Reducibility of Sets in Generalized Settings

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Set Addition

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Given two sets $A, B \subseteq \mathbb{Z}$, *we say that* $A + B = \{a + b \mid a \in A, b \in B\}$.

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A set $S \subseteq \mathbb{Z}$ is ***reducible*** *if $S = A + B$ for two sets A, B such that $|A|, |B| \geq 2$. Otherwise, S is* ***irreducible****.*

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Example

The set $\{0, 1, 2\} = \{0, 1\} + \{0, 1\}$ is reducible. In contrast, $\{0, 1, 3\}$ is irreducible.

Higher Dimensional Irreducibility

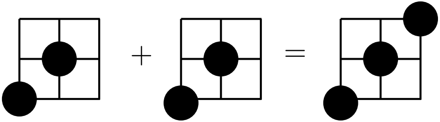
Definition

Let $S \subset \mathbb{Z}^d$. S is reducible iff $S = A + B$ for $A, B \subset \mathbb{Z}^d$ with $|A|, |B| \geq 2$.

Let $S = \{(0,0), (1,1), (2,2)\}$. Then,

$$S = \{(0,0), (1,1)\} + \{(0,0), (1,1)\}$$

so S is reducible.



What is the largest size of an irreducible subset of $[n]^d$?

Definition

Let $[n]^d = \underbrace{\{0,1,\ldots,n\} \times \{0,1,\ldots,n\} \cdots \{0,1,\ldots,n\}}_{d\ copies}.$

Lemma (BDGGPV 2021)

Fix $S \subset \mathbb{Z}^d$ *such that* $|S| \geq 3$ *and* $0 \in S$. *Let* $A = \{0, r\}$ *for* $r \in \mathbb{Z}^d \setminus \{0\}$. $S = A + B$ *for some* $B \subset S$ *iff for all* $s \in S$, $s - r \in S$ *or* $s + r \in S$.

- We consider the minimum size of the complement of an irreducible subset of $[n]^d$.
- The above lemma shows that if $|[n^d] \setminus S|$ is too small then $S = A + B$ for some $A = \{0, r\}$.

The Largest Irreducible Subset of $[n]^d$

Theorem (SMALL 2021)

Let $S \subseteq [n]^d$. Let $k = \left|[n]^d \setminus S\right|$. Then, S is reducible if

$$\frac{k}{d} \ln 2 + H_{\lceil \frac{k}{d} \rceil} + H_{\lceil \binom{k}{2}/d \rceil} < H_{n-1}$$

where H_n is the nth Harmonic number ($H_n \approx \log(n)$).

- This tells us that the size of the complements of irreducible subsets of $[n]^d$ are $\Omega(d \log n)$.

How do you easily show a set is irreducible?

Proposition (Local Irreducibility)

Suppose $S \subset \mathbb{Z}_{\geq 0}^d$ with $0 \in S$ satisfies the following.

1. $(M_S + M_S) \cap S = \varnothing$*, for* M_S *the set of minimal norm elements.*
2. *For each* $s \in S \setminus M_S$ *there is some* $t \in S$ *with* $|t| < |s|$ *and* $s + t \notin S$.

Then, S *is irreducible.*

- Local irreducibility is defined to be easily computer verifiable.
- Irreducibility follows by an iterative contradiction argument.
- In 1-dimension, the minimum size of the complement is $\Theta(\log(n))$.

Local Irreducibility Example

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Suppose $S = \{0, 1, 3\} = A + B$.

- WLOG, $0 \in A, B$ and $A, B \subset S$ by shifting the sets.
- 1 must be in A or B. Suppose $1 \in B$.
- Then, since $1 + 1 \notin S$, $1 \notin A$.
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- So, $A = \{0\}$ and S is irreducible.

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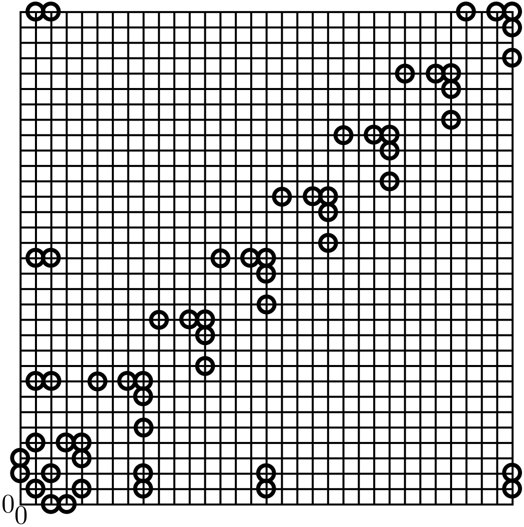
Constructive upper bound of $O(\sqrt{n})$



$\mathbb{Z}^2_{\geq 0}$ with $\circ$ denoting belonging to the set complement.

Lunar Numbers

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- To perform "lunar multiplication" on single digits, take the smaller digit:

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Base 2 numbers correspond to subsets of the natural numbers, with lunar multiplication corresponding to set addition (Gal Gross, 2019).

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This corresponds to the set $\{0, 1, 2, 3, 5\}$, which is $\{0, 1, 3\} + \{0, 2\}$.

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This corresponds to the set $\{0,1,2,3,5\}$, which is $\{0,1,3\}+\{0,2\}$.

Lunar multiplication in arbitrary base can thus be thought of as a generalization of set addition.

Lunar Primes

- We say that a lunar number n is *irreducible* if all of its possible (lunar) factorizations include a number containing only one non-zero digit.
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- If n is base b lunar number, we say that n is *prime* if its only possible factorization is $(b-1) \otimes_b n$.
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- If n is base b lunar number, we say that n is *prime* if its only possible factorization is $(b-1) \otimes_b n$.
  - $(b-1)$ is the unit for lunar multiplication.
- A base b lunar number is prime if and only if it is irreducible, contains the digit $b-1$, and has non-zero final digit.
  - We call lunar numbers which contain the digit $b-1$ and have a non-zero final digit *candidates for primality*.

How many Lunar Primes are there?

Theorem (SMALL 2021)

The proportion of base b, length k candidates for primality which are irreducible (and thus prime) tends to 1 as $k \to \infty$.

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- The base 2 case was proven in 2014 by Yaroslav Shitov.
- The $b = 2$ case implies that the proportion of subsets of $\{0, 1, \ldots, k\}$ containing 0 and k which are irreducible tends to 1 as $k \to \infty$.

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- We call these (possibly infinitely long) strings of digits *lunar digit strings*.
- In the base 2 case, lunar digit strings correspond to sets with a (possibly infinite) number of elements.

Example

The set of even natural numbers corresponds to the string 101010101....

Asymptotic Irreducibility

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Definition

A lunar digit string n is ***asymptotically irreducible*** *if changing any finite number of the digits of n results in an irreducible lunar digit string.*

These definitions correspond in the base 2 case.

Wirsing's Result and our Generalization

Theorem (Eduard Wirsing, 1953)

The proportion of subsets of $\mathbb{N}$ *which are asymptotically irreducible is 1.*

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Wirsing's Result and our Generalization

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Theorem (SMALL 2021)

For $b \geq 2$, the proportion of base b lunar digit strings which are asymptotically irreducible is 1.

Acknowledgments

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