.1. Abelian Categories

We take a quick digression to define limits/colimits in a category.

Definition .1.1

A <u>diagram</u> is a functor $D : I \to \mathscr{C}$.

Definition .1.2

A <u>cone</u> over a diagram $D : I \to \mathscr{C}$ is an object X in $\mathscr{C}$ so that for each $i \in I$ there is a map $\eta_i : X \to D(i)$. Furthermore for every map $i \xrightarrow{f} j$ in I we have the following commuting triangle:

$$\begin{tikzcd} & X \arrow[dl, "\eta_i"'] \arrow[dr, "\eta_j"] & \\ D(i) \arrow[rr, "D(f)"] & & D(j) \end{tikzcd}$$

Example .1.1

Product is the limit of a diagram with index category $I = \{*_1, *_2\}$. Pullback is also a limit over a diagram $\bullet \to \bullet \leftarrow \bullet$. Equalizers are also limits.

Definition .1.3

A <u>limit</u> $\lim D$ over a diagram $D : I \to \mathscr{C}$ is a "universal cone"

That is $\lim D$ is a cone over D, with maps $\eta_i : \lim D \to D(i)$ such that for any other cone T over D with maps $\mu_i : T \to D(i)$ we have that there is a unique arrow $f : T \to \lim D$ such that the following diagram commutes for all i:

$$\begin{tikzcd} T \arrow[d, "f"] \arrow[dd, bend right=60, "\mu_i"'] \\ \lim D \arrow[d, "\eta_i"] \\ D(i) \end{tikzcd}$$

Dually wqe have the notion of a colimit.

Example .1.2

The coproduct, pushouts, and coequalizers are all colimits.

Definition .1.4

$\operatorname{colim} \emptyset$ is called an initial object, as there is a unique arrow $\operatorname{colim} \emptyset \to T$ for every T lying in $\mathscr{C}$. Likewise $\lim \emptyset$ is called a terminal object, as there is a unique arrow $T \to \lim \emptyset$ for every T lying in $\mathscr{C}$

Note: Limits and colimits are only defined up to isomorphism (given by the universal property). However there is only one such isomorphism at the level of cones (i.e. respecting the limiting cones over the diagram).

Definition .1.5

A functor is called <u>right exact</u> when it preserves finite colimits. It is called <u>left exact</u> when it preserves finite limits.

Now we'll think a bit about abelian categories.

Definition .1.6

A category with zero has an initial object and a final object such that the unique morphism $I \to T$ from the initial object to the terminal object is an isomorphism.

In such a category for any X, Y we have an arrow $0 : X \to Y$ given by the unique composition $X \to 0 \to Y$.

Example .1.3

$\mathrm{Ab}, R\text{-Mod}, \mathrm{BasedSpaces}, \mathrm{BasedSets}$.

In a category with zero, we can define kernels and cokernels.

Definition .1.7

In a category with zero, $\ker f$ is the equalizer of $f : X \to Y$ and 0. That is we take the limit over the diagram:

$$X \underset{0}{\overset{f}{\rightrightarrows}} Y$$

Likewise a cokernel $\operatorname{coker} f$ is the coequalizer of $f : X \to Y$ and 0 (that is the colimit of the above diagram.

Definition .1.8

In any category, $f : X \to Y$ is a monomorphism when for any $g, h : Z \to X$ such that $fg = fh$ we have $g = h$.

Likewise $f : X \to Y$ is an epimorphism when for any $u, v : Y \to Z$ such that $uf = vf$ we have $u = v$.

Definition .1.9

An abelian category is an Ab-enriched category with zero and with finite limits and colimits such that every epimorphism is a cokernel and every monomorphism is a kernel.

You can prove a lot of nice properties about abelian categories. Including all the additive properties of abelian groups.

- $X \oplus Y \cong X \prod Y \cong X \coprod Y$.
- $\mathrm{Mor}_{\mathscr{C}}(X, Y)$ is an abelian group and composition is bilinear.

Definition .1.10

Enough projectives in an abelian category $\mathscr{C}$ provided that for all X in $\mathscr{C}$ there exists a projective P and an epimorphism $P \twoheadrightarrow X$.

This gives us projective resolutions and left derived functors. If you have enough injectives (that is for every object X you have a monomorphism $X \hookrightarrow Q$ into an injective) that gives you injective resolutions and right derived functors.

We have enough projectives and injectives in R-Mod. You also have enough injectives in abelian sheaves.

.2. Commutativity of Tor

We want to show that $\mathrm{Tor}_m^R(M, N) \cong \mathrm{Tor}_m^R(N, M)$, and that this isomorphism is canonical.

Idea: Resolve both M, N. Call C a free resolution of M and D a free resolution of N. Redefine $\mathrm{Tor}_m^R(M, N) = H_m(C \otimes_R D)$. We need to define $C \otimes_R D$, and prove that we get the same thing.

Given two chain complexes C and D, we must define their tensor product $C \otimes_R D$.

If D is just an R-module N, then we want $C \otimes_R N$. It's certainly then incorrect to take the componentwise tensor product.

We can take a two-dimensional grid of tensor products:

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_m \otimes D_m \arrow[r] & C_{m-1} \otimes D_m \arrow[r] & \cdots \\
\cdots \arrow[r] & C_m \otimes D_{m-1} \arrow[r] & C_{m-1} \otimes D_{m-1} \arrow[r] & \cdots \\
& \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Definition .2.1

This is a double chain complex $A_{n,m}$. We have two differentials $\partial : A_{n,m} \to A_{n-1,m}$ and $\delta : A_{n,m} \to A_{n,m-1}$ such that:

$$\partial\partial = 0 \qquad \delta\delta = 0 \qquad \partial\delta = \delta\partial$$

Definition .2.2

We define $(C \otimes_R D)_{n,m} = C_n \otimes_R D_m$, the tensor product of two chain complexes, to be the double chain complex with differentials given by $\mathrm{d}^C \otimes \mathrm{Id}_D$ and $\mathrm{Id}_C \otimes \mathrm{d}^D$.

Definition .2.3

Given a double chain complex A, the totalization $|A|$ is a chain complex given by

$$|A|_m = \bigoplus_{k+\ell=m} A_{k,\ell}$$
$$\mathrm{d}x = \partial x + (-1)^k \delta x$$

Reversing roles of k, ℓ gives an isomorphic chain complex. Apply the sign $(-1)^{k\ell}$ to $x \in A_{k,\ell}$.

We can then redefine $\mathrm{Tor}_m^R(M, N) = H_m(|C \otimes_R D|)$ where C, D are projective resolutions of M, N. Nobody writes the totalization, so $H_m(|C \otimes_R D|) = H_m(C \otimes_R D)$.

Homework #4

(3) Suppose we have a double chain complex C such that
- $H_m(C_{k,*}, \partial) = 0$
- $C_{k,\ell} = 0$ if $\ell < 0$ (equiv. $\ell < N$ fixed)

So cut off in the bottom, rows exact. Then $H_m(|C|) = 0$. (Hint: First prove it when there exists a L such that for all k $C_{k,\ell} = 0$ if $\ell > L$. Can induct on L using short exact sequences of chain complexes, which leads to a long exact sequence in homology. Then express C as a colimit of such sequences with L increasing, use commutation of homology with colimits of sequences).

Using this, it's fairly easy to prove that Tor is commutative using the program outlined below. You consider the augmented resolution $\widetilde{C}$ of M given by $\cdots \to C_m \to C_{m-1} \to \cdots \to C_0 \to M \to 0$ for a free resolution C of M. This is exact, and there is actually a short exact sequence $0 \to M[-1] \to \widetilde{C} \to C \to 0$.

By the homework $H_*(\widetilde{C} \otimes_R D) = 0$. By long exact sequence, we then have that:

$$H_m(\widetilde{C} \otimes_R D) = 0 \longrightarrow H_m(C \otimes_R D) \longrightarrow H_{m-1}(M[-1] \otimes_R D) \longrightarrow H_{m-1}(\widetilde{C} \otimes_R D) = 0$$