Notes on MATH 359 (Persistent homology)

2025

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Contents

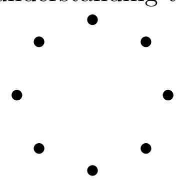
I. Vietoris-Rips Complexes, Morse Theory, and Barcodes

I.1. An Introduction via Speedrun

Persistent Homology intersects with two distinct areas of applied/classical mathematics:

(1) Topological Data Analysis
(2) Calculus of Variations.

Topological Data Analysis is concerned with understanding the topology of "data clouds" For example



The key idea is that with small scales (i.e., taking small neighborhoods around each point), we see a discrete set. On the other hand, with slightly larger scales (i.e., taking a mesh connecting each point to its nearest neighbor) we see a circle. In essence, we think of

$$\text{geometry} + \text{scale} = \text{topology}$$

Let (X, d) be a finite metric space. To this we associate $\{R_t\}_{t\in\mathbb{R}}$ a family of topological spaces for different scales as follows.

Definition I.1.1 (Vietoris-Rips Complex)

The Vietoris-Rits complex R_t of (X, d) is the subcomplex of the complete simplex with vertices in X via the rule

$$\sigma \in X \text{ is a simplex of } R_t \iff \operatorname{diam}(\sigma) < t.$$

Take $X = \{(0,1),(1,0),(1,1),(0,0)\} \subseteq \mathbb{R}^2$, then the Vietoris-Rips complexes associated to this is given in Figure 1.

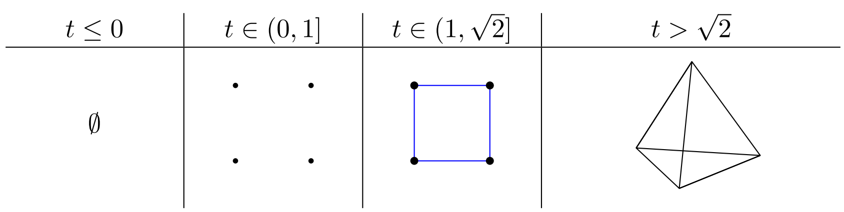


Figure 1. Vietoris-Rips Complex for a Square

For $s \leq t$, we clearly have $R_s \subseteq R_t$. From this we obtain what is called the "topological signature" of (X, d).

On the other hand, the Calculus of Variations is concerned with critical values/critical points of smooth functions on manifolds. It is also concerned with critical points for function spaces; for example, the critical points of the energy functional and geodesics. One question to ask if th count of critical points persist under small C^0-perturbation of a smooth function? The answer is yes if we cut small oscillations!

This idea is closely related to Morse Theory, which mirrors the above picture. Let $f : M \to \mathbb{R}$ be a smooth function, and consider the family of subspaces defined by $R_t = \{x \mid f(x) < t\}$.

Theorem I.1.1 (Morse)

If there is no critical value of f in $[s,t]$ then R_s is homemomorphic (in fact diffeomorphic) to R_t.

Furthermore, Morse describes how the index of the critical point affects the topology. A good reference for this is Milnor's book on Morse theory [Mil63].

We will study these two related objects with Algebraic Tools. Fix a field F (often $\mathbb{R}$ or $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$), then we may define

$$V_t = H_*(R_t; F).$$

We obtain a map $\pi_{s,t} : V_s \to V_t$, and furthermore for $s < t < r$ we have $\pi_{t,r} \circ \pi_{s,t} = \pi_{s,r}$. This gives an algebraic object referred to as a persistence module. To these we will associate a combinatorial object called a barcode, which will suitably classify persistence modules.

Lets return to our example with the square X above, and draw the relevant barcode to give us an idea. Essentially we record when homology appears and when it dies. This is depicted in Figure 2. Similarly, one

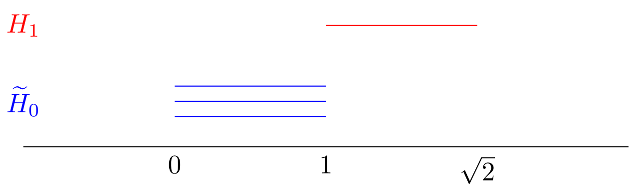


FIGURE 2. Barcode for a Square Metric Space

can draw barcodes for Morse functions.

Exercise I.1.1

Construct a Morse function on the sphere with four critical points and draw the relevant barcode.

Exercise I.1.2

Given a regular 6-gon, draw the barcode of the Vietoris-Rips complex.

I.2. Persistence Modules and Barcodse in Detail

We'll now develop the algebraic theory we need to make the above precise.

Definition I.2.1

A persistence module over a field F is a pair (V, π) of $\{V_t\}_{t\in\mathbb{R}}$ a collection of finite dimensional vector spaces and $\pi_{s,t} : V_s \to V_t$ for $s \leq t$ such that

- $\pi_{t,t} = \text{Id}$.
- $\pi_{t,r} \circ \pi_{s,t} = \pi_{s,r}$.

Furthermore, we assume the following axioms

(1) For all but a finite number of points $t \in \mathbb{R}$ there is a neighborhood U of t so that for all $s, r \in U$ with $s < r$ the map $\pi_{s,r} : V_s \to V_r$ is an isomorphism.

(2) We have semicontinuity. I.e., for all t, there exists an $\varepsilon > 0$ such that for all $s \in (t - \varepsilon, t]$ we have $\pi_{s,t} : V_s \to V_t$ is an isomorphism.

(3) Finally we have $V_t = 0$ for all $t \ll 0$.

Example I.2.1

Here we have interval modules. Let $a < b, b \in \mathbb{R} \cup \{+\infty\}$ and let

$$F(a,b]_t = \begin{cases} F & \text{if } t \in (a,b] \\ 0 & \text{otherwise} \end{cases}.$$

We then define $\pi_{s,t}$ as follows

$$\pi_{s,t} = \begin{cases} \mathrm{Id} & \text{if } s \leq t, s,t \in (a,b] \\ 0 & \text{otherwise} \end{cases}.$$

Definition I.2.2

A barcode is a finite collection of intervals $\{(a_j, b_j], m_j\}$ with $b_j \in \mathbb{R} \cup \{+\infty\}$. The multiplicity records how many lines we draw for each interval. We'll abbreviate this as (I_j, m_j).

Theorem I.2.1 (Normal Form)

For all persistence modules there exists a unique barcode[a]
$B = \{(I_j, m_j)\}$ so that

$$(V_\pi) \cong \bigoplus_j F(I_j)^{\oplus m_j}.$$

[a] These were studied for a discrete parameter in representation theory and were called quivers. These were studied by Gabriel. The normal form theorem follows from thi s work

Ok. Well for this theorem we need to know things about morphisms of persistence modules. One scary thing is that a short exact sequence of persistence modules need not split.

Lets detail the main theorems we will prove about these objects. First, we'll make the above normal form theorem a geometric theorem. We will define distances

(1) Between persistence modules (up to isomorphism), we will have "algebraic interleaving distance."
(2) Between barcodes we will have "combinatorial bottleneck distance."

With these definitions, we will upgrade the above theorem to

Theorem I.2.2 (The Isometry Theorem)

There is an isometry

(persistence modules, algebraic interleaving distance) $\to$ (barcode, combinatorial bottleneck distance).

We will then understand how this depends on the given metric space, for the purpose of Topological Data Analysis.

Theorem I.2.3 (Stability Theorem (1))

The map

$$(\text{metric spaces}, d_{Gromov-Hausdorff}) \to (\text{barcode, bottleneck distance})$$

is 1-Lipschitz.

There is an analogous theorem in Morse theory.

Theorem I.2.4 (Stability Theorem (2))

For a closed manifold M, the map

$$\text{Morse functions on } M \to \text{Barcodes}$$

with the uniform norm on the left and bottleneck distance on the right, is a continuous map.

We can study using these methods even things from physics. For example, take $\mathbb{T}^{2n}$ with coordinates $(p_1, q_1, \ldots, p_n, q_n)$ and its symplectic form $\omega = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$. Then we can consider a Hamiltonian $H(p, q, t)$ and set

$$\dot{p} = -\frac{\partial H}{\partial q}(p, q, t)$$
$$\dot{q} = \frac{\partial H}{\partial p}(p, q, t)$$

We can then consider the time evolution map $f_t : (p(0), q(0)) \mapsto (p(t), q(t))$. This f_t is called the Hamiltonian deformation. We may also look at Hamiltonian self-maps $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ and say we wish to compute the distance $d(\mathrm{Id}, \varphi)$ for φ a Hamiltonian map. This is given as

$$\inf \int_0^1 \|H_t\| \,\mathrm{d}t,$$

where H_t is the Hamiltonian.

This is also related to Floer homology. Namely there is an action functional

$$\mathcal{A} : \mathcal{L}M \to \mathbb{R}$$
$$\mathcal{A}(z) = \int H \,\mathrm{d}t - \int_\Sigma \omega.$$

Then Floer noticed that these have homology thought of as $HF\{\mathcal{A} < t\}$. This allows us to build from any such φ a persistence module $V(\varphi)$ from which we can build a barcode. With this very rough setup, in fact we have.

Theorem I.2.5 (Stability Theorem (3))

The map given above from

$$(\mathrm{Ham}, d_{Hofer}) \to (\text{Floer barcode}, d_{bot}).$$

is a Lipschitz map.

I.3. The Category of Persistence Modules and Interleaving Distance

Last time we discussed persistence modules as objects (see Definition I.2.1). We'll now discuss the morphisms in this category.

Definition I.3.1

Let (V, π^V) and (W, π^W) be persistence modules. A family of linear maps $A_t : V_t \to W_t$ is called a morphism if

$$\begin{array}{ccc} V_s & \xrightarrow{\pi_{st}^V} & V_t \\ \downarrow{\scriptstyle A_s} & & \downarrow{\scriptstyle A_t} \\ W_s & \xrightarrow{\pi_{st}^W} & W_t \end{array}$$

Remark I.3.1

At this point, one might note that if we take a category $(\mathbb{R}, \leq)$ as a poset category, then persistence modules are very similar to functors $\mathbb{R} \to \text{Vect}_F$, and morphisms are natural transformations. However there are additional conditions for the objects for us, namely the continuity-type conditions.

Example I.3.1

Lets try to construct a morphism. Let I, J be intervals and $A : F(I) \to F(J)$ be defined by

$$A_t = \begin{cases} \text{Id} & \text{if } t \in I \cap J \\ 0 & \text{otherwise} \end{cases}$$

There are essentially four cases if $I = (a, b]$ and $J = (c, d]$.

(1) $a < c$. Then A is NOT a morphism, as A_s for $s \in (a, c)$ is 0 but the composition $A_t \circ \pi_{st} = \text{Id}$.
(2) $d \leq b$, $a = c$. Then A is a morphism and is pointwise surjective.
(3) If $b < d$. Then A is NOT a morphism.
(4) If $a > c$ and $b = d$ then A is a pointwise injective morphism.

Now let $a < b < c$. Then there is a short exact sequence

$$0 \to F(b, c] \to F(a, c] \to F(a, b] \to 0.$$

This short exact sequence does not split! By the above discussion.

Exercise I.3.2

Prove that the kernel and image of a morphism are persistence sub-modules.

Definition I.3.2

Let $N_\delta(V, \pi)$ be the number of bars of length $> \delta$.

Theorem I.3.1

FOr all Short Exact Sequences

$$0 \to U \to V \to W \to 0$$

we have that

$$N_{2\delta}(V) \leq N_\delta(U) + N_\delta(W).$$

There is a proof using homological algebra and resolutions. But a question for the class!

Question: Can you find an easy proof?

Example I.3.3

Let (V, π) be a persistence module. We can construct a new persistence module $(V[\delta], \pi[\delta])$ as

$$V[\delta]_t = V(t + \delta)$$

$$\pi[\delta]_{st} = \pi_{s+\delta,t+\delta}.$$

Exercise I.3.4

For $\delta > 0$ there is a map

$$\Phi_\delta^V : V \to V[\delta]$$

$$(\Phi_\delta^V)_t : V_t \xrightarrow{\pi_{t,t+\delta}} V_{t+\delta}.$$

Definition I.3.3

Let (V, π) and (W, θ) be persistence modules. We say that they are δ-interleaved for $\delta > 0$ provided that there exist morphisms

$$f : V \to W[\delta]$$

$$g : W \to V[\delta],$$

such that the diagrams

$$\begin{array}{ccc} V & \xrightarrow{f} & W[\delta] \\ & \searrow^{\Phi_{2\delta}^V} & \downarrow g[\delta] \\ & & V[2\delta] \end{array} \qquad \begin{array}{ccc} W & \xrightarrow{g} & V[\delta] \\ & \searrow^{\Phi_{2\delta}^W} & \downarrow f[\delta] \\ & & W[2\delta] \end{array}$$

commute

Definition I.3.4

The interleaving distance is defined by

$$d_{int}((V, \pi), (W, \theta)) = \inf\{\delta \mid (V, \pi) \text{ and } (W, \theta) \text{ are } \delta\text{-interleaved}\}.$$

Exercise I.3.5

Here are some good exercises to do

(1) Show that $d_{int} < \infty$ if and only if $\dim V_\infty = \dim W_\infty$ (here V_∞, W_∞ are V_s, W_s for s large enough so that $\pi_{st}^V, \pi_{st}^W = \mathrm{Id}$ for all $t > s$).

(2) Show that d_{int} satisfies the triangle inequality.

(3) Difficult exercise (Polterovich doesn't know an easy proof): d_{int} is a genuine distance on the set of persistence modules up to isomorphism.

Example I.3.6

Lets calculate some distances:

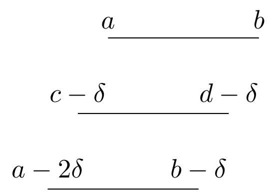
(I) Take $a < b < \infty$ and $c < d < \infty$. Then

$$d_{int}(F(a, b], F(c, d]) \leq \min\left(\max\left(\frac{b-a}{2}, \frac{d-c}{2}\right), \max(|a-c|, |b-d|)\right)$$

And in fact this is equality. The point is there are two strategies to interleave:

(a) Let $\delta > \frac{b-a}{2}, \frac{d-c}{2}$. Then the shift morphisms for $V = F(a,b]$ and $W = F(c,d]$ given by $V \to V[2\delta]$ and $W \to W[2\delta]$ are both zero. Hence we can take $f = 0$ and $g = 0$ in the definition of interleaving distance.

(b) If $\delta > \max(|a-c|, |b-d|)$ then we can align things as follows



and choose the appropriate maps between levels.

(II) On the level of functions. Let M be a closed manifold and $F, G : M \to \mathbb{R}$ be Morse functions. We will start to calculate the distance between persistence modules $H_*(\{F < t\})$ and $H_*(\{G < t\})$ in the next proposition..

Proposition I.3.2

Let M be a closed manifold and $F, G : M \to \mathbb{R}$ two Morse functions. Then for all $\delta > \|F - G\| = \max_x |F(x) - G(x)|$ we have that $H_*(\{F < t\})$ and $H_*(\{G < t\})$ are δ-interleaved.

Proof. We have that $F - 2\delta < G - \delta \leq F$, and hence

$$\{F < t\} \subseteq \{G < t + \delta\} \subseteq \{F < t + 2\delta\},$$

this inclusion gives the interleaving map in one direction.

This is pre-stability theorem. Namely it shows that

$$(C^\infty_{Morse}(M), \text{uniform norm}) \to (\text{persistence modules}, d_{int})$$

is 1-Lipschitz.

We'll now do the same type of calculation for Vietoris-Rips complexes and Gromov-Hausdorff distance.

Definition I.3.5

Let X, Y be finite sets. A surjective correspondence $C : X \rightrightarrows Y$ is a subset $C \subseteq X \times Y$ so that the projection $\text{proj}_X(C) = X, \text{proj}_Y(C) = Y$.

Let (X, d) and (Y, Δ) be metric spaces. The distortion of a surjective correspondence C is

$$\text{dis}(C) = \max_{(x,y),(x',y') \in C} |d(x,x') - \Delta(y,y')|.$$

Note that $\text{dis}(C) = 0$ if and only if C is the graph of an isometry $\varphi : X \to Y$.

Definition I.3.6

The Gromov-Hausdorff distance $d_{GH}(X,Y)$ is given by

$$d_{GH}(X,Y) = \frac{1}{2} \min_{C : X \rightrightarrows Y} \text{dis}(C).$$

Exercise I.3.7

d_{GH} is a genuine distance function.

Recall from last time that we defined for (X, d) a finite metric space the Vietoris-Rips complex $R_t(X)$ the subcomplex of the full simples with vertices X consisting of those simplices with $\operatorname{diam} \sigma < t$. We then defined the persistence module

$$V(X) = H_*(R_t(X), F).$$

Theorem I.3.3 (Chazal-,Silva-Outdots,2009)

$d_{int}(V(X), V(Y)) \leq 2d_{GH}(X, Y)$. In other words the Vietoris-Rips map

$$(\text{finite metric spaces}, d_{GH}) \to (\text{persistence modules up to iso}, d_{int}).$$

Definition I.3.7

Two simplicial maps $H, H' : K \to L$ are <u>contiguous</u> if for each simple $\sigma \in K$ we have that $H(\sigma) \cup H'(\sigma)$ lies in (the same) simplex of K.

Exercise I.3.8

H and H' are then homotopic (see Spanier's book for a proof).

Proof of Theorem I.3.3. Take a surjective correspondence $C : X \rightrightarrows Y$ and $\delta > \operatorname{dis}(C)$. We pick maps $f : X \to Y$ and $g : Y \to X$ so that

$$\operatorname{graph} f \subseteq C \qquad \operatorname{graph} g \subseteq C^T = \{(y, x) \mid (x, y) \in C\}.$$

Note then that

$$\Delta(f(x), f(x')) < d(x, x') + \delta,$$

and so we obtain a simplicial maps $F : R_t(X) \to R_{t+\delta}(Y)$. Similarly we obtain a map $G : R_t(Y) \to R_{t+\delta}(X)$. Now we look at the composition

$$R_t(X) \xrightarrow{F} R_{t+\delta}(Y) \xrightarrow{G[\delta]} R_{t+2\delta}(X).$$

We'll finish the proof by comparing this to the inclusion.

Claim

We claim that the inclusion $\iota : R_t(X) \to R_{t+2\delta}(X)$ is contiguous to $G \circ F$.

Choose $\sigma = [x_0, \ldots, x_k] < R_t(X)$. Then we see that

$$(gf(x_i), f(x_i)) \in C \qquad (x_j, f(x_j)) \in C.$$

By the definition of distortion we see that

$$d(gf(x_i), x_j) < \Delta(f(x_i), f(x_j)) + \delta < \Delta(x_i, x_j) + 2\delta.$$

Hence $\iota(\sigma)$ and $GF(\sigma)$ both belong to the same simplex in $R_{t+2\delta}(X)$. Hence GF and ι are contiguous, and thus they are homotopic. Therefore they correspond to the same morphism on homology, which is exactly what we want to prove to satisfy the definition of interleaving distance.

I.4. **Bottleneck Distance**

We wish to define a distance on barcodes so that the map

$$\text{(Persistence modules, interleaving distance)} \to \text{(Barcodes, ???)}$$

is an isometry. This is essentially due to the fact that interleaving distance is quite difficult to calculate and abstract. On the other hand, barcodes are extremely nice combinatorial objects, so computations with them should be straightfoward.

Definition I.4.1

Let $\mathcal{B}, \mathcal{C}$ be barcodes, $\delta > 0$. We say that $\mathcal{B}$ and $\mathcal{C}$ are δ-matched if after erasing of some finite number of bars of length $< 2\delta$ in both barcodes, the rest of the bars can be matched in a 1-to-1 manner (with multiplicities) so that if

$$(a, b] \in \mathcal{B} \leftrightarrow (c, d] \in \mathcal{C}$$

and $|a - c| < \delta$ and $|b - d| < \delta$.

The bottleneck distance between $\mathcal{B}$ and $\mathcal{C}$ is the infimum among all such δ. This is denoted $d_{bot}(\mathcal{B}, \mathcal{C})$.

Exercise I.4.1

$d_{bot}(\mathcal{B}, \mathcal{C})$ is a genuine distance on the space of barcodes with a given number of infinite rays.

Example I.4.2

Consider $\mathcal{B} = I(a, b]$ and $\mathcal{C} = I(c, d]$. We can consider two algorithms

(I) Let $\delta_{erase} = \max((b-a)/2, (d-c)/2)$

(II) Let $\delta_{align} = \max(|c-a|, |d-b|)$

Then $d_{bot}(\mathcal{B}, \mathcal{C}) = \min(\delta_{erase}, \delta_{align})$.

Theorem I.4.1 (Isometry Theorem)

The map from persistence mdoules to barcodes via the normal form is an isometry for interleaving distance and bottleneck distance, i.e.,

$$(\text{persistence modules / isomorphism}, d_{int}) \to (\text{barcodes}, d_{bot}),$$

is an isometry

The proof in one direction is rather simple – if barcodes have bottleneck distance $< \delta$ then the direct sum modules will δ-interleave. The other direction is much more complicated. We will do this, but it's perhaps not a proof from the Book...

Corollary I.4.2

$d_{int}(I(a, b], I(c, d]) = \min(\delta_{erase}, \delta_{align})$.

Corollary I.4.3 (Stability Theorems)

We have the following

(1) For M a closed manifold, the map

$$\begin{aligned}(C^\infty_{Morse}, d_{uniform}) &\to (\text{Barcodes}, d_{bot})\\ f &\mapsto \mathcal{B}(H_*(\{f < t\}))\end{aligned}$$

is 1-Lipschitz.

(2) The map

$$\begin{aligned}(\text{Finite metric Spaces}, d_{GH}) &\to (\text{Barcodes}, d_{bot})\\ (X, d) &\mapsto \mathcal{B}(H_*(R_t(X, d)))\end{aligned}$$

is 2-Lipschitz.

These stability theorems allow us to pursue topological function theory. Let M be closed manifolds and let f, g be (Morse) functions. We wish to consider the distance

$$\text{dist}_{C^\infty}(f, \text{Diff}^+(M) \cdot g).$$

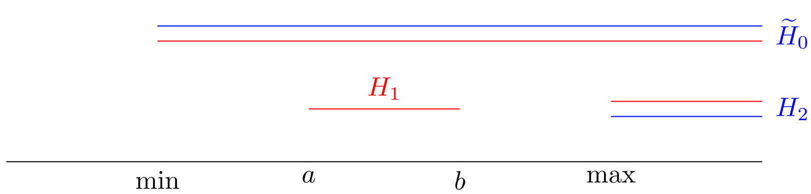
Usual function theory has no way to calculate this, but the above tells us

$$\|f - g \circ \varphi\| \geq d_{bot}(\mathcal{B}(f), \mathcal{B}(g \circ \varphi)) = d_{bot}(\mathcal{B}(f), \mathcal{B}(g)).$$

Example I.4.3

For example, we can consider height maps for S^2. Let f have 4 critical points (dented sphere height) and let g have exactly two critical points (standard sphere height)

The goal is to approximate f by a Morse function with exactly two critical points (i.e., $g\circ\varphi$ for some $\varphi \in \text{Diff}^+(S^2)$). Suppose f has critical points at a, b. Suppose f, g also have the same minimum/maximum heights. The barcodes are



The barcode for f is red and the barcode for g is blue. The bottleneck distance is $b - a$ (erase the one extra bar). Hence $\|f - g \circ \varphi\| \geq b - a$ for all φ.

So what data can be read from a barcode? What about the endpoints of the infinite rays!

$$c_1 \geq c_2 \geq \cdots \geq c_k$$

Exercise I.4.4

Let $\mathfrak{B}_k$ be the set of barcodes with k infinite rays. Show that $c_j : \mathfrak{B}_k \to \mathbb{R}$ is 1-Lipschitz.

This is related to a concept in linear algebra which leads to Lyupanov exponents in dynamics.

Definition I.4.2 (Characteristic Exponent)

Let E be a finite dimensional vector space, $\dim E = n$. A function $c : E \to \mathbb{R} \cup \{-\infty\}$ is called a characteristic exponent (valuation) if

(1) $c(0) = -\infty$, $c(v) \in \mathbb{R}$ for all $v \neq 0$.

(2) $c(\lambda v) = c(v)$ for all $\lambda \in F \setminus \{0\}$.

(3) $c(v_1 + v_2) \leq \max(c(v_1), c(v_2))$.

Observation: For all $\alpha \in \mathbb{R}$, we have that

$$L_\alpha = \{v \in E \mid c(v) < \alpha\}$$

is a linear subspace, and $L_\alpha \subseteq L_\beta$. Hence this gives a flag for E.

Exercise I.4.5

c takes at most n distinct values $\alpha_1 < \ldots < \alpha_k$. Furthermore

$$E_k = \{c \leq \alpha_k\}$$

gives a flag $E_0 = 0 < E_1 < \cdots < E_k = E$ so that $c\big|_{E_k \setminus E_{k-1}} = \alpha_k$.

We obtain the c_j essentially by such a characteristic function as follows.

Example I.4.6

Let (V, π) be a persistence module and V_∞ the space at ∞. We can consider

$$c : V_\infty \to \mathbb{R}$$
$$c(v) = \inf_s \{s \mid v \in \operatorname{im} \pi_{s,\infty}\}.$$

Exercise: c is a characteristic exponent.

Now lets link this back up to Morse theory!

Example I.4.7

Let M be a closed manifold, $f : M \to \mathbb{R}$ a Morse function and $V = H_*(\{f < t\})$. Then $V_\infty = H_*(M)$. For $a \in H_*(M) \setminus \{0\}$ (we can assume is a pure class $a \in H_k(M)$), we have

$$c_f(a) = \inf\{s \mid a \in \operatorname{im} \pi_{s,\infty}\} = \inf_{\substack{\text{cycles } A \\ [A]=a\}}} \max_{x \in A} f(x).$$

This is the "minmax" principle and appears in the calculus of variations.

One can also read off the lengths of the finite bars

$$\beta_1 \geq \beta_2 \geq \cdots$$

If there are $< k$ bars then $\beta_k = 0$.

Definition I.4.3

β_1 is called the boundary depth.

Theorem I.4.4

β_j : Barcodes $\to \mathbb{R}$ is 2-Lipschitz.

Exercise I.4.8

This is referred to as the matching lemma. Let $b_1 \geq \cdots \geq b_N, c_1 \geq \cdots \geq c_N$, and $b_i, c_j \in \mathbb{R}$. Then we have that

$$\min_{\sigma \in S_N} \max \left|b_i - c_{\sigma(i)}\right| = \max |b_i - c_i| .$$

Proof of Theorem. Let $\mathcal{B}$ and $\mathcal{C}$ are δ-matched. We wish to prove that

$$\beta_j(\mathcal{B}) - \beta_j(\mathcal{C}) \leq 2\delta \tag{1}$$

If $\beta_j(B) \leq 2\delta$ then 1 is obvious. Thus, we assume $\beta_j(\mathcal{B}) > 2\delta$.

After removing some bars of length $< 2\delta$ we match the rest with error $< \delta$ at end points. Hence if $I \leftrightarrow J$ we have

$$|\text{length}(I) - \text{length}(J)| < 2\delta.$$

Now look at the matched intervals and denote their length in decreasing order:

$$b_1 \geq \cdots \geq b_N$$
$$c_1 \geq \cdots \geq c_N$$

We now make some remarks

(1) We see as a consequence of $\beta_j(\mathcal{B}) > 2\delta$ that $N \geq j$.
(2) Matching of bars could be different from $b_i \leftrightarrow c_i$.

However, the optimal matching between lengths is the monotone matching by the matching lemma. Hence

$$|b_j - c_j| \leq \left|b_j - c_{\sigma(j)}\right| < 2\delta.$$

We know $b_1 \geq \cdots \geq b_j > 2\delta$ and no bar of length $\geq 2\delta$ was removed. At the same time we know that $c_j \leq \beta_j(\mathcal{C})$ because some bars which are longer than c_j may have been removed. Hence

$$\beta_j(\mathcal{B}) - \beta_j(\mathcal{C}) \leq b_k - c_k < 2\delta.$$

Let $f : V \to W$ be a morphism of persistence modules? How might one define a matching $f_* : \mathcal{B}(V) \to \mathcal{B}(W)$, in order to try to prove the isometry theorem?

Problem: This cannot be done canonically.

But nicely, this can be done canonically for surjections and injections *seperately*. Now decompose $f : V \to W$,

$$V \xrightarrow{f} \operatorname{im} f \hookrightarrow W.$$

The resulting matching is very non-canonical, but we will use it for the proof.

II. Proof of the Isometry Theorem

Today we will prove the isometry theorem. First we set up notation:

- X, Y are multisets, $(x_j, m_j) \in X$ where $m_j \in \mathbb{N}$ is the multiplicity.
- A matching is a bijection $\mu : X' \to Y'$ where $X' \leq X, Y' \leq Y$. X' is denoted by $\operatorname{coim} \mu$ and Y' is denoted $\operatorname{im} \mu$
- For a Barcode $\mathcal{B}$, $\mathcal{B}_\varepsilon$ for $\varepsilon > 0$ will be the set of all bars of length $> \varepsilon$.
- Recall from last time, a δ-matching between barcodes $\mathcal{B}$ and $\mathcal{C}$ is a matching $\mu : \mathcal{B} \to \mathcal{C}$ so that
  (1) $\mathcal{B}_{2\delta} \subseteq \operatorname{coim} \mu$.
  (2) $\mathcal{C}_{2\delta} \subseteq \operatorname{im} \mu$.

The difficult part of the isometry theorem is to prove that if V, W are δ-interleaved persistence modules then $\mathcal{B}(V)$ and $\mathcal{B}(W)$ admit a δ-matching.

II.1. Action of morphisms $V \to W$ on barcodes

We will first construct a non-canonical (in the sense of non-functorial) action of morphisms $\sigma : V \to W$ on the barcodes $\mathcal{B}(V), \mathcal{B}(W)$. Suppose first that $\sigma : V \to W$ is a surjection. We then construct the action on barcodes as follows

(1) For every $b \in \mathbb{R}$, sort intervals in $\mathcal{B}(V)$ as

$$(b, d_1] \supseteq (b, d_2] \supseteq \cdots \supseteq (b, d_k]$$

in decreasing order. Similarly for $\mathcal{B}(W) = \mathcal{C}$

$$(b, c_1] \supseteq (b, c_2] \supseteq \cdots \supseteq (b, c_K].$$

(2) Match these intervals via the longest first principle, obtaining a matching $\mu_{sur} : \mathcal{B}(V) \to \mathcal{B}(W)$.

> **Proposition II.1.1**
> $\operatorname{im} \mu_{sur} = \mathcal{B}(W)$ and $\mu_{sur}(b, d] = (b, e]$ implies $d \geq e$.

We have a similar story for injections. If $\iota : V \to W$ is an injection then we construct the action on barcodes as

(1) For every $d \in \mathbb{R}$, sort intervals in $\mathcal{B}(V)$ as

$$(b_1, d] \supseteq (b_2, d] \supseteq \cdots \supseteq (b_k, d]$$

in decreasing order. Similarly for $\mathcal{B}(W) = \mathcal{C}$

$$(c_1, d] \supseteq (c_2, d] \supseteq \cdots \supseteq (c_K, d].$$

(2) Match these intervals via the longest first principle, obtaining a matching $\mu_{inj} : \mathcal{B}(V) \to \mathcal{B}(W)$.

> **Proposition II.1.2**
> $\operatorname{coim} \mu_{inj} = \mathcal{B}(V)$ and $\mu_{inj}(b, d] = (c, d]$ implies $c \leq b$.

Now if $f : V \to W$ is any morphism, decompose it as

$$V \xrightarrow{\sigma} \operatorname{im} f \xhookrightarrow{\iota} W,$$

then we obtain matchings

$$\mu_{sur} : \mathcal{B}(V) \to \mathcal{B}(\operatorname{im} f)$$
$$\mu_{inj} : \mathcal{B}(\operatorname{im} f) \to \mathcal{B}(W).$$

We now notice that we have

$$\operatorname{im} \mu_{sur} = \mathcal{B}(\operatorname{im} f) = \operatorname{coim} \mu_{inj},$$

and hence we can define $\mu(f) = \mu_{inj} \circ \mu_{surj}$.

Remark II.1.1

Notice the following strange bugs/features of this definition:

(1) $\mu(f)$ depends only on $\operatorname{im} f$.

(2) $\mu(f)$ is functorial separately on surjections/injections, but not functorial in total.

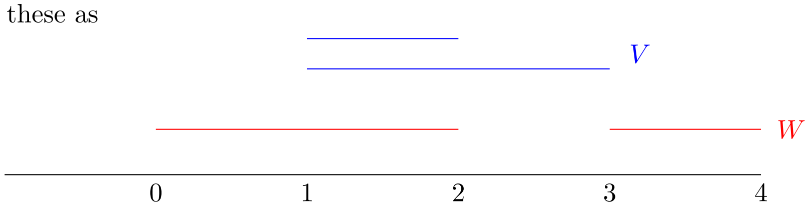
We also should see some examples.

Example II.1.1

Consider the following

$$V = \mathcal{F}(1,3] \oplus \mathcal{F}(1,2]$$
$$W = \mathcal{F}(3,4] \oplus \mathcal{F}(0,2]$$
$$f : V \to W$$
$$f\big|_{\mathcal{F}(1,3]} = 0$$
$$f\big|_{\mathcal{F}(1,2]} = \text{injection to } \mathcal{F}(0,2]$$

Visually we find these as



Now $\operatorname{im} f = \mathcal{F}(1,2]$. We see that μ_{surj} matches

$$\begin{array}{cc} (1,3] & (1,2] \\ \updownarrow & \\ (1,2] & \end{array}$$

while μ_{inj} matches

$$\begin{array}{cc} (1,2] & \\ \updownarrow & \\ (0,2] & (3,4]. \end{array}$$

Hence in total we match $(1,3]$ with $(0,2]$.

II.2. **Proving the Isometry Theorem**

We first recall the definition of a δ-interleaving.

Recall II.2.1

Given (V, π^V) and (W, π^W) we defined the shift $V[\delta]_t = V_{t+\delta}$ and considered the map

$$\Phi_\delta^V : V \to V[\delta]$$
$$V_t \xmapsto{\pi_{t,t+\delta}} V_{t+\delta}.$$

A δ-interleaving was a pair of morphisms $f : V \to W[\delta]$ and $g : W \to V[\delta]$ so that the compositions

$$V \xrightarrow{f} W[\delta] \xrightarrow{g[\delta]} V[2\delta]$$
$$W \xrightarrow{g} V[\delta] \xrightarrow{f[\delta]} W[2\delta]$$

are $\Phi_{2\delta}^V$ and $\Phi_{2\delta}^W$ respectively.

Let us now examine how $\mu(f)$ for $f : V \to W[\delta]$ acts. Decompose this as two maps

$$V \xrightarrow{\sigma} \operatorname{im} f \xrightarrow{\iota} W[\delta].$$

We will prove two lemmas:

Lemma II.2.1

The following hold for $\mu_{sur}(f)$ when (f, g) is a matching:

(1) $\operatorname{coim} \mu_{sur} \supseteq \mathcal{B}(V)_{2\delta}$

(2) $\operatorname{im} \mu_{sur} = \mathcal{B}(\operatorname{im} f)$.

(3) If μ_{sur} takes $(b, d]$ to $(b, d']$ then $d' \in [d - 2\delta, d]$

Lemma II.2.2

The following hold for $\mu_{inj}(f)$ when (f, g) is a matching:

(1) $\operatorname{im} \mu_{inj} \supseteq \mathcal{B}(W[\delta])_{2\delta}$

(2) $\operatorname{coim} \mu_{inj} = \mathcal{B}(\operatorname{im} f)$

(3) If μ_{inj} takes $(b, d]$ to $(b', d]$ then $b' \in [b - 2\delta, b]$.

Proof of Isometry Theorem, Theorem I.2.2. Assuming Lemmas II.2.1 and II.2.2, we build a matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$ as follows given a δ-interleaving (f, g). We have a matching $\Psi_\delta : \mathcal{B}(W[\delta]) \to \mathcal{B}(W)$ just by shifting each bar. We now form a matching from $\mu = \mu(f)$ as

$$\begin{array}{ccccccc} & & & & \mathcal{B}(W[\delta])_{2\delta} & \cong & \mathcal{B}(W)_{2\delta} \\ & & & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \mathcal{B}(V)_{2\delta} & \xrightarrow{\mu_{sur}} & \mathcal{B}(\operatorname{im} f) & \xrightarrow{\mu_{inj}} & \operatorname{im} \mu_{inj} & \xrightarrow{\Psi_\delta} & \mathcal{B}(W) \end{array}$$

We now track where a bar moves under this map

$$(b, d] \mapsto (b, d'] \mapsto (b, d'] \mapsto (b' + \delta, d' + \delta].$$

We now notce that by the lemmas we have that

- Every bar of $\mathcal{B}(V)_{2\delta}$ is matched.
- Every bar in $\mathcal{B}(W)_{2\delta}$ is matched.
- $d - 2\delta \leq d' \leq d$.
- $b - 2\delta \leq b' \leq b$.

Hence we have that

$$|(b' + \delta) - b| \leq \delta$$

$$|(d' + \delta) - d| \leq \delta.$$

Hence we have the required δ-matching.

We now prove Lemma II.2.1, the proof of Lemma II.2.2 is similar. We will use the map g as the critical piece of input to this lemma.

Proof of Lemma II.2.1. We consider that

$$V \xrightarrow{f} W[\delta] \xrightarrow{g[\delta]} V[2\delta]$$

$$V \xrightarrow{f} \operatorname{im} f \xrightarrow{g[\delta]} \operatorname{im} \Phi^V_{2\delta}$$

Functoriality of surjections implies that

$$\mu_{sur}(g[\delta]) \circ \mu_{sur}(f) = \mu_{sur}(\Phi^V_{2\delta}).$$

We know how $\mu_{sur}(\Phi^V_{2\delta})$ acts on bars. If $b - d > 2\delta$ then image contains a non-empty bar, hence

Only bars of length $> 2\delta$ **can survive**.

We then see that

$$\operatorname{coim} \mu_{sur}(f) \supseteq \operatorname{coim} \mu_{sur}(\Phi^{2\delta}_V) = \mathcal{B}(V)_{2\delta}.$$

This proves the first claim of Lemma II.2.1. The second claim was handled in the construction of μ_{sur}.

We now must verify that provided $\mu_{sur}(f)$ matches

$$\mu_{sur}(f) : (b, d] \to (b, d']$$

then $d' \in [d - 2\delta, d]$. There are two cases

- If $d - b \leq 2\delta$ then $b \geq d - 2\delta$ and so

  $$d - 2\delta \leq b < d' \leq d.$$

  where $d' \leq d$ by construction of μ_{sur}.
- Assume $d - b > 2\delta$. We find that

  $$(b, d] \xrightarrow{\mu_{sur}(f)} (b, d'] \xrightarrow{\mu_{sur} g([\delta])} (b, d''].$$

But the total effect is the matching $\mu_{sur}(\Phi^V_{2\delta})$ is

$$\mu_{sur}(\Phi^V_{2\delta}) : (b, d] \to (b, d - 2\delta].$$

Hence $d \geq d' \geq d'' = d - 2\delta$, and so we win.

III. Reading from Barcodes

III.1. Applications to Topological Function Theory

Here are two additional things one can read from barcodes

Definition III.1.1

Let $\mathcal{B}$ be a barcode, and take

$$\nu_\delta(\mathcal{B}) = \#\{\text{finite bars of length } > \delta\}$$
$$\mathcal{N}_\delta(\mathcal{B}) = \#\{\text{bars of length } > \delta\}.$$

We'll now use these invariants to study oscillation in topological function theory. We'll compare two ideas

Topological	Geometric
$\mathcal{B}, \nu_\delta$	Norms of f and its derivatives.

Theorem III.1.1

Let M be a closed d-dimensional manifold (with an auxiliary Riemannian metric). Take $f : M \to \mathbb{R}$ a Morse function with Lipschitz constant $L(f)$. Then

$$\nu_\delta(f) \leq k \cdot \frac{L^d}{\delta^d},$$

where k is a constant depending on the metric.

Remark III.1.1

In some sense we can think of $\nu_\delta(f)$ as measuring the oscillations of the function. This is at least precise when M is a compact interval or S^1 (think about it. . . all bars are finite and represent components). E.g. consider the graph of $\frac{\sin(x^3)}{x}$.

Consider for a Morse function f what $\nu_0(f)$ is, well this is the number of finite bars, and clearly

Exercise III.1.1

We have that

$$\nu_0(f) \leq \frac{\#\{\text{crit points of } f\}}{2}.$$

Well also think about $\beta(M)$ which is the full Betti # of infinite bars.

Let $f : M \to \mathbb{R}$ be a Morse function, where M has a Riemannian metric and $|f(x) - f(y)| \leq L\rho(x, y)$. Call $L(f)$ the Lipschitz constant. With the notation $\nu_\delta(f)$ being the number of finite bars in $\mathcal{B}(f)$ of $V(t) = H_*(\{f < t\}; F)$ of length $> \delta$.

Theorem III.1.2

$\nu_\delta(f) \le k\frac{L^d}{\delta^d}$ where $d = \dim M$ and k depends on the metric on M.

As discussed before, we can take a triangulation Σ and define a simplex $\sigma = (x_0, \ldots, x_k)$ the function

$$u(\sigma) = \max_{x \in \{x_1, \ldots, x_k\}} f(x).$$

We may then define a persistence module $H_*(\Sigma^t; F)$ by $\Sigma^t = \{u < t\}$. Call this $V(\Sigma, u)$.

Proposition III.1.3

$V(\Sigma, u)$ and $V(F)$ are δ-interleaved where $\delta = \operatorname{Osc}(f, \Sigma)$ (the maximum oscilation over any simplex).

Proof. First note $\Sigma^t < M^{t+\delta}$ by definition of the oscillation, since

$$f(vertex) < t \implies f\big|_\sigma < t + \operatorname{Osc}(f, \Sigma).$$

Likewise let $\mathcal{U}$ be the union of all the simplicies which have nontrivial intersection with M^t. Then by a similar argument we have

$$U < \Sigma^{t+\delta},$$

hence $M^t < \Sigma^{t+\delta}$.

Proposition III.1.4

We have that

$$\nu_{2\cdot\operatorname{Osc}(f,\Sigma)}(f) \le \frac{|\Sigma|}{2}.$$

Proof. $V(f)$ and $V(\Sigma, u)$ are δ-interleaved. Hence every fintie bar in $\mathcal{B}(f)$ of length $> 2\delta$ is matched with a bar in $\mathcal{B}(\Sigma, u)$. But the number of finite bars in $\mathcal{B}(\Sigma, u)$ is $\le \frac{|\Sigma|}{2}$.

Proof of Theorem III.1.2. Fix $r > 0$, $r \ll 1$. Then m^d admits a triangulation into kr^{-d} simplices of diameter r, where $d = \dim M$ and k depends on M itself (having to do with its volume). Then $\operatorname{Osc}(f, \Sigma) \le Lr$. Hence we choose $r = \frac{\delta}{2L}$, and we have

$$\frac{|\Sigma|}{2} \le kr^{-d} = k'\frac{L^d}{\delta^d}.$$

Hence

$$\nu_\delta(\mathcal{B}(f)) \le \nu_{2\operatorname{Osc}}(\mathcal{B}(f)) \le k'\frac{L^d}{\delta^d}.$$

as desired.

III.2. Digression

Let T_n be the set of all trigonometric polynomials of degree $\le n$ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Theorem III.2.1 (Chebyshev, alternance/equioscillation)

A trigonometric polynomial $p \in T_{n-1}$ on S^1 provides the best uniform approximation to a continuous function f if and only if there exists

$$0 \leq x_1 < \cdots < x_{2n} \leq 2\pi$$

such that the differences $f(x) - p(x_i)$ reach the maximum value $\pm\|f - p\|_\infty$ with alternating signs.

Example III.2.1

$p = 0 \in T_{n-1}$ provides the best approximation to $f(x) = \cos(nx)$.

We'll do something similar-ish for Morse functions. We'll do simplex and critical point counting to achieve this. In particular we sketch Alternance $\implies$ best approximation.

Lemma III.2.2

Let h, q be two Morse functions on a smooth closed manifold such that for some $c < 0$ q has $< 2\nu_c(h) + b(M)$ critical points where $b(M) = \dim H_*(M)$ then $\|h - q\|_\infty \geq \frac{c}{2}$.

Proof. Assume on the contrary that $\|h - q\|_\infty < \frac{c-\varepsilon}{2}$ for $\varepsilon > 0$ very small and let N be the number of critical points of q. We know that $b(M)$ critical points contribute to infinite rays. Hence the number of finite bars

$$\nu(q, 0) \leq \frac{N - b(M)}{2}.$$

Hence

$$\nu(q, \varepsilon) \leq \frac{N - b(M)}{2} < \nu(h, c).$$

But then by stability we have that

$$\nu(q, \varepsilon) \geq \nu(h, \varepsilon + 2\|h - q\|\}) \geq \nu(h, c).$$

This is a contradiction. ∎

Proof of alternance $\implies$ best approximation. Let $h = f - p$ where f is Morse and p is polynomial. Let $c = \|h\|$.

Exercise III.2.2

$\nu(h, 2c - \varepsilon) = n - 1$ (1 is lost for ∞ ray corresponding to H_0) by alternance.

Now let q be any trigonometric polynomial of degree $n - 1$. It has

$$2n - 2 \leq 2(n-1) + 2 = 2n$$

critical points. Note $n - 1 = \nu(h, 2c - \varepsilon)$ and $2 = b(S^1)$. Hence by the lemma $\|h - q\| \geq c$ and so $\|f - (p + q)\| \geq c$. Since this is true for all q, we obtain the result that p is the best. ∎

There is the following generalization.

Theorem III.2.3 (Lev Buhovsky, Jordan Payette, Iosif Polterovich, Leonid Polterovich, Egor Shelukhin, Vukašin Stojisavljević)

Let M^n be a closed Riemannian manifold, $\|f\|_{k,p}$ be the Sobolev norm i.e. L_p-norm of the k-th derivative. Let $w_\delta(f)$ be all the bars of length $> \delta$ (incliuding infinite bars). Then assuming that

$k > n/p$ we have for some constants c_1 and for all $\delta > 0$ for all $f \in C^\infty(M)$ we have

$$w_\delta(|f|) \leq c_1 \delta^{-n/k} \|f\|_{k,p}^{n/k} + b(M).$$

Example III.2.3

Let $p = \infty$, $k = 1$, then $\|f\|_{k,p} = \max |\nabla f| = L$.

Lets now make an application to spectral geometry. Let (M^n, g) be a closed Riemannian manifold and take

$$\Delta f = -\operatorname{div} \nabla f$$

the Laplace-Beltrami operator. The eigenvalues $\Delta f = \lambda f$ is a discrete spectrum

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

and $\lambda_i \to \infty$.

Example III.2.4

The laplacian $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ on the flat torus $\mathbb{T}^n$.

In general, there is a Weyl law

$$\#\{\text{eigenvalues } \leq \lambda\} \sim \text{constant} \cdot \lambda^{n/2}.$$

For $f \in C^\infty(M)$ the nodal set $Z(f) = \{f = 0\}$. A nodal domain is a connected component of $M \setminus Z(f)$. Let $m_0(f)$ be the number of nodal domains.

Theorem III.2.4 (Courant)

Let f be an eigenfunction of Δ corresponding to $\lambda = \lambda_j$. Then $m_0(f) \leq j + 1$.

Combining these two results,

$$m_0(f_\lambda) \leq c(\lambda + 1)^{n/2}$$

The Courant theorem can be generalized in the following directions

- Taking linear combinations of eigenfunctions of eigenvalues $\leq \lambda$?
- In 1D this above works, because eigenvalues are trigonometric polynomials.
- Viro and Arnold gave some counterexamples to some ideas for gemneralization based on algebraic geometry.

Lets look for a persistent formulation! Let

$$m_r(f, \delta) = \dim \operatorname{im}(H_r\{|f| > \delta\} \to H_r\{|f| > 0\} = H_r(M \setminus Z(f))).$$

Then we have

$$m_0(f, \delta) = \# \text{ nodal domains } U \text{ with } \max_U |f| > \delta.$$

We'll call these "deep nodal domains."

Theorem III.2.5

Let F_λ be the span of the eigenfunctions with eigenvalue $\leq \lambda$. Then for all $\varepsilon > 0$, $k > \frac{n}{2}$, and for all

$f \in F_\lambda$ with $\|f\|_{L^2} = 1$ we have

$$m_r(f, \delta) \leq \frac{c_1}{\delta^{n/k}}(\lambda + 1)^{n/2} + C_2$$

Vaguely the vibe is that $f \in F_\lambda$ is "similar" to a polynomial of degree $\sqrt{\lambda}$ and a "Bezout theorem" gives $\{f_\alpha = 0\}, \{f_\beta = -\}$ should have $\sim \sqrt{\alpha\beta}$ intersection points. The proper formulation of Bezout in this setting requires a persistent intersection count.

IV. **Topological Data Analysis**

Point cloud in High Dimensional Space and you want to learn something about it:

(1) You might have a function on the data, predict its values.
(2) Cluster the data (learning hare vs rabbit).
(3) Dimension
(4) Entropy of dynamical process.
(5) (Ghrist) Coverage by a sensor network.

Most Simplified View: Point cloud is sampled from some "platonic" space X:

- Clusters $\leftrightarrow$ connected components of X.
- dimension $\leftrightarrow$ dimension of X.

Big question: X is what? Often we say X is a

(1) Compact Riemannian Manifold
(2) Polyhedra
(3) Variety
(4) Fractal?

Some difficulties in this story in applications:

- X often isn't a manifold.
- Data has noise – sometimes lots of it. Is topology "just" geometry with noise? Well topology is certainly less sensitive to noise than geometry, but how do you tell the difference between sampling a circle with noise and circling a circle union a point with noise?

References

[Mil63] John Willard Milnor. *Morse theory*. 51. Princeton university press, 1963.