IRREDUCIBILITY OVER THE MAX-MIN SEMIRING (DRAFT)

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ABSTRACT. For sets $A, B \subset \mathbb{N}$, define $A + B = \{a + b : a \in A, b \in B\}$. If we can write a set C as $C = A + B$ with $|A|, |B| \geq 2$, then we say that C is *irreducible*. The question of whether a given set C is irreducible arises naturally in additive combinatorics. Equivalently, we can formulate this question as one about the irreducibility of boolean polynomials, which has been discussed in previous work by K.H. Kim and F.W. Roush (2005) and Y. Shitov (2014). We prove results about the irreducibility of polynomials and power series over the max-min semiring, a natural generalization of the boolean polynomials.

We use combinatorial and probabilistic methods to prove that almost all polynomials are irreducible over the max-min semiring, generalizing work of Y. Shitov (2014) and proving a 2011 conjecture by D. Applegate, M. Le Brun, and N. Sloane. Furthermore, we use measure-theoretic methods and apply Borel's result on normal numbers to prove that almost all power series are asymptotically irreducible over the max-min semiring. This result generalizes work of E. Wirsing (1953).

CONTENTS

1. INTRODUCTION

The max-min semiring is defined as $\mathcal{N} = (\mathbb{N} \cup \{\infty\}, \oplus, \otimes)$, where $a \oplus b = \max(a, b), a \otimes b = \min(a, b)$. Previous work [Kim/Roush, Shitov] has discussed the factorization of polynomials over the *Boolean Semiring*, that is, the subsemiring $\mathcal{B}_2 = \{0, 1\}$. In this restricted case, the two questions we hope to settle have been already resolved.

Definition 1. Let $f \in \mathcal{N}[[x]]$. If $f = gh$ implies that either g or h is a monomial, then f is *irreducible*.

Theorem 1.1 (Shitov, 2014). *As $n \to \infty$, the proportion of degree n polynomials in $\mathcal{B}_2[x]$ which are irreducible tends to 1.*

This result answers a 2005 question of K. H. Kim and F. W. Roush. Remarkably, the proof uses only elementary combinatorics and probability.

Definition 2. Let $f, g \in \mathcal{N}[[x]]$. If f and g differ in only finitely many coefficients, then we say $f \sim g$.

Definition 3. Let $f \in \mathcal{N}[[x]]$. If $f \sim gh$ implies that either g or h is a monomial, then f is *asymptotically irreducible*.

Theorem 1.2 (Wirsing, 1953). *Almost every element of $\mathcal{B}_2[[x]]$ is asymptotically irreducible.*

This proof is measure-theoretic and builds heavily off the work of Borel, in particular the result that almost every number is *normal*[1] in base 2. Interestingly, Wirsing and Shitov phrased these results in two different settings. Wirsing in fact writes that almost every set $A \subset \mathbb{N}$ is asymptotically irreducible, where here irreducible means additively indecomposable.

Definition 4. Let $A, B \subset G$ for some group G. Then $A + B = \{a + b : a \in A, b \in B\}$.

Definition 5. Let $S \subset \mathbb{N}$. If $S = A + B$ implies either A or B is a singleton, then S is irreducible. Similarly, if $S \sim A + B$ implies A or B is a monomial, then S is asymptotically irreducible. Here, $\sim$ denotes difference in finitely many elements just like before.

We then restate Shitov and Wirsing's results in these terms: almost every finite subset of $\mathbb{N}$ is irreducible, and almost every subset of $\mathbb{N}$ is asymptotically irreducible. The semiring of sets under union and set addition is isomorphic to the semiring of boolean polynomials [Gro19], hence these two formulations are equivalent.

Our contribution is to generalize these results to the wider setting of the max-min semiring.

Theorem 1.3. *Fix b, and set $\mathcal{B}_b = \{0, 1, \ldots, b-1\} \subset \mathcal{N}$. Then as $n \to \infty$, the proportion of degree n polynomials in $\mathcal{B}_b[x]$ which are irreducible tends to 1.*

Theorem 1.4. *Almost every element of $\mathcal{B}_b[[x]]$ is asymptotically irreducible.*

Just as products of boolean polynomials correspond to sums of sets, products of min-max polynomials correspond to sums of *multisets*. For a proof of this correspondence, see [Gro19]. We will not, however, discuss the multiset-polynomial correspondence again.

2. PROOF OF THEOREM 1.3

In this section, we generalize Shitov's result to polynomials over the max-min semiring. We begin with some conventions.

Definition 6. Let $f \in \mathcal{N}[x]$. Then $|f|$ denotes the number of nonzero coefficients of f.

Definition 7 ([ALS11]). A *digit map* is a nondecreasing function $\mathbb{N} \to \mathbb{N}$. If d is a digit map and $f = a_0 \oplus a_1 x \oplus a_2 x^2 \oplus \ldots \in \mathcal{N}[[x]]$, then we let $d(f) = d(a_0) \oplus d(a_1)x \oplus d(a_2)x^2 \ldots$.

Proposition 2.1 ([ALS11]). *If d is a digit map, then $d : \mathcal{N}[[x]] \to \mathcal{N}[[x]]$ is a semiring homomorphism. In particular, if $f = gh$ is a nontrivial factorization and $d(1) \geq 0$, then $d(f) = d(g)d(h)$ is a nontrivial factorization of $d(f)$.*

This key idea provides a powerful framework for our proof, allowing us to partially reduce the problem of factoring a polynomial over $\mathcal{B}_b$ to factoring one over $\mathcal{B}_2$.

Definition 8. We define the digit maps maps s_i for each $i \in \mathbb{Z}^+$.

$$s_i(n) = \begin{cases} 0 & n < i \\ i & n \geq i \end{cases} \tag{2.1}$$

For $f \in \mathcal{N}[[x]]$, we additionally define $f_i = s_1(s_i(f))$. These polynomials, which we refer to as the "i-level support of f", are indicator functions for where the coefficients of f are at least i.

In the proof of another lemma, Shitov shows the following statement which will be of great use to us.

Corollary 2.2 ([Shi14]). *For any $d > 0$, the number of pairs boolean polynomials (f, g) satisfying the following conditions is at most $n^{2d+1}2^{(k,n)}$.*

    (1) The constant terms of f, g are nonzero;
    (2) $\deg f = k > 0, \deg g = n - k$;

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[1]reference our later definition of normal

(3) $|f \otimes g| \leq |f| + |g| + d.$

Due to our slightly different convention about irreducibility, we will require a version of this lemma when the condition (1) is not necessarily satisfied. Let's briefly address the opposing conventions of irreducibility.

Our definition of irreducible is slightly broader than the traditional notion of irreducibility over a semiring, and in particular broader than the convention used in [KR05] and [ALS11]. The advantages of our definition are as follows:

- If $f \in \mathcal{B}_b[[x]]$ is irreducible, then f is also irreducible over $\mathcal{N}[[x]]$. In contrast, $f = (b-1) \otimes f$ is trivial factorization over $B_b[[x]]$, but a nontrivial factorization over $\mathcal{N}[[x]]$ as $b-1$ is no longer a unit. Thus, despite introducing no new factors, f is now reducible.
- The Taylor series f such that $g \mapsto f \otimes g$ is an injective endomorphism of any semiring $\mathcal{B}_b[[x]]$ are precisely the monomials. Thus, despite not having multiplicative inverses, multiplication by monomials is invertible in the sense that we can have cancellation.
- This definition lines up with the additive combinatorics. For instance, the set $\{1, 2, 4\}$ is additively indecomposable, but the polynomial $x \oplus x^2 \oplus x^4$ is reducible as $x(1 \oplus x \oplus x^3)$ under previous authors' definitions.

We now generalize Corollary 2.2 to our setting.

Lemma 2.3. *The number of pairs boolean polynomials (f, g) satisfying the following conditions is at most $n^{2d+2}2^k$ for any $d > 0$.*

(1) The constant term of f is nonzero;
(2) $\deg f = k > 1, \deg g = n - k$;
(3) $|f \otimes g| \leq |f| + |g| + d$.

Proof. Write $g = x^j \otimes (1 + \cdots + x^{n-k-j})$ and define $\overline{g}$ by $g = x^j \otimes \overline{g}$. Then clearly $|f \otimes g| = |f \otimes \overline{g}|$ and $|g| = |\overline{g}|$. By Corollary 2.2, there are at most $n^{2d+1}2^{(k,n-j)}$ pairs $(f, \overline{g})$ satisfying the hypotheses of the corollary. Since there are at most n choices for j, the number of pairs (f, g) satisfying the hypotheses of this lemma is at most $\sum_{j=0}^{n-1} n^{2d+1}2^{(k,n-j)} \leq n^{2d+2}2^k$. $\square$

The final ingredient for our proof is Hoeffding's inequality: a probabilistic lemma which Shitov used, in conjunction with Corollary 2.2, to prove Theorem 1.1.

Proposition 2.4 (Hoeffding's Inequality). *Let X_n be a sum of n independent bernoulli random variables X with $\mathbb{E}[X] = p$. Then $P(|X_n - np| > \epsilon n) \leq 2e^{-2\epsilon^2 n}$.*

When we choose a degree $n-1$ polynomial f at random from $\mathcal{B}_b[x]$, the quantity $|f_i|$ is a sum of n independent Bernoulli random variables with probability $\frac{b-i}{b}$. As a consequence, if f is a degree $n-1$ polynomial chosen randomly from $\mathcal{B}_b[x]$, then:

$$P\left(\left||f_i| - \frac{(b-i)n}{b}\right| > \epsilon n\right) \leq 2e^{-2\epsilon^2 n} \tag{2.2}$$

We now have all the tools we need, and are prepared to move to prove a quantitative version of Theorem 1.3.

Proposition 2.5. *Let $b > 1$ and $\alpha = \lfloor \frac{b}{2} \rfloor$ and let $\Sigma_{b,n}$ denote the set of reducible degree $n-1$ polynomials in $\mathcal{B}_b[x]$. Then for any $d, v > 0$:*

$$|\Sigma_{b,n}| \lesssim b^n \left(ne^{-d^2/4(n+1)} + vn^{2d+1}2^v b^{-n} + n^2 2^{-v} + n^{2d+3}2^{\frac{d}{2}-\frac{n}{3}}\right)$$

Proof. We will partition Σ_n into 7 sets $\Sigma \subset E_n^1(d, v) \cup \ldots \cup E_n^7(d, v)$. Our proposition will follow from the bound $|\Sigma_n| \leq \left|E_n^1(d, v)\right| + \cdots + \left|E_n^7(d, v)\right|$

Definition 9. We now detail the partition. Though we will not write this after each set, we stipulate that $h \in E_n^i(d, v)$ only if $h \notin E_n^j(d, v)$ for any $j < i$.

$E_n^1(d, v)$ is the set of polynomials h such that $\left||h_1| - \frac{(b-1)n}{b}\right| > \frac{d}{2}$.

$E_n^2(d, v)$ is those $h = f \otimes g$ such that $\left| |f_1| + |g_1| - \frac{(b-1)(n+1)}{b} \right| > \frac{d}{2}$.

$E_n^3(d, v)$ is those h such that $\left| |h_\alpha| - \frac{(b-\alpha)n)}{b} \right| > \frac{d}{2}$.

$E_n^4(d, v)$ is those $h = f \otimes g$ such that $\left| |f_\alpha| + |g_\alpha| - \frac{(b-\alpha)(n+1)}{b} \right| > \frac{d}{2}$.

$E_n^5(d, v)$ is those $h = f \otimes g$ with $\deg f \leq v$.

$E_n^6(d, v)$ is those $h = f \otimes g$ with $|f_\alpha| \leq 1$ or $|g_\alpha| \leq 1$.

$E_n^7(d, v)$ is all remaining reducible degree $n - 1$ polynomials h.

Now, we will go through and bound the size of each of these sets.

- By Equation (2.2) with $\epsilon = \frac{d}{2n}$, we obtain:

$$\left| E_n^1(d, v) \right|, \left| E_n^3(d, v) \right| \leq 2e^{-d^2/4n} b^n \lesssim e^{-d^2/4(n+1)} b^n$$

- Each pair (f, g) corresponds to only one choice of h, thus it suffices to bound the number of pairs (f, g). If we fix $\deg f = k$, then we must have $\deg g = n - k - 1$ as $\deg(f \otimes g) = n - 1$. The set of pairs $(f, g) \in (\mathcal{B}_b[x])^2$ such that $\deg f = k, \deg g = n - k - 1$ is in bijection with the set of degree n polynomials of $\mathcal{B}_b[x]$, with the bijection given below.

$$\phi(f, g) = f \oplus \left(((b-1)x^{k+1}) \otimes g \right)$$

$$\phi^{-1}(h) = \left(\bigoplus_{j=0}^{k} \gamma_j x^j, \ \bigoplus_{j=k+1}^{n} \gamma_j x^{j-k-1} \right)$$

Moreover, $|f_i| + |g_i| = |(\phi(f, g))_i|$. Thus, choosing $\epsilon = \frac{d}{2n+2}$ and applying Equation (2.2), we obtain that there are at most $2e^{-d^2/4(n+1)} b^{n+1}$ such pairs (f, g). Since there are $\frac{n}{2}$ choices for $\deg f$, we use this bound for each choice and obtain:

$$\left| E_n^2(d, v) \right|, \left| E_n^4(d, v) \right| \leq ne^{-d^2/4(n+1)} b^{n+1} \lesssim ne^{-d^2/4(n+1)} b^n$$

- Let $h = f \otimes g \in E_n^5(d, v)$. Since $h \notin E_n^1(d, v) \cup E_n^3(d, v)$, we have $|h_1| \leq \frac{(b-1)n}{b} + \frac{d}{2}, |f_1| + |g_1| \geq \frac{(b-1)(n+1)}{b} - \frac{d}{2}$. Thus $|h_1| \leq |f_1| + |g_1| + d$. If $|f_1| \leq 1$ or $|g_1| \leq 1$, then f or g is a monomial in contradiction to the assumption that $f \otimes g$ is a nontrivial factorization, hence (f_1, g_1) satisfy every hypothesis of Lemma 2.3. We apply this lemma once for each choice of $1 \leq \deg f \leq v$, and conclude:

$$\left| E_n^5(d, v) \right| \leq \sum_{\deg f = 1}^{v} n^{2d+1} 2^{\deg f} \leq v n^{2d+1} 2^v$$

- Suppose without loss of generality $|f_\alpha| \leq 1$. Then fix $\deg f = k$ Since $h \notin E_n^5(d, v)$, we can assume $k > v$. Then there are $(k + 1)(b - \alpha)(\alpha - 1)^k$ choices[2] for f and b^{n-k} choices for g, hence there are $\leq k + 1(\alpha - 1)^k b^{n-k+1}$ pairs (f, g). There are at most n choices for k, hence:

$$\left| E_n^6(d, v) \right| \leq \sum_{k=v}^{n} k + 1(\alpha - 1)^k b^{n-k+1} \leq n(n + 1)(\alpha - 1)^v b^{n-v+1}$$

$$\lesssim n^2 (\alpha - 1)^v b^{n-v} \leq n^2 \left(\frac{b}{2} \right)^v b^{n-v} \leq n^2 2^{-v} b^n$$

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[2] pick the index of the coefficient to be at least α, then pick its value, then pick the remaining coefficients from $\{0, \ldots, \alpha - 1\}$.

- Let $h = f \otimes g$. Since $h \notin E_n^3(d, v) \cup E_n^4(d, v)$, we have $|h_\alpha| < \frac{(b-\alpha)n}{b} + \frac{d}{2}$ and $|f_\alpha| + |g_\alpha| > \frac{(b-\alpha)(n+1)}{b}$, hence $|h_\alpha| \leq |f_\alpha| + |g_\alpha| + d$. Moreover, as $h \notin E_n^6(d, v)$, neither f_α nor g_α is a monomial and thus the pair (f_α, g_α) is a nontrivial factorization of h_α and satisfies the hypotheses of Lemma 2.3. Thus, using the fact that $(\deg f_\alpha, n - 1) \leq \frac{n-1}{2} \leq \frac{n}{2}$ for $1 \leq \deg f \leq n - 2$, the number of possible choices for h_α is at most:

$$\sum_{\deg f_\alpha = 1}^{n-2} n^{2d+2} 2^{(\deg f_\alpha, n-1)} \leq n^{2d+3} 2^{\frac{n}{2}}$$

Once h_α is known, if $|h_\alpha| = k$, there are $\alpha^{n-k}(b - \alpha)^k$ choices for h. This is because each 0 coefficient of h_α can correspond to any coefficient in $\{0, \ldots, \alpha - 1\}$, and any 1 corresponds to a coefficient in $\{\alpha, \ldots, b - 1\}$. Since $h \notin E_n^3(d, v)$, we can say $k \leq \frac{(b-\alpha)n}{b} + \frac{d}{2}$, a quantity which we will denote s to clean up our expressions. As $\alpha \leq (b - \alpha)$ with equality when b is even, we obtain the following upper bound:

$$\alpha^{n-k}(b-\alpha)^k \leq \alpha^{n-s}(b-\alpha)^s \leq \alpha^{\frac{\alpha n}{b}}(b-\alpha)^{\frac{(b-\alpha)n}{b}}\left(\frac{b-\alpha}{\alpha}\right)^{\frac{d}{2}}$$

$$\leq \alpha^{\frac{n}{2}}(b-\alpha)^{\frac{n}{2}}\left(\frac{b-\alpha}{\alpha}\right)^{\left(\frac{b-\alpha}{b}-\frac{1}{2}\right)n}\left(\frac{b-\alpha}{\alpha}\right)^{\frac{d}{2}}$$

$$\leq \left(\frac{b}{2}\right)^n \left(\frac{b-\alpha}{\alpha}\right)^{\left(\frac{b-\alpha}{b}-\frac{1}{2}\right)n}\left(\frac{b-\alpha}{\alpha}\right)^{\frac{d}{2}}$$

For $b \geq 2$, we have the upper bounds $\frac{b-\alpha}{\alpha} \leq 2$ and $\frac{b-\alpha}{b} - \frac{1}{2} \leq \frac{1}{6}$, both of which are achieved when $b = 3$. Moreover, for $b = 2$, we have $\left(\frac{b-\alpha}{b}^{\frac{b-\alpha}{b}-\frac{1}{2}}\right) = 1$, thus for any $b \geq 2$ we have $\left(\frac{b-\alpha}{b}\right)^{\left(\frac{b-\alpha}{b}-\frac{1}{2}\right)n} \leq 2^{\frac{n}{6}}$. Altogether, this gives us:

$$\left|E_n^7(d, v)\right| \leq n^{2d+3} 2^{\frac{n}{2}} \left(\frac{b}{2}\right)^n \left(\frac{b-\alpha}{\alpha}\right)^{\frac{d}{2}} \left(\frac{b-\alpha}{b}\right)^{\left(\frac{b-\alpha}{b}-\frac{1}{2}\right)n}$$

$$\leq n^{2d+3} 2^{\frac{n}{2}} \left(\frac{b}{2}\right)^n 2^{\frac{n}{6}} 2^{\frac{d}{2}} \leq n^{2d+3} 2^{\frac{d}{2}-\frac{n}{3}} b^n$$

$\square$

With the right choice of d, v, this gives us a proof of Theorem 1.3.

Proof. Our goal is to show that $\frac{|\Sigma_n|}{b^n} \to 0$, from which the result follows. Set $d = 2\sqrt{n+1}\log n$, $v = 3\log_2 n$ and apply Proposition 2.5. We will show that each summand of the upper bound on $\frac{|\Sigma_n|}{b^n}$ vanishes. For bullets (2) and (4), observe that every summand is sublinear except for the one which is negative.

(1) $ne^{-d^2/4(n+1)} = ne^{-(\log n)^2} = n^{1-\log n} \to 0$.

(2) $\log(vn^{2d+1}2^v b^{-n}) = \log v + (4\sqrt{n+1}\log n + 2)\log n + 3\log_2 n \log 2 - n \log b \to -\infty$, hence this quantity vanishes.

(3) $n^2 2^{-v} = n^2 2^{-3\log_2 n} = n^{-1} \to 0$.

(4) $\log\left(n^{2d+3} 2^{-\frac{n}{3}}\left(\frac{b-\alpha}{\alpha}\right)^{\frac{d}{2}}\right) = 4\sqrt{n+1}\log n + (\sqrt{n+1}\log n - \frac{n}{3})\log 2 \to -\infty$, hence this quantity vanishes.

$\square$

This result in hand, we are now prepared to state and prove the conjecture of Applegate, LeBrun, and Sloane. Their conjecture refers to prime elements of the semiring, which we define below, and is in some ways a more natural definition.

Definition 10. A polynomial $h \in \mathcal{B}_b[x]$ is *prime* if $h = f \otimes g$ implies either $f, g = b - 1$.

Conjecture 1. Let $\pi_b(n)$ denote the number of degree $n - 1$ prime polynomials of $\mathcal{B}_b[x]$. Then $\pi_b(n) \sim (b-1)^2 b^{n-2}$.

Motivating their conjecture, Applegate et al. observed that only certain polynomials can be prime.

Definition 11. A *prime candidate* of $\mathcal{B}_b[x]$ is a polynomial with nonzero constant term and maximum coefficient $b - 1$.

It is easy enough to see that a polynomial is prime only if it is a prime candidate. If $h = a_j x^j \oplus \ldots \oplus a_{n-1} x^{n-1}$ for $j > 1$, then $h = (b-1)x^j \otimes (a_j \oplus \ldots \oplus a_{n-1}x^{n-j-1})$ which is a nontrivial factorization in their convention. Moreover, if $c < b - 1$ is the maximum coefficient of h, then $h = c \otimes h$.

They showed that the number of prime candidates is asymptotic to $(b-1)^2 b^{n-2}$, and from their data[3], as $k \to \infty$, almost all prime candidates are in fact prime. As evidence for this fact, Applegate et al. produced the following lower bound:

$$(b-1)^{n-2} + 2(b-2)^{n-2} + \ldots \leq \pi_b(n)$$

Moreover, they observed the following, which we will re-prove here.

Lemma 2.6 ([ALS11]). *An irreducible prime candidate is prime.*

Proof. If h is irreducible, then $h = fg$ implies either f, g is a monomial, without loss of generality, f is. Since the constant term of h is nonzero, we must have that f is a constant. Since the maximum coefficient of h is $b - 1$, we must also have that $f = b - 1$, thus h is prime. $\square$

With this lemma, Conjecture 1 is a simple corollary of Theorem 1.3

Proof. The proportion of degree $n - 1$ prime candidates of $\mathcal{B}_b[x]$ which are irreducible is at most a quantity (eq. (2.3)) which vanishes as $n \to \infty$.

$$\frac{|\Sigma_n|}{(b-1)^2 b^{n-2}} \tag{2.3}$$

It follows that almost all prime candidates are prime. $\square$

3. PROOF OF THEOREM 1.4

Before we prove this, we first must clarify what we mean by "almost all." It turns out, there is a very natural measure to associate to the set $\mathcal{B}_b[[x]]$.

Definition 12. To each element of $\mathcal{B}_b[[x]]$ we associate a real number in $[0, b]$ given by:

$$\rho_b\left(\bigoplus_{k=0}^{\infty} a_k x^k\right) := \sum_{k=0}^{\infty} a_k b^{-n} \tag{3.1}$$

In other words, each power series corresponds to a string of digits in $[0, 1, \ldots, b - 1]$, which we can interpret as the base-b expansion of a number. This allows us to define a probability measure m on $\mathcal{B}_b[[x]]$.

Definition 13. For a set $A \subset \mathcal{B}_b[[x]]$ such that $\rho_b(A)$ is a measurable subset of $\mathbb{R}$, let $m(A) = b^{-1}\mathcal{L}(\rho_b(A))$, where $\mathcal{L}$ denotes the Lebesgue measure.

This reframing allows us to ask and answer questions about these polynomials measure-theoretically. For example, we will use Borel's theorem that every number is normal, regardless of base [citation probably].

We deduce **??** from a second theorem, which goes as follows:

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[3]OEIS sequences (A169912), (A087636) show the number of prime elements of $\mathcal{B}_2[x], \mathcal{B}_{10}[x]$ of each degree n.

Theorem 3.1. *Let $\mathcal{C}^b \subset \mathcal{B}_b[[x]]$ denote the set of reducible polynomials. Then $m(\mathcal{C}^b) = 0$.*

We show first how **??** follows from Theorem 3.1:

Proof. For $f \in \mathcal{B}_b[[x]]$, let $[f]$ denote the set of all g such that $f \sim g$. The set of asymptotically reducible f is precisely the set $[\mathcal{C}^b]$. Fix a natural number n, and notice the set of functions which differ in exactly n coefficients from some element of C^b has measure 0. Thus, $[C^b]$ is a countable union of measure 0 sets, hence it has measure 0 and almost all power series over $\mathcal{B}_b[[x]]$ are irreducible. $\qquad\square$

We now prove Theorem 3.1. Our proof parallels Wirsing's original argument to a great extent, but as the authors are not aware of an English translation of Wirsing's result [Wir53], we reproduce it here for the sake of completeness.

Definition 14. For $n \in \mathbb{N}$ and $f = \bigoplus_{k=0}^{\infty} a_k x^k \in \mathcal{N}[[x]]$, define $f(n) = \bigoplus_{k=0}^{n} a_k x^k \in \mathcal{N}[x]$.

First, partition $\mathcal{C}^b$ into three sets T_1^b, T_2^b, T_3^b:

$$T_1^b := \{h : h = f \otimes g \text{ with } 2 \leq |g_1| < \infty\}$$

$$T_2^b := \left\{h : h = f \otimes g \text{ with } \liminf_{n \to \infty} \frac{|f_1(n)| + |g_1(n)|}{n} < \frac{1}{5} \text{ and } |f_1| = \infty = |f_1|\right\}$$

$$T_3^b := \left\{h : h = f \otimes g \text{ with } \liminf_{n \to \infty} \frac{|f_1(n)| + |g_1(n)|}{n} \geq \frac{1}{5} \text{ and } |f_1| = \infty = |g_1|\right\}$$

Since $T_1^b \cup T_2^b \cup T_3^b = C^b$, it suffices to show that $\mathcal{L}(T_1^b) = \mathcal{L}(T_2^b) = \mathcal{L}(T_3^b) = 0$. In proving that the measures of T_1^b and T_3^b are 0, we will rely extensively on the following idea:

Definition 15. A number $\alpha \in \mathbb{R}$ is *normal* in base b if the base b representation of α contains an equal proportion of each finite sequence of digits base b. That is, if for all positive integers n, all possible strings of n digits have density b^{-n} in the base b representation.

More formally, let $s = (\delta_1, \ldots, \delta_k)$ be a string of digits in $\{0, \ldots, b-1\}$. Fix a real number α and let $N_\alpha(n, s)$ denote the number of occurences of the string s in the first n digits of the base-b expansion of α. Then the following holds:

$$\lim_{n \to \infty} \frac{N_\alpha(n, s)}{n} = b^{-k}$$

An equivalent formulation of this is the following: let $Z \subset \{0, \ldots, b-1\}^k$ and let $N_\alpha(n, Z) = \sum_{s \in Z} N(n, s)$.

$$\lim_{n \to \infty} \frac{N(n, Z)}{n} = |Z| b^{-k} \tag{3.2}$$

Theorem 3.2. *(Borel, 1909 for base 2; Wirsing, 1953 for the general case) For any $b \geq 2$, almost every $\alpha \in \mathbb{R}$ is normal base b.*

Definition 16. Throughout the remainder of this section, let

$$f = \bigoplus_{k=0}^{\infty} \alpha_k x^k \quad g = \bigoplus_{k=0}^{\infty} \beta_k x^k \quad h = \bigoplus_{k=0}^{\infty} \gamma_k x^k \quad \sigma = \bigoplus_{k=0}^{\infty} \delta_k x^k$$

Additionally, set $\alpha_i' = \alpha_i \otimes 1$ and similarly for each other coefficient. This way we have, for instance:

$$f_1 = \bigoplus_{k=0}^{\infty} \alpha_k' x^k \quad g_1 = \bigoplus_{k=0}^{\infty} \beta_k' x^k \quad h_1 = \bigoplus_{k=0}^{\infty} \gamma_k' x^k \quad \sigma_1 = \bigoplus_{k=0}^{\infty} \delta_k' x^k$$

We now state an important lemma with an elementary proof.

Lemma 3.3. *If $h = f \otimes g$, then $\bigoplus_{k=0}^{n} \alpha_i \otimes \beta_j = \gamma_n$.*

Proof. To elucidate this fact, all we need to do is rewrite the product $f \otimes g$.

$$f \otimes g = \bigoplus_{i=0}^{\infty} \alpha_i x^i \bigoplus_{j=0}^{\infty} \beta_j x^j = \bigoplus_{n=0}^{\infty} \left(\sum_{k=0}^{n} \alpha_k \otimes \beta_{n-k} \right) x^n = \bigoplus_{n=0}^{\infty} \gamma_n x^n$$

$\square$

Lemma 3.4. $m(T_1^b) = 0$.

Proof. We show that no element of T_1^b is normal, whence the result follows. Specifically, we claim that the following sequence of digits can never occur in $\rho_b(h)$ for any $h = f \otimes g \in T_1^b$:

$$\underbrace{00\ldots0}_{\deg g+1} 1 \underbrace{00\ldots0}_{\deg g+1}, \tag{3.3}$$

Let $f_1 = \bigoplus_{k=0}^{\infty} \alpha_k x^k$, $g_1 = \bigoplus_{k=0}^{\deg g_1} \beta_k x^k$, $h_1 = \bigoplus_{k=0}^{\infty} \gamma_k x^k$. We can write:

$$h_1 = g_1 \otimes f_1 = \bigoplus_{k=0}^{\deg g} \beta_k x^k \otimes f_1$$

If $\gamma_k = 1$, then by lemma 3.3 there exist i, j such that $\alpha_i = \beta_j = 1$ and $i + j = k$. Since g_1 is not a monomial, there exists another index $j' \neq j$ such that $\beta_{j'} = 1$. Then by lemma 3.3 $1 \leq \gamma_{i+j'}$ and $\gamma'_{i+j'} = 1$. The gap between the two indices $i + j, i + j'$ is at most $\deg g_1$ (but either index can come first), thus $\rho_b(h_1)$ does not have a "1" without another "1" at most $\deg g$ indices away. Thus the string eq. (3.3) does not occur in $\rho_b(h_1)$. $\square$

Lemma 3.5. $m(T_2^b) = 0$.

Proof. We begin by defining a finite counterpart to T_2^b:

$$T_2^b(n) := \left\{ \rho_b(h) : h = f \otimes g : \frac{|f_1(n)| + |g_1(n)|}{n} < \frac{1}{5} \text{ and } |f_1| = \infty = |g_1| \right\}$$

Notice that

$$T_2^b \subseteq \limsup(\{T_2^b(n)\}) = \bigcap_{N \geq 1} \bigcup_{n \geq N} T_2^b(n).$$

By the Borel-Cantelli Lemma, we know that if

$$\sum_{n=1}^{\infty} m(T_2^b(n)) < \infty,$$

then

$$m\left(\limsup_{n \to \infty}(T_2^b(n)) \right) = m(T_2^b) = 0.$$

As such, it suffices to show that $\sum_{n=1}^{\infty} m(T_2^b(n)) < \infty$.

Fix an integer k and consider all possible f and h such that $|f_1(n)| + |g_1(n)| = k$. There are $\binom{2n+2}{k}$ possibilities for $f_1(n)$ and $g_1(n)$: each has $n+1$ coefficients, and we distribute k nonzero coefficients among them. Additionally, for a given choice of $f_1(n)$ and $g_1(n)$, there are $(b-1)^k$ polynomials $f(n)$ and $g(n)$ since each 1 coefficient of f_1 or g_1 can correspond to any value in $\{1, \ldots, b-1\}$. Thus, for a given k, there are at most $(b-1)^k \binom{2n+2}{k}$ possibilities for $f(n) \otimes g(n)$. Therefore, $T_2^b(n)$ is a subset of a union of at most $\sum_{0 \leq k \leq \frac{n}{5}} (b-1)^k \binom{2n+2}{k}$ intervals, each of length b^{-n}.

We then compute:

$$m(T_2^b(n)) \le \frac{1}{b^n} \sum_{0 \le k \le \frac{n}{5}} (b-1)^k \binom{2n+2}{k}$$

$$\le \frac{n}{5b^n} (b-1)^{n/5} \binom{2n+2}{\lfloor n/5 \rfloor}$$

$$\le \frac{n}{b^n} (b-1)^{n/5} \binom{2n}{\lfloor n/5 \rfloor}$$

$$\le \frac{n}{b^n} (b-1)^{n/5} \left(\frac{2ne}{n/5} \right)^{n/5}$$

$$\le \frac{n}{b^n} (10e(b-1))^{n/5}$$

$$\le n \left(\frac{1.94(b-1)^{1/5}}{b} \right)^n$$

Notice that $\frac{1.94(b-1)^{1/5}}{b} < 1$ for $b \ge 2$. Hence, the sum $\sum_{n=1}^{\infty} m(T_2^b(n))$ converges, so $m(T_2^b) = 0$. $\qquad \square$

Lemma 3.6. $m(T_3^b) = 0$.

Proof. As in the case of T_1^b, we will show that no element of T_3^b is normal, from which the result will follow.

Without loss of generality, we know that $\liminf_{n \to \infty} \frac{|f_1(n)|}{n} \ge \frac{1}{10}$. Let k be a positive integer such that

$$\left(\frac{b-1}{b} \right)^k < \frac{1}{10}$$

Pick a positive integer r such that $|g_1(r-1)| = k$. This is equivalent to choosing r such that $\rho_b(g_1(r-1))$ has exactly k ones. Let Z denote the set of degree $r-1$ polynomials in $\sigma \in \mathcal{B}_b[x]$ such that $\sigma_1 \oplus g_1 = \sigma_1$. In other words, Z is the set of degree $r-1$ polynomials of $\sigma \in \mathcal{B}_b[x]$ such that $\beta_i \ne 0 \implies \delta_i \ne 0$. We can compute $|Z|$ using a counting argument: If $\beta_i \ne 0$, then $\delta_i \in \{1, \ldots, b-1\}$, otherwise $\delta_i \in \{0, \ldots, b-1\}$. As $|g_1| = k$ and σ has r coefficients, there are $(b-1)^k b^{r-k}$ possible choices for σ.

If $\rho_b(h)$ is normal, then we expect the digit strings in $\rho_b(Z)$ to occur at a frequency of $\frac{(b-1)^k b^{r-k}}{b^r} = \left(\frac{b-1}{b} \right)^k$ in $\rho_b(h)$. We will show that they instead occur at a frequency of at least $\frac{1}{10}$, from which it follows that $\rho_b(h)$ is not normal.

Suppose $\alpha'_s = 1$. Then from lemma 3.3, it follows that:

$$\left(\gamma'_s x^s \oplus \ldots \oplus \gamma'_{s+r-1} x^{s+r-1} \right) \oplus \left(\alpha'_s x^s \otimes g_1(r-1) \right) = \left(\alpha'_s x^s \otimes g_1(r-1) \right)$$

Thus $\gamma_s \oplus \ldots \oplus \gamma_{s+r-1} x^{r-1} \in Z$. This observation allows us to lower-bound the frequency of these strings in $\rho_b(h)$:

$$\frac{1}{10} \le \liminf_{n \to \infty} \left(\frac{|f_1(n)|}{n} \right) \le \liminf_{n \to \infty} \left(\frac{N_{\rho(h)}(n+r-1, Z)}{n} \right) = \liminf_{n \to \infty} \left(\frac{N_{\rho(h)}(n, Z)}{n} \right)$$

The above contradicts eq. (3.2), thus h is not normal. $\qquad \square$

We now prove theorem 3.1, from which **??** is a corollary.

Proof. By construction, $\mathcal{C}^b = T_1^b \cup T_2^b \cup T_3^b$. As a consequence of lemma 3.4, lemma 3.5, and lemma 3.6:

$$m(\mathcal{C}^b) \le m(T_1^b) + m(T_2^b) + m(T_3^b) = 0$$

$\qquad \square$

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