Notes on
(Moduli Spaces in Algebraic Geometry and Topology)

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Contents

I. The Introduction

I.1. Setting the Stage: Semistability

Throughout we will take the following notation

- Let k be an algebraically closed field of characteristic 0
- Let G be an algebraic group over k which is linear (affine), i.e., there exists a faithful $G \hookrightarrow \mathrm{GL}(V)$ with closed image. We also assume that G is reductive, i.e., given a finite-dimensional representation V of G with a G-invariant subspace $V' \subseteq V$, there exists a G-invariant complement V'' so that $V = V' \oplus V''$.

Let V be a representation of G. We may consider the algebra $k[V]$ of polynomial functions on V. As a representation of the group this decomposes as

$$k[V] = \bigoplus_{n=0}^{\infty} k[V]_n \cong \bigoplus \operatorname{Sym}_n V^*,$$

where $\operatorname{Sym}_n V^*$ is the n-th symmetric power of the dual space and the isomorphism is an isomorphism of G-representations. Taking the G-invariants gives a graded algebra $k[V]^G$.

Theorem I.1.1 (Hilbert)

The graded algebra $k[V]^G$ is finitely generated.

Definition I.1.1

We write the spectrum $\operatorname{Spec} k[V]^G =: G \mathbin{/\!\!/} V$, indicating an orbit space.

Proposition I.1.2

Let $X_0, X_1 \subseteq V$ be closed, G-invariant, and disjoint. Then there exists some $f \in k[V]^G$ such that $f\big|_{X_0} \equiv 0$ and $f\big|_{X_1} \equiv 1$.

Lemma I.1.3

Let G act on an affine-linear space A (by affine-linear transformations). Then G has a fixed point $o \in A$.

Proof. Let $F(A) = \{\text{affine linear functions } A \to k\}$. This is a dimension $n+1$ vector space with a G-representation, where $n = \dim A$. It admits a G-invariant subspace, the constants $k \subseteq F(A)$. Because G is reductive, $k \subseteq F(A)$ must have a G-invariant complement

$$F(A) = V \oplus k,$$

the subspace V defines the desired point in A. Pick a basis $v_1, \dots, v_n$ of V and find a point where $v_i(o) = 0$ for all i.

Proof of Proposition I.1.2. Consider the map

$$k[V] \twoheadrightarrow k[X_0 \cup X_1] \cong k[X_0] \oplus k[X_1],$$

this map is surjective because X_0, X_1 are closed and disjoint. It is a G-equivariant map because X_0, X_1 are G-invariant sets. Suppose that $k[V]_{\leq n}$ has $(0,1)$ in its image. Well $k[V]_{\leq n}$ is a G-represnetation and the preimage of $(0,1)$ is an affine subrepresentation. It contains a fixed point by Lemma I.1.3, completing the proof.

Corollary I.1.4

We have that

(i) The closure of every G-orbit contains a unique *closed* G-orbit.

(ii)

Get from a friend oof

Proof. Let $v \in V$, if $\overline{G \cdot v} \neq Gv$ then choose $v' \in \overline{G \cdot v} \setminus G \cdot v$ and proceed with induction (the dimension must go down) to show existence. If $\overline{Gv}$ contains two closed orbits X_0, X_1 then there exists some $f \in k[V]^G$ from Proposition I.1.2 so that $f \equiv i$ on X_i. But then $f(v) = 0 = 1$, a contradiction (since f is continuous and G-invariant). Exercise: the dimension must go down...

What's the **upshot** of this? Well for the affine algebraic variety $G \mathbin{\backslash\!\!\backslash} V$ we have

$$\text{The closed points of } G \mathbin{\backslash\!\!\backslash} V \leftrightarrow \text{closed } G\text{-orbits in } V.$$

We now define

Definition I.1.2

The unstable locus for the G-representation V is

$$V^{un} := \{v \in V \mid \overline{Gv} \ni o\} \tag{1}$$

Note that $V^{un} = Z(f_1, \ldots, f_r)$ where $k[V]^G$ is generated by homogeneous polynomials $f_1, \ldots, f_r$ of degree $d_i > 0$.

We call $V^{ss} = V \setminus V^{un}$ the semistable locus. A point $v \in V$ is called semistable if it lies in the semistable locus and it is called stable if Gv is closed and $G_v = \mathrm{Stab}_G\, v$ is finite.

Example I.1.1

Let $G = \mathbb{G}_m(k) = k^\times$. This acts on V by scalar multiplication. This is boring for the following reason: every orbit contains the origin in its closure, and so $V = V^{un}$.

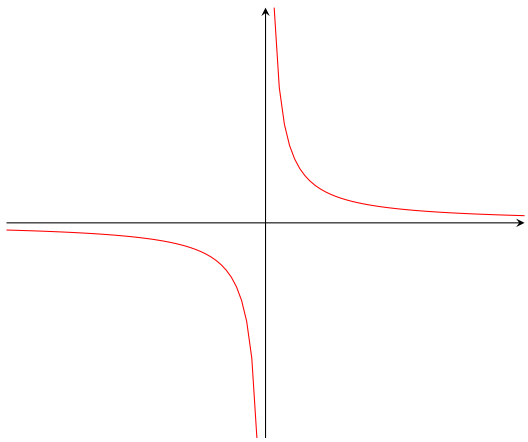
Example I.1.2

Let $\mathbb{G}_m(k)$ act on k^2 as follows

$$\lambda \in k^\times \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

What are the orbits of this action

- A typical orbit is a hyperbola. These are all stable.
- There are two orbits which are unstable, the horizontal and vertical axis.



Lets compare this with $G \mathbin{/\mkern-6mu/} V$, well

$$k[z_1, z_2]^G = k[z_1 z_2]$$
$$G \mathbin{/\mkern-6mu/} V = \operatorname{Spec} k[z_1 z_2] = \mathbb{A}^1$$

Example I.1.3

Let $G = \mathrm{GL}_n(k)$ acting on $\mathrm{End}_n(k) =: V$ acting by conjugation. The charactersiic polynomial

$$\det(T - \lambda \,\mathrm{Id}) = \lambda^n + \chi_1(T)\lambda^{n-1} + \cdots + \chi_n,$$

Note that $\chi_i \in k[V]^G$, and these in fact generate. This follows from Jordan Canonical Form. The unstable locus is given by

$$V^{un} = Z(\chi_1, \ldots, \chi_n) = \text{nilpotent matrices}.$$

There's an observation to make here: there are no stable points! There are no matrices with finite centralizer (i.e., finite stabilizer under this action).

It is helpful to focus on the semistable locus V^{ss}, which we'll assume is nonempty. By definition, V^{ss} does not contain the origin. One can check that V^{ss} is invariant under scalar multiplication. By projectivizing

$$\text{closed orbit in } V^{ss} \to \text{closed orbit in } \mathbb{P}V^{ss} \subseteq \mathbb{P}V.$$

Hence, one can consider the set of closed orbits in $\mathbb{P}V^{ss}$. The question is how to describe this algebro-geometrically. The method is via the Proj construction (look it up!)

$$\text{closed orbits in } \mathbb{P}V^{ss} \leftrightarrow \operatorname{Proj}(k[V]^G).$$

We now have an important theorem.

Theorem I.1.5 (Hilbert-Mumford Criterion)

$v \in V$ is unstable for G if and only if it is unstable for a copy of $\mathbb{G}_m(k)$ in G.

Note that this means you can diagonalize the action of $\mathbb{G}_m(k) \hookrightarrow G$ acting on V. Hence $\lambda \in \mathbb{G}_m(k)$ maps to

$$\begin{pmatrix} t^{d_1} & & 0 \\ & \ddots & \\ 0 & & t^{d_n} \end{pmatrix}.$$

Therefore, if $d_1, \ldots, d_r$ are > 0, then the span of the first r coordinates consists of unstable points.

Example I.1.4

Let W be a k-vector space of dimension 2. And let $G = \mathrm{SL}(W) = \mathrm{SL}_2(k)$. We will let G act on $V = k[W]_n = \mathrm{Sym}_n(W^*)$. We obtain a geometric interpretation by projectivizing. We have that $\mathbb{P}W = \mathbb{P}^1$, and an element f of $k[W]_n$ decomposes into n linear factors, and each factor gives a point in $\mathbb{P}^1$. In other words, $\mathbb{P}V$ consists of degree n divisors on $\mathbb{P}W = \mathbb{P}^1$.

Proposition I.1.6

Let $f \in V$, then

$$\begin{aligned} f \text{ is unstable } &\iff f \text{ has a factor (linear form)}^d \text{ where } d > n/2. \\ f \text{ semistable but not stable } &\iff n \text{ even, } f \text{ has a factor (linear form)}^{n/2} \\ f \text{ has strictly semistable closed orbit } &\iff f = \ell_1^{n/2}\ell_2^{n/2} \text{ where } \ell_1, \ell_2 \text{ linearly indep. linear forms} \end{aligned}$$

Proof of $\Longleftarrow$. Let $\ell \in W^* \setminus \{0\}$ appear in f with multiplicity d. Then $f = \ell^d f_1$ where f_1 is homogeneous of degree $n-d$. We will be inspired the Hilbert-Mumford criterion. Choose a basis (z_1, z_2) for W where $z_1 = \ell$. We then consider the action scaling z_1 by λ and z_2 by λ^{-1} so that

$$f(\lambda z_1, \lambda^{-1} z_2) = \lambda^d z_1^d f_1(\lambda z_1, \lambda^{-1} z_2) = \lambda^{2d-n} z_1^d f_1(\lambda^2 z_1, z_2).$$

Now if $d > n/2$ we find that as $\lambda \to 0$ this takes $f(\lambda z_1, \lambda^{-1} z_2) \to 0$. Hence 0 lies in the orbit closure of f.

If n is even and $d = n/2$ then taking $\lambda \to 0$ we find that $f(z_1, z_2)$ lies in the same orbit closure as $z_1^d f_1(0, z_2)$. But then $f_1(0, z_2)$ must be equal to z_2^d. A quick check shows that the orbit of $z_1^d z_2^d$ is closed.

The remaining orbits are closed.

Question (Faye): Why is $G \cdot v$ open in $\overline{G \cdot v}$? With a basic proposition this will imply $\overline{G \cdot v} \setminus G \cdot v$ is of smaller dimension, allowing us to do the dimension reduction.

Proof. Note that $G \cdot v$ is constructible, by Chevalley's theorem and the fact that G is reductive algebraic, and the representation is algebraic. Thus $G \cdot v$ is a union of locally closed sets by definition. Let $v \in U \cap C$ where U (resp. C) is open (resp. closed) in $\overline{G \cdot v}$. We find that $g \cdot (U \cap C)$ covers $G \cdot v$

We continue the example of last time:

Example I.1.5

Let W be a vector space of dimension 2. We saw that if $V_n = k[W]_n$ then $\mathbb{P}V_n$ contains $\{n$ element subsets of $\mathbb{P}W\}$ as a dense open set. Furthermore, there is a unique strictly semistable closed orbit which has two points of multiplicity $n/2$ and $n/2$, and so n is even.

How can we apply this to moduli problems?

Example I.1.6

Consider the moduli space $\mathcal{M}_{0,n}$ which is the isomoprhism classes of n distinct ordered points on $\mathbb{P}^1$. That is $\mathcal{M}_{0,n}$ consists of pairs $(C \cong \mathbb{P}^1, (p_1, \dots, p_n))$. By choosing an isomorphism $C \cong \mathbb{P}^1$ we introduce a $\mathrm{PSL}(2,k)$-ambiguity. By choosing an ordering we introduce a S_n-ambiguity (by reordering the points).

Hence

$$\begin{tikzcd} \mathcal{M}_{0,n} \arrow[d, "S_n"] & \overline{\mathcal{M}}_{0,n}^{HM} \arrow[d, "S_n"] \\ S_n \times \mathrm{PSL}_2(k) \backslash \operatorname{Conf}_n(\mathbb{P}^1) \arrow[r, hook, "opendense"'] & \operatorname{Proj}(k[V]^{\mathrm{SL}_2(W)}), \end{tikzcd}$$

where Conf_n is the ordered configuration space. Here we are doing a Riemann extension theorem. We have a cover over an open dense subset and we're turning this into a branched cover.

As special cases, lets discuss when $n = 4$ and $n = 5$. For $n = 3$ it's boring because $\mathrm{PSL}_2(k)$ acts 3-transitively on $\mathbb{P}^1$, and so $\mathcal{M}_{0,3} = *$. For $n = 4$, we can find an isomorphism taking (p_1, p_2, p_3, p_4) to $(0, \infty, 1, z)$. This gets rid of the $\mathrm{PSL}(2,k)$ ambiguity.

Remark I.1.1

Here S_4 acts by the exceptional morphism $S_4 \to S_3$ given by the different choices of cross-ratio.

Hence

$$\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

There is only one real way to compactify this!

$$\begin{array}{ccc} \mathcal{M}_{0,4} & = & \mathbb{P}^1 \setminus \{0,1,\infty\} \\ \cap\!\!\!| & & \cap\!\!\!| \\ \overline{\mathcal{M}}_{0,4}^{HM} & = & \mathbb{P}^1. \end{array}$$

By the semistability condition observed earlier, we know that $\mathrm{Proj}(k[V]^{\mathrm{SL}(W)})$ has that if two points coalesce then the other two points must also coalesce (this is the unique strictly semistable closed orbit). There are 3 ways to group 4 points into two groups of 2. This explains the compactification!

Lets do this explicitly, we find that, letting $z = t^2$:

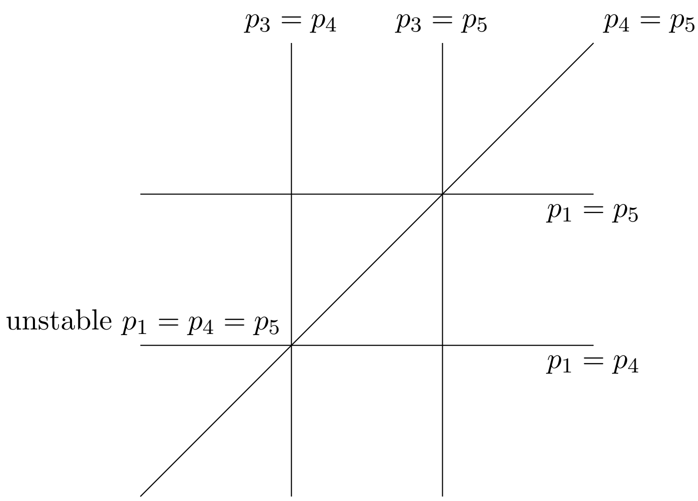
$$(0,\infty,1,t^2) \sim (0,\infty,1/t,t) \xrightarrow{t\to 0} (0,\infty,\infty,0)$$

Now lets do $n = 5$. We take $(p_1, p_2, p_3, p_4, p_5)$ to $(0, \infty, 1, z_1, z_2)$ by the action of $\mathrm{PSL}_2(k)$. Hence

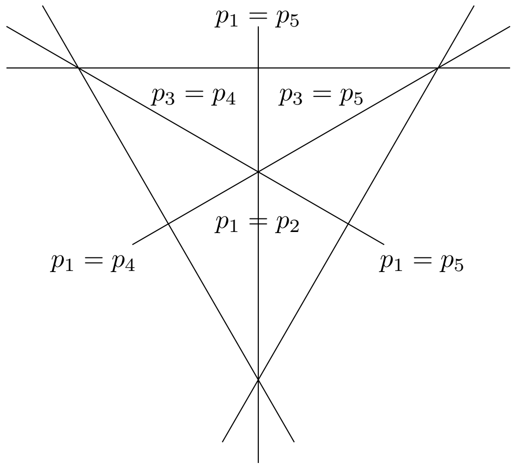
$$\mathcal{M}_{0,5} \cong (\mathbb{A}^1 \setminus \{0,1\})^2 \setminus \text{diagonal} \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2.$$

This almost gives a projective compactification!

Here is the picture in $\mathbb{A}^2$, and each line is removed.



Here is the picture in $\mathbb{P}^2$, where one of the drawn lines is the line at ∞.



We now blow up the 4 triple points. Doing this we obtain 10 copies of $\mathbb{P}^1$ (5 choose 2 is 10) with 15 double points. This gives an intersection graph which is the Peterson graph. This is a Del Pezzo surface of degree 5.

Draw the pictures and put them here

Example I.1.7

Let W be a 3-dimensional k vector space. $\mathbb{P}W \cong \mathbb{P}^2$. We consider $V = k[W]_3$. Then a point in $\mathbb{P}V$ consists of a cubic divisor on $\mathbb{P}W$. Let $G = \mathrm{SL}(W)$. If $Z(F) \subseteq \mathbb{P}W$ is a smooth cubic c curve, then $F \neq 0$. And in fact, in this case F is a stable point under the G-action. We'll delay the proof till later.

Suppose $Z(F)$ is not smooth, say at $p \in \mathbb{P}W$. Choose an affine coordinate system (x, y) with $p = 0$. Then F yields an affine equation

$$f(x, y) = f_2(x, y) + f_3(x, y)$$

of degree 2 and 3 homogeneous parts respectively.

Claim

If f_2 is nondegenerate as a quadratic form, then F is semistable (strictly) its closed orbit contains the cubic XYZ (consiting of three lines).

Proof. Suppose f_2 is nondegenerate. Then f_2 is the product of two independent linear forms. After a change of coordinates this implies that $f_2(x, y) = xy$. Now consider the map

$$g_\lambda = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda^{-2} \end{pmatrix} \in \mathrm{SL}_3(k).$$

Then this takes $F = Zf_2(X,Y) + f_3(X,Y)$ to

$$\lambda^{-2}Zf_2(\lambda X, \lambda Y) + \lambda^3 f_3(X,Y) = Zf_2(X,Y) + \lambda^3 f_3(X,Y).$$

Taking $\lambda \to 0$ gives $Zf_2(X,Y)$. by the change of coordinates we find that this is XYZ. We then need to show that XYZ has a closed orbit (exercise). This shows that F is strictly semistable with orbit closure XYZ (note XYZ has infinite stabilizer).

Now suppose that f_2 is degnerate. Then we can choose coordinates so that $f_2(x,y) = \text{const}\cdot x^2$. So then $F(X,Y,Z)$ is

$$\text{const} \cdot X^2 Z + f_3(X,Y).$$

Then take

$$h_\lambda = \begin{pmatrix} \lambda^3 & & \\ & \lambda^2 & \\ & & \lambda^{-5} \end{pmatrix} \in \mathrm{SL}_3(k).$$

Multiplying by this gives

$$\text{const}\lambda X^2 Z + f_3(\lambda^3, \lambda^2 Y).$$

Taking $\lambda \to 0$ takes this to 0, and so F is unstable.

What does the condition that f_2 is nondegenerate mean in terms of the cubic curves? Well it actually means that p is an ordinary double point! So the strictly semistable orbits are the nodal cubic curves, a quadric witha line, and the union of three lines.

What is not allowed? Unstable curves are like cuspidal cubics or a quadric with a tangent line, or three lines passing through a point. Or a line with multiplicity two, or three.

A final computation is that

$$\dim \mathbb{P}k[W]_3 = 10 - 1 = 9$$
$$\dim \mathrm{SL}(W) = 8$$
$$\dim \mathrm{SL}(W) \backslash\!\backslash \mathbb{P}k[W]_3 = 1.$$

I.2. More on Hypersurfaces

Lets talk more about cubic curves in $\mathbb{P}^2$.

Claim

Smooth cubic plane curves are Hilbert-Mumford stable.

We'll prove something stronger.

Proposition I.2.1

Let W be a k vector space of dimension > 1. Let $V = k[W]_d$. Then $\mathbb{P}V$ parameterizes degree d divisors on $\mathbb{P}W$. We claim that a hypersurface H of degree d which is smooth is Hilbert-Mumford semistable. Furthermore, the locus $\{\text{singular hypersurface}\} \subseteq \mathbb{P}V$ admits an equation which is $\mathrm{SL}(W)$-invariant.

Proof. Note that the latter statement implies the former, since we will construct a polynomial invariant function on V where the smooth hypersurfaces do not vanish. To do the former, we consider the d-fold Veronese embedding

$$\begin{aligned} \mathbb{P}(W) &\hookrightarrow \mathbb{P}(\operatorname{Sym}^d W) \\ w &\mapsto w^d. \end{aligned}$$

Then degree d hypersurfaces in $\mathbb{P}W$ are precisely preimages of hyperplane sections in $\mathbb{P}(\operatorname{Sym}^d W)$. The hypersurface is singular if and only if the intersection in $\mathbb{P}(\operatorname{Sym}^d W)$ of the hyperplane nad $\mathbb{P}(W)$ is not transverse.

We see that these singular hypersurfaces are parameterized (not 1-1) by a smooth variety as follows:

- Pick a point p on the image of the Veronese embedding.
- Pick a tangent line at p to the image of the Veronese embedding.
- Consider any hyperplane containing this tangent line. (Note: a hyperplane in $\mathbb{P}(\operatorname{Sym}^d W)$ is the same as an element of $\mathbb{P}V$).

This gives a codimension 1 condition on $\mathbb{P}V$, and the generic point is that there is one common tangent line and nothing else. In $\mathbb{P}(W)$ this corresponds to a single double point (this is analogous with Morse theory/Morsification).

This gives a hypersurface in $\mathbb{P}(\operatorname{Sym}^d W)$ which parameterizes singular hypersurfaces (meta). Lets call the equation of this locus the discriminant and it has equation $\Delta \in k[V]_m$ for some m. This is unique up to scaling! $\mathrm{SL}(W)$ may not leave this equation invariant, but it scales it. Hence we get a character $\mathrm{SL}(W) \to \mathbb{G}_m(k) = k^\times$. But $\mathrm{SL}(W)$ admits no nontrivial character, and so this must be constant! We're done, because $\mathrm{Id} \mapsto 1$.

Exercise I.2.1

Redo the above proof by using the classical theory of resultants. The author of these notes thinks this is much more down to earth and easy to see...

Before we go on to talk about residues, we need to talk about residues. Take an open subset $U \subseteq \mathbb{C}$ and a meromorphic differential $\alpha = f\,\mathrm{d}z$ on U. We define

$$\operatorname{Res}_p \alpha = a_{-1} = \frac{1}{2\pi i} \int_{\text{small circle about } p} \alpha,$$

if $f = \sum_{k \geq k_0} a_k (z-p)^k$ in a Laurent expansion. This also makes sense on a Riemann surface! It turns out that this can also be done via algebraic geometry. Tate in 1968 gives a direct way to do this for rational 1-forms on smooth curves/k. This has a higher-dimensional generalization: Suppose we have a smooth k-variety M of dimension $n+1$ and a smooth hypersurface (divisor) $D \subseteq M$. Fix a regular $(n+1)$-form α on $M \setminus D$ with a pole of order 1 along D. Then $\operatorname{Res}_D \alpha$ is defined as a regular n-form on D as follows:

Let $p \in D$ and write α as $\frac{\mathrm{d}f}{f} \wedge \beta$ where f is a local equation for D about p. This β will not be unique, but β restricted to D as an n-form *will* be unique.

Exercise I.2.2

Check this is unique.

Example I.2.3

Let $f : \mathbb{A}^{n+1} \to k$ be a regular function whose zero set D is smooth. We define $D_i \subseteq D$ as the locus where $\frac{\partial f}{\partial z_i} \neq 0$. Then $D = D_0 \cup \cdots \cup D_n$ (coordinates are $z_0, \ldots, z_n$). Now consider the following form for some $0 \leq i \leq n$:

$$\mathrm{d}f \wedge \Big((-1)^i \,\mathrm{d}z_0 \wedge \cdots \wedge \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_n \Big) = \frac{\partial f}{\partial z_i} \,\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_n.$$

We now consider

$$\operatorname{Res}_D \frac{\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_n}{f} =: \operatorname{Res}_D \alpha$$

To calculate this, we write

$$\frac{\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_n}{f} = \frac{\mathrm{d}f}{f} \wedge \frac{(-1)^i \,\mathrm{d}z_0 \wedge \cdots \wedge \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_n}{\frac{\partial f}{\partial z_i}}.$$

Call this β_i. We find then that

$$\beta_i\big|_{D_i} = \operatorname{Res}_D \alpha\big|_{D_i}.$$

One checks that these agree on overlaps, and so define a form.

Example I.2.4

Fix $F \in k[Z_0, \ldots, Z_{n+1}]_d$ where $d \geq n+2$ whose zero set X in $\mathbb{P}^{n+1}$ is smooh. Let $G \in k[z_0, \ldots, z_{n+1}]_{d-n-2}$ be relatively prime to F (automatic if $n \geq 1$).

Consider the form

$$\alpha := \frac{G \,\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_{n+1}}{F},$$

this is a rational $n+2$ form on $\mathbb{A}^{n+2} = \P^{n+2} \setminus \P^{n+1}_\infty$. The total expression is homogeneous of degree 0. This implies that α has a pole of order 1 at $\mathbb{P}^{n+1}_\infty$. This can be seen because along a ray this will look like $\frac{\mathrm{d}z}{z}$. We calculate $\operatorname{Res}_{\mathbb{P}^{n+1}_\infty} \alpha$, which will produce an $(n+1)$-form with a simple pole along X so that

$$\mathrm{Res}_X \, \mathrm{Res}_{\mathbb{P}^{n+1}_\infty} \alpha,$$

which is a regular n-form on X with zero divisor given by the "restriction" of G to X. If $d = n+2$ and $G = 1$, then we obtain a regular n-form without zeroes on X. We call this α_F. We find that by scaling

$$\alpha_{\lambda F} = \lambda^{-1}\alpha_F.$$

This will represent a line in $H^{n,0}(X)$. Note that this is SL-invariant in a certain sense as well.

Lets restrict this back to cubic plane curves, and try to understand that case very well.

Example I.2.5

Let $\dim W = 3$ and $F \in k[W]_3$ define a smooth cubic curve $C_F \subseteq \mathbb{P}W \cong \mathbb{P}^2$. We now have the regular 1-form α_F. With $k = \mathbb{C}$, we work with the Euclidean topology for decency's sake.

$$\begin{aligned} \mathbb{Z}^2 \cong H_1(C_F;\mathbb{Z}) &\to \mathbb{C} \\ c &\mapsto \int_c \alpha_F. \end{aligned}$$

The image of $H_1(C_F;\mathbb{Z})$ in $\mathbb{C}$ is given by a lattice. Furthermore

$$\alpha_F \wedge \alpha_F = 0 \qquad \sqrt{-1}\alpha_F \wedge \overline{\alpha_F} \text{ is everywhere } > 0.$$

Call this lattice $L_F \subseteq \mathbb{C}$. Analytically it happens that $(C_F \cong \mathbb{C}/L_F)$. We can consider the space

$$\mathcal{L}(\mathbb{C}) = \{\text{lattices in } \mathbb{C}\}. \qquad \text{cx manifold}$$

The map we've produced above is

$$\begin{aligned} V = k[W]_3 \supseteq V^{sm} &\to \mathcal{L}(\mathbb{C}) \\ F &\mapsto L_F. \end{aligned}$$

This map has some curious properties

(1) $L_{\lambda F} = \lambda^{-1} L_F$.

(2) $F \mapsto L_F$ is constnat on $\mathrm{SL}(W)$-orbits.

We now consider the Eisenstein series

$$G_{2k}(L) = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^{2k}}.$$

We require that $k \geq 2$, so that this will be uniformly convergent. We find that

$$G_{2k}(\lambda L) = \lambda^{-2k} G_{2k}(L).$$

Thus the G_{2k} produce functions which are SL_2-invariant on V^{sm}!!! This is incredible !!!

Last time: Let W be a vector space of dimension 3 over $\mathbb{C}$ and consider $V = \mathbb{C}[W]_3$. Then $\mathrm{SL}(W)$ acts on V and on V^{sm} (the smooth cubics). Given $F \in V^{sm}$ we produced a holomorphic 1-form α_F on $C = Z(F) \subseteq \mathbb{P}W$. In a diagram, we have

$$V \supseteq V^{sm} \ni F \mapsto (C, \alpha_F) \mapsto L_F \in \mathcal{L}(\mathbb{C}),$$

where $\mathcal{L}(\mathbb{C})$ consists of 2×2 lattices in $\mathbb{C}^2$. We also showed that $L_{\lambda F} = \lambda^{-1} F$ for $\lambda \in \mathbb{C}^\times$. Finally we constructed Eisenstein series

$$G_{2k}(L) = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^{2k}},$$

where $k \geq 2$. Furthermore, note that

$$G_{2k}(L_{\lambda F}) = \lambda^{2k} G_{2k}(F).$$

And even better, this is an $\mathrm{SL}(W)$-invariant function on V^{sm}. We will show that it extends to an $\mathrm{SL}(W)$-invariant function on V itself.

For now, we focus on constructing a map $\mathcal{L}(\mathbb{C}) \to V^{sm}$. Fix a lattice $L \subseteq \mathbb{C}$ a lattice and set $C_L \coloneqq \mathbb{C}/L$. This is a genus one Riemann surface with a base point $o \in \mathbb{C}_L$, and $\mathrm{d}z$ gives rise to a 1-form α_L on C_L (by pushforward). Thus, from the lattice L we get a triple of data:

(1) A genus one Riemann Surface C_L,

(2) A point $o \in C_L$,

(3) A 1-form α_L on C_L.

Let A_L be the $\mathbb{C}$-algebra of holomorphic functions $C_L \setminus \{o\} \to \mathbb{C}$ (meromorphic at o).

Claim

A simple pole is not possible.

Proof. Let $f \in A_L$ have a simple pole at o. Consider $f\alpha_L$, we obtain a 1-form with a simple pole at the origin. Of course this has a residue $\mathrm{Res}_o f\alpha_L$. But this must be zero by the residue theorem!

$$\mathrm{Res}_o f\alpha_L = \frac{1}{2\pi i} \int_{\text{circle about } o} f\alpha_L = 0,$$

but this circle is nullhomologous in the torus C_L, and so the integral is zero. This would show that f is holomorphic at o, and so f is a constant (since C_L is compact).

We'll now construct the Weierstrass $\wp$-function. This has a similar flavor to the Eisenstein series:

$$\wp_L(z) = \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

One can check the following.

Proposition I.2.2

The Weierstrass $\wp$-function for a lattice L satisfies:

(1) $\wp_L$ is holomorphic on $\mathbb{C} \setminus L$ and $\wp_L(-z) = \wp_L(z)$.

(2) $\wp_L$ is L-translation invariant, so that $\wp_L : C_L \setminus \{o\} \to \mathbb{C}$ is well-defined. Furthermore, $\wp_L$ has a pole of order 2 at o.

(3) $\wp_L$ has a Laurent expansion

$$\wp_L(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n}(2n+2)(L) z^{2n}.$$

Hence $\wp_L \in A_L$ with a second order pole.

Exercise I.2.6

Do these computations. They are accessible to anyone who has taken complex analysis.

We'll now consider $\wp'_L \in A_L$, which has a pole of order 3. Now any pole order at least two can be realized! Namely $\wp_L^{r-1}\wp'_L$ has pole order $2r+1$ and $\wp_L^r$ has pole order $2r$. But wait! This implies that A_L is generated by $\wp_L, \wp'_L$. Why? Take an element $f \in A_L$ of some pole order, and remove poles by adding/subtracting products of $\wp_L$ or $\wp'_L$. Just continue to do this.

As an explicit example, we see that $\wp_L^3$ and $\wp_L'^2$ both have pole order 6. Then

$$(\wp'_L)^2 - 4\wp_L^3,$$

with pole order ≤ 4. We find that the Laurent series depends on z^2 only. We have two candidates to get rid of the pole orders $\wp_L^2$ and $\wp_L$. Do the work!

Proposition I.2.3

There is a differential equation satisfied by $\wp_L$ of the following form

$$(\wp'_L)^2 = 4\wp_L^3 - g_2(L)\wp_L - g_3(L),$$

for some $g_2(L), g_3(L)$, which we can express as

$$g_2(L) = 60 G_4(L) \qquad\qquad g_3(L) = 140 G_6(L).$$

This algebra A_L separates points (even infinitesimally, i.e., tangent vectors) in C_L. Hence the algebra provides an embedding

$$(\wp_L, \wp'_L) : C_L \setminus \{0\} \hookrightarrow \mathbb{C}^2,$$

and this is actually a proper embedding. The image satisfies the following equation, for (x, y) coordinates on $\mathbb{C}^2$:

$$y^2 = 4x^3 - g_2 x - g_3.$$

But this curve is irreducible! And this is a proper embedding, and so this has to be an isomorphism. Thus the curve $y^2 = 4x^3 - g_2 x - g_3$ is smooth. Smoothness here is equivalent to

$$\operatorname{Disc}(4x^3 - g_2 x - g_3) = g_2^3 - 27g_3^2 \neq 0$$

If we projectively complete in $\mathbb{P}^2$, we obtain all of C_L, and the resulting curve remains smooth. In fact the origin maps to a flex point. There are many classical stories with this. The group law is transferred by the following 2 rules

(1) o maps to the identity element, which we'll call 0.
(2) for points P, Q, R on the curve which are collinear, we have $P + Q + R = 0$.

We can now consider the 1-form $\frac{\mathrm{d}x}{y}$ on the elliptic curve and find

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}\wp_L}{\wp_L'} = \mathrm{d}z.$$

We call the above the Weierstrass form of C_L.

Now identify W with $\mathbb{C}^3$ affine coordinates x, y, z. We find that

$$F \in V^{sm} \mapsto L_F \mapsto \text{Weierstrass form of } C_{L_F} \text{ given by } (g_2, g_3).$$

This constructs a *canonical* representative of F in its $\mathrm{SL}(W)$-orbit. One can show in fact that g_2, g_3 are invariants of F. One may check that they are actually regular functions $g_2, g_3 \in \mathbb{C}[V]^{\mathrm{SL}(W)}$.

We'll now switch gears to the connections with modular forms. Here is the naive defintion of a modular form for $\mathrm{SL}_2 \mathbb{Z}$.'

Definition I.2.1

A modular form for $\mathrm{SL}_2 \mathbb{Z}$ of weight w is a holomorphic function $\Phi : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ which

(1) is homogeneous of degree $-w$, that is $\Phi(\lambda L) = \lambda^{-w}$,
(2) satisfies a boundedness condition that if $\Delta \subseteq \mathbb{C}$ is a disk then Ψ is bounded on all the lattices which meet this disk only at the center of the disk.

Here is a more standard definition. A modular form of weight w is a holomorphic function $f : \mathbb{H}^2 \to \mathbb{H}^2$ satisfying

(1) automorphy, i.e.,

$$f(\gamma \cdot \tau) = (c\tau + d)^w f(\tau),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2 \mathbb{Z}$ acts on $\tau \in \mathbb{H}^2$ in the obvious way.

(2) Boundedness condition: $\mathrm{Im}(\tau)^w \varphi(\tau)$ is bounded above if $\mathrm{Im}(\tau) \geq 1$.

Exercise I.2.7

Show these two definitions are equivalent.

Notice that $\Phi(L) = \Phi(-L) = (-1)^w \Phi(L)$ and so w must be even. To do the exercise, notice the following $\mathrm{SL}_2\,\mathbb{Z}$ cover

$$\begin{tikzcd}
\widetilde{\mathcal{L}}(\mathbb{C}) \arrow[d, "\mathrm{SL}_2\,\mathbb{Z}"'] & = & \{(\omega_0,\omega_1) \in \mathbb{C}^2 : \mathrm{Im}(\omega_1/\omega_0) > 0\} & \cong & \mathbb{C}^\times \times \mathbb{H}^2 \\
 & & (\omega_0,\omega_1) \arrow[rr, mapsto] & & (\omega_0, \omega_1/\omega_0) \\
\mathcal{L}(\mathbb{C}) & & & &
\end{tikzcd}$$

We may then consider

$$\varphi : \mathbb{H}^2 \to \mathbb{C}$$
$$\varphi(\tau) \coloneqq \Phi(\mathbb{Z} \oplus \tau\mathbb{Z}).$$

We then find that

$$\begin{aligned}
\varphi\left(\frac{a\tau+b}{c\tau+d}\right) &= \Phi\left(\mathbb{Z} \oplus \frac{a\tau+b}{c\tau+d}\mathbb{Z}\right) = (c\tau+d)^w \Phi((c\tau+d)\mathbb{Z} \oplus (a\tau+b)\mathbb{Z}) \\
&= (c\tau+d)^w \Phi(\mathbb{Z} \oplus \tau\mathbb{Z}) = (c\tau+d)^w \varphi(\tau).
\end{aligned}$$

Lets explain the equivalence of these two boundedness conditions. We consider the $\mathrm{SL}_2\,\mathbb{Z}$-fundamental domain on $\mathbb{H}^2$. Namely:

$$\{\tau \in \mathbb{H}^2 \mid |\mathrm{Re}(\tau)| \le 1/2, |\tau| \ge 1\}.$$

The boundedness condition actually gives a canonical value at $i\infty$ as the limit $\tau^w\varphi(\tau)$ as $\tau \to i\infty$. Equivalently, Φ takes a well-defined value on a degenerate rank one lattice (which we can always choose to be $\mathbb{Z}$). We call Φ a <u>cusp form</u> if

$$\tau^w \varphi(\tau) \to 0 \text{ as } \tau \to i\infty$$

equivalently $\Phi(\mathbb{Z}) = 0$.

Modular forms make up a graded algebra, and in fact it is generated by g_2 (resp. G_4) and g_3 (resp. G_6). The cusp forms make up a principal ideal generated by

$$\Delta = g_2^3 - 27g_3^2,$$

the discriminant.

Upshot: We can then go back to the beginning! g_2 and g_3 define invariants for the group $\mathrm{SL}(W)$. Furthermore the singular cubics are precisely given by theinvariant $\Delta = 0$.

$$\mathbb{C}[V]^{\mathrm{SL}(W)} = \text{algebra of modular forms} = \mathbb{C}[g_2, g_3],$$

with Δ parameterizing the singular locus. Note that there is a $\mathbb{C}^\times$ action on both sides. The g_2 gets scaled in degree 4 and the g_3 in degree 6. The j-line is given by

$$j = \frac{g_2^3}{\Delta}.$$

Recall from last time for W a 3-dimensional vector space and $V = k[W]_3$ we have a map

$$\begin{tikzcd}
V^{sm} = V^{st} \arrow[rr, "hom.degree-1"] \arrow[dr] \arrow[dd, "/\mathbb{C}^\times"'] & & \mathcal{L}(\mathbb{C}) \arrow[dd, "/\mathbb{C}^\times"] \\
& \mathrm{SL}(W)\backslash V^{st} \arrow[ur, "\cong"] \arrow[dd] & \\
\mathbb{P}(V^{st}) \arrow[dr] & & \mathcal{L}(\mathbb{C})/\mathbb{C}^\times = \mathcal{M}_{1,1} \\
& \mathrm{SL}(W)\backslash \mathbb{P}(V^{st}) \arrow[ur, "\cong"] &
\end{tikzcd}$$

Now the stabilizer of a smooth cubic under the $\mathrm{SL}(W)$-action acts simply transitively on its set of flex points (thus as a stack $\mathrm{SL}(W)\backslash\mathbb{P}(V^{st})$ has a large global interia group compared to $\mathcal{M}_{1,1}$)

Recall I.2.8

Let X be a projective variety (without a particular embedding). Let ℓ be a line bundle on X. Consider $E(\ell^*)$ the total space of its dual.

A section s of $\ell^{\otimes n}$ gives a function on $E(\ell^*)$ which is homogeneous of order n i.e., $s(\lambda e) = \lambda^n s(e)$. This is necessarily zero on the zero section (constant if $n = 0$). The condition that a line bundle is ample is that you can separate all points outside the zero section with such functions s. We give formal definitions below. Before that we need some setup

For $n > 0$ we have a basis $s_0, \ldots, s_N$ on $H^0(X, \ell^{\otimes} n)$ and the map

$$[s_0 : \cdots : s_N] : X \dashrightarrow \mathbb{P}^N.$$

This is everywhere well-defined if for every $p \in X$ there exists a section which is nonzero at p.

Definition I.2.2

We say a line bundle ℓ is <u>ample</u> if for some $n > 0$ the map $X \to \mathbb{P}^N$ is an everywhere well-defined projective embedding. We say it is <u>very ample</u> if n can be chosen to be 1.

A less coordinate specific description is that the map $X \to \widehat{\mathbb{P}}H^0(X, \ell^{\otimes n})$ is a projective embedding (the right hand side the space of hyperplanes).

In that case, we can consider the graded algebra

$$A(\ell) = \bigoplus_{n \geq 0} H^0(X, \ell^{\otimes n}).$$

Under the assumption that ℓ is ample, we find that $A(\ell)$ is finitely generated, and hence there is an associated variety. Consider $\operatorname{Spec} A(\ell)$. The preceding discussion produces a map

$$E(\ell^*) \to \operatorname{Spec} A(\ell).$$

The ampleness translates in geometric terms we have topologically

$$\begin{tikzcd} E(\ell^*) \arrow[rr] \arrow[dr, two heads] & & \operatorname{Spec} A(\ell) \\ & E(\ell^*)/\text{zero section} \arrow[ur] & \end{tikzcd}$$

On the other hand there is an action of $\mathbb{G}_m$. We can instead remove the zero section and mod out by the action of scaling. The ample condition means that $X \xrightarrow{\cong} \operatorname{Proj} A(\ell)$.

Of course you can restrict the tautological line bundle of $\mathbb{P}^N$ to X if X is already embedded in a projective space. These constructions are inverse to each other.

Back to Geometric Invariant Theory. Suppose G is a reductive group acting on X.

Definition I.2.3

A linearization of this G-action consists of an ample line bundle ℓ on X and a lift of the G-action on X to the line bundle ℓ.

Given a linearization, G will act on $E(\ell^*)$, and we give a map

$$E(\ell^*) \to (H^0(X, \ell^{\otimes n}))^*,$$

and G acts on both sides. This map "blows down the zero section" (and that's all it does), and the image is a G-invariant closed cone. On the right hand side we have the ordinary notions of stability/semistability

We then take

Definition I.2.4

A G-orbit in $E(\ell^*)$ is unstable if its closure hits the zero section. Otherwise it is semistable. It is called stable if the G-orbit is closed in $E(\ell^*)$, does not hit the zero section, and has finite stabilizer.

This is compatible with these notions on the vector space $(H^0(X, \ell^{\otimes n}))^*$ equipped with the G-action.

We then can consider invariant rings

$$\operatorname{Spec}(A(\ell)^G)$$

$$0 \neq \text{closed points} \leftrightarrow \text{closed } G\text{-orbit in } E(\ell) \setminus \text{zero section}$$

We cal also conisder for $A(\ell)$ (or any finitely generated graded k-algebra) the Hilbert series

$$\sum_{n \geq 0} \dim A(\ell)_n t^n.$$

From a certain index onwards we have that $\dim A(\ell)_n$ is given by a polynomial in n. The degree of that polynomial is in fact the dimension of X, and it is called the Hilbert polynomial of (X, ℓ).

Example I.2.9

Let C be a smooth genus g curve with $g \geq 2$. This comes with the canonical line bundle Ω_C which is ample (proved using Riemann-Roch). We know that

$$h^0(C, \Omega_C^{\otimes n}) = \dim H^0(C, \Omega_C^{\otimes n}) = (2g-2)n + 1 - g$$

for $n \geq 2$. Otherwise

$$h^0(C, \Omega_C) = g \qquad\qquad h^0(C, \Omega_C^{\otimes 0}) = 1.$$

Thus the Hilbert polynomial for C is given by

$$(2g-2)n + 1 - g.$$

We can use some power of ℓ to obtain an embedding

$$X \hookrightarrow \widehat{\mathbb{P}} H^0(X, \ell^{\otimes n}) \cong \mathbb{P}^N,$$

the right hand embedding is unique up to the $\mathrm{SL}_{N+1}(k)$-action. Hence we get an orbit of subvarieties. We want to make this into as big of a family of objects X moving around as possible. For technical reasons we want the family to be flat.

Naive Idea: Given $X \hookrightarrow \mathbb{P}^N$ we obtain a graded ideal $I(X) \subseteq k[z_0, \ldots, z_N]$. The quotient

$$0 \to I(X) \to k[z_0, \ldots, z_N] \to A(X) \to 0.$$

As we let X move, the graded ideal I will move. To make this move in a family take the degree d be so high that $I(X)_d$ intersects to X. We can consider $I(X)_d \subseteq k[z_0, \ldots, z_N]_d$ and vary the subspace and associate to each deformed subspace the zero set. This is naive but terrible! The number of hypersurfaces that you use to describe X can be way greater than the dimension of X.

Instead we must take all degrees into account at the same time. The way to do this is to look at all the projective subschemes with a given Hilbert polynomial P. Grothendieck proved that this indeed has the structure of a variety itself (in fact a flat family). We'll call this $\mathrm{Hilb}^P(\mathbb{P}^N)$. The construction gives an action of $\mathrm{SL}_{N+1}(k)$.

So now we kill the ambiguity! We consider the GIT quotient $\mathrm{Hilb}^P(\mathbb{P}^N) /\!\!/ \mathrm{SL}_{N+1}(k)$.

II. A long example: The moduli space of curves

The primary reference for this section is work of Deligne–Mumford ("On the irreducibility of the moduli space of curves") and work of Giesler (connecting Deligne–Mumford to GIT).

We'll proceed with the program above for (C, Ω_C). For $n \geq 3$ we get an $\mathrm{SL}_{N+1}(k)$-orbit of curves in $\mathbb{P}^N$ where $N = (2g-2)n - g$. In fact, we get an $\mathrm{SL}_{N+1}(k)$ orbit in the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^N)$

for $P(n) = (2g-2)n - g + 1$. Giesler proved that if $n \geq 5$ then the irreducible component of $\mathrm{Hilb}^P(\mathbb{P}^N$ containing these pluri-canonically (taut. bundle is a tensor power of canonical bundle) embedded curves has

stable $\equiv$ semistable, closed orbits are exactly the Deligne-Mumford stable curves.

To appreciate this, we must be reminded of what Deligne-Mumford stable means.

Definition II.0.1

C is called Deligne-mumford stable of genus $g \geq 2$ provided that

- C is connected,
- C has at worst ordinary double points,
- Every irreducible component of C isomorphic to $\mathbb{P}^1$ contains at least 3 double points.

We then take

$$\overline{\mathcal{M}}_g = \mathrm{SL}_{N+1}(k)\backslash\text{semistable locus},$$

as a projective algebraic variety (or a Deligne-Mumford stack).

III. Automorphic Forms and Ball Quotients

Let W be a $\mathbb{C}$-vector space of dimension $n+1$ equipped with a Hermitian form of signature $(1,n)$. By definition there exists coordinates $(z_0, \dots, z_n)$ such that

$$\langle z, z\rangle = |z_0|^2 - |z_1|^1 \cdots - |z_n|^2 \tag{2}$$

We can now consider the subspace

$$E_W = \{z \in W \mid \langle z, z\rangle > 0\}.$$

Necessarily this means that $z_0 \neq 0$ and so we can divide to obtain

$$\left|\frac{z_1}{z_0}\right|^2 + \cdots + \left|\frac{z_n}{z_0}\right|^2 < 1.$$

This shows that $\mathbb{P}(E_W) \subseteq \mathbb{P}W$ is just a copy of the open unit ball in $\mathbb{C}^n$. We denote this by

$$\mathbb{B}_W = \mathbb{P}(E_W).$$

We find that $U(W)$ (that is $U(1,n)$) acts on W and preserves E_W and $\mathbb{B}_W$. One may show that that the action of $U(W)$ on $\mathbb{B}_W$ is transitive. A point in $\mathbb{B}_W$ is given by a positive definite line L ($L^\perp$ negative). The stabilizer is given as

$$\mathrm{Stab}_{U(W)} L \cong U(L) \times U(L^\perp) \cong U(1) \times U(n),$$

which is compact.

This subgroup is a maximal compact subgroup. Hence $\mathbb{B}$ is what is called a bounded symmetric domain. Consider a discrete subgroup $\Gamma < U(W)$, and suppose further that $\Gamma \backslash U(W)$ is finite volume

Definition III.0.1

We will overload the word "lattice" and call $\Gamma < U(W)$ such that Γ is discrete and $\Gamma \backslash U(W)$ is finite volume a lattice. We will try to make this nonconfusing.

Definition III.0.2

A Γ-automorphic form on $\mathbb{B}$ of weight w is a Γ-invariant holomorphic function $\Phi : E_W \to \mathbb{C}$ which is homogeneous of degree $-w$:

$$\Phi(\lambda e) = \lambda^{-w} \Phi(e),$$

and plus a boundedness condition (turns out to be empty when $n > 1$, and we discussed it for $n = 1$).

Lets get some Hard Facts: Let A_w^Γ be the space of Γ-automorphic forms of weight w.

(1) A_w^Γ is finite dimensional.
(2) $A_\bullet^\Gamma = \bigoplus_w A_w^\Gamma$ is a finitely generated $\mathbb{C}$-algebra.
(3) There is a map $\Gamma \backslash E_W \to \operatorname{Spec}(A_\bullet^\Gamma)$ which inverts the $\mathbb{C}^\times$ action. Projectivizing we obtain a map $\Gamma \backslash \mathbb{B}_W \to \operatorname{Proj}(A_\bullet^\Gamma)$. Call $X_\Gamma \coloneqq \Gamma \backslash \mathbb{B}_W$.
(4) $X_\Gamma \to \operatorname{Proj}(A_\bullet^\Gamma)$ maps isomorphically onto a complement of a finite set in the projective variety $\operatorname{Proj}(A_\bullet^\Gamma)$. The points of that finite set are called cusps.

The case of $\Gamma = \operatorname{SL}_2 \mathbb{Z}$ we have seen before. Lets review

Example III.0.1

Let $n = 1$. Then $\mathbb{B}_W$ is an open disk in $\mathbb{P}^1$. One can see that $U(1,1)$ has center $\operatorname{SU}(1,1)$, and in the uppr half plane coordinates there is an isomorphism $\operatorname{SU}(1,1) \cong \operatorname{SL}_2 \mathbb{R}$. If we consider $\Gamma = \operatorname{SL}_2 \mathbb{Z}$ we find that

$$X_\Gamma = \operatorname{SL}_2 \mathbb{Z} \backslash \mathbb{H}^2,$$

and the projective compactification is given by adding ∞ (corresponds to nodal cubics generically or union of 3 lines for the closed orbit).

One way to view building this compactification is to add $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ to $\mathbb{H}^2$. The space $\mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q})$ is badly topologized, namely it is not locally compact. The neighborhood basis around ∞ for example consists of shifted versions of the upper half plane $+\ \infty$. However, the action of $\operatorname{SL}_2 \mathbb{Z}$ saves you. $\operatorname{SL}_2 \mathbb{Z}$ still acts on $\mathbb{P}^1(\mathbb{Q})$, and the quotient of say one of the elements of this neighborhood basis consits of a cylinder.

The proof of (1) and (2) above rely on the interpretation of (3). To do so, one constructs a fundamental domain for the action on the ball $\mathbb{B}_W$. This was done by Siegel, Satake, Harish-Chandra, and later refined by Bailey–Borel. One can then produce a candidate for $\mathrm{Proj}(A_\Gamma)$ just as a topological space. One carefully studies the sheaf of holomorphic functions on this candidate (where that makes sense) and uses methods of algebraic geometry to then show $A_\bullet^\Gamma$ is finitely generated as a $\mathbb{C}$-algebra.

III.1. **Integrals and Riemann Surfaces**

We'll now turn to integrals and give a geometric approach to algebraic integrals in 1 variable a la Riemann.

Example III.1.1

Lets examine this integral

$$\int_\gamma \frac{\mathrm{d}z}{\sqrt{1-z^2}} = \arcsin(z)\Big|_\gamma.$$

Here's what Riemann suggests to do. Set $w^2 = 1 - z^2$, which defines a plane curve $\mathring{C} \subseteq \mathbb{C}^2$ with a projection $(z, w) \overset{p}{\mapsto} z$. This is a double cover ramified in $0, 1$. This is just the equation of a conic hitting ∞ at 2 distinct points (so it does not branch over ∞). We can then pull back the form to $\mathring{C}$:

$$p^*\left(\frac{\mathrm{d}z}{\sqrt{1-z^2}}\right) = \frac{\mathrm{d}z}{w}\Big|_{\mathring{C}}.$$

Note that $\mathring{C}$ is a copy of $\mathbb{P}^1$ with the two points over ∞ removed. So then there is an isomorphism $\mathring{C} \cong \mathbb{C}^\times$. We find then that

$$\frac{\mathrm{d}z}{z} \mapsto \frac{\mathrm{d}t}{t},$$

and this integrates to $\log t$. In this way we reduce $\int_\gamma \frac{\mathrm{d}z}{\sqrt{1-z^2}}$ to integrals we already understand.

We can soup up the example above by replacing it with

$$\int_\gamma \frac{\mathrm{d}z}{\sqrt{p(z)}},$$

for some polynomial $p(z)$. For degree $p = 3, 4$ this gives you exactly the elliptic integrals, which cannot be expressed in algebraic + exponential + logarithm terms. For arbitrary degree these are called the hyperelliptic integrals.

Unedeterred we go further and replace the square root by an arbitrary root $\sqrt[m]{-}$ for $m > 0$ an integer. Write the polynomial $p(z) = (z - z_0)^{d_0} \cdots (z -_n)^{d_n}$ with $z_0, \ldots, z_n$ distinct roots of p.

$$\int_\gamma \frac{\mathrm{d}z}{\sqrt[m]{p(z)}}.$$

The recipe of Riemann to attack this is to examine the affine curve

$$\begin{aligned} C' = \{(z,w) \in \mathbb{C}^2 \mid w^m = p(z) = (z-z_0)^{d_0} \cdots (z-z_n)^{d_n}\} &\xrightarrow{q} \mathbb{C} \\ (z,w) &\mapsto z. \end{aligned}$$

This is a cyclic cover of degree m. The deck group is $(z,w) \mapsto (z, \zeta_m w)$, and we denote the covering group by $\underline{\mu}_m$. C' is in general not smooth at $(z_0, 0)$. At such a point C' is "essentially" given by $w^m = z'^{d_i}$ where $z' = (z - z_i) \cdot \text{unit}$.

We assume that $d_0, \ldots, d_n, m$ have no common factor. If $r_i = \gcd(m, d_i)$ then the curve germ $w^m = z'^{d_i}$ will have r_i branches. The r_i-th root of unity is what provides ambiguity. We are going to separate the branches and resolve the singularity. Let $\overline{d}_i = d_i/r_i$ and $\overline{m}_i = m/r_i$. Then one branch is given by $w^{\overline{m}} = z'^{\overline{d}_i}$. Since the exponents are coprime we can parameterize this curve germ as follows

$$w = t^{\overline{d}_i} \qquad\qquad z' = t^{\overline{m}_i}.$$

This gives a smooth parameterization of this curve germ.

Now we will examine what happens to the differential $\eta = \frac{\mathrm{d}z}{w}$. Near z_i we find

$$\eta = \frac{\mathrm{d}z'}{w} = (t^{\overline{m}_i})^{-\overline{d}_i/\overline{m}_i}\, \mathrm{d}(t^{\overline{m}_i}) = t^{\overline{m}_i - \overline{d}_i} \frac{\mathrm{d}t}{t}.$$

This form has a zero of order $\overline{m}_i - \overline{d}_i - 1$. Ideally, this is not a pole, and so we want $\overline{m}_i - \overline{d}_i - 1 > 0$. That is we want

$$\frac{\overline{d}_i}{\overline{m}_i} = \frac{d_i}{m} < 1.$$

Lets do this normalization algebraically. Because we should eat our veggies. The ring is given by

$$\mathcal{O} = \mathbb{C}[x,y]/(x^2 - y^3).$$

This is not a normal local ring (i.e. it is not its own integral closure in its field of fractions) – this corresponds to the fact that we're not smooth at this cusp. If you consider the field of fractions $t = \frac{x}{y}$. We find that $t^2 = y$ and $t^3 = x$. Thus t is in the integral closure of $\mathcal{O}$ inside of its fraction field, and that's why we add in this t.

Now write $\eta(z)$ as $(z-z_0)^{-\mu_0} \cdots (z-z_n)^{-\mu_n}\, \mathrm{d}z$ with $\mu_i = \frac{d_i}{m} \in \mathbb{Q} \cap (0,1)$. We have produced now an affine curve C' where η pulls back to a univalued function.

$$\begin{tikzcd} \mathring{C}_{normalized} \arrow[d, "\pi", "\underline{\mu}_m"'] \\ \mathbb{C} \end{tikzcd}$$

where $\widetilde{\eta} = \pi^*\eta$ is represented by a differential without poles. We now want to extend this to the point at ∞. So then we have $C \to \mathbb{P}^1$ and C is smooth. At ∞ the form looks like

$$z^{-(d_0+\cdots+d_n)/m}\,\mathrm{d}z.$$

Replacing $u = 1/z$ we get

$$-u^{(d_0+\cdots+d_n)/m-2}\,\mathrm{d}u.$$

We denote $\mu_\infty = -\frac{d_0+\cdots+d_n}{m} - 2$. We want that

$$\mu_\infty = (-\mu_0 + \cdots + \mu_n) + 2 < 1.$$

These conditions are summarized by

$$\mu_0, \ldots, \mu_n, \mu_\infty \in \mathbb{Q} \cap [0, 1)$$

so that

$$\sum_{i=0}^{n} \mu_i + \mu_\infty = 2.$$

Last time we examined some integrals. Namely for $(d_0, \ldots, d_n, m)$ integers > 0 with no common factor and $z_0, \ldots, z_n$ distinct in $\mathbb{C}$ we studied

$$p(z) = (z - z_0)^{d_0} \cdots (z - z_0)^{d_n}.$$

From this, we obtain a multivalued differential

$$\eta(z) = \frac{\mathrm{d}z}{\sqrt[m]{p(z)}} = (z - z_0)^{-\mu_0} \cdots (z - z_n)^{-\mu_n}\,\mathrm{d}z,$$

where $\mu_i = d_i/m$. There is a compact Riemann surface C over $\mathbb{P}^1$ given by

$$w^m = p(z),$$

and $\widetilde{\eta} \coloneqq \frac{\mathrm{d}z}{w}$ is the pullback of η, giving a well-defined form. In this way we've disambiguated η!

Remark III.1.1

Technically, one normalizes to get rid of the singularities of the curve $w^m = p(z)$. Calling $\widetilde{\eta}$ the pullback of η is a bit silly too, since η is not even well-defined! Nevertheless, locally η is well-defined up to a choice of an m-th root, and $\widetilde{\eta}$ will lift η in a small neighborhood.

Let G be the cyclic group of order m, which is the deck group of C. We have a character $\chi : G \hookrightarrow \mathbb{C}^\times$ so that

$$g \cdot (w, z) = (\chi(g) \cdot w, z).$$

Thinking of w as a function on C, we may compute the pullback $g^*w = \chi(g)^{-1}w$. Therefore

$$g^*\widetilde{\eta} = \chi(g)\widetilde{\eta}.$$

There are irregular G-orbits over $\{z_0, \ldots, z_n\}$ and over z_i we have branching of order $\overline{d}_i = d_i/\gcd(m, d_i)$ with $\overline{m}_i = m/\gcd(m, d_i)$ preimages. Over ∞ we have

$$d_\infty = 2m - (d_0 + \cdots + d_n) \qquad\qquad \mu_\infty = \frac{d_\infty}{m}.$$

Furthermore

$$\mu_0 + \cdots + \mu_n + \mu_\infty = 2.$$

Now assume that $\mu_0, \ldots, \mu_n, \mu_\infty$ are all < 1. This assumption is equivalent to requiring that $\widetilde{\eta}$ is a holomorphic differential. We now write for ∞, d_∞ we just write z_{n+1}, d_{n+1} in case $d_\infty \neq 0$.

Observation: $\widetilde{\eta}$ transforms under the G-action by the conjugate character, so in the character space

$$\widetilde{\eta} \in H^0(C, \Omega_C)^{\overline{\chi}}.$$

Note that, in fact, $H^0(C, \Omega_C)^{\overline{\chi}}$ has dimension 1 and $\widetilde{\eta}$ is a generator.

Proof. Because G is cyclic, $H^0(C, \Omega_C)^{\overline{\chi}}$ has dimension 1. Any other holomorphic differential ζ on C can be written as $\widetilde{f} \cdot \widetilde{\eta}$ for $\widetilde{f}$ some *meromorphic* function on C. Assuming that $\zeta \in H^0(C, \Omega_C)^{\overline{\chi}}$ then $\widetilde{f}$ must be G-invariant.

Therefore $\widetilde{f}$ may be considered as a meromorphic function f on $\mathbb{P}^1$. Furthermore, since η is nonzero away from $z_0, \ldots, z_{n+1}$ we find that f can only have poles along $z_0, \ldots, z_{n+1}$.

Suppose now that f has a simple pole at z_i. The pullback to C also yields a pole over z_i. These poles cannot compensate for the zeros of $\widetilde{\eta}$.

??

We are interested in studying integrals of η like $\int_\gamma \eta$ where γ goes between z_i and z_j (with its interior not hitting branch points). We lift:

$$\int_\gamma \eta \rightsquigarrow \int_{\widetilde{\gamma}} \widetilde{\eta}.$$

We can actually replace $\widetilde{\gamma}$ by a $\mathbb{C}$-valued 1-cycle. Let $g \in G$ and consider another lift

$$\int_{g\widetilde{\gamma}} \widetilde{\eta} = \int_{\widetilde{\gamma}} g^*\widetilde{\eta} = \int_{\widetilde{\gamma}} \chi(g)\widetilde{\eta} = \chi(g) \int_{\widetilde{\gamma}} \widetilde{\eta}$$

Hence

$$\int_{\chi(g)^{-1}g(\widetilde{\gamma}} \widetilde{\eta} = \int_{\widetilde{\gamma}} \widetilde{\eta}.$$

Hence we average $c = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}g(\widetilde{\gamma})$

$$\int_c \widetilde{\eta} = \int_{\widetilde{\gamma}} \widetilde{\eta}.$$

Now c is a $\mathbb{C}$-valued 1-chain. The observation is that c is in fact a 1-cycle. Essentially you hit the preimages of z_i each with a sum of roots of unity. Similarly we find that c lies in $H_1(C;\mathbb{C})^\chi$. Thus, we've converted our integral $\int_\gamma \eta$ into a *period* $\int_c \widetilde{\eta}$ of $\widetilde{\eta}$.

III.2. **Thurston's approach to Deligne–Mostow Theory**

The relevant paper for this is "Shapes and polyhedra." The Hermitian form $|\eta|^2$ on $\mathbb{P}^1$ is unambigious and defines a metric with singularities along $(z_0,\dots,z_{n+1})$. Furthermore, away from $D = \{z_0,\dots,z_{n+1}\}$ we know that η has a local integral, i.e., for $\kappa : U \subseteq \mathbb{P}^1 \to \mathbb{C}$ with $\mathrm{d}\kappa\text{``}=\text{''}\,\eta$. Then this κ is a metric chart for $|\eta|^2$ (i.e., it pulls back the Euclidean metric). Therefore the chart is flat!!!

Hence the metric $|\eta|^2$ is flat outside the singular poitns $z_0,\dots,z_{n+1}$. At a singular point z_i we have that η looks like

$$(z-z_i)^{-\mu_i}\,\mathrm{d}z.$$

Find a local chart κ with $\mathrm{d}(\kappa^{1-\mu_i})\text{``}=\text{''}\,\eta$. The chart κ describes the neighborhood of z_i as a metric cone. Using polar coordinates

$$\kappa = re^{i\theta},$$

where we let θ traverse the interval $[0, 2\pi(1-\mu_i)]$. The angle $2\pi\mu_i$ is the angle defect. Gauss-Bonnet interprets this as the curvature defect at z_i. The total curvature is then of course

$$\sum_{i=0}^{n+1} 2\pi\mu_i = 2\pi\cdot 2 = 4\pi = \text{area(the unit sphere)}.$$

This is exactly Gauss-Bonnet if we normalize η. The only data is $(z_i,\mu_i)_{i=0}^{n+1}$. The metric on $\mathbb{P}^1$ can be obtained as a gluing of polygons. To do so, it suffices to find non-intersecting geodesics between the cone points. We do so via the Voronoi tesselation.

For any $z \in S^2$ we define

$$D_z = \{z_i \mid z_i \text{ realizes } \operatorname{dist}_{|\eta|^2}(z, D)\}$$

Generically $|D_z| = 1$. The set of z where $|D_z| \le 2$ gives a graph called the Voronoi edges. The set where $\{z : |D_z| \ge 3\}$ is a fintie set of points. Taking the convex hull of these points in D_z for such a z gives a tile. The tesselation so obtained paves the 2-sphere, making it a metric combinatorial S^2. The edges of the tiles are called ribs, they are dual to the Voronoi edges.

The length of a rib γ connecting z_i, z_j is precisely

$$\int_\gamma |\mathrm{d}\eta| = \left|\int_\gamma \mathrm{d}\eta\right| = |\text{a period}|\,,$$

which we'll justify next time.

$(\mu_0, \ldots, \mu_{n+1})$	$(1/2, 1/2, 1/2, 1/2)$	tetrahedron
	$\underbrace{(1/4, \ldots, 1/4)}_{8\text{ times}}$	cubic
	$\underbrace{(1/6, \ldots, 1/6)}_{12\text{ times}}$	deformed regular icosahedron

We're going to do some recollections from last time, but change the emphasis a bit. We are given $\underline{\mu} = (\mu_0, \ldots, \mu_{n+1})$ with $\mu_i \in \mathbb{Q} \cap (0, 1)$, $\mu_i = \frac{d_i}{m}$, and $\sum \mu_i = 2$. We considered $z_0, \ldots, z_{n+1} \in \mathbb{P}^1$ distinct with the divisor

$$D = \sum_{i=1}^{n+1} d_i(z_i).$$

Such a divisor gives rise to a multivalued differential $\eta = \frac{\mathrm{d}z}{\sqrt[m]{(z-z_1)^{d_1} \cdots (z-z_{n+1})^{d_{n+1}}}}$. We found that $|\eta|^2$ is a piecewise linear metric which is polyhedral metric copy of S^2 with vertex set $\operatorname{supp}(D)$ and angular defect $2\pi\mu_i$ at z_i.

Claim

The length of the geodesic rib γ between z_i and z_j is given by

$$\int_\gamma |\eta| = \left|\int_\gamma \eta\right|. \tag{3}$$

Proof. To justify this we must consider how the argument of η changes along γ. Take a chart κ in a small neighborhood of γ (with z_i, z_j on the boundary) which is flat with respect to η, i.e., $\mathrm{d}\kappa = \eta$. Because κ is flat with respect to η we have that the geodesic γ will map to an interval in $\mathbb{C}$. The length of the interval is both the left and right hand side of (3).

Last time, we showed that

$$\left|\int_\gamma \eta\right| = \left|\int_{\widetilde{\gamma}} \widetilde{\eta}\right|.$$

In this case, $\widetilde{\eta}$ is a holomorphic differential on a curve $C/\mathbb{P}^1$ and $\widetilde{\gamma}$ is a lift of γ. Furthermore:

$$\int_{\widetilde{\gamma}} \widetilde{\eta} = \int_{\text{1-cycle}} |\widetilde{\eta}|\,.$$

That is a period of $\widetilde{\eta}$. We can further add triangles to polygonal faces to produce a triangulation whose lengths will be periods of $\widetilde{\eta}$.

We will give this entire story the flavor of a Hodge theorem. We will show that

$$\text{periods of } \widetilde{\eta} \rightsquigarrow \text{metric } S^2 \rightsquigarrow (\mathbb{P}^1, D).$$

We'll start topologically with a cell decomposition of $\mathbb{P}^1$. We choose $\gamma_1, \ldots, \gamma_n$ where γ_i goes from z_{i-1} to z_i (notice that z_{n+1} is not included). The 0-cells are $z_0, \ldots, z_n$ with 1-cells $\gamma_1, \ldots, \gamma_n$. There is only one 2-cell, which is the rest of the sphere.

> **Claim**
>
> From this cell decomposition, we produce a cell decomposition of the curve $C/\mathbb{P}^1$ by taking components of the preimage of each (open) cell. The produced cell decomposition is G-invariant.

Note that a cover of an open disk with ramification at a single point still yields a disk, hence the 2-cell containing z_{n+1} is fine.

Choose $\widetilde{z}_i$ a lift over z_i and a lift $\widetilde{\gamma}_i$ over γ_i. Note that $\pi_0(\text{preimage of cell})$ is a G-set and

over z_i	$G/G_{\widetilde{z}_i}$
over 2-cell	$G/G_{\widetilde{z}_{n+1}}$
over 1-cell	regular G-orbit.

Thus we can write down the cell complex of C as G-modules:

$$0 \longrightarrow \mathbb{Z}[G]/G_{\widetilde{z}_{n+1}} \longrightarrow (\mathbb{Z}[G])^n \xrightarrow{\partial} \bigoplus_{i=0}^n \mathbb{Z}[G/G_{\widetilde{z}_i}] \longrightarrow 0. \tag{4}$$

The homology of this compex is $H_\bullet(C)$ as a a $\mathbb{Z}[G]$-module. Recall that we have a primitive faithful character $\chi : G \hookrightarrow \mathbb{C}^\times$. Because G is cyclic all the characters are powers

$$\{\chi^0, \chi^1, \ldots, \chi^{m-1}\}.$$

We find then that

$$\mathbb{C}[G] \xrightarrow{(\chi^0,\ldots,\chi^{m-1})} \mathbb{C}^m,$$

is a $\mathbb{C}$-isomorphism and all characters appear with multiplicity one. Recall also that $\gcd(r, m) = 1$ if and only if χ^r is primitive (faithful).

> **Claim**
>
> Necessarily $\mathbb{C}[G/H]$ cannot contain a primitive character for any subgroup $H < G$, since any primitive character can separate every element of the group (by definition).

Hence, when we complexify (4) and take the eigenspace corresponding to χ^r where $\gcd(r, m) = 1$ we find

$$0 \to 0 \to (V_{\chi^r})^n \to 0 \to 0,$$

and so we have that

$$H_1(C,\mathbb{C})^{\chi^r}$$

has dimension n and has a basis

$$\left\{\sum_{g\in G}\chi^r(g)g^{-1}(\widetilde{\gamma}_i)\right\}_{i=1}^n$$

We now have the symplectic intersection pairing

$$\begin{aligned} H^1(C)\times H^1(C) &\xrightarrow{\cdot} \mathbb{Z} \\ (\alpha,\beta) &\mapsto (\alpha\cup\beta)([C]). \end{aligned}$$

The symplectic intersection pairing gives rise to a Hermitian form in the complexification

$$\begin{aligned} H^1(C;\mathbb{C})\times H^1(C;\mathbb{C}) &\xrightarrow{\langle,\rangle} \mathbb{C} \\ \alpha,\beta &\mapsto \sqrt{-1}\int_C \alpha\wedge\overline{\beta} = \sqrt{-1}(\alpha\cup\overline{\beta})([C]). \end{aligned}$$

Here are some notes on this Hermitian form

- The Hermitian form is nondegenerate.
- The form has Witt index / signature zero (i.e., it's $U(k,k)$).

In terms of the Hodge theory, we have that

$$H^1(C,\mathbb{C}) = H^0(\Omega_C)\oplus\overline{H^0(\Omega_C)}.$$

For two holomorphic differentials we have that in a local chart

$$\sqrt{-1}\alpha\wedge\overline{\alpha} = \sqrt{-1}\,\mathrm{d}z\wedge\mathrm{d}\overline{z} = 2\,\mathrm{d}x\wedge\mathrm{d}y.$$

Therefore the decomposition $H^0(\Omega_C)\oplus\overline{H^0(\Omega_C)}$ is perpendicular with respect to $\langle,\rangle$ and splits it into a positive definite and negative definite part.

We'll now consider the character space containing $\widetilde{\eta}$, that is $H^1(C,\mathbb{C})^{\overline{\chi}}$:

$$\begin{aligned} H^1(C,\mathbb{C})^{\overline{\chi}} &= (H^0(\Omega_C))^{\overline{\chi}}\oplus\overline{H^0(\Omega_C)}^{\overline{\chi}} \\ &= (H^0(\Omega_C))^{\overline{\chi}}\oplus\overline{H^0(\Omega_C)^{\chi}}. \end{aligned}$$

We know the left hand side here has dimension n, by our calculation before. Furthermore we showed that $(H^0(\Omega_C))^{\overline{\chi}}$ has dimension one. Hence $\overline{H^0(\Omega_C)^{\chi}}$ has dimension $n-1$. This is an orthogonal decomposition which is positive def/neg def on each space respectively. Therefore, $H^1(C,\mathbb{C})^{\overline{\chi}}$ has signature $(1,n-1)$.

Great! Therefore we have that $H^1(C,\mathbb{C})^{\overline{\chi}}$ describes an $(n-1)$-dimensional complex ball

$$\mathbb{B}_C \subseteq \mathbb{P}(H^1(C;\mathbb{C})^{\overline{\chi}})$$

which is independent of the complex structure by looking at the elements with positive self-product and projectivizing. This ball is a symmetric domain for the action of the special unitary group. The complex structure gives $H^0(\Omega_C)^{\overline{\chi}}$ (aka $\widetilde{\eta}$) which is a line whose projectivization gives a point in the ball $\mathbb{B}_C$.

We now normalize so that $z_0 = 0, z_1 = 1, z_{n+1} = \infty$ leaving the $n-1$ points $z_2, \dots, z_n$ to vary. Thus we obtain a map

$$\mathrm{Conf}_{n-1}(\mathbb{P}^1 \setminus \{0,1,\infty\}) \to \mathbb{B}_C.$$

We claim this is a local isomorphism, which is a type of local Torelli. That is

$$\{(\mathbb{P}^1, D)\}/\text{Möbius transformation} \to \mathbb{B}_C.$$

This is a bit of a lie, because how do we identify the cohomologies of $H^1(C)$ versus $H^1(C')$ when we have varied the points (monodromy via the mapping class group…).

We begin now a formal approach: Fix an integral symplectic lattice H with G-action which is isomorphic to $H^1(C;\mathbb{Z}) \subseteq H^1(C;\mathbb{C})$ for some C. For this abstract lattice H, we can go through the same constructions as given above. Namely, we have a hermitian form on $H_\mathbb{C}$ which is of type (p,p) and which decomposes according to characters. Furthermore the character space $H_\mathbb{C}^{\overline{\chi}}$ has signature $(1,n)$ and a ball $\mathbb{B} \subseteq \mathbb{P}H_\mathbb{C}^{\overline{\chi}}$ a complex $(n-1)$-dimensional ball.

Definition III.2.1

A choice of isomorphism between H and $H^1(C)$ of symplectic lattices with G-action will be called a G-marking of C.

Proposition III.2.1

Any two such markings "differ" by a G-equivariant symplectic automorphism of H:

$$\mathrm{Sp}(H)^G = \text{centralizer of } G \text{ in } \mathrm{Sp}(H),$$

note that this contains G as a central subgroup.

We define then

$$\widetilde{\widetilde{\mathcal{M}}}(\underline{\mu}) = \text{space of } G\text{-marked curves } C \text{ obtained through this construction.}$$

The space comes equipped with a $\mathrm{Sp}(H)^G$-action (note $\widetilde{\widetilde{\mathcal{M}}}(\underline{\mu})$ need not be connected…). We then have a well-defined map

$$\widetilde{\widetilde{\mathcal{M}}}(\underline{\mu}) \xrightarrow{\text{period of } \widetilde{\eta}} \mathbb{B},$$

which is $\mathrm{Sp}(H)^G$-equivariant.

Theorem III.2.2 (Local Torelli)

The map we've given is a local isomorphism.

Missing Lecture 2025-10-28, get notes from someone

Recall that we have $\underline{\mu} = (\mu_0, \ldots, \mu_{n+1})$ with $\mu_i \in \mathbb{Q} \cap (0,1)$ and $\sum \mu_i = 2$ and common denominator m. Let G be the cylcic group of order n acting on H an integral symplectic lattice as before isomorphic to $H^1(C;\mathbb{Z})$. Then the monodromy lands in $\mathrm{Sp}(H)^G$, we call $\Gamma < \mathrm{Sp}(H)^G$ the image of the monodromy.

The Deligne-Mostow $\frac{1}{2}\mathbb{Z}$-conditions:

(1) If $1 - \mu_i - \mu_j$ is positive then it is the reciprocal of a positive integer $\mu_i \neq \mu_j$.

(2) If $1 - \mu_i - \mu_i$ is positive then it is twice the reciprocal of a positive integer $\mu_i = \mu_j$.

They prove these conditions are equivalent to the isomorphism

$$\begin{tikzcd}
\overline{\mathcal{M}}(\mu)^{st} \arrow[r, "\text{is an iso}", "\Gamma\text{-invariant}"'] \arrow[d, "\Gamma"] & \mathbb{B}(\mu) \arrow[d] \\
\mathcal{M}(\mu)^{st} \arrow[r, dashed, "\cong"'] \arrow[d, phantom, sloped, "\subseteq"] & \Gamma\backslash\mathbb{B}(\mu) \arrow[d, phantom, sloped, "\subseteq"] \\
\mathcal{M}(\mu)^{ss} \arrow[r, dashed, "\cong"'] & \Gamma\backslash\mathbb{B}(\mu)^* & \text{(add cusps)}
\end{tikzcd}$$

Both of these have natural compactifications, as given by the inclusions above in red. Adding the semistable points adds a point for every splitting of $\underline{\mu}$ into point pieces summing to 1. Furthermore, if no such splitting exists, then both sides are already compact and Γ is cocompact.

In particular, Γ acts discretely in $\mathbb{B}(\underline{\mu}) \subseteq \mathbb{P}(H_{\mathbb{C}}^{\overline{\chi}})$.

Example III.2.1

Let $\underline{\mu} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. The corresponding differential form is

$$\eta = \frac{\mathrm{d}z}{\sqrt{z(z-1)(z-\lambda)}},$$

with the final $\frac{1}{2}$ contributed by ∞. We obtain a ball quotient of dimension 1. There are 3 ways to split the tuple into pieces summing to 1. But up to permutation there is only 1 way, so there is 1 cusp. Now $1 - \frac{1}{2} - \frac{1}{2} = 0$, so the isomorphism above holds.

This is exactly associated with the Legendre family and the modular curve, with $\Gamma = \mathrm{PSL}_2\,\mathbb{Z}$. If we label the points, then

$$\Gamma = \ker(\mathrm{PSL}_2\,\mathbb{Z} \to \mathrm{PSL}_2\,\mathbb{F}_2).$$

Example III.2.2

$\underline{\mu} = (\frac{1}{4}, \dots, \frac{1}{4})$ for 8 times. The condition is fulfilled because $1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$. We get a lattice Γ and a ball quotient of dimension 5. Only one cusp for unlabeled points.

Example III.2.3

$\underline{\mu} = (\frac{1}{6}, \dots, \frac{1}{6})$ with 12 of these. Here $1 - \frac{1}{6} - \frac{1}{6} = \frac{2}{3}$ so we satisfy the Deligne–Mostow condition. It has one cusp and is a 9-dimensional ball quotient. It is the highest dimensional case that can be obtained in this manner.

Remark III.2.1

The list of solutions to the Deligne–Mostow conditions is finite with 97 total cases.

Example III.2.4

$\underline{\mu} = \left(\frac{7}{18}, \frac{7}{18}, \frac{7}{18}, \frac{7}{18}, \frac{8}{16}\right)$. We find that $1 - \frac{7}{18} - \frac{7}{18} = \frac{2}{9}$ and $1 - \frac{7}{18} - \frac{8}{18} = \frac{1}{6}$, and so this satisfies the conditions.

It is impossible to have any of these numbers adding up to 1, and so there are no cusps and the corresponding spaces are compact (Γ cocompact).

We'll now take a detour for some additional rep theory on the way to discussing arithemeticity of Γ.

Let G be cyclic of order m with a fixed primitive character $\chi : G \to \mathbb{C}^\times$. Our goal is to discuss the representations of G over $\mathbb{Q}$. Let

$$K_m = \mathbb{Q}(\mu_m) = \mathbb{Q}(\zeta_m)$$

where $\zeta_m = e^{2\pi\sqrt{-1}m}$. We find that K_m is a $\mathbb{Q}$-vector space on which $\mathbb{Q}$ acts. If we complexify $\mathbb{C} \otimes_{\mathbb{Q}} K_m$ decomposes as $\{\chi^r : (r, m) = 1\}$ the set of primitive characters. As a rational represenation (i.e., as a $\mathbb{Q}[G]$-module), K_m is irreducible. All other characters appear as primitive characters of a genuine quotient of G. This exhausts the rational representation theory of G.

In terms of the group algebra

$$\mathbb{Q}[G] = \bigoplus_{r|m} K_r,$$

K_r here is a field treated as a matrix algebra. Now let V be a finite-dimenisonal $\mathbb{Q}$-reprsentation of G. Collect the characters belonging to the same summand. We obtain

$$V = \bigoplus_{r|m} V^r \qquad\qquad V^r = \mathrm{Hom}_G(K_r, V).$$

On the right hand side we obtain a vector space over K_r, and that structure induces the G-action.

If we want to just do this over the reals, then we just have to make sure everything we do is invariant under complex conjugation. The real finite-dimensional irreducible representations of G are planes with a rotation (besides trivial and sign if it exists). This combines the characters $\chi^r, \overline{\chi}^r = \chi^{-r}$ so long as they are distinct. They are distinct if and only if $2r \not\equiv 0$ modulo m.

One may run r over the set $\{1, \dots, \lfloor \frac{m-1}{2} \rfloor\}$, which precisely gives the intersection $\mu_m \cap \mathbb{H}^2$. The corresponding character space yields a 2-dim representation. One thinks of the 1-dimension complex representation χ^r as a 2-dimensional real representation. If $2r \equiv 0 \pmod m$ then $r = 0$ or $r = \frac{m}{2}$ provided that m is even. These correspond to the trivial rep and the sign rep respectively.

Therefore

$$\mathbb{R}[G] \cong \mathbb{C}^{\lfloor \frac{m-1}{2} \rfloor} \oplus \begin{cases} \mathbb{R} & m \text{ odd} \\ \mathbb{R}^2 & m \text{ even} \end{cases}.$$

We return now to H: a symplectic lattice with G-action that is isomorphic to $H^1(C)$ with G acting. We apply the theory above to this situation. We have

$$H_{\mathbb{Q}} = \bigoplus_{r|m} H_{\mathbb{Q}}^r,$$

we interest ourselves in the primitive characters, i.e., $H_{\mathbb{Q}}^{pr} := H_{\mathbb{Q}}^m$ where the group acts faithfully. The symplectic form on H determines a skew-hermitian form on $H_{\mathbb{Q}}^{pr}$. We may also define a lattice $H^{pr} = H_{\mathbb{Q}}^{pr} \cap H$ and the symplectic form will be valued in a number ring on this lattice.

$$\begin{aligned} \langle \bullet, \bullet \rangle^{pr} &: H^{pr} \times H^{pr} \to \mathcal{O}_{K_m} = \mathbb{Z}\text{span of } \mu_m \text{ in } K_m = \text{ring of integers of } K_m \\ \langle \alpha, \beta \rangle^{pr} &:= \sum_{g \in G} (g\alpha \cdot \beta) \chi(g^{-1}) \in \mathcal{O}_{K_m}. \end{aligned}$$

We claim that this is skew-hermitian:

$$\begin{aligned} \langle \beta, \alpha \rangle^{pr} &= \sum (g\beta \cdot \alpha) \chi(g^{-1}) = \sum -(\alpha \cdot g\beta) \chi(g^{-1}) \\ &= -\sum (g\alpha \cdot \beta) \chi(g) = -\sum (g\alpha \cdot \beta) \overline{\chi(g^{-1})} \\ &= -\langle \alpha, \beta \rangle^{pr} \end{aligned}$$

We can make this hermitian by multiplying with a purely imaginary number. Say we multiply with $\zeta_m - \overline{\zeta}_m$ (still $\mathcal{O}_{K_m}$-valued).

Looking at the groups

$$\mathrm{Sp}(H)^G \xrightarrow{\text{restrict}} \mathrm{Sp}(H^{pr})^G \cong U(H^{pr})$$

This has a center, which is essentially the roots of unity $\mu_m \cdot \{\pm 1\}$. Tensoring with $\mathbb{Q}$

$$\mathrm{Sp}(H_{\mathbb{Q}})^G \xrightarrow{\text{restrict}} \mathrm{Sp}(H_{\mathbb{Q}}^{pr})^G \cong U(H_{\mathbb{Q}}^{pr}).$$

Now we have the right hand side as a K_m vector space and a Hermitian form on it. It's *almost* got a signature (must embed K_m into $\mathbb{C}$ to get one). Tensoring with $\mathbb{R}$ we obtain

$$H_{\mathbb{R}}^{pr} = \bigoplus_{r=1}^{\lfloor \frac{m-1}{2} \rfloor} H_{\mathbb{C}}^{\chi^r}$$

The decomposition is orthogonal with respect to the Hermitian form $(\zeta - \overline{\zeta})\langle \bullet, \bullet \rangle^{pr} \otimes \mathbb{R}$. The orthogonal decomposition will have on $H_{\mathbb{C}}^{\chi^r}$ some signature (p_r, n_r).

Therefore

$$\mathrm{Sp}(H_{\mathbb{R}}^{pr})^G \cong \prod_{r=1}^{\lfloor \frac{m-1}{2} \rfloor} U(p_r, q_r).$$

These factors are not naturally defined over $\mathbb{Q}$. This is important because

$$PU(H^{pr}) = U(H^{pr})/\text{center}$$

naturally contains the monodromy group Γ.

An arithmetic group in this situation is a finite index subgroup of $\mathrm{Sp}(H^{pr})^G$ or likewise this group modulo its center.

Question: Is Γ arithmetic?

As before, let H be a unimodular symplectic lattice with G-action (cyclic of order m). As noted before $H \cong H^1(C)$, where C is obtained by

$$y^m = (x - x_1)^{d_1} \cdots (x - x_n)^{d_n},$$

with some normalizations. The action is given by a character $\chi : G \hookrightarrow \mathbb{C}^\times$. The cohomology decomposes as

$$H_{\mathbb{C}} = \bigoplus_{r \in \mathbb{Z}/m} H_{\mathbb{C}}^{\chi^r}$$

$$H_{\mathbb{C}}^{pr} = \bigoplus_{r \in (\mathbb{Z}/m)^\times} H_{\mathbb{C}}^{\chi^r}$$

As discussed last time, $H_{\mathbb{C}}^{pr}$ is defined over $\mathbb{Q}$ and $H_{\mathbb{Q}}^{pr} \subseteq H_{\mathbb{Q}}$ is the smallest $\mathbb{Q}$-subspace whose complexification contains one of the summands. We can also take the lattice $H^{pr} = H \cap H_{\mathbb{C}}^{pr}$.

A relevant group for our purposes is $\mathrm{Sp}(H)^G$. We will focus on $\mathrm{Sp}(H^{pr})^G$. Note that this defines an algebraic group. Given any commutative ring R with 1 we have

$$R \mapsto \mathrm{Sp}(H_R^{pr})^G,$$

in other words this is a functor assigning a ring to a group. One point of view towards schemes is that they give functors from rings to sets, and this completely determines the underlying scheme (when it exists). Given a scheme X, the functor $R \mapsto \{\mathrm{Spec}\, R \to X\}$ is called the functor of points. A group scheme is a schee whose functor of points takes values in groups.

Call the group scheme above $\mathcal{G}$. Let $K_m = \mathbb{Q}(\zeta_m)$ where ζ_m is a primitive m-th root of unity. We obtain

$$\begin{array}{ccccc} \mathcal{G}(\mathbb{Z}) & \subseteq & \mathcal{G}(\mathbb{Q}) & \subseteq & \mathcal{G}(\mathbb{R}) \longleftarrow \text{a real Lie group} \\ & & \cap\!\!| & & \cap\!\!| \\ & & \mathcal{G}(K_m) & \subseteq & \mathcal{G}(\mathbb{C}) \longleftarrow \text{a complex Lie group} \end{array}$$

Note that $\mathcal{G}(\mathbb{Z})$ is discrete in $\mathcal{G}(\mathbb{R})$ (in fact arithmetic).

Theorem III.2.3 (Borel, Harish-Chandra)

Arithmetic groups $\mathcal{G}(\mathbb{Z})$ satisfy $\operatorname{vol}(\mathcal{G}(\mathbb{R})/\mathcal{G}(\mathbb{Z})) < \infty$.

Note, a discrete group in a Lie group with finite covolume is called a lattice.

Last time we introduced a skew-symmetric form on the lattice

$$\langle , \rangle^{pr} : H^{pr} \times H^{pr} \to \mathcal{O}_{K_m} = \mathbb{Z}[\zeta_m]$$
$$\langle \alpha, \beta \rangle = \sum_{g \in G} (g^{-1}\alpha \cdot \beta)\chi(g).$$

We found that this was skew-Hermitian. Multiplying this by the purely imaginary $\zeta_m - \overline{\zeta}_m$ turns this into a Hermitian form. Complexifying, the decomposition

$$H^{pr}_{\mathbb{C}} = \bigoplus_{(r,m)=1} H^{\chi^r}_{\mathbb{C}}$$

is orthogonal with respect to the form. Each of these subspaces has a signature (p_r, q_r). As noted before $H^{\overline{\chi}}_{\mathbb{C}}$ has signature $(p_{-1}, q_{-1}) = (1, n-1)$ (aka, it is hyperbolic).

Over the reals we have an analogous decomposition, but we must group conjugate characters

$$H^{pr}_{\mathbb{R}} = \bigoplus_{(r,m)=1} (H^{\chi^r}_{\mathbb{C}} \oplus H^{\chi^{-r}}_{\mathbb{C}})(\mathbb{R}) = \bigoplus_{\substack{(r,m)=1 \\ 1 \le r < \lfloor m/2 \rfloor}} H^{\chi^r}_{\mathbb{C}} \text{ viewed as a vector space over } \mathbb{R}$$

Here the $(\mathbb{R})$ above means we look at points of $H^{\chi^r}_{\mathbb{C}} \oplus H^{\chi^{-r}}_{\mathbb{C}}$ of the form $(a, \overline{a})$. We now have

$$\mathcal{G}(\mathbb{R}) = \prod_{\substack{(r,m)=1 \\ 1 \le r < \lfloor m/2 \rfloor}} U(H^{\chi^r}_{\mathbb{C}}) \cong \prod_{\substack{(r,m)=1 \\ 1 \le r < \lfloor m/2 \rfloor}} U(p_r, q_r).$$

where $p_r + q_r = n$. Remember, that Γ lies in

$$\Gamma \subseteq PU(H^{\overline{\chi}}_{\mathbb{C}}) \cong PU(1, n-1).$$

Remark III.2.2

For those of us just as confused as I was and need a reminder. Here is the trick – we move around the moduli space $\mathcal{M}(\mu)$ of divisors with coefficients $\mu = (\mu_1, \ldots, \mu_n)$ where $\mu_i \in \mathbb{Q} \cap (0, 1)$. Each such gives a curve

$$C : y^m = (x - x_1)^{d_1} \cdots (x - x_n)^{d_n},$$

where m is a common denominator of $\mu_i = d_i/m$. These have a form $\widetilde{\eta}$ lifting $\eta = \frac{\mathrm{d}x}{\sqrt[m]{(x-x_1)^{d_1}\cdots(x-x_n)^{d_n}}}$. This form $\widetilde{\eta}$ lives in $H_{\mathbb{C}}^{\overline{\chi}}$, and Γ tracks how $\widetilde{\eta}$ changes as you move around this moduli space $\mathcal{M}(\mu)$. We must divide out by the ambiguity of scaling because of the G-action on C.

Of course, $PU(H_{\mathbb{C}}^{\overline{\chi}}) \cong PU(H_{\mathbb{C}}^{\chi}) \cong PU(n-1,1)$.

We claim the red map below is injective.

$$\begin{tikzcd}
\mathcal{G}(\mathbb{R}) \arrow[r] & U(H_{\mathbb{C}}^{\chi}) \\
\cup\!\mid & \\
\mathcal{G}(\mathbb{Q}) \arrow[uur] & \\
\cup\!\mid & \\
\mathcal{G}(\mathbb{Z}) &
\end{tikzcd}$$

Proof. If $\gamma \in \mathcal{G}(\mathbb{Q})$ acts as the identity on $(H_{\mathbb{C}}^{pr})^{\chi}$ then $(H_{\mathbb{C}}^{pr})^{\gamma}$ (where γ acts as the identity) is a subspace of $H_{\mathbb{C}}^{pr}$ defined over $\mathbb{Q}$ which contains $(H_{\mathbb{C}}^{pr})^{\chi}$ and hence must be all of $H_{\mathbb{C}}^{pr}$.

Now $\Gamma \subseteq \mathcal{G}(\mathbb{Z})/\mu_m$

Definition III.2.2

We say Γ is arithmetic if the inclusion in $\mathcal{G}(\mathbb{Z})/\mu_m$ is finite-index.

Now consider that

$$\mathcal{G}(\mathbb{R}) \to U(H_{\mathbb{C}}^{\chi})$$

is propert if and only if each $U(H_{\mathbb{C}}^{\chi^r})$ with $1 < r < \left\lfloor \frac{m}{2} \right\rfloor$ is compact, which holds if and only if one of p_r, q_r is zero. This also holds if and only if $U(H^{\chi^r})$ has finite volume.

But then $\mathcal{G}(\mathbb{Z}) \hookrightarrow U(H_{\mathbb{C}}^{\chi})$ is discrete if and only if we have these finite volume conditions. Lets prove the converse for our health.

Proof. If the image is discrete, there exists a neighborhood N in $U(H_{\mathbb{C}}^{\chi}$ with

$$N \cap \operatorname{im}(\mathcal{G}(\mathbb{Z}) = \{1\},$$

i.e., $\mathcal{G}(\mathbb{Z}) \cap$ preimage of N is the identity. But by the result of Borel, this implies that the preimage of N must have finite volume. Hence all these factors besides $U(H_{\mathbb{C}}^{\chi})$ must be compact.

The conclusion/upshot is that

$$\text{image of } \mathcal{G}(\mathbb{Z}) \text{ in } U(H_{\mathbb{C}}^{\chi}) \text{ discrete} \iff p_r q_r = 0 \text{ for all } 1 < r < \left\lfloor \frac{m}{2} \right\rfloor, (r,m) = 1. \qquad (\dagger\dagger)$$

We know that $\widetilde{\Gamma} < \mathcal{G}(\mathbb{Z})$ (the preimage of Γ). Therefore

$$\Gamma \text{ arithmetic} \iff (\dagger\dagger)$$

The problem now is how do we determine the signatures of these summands! We will analyze this problem. We'll find a recipe. Consider the cyclic cover

$$\begin{tikzcd} C \arrow[d, "G"] \\ \mathbb{P}^1, \end{tikzcd}$$

and $H^1(C) = H$. We obtain a G-equivariant Hodge decomposition

$$H^1(C;\mathbb{C}) = H^0(\Omega_C) \oplus \overline{H^0(\Omega_C)}$$
$$H^1(C;\mathbb{C})^{\chi^r} = H^0(\Omega_C)^{\chi^r} \oplus \overline{H^0(\Omega_C)^{\overline{\chi}^r}}.$$

By definition of the symplectic form giving rise to the Hermitian form we know that $H^0(\Omega_C)$ is positive definite of dimension $g = \mathrm{genus}(C)$ and $\overline{H^0(\Omega_C)}$ is negative definite of dimension g. Hence

$$p_r = \dim H^0(\Omega_C)^{\chi^r} \qquad\qquad q_r = \dim \overline{H^0(\Omega_C)^{\overline{\chi}^r}}.$$

Recall that $H^0(\Omega_C)^{\overline{\chi}}$ is spanned by $\frac{\mathrm{d}z}{w}$ where

$$w^m = (z - z_0)^{d_0} \cdots (z - z_n)^{d-n}$$

with $\mu_i = d_i/m$ for $i = 0, \dots, n$ and $z_{n+1} = \infty$, $\mu_0 + \cdots + \mu_{n+1} = 2$. The action G was defined by $(z, w) \mapsto (z, \chi(g)w)$. If we want to examine

$$H^0(\Omega_C)^{\overline{\chi}^r}$$

we take our cue from this example and take $\frac{\varphi(z)\,\mathrm{d}z}{w^r}$ with φ holomorphic. Problem: this might not be holomorphic on C!

To ensure that the pullback to C is holomorphic, we must examine what the form looks like near a zero. Near z_i we find that

$$\frac{\varphi(z)\,\mathrm{d}z}{w^r} \sim \frac{\varphi(z)\,\mathrm{d}z}{(z - z_i)^{\mu_i r}}.$$

The condition for this to be holomorphic is that

$$(\mathrm{ord}_{z_i}\varphi) + 1 > \mu_i r \iff \frac{\varphi(z)\,\mathrm{d}z}{w} \text{ holomorphic at } z_i.$$

Near $\infty = z_{n+1}$ we must guarantee that

$$(\deg\varphi) + 1 < r\frac{d}{m} \iff \frac{\varphi(z)\,\mathrm{d}z}{w} \text{ holomorphic at } \infty.$$

Exercise III.2.5

Find the dimension of the space of polynomials of this bounded degree with zeroes of the necessary orders of vanishing at z_i. Namely show the dimension of such φ is

$$\lceil \{\mu_0 r\} + \cdots + \{\mu_{n+1} r\} \rceil - 1,$$

where $\{x\}$ is the fractional part of x.

This is everything (take φ meromorphic, then argue it must be holomorphic), and so

$$p_r = \lceil \{\mu_0 r\} + \cdots + \{\mu_{n+1} r\} \rceil - 1.$$

Example III.2.6

Let

$$(\mu_0, \ldots, \mu_4) = \left(\frac{7}{18}, \frac{7}{18}, \frac{7}{18}, \frac{7}{18}, \frac{8}{18}\right).$$

In this case some p_r with $(r, 18) = 1$ has $p_r > 0$ and $q_r > 0$. Take $r = 7$ and you get $p_r = 3$ and $q_r = p_{-r} = 1$. This lies in $U(3,1)$. Hence Γ is non-arithmetic!

IV. Something New from Something Old

A somewhat biased birds eye view of what Deligne–Mostow did:

(1) Begin with a smooth hypersurface (aka a reduced divisor) $D \subseteq \mathbb{P}^1$,

(2) Associate the cyclic cover $C \to \mathbb{P}^1$ with covering group G branched over D,

(3) Look at $H^1(C)$ with its G-action and its Hodge structure,

(4) Obtain a ball quotient.

The question: to what extent can one generalize this? We begin with looking at a smooth hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d given by $F \in \mathbb{C}[z_0, \ldots, z_{n+1}]_d$ for $d \geq 2$. Lets begin with a remark from basic algebraic geometry.

Remark IV.0.1

X being smooth is equivalent to

$$\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}$$

having common zero set exactly $\{0\}$. Equivalently by the Nullstellensatz

$$\langle \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_{n+1}} \rangle \supseteq \langle z_0, \ldots, z_{n+1} \rangle^N,$$

for some N. Equivalently as well

$$A(F) \coloneqq \mathbb{C}[z_0, \ldots, z_{n+1} / \langle \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_{n+1}} \rangle$$

is a finite dimensional $\mathbb{C}$-graded algebra.

Example IV.0.1

X being a Fermat hypersurface $F = \sum_i z_i^d$. Then we find that

$$\langle \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_{n+1}} \rangle = \langle z_0^{d-1}, \ldots, z_{n+1}^{d+1} \rangle,$$

and so

$$A(F) = \mathbb{C}[z_0]/\langle z_0^{d-1}\rangle \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[z_{n+1}]/\langle z_{n+1}^{d-1}\rangle,$$

which gives this dimension $(d-1)^{n+2}$. The element of highest degree is $z_0^{d-1}\cdots z_{n+1}^{d-1}$ which has degree $(n+2)^{d-2}$. In fact, the highest degree is $(n+2)^{d-2}$ for all smooth hypersurfaces X.

Lets return to the general setting. Let $i : X \hookrightarrow \mathbb{P}^{n+1}$ be the inclusion. As we know, the hyperplane class $\eta \in H^2(\mathbb{P}^{n+1})$ generates $H^*(\mathbb{P}^{n+1})$ with $H^*(\mathbb{P}^{n+1}) = \mathbb{Z}[\eta]/(\eta^{n+2})$. The orientation class is given by η^{n+1}.

Theorem IV.0.1 (Weak Lefschetz Hyperplane Theorem)

The pair $(\mathbb{P}^{n+1}, X)$ admits the structure of a cell complex whose cells are all of dimension greater than n. In other words, this does not effect the homology in degree $< n$, and can only kill homology in degree n. It is also essentially equivalent to $\mathbb{P}^{n+1} \setminus X$ admitting a finite cell complex of dimension $\leq n+1$.

Corollary IV.0.2

The cohomology

$$H^k(\mathbb{P}^{n+1}) \xrightarrow{i^*} H^k(X)$$

is an isomorphism for $k < n$ and injective when $k = n$.

Corollary IV.0.3

The Gysin map from Poincaré duals gives

$$H_{2n-k}(X) \cong H^k(X) \xrightarrow{i_!} H^{k+2}(\mathbb{P}^{n+1}) \cong H_{2n-k}(\mathbb{P}^{n+1})$$

is an isomorphism for $k > n$ and surjective when $k = n$.

Remark IV.0.2

Geometrically what the Gysin map does is about intersection. A $2n-k$ cycle in X gives a k cocycle by intersection with k-cycles, but we need 2 more (real) dimensions to intersect with in $\mathbb{P}^{n+1}$.

In other words, the only interesting part of the cohomology of X is in the middle dimension:

$$\begin{tikzcd} H^n(\mathbb{P}^n) \arrow[r, "i^*"] \arrow[dr] & H^n(X) \arrow[d, "i_!"] \\ & H^{n+2}(\mathbb{P}^n) \end{tikzcd}$$

If n is odd the above map is zero. Otherwise the question is where $i_! i^*$ sends η^m where $n = 2m$. Well in fact

$$\eta^m \xrightarrow{i_! i^*} d\eta^{m+1}.$$

Why? Well consider how $m+1$ hyperplanes will meet the intersection of m hyperplanes with X... it's simply the intersection of a hyperplane with X! This is given by the degree.

Call $\ker(i_!)$ the primitive cohomology of $H^n(X)$, denoted $H^n_{pr}(X)$. Then

$$H^n(X;\mathbb{Q}) = H^n_{pr}(X;\mathbb{Q}) \oplus \underbrace{\langle i^*\eta^{n/2}\rangle}_{n \text{ even}},$$

and perpendicular with respect to intersection pairing.

Griffiths gave the Hodge structure of $H^n_0(X)$ in explicit terms. The Gysin sequence for $(\mathbb{P}^{n+1}, X)$ is

$$\longrightarrow H^{n-1}(X) \xrightarrow{i_!} H^{n+1}(\mathbb{P}^{n+1}) \longrightarrow H^{n+1}(\mathbb{P}^{n+1}\setminus X) \xrightarrow{res} H^n(X) \xrightarrow{i_!} H^{n+2}(\mathbb{P}^{n+1}) \longrightarrow$$

this is Poincaré dual to the homology sequence of $(\mathbb{P}^{n+1}, X)$.

Claim

A map fits in here

$$\begin{tikzcd}
{} \arrow[r] & H^{n-1}(X) \arrow[r, "i_!"] & H^{n+1}(\mathbb{P}^{n+1}) \arrow[r] & H^{n+1}(\mathbb{P}^{n+1}\setminus X) \arrow[r, "res"] \arrow[d, "\cong"] & H^n(X) \arrow[r, "i_!"] & H^{n+2}(\mathbb{P}^{n+1}) \arrow[r] & {} \\
 & & & H^n_0(X) \arrow[ur, hook] & & &
\end{tikzcd}$$

Proof. A bit of a diagram chase for you. Now $H^{n-1}(X) \xrightarrow{i_!} H^{n+1}(\mathbb{P}^{n+1})$ has both terms are zero or given by a power of η. Both terms are zero or given by a power of η. In other words $i_!$ is an isomorphism here.

Hence the restriction map res above is injective, and its image is precisely $H^n_0(X)$ by definition of the latter and the exactness at $H^n(X)$.

Now interestingly, the primitive cohomology has been identified with the cohomology of something of an entirely different type! $\mathbb{P}^{n+1}\setminus X$ is an affine variety (embeddable in some $\mathbb{C}^N$ as a closed algebraic subvariety). Algebraically, we have

$$\mathbb{C}[\mathbb{P}^{n+1}\setminus X] = \left(\mathbb{C}[z_0,\dots,z_{n+1}]\left[\frac{1}{F}\right]\right)_0,$$

in other words fractions $\frac{p(z_0,\dots,z_{n+1})}{F^k}$ with total degree 0 (giving a function on $\mathbb{P}^{n+1}$ with poles along X).

The de Rham complex of $\mathbb{P}^{n+1}\setminus X$ denoted $\Omega^p[\mathbb{P}^{n+1}\setminus X]$ (space of regular p-forms on the complement). Explicitly in this case,

$$\Omega^p[\mathbb{P}^{n+1}\setminus X] \cong \bigcup_{r\geq 1}\left\{\frac{G\,\mathrm{d}z_{i_0}\wedge\cdots\wedge\mathrm{d}Z_{i_p}}{F^r} \mid \deg G = rd-p\right\},$$

where the isomorphism here is a residue map about the hyperplane at ∞ of $\mathbb{C}^{n+2}$. Now we have since $\mathbb{P}^{n+1} \setminus X$ is affine that this de Rham complex recovers the cohomology

$$H^k(\Omega^\bullet[\mathbb{P}^n \setminus X, d) = H^k(\mathbb{P}^{n+1} \setminus X; \mathbb{C}).$$

In top degree

$$\Omega^n[\mathbb{P}^{n+1} \setminus X] \xrightarrow{d} \Omega^{n+1}[|PP^{n+1} \setminus X] \to H^{n+1}(\mathbb{P}^{n+1} \setminus X) \cong H^n_{pr}(X; \mathbb{C}) \to 0,$$

is exact. The cokernel here (i.e., $H^n_{pr}(X)$) has a pretty concrete description. Lets do it! Consider

$$G\frac{\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_{n+1}}{F^{r+1}},$$

with $\deg G = (r+1)d - (n+2)$. When can we reduce the pole order? I.e., when is the residue of this exact.

$$\mathrm{d}\left(\sum_i \frac{G_i \,\mathrm{d}z_0 \wedge \cdots \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_{n+1}}{F^r}\right) = \sum_i \frac{(-1)^i \left(\frac{\partial G_i}{\partial z_i} F + G_i \frac{\partial F}{\partial z_i}\right) \mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_{n+1}}{F^{r+1}}.$$

Thus one can compute modulo $\langle \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_{n+1}} \rangle$ in order to reduce pole order. By this, we mean that the only term appearing in pole order $r+1$ is $\frac{G_i \frac{\partial F}{\partial z_i}}{F^{r+1}}$, and so if we compute modulo this ideal we can get a form which is up to

(1) Lower pole order terms,
(2) Exterior derivatives of forms,

equivalent to our original form.

Therefore we have a filtration of $H^{n+1}(\mathbb{P}^{n+1} \setminus X; \mathbb{C})$ by pole order so that

$$\begin{aligned}
&0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \\
F_r/F_{r-1} &= \left\{ \frac{G \,\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_{n+1}}{F^r} \mid \deg G = rd - (n+2)G \mod \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n} \right\} \\
&\cong A(X)_{rd-(n+2)}.
\end{aligned}$$

Theorem IV.0.4 (Griffiths)

The filtration above by pole order is equivalent to the Hodge filtration on $H^n_{pr}(X)$. I.e.,

$$F_r = \bigoplus_{\substack{p+q=n \\ q \leq r-1}} H^{p,q}_{pr}(X)$$

The identification of F_1 with $H^{n,0}_{pr}(X)$ is an earlier computation re: residues near the beginning of this course. Notably,

$$F_{n+1} = H^n_{pr}(X; \mathbb{C}),$$

Now by definition $H^{p,q}(X) \cong H^p(X; \Omega_X^q)$. There is a semi-continuity property of this cohomology (see Voisin), and we know that deformations preserve the Hodge filtration. Thus (again see Voisin), the dimension of $H^{p,q}$ is locally constant.

Example IV.0.2

We will now compute $h_{pr}^{p,q}$ for the Fermat hypersurface, where

$$A(X) = \mathbb{C}[z_0]/(z_0^{d-1}) \otimes \cdots \otimes \mathbb{C}[z_{n+1}]/(z_{n+1}^{d-1}).$$

We start with a non-exciting case, when $d = 2$. Then $A(X) = \mathbb{C} = A(X)_0$. Therefore the only nontrivial quotient is when $2r - (n+2) = 0$, so n is even and

$$\frac{\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_{n+1}}{F^{n/2+1}},$$

gives a form of type $h_{pr}^{n/2,n/2}$. For $n = 1$, we get a conic with $b_1 = 0$. For $n = 2$ we get a quadric isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and $b_2 = 2$ whil $b_2^{pr} = 1$.

For $d = 3$, we find that $A(X)$ is represented by monomials

$$z_0^{\varepsilon_0} \cdots z_{n+1}^{\varepsilon_{n+1}},$$

where $\varepsilon_i \in \{0, 1\}$. Lets examine this when $n = 1$, which is the case we know (smooth cubic curves). Then

(1) The form

$$\frac{\mathrm{d}z_0 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2}{F}$$

is homogeneous of degree 0. So then $h_{pr}^{1,0} = 1$

(2) The form

$$\frac{z_0 z_1 z_2 \,\mathrm{d}z_0 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2}{F^2},$$

and so $h_{pr}^{0,1} = 1$.

Finally we'll do the case of a cubic surface where $n = 2, d = 3$.

(1) The form

$$z_i z_j \frac{\mathrm{d}z_0 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3}{F^2},$$

for $0 \leq i < j \leq 3$ which has dimension $\binom{4}{2} = 6$, and this gives us $h_{pr}^{1,1} = 6$.

Exercise IV.0.3

Do this for $n = 2, d = 4$, i.e., the case of a K3 surface.

Get notes for last lecture

Takeaways from last time: let $\mathcal{H}_d(V)$ be the projective space of all hypersurfaces of degree d in $\mathbb{P}(V)$. Choose a basepoint $[X] \in \mathcal{H}_d(V)$ which is a smooth hypersurface. As we know $\mathrm{GL}(V)$ acts on $\mathcal{H}_d(V)$, factoring through $\mathrm{PGL}(V)$. We wish to compute $T_{[X]}\mathcal{H}_d(V)$. We found that

$$T_{[X]}\mathcal{H}_d(V) = \mathbb{C}[V]_d/\mathbb{C},$$

where $F \in \mathbb{C}[V]_d$ defines X. Likewise we identified

$$T_{[X]}\,\mathrm{PGL}(V)\text{-orbit of } [X] \subseteq T_{[X]}\mathcal{H}_d(V),$$

and the quotient (i.e., the normal space) as $A(X)_d$ where

$$A(X) \coloneqq \mathbb{C}[V]/\mathfrak{J}(F),$$

where $\mathfrak{J}(F)$ is the Jacobian ideal of F, i.e., the ideal generated by the partial derivatives

$$\mathfrak{J}(F) = \langle \frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n} \rangle$$

(exercise: $\mathfrak{J}(F)$ does not depend on the choice of coordinates $z_0, \dots, z_n$).

Call a <u>slice</u> S a transversal complex manifold to the $\mathrm{PGL}(V)$-orbit through $[X]$. Formally, S is a complex manifold with a point $o \in S$ and a map

$$\begin{aligned} S &\to \mathcal{H}_d(V) \\ o &\mapsto [X], \end{aligned}$$

which is an immersion at 0 and transverse to the $\mathrm{PGL}(V)$-orbit, so that $\varphi^{-1}([X]) = o$.

This implies we have a family of hypersurfaces

$$\begin{tikzcd} \mathscr{X} \arrow[d] & \subseteq & \mathbb{P}(V) \times S \arrow[dll] \\ S & & \end{tikzcd}$$

By making S small we can assume that this is a smooth family.

Definition IV.0.1

In the case above, the family $\mathscr{X} \to S$ is a universal at $o \in S$.

We will assume that the degree $d > 2$, so that the isotropy groups are finite. If the isotropy groups are trivial, then S is a chart for the orbit space. If the isotropy group is finite, then S is an orbifold chart for the orbit space. We can then consider

$$T_oS \to \{\text{normal space to } \mathrm{PGL}(V)\text{-orbit of } [X]\} \cong A(X)_d.$$

If

$$A(X)_d \to \bigoplus_{r=1}^{n} \mathrm{Hom}_{\mathbb{C}}(A_{rd-(n+2)}, A_{(r+1)d-(n+2)})$$

then there is an "infinitesimal torelli theorem" at X!

IV.1. **Cubic 3-folds**

Let $F \in \mathbb{C}[z_0, \dots, z_4]_3$ be the defining equation of a smooth cubic 3-fold $X \subseteq \mathbb{P}^4$. Look at

$$G \frac{\mathrm{d}z_0 \wedge \cdots \wedge \mathrm{d}z_4}{F^r}$$

where $G \in \mathbb{C}[z_0, \dots, z_4]$ so that the whole expression has degree 0. Of course the total degree of the expression is

$$5 - 3r + \deg G$$

For $r = 1$ this will not produce anything. In other words

$$H^{3,0}_{pr}(X) = 0.$$

If $r = 2$, then $\deg G = 1$ and $G \in \operatorname{span}_{\mathbb{C}}(z_0, \dots, z_4)$. Thus

$$\dim H^{2,1}_{pr}(X) = 5.$$

The hodge numbers are symmetric, and so we are done with the middle dimension (the most important). For $r = 3$, $A(X)_4$ has degree 5.

Therefore, the 3-dimensional cohomology has dimension 10, and the Hodge structure "looks like" a projective complex curve of genus 5, just shifted up a dimension (must correct weights). This is called an "intermediate Jacobian."

Clemens–Griffiths studied this in detail. They asked if these cubic 3-folds are rational or not. The answer was no, and it's based on proving that this Hodge structure cannot arise from blowing down/blowing up a curve in any way.

One can check (do it for Fermat 3-fold) that infinitesimal Torelli holds.

IV.2. **Cubic Surfaces**

As previously discussed, cubic surfaces have absolutely boring Hodge structures. Thus it's difficult to extract information from Hodge theory. But what we can do instead is consider cyclic covers of $\mathbb{P}^3$ branched along the hypersurface

Let $Y \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq 2$. We will construct cyclic covers of $\mathbb{P}^n$ ramified at Y. In affine coordinates $z_1, \dots, z_n$ with $Y \cap \mathbb{C}^n$ given by $f \in \mathbb{C}[z_1, \dots, z_n]_{\leq d}$ then for order m we consider

$$w^m = f(z_1, \dots, z_n).$$

The solutions of this equation give a μ_m-cover of $\mathbb{C}^n$ ramified along $Y \cap \mathbb{C}^n$. The problem is at the hyperplane at ∞. The cover extends across the hyperplane at ∞ with a smooth total space precisely when $m \mid d$. On the level of algebraic topology we can see that this is necessary because a cover will give a map

$$\mathbb{Z}/d \cong H_1(\mathbb{P}^n \setminus Y; \mathbb{Z}) \twoheadrightarrow \mu_m.$$

This is realizable for $m = d$ by the homogeneous equation

$$W^d = F(Z_0, \dots, Z_n),$$

giving you a degree d hypersurface in $\mathbb{P}^{n+1}$. The μ_d-action is via scalar multiplication on W. If $d' \mid d$, then we can quotient by the subgroup $\mu_{d/d'}$. We

For the case of a cubic surface Y, we thus produce a cubic 3-fold X so that

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r, phantom, "\subseteq"] & \mathbb{P}^3 \end{tikzcd}$$

is a branched cover with deck group $\mathbb{Z}/3$. Callthe μ_3-action to be given by a character $\chi : \mu_3 \subseteq \mathbb{C}^\times$. Then $H^3(X; \mathbb{C})$ decokposes according to characters:

$$H^3(X; \mathbb{C})^{\chi^0} = H^3(X/\mu_3; \mathbb{C}) = H^3(\mathbb{P}^3; \mathbb{C}) = 0.$$

Thus $H^3(X; \mathbb{C})$ decomposes according to $\chi, \overline{\chi}$. We proceed as in Deligne-Mostow – we have a Hermitian form on $H^3(X; \mathbb{C})$:

$$\langle \alpha, \beta \rangle := -\sqrt{-1} \int_X \alpha \wedge \overline{\beta}.$$

A computation in local coordinates shows that this is nondegenerate. Namely

$$-\sqrt{-1}(\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\overline{x_3}) \wedge (\mathrm{d}\overline{x_1} \wedge \mathrm{d}\overline{x_2} \wedge \mathrm{d}x_3) > 0.$$

Therefore this form is positive definite on $H^{2,1}(X)$ and negative definite on $H^{1,2}(X)$. We know that

$$H^{2,1}(X; \mathbb{C}) = \langle \frac{z_i \, \mathrm{d}z_0 \wedge \cdots \mathrm{d}z_3 \wedge \mathrm{d}w}{\widetilde{F}^2}, \frac{w \, \mathrm{d}z_0 \wedge \cdots \mathrm{d}z_3 \wedge \mathrm{d}w}{\widetilde{F}^2} \rangle,$$

where $\widetilde{F} = F - W^3$ and F is the defining equation for Y. The μ_3-action acts by χ on the w component. This really looks like the Deligne-Mostow situation! And we can compute things!

$$\begin{aligned} H^{2,1}(X; \mathbb{C})^{\overline{\chi}} &\leftrightarrow \langle z_0, z_1, z_2, z_3 \rangle \\ H^{2,1}(X; \mathbb{C})^{\chi} &\leftrightarrow \langle w \rangle. \end{aligned}$$

At this point, we are trained to see a ball quotient! Namely

$$\begin{aligned} H^3(X; \mathbb{C})^\chi &= H^{2,1}(X)^\chi \oplus H^{1,2}(X)^\chi \\ &= H^{2,1}(X)^\chi \oplus \overline{H^{2,1}(X)^{\overline{\chi}}}, \end{aligned}$$

the left hand side is of dimension 1, and the right hand side is of dimension 4. Positive definite on the left and negative definite on the right. Hence, this defines a point in the 4-dimensional ball quotient.

To do this all formally, as before we fix a symplectic lattice H of rank 10 which is unimodular with a μ_3-action isomorphic to $H^3(X)$ with its μ_3-action. As before, we consider cubic surfaces Y equipped markings on the cubic 3-fold X obtained by branched covers.

We obtain the space

$$\mathbb{P}(H^{\chi}_{\mathbb{C}}) \supseteq \mathbb{B} := \{\text{positivity of Hermitian form}\},$$

and the arithmetic group $\Gamma = \mathrm{Sp}(H)^{\mu_3}$ acts properly discontinuously on $\mathbb{B}$. We then get a map

$$\mathcal{M} \to \Gamma \backslash \mathbb{B}$$

where $\mathcal{M}$ is the moduli space of smooth cubic surfaces Y. The infinitesimal Torelli theorem holds, so this is a local isomorphism of orbifolds.

We can compactify and to do so we think about the geometric invariant theory of cubic surfaces. FACTS:

A cubic surface Y is GIT-stable $\iff$ Y has A_1-singularity at worst (local equation like $w_1^2 + w_2^2 + w_3^2 = 0$) A cubic surface Y is GIT-semistable $\iff$ Y has A_2-singularities at worst (local equation like $w_1^3 + w_2^2 + w_3^2 = 0$)

This has only one closed strictly stable orbit with 3 A_2-singularities. This is given by the equation

$$Z_0 Z_1 Z_2 = Z_3^3,$$

The affine equation for $Z_0 = 1$ is $z_2 z_2 = z_3^3$ which is an A_2-singularity.

The maximum number of A_1 singularities is 4. It is given by the Cayley nodal surface.

Get notes

We return to cubic surfaces. We have $Y \hookrightarrow \mathbb{P}^2$ a cubic surface and a cubic threefold in $\mathbb{P}^4$ given by a branched cover of degree 3 at Y. Now $H^3(X)$ has a symplectic pairing with G-action. Let H be a symplectic lattice with G-action isomorphic to this.

Let $\mathrm{Sp}(H)^G := \Gamma$ and $\mathcal{M}(H)$ be H-marked cubic suraces.

Question: How does the H-marking determine the H'-marking where H' is isomorphic to a lattice given by $H^2(Y)$. This asks about the relative behavior of Y in X.

The Gysin sequence for $Y = X^G \subseteq X$ is

$$\begin{array}{ccccccccccc} 0 & = & H^1(Y) & \longrightarrow & H^3(X) & \longrightarrow & H^4(X \setminus Y) & \longrightarrow & H^2(Y) & \xrightarrow{onto} & H^4(X) \\ & & \cong & & \cong & & \cong & & \cong & & \cong \\ & & H_3(Y) & \longrightarrow & H_3(X) & \longrightarrow & H_3(X, Y) & \longrightarrow & H_2(Y) & \longrightarrow & H_2(X) \end{array}$$

We note that the line class in $H_2(Y)$ maps to the line class in $H_2(X)$ and generates. This gives the surjectivity of the final map, and the kernel is $H^2_{pr}(Y)$. Thus we obtain a short exact sequence

$$0 \longrightarrow H^3(X) \longrightarrow H^3(X \setminus Y) \xrightarrow{r} H^2_{pr}(Y) \longrightarrow 0.$$

Note this sequence has G-action and $H^2_0(Y)$ carries the trivial G-action. In fact, the sequence splits after tensoring with $\mathbb{C}$ by taking the G-eigenspaces.

We produce a G-invariant section of r after tensoring with $\mathbb{Q}$. Let $\alpha \in H^2_{pr}(Y)$. Choose $\widetilde{\alpha} \in H^3(X \setminus Y)$ mapping to α. We take the part of $\widetilde{\alpha}$ fixed by G:

$$s(\alpha) = \frac{\widetilde{\alpha} + g\widetilde{\alpha} + g^2\widetilde{\alpha}}{3}.$$

This is unique because $H^3(X)$ contains no G-fixed vectors besides 0. The lifting requires a $\frac{1}{3}$ so really we can work over $\mathbb{Z}\left[\frac{1}{3}\right]$. Consider

$$\widetilde{\alpha} - \frac{\widetilde{\alpha} + g\widetilde{\alpha} + g^2\widetilde{\alpha}}{3}.$$

We find that this is the part of $\widetilde{\alpha}$ with eigenvalues the primitive third roots of unity $\zeta_3, \overline{\zeta}_3$. Multiplying by 3 gives

$$2\widetilde{\alpha} - g\widetilde{\alpha} - g^2\widetilde{\alpha} \in H^3(X).$$

Now $\widetilde{\alpha}$ is unique up to an element of $\beta \in H^3(X)$. Then, observing that G acts by multiplying by a primitive third root of unity in $H^3(X)$ that

$$3\beta - (\beta + g\beta + g^2\beta) = 3\beta.$$

Therefore, we obtain a well-defined map

$$\begin{tikzcd} H^2_{pr}(Y) \arrow[rr] \arrow[dr] & & H^3(X;\mathbb{F}_3) \\ & H^2_{pr}(Y;\mathbb{F}_3) \arrow[ur] & \end{tikzcd}$$

We'll first examine the map $H^2_{pr}(Y;\mathbb{F}_3) \to H^3(X;\mathbb{F}_3)$. Let $L = H^2_{pr}(Y)$, which as we recall is the orthogonal complement of the hyperplane class $\eta_Y \in H^2(Y)$. Recall also that the $\mathbb{F}_3$-vector space $H^2_{pr}(Y;\mathbb{F}_3)$ is an $\mathbb{F}_3$-vector space of dimension six. Notably

$$3H^2(Y) \cap H^2_{pr}(Y) \underbrace{\supseteq}_{index 3} 3H^2_{pr}(Y).$$

This is because

$$L \supseteq 3L^* \supseteq 3L,$$

where $L^* = \mathrm{Hom}(L;\mathbb{Z}) \hookrightarrow L \otimes \mathbb{Q}$, which gives us that

$$L/3L \supseteq 3L^*/3L,$$

and so $L/3L^*$ is a 5-dimensional $\mathbb{F}_3$-vector space.

Thus $H' \otimes \mathbb{F}_3 \to H \otimes \mathbb{F}_3$ has image an $\mathbb{F}_3$-subspace of dimension 5 invariant under $\mathrm{Sp}(H)^G$ and the induced action reproduces the $W(E_6)$-action on H' (faithfully). Hence this gives us a map

$$\Gamma = \mathrm{Sp}(H)^G \to W(E_6).$$

Both objects under consideration have compactifications.

$$\begin{tikzcd}
\mathcal{M}(H) \arrow[d, "\Gamma"] & \subseteq & \mathcal{M}^{st}(H) \arrow[r, "\cong"] \arrow[d, "\Gamma"] & \mathbb{B} \arrow[d] \\
\mathcal{M} & \subseteq & \mathcal{M}^{st} \arrow[r, "\cong"] & \Gamma\backslash\mathbb{B} \\
 & & \cap\!\!\mid & \cap\!\!\mid \\
 & & \mathcal{M}^{GIT} \arrow[r, "\cong"] & (\Gamma\backslash\mathbb{B})^*
\end{tikzcd}$$

V. Quartic Curves

Summary of what has been done:

(1) $\mathcal{M}^{st}$ the moduli space of quartic curves. One then blows up the locus represented by the double conic (these are the hyperelliptics living in a $\mathbb{P}^5$).

(2) As with cubic surfaces we construct a Γ-equivariant map (for Γ an arithmetic group):

$$\begin{tikzcd}
\widetilde{\mathcal{M}}^{st}_{hyp} \subseteq \widetilde{\mathcal{M}}^{st} \arrow[r, "\Gamma", "\cong"'] \arrow[d, "\Gamma"'] & \mathbb{B} \arrow[d] \\
\widetilde{\mathcal{M}}^{st}_{hyp} \subseteq \mathcal{M}^{st} \arrow[r] & \Gamma\backslash\mathbb{B}
\end{tikzcd}$$

where $\mathbb{B}$ is a 4-dimensional ball. The hyperelliptic locus is $\mathcal{M}^{st}_{hyp}$. Each irreducible component maps to a 3-dimensional subballs of Deligne-Mostow type.

(3) In $\mathbb{P}^2$ the hyperelliptic points are represented by a double conic. A nearby quartic intersects the conic in 8 points, and the corresponding hyperelliptic curve is given by branch covering the conic ($\cong \mathbb{P}^1$) over these 8 points. This is assigned to the Deligne-Mostow ball with coefficients $(1/4, \ldots, 1/4)$ (8 copies of $1/4$).

The similarity with the cubic surface case is not accidental. Let $C \subseteq \mathbb{P}^2$ be a quartic curve, we obtain a branched cover

$$\begin{tikzcd}
Y \arrow[dd, "\mu_4"'] \arrow[dr] & \\
 & X = \mu_2\backslash Y \arrow[dl, "\mu_2"] \\
C \subseteq \mathbb{P}^2 &
\end{tikzcd}.$$

The obtained surface X is a degree 2 del Pezzo surface. We'll now review the theory of such, with some general review as well.

V.1. **del Pezzo Surfaces**

Let X be a general complex manifold of dimension n. The canonical bundle $\omega_X \coloneqq \Omega_X^n$ is a holomorphic line bundle. The zeroes of sections are called canonical divisors. The dual $\omega_X^\vee$ is the anticanonical bundle, and the zeros of its sections are the so called anticanonical divisors. Notably

$$c_1(\omega_X^\vee) = c_1(TX) = c_1(X).$$

Example V.1.1

Consider $X = \mathbb{P}^n \supseteq \mathbb{C}^n$. There is an n-form $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ with a pole at ∞ of order $n+1$ (a meromorphic section of ω_X. Dualizing, we obtain a zero of order $n+1$ at the hyperplane $\eta_{\mathbb{P}^n}$:

$$\omega_{\mathbb{P}^n}^\vee \rightsquigarrow \text{anticanonical divisor} = (n+1)\eta_{\mathbb{P}^n}.$$

Definition V.1.1

If X is compact and $\omega_X^\vee$ is ample, then we say X is a Fano manifold. If $\dim X = 2$, then X is called a del Pezzo surface (for historical reasons).

We define the degree of a del Pezzo surface as

$$c_1(\omega_X^\vee) \cdot c_1(\omega_X^\vee) = c_1(\omega_X) \cdot c_1(\omega_X).$$

Recall V.1.2

As mentioned before, a bundle is ample provided that a tensor power has enough sections to embed X in $\mathbb{P}^N$. An equivalent criterion for ampleness is that every anticanonical divisor ≥ 0 hits every curve in X. (one can see that this is necessary by direct computation)

There are a finite list of examples of del Pezzo surfaces. We list them all here.

Example V.1.3 (Basic Examples)

Let's roll!

(1) Clearly $X = \mathbb{P}^2$ is del Pezzo. The anticanonical system is all cubics (9-dimensional projective space). This defines an embedding

$$\mathbb{P}^2 \hookrightarrow \mathbb{P}^9,$$

of degree also 9.

(2) We can consider $\mathbb{P}^1 \times \mathbb{P}^1$. The anticanonical system consists of curves of bidegree $(2, 2)$. All such curves have artihmetic genus 1 (so the smooth ones are elliptic curves).

Example V.1.4

We can blow up a point in $\mathbb{P}^2$ and see how many sections survive

(3) Consider $X = \mathcal{B}\ell_p\mathbb{P}^2 \supseteq E \to \mathbb{P}^2$. We find that

$$\omega_X = \pi^*\omega_{\mathbb{P}^2}(E),$$

where $\pi^*\omega_{\mathbb{P}^2}(E)$ corresponds to twisting by E (allowing a pole at E). Thus

$$\omega_X^\vee = \pi^*\omega_{\mathbb{P}^2}^\vee(-E).$$

Then the anticanonical divisors must come from anticanonical divisors of $\mathbb{P}^2$ passing through p. That is a codimension one condition, so we get an 8-dimensional linear system.

However, it is still ample! The cubics passing through a single point are still able to separate every point in $\mathbb{P}^2$ (even infinitesimally). Furthermore, it separates the exceptional curve E in the blowup. To compute the anticanonical divisor, we find that if C is a cubic passing through p then

$$\pi^*C = C_X + E,$$

where C_X is the strict transform of C. Thus

$$C_X = 3\ell - e.$$

The degree is $C_X \cdot C_X = 9 + (-1) = 8$.

Recall V.1.5

To see why the anticanonical class transforms as indicated. Consider a 2-form $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ on $\mathbb{C}^2$ and blow up at zero. The blow-up has coordinates w_1, w_2 so that $z_1 = w_1w_2$ and $z_2 = w_2$. Thus the pullback is

$$\pi^*(\mathrm{d}z_1 \wedge \mathrm{d}z_2) = w_2\,\mathrm{d}w_1 \wedge \mathrm{d}w_2,$$

and this is zero when $w_2 = 0$

From this, one can see that you can try to blow up more times. You won't always succeed

Non-Example V.1.6

Lets blow up at 3 points p_1, p_2, p_3 which lie along a line ℓ. $X = \mathcal{B}\ell_{p_1,p_2,p_3}\mathbb{P}^2$. Any cubic curve C passing through p_1, p_2, p_3 must have different tangents at the three points compared to ℓ.

Hence the strict transforms $\widehat{\ell}, \widehat{C}$ do not intersect, which obstructs ampleness.

Non-Example V.1.7

For the same reason, we cannot have 6 points on a conic. Find a cubic C hitting those 6 points. Blowing up at six points separates the strict transform of the conic from the strict transform of a cubic C.

Non-Example V.1.8

We can have no 8 points on a singular cubic, one of the points being the singular point. Otherwise a generic cubic intersecting at the singular point will be separated in the strict transform (the nodal cubic becomes a smooth curve).

Now these are all the examples. You can blow up at up to 8 points on $\mathbb{P}^2$ that do not have

(1) 3 on a line.
(2) 6 on a conic.
(3) 8 on a nodal cubic, one of which is a singular point.

This means we obtain del Pezzo surfaces of degree $d = 1, 2, \ldots, 8, 9$ given by blowing up at $r = 9 - d$ points. We also have the del Pezzo surface $\mathbb{P}^1 \times \mathbb{P}^1$ (which is of degree 8).

When $d = 3$ and $r = 6$, and the six points are in general position, then we obtain an embedding

$$\mathcal{B}\ell_{p_1,\ldots,p_6}\mathbb{P}^2 \hookrightarrow \mathbb{P}^3,$$

a cubic surface. The lines on the cubic surface are easily recognizable as exceptional curves

$$e_1, \ldots, e_6$$

$$2\ell - e_1 - e_2 - e_3 - e_4 - e_5 \text{ strict transform of conic through five points (six of them)}$$

$$\ell - e_1 - e_2 \text{ (strict transform of line through two points).}$$

This gives $6 + 6 + \binom{6}{2} = 12 + 15 = 27$ lines on the cubic surface.

When $d = 2$ and $r = 7$, we obtain a linear system of dimension 2 giving a map

$$X = \mathcal{B}\ell_{p_1,\ldots,p_7} \to \mathbb{P}^2,$$

a map of degree 2 which is ramified along quartic curve. Note, we can no longer separate points (but a multiple of the anticanonical divisor will).

Upshot: the quartic curves and cubic surfaces appear under the same roof.

Question: The preimage $f^{-1}(line)$ will give us an anticanonical divisor.

What is the analogue of the 27 lines on a cubic surface? The exceptional curves. We obtain $28 \cdot 2 = 56$ exceptional curves (this relates to bitangents on a quartic curve). Why do they relate?

Well an exceptional curve must be hit by an anticanonical divisor with intersection number 1 (by the adjunction formula). Thus the image in $\mathbb{P}^2$ is a line! Which line is it? Well it's a bitangent to the quartic curve $C \subseteq \mathbb{P}^2$. Why? The preimage of the line has two irreducible components (you obtain no branching because of the tangent behavior), giving you exceptional curves λ', λ''.

We can also connect this more directly with cubic surfaces!!! Take $p \in X' \subseteq \mathbb{P}^3$ a cubic surface. We find that X itself is obtained by blowing up $\mathcal{B}\ell_p X'$. Projecting away from p to a plane $\mathbb{P}^2$

(thought of as the visual sphere about p) we have a map

$$X' \setminus \{p\} \to \mathbb{P}^2,$$

which is generically 2 to 1. A line in X' maps to a line in this $\mathbb{P}^2$. By blowing up we obtain a map $\mathcal{B}\ell_p X' \to \mathbb{P}^2$ which again is generically 2 to 1. It branches when the line through p is tangent to the cubic surface X', this defines a quartic in $\mathbb{P}^2$ as desired.

Todo list