Before we defined the R-modules $\mathrm{Ext}_R^m, \mathrm{Tor}_m^R$ for a commutative ring R. If R is not commutative, then $\mathrm{Ext}_R^m(M,N)$ is defined if M, N are both left R-modules. In general, then, $\mathrm{Ext}_R^m(M,N)$ is just an abelian group. Then Tor_R^m is defined when M is a right R-module and N is a left R-module, and it is only an abelian group.

Example .0.1

Let G be a group. The group ring $\mathbb{Z}[G]$ is the free abelian group on G wiht multiplication given by the multiplication in G (and extended by distributivity).

For example, if $G = \mathbb{Z}/2\mathbb{Z}$. Then we can consider $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$. Let $G = \{1, \alpha\}$ be the particular representative, with $\alpha \cdot \alpha = 1$. Then:

$$(k + \ell\alpha)(n + m\alpha) = (kn + \ell m) + (km + \ell n)\alpha$$

Bad habit (in general G): A $\mathbb{Z}[G]$ module is called a "G-module." This clashes with other terminology. This really means that G acts on M by linear maps. And of course a left G-action and a right action are equivalent by $gm \leftrightarrow mg^{-1}$.

Definition .0.1

Let M be a G-module. We define the homology and cohomology of G with coefficients in M by:

$$H_m(G; M) := \mathrm{Tor}_m^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$
$$H^m(G; M) := \mathrm{Ext}_{\mathbb{Z}[G]}^m(\mathbb{Z}, M)$$

Back to topology.

Definition .0.2

Let G be a group, A G-CW-complex is a G-equivariant space (space with a G-action) X where $X = \bigcup_{n\in\mathbb{N}_0}$ where $X_{-1} = \emptyset$ (indeed we can take X_{-1} to be a G-space to get a G-CW-pair).

I_m (the set of m-cells) is a G-set (set with a G-action). Furthermore, $f_m : I_m \times S^{m-1} \to X_{m-1}$ is a G-equivariant map (when tking the G-action to be trivial on the sphere). Then X_m is a pushout:

$$\begin{tikzcd}
I_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d, hook] & X_{m-1} \arrow[d, hook] \\
I_n \times D^{n-1} \arrow[r] & X_m
\end{tikzcd}$$

Suppose we have a G-space X which is both free (all the I_m are free G-sets, aka $gx = x \implies g = 1$) and $X \simeq *$ non-equivariantly..

Then $C^{\mathrm{cell}}X$ is a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$. Why? Well because $\mathbb{Z}I_m$ is a free $\mathbb{Z}[G]$-module, and we have an exact sequence (because $X \simeq *$) given by:

$$\cdots \longrightarrow \mathbb{Z}I_m \longrightarrow \mathbb{Z}I_{m-1} \longrightarrow \cdots \longrightarrow \mathbb{Z}I_0 \longrightarrow \mathbb{Z}$$

So we have that X/G is CW-complex (with set of m-cells I_m/G). Furthermore $C^{\mathrm{cell}}(X/G) = C^{\mathrm{cell}}(X) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. More generally, using that I_m is G-free:

$$\mathbb{Z}[I_m] \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}[I_m/G]$$

Likewise,

$$C_{\mathrm{cell}}(X/G) = \mathrm{Hom}_{\mathbb{Z}[G]}(C^{\mathrm{cell}}(X), \mathbb{Z})$$
$$\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}I_m, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}[I_m/G], \mathbb{Z})$$

We can conclude that:

$$H_m(G;\mathbb{Z}) = H_m(X/G)$$
$$H^m(G;\mathbb{Z}) = H^m(X/G)$$

We sometimes write $BG = X/G$, where $X* \simeq$ (non-equivariantly) is a free G-CW-complex. This is also sometimes called the classifying space of G.

We call $X = EG$, and it is the universal cover of BG via the quotient map. Therefore $\pi_1(BG) = G$ and $\pi_k(EG) = \pi_k(BG) = 0$ for $k > 1$. We will come back to this in more detail.

Example .0.2

Let $G = \{1, \alpha\} \cong \mathbb{Z}/2\mathbb{Z}$. Consider $\mathbb{RP}^\infty := \bigcup_n \mathbb{RP}^n$ (a CW-complex with the union topology). Then $\mathbb{RP}^\infty$ is ain fact a BG. Why?

Well the universal cover of $\mathbb{RP}^\infty$ is S^∞, which we know to be contractible (and will be a EG). To see this think of a homotopy:

$$h_t(x_0, x_1, \ldots) = \frac{t(x_0, x_1, \ldots) + (1-t)(0, x_0, 0, x_1, 0, x_2, \ldots)}{\|t(x_0, x_1, \ldots) + (1-t)(0, x_0, 0, x_1, 0, x_2, \ldots)\|}$$

Thus the identity is homotopic to $(x_0, x_1, \ldots) \mapsto (0, x_0, 0, x, 0, x_2, \ldots)$. Then we can use the straight line homotopy to $(1, 0, 0, \ldots)$. This gives a homotopy from the identity to the constant map.

The map $S^\infty \to \mathbb{RP}^\infty$ is the quotient map identifying antipodal points. Thus letting α act on S^∞ by the antipodal map we have the desired structure. We then see that:

$$H_k(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = H_k\mathbb{RP}^\infty = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k > 0 \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = H^k\mathbb{RP}^\infty = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k > 0 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Homework #4

1a) Prove that the following is a free $\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$-resolution of $\mathbb{Z}$ (with the trivial action), where $k \in \mathbb{Z} \setminus \{0\}$:

$$\cdots \longrightarrow \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}] \xrightarrow{T} \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}] \xrightarrow{T} \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$$

Let $\mathbb{Z}/k\mathbb{Z} = \{1, \alpha, \alpha^2, \ldots, \alpha^{k-1}\}$ with $\alpha^k = 1$. Then we can set $T(1) = 1 - \alpha$, Then $N(1) = 1 + \alpha + \alpha^2 + \cdots + \alpha^{k-1}$.

1b) Calculate $H_n(\mathbb{Z}/k\mathbb{Z};\mathbb{Z})$ and $H^n(\mathbb{Z}/k\mathbb{Z};\mathbb{Z})$ when $k \neq 0$.

Note: S^∞ can be considered as the unit sphere in $\mathbb{C}$, namely

$$\{(z_0, z_1, \ldots) \mid z_m \in \mathbb{C}, \text{finitely many nonzero}, \sum |z_m|^2 = 1\}.$$

Then $\mathbb{Z}/k\mathbb{Z} \hookrightarrow S^1$ via the k-th roots of unity. One can then make S^∞ a $\mathbb{Z}/k\mathbb{Z}$-CW-complex via a bit of work. Then $B\mathbb{Z}/k\mathbb{Z} = S^\infty/(\mathbb{Z}/k\mathbb{Z})$. This is sometimes known as an infinite lens space.

One can make it in such a way that $C^{\text{cell}}E\mathbb{Z}/k\mathbb{Z}$ can be chosen to be precisely the resolution given in Homework #4 1a.

Next time: Correctness of definition of Tor, Ext.