Definition .0.1

The mapping cocylinder of a map $f : X \to Y$ is

$$Nf := \{(x, \omega) \mid x \in X,\ \omega : [0,1] \to Y,\ \omega(0) = f(x)\}$$

The projection $(x, \omega) \mapsto x$ is a homotopy equivalence $N(f) \simeq X$. Furthermore $(x, \omega) \mapsto \omega(1)$ is a fibration. This leads to a way to replace maps by fibrations

$$\begin{tikzcd} Cf \arrow[r] & Nf \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & Y \arrow[d, "\mathrm{Id}"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

The dual version with $f : Y \to X$ and the mapping cylinder $Mf = (Y \times [0,1]) \coprod X/(y,0) \sim f(y)$ we have $y \mapsto (y,1)$ is a cofibration.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\simeq"] & Mf \arrow[r] \arrow[d] & Cf \\ Y \arrow[r, "f"'] & X & \end{tikzcd}$$

Theorem .0.1 (Serre Spectral Sequence)

Let $F \to X \xrightarrow{f} Y$ be a fibration with $\pi_0 Y, \pi_1 Y = 0$. Then there is a spectral sequence

$$E^2_{pq} = H_p(Y, H_q(F; A)) \Rightarrow H_{p+q}(X; A)$$

(Y general also works. Have to use homology with local coefficients).

Recall that

$$\mathrm{d}^r : E^r_{pq} \to E^r_{p-r,q+r-1} E^{r+1} \qquad = H(E^r, \mathrm{d}^r)$$

$$E^\infty_{pq} = \operatorname{colim} E^r_{pq}$$

There is an exhaustive filtration $F_{-1} = 0$, $F_p H_m(X; A)$ with $\bigcup F_p H_m(X; A) = H_m(X; A)$.

We don't have to worry about convergence because the serre spectral sequence exists entirely in the first quadrant. Recall also that

$$F_p H_{p+q} X / F_{p-1} H_{p+q}(X) = E^\infty_{pq}$$

Also there is a cohomological spectral sequence:

$$E_2^{pq} = H^p(Y; H^q(F; A)) \Rightarrow H^{p+q}(X; A)$$

In this case we have

$$\mathrm{d}_r : E_r^{pq} \to E_r^{p+r,q-r+1}$$

$$E_{r+1} = H(E_r, \mathrm{d}_r)$$

$$F_0 H^m(X; A) = H^m(X; A)$$

$$F_0 \supseteq F_1 \supseteq \cdots$$

And for $N \gg 0$ we have $F_N H^m(X; A) = 0$. We then also have

$$E_\infty^{pq} = \operatorname{colim} E_r^{pq} = F^p H^{p+q}(X; A)/F^{p+1} H^{p+q}(X; A).$$

> If A is a commutative ring, then E_r is a spectral sequence of rings. That is E_r are bigraded rings, graded-commutative with respect to the total degree $p+q$ and d_r satisfies
>
> $$\mathrm{d}_r(xy) = (\mathrm{d}_r x)\cdot y + (-1)^{|x|} x \cdot \mathrm{d}_r(y)$$
>
> Where $|x|$ is the total degree of x.
>
> This is in Serre's thesis (paraphrased less rigorously in Spanier's book).

Homework #7

(1) Calculate completely the homological and cohomological Serre spectral sequence of the Hopf fibration $S^1 \to S^3 \to S^2$ with coefficients in $\mathbb{Z}$. (Note this is unreduced, so we have to worry about degree zero).

…back to generalized cohomology. In the based version for $X \xrightarrow{f}\xrightarrow{i} Cf$ we have

(1) $\widetilde{E}^m Cf \xrightarrow{i^*} \widetilde{E}^m Y \xrightarrow{f^*} \widetilde{E}^m X$ is exact

(2) $\widetilde{E}^m X \cong \widetilde{E}^{m+1}\Sigma X$.

$*$ $\widetilde{E}^m \bigvee_i X_i \cong \prod_i \widetilde{E}^m(X_i)$

If I have a sequence of based spaces Z_m, $m \in \mathbb{Z}$ and $Z_m \xrightarrow{\simeq} \Omega$ then I can put $\widetilde{E}^m X := [X, Z_m]$. It turns out (in a proper setting) to be an if and only if, every generalized cohomology (satisfying $3*$) is given like this.

…Something is being swept under the rug. Namely $C_{\text{based}} f$ may not be homotopy equivalent to $C_{\text{unbased}} f$ and $\Sigma X \simeq SX$ might not hold. We should require that $* \hookrightarrow X, Y$ are cofibrations for the unbased and based constructions to agree.

> **Example .0.1**
>
> "Ordinary" cohomology = singular cohomology with coefficients in A (abelian group). What are the spaces Z_m? Well test it for $X = S^k$. Then
>
> $$\pi_k Z_m = [S^k, Z_m] = \widetilde{H}^m(S^k; A) = \begin{cases} A & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$
>
> This is called an <u>Eilenberg-MacLane space</u>, $K(A,m)$ has $\pi_m(K(A,m)) = A$ and $\pi_n(K(A,m))$ whenever $n \neq m$. (for $m = 1$, $K(A,1) \simeq BA$)

There are loose ends to tie up

(1) Are the spaces $K(A,m)$ unique up to homotopy equivalence?

(2) Do we automatically have $K(A, m-1) \simeq \Omega K(A,m)$?

(3) For what (based) spaces X is $\widetilde{H}^m(X;A)$ determined just by the fact that $Z_m = K(A,m)$?

<u>An answer</u>: For X a CW-complex $\widetilde{H}^m(X;A)$ is determined. Namely $X_m/X_{m-1} = \bigvee_{I_m} S^m$, and we know the cohomology of this. Functoriality hands us $C^*_{\text{cell}}(X;A)$, telling us $H^m_{\text{cell}}(X;A) = H^m(X;A)$.

<u>The moral of the story</u>:

- Develop a notion of "equivalence of spaces" using <u>homotopy groups</u>.
- Approximate spaces by CW-complexes with respect to this equivalence
- An equivalence of CW-complexes is a homotopy equivalence.

Next time: Another (first "nontrivial") example of generalized cohomology, K-theory. Based on

$$U(m) = m \times n \text{ complex matrices whose columns form an orthonormal basis of } \mathbb{C}^m.$$

Then $U(m) \subseteq U(m+1)$ with

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

Then $U := \bigcup_m U(m)$ with the union topology.

Theorem .0.2 (Bott)

Bott periodicity tells us that $\Omega^2 U \simeq U$.

The corresponding generalized cohomology theory for $Z_{2m+1} := U$, $Z_{2m} := \Omega U$ is called (topological complex) K-theory.