

Biases in Parts Among k -regular and k -indivisible Partitions

by Faye Jackson and Misheel Otgonbayar

University of Virginia

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Partitions

Definition

- A *partition* of n , denoted $\lambda \vdash n$, is a non-increasing sequence of positive integers summing to n ,

$$\lambda = (\lambda_1, \dots, \lambda_k) \qquad \sum_{i=1}^k \lambda_i = n$$

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Example

We have $p(4) = 5$ since

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Hardy-Ramanujan's Asymptotic

Question

How does $p(n)$ grow as $n \rightarrow \infty$?

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Theorem (Hardy-Ramanujan)

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Parts in Congruence Classes

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The work of Beckwith and Mertens

Theorem (Beckwith and Mertens)

We have that

$$T(r, t; n) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\pi tn^{1/2}\sqrt{2}} \left[\log(n) - 2\psi\left(\frac{r}{t}\right) + \alpha_t + O\left(n^{-\frac{1}{2}} \log(n)\right) \right],$$

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- Parts are asymptotically *equidistributed*.
- The *second order term* implies a *BIAS*.
- If $r < s$, we eventually have $T(r, t; n) > T(s, t; n)$.

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$$D(r, t; n) = \sum_{\substack{\lambda \vdash n \\ \lambda \in \mathcal{D}}} \#\{\lambda_j \mid \lambda_j \equiv r \pmod{t}\}.$$

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- And for $2 \leq t \leq 10$, $r < s$, $D(r, t; n) \geq D(s, t; n)$ when $n \geq 8$.

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$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

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- If $2 \leq k \leq 10, 2 \leq t \leq 10$, then $r < s$ implies $D_k(r, t; n) > D_k(s, t; n)$ for $n \geq 17$.

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- Relies on explicit error terms and a finite computer check.
- *Better* explicit error terms than Craig when $k = 2$.

Introduction
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Our Results
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Conjectures/Examples
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The Circle Method
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Understanding $\psi_{k,t}$
ooooo

Wrap-Up
ooo

Stable k -regular counterexamples

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The case $(n, r, s, t) = (t, t - 1, t, t)$ is always a counterexample to the *strict* inequality for any k . The only partitions that contain such parts are,

(t) and $(t - 1, 1)$.

Stable counterexamples for small k

Example

For $k = 3$, $(n, r, s, t) = (t + 2, t - 1, t, t)$ is also a counterexample:

$$\begin{array}{cc} (t, 2) & (t, 1, 1) \\ (t - 1, 3) & (t - 1, 2, 1) \end{array}$$

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Remark

- **Asymptotic equidistribution**
- **UNPREDICTABLE BIASES!** If $\psi_{k,t}(r) > \psi_{k,t}(s)$, then eventually $D_k^\times(r, t; n) > D_k^\times(s, t; n)$.

The ordering $\prec_{k,t}$

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Let $\mathcal{O}(t)$ be the *number* of such orderings on $\{1, \dots, t\}$ induced by $\prec_{k,t}$ for $k, t \geq 2$ coprime.

A Glimpse of k -indivisible Biases

$k = 2$	1	3	5	7	2	4	6
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Figure: Biases among congruence classes mod t for k -indivisible partitions when $t = 7$, from most common to least common.

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$k = 3$	1	2	4	5	7	3	6
$k = 4$	1	2	3	5	6	7	4
$k = 5$	1	2	3	4	6	7	5

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$k = 5$	1	2	3	4	6	7	5
$k = 6, 10, 13, 20$	1	2	3	4	5	7	6
$k = 12$	1	2	3	4	6	5	7

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All other k	1	2	3	4	5	6	7

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- Let $t > 2$, then $\mathcal{O}(t) \geq \frac{\varphi(t)}{2}$, where $\varphi(t)$ is Euler's φ -function.

Conjectures/Examples

Looking at $t = 7$ in a new light

$k = 2$	1	3	5	7	2	4	6
$k = 3$	1	2	4	5	7	3	6
$k = 4$	1	2	3	5	6	7	4
$k = 5$	1	2	3	4	6	7	5
$k = 6, 10, 13, 20$	1	2	3	4	5	7	6
$k = 12$	1	2	3	4	6	5	7
All other k	1	2	3	4	5	6	7

Figure: Biases among congruence classes mod t for k -indivisible partitions when $t = 7$, from most common to least common.

Plot versus the Digamma Function for $k = 3, t = 7$

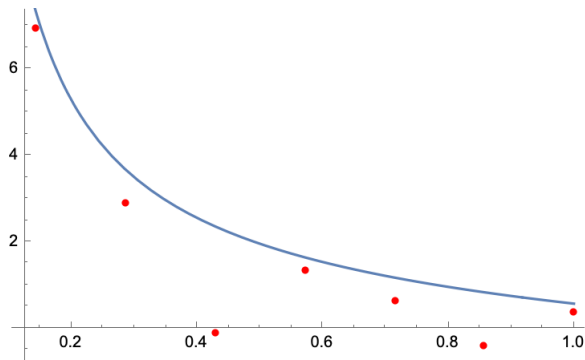


Figure: Second order term of each $1 \leq r \leq 7$ when $k = 3, t = 7$

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Plot versus the Digamma Function for $k = 10, t = 7$

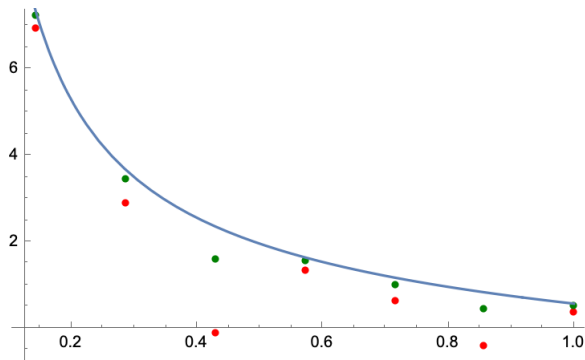


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Plot versus the Digamma Function for $k = 17, t = 7$

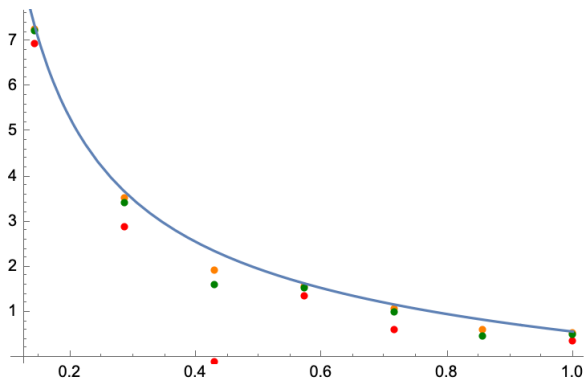


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A Conjecture concerning $\mathcal{O}(t)$

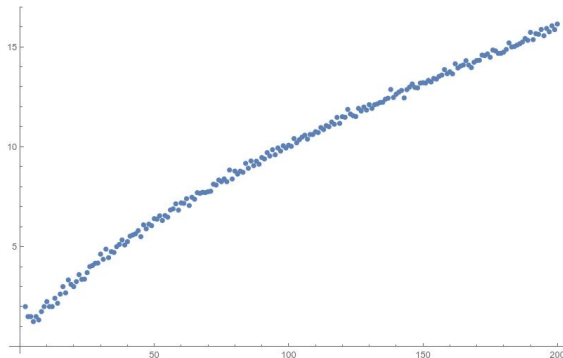


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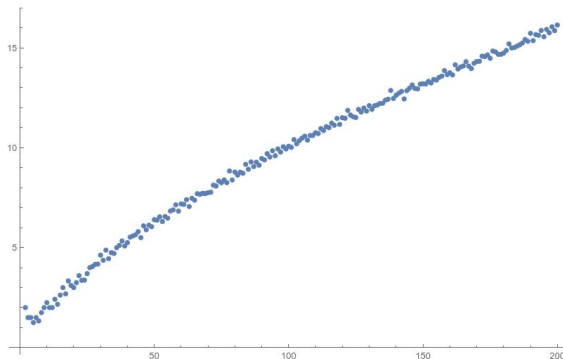


Figure: $\frac{\mathcal{O}(t)}{\varphi(t)}$ vs. t .

Conjecture (J.-O.)

$\frac{\mathcal{O}(t)}{\varphi(t)}$ grows sublinearly as well as superlogarithmically

There are no Ties

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- Concerns the linear independence of $\{\psi(a/t) \mid \gcd(a, t) = 1\}$ over number fields.
- Shows there exists a t_0 such that if $\gcd(t, t_0) = 1$, and $\gcd(r, t) = \gcd(s, t) = 1$, then the conjecture holds.

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Example

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$$k = 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, \dots, 42, 43, 54, 65.$$

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- Combinatorial heuristics aren't enough to see orderings*

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Idea

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$$F(q) = \sum_{n=0}^{\infty} a(n)q^n.$$

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How do we estimate the integral?

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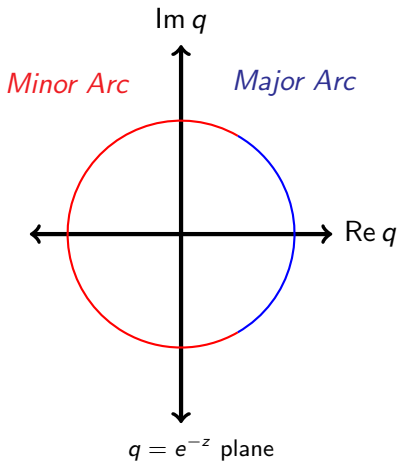
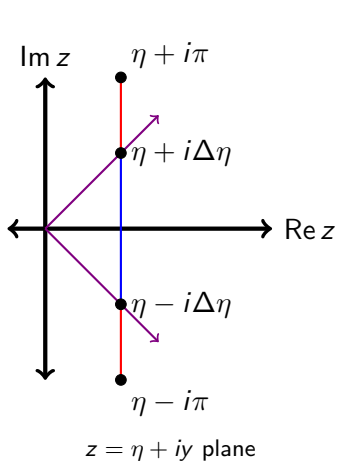
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- The main contribution of the integral is the **major arc**, and the error term is the **minor arc**.

Path of Integration



The Generating Function for k -regular/ k -indivisible Partitions

Proposition

The generating function for k -regular/ k -indivisible partitions is

$$\xi_k(q) := \prod_{m=1}^{\infty} \frac{1 - q^{mk}}{1 - q^m} = \frac{\mathcal{P}(q)}{\mathcal{P}(q^k)}.$$

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Choice of term in the product $\leftrightarrow$ How many parts have size m . $\square$

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How do we get generating functions for $D_k(r, t; n)$ and $D_k^\times(r, t; n)$?

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Generating functions for $D_k(r, t; n)$, $D_k^\times(r, t; n)$

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For $q = e^{-z}$ and $\varepsilon := \exp\left(-\frac{4\pi^2}{kz}\right)$, we have that

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Proposition (Euler's Pentagonal Number Theorem)

$$\mathcal{P}(q)^{-1} = 1 + \sum_{m \geq 1} (-1)^m \left(q^{m(3m+1)/2} + q^{m(3m-1)/2} \right)$$

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- Zagier gives an asymptotic version estimating $\sum_{m \geq 1} f(mz)$ under mild conditions on f .

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as $x \rightarrow 0^+$.

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If $f(x) \sim \sum_{n \geq 0} c_n x^n$ and its derivatives have rapid decay at ∞ , then

$$\sum_{m \geq 1} f(mx) \sim \frac{1}{x} \int_0^\infty f(u) du + \sum_{n=0}^\infty c_n \frac{B_{n+1}}{n+1} (-x)^n$$

as $x \rightarrow 0^+$.

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- Apply variant of Euler-Maclaurin, **without a pole so no ψ** . □

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- Apply Euler-Maclaurin twice, with a pole, giving ψ .



Minor Arc Estimates for $L_k(r, t; q), L_k^\times(r, t; q)$

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- Apply two variants of Wright's Circle Method.



Understanding $\psi_{k,t}$

$$\psi_{k,t}(r) = -\psi\left(\frac{r}{t}\right) + \frac{1}{k}\psi\left(\frac{\bar{r}}{t}\right)$$

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For $0 < a < 1$, we have

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Estimating $\psi(a) - \psi(b)$ for $0 < b < a < 1$

Lemma (J.-O.)

For $0 < b < a < 1$, we have

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- Rewrite as $\psi\left(\frac{s}{t}\right) - \psi\left(\frac{r}{t}\right) > \frac{1}{k} \left(\psi\left(\frac{\bar{s}}{t}\right) - \psi\left(\frac{\bar{r}}{t}\right) \right)$.

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- Apply the previous two lemmas and rearrange.



Summary of Methods

Idea

- Want: Asymptotics for parts lying in residue classes.
- Use the **circle method**. How do we estimate?
 - Modular Transformation Laws.
 - Euler-Maclaurin Summation when you're not so lucky.
- $\psi_{k,t}$ is **intricate**. How do we understand it?
 - Approximate with a square sum.
 - Make the weighting factor $\frac{1}{k}$ so small it washes everything out.
 - Fix $k \bmod t$ (i.e., $k = mt - 1$).

Summary of Results

Theorem (J.-O.)

Asymptotics for parts in k -regular/ k -indivisible partitions of n which are $r \bmod t$ (notationally, $D_k(r, t; n)$, $D_k^\times(r, t; n)$).

Corollary (J.-O.)

- *Bias* towards lower congruence classes for $D_k(r, t; n)$.
- *Intricate Bias* for $D_k^\times(r, t; n)$.

Theorem (J.-O.)

Basic properties of the biases in $D_k^\times(r, t; n)$

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- The authors would like to thank Ken Ono, the director, as well as their graduate student mentor William Craig.
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