.1. **Derived Categories**

Definition .1.1

Setup: C is a category $E \subseteq C$ is a subcategory (we call the morphisms of E equivalences and write $f : X \xrightarrow{\sim} Y$). We assume "2 out of three property" that for $f : X \to Y$, $g : Y \to Z$ that if two out of f, g, gf are equivalences then so is the third. Also we assume that all isomorphisms belong to E.

A derived category (if one exists) with respect to this data is a category DC together with a functor $\Phi : C \to DC$ which is universal among functors $F : C \to Q$ which take all morphisms in E into isomorphisms.

Strict version: $\Phi(e)$ is an isomoprhism for all $e \in \mathrm{Mor}(E)$. For all $F : C \to Q$ satisfying $F(e)$ is an isomoprhism for all $e \in \mathrm{Mor}(E)$ there exists a uique $G : DC \to Q$ with $G\Phi = F$.

Lax version: For all $e \in \mathrm{Mor}(E)$, $\Phi(e)$ is an isomorphism. For all $F : C \to Q$ satisfying this same property there exists a $G : DC \to Q$ togehter with a natural isomorphism $\kappa : G \circ \Phi \to F$. For any other functor $H : DC \to Q$ together with a natural isomorphism $\lambda : H \circ \Phi \to F$ there exists a unique $\mu : H \to G$ with $\lambda = \kappa \circ (\mu\Phi)$.

$$
\begin{tikzcd}
C \arrow[rr, "F"] \arrow[dr, "\Phi"'] & & Q \\
& DC \arrow[ur, dashed, "G"'] \arrow[u, phantom, "\kappa\Uparrow"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "F"] \arrow[dr, "\Phi"'] & & Q \\
& DC \arrow[ur, dashed, "G"] \arrow[ur, dashed, bend right=50, "H"'] \arrow[u, phantom, "\kappa\Uparrow\ \Uparrow\lambda"] \arrow[ur, phantom, bend right=25, "\mu\Uparrow"] &
\end{tikzcd}
$$

Observation: The two definitions are equivalent if $\Phi : C \to DC$ is bijective on objects.

Concrete construction of derived categories:

where we have an analog of the Whitehead Theorem (or its dual) (given above definition)

An object $Z \in \mathrm{Ob}(C)$ is called co-local if for every $e : X \xrightarrow{\sim} Y$ we have

$$\mathrm{Mor}_C(Z, X) \xrightarrow[\cong]{\mathrm{Mor}_C(Z,e)} \mathrm{Mor}_C(Z, Y)$$

is a bijection. For example, CW-complexes are co-local. We say that we have co-localization if for every $X \in \mathrm{Ob}(C)$ there exists X' which is co-local and an equivalence $X' \xrightarrow{\sim} X$. If always we have $X' \in B$ for some $B \subseteq \mathrm{Ob}(C)$ (with every object of B colocal), we say this is colocalization by B.

This situation is exactly the content of Whitehead's theorem in hTop.

Theorem .1.1

If C, E are as above and we have a colocalization by $B \subseteq \mathrm{Ob}(C)$ then the derived category DC exists and it is equivalence to the full subcategory of C on B.

Proof. $\Phi : C \to DC$ given by $X \mapsto X'$. This is functorial because for $f : X \to Y$ we can use colocality of X' to get a unique map $X' \to Y'$.

The other checks are similarly trivial.

Definition .1.2

A cell complex is a space $X = \bigcup X_{(m)}$

$\emptyset = X_{(-1)} \subseteq X_{(0)} \subseteq \cdots$.

$X_{(m)}$ is obtained from$X_{(m-1)}$ by attaching cell in any dimension $J_m, d_n : J_m \to \mathbb{N}_0$ with

$$f_m : \coprod_{j \in J_m} S^{d_j - 1} \to X_{(m)-1}$$

$X_{(m)}$ is the pushout

$$\begin{tikzcd} \coprod_{j\in J_m} S^{d_j-1} \arrow[r, "f_m"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X_{(m-1)} \arrow[d] \\ \coprod_{j\in J_m} D^{d_j} \arrow[r] & X_{(m)} \end{tikzcd}$$

Observe that cell complexes also satisfy the Whitehead Theorem.

Homework #9

(1) Prove that every cell complex is homotopically equivalent to a CW-complex.

Let A be an abelian category with coproducts and enough projectives

For all $X \in \mathrm{Ob}(A)$ there exists a projective P and an epimorphism $P \twoheadrightarrow X$.

We can look at h-A-Chain. We can define a <u>cell chain complex</u>

$$0 = X_{(-1)} \subseteq X_{(0)} \subseteq \cdots$$

We say $P_{(m)}$ is a projective chain complex with zero differentials.

We then can take

$$CP_{(m)} \qquad P_{(m)} \longrightarrow P_{(m)k} \oplus P_{(m)(k-1)} \longrightarrow P_{(m)(k-1)} \oplus P_{(m)(k-2)} \longrightarrow \cdots$$

$H_* CP_{(m)} = 0$.

We require $X_{(m)}$ is a pushout

$$\begin{tikzcd} P_{(m)} \arrow[r, "f_m"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X_{(m-1)} \arrow[d] \\ CP_{(m)} \arrow[r] & X_{(m)} \end{tikzcd}$$

One can prove the Whitehead Theorem precisely analogously

The equivalences are quasiisomprhisms (chain maps which induce isomorphisms in homology),

Theorem .1.2

Cell chain complexes are colocal in h-A-Chain with respect to quasiisomorphisms and one has colocalization by cell chain complexes.

We define

$$DA := Dh\text{-A-Chain}$$

called the "derived category of the abelian category A."