I. Operads

Definition I.0.1

A symmetric monoidal category $\mathcal{C}$ has a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

along with a unit $1 \in \operatorname{Ob} \mathcal{C}$. We wish for this to be commutative, associative, and unital. We need this in the 2-categorical sense. Namely we need natural isomorphisms

$$A \otimes B \cong_{\iota_{AB}} B \otimes A$$
$$A \otimes (B \otimes C) \cong_{\alpha_{ABC}} (A \otimes B) \otimes C$$
$$A \cong_{\mu_A} A \otimes 1.$$

We also need some axioms. These are called coherence diagrams. Consider a word in the operator an units in a commutative monoid such as

$$((a \cdot b) \cdot c) \cdot d \to (a \cdot b) \cdot (\gamma \cdot d) \to a \cdot (b \cdot (c \cdot d))$$

But we can also do it in a different way

$$((a \cdot b) \cdot c) \cdot d \to (a \cdot (b \cdot c)) \cdot d \to a \cdot ((b \cdot c) \cdot d) \to a \cdot (b \cdot (c \cdot d))$$

Any time I can do this in two different ways, I get a coherence diagram for symmetric monoidal categories. This example is known as the pentagram diagram.

The actual coherence diagrams may be found on wikipedia.

Definition I.0.2

A closed symmetric monoidal category is one where for every object X, $X \otimes ? : \mathcal{C} \to \mathcal{C}$ has a right adjoint $\operatorname{Hom}(X, ?)$.

Example I.0.1

Set, $\times$; compactly generated spaces, $\times$; R-Mod, $\otimes$; R-Chain, $\otimes$.

Definition I.0.3

In a symmetric monoidal category one can define an operad. This is a collection of objects $\mathcal{D}(m)$ for $m \in \mathbb{N}_0$ satisfying the same formal properties as $\operatorname{Hom}(X^{\otimes m}, X)$ for some object X.

What structure maps do we have?

(1) $1 \xrightarrow{\iota} \mathcal{D}(1)$
(2) Σ_m acts on $X^{\otimes m}$ by permutation. Thus we require Σ_m acts on $\mathcal{D}(m)$.
(3) There is a map $\operatorname{Hom}(X^{\otimes m}, X) \otimes \bigotimes_{i=1}^m \operatorname{Hom}(X^{\otimes k_i}, X) \to \operatorname{Hom}(X^{\otimes \sum_i k_i}, X)$. Thus we require a map $\mathcal{D}(m) \otimes \bigotimes_{i=1}^m \mathcal{D}(k_i) \xrightarrow{\gamma} \mathcal{D}(k_1 + \cdots + k_m)$.

Axioms: Associativity, permutations, two unitalities.

Recommend the book by May. Geometry of Iterated Loop Spaces. [1].

There is an obvious notion of homomorphism of operads $\mathcal{D}_1 \to \mathcal{D}_2$ given by maps $\mathcal{D}_1(m) \to \mathcal{D}_2(m)$ preserving the above operations.

The operad $\operatorname{Hom}(X^{\otimes m}, X)$ is called the endomorphism operad $\operatorname{End}(X)$.

Definition I.0.4

An object X is called a $\mathcal{D}$-algebra (for an operad $\mathcal{D}$) if we are given a homomorphism of operads $\mathcal{D} \to \mathrm{End}(X)$. Equivalently in terms of maps $\mathcal{D}(m) \otimes X^{\otimes m} \to X$ satisfying some diagrams.

Back to cochains (with coefficients in $\mathbb{F}_p$). There is an operad $\mathcal{E}$ in $\mathbb{F}_p$-chain such that

(1) $\mathcal{E}(m) \simeq \mathbb{F}_p$ by chain homotopy (chain contractible).
(2) $\mathcal{E}(m)$ is a chain complex of free $\mathbb{F}_p(\Sigma_m)$-modules. This is the same thing as a linear action of the group.

Such an operad is called an E_∞-operad. An algebra of such an operad is called an E_∞-algebra. Running these through a colocalization game, we get a unique derived category of E_∞ algebras.

Theorem I.0.1 (Hinich-Schectman)

For a space X, $C^*(X;\mathbb{F}_p)$ has a natural structure of an E_∞-algebra.
(Proof: a souped up version of Eilenberg-Zilber theorem).

Example I.0.2

Another example of an operad on $\mathbb{F}_p$-chain given by $\mathcal{N}(m) = \mathbb{F}_p$. Then an $\mathcal{N}$-algebra is the same thing as a graded commutative DGA.

The structure maps are $\mathbb{F}_p \otimes X \otimes \cdots \otimes X \to X$. The $\mathbb{F}_p$ is a unit so it gets killed, and the signs of a graded commutative DGA come from the signs in the chain complex tensor product.

Remark I.0.1

Just as we defined Steenrod operations, we can define operations in the homology of an E_∞-$\mathbb{F}_p$-algebra.
Caution: This time, $Sq^0 = 1$ and $Sq^i = 0$ for $i < 0$ do not hold.

A convention in the context of E_∞-algebras is to put $Q^i = Sq^{-i}$. These are called Dyer-Landof operations.

Homework #11

(1) Write down the axiom diagrams for an operad (Ok to use reference, but adapt it exactly to the concept covered in class).

Note: We have not constructed any example of an E_∞-operad yet!
One method is to construct an E_∞-operad in spaces, and apply chains.
E_∞-operad in Top would satisfy

(1) $\mathcal{E}(m) \simeq *$
(2) $\mathcal{E}(m)$ has the homotopy type of a CW-complex with Σ_m acting freely on cells.

Note: This requires constructing a map $C_*(X) \otimes C_*(Y) \xrightarrow{\varphi} C_*(X \times Y)$ which is commutative, associative, and unital strictly (on the nose).

There is such a map (not in the opposite direction becuase of Steenrod operations being nonzer) called the shuffle map (standard transformation of a product of two standard simplices).

E_∞-algebras in spaces were in fact discovered first, and have a very close conection with generalized cohomology. This is called Infinite Loop Space Theory.