

How large is the braid monodromy of low-genus Lefschetz fibrations?

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ABSTRACT. Given a genus g smooth Lefschetz fibration $\pi : M \rightarrow S^2$ with singular locus $\Delta \subseteq S^2$, we describe the subgroup $\text{Br}(\pi)$ of the spherical braid group $\text{Mod}(S^2, \Delta)$ consisting of braids admitting a lift to a fiber-preserving diffeomorphism of M . We develop general methods for showing that the index $[\text{Mod}(S^2, \Delta) : \text{Br}(\pi)]$ is infinite. As an application of our methods, we prove that $[\text{Mod}(S^2, \Delta) : \text{Br}(\pi)] = \infty$ when $g = 1$, when π is expressible as a self-fiber sum when $g \geq 2$, or when π is a holomorphic genus $g = 2$ Lefschetz fibration whose vanishing cycles are nonseparating. In the genus $g = 1$ case, we relate the subgroup $\text{Br}(\pi)$ to the action of $\text{Mod}(S^2, \Delta)$ on the SL_2 -character variety for $S^2 \setminus \Delta$ and provide an alternate proof of the first application via recent work of Lam–Landesman–Litt [LLL23].

1. Introduction

This paper relates smooth Lefschetz fibrations of 4-manifolds, the spherical braid group, and SL_2 -character varieties. Given a smooth Lefschetz fibration¹ $\pi : M \rightarrow S^2$ whose generic fibers have genus g , let

$$\text{Diff}^+(\pi) := \{F \in \text{Diff}^+(M) \mid F \text{ takes fibers of } \pi \text{ to fibers of } \pi\}.$$

The smooth mapping class group of π is

$$\text{Mod}(\pi) := \pi_0(\text{Diff}^+(\pi)).$$

Tracking where an element of $\text{Diff}^+(\pi)$ takes the fibers gives a spherical braid² monodromy representation

$$\rho : \text{Mod}(\pi) \rightarrow \text{Mod}(S^2, \Delta_\pi),$$

where $\Delta_\pi \subseteq S^2$ is the singular locus of π and $\text{Mod}(S^2, \Delta_\pi) = \pi_0(\text{Diff}^+(S^2, \Delta_\pi))$. In this paper, we study the image

$$\text{Br}(\pi) := \text{im}(\rho) < \text{Mod}(S^2, \Delta_\pi), \tag{1.1}$$

of the braid monodromy ρ . The subgroup $\text{Br}(\pi) < \text{Mod}(S^2, \Delta_\pi)$ consists of those braids that lift to a fiber-preserving diffeomorphism of M ; we call these braids *liftable braids* or *liftable mapping classes*. Hence $\text{Br}(\pi)$ associates to each Lefschetz fibration π a subgroup of the spherical braid group. See Figure 1 for an example of two braids whose marked points are the singular values of rational elliptic fibration $\pi : M \rightarrow S^2$. The top braid lifts while the bottom braid does not.

The main goal of this paper is to understand how large $\text{Br}(\pi)$ is in comparison to the full spherical braid group $\text{Mod}(S^2, \Delta)$. The coarsest such measure is the index of $\text{Br}(\pi)$ inside $\text{Mod}(S^2, \Delta)$. We give a general method for determining when the index $[\text{Mod}(S^2, \Delta) : \text{Br}(\pi)]$ is infinite which is applicable to a wide range of Lefschetz fibrations. Applications include Theorem 1.1, Theorem 1.2, and Theorem 1.3 (see below).

The Index of $\text{Br}(\pi)$. Our first result applies when π is a self-fiber sum or when the fiber genus $g = 1$. For the definition of the fiber sum of two Lefschetz fibrations, we refer the reader to Gompf–Stipsicz [GS23, §7.1]

Theorem 1.1 (Index of $\text{Br}(\pi)$). *Let $\pi : M \rightarrow S^2$ be a nontrivial Lefschetz fibration of genus $g \geq 1$ with singular locus $\Delta_\pi \subseteq S^2$. Then*

¹For us, we consider only Lefschetz fibration $\pi : M \rightarrow S^2$ with simple singularities, i.e., with at most one critical point in each fiber. We also work in the smooth category unless otherwise stated.

²There are two distinct notions of the spherical braid group, one is $B_n(S^2) := \pi_1(\text{Conf}_n(S^2))$, the fundamental group of n -point configurations in S^2 , the other is $\text{Mod}(S^2, \{p_1, \dots, p_n\})$, a marked mapping class group. These differ by an exact sequence $1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow B_n(S^2) \rightarrow \text{Mod}(S^2, \{p_1, \dots, p_n\}) \rightarrow 1$ [FM12, §9.1.4]. In this paper we restrict ourselves to the latter notion.

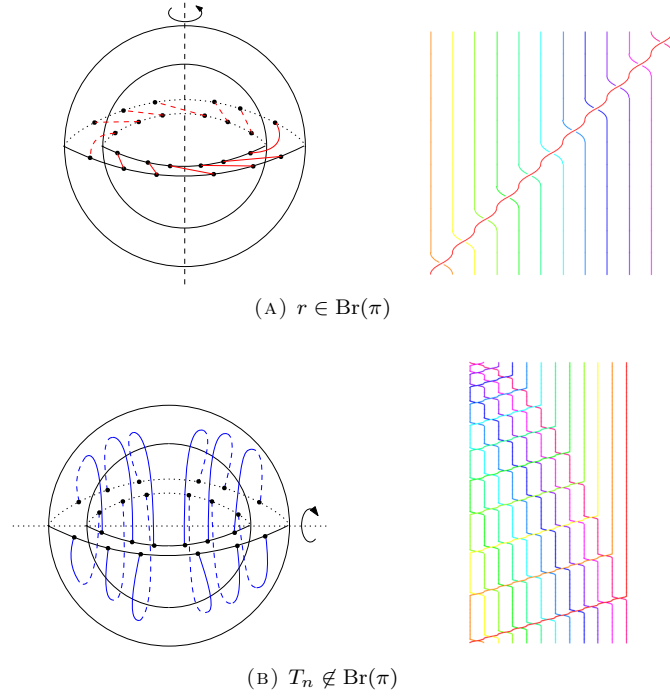


FIGURE 1. The braids $r = \sigma_1 \cdots \sigma_{n-1}$ given by rotation about the equator and the Garside half-twist $T_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1$ given by rotation about a longitude for $n = 12$. r lies in $\text{Br}(\pi)$ while T_n does not. In each case, the left figure depicts a spherical braid with its associated mapping class and the right figure depicts the corresponding element in the braid group B_n .

(1) If $g = 1$ then $[\text{Mod}(S^2, \Delta_\pi) : \text{Br}(\pi)] = \infty$.

(2) If π can be expressed as a self fiber-sum $\pi = p \#_F p$ then $[\text{Mod}(S^2, \Delta_\pi) : \text{Br}(\pi)] = \infty$.

Remarks 1.1.

- (i) Part (1) of Theorem 1.1 can be deduced from Part (2). Explicitly, every genus 1 Lefschetz fibration is expressed as a self-fiber sum of the rational elliptic surface given by the blowup $\mathcal{B}\ell_{\{p_1, \dots, p_9\}} \mathbb{P}^2$ of $\mathbb{P}^2$ along the intersection points of two cubics. The case of the rational elliptic surface itself can be addressed by a similar argument to that used in the proof of Part (2).
- (ii) Part (1) of Theorem 1.1, the genus one case, can also be deduced from recent work of Lam, Landesman, and Litt classifying certain³ SL_2 -local systems on (S^2, Δ_π) with finite mapping class group orbit, see Section 2 [LLL23]. Their classification relies on non-abelian Hodge theory and is inspired by Katz's classification of rigid local systems. Our argument has the advantages of applying in genus $g \geq 2$ and being elementary; however, it relies heavily on our assumption that the singularities of our genus one fibrations are nodal. Lam–Landesman–Litt allows one to reproduce a variant of Part (1) for a wider variety of genus one fibrations allowing other types of singularities besides nodal cubics.

Using Chakiris' classification of holomorphic genus 2 Lefschetz fibrations whose vanishing cycles are nonseparating [Cha83], we are also able to prove the following.

³Lam–Landesman–Litt's work only directly applies to those SL_2 -local systems where the monodromy about at least one puncture has infinite order and the image is Zariski dense in $\text{SL}_2 \mathbb{C}$. However, the local systems arising from genus one fibrations all satisfy these conditions due to Moishezon's classification (see [Moi06, Theorem 9]). Indeed, the image of the monodromy is $\text{SL}_2 \mathbb{Z}$ and the monodromy about a puncture is parabolic.

Theorem 1.2. *Let $\pi : M \rightarrow S^2$ be a nontrivial holomorphic Lefschetz fibration with fibers of genus $g = 2$ and singular locus Δ . If π only has nonseparating vanishing cycles then*

$$[\text{Mod}(S^2, \Delta) : \text{Br}(\pi)] = \infty.$$

As we will later see, given two Lefschetz fibrations π and π' such that $\text{Br}(\pi)$ has infinite index in $\text{Mod}(S^2, \Delta_\pi)$, the liftable braid group $\text{Br}(\pi \#_F \pi')$ of their fiber sum has infinite index in $\text{Mod}(S^2, \Delta_{\pi \#_F \pi'})$. Thus Theorem 1.2 is equivalent to proving that three explicit fibrations satisfy the infinite index property due to Chakiris' classification [Cha83].

The Hurwitz Action. To prove Theorems 1.1 and 1.2, we analyze the *Hurwitz action* of $\text{Mod}(S^2, \Delta_\pi)$ on the set

$$\mathfrak{X}_g(S^2, \Delta_\pi) := \text{Hom}(\pi_1(S^2 \setminus \Delta_\pi), \text{Mod}(\Sigma_g)) / \text{Mod}(\Sigma_g),$$

where $\text{Mod}(\Sigma_g)$ acts by conjugation and $\text{Mod}(S^2, \Delta_\pi)$ acts by precomposition through its outer action on π_1 . The orbit of a representation under this action is called its *Hurwitz orbit*. In genus $g = 1$, there is a map from $\mathfrak{X}_1(S^2, \Delta_\pi)$ to the SL_2 -character variety:

$$\mathfrak{X}_1(S^2, \Delta_\pi) \rightarrow \text{Hom}(\pi_1(S^2 \setminus \Delta_\pi), \text{SL}_2 \mathbb{C}) // \text{SL}_2 \mathbb{C} =: \mathfrak{Y}(S^2, \Delta_\pi)(\mathbb{C}),$$

via the identification $\text{Mod}(\Sigma_1) \cong \text{SL}_2 \mathbb{Z}$. For SL_2 -character varieties, the Hurwitz orbits are well-studied but not well-understood in general, with applications ranging from Diophantine equations via Markoff surfaces to algebraic solutions of ordinary differential equations. Of particular interest to the present paper are large orbits under the action on the mod p points $\mathfrak{Y}(S^2, \Delta_\pi)(\mathbb{F}_p)$ as in [BGS16] or finite orbits over $\mathbb{C}$ as in [LLL23]. Motivated by the relationship to classical character varieties in genus one, we call $\mathfrak{X}_g(S^2, \Delta_\pi)$ the *modular character variety*.

Given a Lefschetz fibration $\pi : M \rightarrow S^2$, we consider its associated monodromy representation

$$\phi_\pi : \pi_1(S^2 \setminus \Delta_\pi, b) \rightarrow \text{Mod}(\pi^{-1}(b)) \cong \text{Mod}(\Sigma_g), \quad (1.2)$$

for $b \in S^2 \setminus \Delta_\pi$ and Σ_g a genus g connected oriented surface. Note that the monodromy representation ϕ_π is only well-defined up to conjugacy, due to the choice of identification of the fiber $\pi^{-1}(b)$ with Σ_g . Explicitly, let $\psi : \pi^{-1}(b) \rightarrow \Sigma_g$ be such an identification and let $\gamma : [0, 1] \rightarrow S^2 \setminus \Delta_\pi$ satisfy $\gamma(0) = \gamma(1) = b$. The bundle $\gamma^* \pi$ admits a trivialization

$$\Psi : \gamma^* \pi \rightarrow [0, 1] \times \Sigma_g.$$

Writing $\Psi_t : \pi^{-1}(\gamma(t)) \rightarrow \Sigma_g$ for the induced map on fibers, we choose Ψ so that $\Psi_0 = \psi : \pi^{-1}(b) \rightarrow \Sigma_g$ and we define $\phi_\pi([\gamma])$ by the composition

$$\phi_\pi([\gamma]) = [\Psi_1 \circ \psi^{-1}] : \Sigma_g \xrightarrow{\psi^{-1}} \pi^{-1}(b) \xrightarrow{\Psi_1} \Sigma_g.$$

The resulting map is a homomorphism as claimed, where $\text{Mod}(\Sigma_g)$ acts on the left and path concatenation in $\pi_1(S^2 \setminus \Delta_\pi, b)$ is read left to right, i.e., $\gamma \delta$ traverses γ first (see Remark 2.1 regarding conflicting conventions in the existing literature). Taking simple loops $\gamma_1, \dots, \gamma_n$ about each puncture whose product $\gamma_1 \cdots \gamma_n$ is nullhomotopic, we also obtain a factorization

$$\phi_\pi(\gamma_1) \cdots \phi_\pi(\gamma_n) = \text{Id}$$

in the mapping class group. According to the convention above on the monodromy representation, Picard–Lefschetz theory implies that $\phi_\pi(\gamma_i) \in \text{Mod}(\Sigma_g)$ is a (left-handed) Dehn twist about some simple closed curve. The choice of left-handed Dehn twists may seem odd to those familiar with the subject of Lefschetz fibrations. We direct the reader to Remark 2.1 for details on how our conventions in this paper differ from

the standard conventions and how to reconcile this difference. The ordered n -tuple $(\phi_\pi(\gamma_i))$ is called the *monodromy factorization of π* .

In this way, any Lefschetz fibration canonically determines a point $[\phi_\pi] \in \mathfrak{X}_g(S^2, \Delta_\pi)$. A liftable braid $[f] \in \text{Br}(\pi) < \text{Mod}(S^2, \Delta_\pi)$ must fix this point under the Hurwitz action because a lift $F \in \text{Diff}^+(\pi)$ realizes a bundle isomorphism between π and $f \circ \pi$ over $S^2 \setminus \Delta_\pi$. In general, the monodromy group is expected to be as large as possible, limited only by obvious geometric constraints. This slogan is referred to as “big monodromy.” Confirming this philosophy in our particular case, we realize $\text{Br}(\pi) < \text{Mod}(S^2, \Delta_\pi)$ as the stabilizer

$$\text{Br}(\pi) = \text{Stab}_{\text{Mod}(S^2, \Delta_\pi)}[\phi_\pi] \quad (1.3)$$

in Section 2, with respect to the Hurwitz action. The identification (1.3) can be inferred from work of Moishezon in genus $g = 1$ and Matsumoto in genus $g \geq 2$ (see Theorem 2.2 below) [Moi06; Mat96]. Their work also shows that two Lefschetz fibrations π, π' are isomorphic if and only if $[\phi_\pi]$ and $[\phi_{\pi'}]$ lie in the same Hurwitz orbit. In this case $\text{Br}(\pi)$ is conjugate to $\text{Br}(\pi')$.

Finite Hurwitz Orbits over the Disk. We now describe our results on genus one Lefschetz fibrations $\pi : M \rightarrow D^2$ over the disk. The subgroup $\text{Br}(\pi)$ is defined analogously to (1.1) above (see Section 2 for details). In this case, $\text{Br}(\pi)$ is realized as a subgroup of the classical braid group $\text{Mod}(D^2, \Delta_\pi) \cong B_n$, where $n = |\Delta_\pi|$ is the number of singular fibers.

Let $q_n : M_n \rightarrow D^2$ be the genus one Lefschetz fibration with n singular fibers and monodromy representation

$$\begin{aligned} \phi_{q_n} : \pi_1(D^2 \setminus n \text{ points}) &\rightarrow \text{Mod}(\Sigma_1) \\ \phi_{q_n}(\gamma_{2i+1}) &= T_\alpha, \quad \phi_{q_n}(\gamma_{2i}) = T_\beta, \end{aligned} \quad (1.4)$$

where $\gamma_1, \dots, \gamma_n$ are a standard system of generators for π_1 and α, β are simple closed curves in Σ_1 with geometric intersection number $i(\alpha, \beta) = 1$. Using the general criteria developed in Section 5, we prove the following theorem.

Theorem 1.3. *Let $q_n : M_n \rightarrow D^2$ be the genus one Lefschetz fibration with monodromy ϕ_{q_n} given in (1.4). If $n \geq 5$, then $[B_n : \text{Br}(q_n)] = \infty$. The indices for $n \leq 4$ are*

$$[B_2 : \text{Br}(q_2)] = 3 \qquad [B_3 : \text{Br}(q_3)] = 8 \qquad [B_4 : \text{Br}(q_4)] = 27.$$

The Hurwitz orbits of the monodromy representation ϕ_{q_n} for $n = 3, 4$ are displayed in Figure 2, and the orbit for $n = 2$ is merely a triangle. The orbits of ϕ_{q_3} and ϕ_{q_4} are computed using a program developed by the author in SAGE/Python (see (††)). The implemented algorithm is able to compute the Hurwitz orbit whenever it is finite and the fiber of the Lefschetz fibration $\pi : M \rightarrow D^2$ is a genus g surface with one marked point. The input to the program is the monodromy representation of π .

For $\text{Br}(q_3) < B_3$ and $\text{Br}(q_4) < B_4$, we use GAP to compute generating sets for these subgroups. Call an arc between two singular fibers *equinodal* if the vanishing cycles (in the sense of Picard–Lefschetz theory) for the singular fibers about these arcs are equal. The arc is called *antinodal* if the vanishing cycles have geometric intersection number one. A direct computation shows that the half-twist about an equinodal arc lies in $\text{Br}(\pi)$ and the third power of the half-twist about an antinodal arc lies in $\text{Br}(\pi)$, for any genus g fibration π . Our computation proves the following proposition.

Proposition 1.4. *$\text{Br}(q_3)$ and $\text{Br}(q_4)$ are generated by half-twists about finitely many equinodal arcs along with the cubes of half-twists about finitely many antinodal arcs.*

Relation to Previous Work. For algebraic genus one fibrations $\pi : X \rightarrow \mathbb{P}^1$, Lönne defines the *bifurcation braid monodromy* of an algebraic family $\mathcal{X} \rightarrow \mathcal{B}$ of Lefschetz fibrations over $\mathbb{P}^1$ containing π . Formally, such a

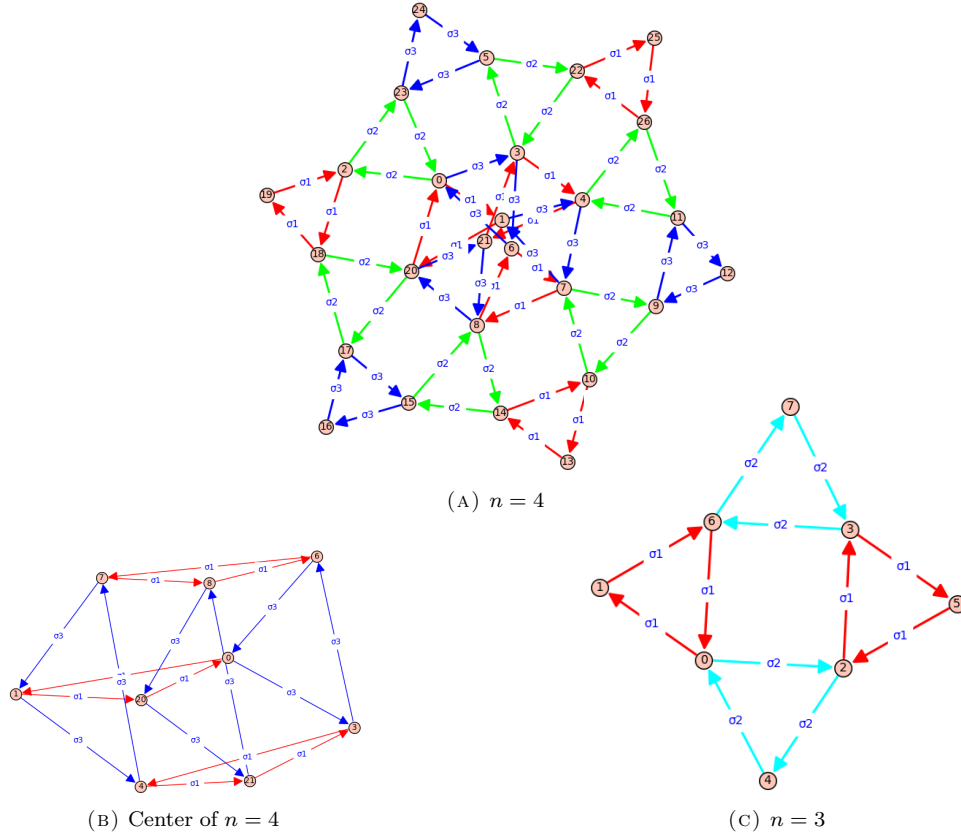


FIGURE 2. Hurwitz orbits of ϕ_{q_n} for $n = 3$ and 4 . Vertex 0 is ϕ_{q_n} , and missing edges indicate representations fixed by the corresponding σ_i . When $n = 3$ the orbit has size 8 , and when $n = 4$ it has size 27 . Figure (B) is the center portion of Figure (A).

family is equipped with a map $f : \mathcal{X} \rightarrow \mathcal{P}$, where $\mathcal{P}$ is a $\mathbb{P}^1$ -bundle over $\mathcal{B}$ so that the restriction $f_b : X_b \rightarrow \mathcal{P}_b$ is a genus one Lefschetz fibration and the restriction to some $b_0 \in \mathcal{B}$ is π itself. In this setting, the *bifurcation braid monodromy* of $\mathcal{X} \rightarrow \mathcal{B}$ is the composition

$$\pi_1(\mathcal{B}) \rightarrow \text{Mod}(\pi) \rightarrow \text{Br}(\pi).$$

Lönne studies in detail the group $\text{Br}^{alg}(\pi)$ generated by the images of all such monodromies for all families $\mathcal{X} \rightarrow \mathcal{B}$ containing π [Lön06]. Lönne shows that, when π is a genus one fibration, there is some $b \in \text{Mod}(S^2, \Delta)$ so that

$$\text{Br}^{alg}(\pi) = b \langle \sigma_{ij}^{m_{ij}} \mid m_{ij} = 1 \text{ if } i \equiv j \pmod{2} \text{ or } m_{ij} = 3 \text{ if } i \not\equiv j \pmod{2} \rangle b^{-1}$$

where σ_{ij} is a simple braid exchanging punctures i, j on the sphere [Lön06, Main Theorem]. Lönne's generating set for $\text{Br}^{alg}(\pi)$ when π is a genus one fibration agrees with the generating sets given in Proposition 1.4 for finite-index genus one fibrations over the disk. Lönne leaves open the question of whether $\text{Br}^{alg}(\pi) = \text{Br}(\pi)$ for genus one fibrations with nodal fibers. However, he does show that $\text{Br}^{alg}(\pi) = \text{Br}(\pi)$ for genus one fibrations with six nodal fibers and an odd number of singular fibers of type I_0^* .

Independently, Lönne and Ito have previously examined the Hurwitz orbits of representations from the free group F_n to the braid group B_3 [Ito11; Lön10]. In particular, Ito provides criteria for these orbits to be finite, and upon applying the morphism $B_3 \rightarrow \text{SL}_2 \mathbb{Z}$, his examples induce genus one Lefschetz fibrations over

the disk with finite Hurwitz orbit. E.g., the Hurwitz orbit of q_3 , displayed here as Figure 2, can be found as [Ito11, Fig. 6]. Lönne’s work analyzes these finite Hurwitz orbits and derives consequences for degenerations of plane polynomials [Lön10, Theorem 8]. In particular, our results on $\text{Br}(q_n)$ for $n \leq 4$ follow from Lönne’s work.

Further Questions. We hope this paper encourages further research into $\text{Br}(\pi)$. To do so, we have compiled a list of questions. Given a Lefschetz fibration $\pi : M \rightarrow S^2$, we ask:

Question 1: When is $\text{Br}(\pi)$ finitely generated/finitely presented?

Question 2: Do there exist Lefschetz fibrations of genus g over the sphere so that $\text{Br}(\pi)$ is finite index?

Question 3: Can one classify the torsion elements of $\text{Br}(\pi)$?

Resolving Question 1 in the affirmative for genus one Lefschetz fibrations would imply that $\text{Mod}(\pi)$ is finitely generated (resp. finitely presented), as Farb–Looijenga have previously shown that in genus one $\ker(\text{Mod}(\pi) \rightarrow \text{Mod}(S^2, \Delta))$ is free abelian of rank $|\Delta| - 4$ [FL24, Theorem 1.2]. Question 3 has prospective applications to the Nielsen Realization Problem. The author’s previous paper addresses Question 3 in the case of elliptic fibrations and where the torsion elements in $\text{Br}(\pi)$ have orders $|\Delta|$, $|\Delta| - 1$, and $|\Delta| - 2$ [Jac25].

Organization of the Paper. Section 2 discusses the relationship between $\text{Br}(\pi)$, i.e., the liftable mapping classes, and the monodromy representation of a Lefschetz fibration. We also give a proof of Part (1) of Theorem 1.1 based on the connection to Lam–Landesman–Litt’s work and SL_2 -character varieties for genus one fibrations. We begin Section 3 by proving Theorem 1.1 which correspond to the $g = 1$ and self-fiber sum cases. We then explore the exceptional finite-index examples over the disk in Section 4, proving Theorem 1.3 when $n \leq 4$ and Proposition 1.4. In Section 5, we develop general criteria for showing $\text{Mod}_\pi = \text{im}(\text{Mod}(\pi) \rightarrow \text{Mod}(B, \Delta))$ has infinite index in $\text{Mod}(B, \Delta)$ for Lefschetz fibrations over arbitrary base and prove Theorem 1.3 when $n \geq 5$. Using these general criteria, we prove Theorem 1.2 in Section 6, showing that holomorphic genus $g = 2$ Lefschetz fibrations with nonseparating vanishing cycles satisfy $[\text{Mod}(S^2, \Delta) : \text{Br}(\pi)] = \infty$.

Code/Data Availability. A program in Sage/Python was implemented to do the necessary computations in Section 4. This program can be obtained from GitHub at

<https://github.com/FayeAlephNil/solomon> (††)

or, upon request, from the author.

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2. The Hurwitz Action and $\text{Br}(\pi)$

Let $\pi : M \rightarrow S$ be a genus g Lefschetz fibration⁴ over a connected oriented surface S with at most one boundary component and with no singular values along the boundary ∂S . We assume throughout that

- (1) each singular fiber of π has one critical point i.e., that the singularities are simple,
- (2) the Lefschetz fibrations are relatively minimal, i.e., no singular fiber contains a sphere of self-intersection -1 .

Let $\Delta_\pi \subseteq S \setminus \partial S$ denote the set of singular values of π . We call π *nontrivial* provided that Δ_π is nonempty. For convenience, when the fibration is clear from context, we will omit π from the notation and simply write Δ . As in [FL24] we let,

$$\text{Diff}^+(\pi) := \{F \in \text{Diff}^+(M) \mid F|_{\pi^{-1}(\partial S)} = \text{Id}, F \text{ sends fibers of } \pi \text{ to fibers of } \pi\} \quad (2.5)$$

$$\text{Mod}(\pi) := \pi_0 \text{Diff}^+(\pi). \quad (2.6)$$

There is a natural map $\text{Mod}(\pi) \rightarrow \text{Mod}(S, \Delta)$ given by tracking where F takes each fiber, notably, singular fibers must be sent to singular fibers. The image of $\text{Mod}(\pi) \rightarrow \text{Mod}(S, \Delta)$ consists precisely of those mapping classes which lift to fiber-preserving diffeomorphisms of M acting as the identity on $\pi^{-1}(\partial S)$. We call such mapping classes *liftable mapping classes*, and denote the subgroup of liftable mapping classes by

$$\text{Mod}_\pi := \text{im}(\text{Mod}(\pi) \rightarrow \text{Mod}(S, \Delta)) \quad (2.7)$$

When $S = S^2$ or $S = D^2$, we adopt the notation of the introduction

$$\text{Br}(\pi) := \text{Mod}_\pi = \text{im}(\text{Mod}(\pi) \rightarrow \text{Mod}(S^2, \Delta)) < \text{Mod}(S^2, \Delta) \quad (2.8)$$

$$\text{Br}(\pi) := \text{Mod}_\pi = \text{im}(\text{Mod}(\pi) \rightarrow \text{Mod}(D^2, \Delta)) < B_n, \quad (2.9)$$

where $n = |\Delta|$ is the number of singular fibers. In this way we think of $\text{Br}(\pi)$ as a subgroup of the (spherical) braid group which is associated to π .

We now give a description of the liftable mapping classes in terms of the monodromy of the Lefschetz fibration, essentially due to Moishezon in genus $g = 1$ and Matsumoto in genus $g \geq 2$. To do so, when $\partial S \neq \emptyset$, let $b \in \partial S$ (resp. $b \in S \setminus \Delta$ if $\partial S = \emptyset$), then let

$$\phi_\pi : \pi_1(S \setminus \Delta, b) \rightarrow \text{Mod}(\pi^{-1}(b)) \cong \text{Mod}(\Sigma_g) \quad (2.10)$$

be the monodromy representation of π , choosing some identification $\pi^{-1}(b)$ with the standard Σ_g once and for all. For later use, recall that given an arc ℓ from b to the image $p \in \Delta$ of some singular fiber there is an associated *vanishing cycle* $\alpha \subseteq \pi^{-1}(b)$, which can be represented by a simple closed curve. Furthermore, if γ is obtained by adjoining a small counterclockwise loop about p to ℓ , then $\phi_\pi(\gamma) = T_\alpha$, the (left-handed) Dehn twist about α .

Remark 2.1. The convention on monodromy adopted above (see the definition following (1.2)) differs slightly from the conventions adopted by Gompf–Stipsicz as well as Endo’s survey [GS23, pg. 291], [End21, Section 2]. In both of these sources, right-handed Dehn twists are taken instead of left-handed Dehn twists. The incompatibility with our convention can be explained by the fact that in Gompf–Stipsicz the monodromy is an *anti-homomorphism* and in Endo’s survey (as well as Moishezon’s work) the mapping class group $\text{Mod}(\Sigma_g)$ acts on the right (and hence composition runs in the opposite order).

⁴For a survey of the theory of Lefschetz fibrations we refer the reader to [End21].

The convention we adopt here has the advantage of turning ϕ_π into a genuine homomorphism. Furthermore, it allows us to directly reference the literature on mapping class groups, which for the most part uses left-handed Dehn twists (see [FM12, pg. 65]). To recover the homomorphism ϕ_π from the anti-homomorphism ϕ_π^{GS} defined by Gompf–Stipsicz, one simply applies the anti-homomorphism $\psi \mapsto \psi^{-1}$ to $\text{Mod}(\Sigma_g)$ [GS23, pg. 261]. Note that this turns right-handed Dehn twists into left-handed twists and vice versa.

Hence, over $S = S^2$, we may choose simple loops $\gamma_1, \dots, \gamma_n$ about the punctures so that $\gamma_1 \cdots \gamma_n = 1$ and the monodromy representation takes the form of a factorization of the identity

$$T_{\alpha_1} \cdots T_{\alpha_n} = \text{Id}$$

in the mapping class group $\text{Mod}(\Sigma_g)$. The tuple $(T_{\alpha_1}, \dots, T_{\alpha_n})$ is called the *monodromy factorization* associated to π .

When $\partial S \neq \emptyset$, we find that $\text{Mod}(S, \Delta)$ acts on $\pi_1(S \setminus \Delta, b)$, and hence on the set of all such representations by precomposition. When $\partial S = \emptyset$, the action of $\text{Mod}(S, \Delta)$ on π_1 is merely an outer action, and so it acts on the set of all such representations up to conjugacy. For $S = S^2$, the action by precomposition is the familiar *Hurwitz action* from the theory of Lefschetz fibrations. Equipped with this notation, we state the theorems of Moishezon/Matsumoto.

Theorem 2.2 (Moishezon [Moi06], Matsumoto [Mat96]). *Let $\pi : M \rightarrow S$ be a Lefschetz fibration of genus g over a connected oriented base S , with singular locus $\Delta \subseteq S$. Suppose that ∂S is nonempty (resp. $\partial S = \emptyset$) and has a single component, then Mod_π is the stabilizer of ϕ_π (resp. up to conjugation) under the action of $\text{Mod}(S, \Delta)$.*

Remark 2.3. Originally, Moishezon and Matsumoto’s theorems are phrased as a classification result, i.e., that a Lefschetz fibration of genus g is classified by its monodromy factorization (equivalently, its monodromy representation). The above statement is a maps version of their results, which follows immediately from their proofs. See [End21, Theorem 3.3] for a similar statement in the literature. Namely, $[f] \in \text{Stab}_{\text{Mod}(S, \Delta)}[\phi_\pi]$ if and only if the fibrations $\pi : M \rightarrow S$ and $f \circ \pi : M \rightarrow S$ have strictly equivalent monodromy representation. By Theorem 3.3 of [End21], this holds if and only if π and $f \circ \pi$ themselves are strictly equivalent. Finally, π and $f \circ \pi$ are strictly equivalent if and only if $[f] \in \text{Mod}_\pi$.

Using Theorem 2.2, Moishezon was able to classify Lefschetz fibrations in genus 1 by classifying homomorphisms $\pi_1(S^2 \setminus \Delta) \rightarrow \text{Mod}(\Sigma_1) \cong \text{SL}_2 \mathbb{Z}$ which take simple loops about the punctures to Dehn twists.

Theorem 2.4 (Moishezon [Moi06, Theorem 9]). *Any nontrivial Lefschetz fibration $\pi : M \rightarrow S^2$ of genus one has $12d$ singular fibers for some integer $d \geq 1$. Furthermore, the number of singular fibers determines π in the following sense: given genus one fibrations $\pi, \pi' : M, M' \rightarrow S^2$ with the same number of singular fibers, there are diffeomorphisms F, f making the following diagram commute*

$$\begin{array}{ccc} M & \xrightarrow{F} & M' \\ \downarrow & & \downarrow \\ (S^2, \Delta_\pi) & \xrightarrow{f} & (S^2, \Delta_{\pi'}) \end{array}$$

In general, following [End21, Definition 3.1], we call two Lefschetz fibrations $\pi, \pi' : M, M' \rightarrow S$ of genus g *weakly isomorphic* provided there exists such a pair (F, f) of diffeomorphisms so that

$$\begin{array}{ccc} M & \xrightarrow{F} & M' \\ \downarrow & & \downarrow \\ (S, \Delta_\pi) & \xrightarrow{f} & (S, \Delta_{\pi'}) \end{array}$$

commutes. Changing π by a weak isomorphism changes Mod_π by conjugation in $\text{Mod}(S, \Delta_\pi)$. Furthermore $\text{Mod}(\pi)$ consists of the isotopy classes of weak isomorphisms from π to itself.

Equipped with Theorems 2.2 and 2.4, we now deduce Part (1) of Theorem 1.1 from Lam–Landesman–Litt’s work on classifying finite mapping class group orbits of rank 2 local systems. As a corollary of their work, they show that any representation $\rho : \pi_1(S^2 \setminus \Delta) \rightarrow \text{SL}_2 \mathbb{C}$ such that

- (1) ρ has Zariski-dense image,
- (2) $\rho(\gamma)$ is infinite order for some simple loop γ about a single puncture, and
- (3) $|\Delta| \geq 7$,

has infinite orbit under the Hurwitz action of $\text{Mod}(S^2, \Delta)$, where two representations which are simultaneously conjugate over $\text{SL}_2 \mathbb{C}$ are considered equivalent [LLL23, Corollary 1.1.8]. Because the action of $\text{Mod}(\Sigma_1)$ on the homology $H_1(\Sigma_1, \mathbb{Z})$ provides an isomorphism $\text{Mod}(\Sigma_1) \xrightarrow{\cong} \text{SL}_2 \mathbb{Z}$, we find that for π a genus one Lefschetz fibration over the sphere with singular locus Δ , the subgroup $\text{Br}(\pi) = \text{Mod}_\pi$ is the stabilizer of the conjugacy class of the monodromy representation $\phi_\pi : \pi_1(S^2 \setminus \Delta) \rightarrow \text{SL}_2 \mathbb{Z}$ under the Hurwitz action. Using Moishezon’s classification theorem [Moi06, Theorem 9], we may directly verify that the monodromy ϕ_π satisfies (1)–(3) since

- (1) $\text{im}(\phi_\pi) = \text{SL}_2 \mathbb{Z}$, which is Zariski-dense in $\text{SL}_2 \mathbb{C}$,
- (2) $\phi_\pi(\gamma_1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ where γ_1 is some simple loop surrounding a single puncture, and
- (3) $(\text{SL}_2 \mathbb{Z})^{ab} \cong \mathbb{Z}/12\mathbb{Z}$ and so $|\Delta|$ is divisible by 12 (hence at least 7).

Therefore the conjugacy class $[\phi_\pi]$ (under conjugation by $\text{SL}_2 \mathbb{C}$) has infinite Hurwitz orbit. By Theorem 2.2, $\text{Br}(\pi)$ has infinite index if and only if the orbit of $[\phi_\pi]$ (under conjugation by $\text{SL}_2 \mathbb{Z}$) is infinite. But the latter condition is weaker, and so we have proved Part (1) of Theorem 1.1.

Remark 2.5. This proof may alternatively be phrased in terms of character varieties and local systems. Let

$$Y_\Delta(\mathbb{Z}) := \text{Hom}(\pi_1(S^2 \setminus \Delta), \text{SL}_2 \mathbb{Z}) / \text{SL}_2 \mathbb{Z},$$

$$Y_\Delta(\mathbb{C}) := \text{Hom}(\pi_1(S^2 \setminus \Delta), \text{SL}_2 \mathbb{C}) / \text{SL}_2 \mathbb{C},$$

where $\text{SL}_2 \mathbb{Z}$ and $\text{SL}_2 \mathbb{C}$ act by conjugation. $Y_\Delta(\mathbb{C})$ parameterizes rank two local systems on $S^2 \setminus \Delta$ with trivial determinant bundle, and the GIT quotient

$$\text{Hom}(\pi_1(S^2 \setminus \Delta), \text{SL}_2 \mathbb{C}) // \text{SL}_2 \mathbb{C}$$

is the character variety for such local systems. Because the natural map $Y_\Delta(\mathbb{Z}) \rightarrow Y_\Delta(\mathbb{C})$ is $\text{Mod}(S^2, \Delta)$ -equivariant, it follows immediately that $\text{Br}(\pi)$ has infinite index inside $\text{Mod}(S^2, \Delta)$ provided that the $\mathbb{C}$ -linear local system has infinite mapping class group orbit.

3. Index of the Braid Monodromy

Let $\pi : M \rightarrow S$ be a Lefschetz fibration of genus g with Δ its singular locus. As previously discussed, π induces a subgroup $\text{Mod}_\pi < \text{Mod}(S, \Delta)$ of mapping classes which lift to a fiber-preserving diffeomorphism of M (see eq. (2.7)). At this point, we are equipped to prove Theorem 1.1. For convenience, we restate part (2) of Theorem 1.1 here.

Theorem 3.1. *Let $\pi : M \rightarrow S^2$ be a genus $g \geq 1$ Lefschetz fibration over the sphere with singular locus $\Delta_\pi \subseteq S^2$. Then for the fiber-sum $\pi \#_F \pi : M \#_F M \rightarrow S^2$, $\text{Br}(\pi \#_F \pi)$ has infinite index in $\text{Mod}(S^2, \Delta_{\pi \#_F \pi})$.*

Proof. Let $\delta_1, \dots, \delta_n$ be vanishing cycles corresponding to the monodromy factorization $T_{\delta_1} \cdots T_{\delta_n} = \text{Id}$ of π . Furthermore let $\text{Mon}(\pi) := \langle T_{\delta_1}, \dots, T_{\delta_n} \rangle$. We apply Auroux’s lemma, which shows that if $T_{\delta_1}, \dots, T_{\delta_n}$

is a factorization of a central element in $\text{Mod}(\Sigma_g)$ then $T_{\delta_1}, \dots, T_{\delta_n}$ is Hurwitz equivalent to $T_{f\delta_1}, \dots, T_{f\delta_n}$ for any $f \in \text{Mon}(\pi)$ [Aur05, Lemma 6(b)]. Since $T_{\delta_1} \cdots T_{\delta_n} = \text{Id}$, there must be some pair δ_i, δ_j so that $i(\delta_i, \delta_j) \geq 1$, as otherwise these Dehn twists would generate a free abelian group. Let $f_k = T_{\delta_i}^k \in \text{Mon}(\pi)$, then by Auroux's lemma there is a Hurwitz equivalence

$$T_{\delta_1}, \dots, T_{\delta_n}, T_{\delta_1}, \dots, T_{\delta_n} \sim T_{\delta_1}, \dots, T_{\delta_n}, T_{f_k \delta_1}, \dots, T_{f_k \delta_n}$$

where $T_{\delta_1}, \dots, T_{\delta_n}, T_{\delta_1}, \dots, T_{\delta_n}$ is the monodromy factorization corresponding to $\pi \#_F \pi$. Thus each of these factorizations lie in the same Hurwitz orbit. To see that $\text{Br}(\pi \#_F \pi)$ has infinite index in $\text{Mod}(S^2, \Delta_{\pi \#_F \pi})$ it suffices to show that the Hurwitz orbit of $\phi_{\pi \#_F \pi}$ is infinite up to simultaneous conjugacy by Theorem 2.2. A conjugation by $h \in \text{Mod}(\Sigma_g)$ will not change the intersection number of vanishing cycles, and hence two such factorizations are distinguished by $i(\delta_j, T_{\delta_i}^k \delta_j) = |k| i(\delta_i, \delta_j)^2$ [FM12, Proposition 3.2]. $\square$

Part (1) of Theorem 1.1, restated here for convenience, follows as a corollary.

Corollary 3.2. *For any nontrivial genus one Lefschetz fibration $\pi : M \rightarrow S^2$, $\text{Br}(\pi)$ has infinite index in $\text{Mod}(S^2, \Delta)$, where $\Delta \subseteq S^2$ is the singular locus.*

Proof. Genus one Lefschetz fibrations are classified as iterated fiber sums of the standard elliptic fibration $E(1)$ with 12 nodal fibers⁵ due to Moishezon (see Theorem 2.4). Thus, it suffices to verify the result for $E(1)$. The monodromy of $E(1)$ generates $\text{Mod}(T^2) \cong \text{SL}_2 \mathbb{Z}$. Furthermore, identifying $H_1(T^2)$ with $\mathbb{Z}^2$ we may fix simple closed curves α, β in the homology classes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. Under this identification, the monodromy factorization of $E(1)$ is $T_\alpha, T_\beta, \dots, T_\alpha, T_\beta$ of length 12. Note that Auroux's lemma applies for the subfactorization $(T_\alpha T_\beta)^3$ which acts as $-\text{Id}$ on $H_1(T^2; \mathbb{Z})$, and hence is central in $\text{Mod}(T^2) \cong \text{Aut}(H_1(T^2; \mathbb{Z}))$. The proof of Theorem 3.1 thus generalizes exactly to this setting. $\square$

Remark 3.3. The proof of Theorem 3.1 given above is inspired by a proof of Lee and Serván. They show that for any Lefschetz fibration π of genus $g \geq 2$ over the sphere the fiber sum $\pi \#_F \pi$ admits infinitely many homologically distinct sections [LS24].

In Section 5, we will extend Theorem 3.1 to base surfaces with genus and/or boundary. Before doing so, we will first examine some exceptions where $\text{Br}(\pi)$ is finite index over the disk.

4. Finite-Index Exceptions

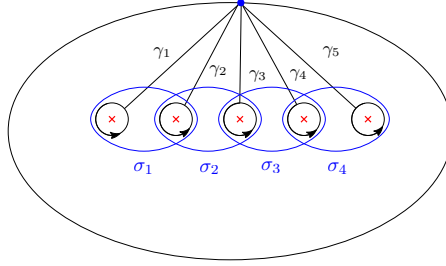
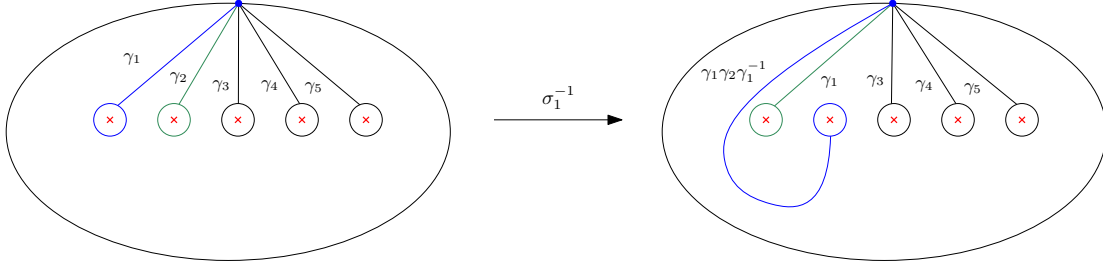
Let D be the unit disk with boundary, D_n the disk with n punctures, and fix some base point $b \in \partial D$. Furthermore let α, β be simple closed curves on the torus T^2 in the homology classes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under some identification $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$. Consider the family $q_n : M_n \rightarrow D$ of Lefschetz fibrations given by the monodromy representations

$$\phi_n : \pi_1(D_n, b) \rightarrow \text{Mod}(T^2) \cong \text{SL}_2 \mathbb{Z}$$

$$\begin{aligned} \gamma_{2i+1} &\mapsto T_\alpha \cong \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ \gamma_{2i} &\mapsto T_\beta \cong \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \end{aligned}$$

where γ_i is a standard system of generators for $\pi_1(D_n, b)$, i.e., each γ_i surrounds a single puncture counterclockwise and their product is homotopic to the boundary ∂D_n (see Figure 3). A representation $\phi : \pi_1(D_n, b) \rightarrow \text{Mod}(T^2)$ arising from a genus one fibration can be identified with a tuple $(\phi(\gamma_i))_{i=1}^n =$

⁵For a definition of $E(1)$, see [GS23, §3.1]

FIGURE 3. A standard set of generators for B_n and $\pi_1(D_n, b)$ FIGURE 4. The Hurwitz action of σ_1 , visualized as the action of σ_1^{-1} on $\pi_1(D_n, b)$

$(T_{\delta_1}, \dots, T_{\delta_n})$ of Dehn twists. The braid group $B_n = \text{Mod}(D_n, \partial D_n)$ acts by Hurwitz moves on the representation ϕ by precomposing with its action on π_1 (after an inversion to make it a left action). Fix generators σ_i of B_n given by half-twists inside the interior of a simple closed curve representing $\gamma_i \gamma_{i+1}$ (see Figure 3 and Figure 4). The Hurwitz action of σ_i on such a representation ϕ (given as a tuple of Dehn twists) is

$$\sigma_i : (T_{\delta_1}, \dots, T_{\delta_n}) \mapsto (T_{\delta_1}, \dots, T_{\delta_i} T_{\delta_{i+1}} T_{\delta_i}^{-1}, T_{\delta_i}, \dots, T_{\delta_n}) = (T_{\delta_1}, \dots, T_{T_{\delta_i} \delta_{i+1}}, T_{\delta_i}, \dots, T_{\delta_n}). \quad (4.11)$$

Given the setup above, we show Theorem 1.3, restated here for convenience.

Theorem 4.1. *For $n \leq 4$, $\text{Br}(q_n)$ is finite index in B_n , with indices*

$$[B_2 : \text{Br}(q_2)] = 3 \qquad [B_3 : \text{Br}(q_3)] = 8 \qquad [B_4 : \text{Br}(q_4)] = 27.$$

For $n \geq 5$, $\text{Br}(q_n)$ has infinite index in B_n .

We delay the proof of the second statement until Section 5, where we will prove a more general result.

Proof when $n \leq 4$. For $n \leq 4$, it suffices to compute the Hurwitz orbit. One can do so explicitly by constructing a directed graph whose edges are generators and whose vertices are representations of $\pi_1(D_n) \cong F_n$. The orbit is complete provided that the in/out-degree of each vertex is $n - 1$, the number of generators of B_n . This computation is performed in Sage/Python, the code for which can be found at (††). These orbits are displayed in Figure 2 for $n = 3, 4$, the orbit for $n = 2$ is simply a triangle. $\square$

We will show in Section 5 that $[B_n : \text{Br}(q_n)]$ is infinite for $n \geq 5$. In this section we dedicate ourselves to the analysis of these finite-index exceptions where $n \leq 4$. Let σ_{ij} be the braid exchanging puncture i and j along the interiors of γ_i and γ_j , as usual we abbreviate $\sigma_i = \sigma_{i(i+1)}$. Explicitly

$$\sigma_{ij} := (\sigma_{j-2} \cdots \sigma_i)^{-1} \sigma_{j-1} (\sigma_{j-2} \cdots \sigma_i),$$

when $j \geq i + 2$. If $j < i$ we set $\sigma_{ij} = \sigma_{ji}^{-1}$ and if $i = j$ then $\sigma_{ii} = \text{Id}$. It is well-known that $\sigma_{ij} \in \text{Br}(q_n)$ when $i \equiv j \pmod{2}$, and that $\sigma_{ij}^3 \in \text{Br}(q_n)$ when $i \not\equiv j \pmod{2}$ (see [FL24, Remark 5.4] and [Lön06, Main

Theorem]). Using the orbit graphs in Figure 2, one can find representatives for the cosets of $\text{Br}(q_n)$ in B_n for $n \leq 4$, and subsequently compute a presentation for $\text{Br}(q_n)$ in GAP for $n \leq 4$ [GAP24; Bar24].

Proposition 4.2. *$\text{Br}(q_3)$ and $\text{Br}(q_4)$ are generated as follows:*

$$\text{Br}(q_3) = \langle \sigma_1^3, \sigma_{13} \rangle \quad \text{Br}(q_4) = \langle \sigma_1^3, \sigma_{13}, \sigma_{24} \rangle. \quad (4.12)$$

As a consequence, one finds that the abelianizations $\text{Br}(q_3)^{ab}, \text{Br}(q_4)^{ab}$ are free abelian of rank 2 and that $[PB_n : PB_n \cap \text{Br}(q_n)] = [B_n : \text{Br}(q_n)]$ for $n \leq 4$ (here PB_n is the pure braid group). Moreover, $\text{Br}(q_3)$ admits the presentation

$$\text{Br}(q_3) = \langle \sigma_1^3, \sigma_{13} \mid (\sigma_{13}\sigma_1^3)^3 = (\sigma_1^3\sigma_{13})^3 \rangle.$$

Remark 4.3. The analogous presentation of $\text{Br}(q_4)$ is quite complex, requiring 13 relations where the maximum length relation has word length 36.

Proposition 1.4 from the introduction is simply the first part of Proposition 4.2. Notably, these generating sets agree with the generating sets derived in the algebraic category by Lönne. Having analyzed the finite-index exceptions, we turn our attention back to the general case.

5. General Criteria

We'll now devote ourselves to extending Theorem 3.1 over arbitrary compact connected oriented base surfaces B . We develop multiple criteria guaranteeing that $\text{Mod}_\pi = \text{im}(\text{Mod}(\pi) \rightarrow \text{Mod}(B, \Delta))$ has infinite index for a Lefschetz fibration $\pi : M \rightarrow B$ with singular fibers along $\Delta \subseteq B \setminus \partial B$. To do so succinctly, we require some additional language.

Definition 5.1 (Moishezon spider). Let $\pi : M \rightarrow B$ be a Lefschetz fibration of genus g over a compact connected oriented base B with singular locus Δ_π . A *Moishezon spider* for π consists of the following data

- (1) An embedded disk D in B , so that $\partial D \cap \Delta_\pi = \emptyset$.
- (2) A basepoint $b \in \partial D$, where if B has boundary we require $b \in \partial D \cap \partial B$.
- (3) A collection of arcs $a_1, \dots, a_k$ in D beginning at b and ending at distinct singular values $p_1, \dots, p_k \in \text{int}(D) \cap \Delta_\pi$ of π , disjoint from ∂D except at b . The arcs $a_1, \dots, a_k$ are ordered counterclockwise by how they meet a sufficiently small neighborhood of b .

The *topological type* of a Moishezon spider is the collection of vanishing cycles $(\delta_1, \dots, \delta_k)$ associated to $a_1, \dots, a_k$ via π in the fiber $\Sigma_g \cong \pi^{-1}(b)$ and is well-defined up to the action of $\text{Mod}(\Sigma_g)$. The arcs $a_1, \dots, a_k$ are called the *legs of the spider*.

Our first result concerns spiders of type (δ_1, δ_2) where the geometric intersection number $i(\delta_1, \delta_2) \geq 2$.

Lemma 5.2. *Suppose that a genus g Lefschetz fibration $\pi : M \rightarrow B$ over a compact connected oriented surface B with one boundary component admits a Moishezon spider with topological type (δ_1, δ_2) , where $i(\delta_1, \delta_2) \geq 2$. Then Mod_π has infinite index in $\text{Mod}(B, \Delta_\pi)$.*

Proof. Let $b \in \partial B$ and take $\gamma_1, \gamma_2 \in \pi_1(B \setminus \Delta_\pi, b)$ simple loops about critical values p_1, p_2 of π with vanishing cycles δ_1, δ_2 respectively. A monodromy factorization W of π thus contains $W' = T_{\delta_1}T_{\delta_2}$ as a subword. Let $\sigma \in \text{Mod}(B, \Delta_\pi)$ be the braid which exchanges p_1, p_2 and is supported inside the disk D defining the spider. More precisely, σ is the image of the generator of $\text{Mod}(D, \{p_1, p_2\}, \partial D) \cong B_2 \cong \mathbb{Z}$ where D is the embedded disk of the spider under the inclusion of D into B . We show that $\sigma^k \cdot W$ are distinct for all k , and hence the

Hurwitz orbit is infinite, proving the lemma. To do so, note that σ acts only on the subword W' . That is σ acts on the representation

$$\phi' = \phi|_{\langle \gamma_1, \gamma_2 \rangle} : \langle \gamma_1, \gamma_2 \rangle \rightarrow \langle T_{\delta_1}, T_{\delta_2} \rangle,$$

where $\phi : \pi_1(B \setminus \Delta_\pi, b) \rightarrow \text{Mod}(\Sigma_g)$ is the monodromy representation of π . By assumption, $i(T_{\delta_1}, T_{\delta_2}) \geq 2$ and so $\langle T_{\delta_1}, T_{\delta_2} \rangle \cong F_2$, the free group on two elements [FM12, Theorem 3.14]. Let $\eta : F_2 \rightarrow F_2$ be the automorphism

$$\eta(T_{\delta_1}) = T_{\delta_1} T_{\delta_2} T_{\delta_1}^{-1} \quad \eta(T_{\delta_2}) = T_{\delta_1}.$$

We find that by definition of the Hurwitz action $\sigma \cdot \phi' = \eta \circ \phi'$. Hence $\sigma^k \cdot \phi = \sigma^j \cdot \phi$ if and only if $\eta^k \circ \phi' = \eta^j \circ \phi'$. Since ϕ' is surjective, this implies that $\eta^{k-j} = \text{Id}$. But η has infinite order, since the Artin representation $B_2 \rightarrow \text{Aut}(F_2)$ mapping σ to η is faithful [Art47, Theorem 14]. $\square$

The proof generalizes directly to the following lemma.

Lemma 5.3. *Let $\pi : M \rightarrow B$ be a Lefschetz fibration over a compact connected, oriented base B with at most one boundary component containing a Moishezon spider of type $(\delta_1, \delta_2, \delta_1, \delta_2)$ or $(\delta_1, \delta_2, \delta_2, \delta_1)$ over singular values p_1, p_2, p_3, p_4 such that $i(\delta_1, \delta_2) \geq 2$. Then Mod_π has infinite index.*

Proof. It suffices to show the Hurwitz orbit of the monodromy factorization is infinite up to global conjugation. Suppose π contains a spider of type $(\delta_1, \delta_2, \delta_1, \delta_2)$ with $i(\delta_1, \delta_2) \geq 2$. We apply the proof of Lemma 5.2 where σ exchanges p_1, p_2 and note that if $\sigma^k \cdot \phi$ is conjugate to $\sigma^j \cdot \phi$ by $f \in \text{Mod}(\Sigma_g)$, then $f(\delta_1) = \delta_1, f(\delta_2) = \delta_2$. Hence f commutes with $\langle T_{\delta_1}, T_{\delta_2} \rangle$, and so $\sigma^k \cdot \phi = f \sigma^j \phi f^{-1}$ if and only if $\eta^k \circ \phi' = \eta^j \circ \phi'$ as before. The proof when π contains a spider of type $(\delta_1, \delta_2, \delta_2, \delta_1)$ is completely analogous. $\square$

Lemma 5.4. *Suppose a genus g Lefschetz fibration $\pi : M \rightarrow B$ over a compact, connected oriented base B with at most one boundary component contains a Moishezon spider of type $(\alpha, \beta, \alpha, \beta, \alpha)$ over singular values $p_1, \dots, p_5$ where $i(\alpha, \beta) = 1$. Then π contains a Moishezon spider of type $(T_\alpha \beta, T_\beta \alpha)$.*

Proof. Let $\gamma_1, \dots, \gamma_5$ be simple loops from $b \in B$ (if B has boundary, take $b \in \partial B$) surrounding $p_1, \dots, p_5$ respectively so that the monodromy about γ_{2i+1} is T_α and the monodromy about γ_{2i} is T_β . Take a small disk containing $\gamma_1, \dots, \gamma_5$ and $p_1, \dots, p_5$ embedded in B . Then the braid group B_5 acts on the monodromy restricted to $\langle \gamma_1, \dots, \gamma_5 \rangle$ (equivalently, the subword of the monodromy factorization).

We then perform a local calculation over this disk and the torus-with-boundary Σ_1^1 formed from a neighborhood of α, β in the fiber Σ_g over b . The allowed moves are to apply the braid σ_i to the vanishing cycles $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ and obtain

$$\sigma_i : \delta_1, \dots, \delta_i, \delta_{i+1}, \dots, \delta_5 \mapsto \delta_1, \dots, T_{\delta_i} \delta_{i+1}, \delta_i, \dots, \delta_5,$$

and the inverse. We apply the word $\sigma_2 \sigma_1^2 \sigma_4$

$$\begin{aligned} (\alpha, \beta, \alpha, \beta, \alpha) &\xrightarrow{\sigma_4} (\alpha, \beta, \alpha, T_\beta \alpha, \beta) \\ &\xrightarrow{\sigma_1^2} (T_\alpha T_\beta T_\alpha^{-1} \alpha, T_\alpha \beta, \alpha, T_\beta \alpha, \beta) = (\beta, T_\alpha \beta, \alpha, T_\beta \alpha, \beta) \\ &\xrightarrow{\sigma_2} (\beta, \beta, T_\alpha \beta, T_\beta \alpha, \beta). \end{aligned}$$

Standard computations then show that $i(T_\alpha \beta, T_\beta \alpha) = 2$. See Figure 5 for a purely visual computation of $\sigma_2 \sigma_1^2 \sigma_4$ applied to the monodromy factorization—note that, in Figure 5, the curves are labeled $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ in each step to keep track of their indexing. Alternatively, one can identify $\text{Mod}(\Sigma_1^1)$ with B_3 to perform the calculation [FM12, Section 3.6.4]. This completes the proof. $\square$

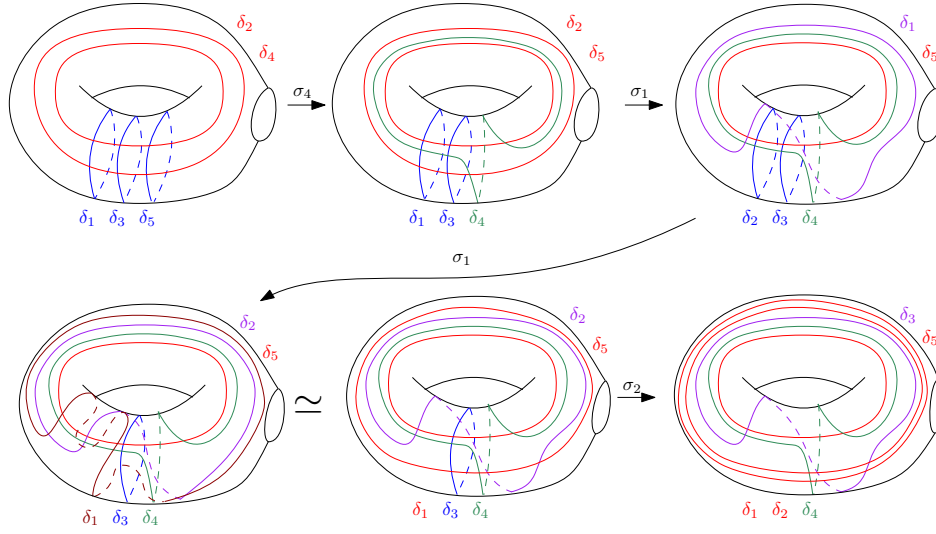


FIGURE 5. The Local Calculation of Lemma 5.4.

Remark 5.5. Direct calculation shows that $i(T_\alpha\beta, T_\beta\alpha) = 2$. We often use Lemma 5.4 to construct Moishezon spiders to apply Lemmas 5.2 and 5.3.

Combining Lemma 5.2 and Lemma 5.4 yields the following corollary promised in Section 4. Note that this is a restatement of Theorem 4.1 for $n \geq 5$.

Corollary 5.6. *Let $n \geq 5$ and $q_n : M_n \rightarrow D^2$ be the genus one Lefschetz fibration with monodromy*

$$\phi_n : \pi_1(D^2 \setminus \{p_1, \dots, p_n\}) \rightarrow \mathrm{SL}_2 \mathbb{Z}$$

$$\phi_n(\gamma_{2i+1}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \phi_n(\gamma_{2i}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then

$$[B_n : \mathrm{Br}(q_n)] = \infty$$

Lemmas 5.3 and 5.4 also yield a new proof of Part (1) of Theorem 1.1.

Alternate proof of Part (1) of Theorem 1.1. Via Moishezon's Theorem [Moi06], any genus one Lefschetz fibration contains a Moishezon spider of type

$$(\alpha, \beta, \dots, \alpha, \beta)$$

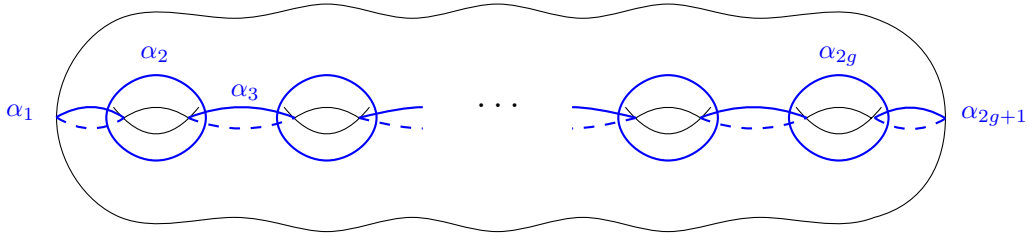
with 12 legs, such that $i(\alpha, \beta) = 1$. One then applies Lemma 5.4 twice before applying Lemma 5.3. $\square$

Finally, we obtain a mild generalization of Part (2) of Theorem 1.1 to arbitrary base.

Corollary 5.7. *Let $\pi : M \rightarrow B$ be a nontrivial Lefschetz fibration of genus g , where B is any connected, oriented surface, which admits a Moishezon spider of topological type (α, β) so that $i(\alpha, \beta) \neq 0$, then $\mathrm{Mod}_{\pi \#_F \pi \#_F \pi \#_F \pi}$ has infinite index in $\mathrm{Mod}(B \# B \# B \# B, \Delta_{\pi \#_F \dots \#_F \pi})$.*

The proof is similar to the proof above of Part (1) of Theorem 1.1. If $i(\alpha, \beta) \geq 2$, one may invoke Lemma 5.3 directly. Otherwise, three of the fiber sums are used to produce the five vanishing cycles of Lemma 5.4. The remaining two vanishing cycles serve as a marking.

⁶As one can guess, this is much easier in practice than directly computing with curves. In particular, one does not make so many orientation mistakes performing Dehn twists.

FIGURE 6. Curves $\alpha_1, \dots, \alpha_{2g+1}$ on Σ_g

6. Proof of Theorem 1.2

Let Σ_g be a connected oriented surface of genus $g \geq 2$. Fix curves $\alpha_1, \dots, \alpha_{2g+1}$ on Σ_g as in Figure 6. With $a_i = T_{\alpha_i}$, the following are monodromy factorizations in $\text{Mod}(\Sigma_g)$

$$(a_1 \cdots a_{2g+1}^2 \cdots a_2 a_1)^2 = 1 \quad (6.13)$$

$$(a_1 \cdots a_{2g})^{2(2g+1)} = 1 \quad (6.14)$$

$$(a_1 \cdots a_{2g+1})^{2g+2} = 1, \quad (6.15)$$

see [FM12, Proposition 4.12, §5.1.4]. Each of these monodromy factorizations gives rise to a Lefschetz fibration:

$$\eta_1 : X(1) \rightarrow S^2 \quad (6.16)$$

$$\eta_2 : X(2) \rightarrow S^2 \quad (6.17)$$

$$\eta_3 : X(3) \rightarrow S^2, \quad (6.18)$$

where $X(1), X(2), X(3)$ are the total spaces of these fibrations. Note that the choices of η_i are determined by the factorizations (6.13), (6.14), (6.15) by work of Matsumoto [Mat96]. Theorem 1.2 follows from the following theorem.

Theorem 6.1. *For any genus $g \geq 2$, each of the three Lefschetz fibrations $\eta = \eta_1, \eta_2, \eta_3$ satisfies*

$$[\text{Mod}(S^2, \Delta_\eta) : \text{Br}(\eta)] = \infty.$$

We first sketch the implication of Theorem 1.2 from Theorem 6.1.

Theorem 6.1 implies Theorem 1.2. Let $\pi : M \rightarrow S^2$ be a nontrivial genus $g = 2$ holomorphic Lefschetz fibration whose vanishing cycles are all nonseparating. By Chakiris' classification⁷, π must be a fiber sum $\eta_1^{\#_F k_1} \#_F \eta_2^{\#_F k_2} \#_F \eta_3^{\#_F k_3}$, where $\eta_i^{\#_F k}$ denotes the fiber sum of η_i with itself k times [Cha83, Main Theorem]. Let $\rho_\pi : \pi_1(S^2 \setminus \Delta_\pi) \rightarrow \text{Mod}(\Sigma_g)$ be the monodromy representation of π . Because π is a fiber sum, the monodromy factorization for π contains some subword equivalent to a monodromy factorization for η_i for some $1 \leq i \leq 3$. By restricting our attention to this subword, we find that the Hurwitz orbit of $[\rho_\pi]$ contains an isomorphic copy of the Hurwitz orbit for $[\eta_i]$, which is infinite by Theorem 6.1. Hence the Hurwitz orbit of $[\rho_\pi]$ is infinite. The result follows as $\text{Br}(\pi) = \text{Stab}[\rho_\pi]$ by Theorem 2.2. $\square$

With the implication above in hand, we conclude by proving Theorem 6.1.

Proof of Theorem 6.1. Let

$$\rho : \pi_1(S^2 \setminus \Delta_{\eta_1}) \rightarrow \text{Mod}(\Sigma_g)$$

⁷The classification of Chakiris was independently discovered by Ivan Smith [Smi99, Theorem 5.5].

be the monodromy representation of the Lefschetz fibration η_1 . It suffices to show that the conjugacy class $[\rho]$ has infinite orbit under the Hurwitz action of $\text{Mod}(S^2, \Delta_{\eta_1})$ by Matsumoto's theorem (see Theorem 2.2 above). By inspection of the monodromy factorization (6.13), η_1 contains a Moishezon spider of type

$$(\delta_1, \dots, \delta_8) = (\alpha_1, \alpha_2, \alpha_2, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_1), \quad (6.19)$$

where $i(\alpha_1, \alpha_2) = 1$. By applying the braid $\sigma_1^2 \sigma_6^{-1} \sigma_7$ to (6.19) a new Moishezon spider of type

$$(\delta'_1, \dots, \delta'_8) = (\alpha_2, T_{\alpha_1} \alpha_2, \alpha_2, \alpha_1, \alpha_1, T_{\alpha_2} \alpha_1, T_{\alpha_2} T_{\alpha_1}^{-1}(\alpha_2), \alpha_2), \quad (6.20)$$

where notably $i(\alpha_2, \delta'_7) = 1$ (working on the torus with boundary as a local model). Let $\gamma_1, \dots, \gamma_8$ be simple loops about single punctures with monodromies $\rho(\gamma_i) = T_{\delta'_i}$, the Dehn twist about δ'_i . Let σ be the braid exchanging the punctures corresponding to γ_2, γ_6 , that is

$$\sigma = (\sigma_4 \sigma_3 \sigma_2)^{-1} \sigma_5 (\sigma_4 \sigma_3 \sigma_2).$$

It suffices to show that if $\sigma^k \cdot \rho$ is conjugate to $\sigma^j \cdot \rho$ then $k = j$. Suppose there exists some $f \in \text{Mod}(\Sigma_g)$ so that

$$f(\sigma^k \cdot \rho) f^{-1} = \sigma^j \cdot \rho. \quad (6.21)$$

Evaluating both sides of (6.21) on γ_1, γ_7 yields

$$f T_{\alpha_2} f^{-1} = f \rho(\gamma_1) f^{-1} = \rho(\gamma_1) = T_{\alpha_2} \quad f T_{\delta'_7} f^{-1} = f \rho(\gamma_7) f^{-1} = \rho(\gamma_7) = T_{\delta'_7},$$

since σ is supported away from γ_1, γ_7 . Note that $\langle T_{\alpha_1}, T_{\alpha_2} \rangle = \langle T_{\alpha_2}, T_{\delta'_7} \rangle$ since these both generate the mapping class group of the torus-with-boundary formed from a tubular neighborhood of α_1, α_2 . Now consider the restrictions

$$f(\sigma^k \cdot \rho) f^{-1}|_{\langle \gamma_2, \gamma_6 \rangle} = (\sigma^j \cdot \rho)|_{\langle \gamma_2, \gamma_6 \rangle} : \langle \gamma_2, \gamma_6 \rangle \rightarrow \langle T_{T_{\alpha_1} \alpha_2}, T_{T_{\alpha_2} \alpha_1} \rangle \cong F_2,$$

where the final isomorphism holds because $i(T_{\alpha_1} \alpha_2, T_{\alpha_2} \alpha_1) \geq 2$ [FM12, Theorem 3.14]. Applying the equality $T_{T_{\alpha_i} \alpha_j} = T_{\alpha_i} T_{\alpha_j} T_{\alpha_i}^{-1}$ and the fact that f acts by the identity via conjugation on $\langle T_{\alpha_1}, T_{\alpha_2} \rangle$ yields that

$$\sigma^k \cdot \rho|_{\langle \gamma_2, \gamma_6 \rangle} = \sigma^j \cdot \rho|_{\langle \gamma_2, \gamma_6 \rangle} : \langle \gamma_2, \gamma_6 \rangle \rightarrow F_2.$$

As in the proof of Lemma 5.2, the equality $\sigma^k \cdot \rho = \sigma^j \cdot \rho$ must then imply that $k = j$, by the faithfulness of the Artin representation $B_2 \rightarrow \text{Aut}(F_2)$ [Art47, Theorem 14]. Thus $[\rho]$ has infinite Hurwitz orbit, and $[\text{Mod}(S^2, \Delta_{\eta_1}) : \text{Br}(\eta_1)] = \infty$.

We now proceed to η_2 and η_3 respectively. By inspection of the monodromy factorization (6.14) (resp. (6.15)), we see that η_2 (resp. η_3) contains a Moishezon spider of type

$$(\delta_1, \dots, \delta_{10}) = (\alpha_1, \alpha_2, \alpha_1, \alpha_2, \dots, \alpha_1, \alpha_2).$$

Applying Lemma 5.4 twice yields a Moishezon spider of type

$$(T_{\alpha_1} \alpha_2, T_{\alpha_2} \alpha_1, T_{\alpha_2} \alpha_1, T_{\alpha_1} \alpha_2).$$

Therefore, Lemma 5.3 implies that $\text{Br}(\eta_2)$ (resp. $\text{Br}(\eta_3)$) has infinite index inside of $\text{Mod}(S^2, \Delta_{\eta_2})$ (resp. $\text{Mod}(S^2, \Delta_{\eta_3})$). $\square$

References

- [Art47] E. Artin. “Theory of Braids”. In: *Annals of Mathematics* 48.1 (1947), pp. 101–126. ISSN: 0003-486X. DOI: [10.2307/1969218](https://doi.org/10.2307/1969218). JSTOR: [1969218](https://www.jstor.org/stable/1969218). (Visited on 01/16/2025).
- [Aur05] D. Auroux. “A Stable Classification of Lefschetz Fibrations”. In: *Geometry & Topology* 9.1 (Jan. 2005), pp. 203–217. ISSN: 1465-3060, 1364-0380. DOI: [10.2140/gt.2005.9.203](https://doi.org/10.2140/gt.2005.9.203). (Visited on 05/04/2025).

- [Bar24] L. Bartholdi. *FR, Computations with Functionally Recursive Groups, Version 2.4.13*. Jan. 2024.
- [BGS16] J. Bourgain, A. Gamburd, and P. Sarnak. “Markoff Triples and Strong Approximation”. In: *Comptes Rendus Mathématique* 354.2 (Feb. 2016), pp. 131–135. ISSN: 1631-073X. DOI: [10.1016/j.crma.2015.12.006](https://doi.org/10.1016/j.crma.2015.12.006). (Visited on 08/11/2025).
- [Cha83] K. N. Chakiris. “The monodromy of genus two pencils”. PhD thesis. Columbia University, 1983.
- [End21] H. Endo. “Lefschetz Fibrations”. In: *Sugaku Expositions* 34.2 (Dec. 2021), pp. 175–204. ISSN: 0898-9583, 2473-585X. DOI: [10.1090/suga/462](https://doi.org/10.1090/suga/462). (Visited on 08/19/2024).
- [FL24] B. Farb and E. Looijenga. *The Smooth Mordell-Weil Group and Mapping Class Groups of Elliptic Surfaces*. May 2024. DOI: [10.48550/arXiv.2403.15960](https://doi.org/10.48550/arXiv.2403.15960). arXiv: [2403.15960](https://arxiv.org/abs/2403.15960) [math]. (Visited on 05/20/2024).
- [FM12] B. Farb and D. Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton University Press, 2012. ISBN: 978-0-691-14794-9. JSTOR: [j.ctt7rkjw](https://www.jstor.org/stable/j.ctt7rkjw). (Visited on 03/30/2024).
- [GAP24] GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.14.0*. Manual. 2024.
- [GS23] R. E. Gompf and A. I. Stipsicz. *4-Manifolds and Kirby Calculus*. American Mathematical Society, Aug. 2023. ISBN: 978-1-4704-7455-3.
- [Ito11] T. Ito. “Finite Orbits of Hurwitz Actions on Braid Systems”. In: *Osaka Journal of Mathematics* 48.3 (Sept. 2011), pp. 613–632. ISSN: 0030-6126. (Visited on 05/08/2025).
- [Jac25] F. Jackson. *Torsion in the Braid Monodromy of Elliptic Fibrations*. 2025. arXiv: [2509.14454](https://arxiv.org/abs/2509.14454) [math.GT]. URL: <https://arxiv.org/abs/2509.14454>.
- [LLL23] Y. H. J. Lam, A. Landesman, and D. Litt. *Finite Braid Group Orbits on SL_2 -Character Varieties*. Aug. 2023. DOI: [10.48550/arXiv.2308.01376](https://doi.org/10.48550/arXiv.2308.01376). arXiv: [2308.01376](https://arxiv.org/abs/2308.01376) [math]. (Visited on 01/08/2025).
- [Lön06] M. Lönne. “On Bifurcation Braid Monodromy of Elliptic Fibrations”. In: *Topology* 45.4 (July 2006), pp. 785–806. ISSN: 0040-9383. DOI: [10.1016/j.top.2006.03.001](https://doi.org/10.1016/j.top.2006.03.001). (Visited on 04/16/2025).
- [Lön10] M. Lönne. “GEOMETRIC AND ALGEBRAIC SIGNIFICANCE OF SOME HURWITZ STABILISERS IN Brn”. In: *International Electronic Journal of Algebra* 8.8 (2010), pp. 1–17. URL: <https://izlik.org/JA95AB87JU>.
- [LS24] S. E. B. Lee and C. A. Serván. *Lefschetz Fibrations with Infinitely Many Sections*. Sept. 2024. DOI: [10.48550/arXiv.2409.15265](https://doi.org/10.48550/arXiv.2409.15265). arXiv: [2409.15265](https://arxiv.org/abs/2409.15265) [math]. (Visited on 05/06/2025).
- [Mat96] Y. Matsumoto. “Lefschetz Fibrations of Genus Two - a Topological Approach”. In: *Topology and Teichmüller Spaces*. WORLD SCIENTIFIC, Nov. 1996, pp. 123–148. ISBN: 978-981-02-2686-2. DOI: [10.1142/9789814503921_0008](https://doi.org/10.1142/9789814503921_0008). (Visited on 01/18/2025).
- [Moi06] B. Moishezon. *Complex Surfaces and Connected Sums of Complex Projective Planes*. Springer, Nov. 2006. ISBN: 978-3-540-37276-9.
- [Smi99] I. Smith. “Lefschetz Fibrations and the Hodge Bundle”. In: *Geometry & Topology* 3.1 (July 1999), pp. 211–233. ISSN: 1364-0380. DOI: [10.2140/gt.1999.3.211](https://doi.org/10.2140/gt.1999.3.211). (Visited on 06/15/2024).