.1. Constructing E_∞ Operads

An E_∞ operad in spaces consists of the following

(1) $\mathcal{C}(m) \simeq$ CW-complex, Σ_m-equivariantly, and Σ_m acts freely on the cells (when G acts on the sets of cells of a CW-complex we call this a G-CW-complex).

(2) $\mathcal{C}(m) \simeq *$ (non-equivariantly).

Start with any operad $\mathcal{M}$ satisfying (1). For example $\mathcal{M}(m) = \Sigma_m$. Then a $\mathcal{M}$-algebra is a monoid (an associative, unital).

Čech resolution If X is an object of a category $\mathcal{G}$ (with product), then this builds a simplicial object EX in the same category $\mathcal{G}$, that is a functor $\Delta^{\text{op}} \to \mathcal{G}$.

Then we set $EX_m = \underbrace{X \times \cdots \times X}_{m+1 \text{ times}}$. Labeling these coordinates $0, \ldots, m$ then the i-th face map $\{0, \ldots, m-1\} \to \{0, \ldots, m\}$ gets mapped to the projection away from the i-th coordinate.

The degeneracies are given by applying the diagonal $X \xrightarrow{\Delta} X \times X$ in the appropriate coordinate given by $\{0, \ldots, m+1\} \to \{0, \ldots, m\}$. Namely this sends $i, i+1$ to i, so apply the diagonal to the i-th coordinate. In some sense we have "$EX = X^\Delta$," or as a right Kan Extension along $\Delta \to *$.

In Set, Top (compactly generated weakly Hausdorff spaces see [**may**]). Here we have the geometric realization. If $Y_\bullet$ is a simplicial space (simplicial object in Top, then

$$|Y_\bullet| = \coprod Y_m x x \Delta^m/(y, \alpha t) \sim (Y_\bullet(\alpha) y, t) \qquad (\alpha \in \mathrm{Mor}(\Delta))$$

It suffices to just take faces and degeneracies (the generators).

Proposition .1.1

If $X \neq \emptyset$, then $|EX| \simeq *$.

Proof sketch. We have some basepoint $* \in X$. Then we have that

$$|EX| = \coprod_{m \geq 0} X^{\{0, \ldots, m\}} \times \Delta^m/(y, \alpha t) \sim (EX(\alpha) y, t).$$

We have a map $h_s : |EX| \to |EX|$ given by

$$h_s((x_0, \ldots, x_m), [t_0, \ldots, t_m]) = ((x_0, \ldots, x_m, *), [(1-s)t_0, \ldots, (1-s)t_m, s]).$$

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(2) Verify that this definition is compatible with face and degeneracy identification, proving that for a non-empty space X, $|EX| \simeq *$.

If $s = 0$, then $h_0 = \mathrm{Id}$, and if $s = 1$¡ then h_1 is constant at $(*, 1)$ by face/degeneracy identifications.

Geometric realization preserves products (triangulation of $\Delta^m \times \Delta^n$ by shuffles). If $\mathcal{D}$ is a simplicial operad in spaces, then $|\mathcal{D}_\bullet|$ is also an operad. This shows us by definition then that $|E\mathcal{M}|$ is an E_∞ operad.

Definition .1.1

An E_∞-space is an algebra over an E_∞-operad in spaces.

We can play the game to show that $\mathcal{D}$-algebras have colocalization, giving a derived category.

Theorem .1.2

The derived category does not depend on the particular E_∞-operad chosen.

Proof sketch. If $\mathcal{D}$, ξ are E_∞-operads then there is a diagram

$$\begin{tikzcd} & \mathcal{D} \times \xi \arrow[dl, "\text{proj. } \pi_1"'] \arrow[dr, "\text{proj. } \pi_2"] & \\ \mathcal{D} & & \xi \end{tikzcd}$$

For a homomorphism of operads $f : \xi \to \mathcal{D}$ we have a pullback functor $f^* : \mathcal{D}$-algebra $\to \xi$-algebra, one proves that π_1^*, π_2^* induce equivalence of derived categories of algebra.s

[**mayGeometryIterated**] does this more concretely without derived categories.

.2. Infinite Loop Space Theory

Recall that a generalized cohomology theory is determined by some based spaces Z_n where $n \in \mathbb{Z}$ equipped with weak equivalences

$$Z_n \xrightarrow{\sim} \Omega Z_{n+1}. \tag{$\star$}$$

In fact $\mathbb{N}_0$ would do. GivenZ_0, define $Z_{-m} = \Omega^m Z_0$.

The spaces Z_m of $(*)$ are called infinite loops spaces. Peter May notices that infinite loop spaces (up to $\simeq$) are E_∞-spaces, and connected E_∞-spacesm are infinite loop spaces.

Application: Construction of generalized cohomology theories. For example, we can consider alegbraic K-theory.

Why are infinite loop spaces E_∞-spaces. Consider that E_∞-spaec are commutative monoids up to homotopy and all reaonsonable higher homotopies.

What does this have to do with loops: π_m is commutative for $m \geq 2$. Consider a space of the form $\Omega^* X$, X is a based space, and $\Omega^m X$ is $\mathrm{Hom}(([0,1]^m, \partial[0,1]^m), (X, *))$.

Peter May invented an operad so that m-loop spaces are E_∞ algebras over this operad ξ_m.

The litte n-cubes operad $\mathcal{E}_m(k)$ is merely a configuration of k cubes in $[0,1]^m$ with disjoint images.

It is obvious then that $\Omega^m X$ (as defined above) is a $\mathcal{C}_m$-aglgebra (same as our proof of commutativity of $\pi,$).

Inclusions of operads

$$\mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \qquad \cdots.$$

Take a little cubes $\times[0,1]$ Then

$$\mathcal{C}_\infty = \bigcup \mathcal{C}_n.$$

May tells us that $\mathcal{C}_\infty$ is a C_∞-algebra, that is an E_∞-operad algebra.