Definition .0.1

Lets introduce a new category Pairs whose objects are pairs of spaces (X, Y) where Y is a subspace of X. A morphism $(X_1, Y_1) \to (X_2, Y_2)$ is a continuous map such that $f(Y_1) \subseteq Y_2$.

If A is an abelian group, there are functors $H_m(?; A) : \text{Pairs} \to \text{Ab}$ and $H^m(?; A) \to \text{Pairs}^{\text{op}} \to \text{Ab}$.

To do this we define $C(X, Y) := C(X)/C(Y)$. That is $C_m(X, Y) := C_m(X)/C_m(Y)$, and this will also be a chain complex of free abelian groups by some basic homological algebra. This follows by the principal that if $T \subseteq S$ then $\mathbb{Z}S/\mathbb{Z}T \cong \mathbb{Z}(S \setminus T)$.

From this point we cna just define $C(X, Y; A) := C(X, Y) \otimes A$ and $C^*(X, Y; A) := \text{Hom}(C(X, Y), A)$. Taking homology of the chain complex gives homologies $H_m(X, Y; A)$ and $H^m(X, Y; A)$.

There is a short exact sequence

$$0 \longrightarrow C(Y) \longrightarrow C(X) \longrightarrow C(X, Y) \longrightarrow 0$$

Note here that an exact sequence is a chain complex with homology zero (although we stop the convention that if it stops we fill in with zeros, so there is no condition on the first/last maps).

We also have short exact sequences

$$0 \longrightarrow C(Y; A) \longrightarrow C(X; A) \longrightarrow C(X, Y; A) \longrightarrow 0$$

$$0 \longrightarrow C^*(X, Y; A) \longrightarrow C^*(X; A) \longrightarrow C^*(Y; A) \longrightarrow 0$$

Note! $? \otimes A$ and $\text{Hom}(?, A)$ are <u>not exact</u>. That is they do not preserve exact sequences. However, they do behave well with direct products, as

$$\left(\bigoplus_i B_i\right) \otimes \cong \bigoplus_i (B_i \otimes A)$$

$$\text{Hom}\left(\bigoplus_i B_i, A\right) \cong \prod_i \text{Hom}(B_i, A)$$

So $? \otimes A$ and $\text{Hom}(?, A)$ preserve <u>split</u> exact sequences. For completeness we recall this definition

Definition .0.2

A split exact sequence has the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "j"] & C \arrow[r] \arrow[l, bend left, dashed, "s"] & 0 \end{tikzcd}$$

This exhibits $(i, s) : A \oplus C \xrightarrow{\cong} B$, and so $\text{Hom}(?, A)$ and $? \otimes A$ preserves this.

A short exact sequence at the level of the chain complexes induces a long exact sequence in homology. I.E. if C^1, C^2, C^3 are chain complexes with a short exact sequence:

$$0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow C^3 \longrightarrow 0$$

Then there is a long exact sequence in homology

$$\cdots \longrightarrow H_n(C^1_*) \longrightarrow H_n(C^2_*) \longrightarrow H_n(C^3_*) \xrightarrow{\partial} H_{n-1}(C^1_*) \longrightarrow \cdots$$

Where the morphisms between n-th homology are the induced maps and the ∂ morphism is complicated (see [1])

Definition .0.3

Let $F, G : C \to D$ be functors. A **natural transformation** $\eta : F \Rightarrow G$ consists of a collection of maps $\eta_X : F(X) \to G(X)$ for every object X in C so that for any map $f : X \to Y$ the diagram below commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

Great!

Definition .0.4

Two categories C, D are equivalent when there are functors $F : C \to D$ and $G : D \to C$ such that $F \circ G \cong \mathrm{Id}_D$ and $G \circ F \cong \mathrm{Id}_C$. Here $\cong$ denotes a natural isomorphism.

A long exact sequence in homology of a space (X, Y) with coefficinets in A is given below

$$\cdots \longrightarrow H_m(Y;A) \longrightarrow H_m(X;A) \longrightarrow H_m(X,Y;A) \xrightarrow{\partial} H_{m-1}(Y;A) \longrightarrow \cdots$$

And in cohomology we have

$$\cdots \longrightarrow H^m(X,Y;A) \longrightarrow H^m(X;A) \longrightarrow H^m(Y;A) \xrightarrow{\delta} H^{m+1}(X,Y;A) \longrightarrow \cdots$$

Both ∂, δ are natural.

.1. Eilenberg-Steenrod Axioms

We now list the Eilenberg-Steenrod axioms for homology (cohomology). First $H_n(?;A)$ and $H^m(?;A)$ are covariant/contravariant functors respectively from Top or Pairs into Ab.

Homotopy Axiom

We also require that homotopic maps in Top or Pairs induce the same map in (co)homology.

We can define categories hTop and hPairs whose objects are the same as Top and Pairs and whose morphisms are equivalence classes of maps up to homotopy.

Then the above condition is the same as requiring that $H^m(?;A)$ and $H^m(?;A)$ are covariant/contravariant functors from hTop or hPairs into Ab.

The key idea to providng this axiom is something called a chain homotopy.

Definition .1.1

Let $f, g : C \to D$ be chain maps. A **chain homotopy** is a sequence of homomorphisms of abelian groups $h_m : C_m \to D_{m+1}$ satisfying

$$dh + hd = f - g$$

One can then define hChain, whose objects are chain complexes and whose morphisms are chain-homotopy classes of chain maps.

Excision Axiom

Let $Z \subseteq Y \subseteq X$ where $\mathrm{Closure}_X(Z) \subseteq \mathrm{Interior}_X(Y)$.

Then there is a map of pairs $(X \setminus Z, Y \setminus Z) \subseteq (X, Y)$ given by the inclusion. This induces an isomorphism on $H_m(?;A)$, $H^m(?;A)$.

Limit Axioms

Take a collection os spaces X_i. Then the inclusions $X_i \hookrightarrow \coprod_i X_i$ induces isomorphisms:

$$\bigoplus_i H_m(X_i; A) \to H_m\left(\coprod_i X_i; A\right)$$
$$H^m\left(\coprod_i X_i; A\right) \to \prod_i H^m(X_i; A)$$

More generally we have something nice that holds for homology and not for cohomology if you know about limits of diagrams $F : J \to \text{Pairs}$.

$$H_m(\lim F; A) \cong \lim H_m(F; A)$$

Exactness Axiom

Each pair (X, A) induces a long exact sequence via the inclusions as above in (co)homology

$$\cdots \longrightarrow H_m(Y; A) \longrightarrow H_m(X; A) \longrightarrow H_m(X, Y; A) \xrightarrow{\partial} H_{m-1}(Y; A) \longrightarrow \cdots$$

And in cohomology we have

$$\cdots \longrightarrow H^m(X, Y; A) \longrightarrow H^m(X; A) \longrightarrow H^m(Y; A) \xrightarrow{\delta} H^{m+1}(X, Y; A) \longrightarrow \cdots$$

At this point if we replace H_m by E_m and H^m by E_m we obtain what are called **generalized** (co)homology theories.

Dimension Axiom

To get **ordinary** (co)homology, we require that $H_m(*) = H^m(*) = 0$ for $m \neq 0$.

Homework 2021-09-07

Define $E_m(X) = E_m(X, \emptyset)$ and $\widetilde{E}_m(X) \coloneqq E_m(X, *)$ where $*$ is a basepoint.

(4a) Using the long exact sequence, prove that for any generalized (co)homology and a based space X

$$E_m(X) = \widetilde{E}_m(X) \oplus E_m(*)$$
$$E^m(X) = \widetilde{E}^m(X) \oplus \widetilde{E}^m(*).$$