathbb{C}$ with ∞

- We want to extend the methods of the complex plane to a structure that includes ∞.
- In topology you might learn that the "one-point compactification" of a plane is the sphere. This will be our model.

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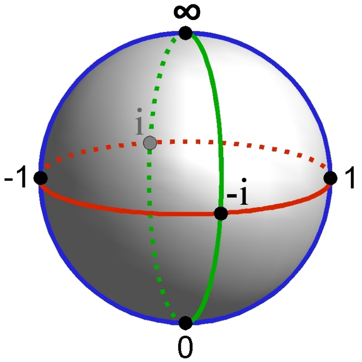
- We want to extend the methods of the complex plane to a structure that includes ∞.
- In topology you might learn that the "one-point compactification" of a plane is the sphere. This will be our model.
- The Riemann Sphere, imagine taking the plane and wrapping it up to meet the point at infinity.



How to go back and forth from $\hat{\mathbb{C}}$ and $\mathbb{C}$

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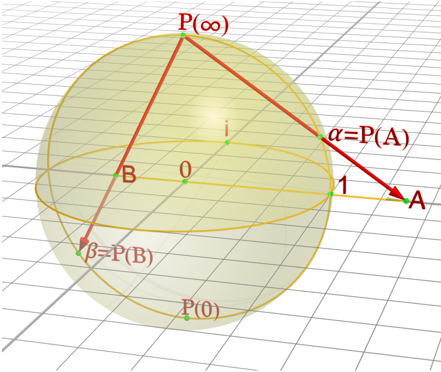
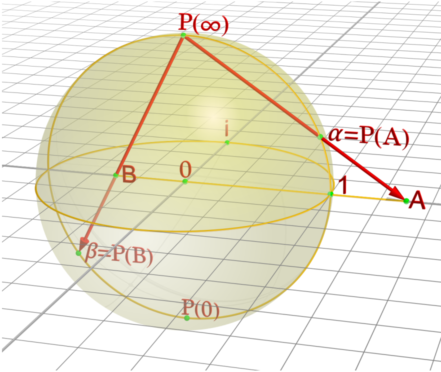
- The key is projection, this allows us to identify the complex plane with the sphere without ∞



How to go back and forth from $\hat{\mathbb{C}}$ and $\mathbb{C}$

- It would be nice to use what we know about the complex plane to understand the Riemann Sphere
- The key is projection, this allows us to identify the complex plane with the sphere without ∞
- But wait! Our choice of which point to use is arbitrary! What if we use 0! These are called *coordinates on the sphere*



Degree on the Riemann Sphere

The Idea

We say a sufficiently nice function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has degree d provided that at a generic point $y \in \hat{\mathbb{C}}$ there are d distinct points $x_1, \ldots, x_d$ so that $f(x_i) = y$ for $1 \leq i \leq d$.

The Idea

To deal with non-generic points we should say that for a sufficiently nice function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, a point $a \in \hat{\mathbb{C}}$ has multiplicity m provided that there are local coordiinates around a in which f can be expressed as a polynomial with a root at a with multiplicity m. There is a more complicated and full definition with power series, but it requires more technicality.

Degree on the Riemann Sphere

Example

Let's work through an example. Take the function $z \overset{f}{\mapsto} \frac{1}{z^2}$ on the Riemann Sphere. Then consider $a = \infty$. Our choice of coordinates is the projection from 0 onto the plane. After calculation with projections one sees that the coordinate function $f_\star : \mathbb{C} \to \mathbb{C}$ is the polynomial $f_\star(z) = z^2$. Thus f has degree 2 at $a = \infty$, further calculations hows that any point on the sphere gives the same degree. In fact any rational function $z \mapsto \frac{p(z)}{q(z)}$ with p and q polynomials so that p and q share no roots has a well defined degree as a function on the Riemann sphere.

But What *Are* Coordinates?

- I've been intentionally vague about coordinates because they require some background knowledge to understand.
- I think of it as making your space by patching together pieces of the Complex Plane. However, you need some compatibility condition on how those pieces interact where they intersect.

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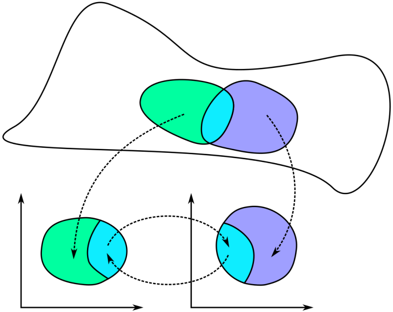
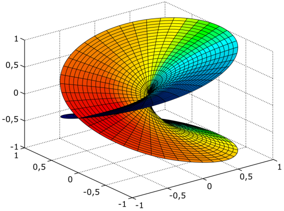


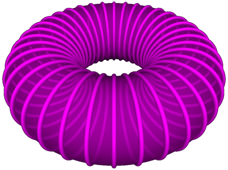
A More General Discussion

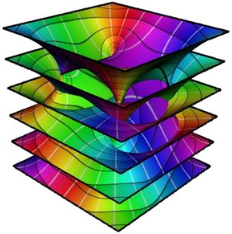
Definition

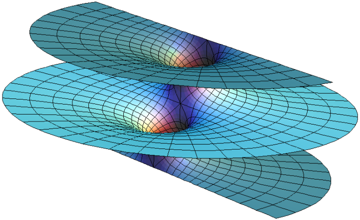
A *Riemann Surface* is a space X where you can assign coordinates around every point $x \in X$ which look like the complex plane. Furthermore, these coordinates should be compatible in some sense (holomorphic transition maps and complex manifolds for anyone who's experienced with these ideas)

A Zoo of Surfaces









Degree on General Surfaces

Theorem (An Amazing Result)

For X and Y compact Riemann Surfaces, there is an appropriate definition of degree so that for any holomorphic and nonconstant function $f : X \to Y$, the degree of f is constant over the entire surface. Furthermore, similar results like the Fundamental Theorem of Algebra hold.

The Idea

This allows us to do amazing things with the theory of compact Riemann Surfaces, and makes a lot of problems much easier. Just like how polynomials are easier to work with than other functions.

Primary Reference

Algebraic Curves and Riemann Surfaces by Rick Miranda.

Thank you so much!!!