Remark .0.1

In most (homological) spectral sequences, the pages are denoted by E^r_{pq}. This has nothing to do with the generalized homology theory E in last class. On Homework, generalized homology theory is called K, the spectral sequence terms should still be denoted by E^r_{pq}.

Today we show that the groups $H_m(X, Y; A), H^m(X, Y; A)$ are completely determined (algebraically) by $H_m(X, Y) = H_m(X, Y; \mathbb{Z})$ and $H^m(X, Y) = H^m(X, Y; \mathbb{Z})$.

However, the way they are determined is not completely functorial. The key point: $C(X, Y)$ (singular chain complex) are chain complexes of free abelian groups (terms are free abelian groups).

Theorem .0.1

Any chain complex of free abelian groups can be written as follows:

$$C \cong \bigoplus_{m\in\mathbb{Z}} \mathscr{H}_m[m]$$

The brackets denote a "shift" of a chain complex by m, and $\mathscr{H}_m$ is a complex of the form:

$$\begin{array}{cccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & B_m & \longrightarrow & Z_m & \longrightarrow & 0 & \longrightarrow \cdots \\ \cdots & & 2 & & 1 & & 0 & & -1 & \quad \cdots \end{array}$$

Where the map $B_m \to Z_m$ is injective and $H_0\mathscr{H}_m = H_m(C)$ (note that $H_k\mathscr{H}_m = 0$ for $k \neq 0$).

Proof. $C : \cdots \to C_{m+1} \to C_m \to C_{m-1} \to \cdots$. We let $Z_m \coloneqq \ker \mathrm{d}_m$ and $B_m \coloneqq \operatorname{im} \mathrm{d}_{m+1}$. Then $H_0\mathscr{H}_m = H_m C \coloneqq Z_m/B_m$.

Note that we have a short exact sequence:

$$0 \longrightarrow Z_m \xrightarrow{\subseteq} C_m \longrightarrow B_{m-1} \longrightarrow 0$$

Now $B_{m-1} \subseteq Z_{m-1} \subseteq C_{m-1}$. C_{m-1} is free abelian, so B_{m-1} is free abelian as well (see [1] for the algebra).

Thus this splits (say by s_{m-1}), and we have that:

$$\begin{tikzcd} \cdots \arrow[r] & C_{m+1} \arrow[r] & C_m \arrow[r] & C_{m-1} \arrow[r] & C_{m-2} \arrow[r] & \cdots \\ \cdots \arrow[r] & Z_{m+1} \oplus B_m \arrow[r] \arrow[u, "\subseteq\oplus s_m"] & Z_m \oplus B_{m-1} \arrow[r] \arrow[u, "\subseteq\oplus s_{m-1}"] & Z_{m-1} \oplus B_{m-2} \arrow[r] \arrow[u, "\subseteq\oplus s_{m-2}"] & Z_{m-2} \oplus B_{m-3} \arrow[r] \arrow[u, "\subseteq\oplus s_{m-3}"] & \cdots \end{tikzcd}$$

Given the simple complex $\mathscr{H} : B \subseteq Z$, we have $H_0\mathscr{H} = H$. What is $H_*(\mathscr{H} \otimes A)$? Well $H_0(\mathscr{H} \otimes A) = H \otimes A$. Why? Well $\otimes$ is right exact so:

$$0 \longrightarrow B \longrightarrow Z \longrightarrow H \longrightarrow 0$$

$$B \otimes A \longrightarrow Z \otimes A \longrightarrow H \otimes A \longrightarrow 0$$

And then we set $\operatorname{Tor}_1^{\mathbb{Z}}(H, A) \coloneqq H_1(\mathscr{H} \otimes A)$. Is this well-defined? For this to be well-defined, the answer needs to depend only on H, not on $\mathscr{H}$. We'll postpone this for now, and we'll prove it later in greater generality.

Similarly, what is the cohomology of $\operatorname{Hom}(\mathscr{H}, A)$? Well we have left exactness so:

$$0 \longrightarrow B \longrightarrow Z \longrightarrow H \longrightarrow 0$$

$$\operatorname{Hom}(B, A) \longleftarrow \operatorname{Hom}(Z, A) \longleftarrow \operatorname{Hom}(H, A) \longleftarrow 0$$

Then $H^0 \operatorname{Hom}(X, A) = \operatorname{Hom}(H, A)$. We then set $\operatorname{Ext}^1_{\mathbb{Z}}(H, A) \coloneqq H^1 \operatorname{Hom}(X, A)$.

Example .0.1

Let $H = \mathbb{Z}/2\mathbb{Z}$ and $A = \mathbb{Z}$. Then $\mathscr{H} : \mathbb{Z} \xrightarrow{2} \mathbb{Z}$. Homming into $\mathbb{Z}$ we have:

$$\operatorname{Hom}(\mathscr{H}, A) : \mathbb{Z} \xleftarrow{2} \mathbb{Z}$$

Then $\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0 = \operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$. And $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

From the structure theorem of chain complexes of free abelian groups, commutation of homology with shifts, and direct sum, we get the following wonderful result

Theorem .0.2 (The Universal Coefficient Theorem)

We have that

$$H_m(X, Y; A) \cong (H_m(X, Y) \otimes A) \oplus \operatorname{Tor}^{\mathbb{Z}}_1(H_{m-1}(X, Y), A)$$
$$H^m(X, Y; A) \cong \operatorname{Hom}(H_m(X, Y), A) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{m-1}(X, Y), A)$$

Thus we've reduced the problem to figuring out how to calculate $\operatorname{Tor}^{\mathbb{Z}}_1$ and $\operatorname{Ext}^1_{\mathbb{Z}}$.

A slight catch: This is not completely functorial, namely the splittings are not natural transformations. Functorially, we only have short exact sequences:

$$0 \longrightarrow H_m(X, Y) \otimes A \longrightarrow H_m(X, Y; A) \longrightarrow \operatorname{Tor}^{\mathbb{Z}}_1(H_{m-1}(X, Y), A) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_{m-1}(X, Y), A) \longrightarrow H^m(X, Y; A) \longrightarrow \operatorname{Hom}(H_m(X, Y), A) \longrightarrow 0$$

These split, but not naturally.

This actually works for any chain complex of free abelian groups.

Homework #3

3a) Calculate $\operatorname{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, where $m, n \in \mathbb{Z}$.

3b) Calculate $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ where $m, n \in \mathbb{Z}$.

The cases where one of them is 0 may need special care.

A headstart on next class–the general theory of all this. Namely, resolutions.

Definition .0.1

Let R be any commutative ring, and let M be an R-module. A **free R-resolution of M** is a chain complex of free R-modules

$$\mathscr{C} \qquad \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0$$

Then $H_0\mathscr{C} = M$, $H_k\mathscr{C} = 0$ for $k \neq 0$.

Now let N be any other R-module. We define:

$$\operatorname{Tor}_m^R(M, N) \coloneqq H_m(\mathscr{C} \otimes_R N)$$
$$\operatorname{Ext}_R^m(M, N) \coloneqq H^m \operatorname{Hom}_R(\mathscr{C}, N)$$

It is still true that:

$$\operatorname{Tor}_0^R(M, N) = M \otimes_R N$$
$$\operatorname{Ext}_R^0(M, N) = \operatorname{Hom}_R(M, N)$$