Torsion in the Braid Monodromy of Elliptic Fibrations

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Abstract. Given an elliptic fibration $\pi : M \to S^2$ with singular locus $\Delta \subseteq S^2$, let $\mathrm{Br}(\pi) < \mathrm{Mod}(S^2, \Delta)$ be the subgroup of the spherical braid group consisting of those braids that lift to a fiber-preserving diffeomorphism of M. We classify the order $n = |\Delta|$ elements of $\mathrm{Br}(\pi)$ up to conjugacy in $\mathrm{Br}(\pi)$. To do so, we relate these conjugacy classes to special points on the SL_2-character variety for (S^2, Δ) that correspond naturally to the exceptional elliptic curves $\mathbb{C}/\mathbb{Z}[\omega]$ and $\mathbb{C}/\mathbb{Z}[i]$ with their associated norms on $\mathbb{Z}[\omega]$ and $\mathbb{Z}[i]$. We also show that there are no elements of order $n-1$ or $n-2$ in $\mathrm{Br}(\pi)$, as there are in $\mathrm{Mod}(S^2, \Delta)$.

1. Introduction

A *smooth elliptic fibration* $\pi : M \to S^2$ of an oriented 4-manifold M is a smooth map $M \to S^2$ with finitely many critical values $\Delta \subseteq S^2$ such that the fibers $\pi^{-1}(b)$ are smooth elliptic curves for $b \in S^2 \setminus \Delta$ and nodal elliptic curves for $b \in \Delta$. Note that this equips π with a section[1] $s : S^2 \to M$ picking out the basepoint of each elliptic curve. Examples include the rational elliptic surface, given by blowing up $\mathbb{P}^2$ along the intersection points of two generic cubics, and the elliptically fibered K3 surfaces. Given an elliptic fibration $\pi : M \to S^2$, let

$$\mathrm{Diff}^+(\pi) := \{F \in \mathrm{Diff}^+(M) \mid F \text{ takes fibers of } \pi \text{ to fibers of } \pi\}.$$

The *smooth automorphism group* of π as defined by Farb–Looijenga (see [FL24]) is

$$\mathrm{Mod}(\pi) := \pi_0(\mathrm{Diff}^+(\pi)),$$

Tracking the location of the singular fibers under an element of $\mathrm{Diff}^+(\pi)$ gives a *braid monodromy representation*:

$$\rho : \mathrm{Mod}(\pi) \to \mathrm{Mod}(S^2, \Delta),$$

where $\Delta \subseteq S^2$ is the singular locus of π and $\mathrm{Mod}(S^2, \Delta) := \pi_0(\mathrm{Diff}^+(S^2, \Delta))$, with $\mathrm{Diff}^+(S^2, \Delta)$ consisting of diffeomorphisms of S^2 mapping Δ to itself setwise. We study the image

$$\mathrm{Br}(\pi) := \mathrm{im}(\rho) < \mathrm{Mod}(S^2, \Delta), \tag{1.1}$$

of the braid monodromy ρ. Thus $\mathrm{Br}(\pi)$ associates to each elliptic fibration π a subgroup of the spherical braid group[2] $\mathrm{Mod}(S^2, \Delta)$. The subgroup $\mathrm{Br}(\pi) < \mathrm{Mod}(S^2, \Delta)$ consists of those braids that lift to a fiber-preserving diffeomorphism of M. For convenience, let $n := |\Delta|$ be the number of singular fibers. In this paper, we classify the conjugacy classes of elements of order n in $\mathrm{Br}(\pi)$ and show that there are no elements of order $n-1$ or $n-2$, as there are in $\mathrm{Mod}(S^2, \Delta)$.

Classifying torsion elements in $\mathrm{Br}(\pi)$ up to $\mathrm{Br}(\pi)$ conjugacy is intimately tied to a theorem of Murasugi which classifies the finite order elements of $\mathrm{Mod}(S^2, \Delta)$ up to conjugacy in $\mathrm{Mod}(S^2, \Delta)$ [Mur82]. Murasugi shows that every finite order element of $\mathrm{Mod}(S^2, \Delta)$ is conjugate to a power of one of the following:

(1) The order n rotation $\sigma_1 \cdots \sigma_{n-1}$.
(2) The order $n-1$ rotation, achieved by placing one marked point at the north pole and the remainder along the equator.

[1]For us, we consider only elliptic fibrations $\pi : M \to S^2$ with nodal singularities without multiples. Further, the role of a section is auxiliary to our results, and so one may instead consider genus one fibrations.

[2]There are two distinct notions of the spherical braid group, one is $B_n(S^2) := \pi_1(\mathrm{Conf}_n(S^2))$, the fundamental group of n-point configurations in S^2, the other is $\mathrm{Mod}(S^2, \{p_1, \ldots, p_n\})$, a marked mapping class group. These differ by an exact sequence $1 \to \mathbb{Z}/2\mathbb{Z} \to B_n(S^2) \to \mathrm{Mod}(S^2, \{p_1, \ldots, p_n\}) \to 1$ [FM12, §9.1.4]. In this paper we restrict ourselves to the latter notion.

(3) The order $n-2$ rotation, achieved by placing one marked point at the north pole, another at the south pole, and the remainder along the equator.

A similar classification in $\mathrm{Br}(\pi)$ does not follow formally, however. First, each conjugate bgb^{-1} of a finite order element $g \in \mathrm{Mod}(S^2, \Delta)$ for $b \in \mathrm{Mod}(S^2, \Delta)$ may or may not appear in the subgroup $\mathrm{Br}(\pi) < \mathrm{Mod}(S^2, \Delta_\pi)$. The conditions for whether such an element bgb^{-1} lies in $\mathrm{Br}(\pi)$ are given in Section 2 and further explained in Section 3. Second, a single conjugacy class in $\mathrm{Mod}(S^2, \Delta_\pi)$ may split into multiple distinct conjugacy classes in $\mathrm{Br}(\pi)$. The relationship between these classifications is further complicated by the fact that the index $[\mathrm{Mod}(S^2, \Delta_\pi) : \mathrm{Br}(\pi)] = \infty$, which we show in [Jac25]. Nonetheless, we prove the following.

Theorem 1.1 (**Torsion in** $\mathrm{Br}(\pi)$ **of order** $n, n-1$, **and** $n-2$)**.** *Let $\pi : M \to S^2$ be an elliptic fibration with n nodal fibers. The following hold:*

(a) There are exactly two distinct conjugacy classes in $\mathrm{Br}(\pi)$ of elements with order n. Furthermore, up to replacing $\mathrm{Br}(\pi)$ by a conjugate subgroup in $\mathrm{Mod}(S^2, \Delta)$, these two classes are represented by $r = \sigma_1 \cdots \sigma_{n-1} \in \mathrm{Br}(\pi)$ and $r^{-1} = T_n r T_n^{-1} \in \mathrm{Br}(\pi)$, where T_n is the Garside half-twist defined by

$$T_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)\sigma_1,$$

and $\sigma_1, \ldots, \sigma_{n-1}$ are standard half-twist generators of $\mathrm{Mod}(S^2, \Delta_\pi)$.

(b) There are no elements of order $n-1$ in $\mathrm{Br}(\pi)$,

(c) There are no elements of order $n-2$ in $\mathrm{Br}(\pi)$.

Note that, despite part (c) of Theorem 1.1 ruling out elements of order $n-2$, there do exist elements whose orders divide $n-2$ and do not divide $n-1$ or n itself. For an example, see (7.10) in Section 7. The proof of Theorem 1.1 relies on an understanding of the Hurwitz action of $\mathrm{Mod}(S^2, \Delta)$ on the SL_2-character variety for (S^2, Δ), which we now recall.

$\mathbf{Br}(\boldsymbol{\pi})$ and $\mathbf{SL_2}$-character varieties. We prove Theorem 1.1 by showing that each part is equivalent to a corresponding statement classifying simultaneous conjugacy classes of n-tuples of matrices in $\mathrm{SL}_2 \mathbb{Z}$ which satisfy particular conditions. We first detail the relationship between $\mathrm{Br}(\pi)$ and the Hurwitz action on the SL_2-character variety.

Let $\pi : M \to S^2$ be an elliptic fibration. A choice of basepoint $b \in S^2 \setminus \Delta$ gives an associated monodromy representation

$$\phi_\pi : \pi_1(S^2 \setminus \Delta, b) \to \mathrm{Mod}(\pi^{-1}(b)) \cong \mathrm{Mod}(\Sigma_1) \cong \mathrm{SL}_2 \mathbb{Z},$$

by identifying $\pi^{-1}(b)$ with the standard torus Σ_1. Note that changing the basepoint b or the identification of $\pi^{-1}(b)$ with Σ_1 changes ϕ_π by conjugation, and hence associated to π is a point $[\phi_\pi]$ in the *integral character variety*[3]

$$\mathfrak{X}_{\mathbb{Z}}(S^2, \Delta) := \mathrm{Hom}(\pi_1(S^2 \setminus \Delta), \mathrm{SL}_2 \mathbb{Z}) / \mathrm{SL}_2 \mathbb{Z},$$

where $\mathrm{SL}_2 \mathbb{Z}$ acts by simultaneous conjugation. The spherical braid group $\mathrm{Mod}(S^2, \Delta)$ naturally acts on $\mathfrak{X}_{\mathbb{Z}}(S^2, \Delta)$ via precomposition by (outer) automorphisms on π_1. The induced action of $\mathrm{Mod}(S^2, \Delta)$ on $\mathfrak{X}_{\mathbb{Z}}(S^2, \Delta)$ is referred to as the *Hurwitz action*. A theorem of Moishezon (see Theorem 2.1 below) implies that the liftable braids are precisely the stabilizers of $[\phi_\pi] \in \mathfrak{X}_{\mathbb{Z}}(S^2, \Delta)$; that is

$$\mathrm{Br}(\pi) = \mathrm{Stab}_{\mathrm{Mod}(S^2, \Delta)}[\phi_\pi]. \tag{1.2}$$

We discuss this further in Section 2. For now, recall that choosing generators $\gamma_1, \ldots, \gamma_n$ for $\pi_1(S^2, \Delta)$ surrounding each puncture counterclockwise realizes the representation ϕ_π as a *factorization* in the mapping

[3] $\mathfrak{X}_{\mathbb{Z}}(S^2, \Delta)$ itself is not a variety. However, the GIT quotient $\mathrm{Hom}(\pi_1(S^2 \setminus \Delta), \mathrm{SL}_2 \mathbb{C}) /\!\!/ \mathrm{SL}_2 \mathbb{C})$ does form a variety, commonly referred to as the character variety. We refer to $\mathfrak{X}_{\mathbb{Z}}(S^2, \Delta)$ as the integral character variety for convenience and for context.

class group

$$\phi_\pi(\gamma_1)\cdots\phi_\pi(\gamma_n) = \mathrm{Id}\,.$$

By Picard–Lefschetz theory, each $\phi_\pi(\gamma_i)$ is a Dehn twist about a simple closed curve in $\pi^{-1}(b) \cong \Sigma_1$ [End21]. We refer to the ordered tuple $(\phi_\pi(\gamma_i)) \in (\mathrm{SL}_2\,\mathbb{Z})^n$ as the *monodromy factorization* associated to the fibration π.

We now return to Theorem 1.1. By Murasugi's theorem, an order n element of $\mathrm{Br}(\pi)$ is given as a conjugate $brb^{-1} \in \mathrm{Br}(\pi)$ with $b \in \mathrm{Mod}(S^2, \Delta)$, where r is the order n rotation. Applying (1.2) shows that $rb^{-1} \cdot [\phi_\pi] = b^{-1} \cdot [\phi_\pi]$, i.e., that these representations are conjugate. We call $b^{-1} \cdot [\phi_\pi]$ a *rotation invariant monodromy representation*. By choosing generators $\gamma_i \in \pi_1(S^2 \setminus \Delta)$ appropriately, we obtain a notion of a *rotation invariant monodromy factorization*. In Section 3, we show that part (a) of Theorem 1.1 is equivalent to the following theorem classifying rotation invariant monodromy factorizations up to conjugacy.

Theorem 1.2 (**Rotation Invariant Monodromy Factorizations**). *Call an ordered n-tuple $(X_1, \dots, X_n)$ in $(\mathrm{SL}_2\,\mathbb{Z})^n$ a rotation invariant monodromy factorization provided that for some $C \in \mathrm{SL}_2\,\mathbb{Z}$:*

(1) $X_1 = T_v$, where $T_v(x) = x + (x, v)v$ is a symplectic transvection in a primitive vector $v \in \mathbb{Z}^2$. Equivalently, X_1 is conjugate to $\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)$.

(2) $X_{i+1} = CX_iC^{-1}$ for $i \in \mathbb{Z}/n\mathbb{Z}$

(3) $\prod_{i=1}^n X_i = \mathrm{Id}$

Then there are exactly two rotation invariant monodromy factorizations $(X_1, \dots, X_n)$ in $(\mathrm{SL}_2\,\mathbb{Z})^n$ up to simultaneous conjugation. Explicitly, for any such sequence $(X_1, \dots, X_n)$, n is divisible by 12 and there is some matrix $D \in \mathrm{SL}_2\,\mathbb{Z}$ so that exactly one of the following holds

(a) $$(DX_1D^{-1}, \dots, DX_nD^{-1}) = \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \dots\right) \text{ or}$$

(b) $$(DX_1D^{-1}, \dots, DX_nD^{-1}) = \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \dots\right)$$

The tuples falling into case (a) are said to have period 2 and the tuples falling into case (b) are said to have period 3.

Experts may notice the similarity between Theorem 1.2 and a theorem of Moishezon and Seiler, which states that any two tuples $(X_1, \dots, X_n)$ satisfying $\prod_{i=1}^n X_i = \mathrm{Id}$ and with $X_i \in \mathrm{SL}_2\,\mathbb{Z}$ conjugate to $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$ lie in the same Hurwitz orbit (up to simultaneous conjugation) [Moi06, Lemma 8]. However, in contrast, Theorem 1.2 states that there are only two rotation invariant monodromy factorizations up to simultaneous conjugacy within the Hurwitz orbit.

We delay the precise statements of the analogous classifications corresponding to parts (b) and (c) of Theorem 1.1 to Section 6, given there as as Theorem 6.1 and Proposition 6.2.

Geometric interpretation of r and $r^{-1} = T_n r T_n^{-1}$. Let $\pi : M \to S^2$ be an elliptic fibration with n singular fibers. Observing that $(\mathrm{SL}_2\,\mathbb{Z})^{ab} = \mathbb{Z}/12\mathbb{Z}$ and that all Dehn twists are conjugate in $\mathrm{SL}_2\,\mathbb{Z}$, we find that $n = 12d$ for some $d \geq 1$. The two conjugacy classes r and $r^{-1} = T_n r T_n^{-1}$ in $\mathrm{Br}(\pi)$ are associated to two distinct monodromy factorizations for π, given by different choices of simple closed curves γ_i (resp. γ_i') enclosing each puncture. In Theorem 1.2, $\rho_\pi(\gamma_i) \in \mathrm{SL}_2\,\mathbb{Z}$ (resp. $\rho_\pi(\gamma_i')$) appear as the matrices given in (a) and (b). Interpreted geometrically, these factorizations are given by

$$(T_\alpha \cdot T_\beta)^{6d} = \mathrm{Id} \qquad\qquad (T_\alpha \cdot T_\delta \cdot T_\beta)^{4d} = \mathrm{Id}, \tag{1.3}$$

respectively, where the curves α, β, δ are the vertices of a triangle labeled counterclockwise on the Farey tesselation (see Figure 2 in Section 7 below) and $T_\alpha, T_\beta, T_\delta$ are the corresponding Dehn twists. Each of these monodromy factorizations are fixed up to conjugacy by r under the Hurwitz action, and the latter is obtained from $(T_\alpha \cdot T_\beta)^{6d}$ by applying the Garside half-twist T_n. We can associate the first monodromy factorization to the order 4 automorphism of the elliptic curve $E_i = \mathbb{C}/\mathbb{Z}\langle 1, i\rangle$, and the second monodromy factorization to the order 6 automorphism of the elliptic curve $E_\omega = \mathbb{C}/\mathbb{Z}\langle 1, \omega\rangle$ with $\omega = e^{2\pi i/3}$. The automorphisms of these elliptic curves naturally appear in the proof of Theorem 1.1. Unsurprisingly, the number rings $\mathbb{Z}[\omega]$ and $\mathbb{Z}[i]$ also appear and play a pivotal role.

Method of Proof for Theorem 1.2. To prove Theorem 1.2 we first prove that the conjugating matrix C has finite order. Thus $T_v T_{Cv} \cdots T_{C^{k-1}v}$ has finite order, where $C^k = -\operatorname{Id}$. The second step of the proof consists of analyzing and computing the trace polynomial

$$f(v) = \operatorname{tr}(T_v T_{Cv} \cdots T_{C^{k-1}v}).$$

We then show that the level sets of this polynomial are ellipses, at which point we can simply identify the integral points $v \in \mathbb{Z}^2$ where $|f(v)| \leq 2$. The level sets of the trace polynomials naturally correspond to integral points in $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ with specified norms. We display these here as Figure 1, and delay the explanation for the proof. The method of proof for Theorem 6.1 and Proposition 6.2 is similar and relies on analyzing the conjugating matrix C as well.

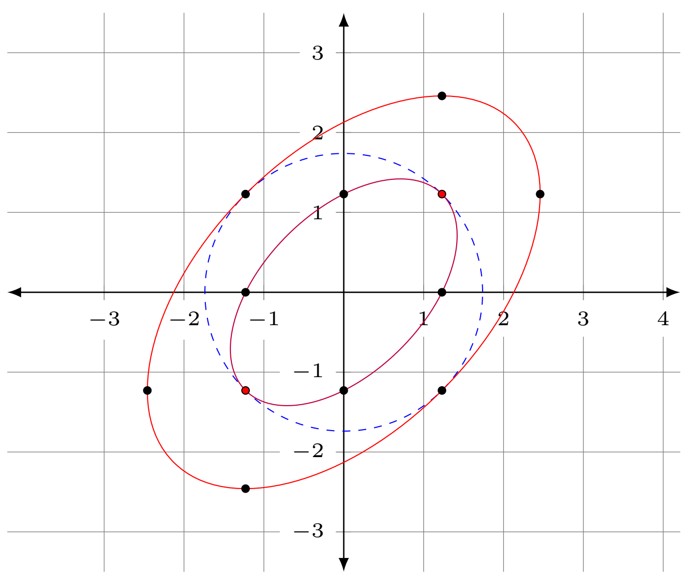

FIGURE 1. The three conics $N(p + q\omega) = 1, N(p + qi) = 2$, and $N(p + q\omega) = 3$, where N is the norm of the corresponding number ring.

Organization of the Paper. In Section 2 we explain Moishezon's theorem classifying elliptic fibrations, and review a maps version of the theorem which allows us to determine when a braid lifts to $\operatorname{Diff}^+(\pi)$ via the monodromy representation. Using this description, we prove the equivalence of Theorem 1.1 part (a) and Theorem 1.2 in Section 3. After doing so, we directly compute the aforementioned trace polynomial and prove Theorem 1.2 in Section 4. Following this, we give an algebraic motivation for the appearance of the norms on $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ in Section 5. In Section 6, we state and prove Theorem 6.1 and Proposition 6.2, which are equivalent to part (b) and part (c) of Theorem 1.1 respectively. Finally, we indicate in Section 7 the difficulties that orders smaller than $n, n-1$, and $n-2$ present for our proof.

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2. Monodromy and Lifting Mapping Classes

Let $\pi : M \to S^2$ be an elliptic fibration. We assume throughout that each singular fiber of π has one critical point i.e., that the singularities are stable. For convenience, let $\Delta_\pi \subseteq S^2$ denote the set of singular values of π; when the fibration π is unambiguous we will instead write $\Delta = \Delta_\pi$. As discussed in the introduction, the braids in $\mathrm{Mod}(S^2, \Delta)$ which lift to a fiber-preserving diffeomorphism of M are referred to as the *liftable braids*, and the subgroup consisting of liftable braids is denoted $\mathrm{Br}(\pi)$.

We now give a description of the liftable braids in terms of the Hurwitz action, essentially due to Moishezon. Let

$$\phi_\pi : \pi_1(S \setminus \Delta, b) \to \mathrm{Mod}(\pi^{-1}(b)) \cong \mathrm{Mod}(\Sigma_1) \cong \mathrm{SL}_2\,\mathbb{Z} \tag{2.4}$$

be the monodromy representation of π, choosing some identification $\pi^{-1}(b)$ with the standard torus Σ_1 once and for all. As discussed in the introduction, the action of $\mathrm{Mod}(S^2, \Delta)$ acts on the collection of conjugacy classes of all such representations via its outer action on π_1. The induced action is the familiar *Hurwitz action* from the theory of Lefschetz fibrations, and can be phrased in terms of the monodromy factorization. The orbit of $[\phi_\pi]$ under the Hurwitz action is referred to as the *Hurwitz orbit*. Equipped with this notation, we may state the theorem of Moishezon.

Theorem 2.1 (Moishezon [Moi06, Lemma 7a])**.** *Let $\pi : M \to S^2$ be an elliptic fibration, then* $\mathrm{Br}(\pi)$ *is the stabilizer of $[\phi_\pi] \in \mathfrak{X}_{\mathbb{Z}}(S^2, \Delta)$, i.e, the conjugacy class of the monodromy representation ϕ_π, under the action of* $\mathrm{Mod}(S^2, \Delta)$.

Remark 2.2. Moishezon original phrasing of Theorem 2.1 is slightly different. However, the statement in Theorem 2.1 follows immediately. See Endo's survey for a similar statement in the literature [End21, Theorem 3.3].

Using Theorem 2.1, Moishezon classified elliptic fibrations by showing that all conjugacy classes of homomorphisms $\pi_1(S^2 \setminus \Delta) \to \mathrm{Mod}(\Sigma_1) \cong \mathrm{SL}_2\,\mathbb{Z}$ that take simple loops about the punctures to Dehn twists lie in a single Hurwitz orbit. Note that the presence of a single such homomorphism implies that $n = |\Delta|$ is a multiple of 12, since $(\mathrm{SL}_2\,\mathbb{Z})^{ab} = \mathbb{Z}/12\mathbb{Z}$ and each Dehn twist is sent under the abelianization map to $1 \in \mathbb{Z}/12\mathbb{Z}$.

Theorem 2.3 (Moishezon [Moi06, Theorem 9])**.** *The number $n = 12d$ of singular fibers of a genus one Lefschetz fibration $\pi : M \to S^2$ determines π in the following sense: given two genus one fibrations $\pi, \pi' : M, M' \to S^2$ with the same number of singular fibers, there are diffeomorphisms F, f making the following diagram commute*

$$\begin{array}{ccc} M & \xrightarrow{F} & M' \\ \downarrow & & \downarrow \\ (S^2, \Delta_\pi) & \xrightarrow{f} & (S^2, \Delta_{\pi'}). \end{array}$$

Furthermore, simple loops $\gamma_1, \dots, \gamma_n$ on (S^2, Δ) may be chosen so that the monodromy factorization of π is

$$(T_\alpha T_\beta)^{6d} = \mathrm{Id},$$

where α, β are curves with geometric intersection number $i(\alpha, \beta) = 1$ and T_α, T_β denote the respective Dehn twists about these curves.

In general, following Endo, we call two genus one fibrations $\pi, \pi' : M, M' \to S^2$ *weakly isomorphic* provided there exists such a pair (F, f) of diffeomorphisms so that

$$\begin{array}{ccc} M & \xrightarrow{F} & M' \\ \downarrow & & \downarrow \\ (S^2, \Delta_\pi) & \xrightarrow{f} & (S^2, \Delta_{\pi'}) \end{array}$$

commutes [End21, Definition 3.1]. Changing π by a weak isomorphism changes $\mathrm{Br}(\pi)$ by conjugation in $\mathrm{Mod}(S^2, \Delta_\pi)$. Furthermore $\mathrm{Mod}(\pi)$, as defined in the introduction, consists of the isotopy classes of weak isomorphisms from π to itself.

3. Equivalence of Theorem 1.1 part (a) and Theorem 1.2

Let $\pi : M \to S^2$ be an elliptic fibration with n nodal singular fibers located along $\Delta \subseteq S^2$. Theorem 2.1 identifies the subgroup $\mathrm{Br}(\pi)$ of liftable braids with the stabilizer of the conjugacy class of the monodromy representation. This identification is the essential ingredient in showing part (a) of Theorem 1.1 and Theorem 1.2 are equivalent. Let

$$\phi_\pi : \pi_1(S^2 \setminus \Delta) \to \mathrm{Mod}(T^2) \cong \mathrm{SL}_2 \mathbb{Z}$$

be the monodromy representation of π. By Moishezon's classification of elliptic fibrations (see Theorem 2.3 above), there are simple loops $\gamma_1, \dots, \gamma_n$ about each critical value of Δ so that

$$\begin{aligned} \gamma_1 \cdots \gamma_n &= \mathrm{Id} \\ \phi_\pi(\gamma_{2i+1}) &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ \phi_\pi(\gamma_{2i}) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \end{aligned} \tag{3.5}$$

Furthermore, by Theorem 2.1, a mapping class f lies in $\mathrm{Br}(\pi)$ if and only if $\phi_\pi \circ f_*$ is conjugate to ϕ_π for any lift of f to an automorphism of $\pi_1(S^2 \setminus \Delta)$ (i.e., via the Hurwitz action). We will denote that two representations ϕ, ϕ' are conjugate by $\phi \simeq \phi'$. To simplify notation, let

$$r \coloneqq \sigma_1 \cdots \sigma_{n-1} \in \mathrm{Mod}(S^2, \Delta) \tag{3.6}$$

be the $2\pi/n$ rotation of (S^2, Δ), where σ_i is a half-twist supported on a neighborhood of the once-punctured disks on the interiors of γ_i and γ_{i+1}. Note that, by the theorem of Murasugi mentioned in the introduction, the rotation r represents the unique order n conjugacy class in $\mathrm{Mod}(S^2, \Delta)$ [Mur82]. Before we proceed with the proof that part (a) of Theorem 1.1 and Theorem 1.2 are equivalent, we require an elementary fact about centralizers in spherical braid groups.

Lemma 3.1. *If $n \coloneqq |\Delta| > 2$, then the centralizer $Z_{\mathrm{Mod}(S^2,\Delta)}(r)$ of $r = \sigma_1 \cdots \sigma_{n-1}$ in $\mathrm{Mod}(S^2, \Delta)$ is the cyclic group $\langle r \rangle$.*

Proof. We deduce this from the analogous fact in the braid group. Let B_n be the braid group on n strands. Applying the capping homomorphism (see [FM12, Section 3.6.2]), gives an exact sequence

$$1 \to \langle \widetilde{T}_n^2 \rangle \to B_n \to \mathrm{Mod}(S^2, \Delta) \to 1,$$

with $\widetilde{T}_n$ the lift of T_n, defined by

$$\widetilde{T}_n = (\widetilde{\sigma}_1 \cdots \widetilde{\sigma}_{n-1})(\widetilde{\sigma}_1 \cdots \widetilde{\sigma}_{n-2}) \cdots (\widetilde{\sigma}_1 \widetilde{\sigma}_2)\widetilde{\sigma}_1,$$

where each $\widetilde{\sigma}_i$ is a lift of σ_i. Let $\widetilde{r}$ be the lift of r which rotates the n marked points around the center of the disk. Let $g \in \mathrm{Mod}(S^2, \Delta)$ centralize r, and choose a lift $\widetilde{g} \in B_n$. Then $\widetilde{g}\widetilde{r}\widetilde{g}^{-1} = \widetilde{r}T_n^{2k}$ for some $k \in \mathbb{Z}$. Furthermore, $\widetilde{r}^n = T_n^2$, and so this implies that $\widetilde{g}\widetilde{r}\widetilde{g}^{-1} = \widetilde{r}^{1+nk}$. Conjugation by $\widetilde{g}$ thus restricts to an automorphism of $\langle \widetilde{r} \rangle$, implying that $1 + nk = \pm 1$. Since $k \in \mathbb{Z}$ and $n > 2$, the only solution to this equation is $k = 0$. Therefore $\widetilde{g}$ lies in the centralizer of $\widetilde{r}$. Proposition 3.3 of [GW04] shows that $Z_{B_n}(\widetilde{r}) = \langle \widetilde{r} \rangle$, and so $\widetilde{g} \in \langle \widetilde{r} \rangle$. It follows that $g \in \langle r \rangle$ as claimed. □

With the above in hand we can now prove the equivalence of Theorem 1.2 and part (a) of Theorem 1.1.

Proof that Theorem 1.2 implies part (a) of Theorem 1.1. Let $r \in \mathrm{Mod}(S^2, \Delta)$ be the order n rotation. By Murasugi's theorem (see the introduction), any order n element of $\mathrm{Br}(\pi)$ is conjugate to r by some $b \in \mathrm{Mod}(S^2, \Delta)$. Thus, it suffices to consider elements of the form brb^{-1} for $b \in \mathrm{Mod}(S^2, \Delta)$. Choose loops $\gamma_1, \ldots, \gamma_n$ realizing the standard monodromy as in (3.5). A direct calculation shows that $\phi_\pi \circ r_*$ is conjugate to ϕ_π via the conjugating matrix $C_4 \coloneqq \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ exchanging the simple closed curves corresponding to $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. By the aforementioned theorem of Moishezon (see Theorem 2.1), the conjugate brb^{-1} belongs to $\mathrm{Br}(\pi)$ if and only if $\phi_\pi \circ (brb^{-1})_*$ is conjugate to ϕ_π by some matrix in $\mathrm{SL}_2 \mathbb{Z}$. Adopting the notation $\simeq$ for conjugacy between maps, we may rewrite this as

$$\phi_\pi \circ b_* \circ r_* \circ b_*^{-1} \simeq \phi_\pi \tag{3.7}$$

$$\phi_\pi \circ b_* \circ r_* \simeq \phi_\pi \circ b_*. \tag{3.8}$$

Let $X_i = \phi_\pi(b_*(\gamma_i))$. Then $X_1 \cdots X_n = \mathrm{Id}$ since b_* is an automorphism of π_1. Because b_* is represented by a diffeomorphism, each $b_*(\gamma_i)$ is represented by a simple loop around a single puncture, and so Picard-Lefschetz theory implies that $X_i = T_v \in \mathrm{SL}_2 \mathbb{Z}$ for some primitive vector $v \in \mathbb{Z}^2$. Direct calculation gives $r(\gamma_i) = \gamma_{i+1}$, so that

$$(\phi_\pi \circ b_* \circ r_*)(\gamma_i) = X_{i+1},$$

and so the conjugacy (3.8) implies that for some $C \in \mathrm{SL}_2 \mathbb{Z}$ we have $X_{i+1} = CX_iC^{-1}$ for all i. In summary, $C, X_1, \ldots, X_n$ satisfy assumptions (1) to (3) in Theorem 1.2:

(1) $X_1 \in \mathrm{SL}_2 \mathbb{Z}$ is given by symplectic transvection in the vanishing cycle, by Picard-Lefschetz theory.
(2) $X_{i+1} = CX_iC^{-1}$.
(3) $\prod_{i=1}^n X_i = \mathrm{Id}$.

Thus, assuming that Theorem 1.2 holds, there exists some conjugating matrix $D \in \mathrm{SL}_2 \mathbb{Z}$ so that either $X_i = DY_iD^{-1}$ for all i or $X_i = DZ_iZ^{-1}$ for all i, where

$$(Y_1, Y_2, Y_3, \ldots) \coloneqq \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ldots \right)$$

$$(Z_1, Z_2, Z_3, Z_4, \ldots) \coloneqq \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ldots \right).$$

If $(X_1, \dots, X_n)$ is simultaneously conjugate to $(Y_1, \dots, Y_n)$, then $\phi_\pi \circ b_* \simeq \phi_\pi$ and so $b \in \mathrm{Br}(\pi)$. Therefore $brb^{-1} \in \mathrm{Br}(\pi)$ is an order n element in the conjugacy class of r. If instead $(X_1, \dots, X_n)$ is simultaneously conjugate to $(Z_1, \dots, Z_n)$, then for the Garside twist T_n defined by

$$T_n = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)\sigma_1$$

we have that $\phi_\pi \circ b_* \simeq \phi_\pi \circ (T_n)_*$. Therefore $(T_n b^{-1})$ lies in $\mathrm{Br}(\pi)$ and so brb^{-1} lies in the same conjugacy class as $r^{-1} = T_n r T_n^{-1}$. It is clear that r^{-1} itself lies in $\mathrm{Br}(\pi)$, since r does.

Finally, it suffices to see that the conjugacy classes of r and $r^{-1} = T_n r T_n^{-1}$ in $\mathrm{Br}(\pi)$ are distinct. Suppose not. Then there is some $b \in \mathrm{Br}(\pi)$ so that $brb^{-1} = T_n r T_n^{-1}$. Then $T_n^{-1} b$ centralizes r, and so applying Lemma 3.1, we have that $T_n^{-1} b = r^k$ for some k. However, this would imply that $T_n = br^{-k} \in \mathrm{Br}(\pi)$, and we know that $(T_n \cdot \phi_\pi)(\gamma_i) = Z_i$ by direct calculation. The sequence Z_i cannot be simultaneously conjugated to Y_i, and thus $T_n \notin \mathrm{Br}(\pi)$, giving us a contradiction. □

The proof that part (a) of Theorem 1.1 implies Theorem 1.2 is similar to the above, and we omit it for brevity (and since we do not need it). The essential realization is that any matrices X_i satisfying the assumptions of Theorem 1.2 induce a point in the Hurwitz orbit of ϕ_π which is fixed by the rotation r.

4. **Proving Theorem 1.2 via the Trace Polynomial**

Proof of Theorem 1.2. Let $C, X_1, \dots, X_n \in \mathrm{SL}_2 \mathbb{Z}$ satisfy Items 1 to 3 of Theorem 1.2. For convenience, we recall Items 1 to 3 here:

(1) $X_1 = T_v$, where T_v is the symplectic transvection $T_v(x) = x + (x, v)v$ for some primitive vector $v \in \mathbb{Z}^2$,
(2) $X_{i+1} = C X_i C^{-1}$ for $i \in \mathbb{Z}/n\mathbb{Z}$,
(3) $\prod_{i=1}^n X_i = \mathrm{Id}$.

Note that $X_i = T_{C^{i-1}v}$, as $C T_w C^{-1} = T_{Cw}$ for any primitive vector $w \in \mathbb{Z}^2$.

Step 1: The conjugating matrix C has order 3, 4, or 6.

It is well known that $T_v = T_w$ if and only if $v = \pm w$, and so $T_v = T_{C^n v}$ implies $C^n v = \pm v$. As a consequence, each eigenvalue of C is a root of unity. Since $C \in \mathrm{SL}_2 \mathbb{Z}$, the eigenvalues of C satisfy a monic degree two polynomial with integral coefficients, and so must be one of $1, -1, i, -i, \omega$, or $-\omega$, where $\omega = e^{2\pi i/3}$ is a primitive 3rd root of unity. In the latter four cases, there are two distinct eigenvalues, and so $C^3, C^4, C^6 = \mathrm{Id}$ respectively in each case (by diagonalizing C). In the first case, where C has eigenvalue 1 with algebraic multiplicity 2, this implies either $C = \mathrm{Id}$ or $Cv = v$, hence $X_i = T_v$ for all i, and so $\prod_{i=1}^n X_i = T_v^n \neq \mathrm{Id}$. Similarly, if C has eigenvalue -1 then $X_i = T_v$, and the product cannot be the identity.

Note further that if C has order 4 then $C^2 = -\mathrm{Id}$, and hence X_i is periodic with period 2. Otherwise, X_i has period 3. Note also that $(\mathrm{SL}_2 \mathbb{Z})^{ab} = \mathbb{Z}/12\mathbb{Z}$ and T_v maps to a generator of $\mathbb{Z}/12\mathbb{Z}$ under the abelianization map (see [FM12, p. 123]). As a consequence, n must be divisible by 12. Since the period of $(X_1, \dots, X_n)$ is $k \in \{2, 3\}$, Item 3 may be rewritten as

$$(X_1 \cdots X_k)^{n/k} = \mathrm{Id}\,.$$

We define the *trace polynomial* as

$$f(p, q) \coloneqq \mathrm{tr}(X_1 \cdots X_k) = \mathrm{tr}(T_v \cdots T_{C^{k-1}v}) \in \mathbb{Z}[p, q],$$

in the coordinates $v = \binom{p}{q}$. Since $X_1 \cdots X_k$ is finite order, its trace has absolute value at most two, i.e., $|f(p,q)| \leq 2$.

Step 2: Up to a change of coordinates in $\mathrm{SL}_2 \mathbb{Z}$*,* f *is given by an explicit polynomial of the norm*

$$N(p + q\lambda) \coloneqq (p + q\lambda)(p + q\overline{\lambda})$$

where λ *is a 3rd or 4th root of unity.*

Because C has order 3,4, or 6, it must be conjugate in $\mathrm{SL}_2 \mathbb{Z}$ to one of the three matrices[4]

$$C_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, C_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, C_6 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

with orders 3, 4, and 6 respectively, note that $C_3 = C_6^2$ (see [FM12, p. 201]). Hence, by applying a global conjugacy, one may assume that C is one of these three matrices. At this point, one can explicitly compute the polynomials in each case, which we will denote by f_3, f_4, and f_6 respectively:

$$\begin{aligned} f_3(p,q) &= -(N_3 + 1)(N_3^2 + 2N_3 - 2) \\ f_4(p,q) &= -2(N_4 + 1)(N_4 - 1) \\ f_6(p,q) &= (N_3 - 1)(N_3^2 - 2N_3 - 2), \end{aligned}$$

where $N_j \coloneqq N(p + q\zeta_j)$ and $\zeta_j = e^{2\pi i/j}$

Step 3: Determining when $|f_j(p,q)| \leq 2$ *and* $T_v \cdots T_{C^{k-1}v} = \mathrm{Id}$.

Take some $h \in \mathrm{SL}_2 \mathbb{Z}$ so that $hCh^{-1} = C_j$ for $j \in \{3,4,6\}$. Then for $Y_i = hX_ih^{-1}$ we have $Y_1 = T_{hv}$, $Y_i = C_jY_{i-1}C_j^{-1}$, and $\prod_{i=1}^n Y_i = \mathrm{Id}$. Thus we can assume without loss of generality that $C = C_j$, and instead classify the possible $v = \binom{p}{q}$ such that $|f_j(p,q)| \leq 2$. By direct computation, we find that

$$\begin{aligned} |\mathrm{tr}(X_1X_2X_3)| = |f_3(p,q)| \leq 2 &\implies N_3 = N(p + q\omega) \leq 1 && (\text{if } C = C_3) \\ |\mathrm{tr}(X_1X_2)| = |f_4(p,q)| \leq 2 &\implies N_4 = N(p + qi) < 2 && (\text{if } C = C_4) \\ |\mathrm{tr}(X_1X_2X_3)| = |f_6(p,q)| \leq 2 &\implies N_3 = N(p + q\omega) \leq 3, && (\text{if } C = C_6) \end{aligned}$$

where $\omega = e^{2\pi i/3}$. These regions are the interiors of conics in the (p,q) plane, as displayed in Figure 1. There are then a finite number of possibilities of $v \in \mathbb{Z}^2$ for each C. When $C = C_3$, there is one C-orbit of primitive vectors up to sign which satisfy $|f_3(p,q)| \leq 2$:

$$v = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Cv = \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C^2v = \pm \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

In this case, direct computation shows that $T_vT_{Cv}T_{C^2v}$ is a parabolic element of infinite order. When $C = C_4$, there is again one orbit up to sign

$$v = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Cv = \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

in this case T_vT_{Cv} has order 6, and T_v, T_{Cv} is precisely the pair Y_1 and Y_2 identified in the statement of the theorem,

$$Y_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad Y_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

[4]These matrices correspond precisely to those regular polygons tiling the plane as well as the number rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$

Finally, we turn to when $C = C_6$, there are then two C-orbits of primitive vectors up to sign where $N(p + q\omega) \leq 3$.

$$v = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Cv = \pm \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad C^2 v = \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$w = \pm \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad Cw = \pm \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad C^2 w = \pm \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

A simple calculation verifies that $T_v T_{Cv} T_{C^2 v}$ has order 4, and corresponds to the triple identified in the theorem statement

$$Z_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad Z_2 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \qquad Z_3 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Furthermore, the product $T_w T_{Cw} T_{C^2 w}$ is a parabolic element of infinite order. Because $(X_1 \cdots X_k)$ must have finite order, these calculations imply that $X_1, \dots, X_n$ must be equal to one of $Y_1, \dots, Y_n$ or $Z_1, \dots, Z_n$, depending on whether $C = C_4$ or $C = C_6$, completing the proof. $\square$

Granted the equivalence of Theorem 1.2 and part (a) of Theorem 1.1 proved above, we have then completed the classification of order n elements in $\mathrm{Br}(\pi)$ for $\pi : M \to S^2$ a genus one Lefschetz fibration with n singular fibers.

5. An Algebraic Approach to the Trace Polynomial

Before proving parts (b) and (c) of Theorem 1.1 and discussing the difficulties which arise from the approach above for orders less than $n - 2$, we will describe an invariant theory approach to Step 2. This alternate description explains why one might expect the polynomials f_3, f_4, f_6 to be polynomials in the norms N_4, N_3 in more algebraic terms. To begin, we note the following proposition.

Proposition 5.1. *Let $C, X_1, \dots, X_n \in \mathrm{SL}_2 \mathbb{Z}$ satisfy Items 1 to 3 of Theorem 1.2, and let $(X_1, \dots, X_n)$ have period k. Then*

$$f(p, q) = \mathrm{tr}(T_v T_{Cv} \cdots T_{C^{k-1} v}) \in \mathbb{Z}[p, q],$$

for $v = \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$, is invariant under the centralizer $Z_{\mathrm{GL}_2 \mathbb{Z}}(C)$ of C in $\mathrm{GL}_2 \mathbb{Z}$ acting on $\mathbb{Z}[p, q]$. Furthermore, $Z_{\mathrm{GL}_2 \mathbb{Z}}(C) \cong D_{2k}$, the dihedral group with $2k$ elements.

Because D_{2k} is a Coxeter group, standard methods such as the Chevalley-Shephard-Todd Theorem, which states that the invariant ring $\mathbb{C}[p, q]^{D_{2k}}$ is a free polynomial algebra, apply over $\mathbb{C}$. In this case, we are actually able to compute the full invariant ring over $\mathbb{Z}$, since there is an integral choice of generators for the invariant ring over $\mathbb{C}$.

Proposition 5.2. *Let $C \in \mathrm{SL}_2 \mathbb{Z}$ have order $3, 4,$ or 6 and let H be the centralizer of C in $\mathrm{GL}_2 \mathbb{Z}$, then*

$$\mathbb{Z}[p, q]^H \cong \begin{cases} \mathbb{Z}[N(x + y\omega), (x + y\omega)^6 + (x + y\overline{\omega})^6] & C \text{ has order } 3 \text{ or } 6 \\ \mathbb{Z}[N(x + yi), x^2 y^2] & C \text{ has order } 4 \end{cases}$$

where the isomorphism is by an $\mathrm{SL}_2 \mathbb{Z}$ change of coordinates.

Proof. By applying a global conjugation, let C be one of C_3, C_4 or C_6. The conjugating matrix L so that $C = L C_j L^{-1}$ induces the coordinates x, y given in the theorem. One verifies that the centralizer is generated by $C, -\mathrm{Id}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and is isomorphic to D_{2k}, where $k = 2$ if C has order 4 and $k = 3$ if C has order 3 or 6. If $C = C_3$ then C_6 lies in the centralizer as well. A direct computation shows that the generators claimed above are in the invariant ring. The Jacobian criterion states that polynomials $g_1, \ldots, g_n \in \mathbb{C}[x_1, \ldots, x_m]$ are algebraically independent provided that the differential $\mathrm{d}g_1 \wedge \cdots \mathrm{d}g_n$ is not identically zero [Hum90, §3.10]. When $n = m$, we identify the wedge product with the determinant and write $J(g_1, \ldots, g_n) = \det \mathcal{J}(g_1, \ldots, g_n)$, where $\mathcal{J}(g_1, \ldots, g_n)$ is the matrix of partial derivatives. Applying the Jacobian criterion in this case verifies algebraic independence:

$$\begin{aligned} J(N(x+y\omega), (x+y\omega)^6 + (x-y\omega)^6) &= \det \begin{pmatrix} 2x-y & 6(2x-y)(x^4 - 2x^3y - 6x^2y^2 + 7xy^3 + 4) \\ 2y-x & 6(2y-x)(y^4 - 2y^3x - 6y^2x^2 + 7yx^3 + 4) \end{pmatrix} \\ &= 6(2x-y)(2y-x)(y^4 - 9xy^3 + 9x^3y - x^4) \neq 0 \\ J(N(x+yi), x^2y^2) = J(x^2+y^2, x^2y^2) &= \det \begin{pmatrix} 2x & 2xy^2 \\ 2y & 2x^2y \end{pmatrix} = 2x^3y - 2xy^3 \neq 0. \end{aligned}$$

Let these two proposed generators be referred to as F_1, F_2. One may check that

$$|D_{2k}| = 4k = \deg(F_1)\deg(F_2).$$

Hence, these are generators for $\mathbb{C}[x,y]^H$ (see [Hum90, p. 67]). Because $F_1, F_2 \in \mathbb{Z}[x,y]$ are primitive polynomials, we in fact obtain that these are generators for $\mathbb{Z}[x,y]^H$ as desired. □

Note that the trace polynomial f has total degree 4 or 6 depending on whether $(X_1, \ldots, X_k)$ has period $k = 2$ or $k = 3$. To show that $f(p,q)$ is a polynomial in the norm $N_4 = N(x+yi)$ or $N_3 = N(x+y\omega)$ respectively it thus suffices by Proposition 5.2 to determine the degree $2k$ homogeneous part of f. This computation must be carried out directly, and yields $-2N_4^2$ and $\pm N_3^3$ respectively.

6. Orders $n-1$ and $n-2$

Let $\tau \in \mathrm{Mod}(S^2, \Delta)$ be the order $n-1$ rotation and $\eta \in \mathrm{Mod}(S^2, \Delta)$ be the order $n-2$ rotation. For convenience, we modify η by isotopy so that it fixes a neighborhood of the south pole, and choose a basepoint within this neighborhood. When dealing with τ, we can choose the south pole itself as the basepoint, and let the north pole be one of the punctures. One can choose generators $\gamma_1', \ldots, \gamma_{n-1}', \delta$ and $\gamma_1'', \ldots, \gamma_{n-2}'', \nu_1, \nu_2$ for $\pi_1(S^2 \setminus \Delta)$ so that

$$\begin{aligned} \tau(\gamma_i') &= \gamma_{i+1}' && (\text{for } i \in \mathbb{Z}/(n-1)\mathbb{Z}) \\ \tau(\delta) &= \gamma_1^{-1}\delta\gamma_1 \\ \eta(\gamma_i'') &= \gamma_{i+1}'' && (\text{for } i \in \mathbb{Z}/(n-2)\mathbb{Z}) \\ \eta(\nu_1) &= \nu_1 \\ \eta(\nu_2) &= \gamma_1^{-1}\nu_2\gamma_1, \end{aligned}$$

where γ_i' and γ_i'' are loops about the equatorial punctures, δ and ν_1 are loops about the north pole, and ν_2 is a loop about the south pole. Note that, choosing $\gamma_i', \gamma_i'', \delta, \nu_1$, and ν_2 appropriately gives

$$\delta\gamma_1' \cdots \gamma_{n-1}' = 1 \qquad\qquad \nu_1\gamma_1'' \cdots \gamma_{n-2}''\nu_2 = 1.$$

Similarly to Section 3, to prove parts (b) and (c) of Theorem 1.1 it suffices to show that there are no monodromy representations which are fixed up to conjugacy by τ and η respectively. By evaluating such a representation $\rho_\pi : \pi_1(S^2 \setminus \Delta) \to \mathrm{SL}_2\,\mathbb{Z}$ at our choice of generators, we reduce to showing that there are no monodromy factorizations fixed by τ or η. Thus, parts (b) and (c) of Theorem 1.1 are implied by the following statements.

Theorem 6.1. *Let $n \geq 1$, then there are no tuples $(X_1, \ldots, X_{n-1}, L) \in (\mathrm{SL}_2 \mathbb{Z})^n$ of matrices satisfying the following for some $C \in \mathrm{SL}_2 \mathbb{Z}$:*

(1') X_1 and L are symplectic transvections,
(2') $L \prod_{i=1}^{n-1} X_i = \mathrm{Id}$,
(3') $CX_iC^{-1} = X_{i+1}$ and $CLC^{-1} = X_1^{-1} L X_1$.

Proposition 6.2. *Let $n \geq 1$, then there are no tuples $(X_1, \ldots, X_{n-2}, L_1, L_2) \in (\mathrm{SL}_2 \mathbb{Z})^n$ satisfying the following for some $C \in \mathrm{SL}_2 \mathbb{Z}$:*

(1") X_1, L_1, L_2 are symplectic transvections,
(2") $L_1 \left(\prod_{i=1}^{n-2} X_i\right) L_2 = \mathrm{Id}$,
(3") $CX_iC^{-1} = X_{i+1}$, $CL_1C^{-1} = X_1^{-1} L_1 X_1$, and $CL_2C^{-1} = L_2$.

We now show that Theorem 6.1 and Proposition 6.2 imply parts (b) and (c) of Theorem 1.1 respectively.

Proof that Theorem 6.1 implies part (b) of Theorem 1.1. j Let $\pi : M \to S^2$ be an elliptic fibration, and let $\tau \in \mathrm{Mod}(S^2, \Delta)$ be the order $n-1$ rotation. By Murasugi's theorem (see the introduction), τ represents the unique order $n-1$ conjugacy class in $\mathrm{Mod}(S^2, \Delta)$ [Mur82]. Hence any order $n-1$ element of $\mathrm{Br}(\pi)$ is represented by $b\tau b^{-1} \in \mathrm{Br}(\pi)$ for some $b \in \mathrm{Mod}(S^2, \Delta)$. Let

$$\rho_\pi : \pi_1(S^2 \setminus \Delta) \to \mathrm{SL}_2 \mathbb{Z}$$

be the monodromy representation of π. By Moishezon's theorem (see Theorem 2.1 above), we know that $b\tau b^{-1} \in \mathrm{Br}(\pi)$ if and only if $\rho_\pi \circ (b\tau b^{-1})_*$ is conjugate to ρ_π. We choose generators $\gamma_1', \ldots, \gamma_{n-1}', \delta$ so that

$$\tau(\gamma_i') = \gamma_{i+1}' \qquad \text{(for } i \in \mathbb{Z}/(n-1)\mathbb{Z})$$
$$\tau(\delta) = \gamma_1^{-1} \delta \gamma_1$$

where γ_i' are loops about the equatorial punctures and δ is a loop about the marked fixed point of τ. Note that $\delta\gamma_1' \cdots \gamma_{n-1}' = \mathrm{Id}$. Therefore, letting $X_i = \rho_\pi(b_*(\gamma_i'))$ and $L = \rho_\pi(b_*(\delta))$ we conclude that there is some matrix $C \in \mathrm{SL}_2 \mathbb{Z}$ so that

$$CX_iC^{-1} = C\rho_\pi(b_*(\gamma_i'))C^{-1} = \rho_\pi(b_*(\tau_*(\gamma_i'))) = \rho_\pi(b_*(\gamma_{i+1}')) = X_{i+1}$$
$$CLC^{-1} = C\rho_\pi(b_*(\delta))C^{-1} = \rho_\pi(b_*(\tau_*(\delta))) = \rho_\pi(b_*(\gamma_1^{-1}\delta\gamma_1)) = X_1^{-1} L X_1.$$

Furthermore, because ρ_π and b_* are group homomorphisms, $LX_1 \cdots X_n = \mathrm{Id}$. By Picard-Lefschtz theory, we also know that X_i, L are symplectic transvections. Therefore, $(X_1, \ldots, X_{n-1}, L) \in (\mathrm{SL}_2 \mathbb{Z})^n$ is a tuple satisfying the conditions of Theorem 6.1. No such tuple exists, and so there can be no such $b \in \mathrm{Mod}(S^2, \Delta)$ so that $b\tau b^{-1} \in \mathrm{Br}(\pi)$. □

The proof that Proposition 6.2 implies part (c) of Theorem 1.1 is identical, and so we omit it for brevity. With this motivation, we prove Theorem 6.1 and Proposition 6.2.

Proof of Theorem 6.1. Let $(X_1, \ldots, X_{n-1}, L)$ satisfy the conditions of the theorem and let $X_1 = T_v$ and $L = T_w$ for primitive vectors $v, w \in \mathbb{Z}^2$. As before, $n = 12d$ for some $d \geq 1$ since $(\mathrm{SL}_2 \mathbb{Z})^{ab} = \mathbb{Z}/12\mathbb{Z}$. Note that C^{n-1} fixes X_1 by conjugation. Hence $C^{n-1}(v) = \pm v$. As argued in Step 1 in Section 4, either $C \cdot v = \pm v$ or C has order 3, 4, or 6. If $C \cdot v = \pm v$, then $X_1 = X_i$ for all i. Therefore $L = X_1^{-n+1}$. However, since $n \geq 12$, this is impossible, since the only symplectic transvection which is a power of X_1 is X_1 itself.

If $C^2 = -\operatorname{Id}$, then

$$T_w = C^2 T_w C^{-2} = X_2^{-1} X_1^{-1} T_w X_1 X_2 = T_{(X_1 X_2)^{-1} w}.$$

and so $X_1 X_2(w) = \pm w$. Therefore $X_1 X_2 = \pm L^k$ for some $k \in \mathbb{Z}$, and so

$$L(X_1 X_2)^{6d} X_2^{-1} = \operatorname{Id},$$

but then $\pm L^{6dk+1} = X_2$. Because X_2 and L are both symplectic transvections, this is impossible unless $6dk + 1 = 1$. However, this implies that $k = 0$, and so $X_1 X_2 = \operatorname{Id}$. Applying the abelianization map, this would imply that $2 = 0$ in $(\mathrm{SL}_2 \mathbb{Z})^{ab} \cong \mathbb{Z}/12\mathbb{Z}$, a contradiction.

Otherwise, $C^3 = \pm \operatorname{Id}$ and

$$T = C^3 T_w C^{-3} = X_3^{-1} X_2^{-1} X_1^{-1} T_w X_1 X_2 X_3 = T_{(X_1 X_2 X_3)^{-1} w}$$

and so $X_1 X_2 X_3(w) = \pm w$. As before, $X_1 X_2 X_3 = \pm L^k$ for some $k \in \mathbb{Z}$ and so

$$L(X_1 X_2 X_3)^{4d} X_3^{-1} = \operatorname{Id},$$

which implies $\pm L^{4dk+1} = X_3$. As before, this implies that $3 = 0$ in $\mathbb{Z}/12\mathbb{Z}$. □

Proof of Proposition 6.2. Let $(X_1, \ldots, X_{n-2}, L_1, L_2)$ satisfy the conditions of the proposition. Conjugating Item (2") by the given $C \in \mathrm{SL}_2 \mathbb{Z}$ gives

$$X_1^{-1} L_1 X_1 \cdots X_{n-2} X_1 L_2 = \operatorname{Id}$$
$$X_1^{-1} L_2^{-1} X_1 L_2 = \operatorname{Id},$$

and so $L_2^{-1} X_1 L_2 = X_1$. Let $X_1 = T_v$, where $v \in \mathbb{Z}^2$ is some primitive vector. Then $L_2 \cdot v = \pm v$, and since L_2 is a symplectic transvection $L_2 = X_1$. Because $C L_2 C^{-1} = L_2$, we also have that $X_i = X_1$ for all i. Therefore

$$L_1 X_1^{n-1} = L_1 X_1 \cdots X_{n-2} L_2 = \operatorname{Id},$$

and so $L_1 = X_1^{1-n}$. Since L_1 and X_1 are both symplectic transvections, this is only possible if $1 - n = 1$. But then $n = 0$, and so the only such tuple is the empty tuple. □

Hence, we have proved parts (b) and (c) of Theorem 1.1.

7. Difficulties Inherent in Classifying Small Orders

We now explain why the methods above do not generalize to classifying elements of order n/a for $a \mid n$. As an example of an order $n/2$ element of $\mathrm{Br}(\pi)$, one may consider simple closed curves $\alpha, \beta, \gamma, \delta$ forming the vertices of two triangles (α, γ, β) and (α, β, δ) labeled clockwise in the Farey complex (see Figure 2). The monodromy ϕ corresponding to the factorization

$$((T_\alpha T_\gamma T_\beta)(T_\beta T_\delta T_\alpha))^{n/6} = \operatorname{Id} \tag{7.9}$$

is not conjugate to $r \cdot \phi$, where r is the order n rotation. However it is conjugate to $r^3 \cdot \phi$. Hence the monodromy factorization (7.9) corresponds to a "new" order $n/3$ conjugacy class not arising as the cube of either of the order n conjugacy classes discussed in Theorem 1.1, and so the order $n/2$ element corresponding to (7.9) does not belong to a $\mathbb{Z}/n\mathbb{Z}$ subgroup of $\mathrm{Br}(\pi)$.

When $a = 2$, a similar argument to the proofs of Theorems 1.1 and 1.2 shows that classifying elements of order $n/2$ is essentially equivalent to classifying primitive vectors $v, w \in \mathbb{Z}^2$ so that

$$|\operatorname{tr} T_v T_w T_{Cv} T_{Cw} \cdots T_{C^{n/2-1} v} T_{C^{n/2-1} w}| \le 2.$$

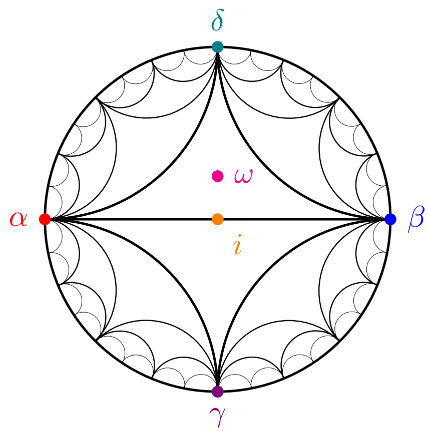
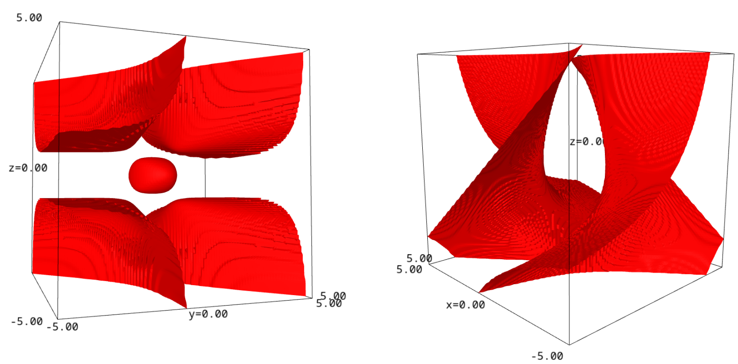


FIGURE 2. Farey Tesselation labeled α, δ, β as well as points $i, \omega = e^{2\pi i/3}$ under the identification of the Poincaré disk with $\mathbb{H}^2$.

As before, C is of finite order, and so can be conjugated to one of C_3, C_4, C_6. Thus we obtain three polynomials $g_3(v, w), g_4(v, w), g_6(v, w)$, and we wish to find the lattice points in $\mathbb{Z}^4$ satisfying $|g_j(v, w)| \leq 2$. Two three-dimensional slices of $g_6 = 0$ are displayed in Figure 3.



(A) $w_2 = 0$ (B) $w_2 = -4$

FIGURE 3. Noncompact 3-dimensional slices of $g_6 = 0$ where $w = (w_1, w_2)$ is restricted to a hyperplane.

There are two primary obstructions to generalizing the proof of Theorem 1.1 above, which are implicitly linked:

(1) The invariant ring $\mathbb{Z}[v, w]^H$, where H is the centralizer of C acting diagonally on the v, w coordinates, is significantly more complex than the invariant ring computed in Proposition 5.2.
(2) The real algebraic locus of $|g_j(v, w)| \leq 2$ in $\mathbb{R}^4$ is not compact, as can be seen in Figure 3.

Similar difficulties arise when considering which powers of τ and η lie in some conjugate of $\mathrm{Br}(\pi)$, where τ and η are the order $n-1$ and $n-2$ rotation respectively. For example, consider the following monodromy factorization for an elliptic fibration π with 24 singular fibers, i.e., of a K3 surface:

$$T_\alpha (T_\beta T_\alpha)^6 (T_\beta T_\alpha)^5 T_\beta = \mathrm{Id}, \tag{7.10}$$

where α, β are curves with intersection number $i(\alpha, \beta) = 1$. The monodromy factorization (7.10) is fixed by η^{12}, which can be verified using the generating set for $\pi_1(S^2 \setminus \Delta)$ given in Section 6. Hence (7.10) represents an element of order 11 in $\mathrm{Br}(\pi)$, while there is no element of order 22, contrasting sharply with Murasugi's theorem. Despite these difficulties, the symmetry involved in the construction of the monodromy factorizations corresponding to the two order n conjugacy classes and the conjugacy class of order $n/2$ displayed in (7.9) suggests leveraging the geometry of the Farey complex as a general approach.

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