$\mathcal{C}_m$ is the little m-cubes operad. It acts on the loop space $\Omega^m X$ so that it is a $\mathcal{C}_m$-algebra.

We can then take $\mathcal{C}_\infty = \bigcup_m \mathcal{C}_m$ We wish to see that $\mathcal{C}_\infty$ acts on an infinite loop space $Z_m \xrightarrow{\sim} \Omega Z_{m+1}$.

Definition .0.1

A collection of based spaces (Z_m), $m \subseteq \mathbb{N}_0$ together with based homeomorphisms $\rho_m : Z_m \xrightarrow{\cong} \Omega Z_{m+1}$ is called a (May) spectrum.

A morphism of spectra $(Z_m) \to (T_m)$ is a collection of based maps $f_m : Z_m \to T_m$ with commutative diagrams

$$\begin{tikzcd} Z_m \arrow[r, "f_m"] \arrow[d] & T_m \arrow[d] \\ \Omega Z_{m+1} \arrow[r, "\Omega f_{m+1}"'] & \Omega T_{m+1} \end{tikzcd}$$

Therefore if (Z_m) is a May spectrum, then obviously $\mathcal{C}_\infty$ acts on each Z_m (i.e. each Z_m is an E_∞-space).

Can we make a spectrum out of $Z_m \xrightarrow{\sim} \Omega Z_{m=1}$ that would give the same generalized cohomology theory on CW-complexes?

Definition .0.2

A prespectrum is defined the same way as a spectrum, except no condition is given on the continuous map ρ_m (besides being a based continous map)

Thus there is a forgetful functor Spectra $\to$ Prespectra. One can prove that there is a left adjoint (i.e., a free functor) L : Prespectra $\to$ Spectra, which we call spectrification. This was proved by Freyd-Kelly in a transfinite argument.

For the moment we should not we're working with the following convenient category of spaces (see [1]).

- Weakly Hausdorff, compactly generated spaces
- Closed symmetric monoidal category under $\times$.

$L(D_n)_k$ can be described explicitly if (D_n) is an inclusion prespectrum which means that $\rho_n : D_n \to \Omega D_{n+1}$.

Then we have that

$$(L(D_n))_k = \operatorname{colim} \Omega^m D_{k+m} = \operatorname{colim}(D_m \hookrightarrow \Omega D_{m+1} \hookrightarrow \Omega^2(D_{m+2}) \hookrightarrow \cdots).$$

The issue with non-inclusions: Ω commutes past colimit of a sequence of inclusions, but not an arbitrary sequence.

Then for general $Z_m \xrightarrow{\sim} \Omega Z_{m+1}$, we replace them by inclusions by looking at the based mapping cylinder $\Sigma Z_m \to Z_{m+1}$ recursively.

Theorem .0.1 (May)

A connected E_∞-space ($\simeq$ CW-complex) is $\simeq$ to an infinite loop space (Z_0 for some spectrum (Z_m)). [2]

Note: since it doesn't matter which E_∞-operad we are using. We may as well use $E\mathcal{M}$ where $\mathcal{M}(k) = \Sigma_k$. By construction then we have a map of operads $\mathcal{M} \to E\mathcal{M}$. An $\mathcal{M}$-algebra is a topological monoid. Thus an $E\mathcal{M}$-space (E_∞-space) is a topological monoid.

Since $E\mathcal{M}_k \simeq *$, this topological monoid is commutative up to homotopy (and higher homotopies). In particular, π_0 for such a space is a commutative monoid. If X was an infinite loop space, $\pi_0 X$ would be forced to be an abelian group (associated generalized cohomology theory gives E has $E^0(*) = \pi_0 X$).

One can construct a "group completion" of X, say $X \to \overline{X}$ which satisfies

- On π_0 is the K-groupification (the universal abelian group on this commutative monoid, $K\pi_0 X$).

- $H_*(\overline{X};\mathbb{Z}) = [\pi_0 X]^{-1} H_*(X;\mathbb{Z})$. Where $H_*(X;\mathbb{Z})$, homology of a commutative monoid is a graded commutative ring, using the product $\mu : X \times X \to X$ and chain-approximation $CX \otimes CX \to CX$.

.0.1. Homework #11

(3a) Prove that a path-connected topological monoid X is a simple space. Namely $\pi_1 X$ is commutative and acts trivially on $\pi_m X$ for $m > 1$.

> **Recall .0.1**
>
> We should recall how $\pi_1 X$ acts on $\pi_m X$. 2

(3b) Prove that $S^2 \vee S^1$ is not homotopy equivalent to a topological monoid. (Consider how π_1 acts on π_2).

Let first G be a (discrete) group. Recall the Čech resolution $EG = |EG|$, where $EG_\bullet$ is the simplicial set $EG_m = G^{\{0,\dots,m\}}$.

G acts on EG as $g(g_0, \dots, g_m) = (gg_0, \dots, gg_m)$, and it acts freely, properly discontinuously, and all the nice things.

We then call $BG \coloneqq EG/G$. Thus we have $G \to EG \to BG$ as a fibration (where $EG \to BG$) is a universal covering. This shows another construction of BG because $\pi_1 BG = G$ and $\pi_m BG = 0$ for $m > 1$.

Another description of BG which can be generalized. Again we have $BG = |BG_\bullet|$. And we define

$$BG_m = G^m = \{(h_1, \dots, h_m)\}.$$

Then

$$(g_0, g_1, \dots, g_m) \mapsto (h_1, \dots, h_m)$$

Where we put $h_i = g_{i-1}^{-1} g_i$. This is called dehomogenization (and is an isomorphism).

$$g_0^{-1} g_1 = (gg_0)^{-1}(gg_1).$$

We wish to describe the faces and degeneracies directly in terms of $(h_1, \dots, h_m)$.

- For the 0-face, drop h_1 to get $(h_2, \dots, h_m)$
- For the i-th face for $1 \le i \le m-1$ we get $(h_1, \dots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \dots, h_m)$
- For the m-th face, drop h_m to get $(h_1, \dots, h_{m-1})$.
- For degeneracies, insert a unit. $(h_1, \dots, h_{i-1}, 1, h_i, \dots, h_n)$.

This is BG (Bar construction = classifying space = nerve). We can already see that we can do this for a monoid, getting us BM for a monoid M.

Now we see that an E_∞-space X is a topological monoid. The group completion is constructed by

$$\overline{X} = \Omega BX$$