

Definition .0.1

Lets introduce a new category Pairs whose objects are pairs of spaces (X, Y) where Y is a subspace of X . A morphism $(X_1, Y_1) \rightarrow (X_2, Y_2)$ is a continuous map such that $f(Y_1) \subseteq Y_2$.

If A is an abelian group, there are functors $H_m(?; A) : \text{Pairs} \rightarrow \text{Ab}$ and $H^m(?; A) : \text{Pairs}^{\text{op}} \rightarrow \text{Ab}$.

To do this we define $C(X, Y) := C(X)/C(Y)$. That is $C_m(X, Y) := C_m(X)/C_m(Y)$, and this will also be a chain complex of free abelian groups by some basic homological algebra. This follows by the principal that if $T \subseteq S$ then $\mathbb{Z}S/\mathbb{Z}T \cong \mathbb{Z}(S \setminus T)$.

From this point we can just define $C(X, Y; A) := C(X, Y) \otimes A$ and $C^*(X, Y; A) := \text{Hom}(C(X, Y), A)$. Taking homology of the chain complex gives homologies $H_m(X, Y; A)$ and $H^m(X, Y; A)$.

There is a short exact sequence

$$0 \longrightarrow C(Y) \longrightarrow C(X) \longrightarrow C(X, Y) \longrightarrow 0$$

Note here that an exact sequence is a chain complex with homology zero (although we stop the convention that if it stops we fill in with zeros, so there is no condition on the first/last maps).

We also have short exact sequences

$$0 \longrightarrow C(Y; A) \longrightarrow C(X; A) \longrightarrow C(X, Y; A) \longrightarrow 0$$

$$0 \longrightarrow C^*(X, Y; A) \longrightarrow C^*(X; A) \longrightarrow C^*(Y; A) \longrightarrow 0$$

Note! $? \otimes A$ and $\text{Hom}(?, A)$ are not exact. That is they do not preserve exact sequences. However, they do behave well with direct products, as

$$\begin{aligned} \left(\bigoplus_i B_i \right) \otimes A &\cong \bigoplus_i (B_i \otimes A) \\ \text{Hom} \left(\bigoplus_i B_i, A \right) &\cong \prod_i \text{Hom}(B_i, A) \end{aligned}$$

So $? \otimes A$ and $\text{Hom}(?, A)$ preserve split exact sequences. For completeness we recall this definition

Definition .0.2

A split exact sequence has the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

This exhibits $(i, s) : A \oplus C \xrightarrow{\cong} B$, and so $\text{Hom}(?, A)$ and $? \otimes A$ preserves this.

A short exact sequence at the level of the chain complexes induces a long exact sequence in homology. I.E. if C^1, C^2, C^3 are chain complexes with a short exact sequence:

$$0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow C^3 \longrightarrow 0$$

Then there is a long exact sequence in homology

$$\cdots \longrightarrow H_n(C_*^1) \longrightarrow H_n(C_*^2) \longrightarrow H_n(C_*^3) \xrightarrow{\partial} H_{n-1}(C_*^1) \longrightarrow \cdots$$

Where the morphisms between n -th homology are the induced maps and the ∂ morphism is complicated (see [hatcher])

Definition .0.3

Let $F, G : C \rightarrow D$ be functors. A **natural transformation** $\eta : F \Rightarrow G$ consists of a collection of maps $\eta_X : F(X) \rightarrow G(X)$ for every object X in C so that for any map $f : X \rightarrow Y$ the diagram below commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow F(f) & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

Great!

Definition .0.4

Two categories C, D are equivalent when there are functors $F : C \rightarrow D$ and $G : D \rightarrow C$ such that $F \circ G \cong \text{Id}_D$ and $G \circ F \cong \text{Id}_C$. Here \cong denotes a natural isomorphism.

A long exact sequence in homology of a space (X, Y) with coefficients in A is given below

$$\cdots \longrightarrow H_m(Y; A) \longrightarrow H_m(X; A) \longrightarrow H_m(X, Y; A) \xrightarrow{\partial} H_{m-1}(Y; A) \longrightarrow \cdots$$

And in cohomology we have

$$\cdots \longrightarrow H^m(X, Y; A) \longrightarrow H^m(X; A) \longrightarrow H^m(Y; A) \xrightarrow{\delta} H^{m+1}(X, Y; A) \longrightarrow \cdots$$

Both ∂, δ are natural.

.1. Eilenberg-Steenrod Axioms

We now list the Eilenberg-Steenrod axioms for homology (cohomology). First $H_n(?; A)$ and $H^m(?; A)$ are covariant/contravariant functors respectively from Top or Pairs into Ab.

Homotopy Axiom

We also require that homotopic maps in Top or Pairs induce the same map in (co)homology.

We can define categories hTop and hPairs whose objects are the same as Top and Pairs and whose morphisms are equivalence classes of maps up to homotopy.

Then the above condition is the same as requiring that $H^m(?; A)$ and $H_m(?; A)$ are covariant/contravariant functors from hTop or hPairs into Ab.

The key idea to providing this axiom is something called a chain homotopy.

Definition .1.1

Let $f, g : C \rightarrow D$ be chain maps. A **chain homotopy** is a sequence of homomorphisms of abelian groups $h_m : C_m \rightarrow D_{m+1}$ satisfying

$$dh + hd = f - g$$

One can then define hChain, whose objects are chain complexes and whose morphisms are chain-homotopy classes of chain maps.

Excision Axiom

Let $Z \subseteq Y \subseteq X$ where $\text{Closure}_X(Z) \subseteq \text{Interior}_X(Y)$.

Then there is a map of pairs $(X \setminus Z, Y \setminus Z) \subseteq (X, Y)$ given by the inclusion. This induces an isomorphism on $H_m(?; A)$, $H^m(?; A)$.

Limit Axioms

Take a collection of spaces X_i . Then the inclusions $X_i \hookrightarrow \coprod_i X_i$ induces isomorphisms