I. Introduction to the Class

Logistical Announcements

- Homework
  - Gradescope Invitation Code: ERGX7Y.
  - HW due on Mondays 8PM (except when said otherwise. Next week HW due. Tuesday 9/7 8PM).
  - HW assigned in class.
  - Homework is less stringent. More about understanding concepts and a way of thinking. This does not mean the class is any easier.
- Notes on Professor Kriz's web page.

  http://www.math.lsa.umich.edu/ ikriz/math2021695.html
- Office Hours: MWF: 11-12pm.
- A nice reference is [1]

Goals and Philosophy

First version of 695: Homology with coefficients, cohomology, products, and duality. From today's point of view, this is not nearly enough. This fits the original goal of algebraic topology, which is telling spaces apart.

Today: Focus is more on the method than the original goal. Why?

- There aren't enough examples.
- Constructing interesting spaces is as fundamental as telling them apart.
- Information is not contained just in algebra.

II. Singular (co)homology

II.1. The Basic Definitions

Definition II.1.1

The **standard simplex** is $\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 0, t_i \geq 0\}$. Thi sis sometimes written $[t_0, \ldots, t_n]$.

Definition II.1.2

One may define the **free group with coefficients in** A **generated by a set** S as

$$AS := \{a : S \to A \mid \exists F \subseteq S \text{ finite } a(s) = 0 \text{ when } s \notin F\} = \bigoplus_{s \in S} A$$

The **free group** is $\mathbb{Z}S$. Note that $AS = \mathbb{Z}S \otimes A$. Because we have that

$$\left(\bigoplus_{i \in I} A_i\right) \otimes B \cong \bigoplus_{i \in I} (A_i \otimes B)$$

Definition II.1.3

An n**-simplex** in a space X is a continuous (default assumption) map $\sigma : \Delta^n \to X$.

Let $S_m X$ be the set of all n-simplices in X. We then define $C_m X = \mathbb{Z}S_m X$ to be the free abelian group on $S_m X$, and this is the group of n**-chains in** X

Definition II.1.4

If A is an abelian group then $C_m(X;A) = AS_mX$ is the group of singular n-chains with coefficients in A.

Definition II.1.5

If A is an abelian group, then $C^m(X;A) \coloneqq \mathrm{Hom}(C_mX, A)$. Equivalently this is the set of all functions $S_m(X) \to A$, which we denote $\mathrm{Map}(S_m(X), A)$.

Notice that $AS \subsetneq \mathrm{Map}(S, A)$, with the finiteness condition of AS being the key difference.

To define (co)homology we need some standard maps between standard simplices.

Definition II.1.6

The j-th face map $\partial_j \mid \Delta^{m-1} \to \Delta^m$ is defined by taking the tuple $(t_0, \ldots, t_{m-1})$ and inserting a zero into the j-th place:

$$(t_0, \ldots, t_{m-1}) \mapsto (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{m-1})$$

If $0 \leq i \leq j \leq m$, then we have that $\partial_i \partial_j = \partial_{j+1}\partial_i$.

We define $\mathrm{d} : C_mX \to C_{m-1}X$. It suffices to define $\mathrm{d}\big|_{S_mX}$. Let $\sigma : \Delta^m \to X$. Then

$$\mathrm{d}\sigma = \sum_{i=0}^{m} (-1)^i (\sigma \circ \partial_i)$$

This corresponds to restricting to the boundary simplices and with signs corresponding to a sense of orientation.

Lemma II.1.1

The key point is that $\mathrm{d}^2 = 0$. This follows via a calculation

$$\begin{aligned} \mathrm{d}^2\sigma &= \mathrm{d}\left(\sum_{i=0}^{m} (-1)^i \sigma \circ \partial_i\right) \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{m} (-1)^{i+j} \sigma \circ \partial_j \circ \partial_i \end{aligned}$$

This follows by dividing up to when $j \leq i$, and using the crucial formula $\partial_j \partial i = \partial_{i+1}\partial_j$.

Definition II.1.7

The **chain complex** $C_\bullet X$ is defined to be

$$\cdots \longrightarrow C_mX \xrightarrow{\mathrm{d}_m} C_{m-1}X \xrightarrow{\mathrm{d}_{m-1}} C_{m-2}X \longrightarrow \cdots$$

Using the fact that $d_{m-1} \circ \mathrm{d}_m = 0$, this is a chain complex as in algebra.

Definition II.1.8

If C is a chain complex, define the m-th **homology group**:

$$H_mC = \ker(\mathrm{d}_m) / \operatorname{im}(d_{m+1})$$

We call the elements of $\ker(\mathrm{d}_m)$ the m**-cycles** and $\operatorname{im}(d_{m+1})$ the m**-boundaries**

Homework 2021-08-30

(1) Show that $\mathbb{Z}S \otimes A \cong AS$. Try to recall and use the universal property of $\otimes$.
(2) Compute $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$ for $n, m \in \mathbb{Z}$.