Complexity of the Zeckendorf Graph Game

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Roadmap

- Explanation of Zeckendorf's Theorem and the Zeckendorf Game
- Generalization to the Zeckendorf Graph Game
- PSPACE-completeness
- Extensions and Further Questions

The Fibonacci Numbers

Definition

The Fibonacci Numbers are a recursively defined sequence so that $F_0 = 1, F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Theorem (Zeckendorf, 1972)

Every integer $m > 0$ may be written uniquely as a sum of non-adjacent Fibonacci numbers.

Example

$$2021 = 1597 + 377 + 34 + 13 = F_{15} + F_{12} + F_7 + F_5$$

Definition of the Zeckendorf Game

Definition (Baird-Smith, Epstein, Flint, Miller)

Consider bins labeled by each Fibonacci number $F_0 = 1, F_1 = 2, \dots$. Begin with n summands in the bin labeled F_0. The combine move is:

$$F_i^{(1)} \wedge F_{i+1}^{(1)} \to F_{i+2}^{(1)}$$
$$(1,1,0) \to (0,0,1).$$

We also have a splitting move:

$$F_i^{(2)} \to F_{i-2}^{(1)} \wedge F_{i+1}^{(1)}$$
$$(0,0,2,0) \to (1,0,0,1).$$

And boundary moves $F_0^{(2)} \to F_1^{(1)}$, $F_1^{(2)} \to F_0^{(1)} \wedge F_2^{(1)}$. Players take turns making moves, and the last player to move wins.

Playing the Zeckendorf Game

1	2	3	5	8
9	0	0	0	0

Playing the Zeckendorf Game

MERGE ONES

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0

Playing the Zeckendorf Game

COMBINE

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0
6	0	1	0	0

Playing the Zeckendorf Game

MERGE ONES

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0
6	0	1	0	0
4	1	1	0	0

Playing the Zeckendorf Game

COMBINE

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0
6	0	1	0	0
4	1	1	0	0
3	0	2	0	0

Playing the Zeckendorf Game

SPLIT

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0
6	0	1	0	0
4	1	1	0	0
3	0	2	0	0
4	0	0	1	0

Playing the Zeckendorf Game

MERGE ONES

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0
6	0	1	0	0
4	1	1	0	0
3	0	2	0	0
4	0	0	1	0
2	1	0	1	0

Playing the Zeckendorf Game

COMBINE

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0
6	0	1	0	0
4	1	1	0	0
3	0	2	0	0
4	0	0	1	0
2	1	0	1	0
1	0	1	1	0

Playing the Zeckendorf Game

COMBINE

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0
6	0	1	0	0
4	1	1	0	0
3	0	2	0	0
4	0	0	1	0
2	1	0	1	0
1	0	1	1	0
1	0	0	0	1

Playing the Zeckendorf Game

1	2	3	5	8
9	0	0	0	0
7	1	0	0	0
6	0	1	0	0
4	1	1	0	0
3	0	2	0	0
4	0	0	1	0
2	1	0	1	0
1	0	1	1	0
1	0	0	0	1

Player One Wins! Notice that $9 = F_4 + F_0 = 8 + 1$, the Zeckendorf Decomposition.

Results about the Zeckendorf Game

- All games terminate in $O(n \log n)$ moves. [Bai+20]
- The shortest game (greedy algorithm) takes $n - Z(n)$ moves where $Z(n)$ is the number of summands in the Zeckendorf Decomposition. [Bai+20]
- There is a non-constructive winning strategy for the second player for all $n \geq 10$ using a strategy stealing argument. [Bai+20]
- How difficult is the Zeckendorf Game?

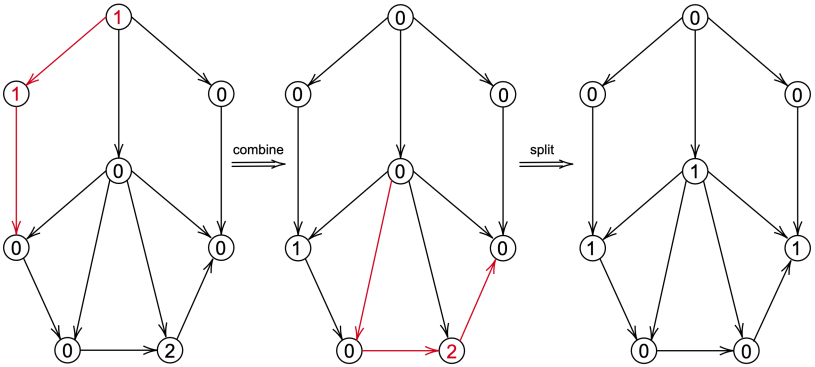
The Zeckendorf Graph Game

We may reenvision the Zeckendorf Game as being played on a **directed path graph** rather than on a tuple:

0 | 0 | 1 | 0 | 7 | 1 | 0 | 0 | 0

We can also think of playing the game on different Directed Graphs.

Playing the ZGG



PSPACE

Definition

A problem is in **PSPACE** if it can be solved by a machine which has polynomial-size memory.

Definition

A problem is **PSPACE-hard** if any instance of any other game in PSPACE can be reduced to to an instance of the game in question, and **PSPACE-complete** if it is both PSPACE-hard and in PSPACE.

A Wish List for Graphs

Our result is one about PSPACE-completness for the ZGG played on a wide family of digraphs. Before we state the family, let's state what we want to be true about this family.

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- Termination: Acyclic
- Polynomial Termination: Leveled

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- Termination: Acyclic
- Polynomial Termination: Leveled
- Legible Gameboards: Planar

A Wish List for Graphs

Our result is one about PSPACE-completness for the ZGG played on a wide family of digraphs. Before we state the family, let's state what we want to be true about this family.

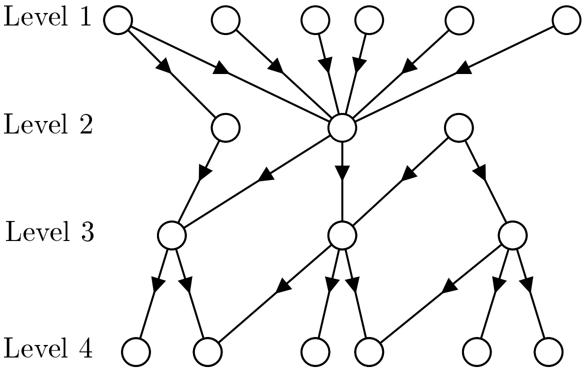
- Termination: Acyclic
- Polynomial Termination: Leveled
- Legible Gameboards: Planar
- PSPACE-hardness: Successfully Reducible

Leveled Digraphs

Definition

A directed graph is leveled if every node has a level and each edge points down exactly one level.

Example



The Projection Lemma

Lemma (Small, 2021)

The Zeckendorf Game terminates in polynomial time on leveled digraphs.

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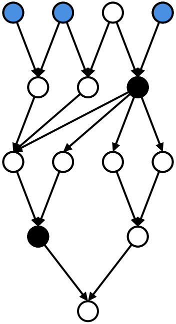

F_0	6
F_1	1
F_2	0
F_3	1
F_4	0

Key: White 0, Black 1, Blue 2.

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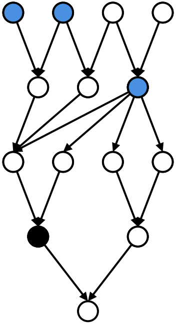

F_0	4
F_1	2
F_2	0
F_3	1
F_4	0

Key: White 0, Black 1, Blue 2.

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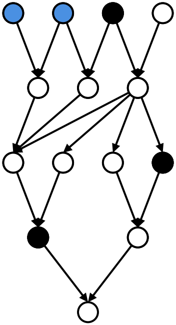

F_0	5
F_1	0
F_2	1
F_3	1
F_4	0

Key: White 0, Black 1, Blue 2.

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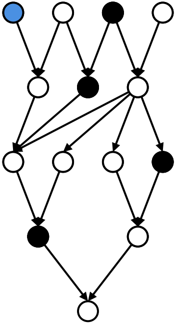

F_0	3
F_1	1
F_2	1
F_3	1
F_4	0

Key: White 0, Black 1, Blue 2.

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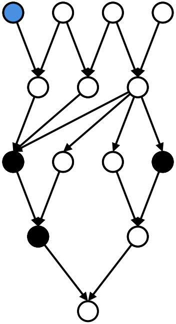

F_0	2
F_1	0
F_2	2
F_3	1
F_4	0

Key: White 0, Black 1, Blue 2.

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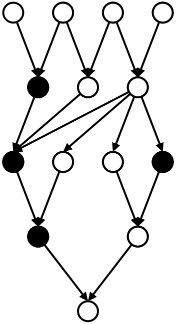

F_0	0
F_1	1
F_2	2
F_3	1
F_4	0

Key: White 0, Black 1, Blue 2.

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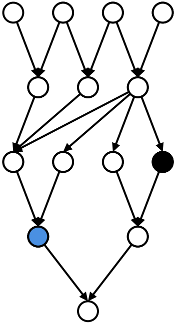
The Zeckendorf Game terminates in polynomial time on leveled digraphs.



F_0	0
F_1	0
F_2	1
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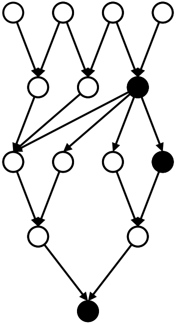

F_0	0
F_1	1
F_2	1
F_3	0
F_4	1

Key: White 0, Black 1, Blue 2.

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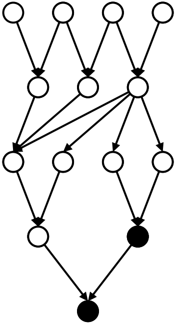
The Zeckendorf Game terminates in polynomial time on leveled digraphs.



F_0	0
F_1	0
F_2	0
F_3	1
F_4	1

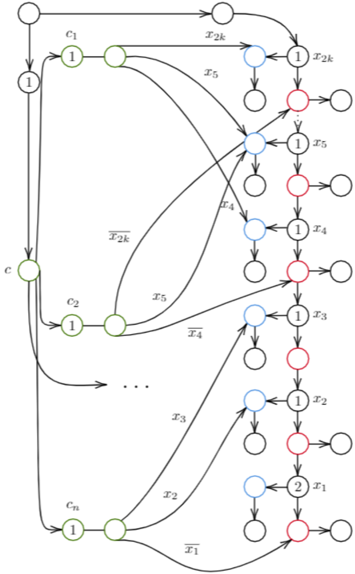
Key: White 0, Black 1, Blue 2.

PSPACE-Completeness

Theorem (SMALL, 2021)

The Zeckendorf Graph Game (ZGG) on planar leveled digraphs is PSPACE-Complete.

Reduction to PSPACE-completeness



The PLRS-ZGG

Definition

We say a sequence (a_n) is a Positive Linear Recurrence Sequence (PLRS) if $a_0, \ldots, a_{k-1}$ are specified, and for $n \geq k$ it is given by a linear recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$$

with $c_i > 0$. We say it is non-increasing if $c_1 \geq c_2 \geq \cdots \geq c_k > 0$.

Example

Let $a_0 = a_1 = a_2 = 1$, and for $n \geq 3$, $a_n := 3a_{n-1} + 2a_{n-2} + a_{n-3}$.
1, 1, 1, 6, 21, 76, 276, ...

PLRS-ZGG

The Zeckendorf Game and Zeckendorf Graph Game are based on the Fibonacci recurrence relation. We may define analogous games based on non-increasing PLRS with generalized move sets.

Theorem (SMALL, 2021)

These generalized games, played on the doubly-infinite path graph, terminate in $O(n^2)$ moves where n is the number of starting chips.

Theorem (SMALL, 2021)

Each non-increasing PLRS game on planar leveled digraphs is PSPACE-Complete.

Further Questions

- Given that the ZGG is very algorithmically hard, can we show that the Zeckendorf Game is, too?
- How far can the results on the ZGG be pushed to PLRS-ZGGs or even other classes of recurrences all-together?

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Michael Sipser. *Introduction to the Theory of Computation*. 1st. International Thomson Publishing, 1996. ISBN: 053494728X.

The Formula Game

Definition

In **The Formula Game**, two players take turns choosing truth values for a finite set of quantifiers $x_1, x_2, x_3, \ldots, x_m$, given a statement of the form

$$(y_{11} \vee y_{12} \vee y_{13}) \wedge (y_{21} \vee y_{22} \vee y_{23}) \wedge \cdots \wedge (y_{n1} \vee y_{n2} \vee y_{n3}).$$

Each y_{ij} represents either $x_k, \bar{x}_k$ for some $1 \leq k \leq m$. Player 1 wins if at the end, the statement is True.

Theorem ([Sip96], Theorem 8.11)

The Formula Game is PSPACE-complete.

Reduction to PSPACE-completeness