Notes on
MATH 695
(Algebraic Topology II)
Syllabus

December 14, 2021

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Contents

I. Introduction to the Class

Logistical Announcements

- Homework
  - Gradescope Invitation Code: ERGX7Y.
  - HW due on Mondays 8PM (except when said otherwise. Next week HW due. Tuesday 9/7 8PM).
  - HW assigned in class.
  - Homework is less stringent. More about understanding concepts and a way of thinking. This does not mean the class is any easier.
- Notes on Professor Kriz's web page.

  http://www.math.lsa.umich.edu/ ikriz/math2021695.html
- Office Hours: MWF: 11-12pm.
- A nice reference is [11]

Goals and Philosophy

First version of 695: Homology with coefficients, cohomology, products, and duality. From today's point of view, this is not nearly enough. This fits the original goal of algebraic topology, which is telling spaces apart.

Today: Focus is more on the method than the original goal. Why?

- There aren't enough examples.
- Constructing interesting spaces is as fundamental as telling them apart.
- Information is not contained just in algebra.

II. Singular (co)homology

II.1. The Basic Definitions

Definition II.1.1

The **standard simplex** is $\Delta^n \coloneqq \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 0, t_i \geq 0\}$. Thi sis sometimes written $[t_0, \dots, t_n]$.

Definition II.1.2

One may define the **free group with coefficients in A generated by a set S** as

$$AS \coloneqq \{a : S \to A \mid \exists F \subseteq S \text{ finite } a(s) = 0 \text{ when } s \notin F\} = \bigoplus_{s \in S} A$$

The **free group** is $\mathbb{Z}S$. Note that $AS = \mathbb{Z}S \otimes A$. Because we have that

$$\left(\bigoplus_{i \in I} A_i\right) \otimes B \cong \bigoplus_{i \in I} (A_i \otimes B)$$

Definition II.1.3

An **n-simplex** in a space X is a continuous (default assumption) map $\sigma : \Delta^n \to X$.

Let $S_m X$ be the set of all n-simplices in X. We then define $C_m X = \mathbb{Z}S_m X$ to be the free abelian group on $S_m X$, and this is the group of **n-chains in X**

Definition II.1.4

If A is an abelian group then $C_m(X;A) = AS_mX$ is the group of singular n-chains with coefficients in A.

Definition II.1.5

If A is an abelian group, then $C^m(X;A) \coloneqq \mathrm{Hom}(C_mX, A)$. Equivalently this is the set of all functions $S_m(X) \to A$, which we denote $\mathrm{Map}(S_m(X), A)$.

Notice that $AS \subsetneq \mathrm{Map}(S, A)$, with the finiteness condition of AS being the key difference.

To define (co)homology we need some standard maps between standard simplices.

Definition II.1.6

The j-th face map $\partial_j \mid \Delta^{m-1} \to \Delta^m$ is defined by taking the tuple $(t_0, \dots, t_{m-1})$ and inserting a zero into the j-th place:

$$(t_0, \dots, t_{m-1}) \mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{m-1})$$

If $0 \leq i \leq j \leq m$, then we have that $\partial_i \partial_j = \partial_{j+1}\partial_i$.

We define $\mathrm{d} : C_mX \to C_{m-1}X$. It suffices to define $\mathrm{d}\big|_{S_mX}$. Let $\sigma : \Delta^m \to X$. Then

$$\mathrm{d}\sigma = \sum_{i=0}^{m} (-1)^i (\sigma \circ \partial_i)$$

This corresponds to restricting to the boundary simplices and with signs corresponding to a sense of orientation.

Lemma II.1.1

The key point is that $\mathrm{d}^2 = 0$. This follows via a calculation

$$\begin{aligned} \mathrm{d}^2\sigma &= \mathrm{d}\left(\sum_{i=0}^{m} (-1)^i \sigma \circ \partial_i\right) \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{m} (-1)^{i+j} \sigma \circ \partial_j \circ \partial_i \end{aligned}$$

This follows by dividing up to when $j \leq i$, and using the crucial formula $\partial_j \partial i = \partial_{i+1}\partial_j$.

Definition II.1.7

The **chain complex** $C_\bullet X$ is defined to be

$$\cdots \longrightarrow C_mX \xrightarrow{\mathrm{d}_m} C_{m-1}X \xrightarrow{\mathrm{d}_{m-1}} C_{m-2}X \longrightarrow \cdots$$

Using the fact that $d_{m-1} \circ \mathrm{d}_m = 0$, this is a chain complex as in algebra.

Definition II.1.8

If C is a chain complex, define the m-th **homology group**:

$$H_mC = \ker(\mathrm{d}_m)/\operatorname{im}(d_{m+1})$$

We call the elements of $\ker(\mathrm{d}_m)$ the **m-cycles** and $\operatorname{im}(d_{m+1})$ the **m-boundaries**

Homework 2021-08-30

(1) Show that $\mathbb{Z}S \otimes A \cong AS$. Try to recall and use the universal property of $\otimes$.
(2) Compute $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$ for $n, m \in \mathbb{Z}$.

Now we take a digression to quickly review some category theory which will help us in our definitions. For a reference see Category Theory in Context by Emily Riehl [13].

Definition II.1.9

A category $\mathscr{C}$ has a class of objects $\operatorname{Ob}\mathscr{C}$ and of morphisms $\operatorname{Mor}\mathscr{C}$. There are maps $S, T : \operatorname{Mor}\mathscr{C} \to \operatorname{Ob}\mathscr{C}$ which stand for source and target as well as $\operatorname{Id} : \operatorname{Ob}\mathscr{C} \to \operatorname{Mor}\mathscr{C}$, and notably $S \circ \operatorname{Id}, T \circ \operatorname{Id}$ are both the identity on objects.

We call $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ the class of all $f \in \operatorname{Mor}\mathscr{C}$ such that $S(f) = X$ and $T(f) = Y$. This is sometimes also denoted by $\mathscr{C}(X, Y)$, and we usually assume that this is a set. We also sometimes write $f : X \to Y$ to mean that $f \in \operatorname{Hom}_{\mathscr{C}}(X, Y)$ when the ambient category is clear.

Furthermore if $f : X \to Y$ and $g : Y \to Z$ then we define $g \circ f : X \to Z$. This is associative when defined and for $f : X \to Y$:

$$\operatorname{Id}_Y \circ f = f = f \circ \operatorname{Id}_X$$

Example II.1.1

There are a variety of examples:

Name	Objects	Morphisms
Set	sets	functions
Grp	groups	homomorphisms
Ab	abelian groups	homomorphisms
Top	spaces	continuous maps

Also given any category $\mathscr{C}$ there is a category $\mathscr{C}^{\mathrm{op}}$ which has the same objects as $\mathscr{C}$ and the morphisms point in the opposite direction with composition also reversed.

Definition II.1.10

A functor $F : \mathscr{C} \to \mathscr{D}$ is a map of objects and of morphisms which preserves Id, S, T, and composition. One can of course compose functors.

Why are we concerned about this? Well we have functors in algebraic topology

$$C_m : \mathrm{Top} \to \mathrm{Ab}$$
$$C : \mathrm{Top} \to \mathrm{Chain}$$

This category Chain has objects chain complexes, and the maps are collections of group homomorphisms $f_n : A_n \to B_n$ satisfying the commutative diagram below

$$\begin{tikzcd} A_n \arrow[r, "d_n^A"] \arrow[d, "f_n"] & A_{n-1} \arrow[d, "f_{n-1}"'] \\ B_n \arrow[r, "d_n^B"] & B_{n-1} \end{tikzcd}$$

Furthermore H_m : Chain $\to$ Ab is a functor, and so we may define the composition H_m : Top $\to$ Ab, overloading notation.

Even better, $?\otimes? : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$ is a functor, where the product of categories is appropriately defined.

For a specified abelian group A, $? \otimes A : \mathrm{Ab} \to \mathrm{Ab}$ is a functor, which is defined on morphisms as

$$(f \otimes A)(x \otimes a) = f(x) \otimes a$$

This will allow us to construct a homology with coefficients functor via "abstract nonsense." Namely, if C is a chain complex then $C \otimes A$ is a chain complex given below:

$$\cdots \longrightarrow C_m \otimes A \xrightarrow{\mathrm{d}_m \otimes A} C_{m-1} \otimes A \xrightarrow{\mathrm{d}_{m-1} \otimes A} \cdots$$

Perfect! Thus with A an abelian group, $? \otimes A : \mathrm{Chain} \to \mathrm{Chain}$ is a functor. We then know that $C(X; A) :- (CX) \otimes A$, and this will be a functor.

Therefore $H_m(X; A) := H_m C(X; A)$ is a functor as well.

At first, this seems strange, as homology with coefficients is determined by homology, and so it cannot contain new information. However, it contains some new and interesting information.

For cohomology, recall that we defined $C^m(X; A) = \mathrm{Hom}(C_m X, A)$. Notice that $\mathrm{Hom}(?, A) : \mathrm{Ab} \to \mathrm{Ab}^{\mathrm{op}}$ is a functor. It is defined on objects via pointwise addition, and it is defined on morphisms as follows.

Let $f : B \to D$ be a morphism of abelian groups. Then we define $\mathrm{Hom}(f, A) : \mathrm{Hom}(D, A) \to \mathrm{Hom}(B, A)$ as foloows. If we have a morphism $h : D \to A$ then:

$$\begin{tikzcd}
D \arrow[r, "h"] & A \\
B \arrow[u, "f"] \arrow[ur, dashed, bend right, "\mathrm{Hom}(f,A)(h)=h\circ f"'] &
\end{tikzcd}$$

Definition II.1.11

A functor $F : C^{\mathrm{op}} \to D$ is called a contravariant functor from C to D. A "normal" functor is called covariant. Hom is covariant in the first coordinate, aka if A is fixed then $\mathrm{Hom}(A, ?)$ is a covariant functor from $\mathrm{Ab} \to \mathrm{Ab}$.

Now say we have a chain complex

$$\cdots \longrightarrow C_m \xrightarrow{\mathrm{d}_m} C_{m-1} \xrightarrow{\mathrm{d}_{m-1}} \cdots$$

Then we may apply $\mathrm{Hom}(?, A)$ everywhere, and we get a chain complex in the "reverse" direction:

$$\cdots \longleftarrow \mathrm{Hom}(C_m, A) \xleftarrow{\mathrm{Hom}(\mathrm{d}_m, A)} \mathrm{Hom}(C_{m-1}, A) \xleftarrow{\mathrm{Hom}(\mathrm{d}_{m-1}, A)} \cdots$$

We say that Cochain the category of such "reversed" chains. If C^* is a cochain complex, then of course defining the chain complex $C_m := C^{-m}$ gives us an equivalence Chain $\cong$ Cochain. We may also define cohomology of a cochain complex as $H^m(C) = \ker \mathrm{d}^m / \operatorname{im} \mathrm{d}^{m-1}$.

Great! This means we may define $C^*(X; A) = \mathrm{Hom}(CX, A)$ as a cochain complex and then:

$$H^m(X; A) = H_{-m}(C^*(X; A)) = H_{-m}(\mathrm{Hom}(CX, A))$$

Functoriality of cohomology then just follows by composing functors:

$$\mathrm{Top} \xrightarrow{C} \mathrm{Chain} \xrightarrow{\mathrm{Hom}(?,A)} \mathrm{Cochain}^{\mathrm{op}} \xrightarrow{H^*} \mathrm{Ab}$$

So cohomology is a contravariant functor. Why do we care about cohomology?

- We encounter it in geometry (de Rham)
- Cohomology has additional structure. if R is a commutative ring, then $H^*(?; R)$ is a functor from spaces into commutative rings.

Homework due 2021-09-07

(3a) Write down the differential in $C(X; A)$, $C^*(X; A)$ in elements.

(3b) Say in a few words why d_m needs (and preserves) the finiteness condition and why d^m does not.

Definition II.1.12

Lets introduce a new category Pairs whose objects are pairs of spaces (X, Y) where Y is a subspace of X. A morphism $(X_1, Y_1) \to (X_2, Y_2)$ is a continuous map such that $f(Y_1) \subseteq Y_2$.

If A is an abelian group, there are functors $H_m(?; A) : \mathrm{Pairs} \to \mathrm{Ab}$ and $H^m(?; A) \to \mathrm{Pairs}^{\mathrm{op}} \to \mathrm{Ab}$.

To do this we define $C(X, Y) \coloneqq C(X)/C(Y)$. That is $C_m(X, Y) \coloneqq C_m(X)/C_m(Y)$, and this will also be a chain complex of free abelian groups by some basic homological algebra. This follows by the principal that if $T \subseteq S$ then $\mathbb{Z}S/\mathbb{Z}T \cong \mathbb{Z}(S \setminus T)$.

From this point we cna just define $C(X, Y; A) \coloneqq C(X, Y) \otimes A$ and $C^*(X, Y; A) \coloneqq \mathrm{Hom}(C(X, Y), A)$. Taking homology of the chain complex gives homologies $H_m(X, Y; A)$ and $H^m(X, Y; A)$.

There is a short exact sequence

$$0 \longrightarrow C(Y) \longrightarrow C(X) \longrightarrow C(X, Y) \longrightarrow 0$$

Note here that an exact sequence is a chain complex with homology zero (although we stop the convention that if it stops we fill in with zeros, so there is no condition on the first/last maps).

We also have short exact sequences

$$0 \longrightarrow C(Y; A) \longrightarrow C(X; A) \longrightarrow C(X, Y; A) \longrightarrow 0$$

$$0 \longrightarrow C^*(X, Y; A) \longrightarrow C^*(X; A) \longrightarrow C^*(Y; A) \longrightarrow 0$$

Note! $? \otimes A$ and $\mathrm{Hom}(?, A)$ are not exact. That is they do not preserve exact sequences. However, they do behave well with direct products, as

$$\left(\bigoplus_i B_i\right) \otimes \cong \bigoplus_i (B_i \otimes A)$$

$$\mathrm{Hom}\left(\bigoplus_i B_i, A\right) \cong \prod_i \mathrm{Hom}(B_i, A)$$

So $? \otimes A$ and $\mathrm{Hom}(?, A)$ preserve split exact sequences. For completeness we recall this definition

Definition II.1.13

A split exact sequence has the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "j"] & C \arrow[l, bend left, dashed, "s"] \arrow[r] & 0 \end{tikzcd}$$

This exhibits $(i, s) : A \oplus C \xrightarrow{\cong} B$, and so $\mathrm{Hom}(?, A)$ and $? \otimes A$ preserves this.

A short exact sequence at the level of the chain complexes induces a long exact sequence in homology. I.E. if C^1, C^2, C^3 are chain complexes with a short exact sequence:

$$0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow C^3 \longrightarrow 0$$

Then there is a long exact sequence in homology

$$\cdots \longrightarrow H_n(C^1_*) \longrightarrow H_n(C^2_*) \longrightarrow H_n(C^3_*) \xrightarrow{\partial} H_{n-1}(C^1_*) \longrightarrow \cdots$$

Where the morphisms between n-th homology are the induced maps and the ∂ morphism is complicated (see [3])

Definition II.1.14

Let $F, G : C \to D$ be functors. A **natural transformation** $\eta : F \Rightarrow G$ consists of a collection of maps $\eta_X : F(X) \to G(X)$ for every object X in C so that for any map $f : X \to Y$ the diagram below commutes

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

Great!

Definition II.1.15

Two categories C, D are equivalent when there are functors $F : C \to D$ and $G : D \to C$ such that $F \circ G \cong \mathrm{Id}_D$ and $G \circ F \cong \mathrm{Id}_C$. Here $\cong$ denotes a natural isomorphism.

A long exact sequence in homology of a space (X, Y) with coefficinets in A is given below

$$\cdots \longrightarrow H_m(Y; A) \longrightarrow H_m(X; A) \longrightarrow H_m(X, Y; A) \xrightarrow{\partial} H_{m-1}(Y; A) \longrightarrow \cdots$$

And in cohomology we have

$$\cdots \longrightarrow H^m(X, Y; A) \longrightarrow H^m(X; A) \longrightarrow H^m(Y; A) \xrightarrow{\delta} H^{m+1}(X, Y; A) \longrightarrow \cdots$$

Both ∂, δ are natural.

II.2. Eilenberg-Steenrod Axioms

We now list the Eilenberg-Steenrod axioms for homology (cohomology). First $H_n(?; A)$ and $H^m(?; A)$ are covariant/contravariant functors respectively from Top or Pairs into Ab.

Homotopy Axiom

We also require that homotopic maps in Top or Pairs induce the same map in (co)homology.

We can define categories hTop and hPairs whose objects are the same as Top and Pairs and whose morphisms are equivalence classes of maps up to homotopy.

Then the above condition is the same as requiring that $H^m(?; A)$ and $H^m(?; A)$ are covariant/contravariant functors from hTop or hPairs into Ab.

The key idea to providng this axiom is something called a chain homotopy.

Definition II.2.1

Let $f, g : C \to D$ be chain maps. A **chain homotopy** is a sequence of homomorphisms of abelian

> groups $h_m : C_m \to D_{m+1}$ satisfying
> $$dh + hd = f - g$$
> One can then define hChain, whose objects are chain complexes and whose morphisms are chain-homotopy classes of chain maps.

Excision Axiom

Let $Z \subseteq Y \subseteq X$ where $\text{Closure}_X(Z) \subseteq \text{Interior}_X(Y)$.

Then there is a map of pairs $(X \setminus Z, Y \setminus Z) \subseteq (X, Y)$ given by the inclusion. This induces an isomorphism on $H_m(?; A)$, $H^m(?; A)$.

Limit Axioms

Take a collection os spaces X_i. Then the inclusions $X_i \hookrightarrow \coprod_i X_i$ induces isomorphisms:

$$\bigoplus_i H_m(X_i; A) \to H_m\left(\coprod_i X_i; A\right)$$
$$H^m\left(\coprod_i X_i; A\right) \to \prod_i H^m(X_i; A)$$

More generally we have something nice that holds for homology and not for cohomology if you know about limits of diagrams $F : J \to \text{Pairs}$.

$$H_m(\lim F; A) \cong \lim H_m(F; A)$$

Exactness Axiom

Each pair (X, A) induces a long exact sequence via the inclusions as above in (co)homology

$$\cdots \longrightarrow H_m(Y; A) \longrightarrow H_m(X; A) \longrightarrow H_m(X, Y; A) \xrightarrow{\partial} H_{m-1}(Y; A) \longrightarrow \cdots$$

And in cohomology we have

$$\cdots \longrightarrow H^m(X, Y; A) \longrightarrow H^m(X; A) \longrightarrow H^m(Y; A) \xrightarrow{\delta} H^{m+1}(X, Y; A) \longrightarrow \cdots$$

At this point if we replace H_m by E_m and H^m by E_m we obtain what are called **generalized** (co)homology theories.

Dimension Axiom

To get **ordinary** (co)homology, we require that $H_m(*) = H^m(*) = 0$ for $m \neq 0$.

Homework 2021-09-07

Define $E_m(X) = E_m(X, \emptyset)$ and $\widetilde{E}_m(X) \coloneqq E_m(X, *)$ where $*$ is a basepoint.

(4a) Using the long exact sequence, prove that for any generalized (co)homology and a based space X

$$E_m(X) = \widetilde{E}_m(X) \oplus E_m(*)$$
$$E^m(X) = \widetilde{E}^m(X) \oplus \widetilde{E}^m(*).$$

Using reduced (co)homology, we can simplify to talking about based spaces instead of about pairs. However, $E_m(X,A) \ncong \widetilde{E}_m(X/A)$, where X/A is the quotient space (even made into a Hausdorff space). Although this holds for special classes of pairs (X,A), we cannot use it to reduce.

We can get rid of this problem by defining some new constructions.

Definition II.2.2

The **mapping cone** CY of a space Y is definde to be

$$CY \coloneqq (Y \times [0,1])/(Y \times \{1\})$$

The **mapping cone** Cf of a map $f : Y \to X$ is defined to be

$$Cf \coloneqq (X \coprod CY)/(y,0) \sim f(y)$$

The quotient topology here is universal. That is a map $Cf \to Z$ is in a natural bijection with maps $g : X \to Z$ such that $g \circ f$ is nullhomotopic.

Definition II.2.3

Given a space Y, its **suspension** SY is defined by

$$SY = (Y \times [0,1])/(y,0) \sim (y',0), (y,1) \sim (y',1)$$

The upshot of mapping cones?

Proposition II.2.1

For an inclusion $f : Y \to X$, $\widetilde{E}_m(Cf) \cong E_m(X,Y)$, and likewise $\widetilde{E}^m(Cf) \cong E^m(X,Y)$.

Proof. This is just some simple arguments from the Ellenberg-Steenrod axioms

$$\begin{aligned}\widetilde{E}_m(Cf) &\cong E_m(Cf,*) \cong E_m(Cf,CY)\\ &\cong E_m(C_-f, C_-Y) \cong E_m(X,Y)\end{aligned}$$

Where we define:

$$C_-Y \coloneqq Y \times [0,1/2]$$

$$C_-f \coloneqq (X \coprod C_-Y)/(y,0) \sim f(y)$$

The third isomorphism above follows by excision on $CY \setminus C_-Y \subseteq CY \subseteq Cf$, and the others follow by homotopy equivalences between pairs $(Cf,*) \simeq (Cf,CY)$ and $(C_-f, C_-Y) \simeq (X,Y)$.

Similarly for cohomology.

We always have an inclusion $X \xrightarrow{\iota} Cf$. We can then ask what is Ci? Well

$$Ci \cong (CX \coprod CY)/(y,0) \sim (f(y),0)$$

This then allows us to see that $Ci \simeq SY$, where SY is the suspension (see Definition II.2.3). This is visualized by Figure 1 Why is this? Well there are maps $SY \to C\iota \to SY = C\iota/CX$ which give a homotopy equivalence. Explicitly for $SY \to C\iota$, we map

$$\begin{aligned}(y,t) &\mapsto (f(y), 1-2t) && (0 \le t \le 1/2)\\ (y,t) &\mapsto (y, 2t-1) && (1/2 \le t \le 1)\end{aligned}$$

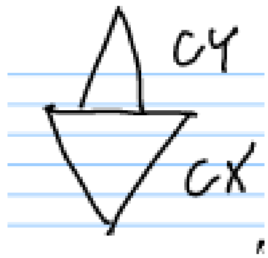

FIGURE 1. $C\iota$ for the inclusion $\iota : X \to Cf$

This suggests that $\widetilde{E}_m(X) \cong \widetilde{E}_{m+1}(SX)$ (which will be on homework).

It also suggests an alternative formulation of the Ellenberg-Steenrod axioms.

Definition II.2.4

Functors $\widetilde{E}_m : \mathrm{hBased} \to \mathrm{Ab}$ are called a generalized based homology theory provided that:

(1) We have an exact sequence for every map of spaces $f : Y \to X$:

$$\widetilde{E}_m(Y) \xrightarrow{\widetilde{E}_m(f)} \widetilde{E}_m(X) \xrightarrow{\widetilde{E}_m(\iota)} \widetilde{E}_m(Cf)$$

where $\iota : X \hookrightarrow Cf$ is the inclusion.

(2) There is a natural isomorphism

$$\widetilde{E}_m(X) \cong \widetilde{E}_{m+1}(SX)$$

for all $m \in \mathbb{Z}$.

Similarly for cohomology. The product axiom involves the wedge sum.

Definition II.2.5

Given based spaces X_i we define their **wedge sum** by:

$$\bigvee_{i \in I} X_i \coloneqq \coprod_i X_i / *_i \sim *_j$$

Definition II.2.6

We call a generalized based homology theory $\widetilde{E}_m$ **additive** provided that the inclusions provide an isomorphism

$$\bigoplus_{i \in I} \widetilde{E}_m X_i \to \widetilde{E}_m \left(\bigvee_{i \in I} X_i \right)$$

Likewise, a generalized based cohomology theory $\widetilde{E}^m$ is called **additive** provided that the inclusions induce an isomorphism

$$\prod_{i \in I} \widetilde{E}_m X_i \leftarrow \widetilde{E}_m \left(\bigvee_{i \in I} X_i \right)$$

THe based and unbased sets of axioms are equivalence. Why? Well given an unbased theory E_m we may define $\widetilde{E}_m(X) \coloneqq E_m(X, *)$ and prove the suspension axiom as well as exactness.

Likewise, given a based theory $\widetilde{E}_m$ we may define $E_m(X) \coloneqq \widetilde{E}_m(X_+)$ where $X_+ \coloneqq X \coprod \{*\}$. For $f : Y \hookrightarrow X$ we define $E_m(X, Y) \coloneqq \widetilde{E}_m(Cf)$.

We then can prove a long exact sequnece from $C\iota \simeq SY$ for $\iota : X \to Cf$ and the suspension axiom.

Similarly for cohomology

II.3. Computing ordinary (co)homology

How do we actually compute it? Well we need a nice category of spaces. The CW-complexes.

Definition II.3.1

Let $X = \bigcup_{i \geq -1} X_i$, where

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$$

are given the subspace topology, and $Z \subseteq X$ is closed if and only if $Z \cap X_i$ is closed in X_i for each i. We say X is a **CW-complex**.

We mandate that X_m is built from X_{m-1} by adjoining m-cells along their boundaries to X_{m-1}. For clarity recall the definitions of an m-cell D^m and its boundary $S^{m-1} = \partial D^m$.

$$D^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid \sum x_i^2 \leq 1\} S^{m-1} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i^2 = 1\}$$

More formally, we are given a set I_m of m-cells, and there is a map $f_m : I_m \times S^{m-1} \to X_{m-1}$ called the **attaching map** so that the following is a pushout diagram

$$\begin{tikzcd} I_m \times S^{m-1} \arrow[r, "f_m"] \arrow[d, hook] & X_{m-1} \arrow[d, hook] \\ I_m \times D^m \arrow[r] & X_m \end{tikzcd}$$

This gives a formula for X_m as follows:

$$X_m = (X_{m-1} \coprod (I_m \times D^m))/(i, y) \sim f_m(i, y)$$

Often X_m is called the m**-skeleton.**

Definition II.3.2

A **CW-pair** is defined the same way except $X_{-1} = Z$ instead of $\emptyset$.

Homework #2

(1) There is a long exact sequence in reduced homology for any based inclusion $i : Y \to X$

$$\cdots \longrightarrow \widetilde{E}_m(Y) \longrightarrow \widetilde{E}_m(X) \longrightarrow E_m(X, Y) \longrightarrow \widetilde{E}_{m-1}(Y) \longrightarrow \cdots$$

Hint: a long exact sequence is a chain complex with homology 0. Consider the LES of the inclusion $* \to *$ and map it into the unbased LES of i. Then consider the "quotient chain complex"

(2) Show that $\widetilde{E}_m(X) \cong \widetilde{E}_{m+1}(SX)$.

This essentially follows by the following, letting

$$S_+X := X \times [1/2, 1]/(x, 1) \sim (x', 1) \cong CX \simeq *$$
$$S_0X := X \times [1/2, 3/4]$$
$$S_-X := X \times [0, 3/4]/(x, 0) \sim (x', 0) \cong CX \simeq *$$

Apply the long exact sequence of a pair to show

$$E_{m+1}(S_-X, S_0X) \cong \widetilde{E}_m(S_0X) \cong \widetilde{E}_m(X).$$

Then apply excision and homotopy equivalence to show that

$$E_{m+1}(SX, *) \cong E_{m+1}(SX, S_+X) \cong E_{m+1}(S_-X, S_0X).$$

Definition II.3.3

A **cell map** (cellular map, CW-map) between CW-pairs $f : (X, Z) \to (Y, T)$ is a continuous map which preserves skeleta. That is $f(X_n) \subseteq Y_n$.

Theorem II.3.1

Every (continuous) map between CW-pairs is homotopic to a cell map.

Proof in Hatcher (Theorem 4.8 [3]). An elaboration of the proof. Why is every map $f : S^k \to S^m$ for $k < m$ homotopic to the constant map. It's clear if images misses a point $S^m \setminus \{*\} \simeq *$. But $f \simeq$ smooth map, which always misses a point.

Proposition II.3.2

If (X, Z) is a CW-pair, then $X/Z \simeq C\iota$ where $\iota : Z \hookrightarrow X$. As a consequence

$$E_m(X, Z) \cong \widetilde{E}_m(C\iota) \cong \widetilde{E}_m(X/Z)$$

This works more generally when $Z \hookrightarrow X$ has the homotopy extension property (HEP), which holds for CW-pairs)

Definition II.3.4

The mapping cylinder Mf of a map $f : Y \to X$ is given as

$$Mf : ((Y \times [0, 1]) \coprod X)/(y, 0) \sim f(y)$$

Definition II.3.5

A map $f : Z \to X$ is a **cofibration** (satisfies HEP) if there is a left inverse $r : X \times [0, 1] \to Mf$ of the map

$$\overline{f} : Mf \to X \times [0, 1]$$
$$(y, t) \mapsto (f(y), t)$$
$$x \mapsto (x, 0)$$

A more explicit definition is given by the commuting diagram below, which means that if we have g_0 and g_t commuting then there must exist a $\widetilde{g}_t$.

$$\begin{tikzcd}
Z \arrow[rr, "\iota_0"] \arrow[dd, "f"'] & & Z \times I \arrow[dl, "g_t"'] \arrow[dd, "f \times \mathrm{Id}"] \\
& Y & \\
X \arrow[ur, "g_0"] \arrow[rr, "\iota_0"'] & & X \times I \arrow[ul, dashed, "\widetilde{g}_t"']
\end{tikzcd}$$

See [6] Chapter 6 for details.

A CW-pair is a cofibration. Only need to observe that $S^{m-1} \subseteq D^m$ is a cofibration, because cofibrations do well with pushouts. This means we need a retract of

$$S^{m-1} \times [0,1] \cup D^m \times \{0\} \hookrightarrow D^m \times [0,1]$$

But this is homeomorphic to

$$D^m \times \{0\} \hookrightarrow D^m \times [0,1]$$

And this has a retract given by taking every (x,t) to $(x,0)$..

If $\iota : Z \hookrightarrow X$ is a cofibration then $C\iota \simeq X/Z$. We know that $M\iota \overset{j}{\subseteq} X \times [0,1]$ has a left inverse. We can perform $M\iota/(Z \times \{1\})$, and this gives $C\iota \overset{j'}{\subseteq} X \times [0,1]/Z \times \{1\}$ has a left inverse r'.

Restrict r' to $X/Z \cong X \times \{1\}/Z \times \{1\} \xrightarrow{\ell} C\iota$.

We claim that ℓ is a homotopy inverse to $c : C\iota \to X/Z$. The details of this will be on the homework

Calculating (Co)homology of CW-pairs

First we'll look at Ordinary homology with coefficients in $\mathbb{Z}$. Make a chain complex $C^{\text{cell}}(X, Z)$. Namely, look at the homology

$$H_k(X_m, X_{m-1}) \cong \widetilde{H}_k(X_m/X_{m-1}) \cong \widetilde{H}_k\left(\bigvee_{I_m} S^m\right) = \bigoplus_{I_m} \widetilde{H}_k(S^m) = \begin{cases} 0 & \text{if } k \neq m \\ \mathbb{Z}I_m & \text{if } k = m \end{cases}$$

We can calculate $\widetilde{H}_k(S^m)$ by noting it is the m-fold suspension of $S^0 = \{*, *'\}$.

That is $H_m(X_m, X_{m-1}) = \mathbb{Z}I_m$ is the free abelian group on the set of m-cells. We then have from the long exact sequence of a pair the map ∂_m below, which we can combine with the inclusion j_{m-1}:

$$H_m(X_m, X_{m-1}) \xrightarrow{\partial_m} H_{m-1}(X_{m-1}) \xrightarrow{j_{m-1}} H_{m-1}(X_{m-1}, X_{m-2})$$

We can set $\mathrm{d}^{\text{cell}}_m = j_{m-1} \circ \partial_m$. Some calculations with long exact sequences of pairs shows that this gives a chain complex.

This allows us to define $C^{\text{cell}}(X)$ as

$$\cdots \longrightarrow \mathbb{Z}I_m \xrightarrow{\mathrm{d}^{\text{cell}}} \mathbb{Z}I_m \longrightarrow \cdots$$

And we can of course define

$$C^{\text{cell}}(X; A) = C^{\text{cell}}(X) \otimes A$$
$$C_{\text{cell}}(X; A) = \text{Hom}(C^{\text{cell}}(X), A)$$

Theorem II.3.3

We in fact have

$$H_m(X; A) = H_m(C^{\text{cell}}(X; A))$$
$$H^m(X; A) = H^m(C_{\text{cell}}(X; A))$$

The proof will be later.

Next time: How to calculate d^{cell}.

Homework #2

(3) Prove that if $Z \overset{\iota}{\hookrightarrow} X$ is a cofibration then $X/Z \simeq C\iota$. (detailed hint in lecture).

Last time, we defined for a CW-complex X [more generally a CW-pair (X, Z)], a chain complex $C_m^{\text{cell}}(X, Z) \coloneqq \mathbb{Z}[I_m]$, where I_m is the set of m-cells.

We also observed that $\mathbb{Z}[I_m] = \widetilde{H}_m(X_m/X_{m-1}) = H_m(X_m, X_{m-1})$.

This allows us to build a chain complex with coefficients or a cochain cell complex via $? \otimes A$ and $\text{Hom}(?, A)$.

Furthermore, the differential $\mathrm{d}_m^{\text{cell}} : \mathbb{Z}[I_m] \to \mathbb{Z}[I_{m-1}]$ is obtained as a connecting map composed with an inclusion:

$$H_m(X_m, X_{m-1}) \xrightarrow{\partial} H_{m-1}(X_{m-1}) \longrightarrow H_{m-1}(X_{m-1}, X_{m-2})$$

This can be shown to give a chain complex as desired (see 592 Notes).

How do we actually compute $\mathrm{d}_m^{\text{cell}}$? Well it's 0 if $m = 0$. Then if $m = 1$, the 1-cells are oriented line segments, and:

$$\mathrm{d}_1^{\text{cell}}(e) = \text{beginning point} \ - \ \text{end point}$$

Now for $e \in I_m$ with $m > 1$ we compute $\mathrm{d}_m^{\text{cell}}(e)$ differently. Namely we have a map $f_m : S^{m-1} \times I_m \to X_{m-1}$. We can then write:

$$S^{m-1} \xrightarrow{f|_{S^{m-1} \times \{e\}}} X_{m-1} \qquad X_{m-1}/X_{m-2} = \bigvee_{I_{m-1}} S^{m-1}$$

We take this map in homology (apply $H_{m-1}(?; \mathbb{Z})$). It gives a map:

$$\mathbb{Z} \to \mathbb{Z}[I_{m-1}]1 \qquad\qquad \mapsto \mathrm{d}_m^{\text{cell}}(e)$$

We are using the fact that:

$$\widetilde{H}_{m-1}\left(\bigvee_{I_{m-1}} S^{m-1}\right) \cong \bigoplus_{i \in I_{m-1}} \widetilde{H}_{m-1}(S^{m-1}).$$

However, we could also just project this map down, sending every cell except c to the basepoint and mapping c by the identity:

$$\bigvee_{I_{m-1}} S^{m-1} \to S^{m-1}$$

And then take homology.

We are then given another problem! Given a continuous map $f : S^k \to S^k$ for $k = m - 1 \geq 1$, what does it induce in homology?

$$H_k(S^k) \longrightarrow H_k(S^k)$$

$$\mathbb{Z} \longrightarrow \mathbb{Z}$$

$$1 \longmapsto \deg(f)$$

We may homotope f to a smooth map, so let us assume $x \in S^k$ and there exists an open neighborhood U of x so that:

$$f^{-1}(U) = \coprod_{i=1}^{\ell} V_i$$

Such that $f : V_i \to U$ is a diffeomorphism (C^1), and we let $f : y_i \in V_i \mapsto x$.

Theorem II.3.4

$\deg(f) = \sum_{i=1}^{\ell} \sigma_i$, where $\sigma_i = 1$ if $f\big|_{V_i}$ preserves orientation and $\sigma_i = -1$ if $f\big|_{V_i}$ reverses orientation. A good book for this material is Milnor's Topology from a differential viewpoint [9].

Example II.3.1

Consider $\mathbb{R}\mathrm{P}^m$, which is the space of all lines through the origin in $\mathbb{R}^{m+1}$, or $S^m/x \sim -x$.
This has a CW-complex structure. We

$$\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \cdots \subseteq \mathbb{R}^{m+1}$$
$$\mathbb{R}\mathrm{P}^0 \subseteq \mathbb{R}\mathrm{P}^1 \subseteq \cdots \subseteq \mathbb{R}\mathrm{P}^m$$

This is a CW-filtration, and $\mathbb{R}\mathrm{P}^m$ is an m-dimensional CW-complex (meaning it only has cells up to dimension m).

For $\mathbb{R}\mathrm{P}^2$ we have the 2-cell v_0 as the top hemisphere, in general the m-cell is $\{(x_0, \dots, x_m) \in S^m \mid x_m \geq 0\}$. the boundary is exactly when $x_m = 0$, whihc is S^{m-1}. The attaching map is then the quotient $S^{m-1} \to \mathbb{R}\mathrm{P}^{m-1}$.

So then we have that:

$$\begin{array}{cccccccc} C^{\text{cell}}(\mathbb{R}\mathrm{P}^m) & \mathbb{Z} & \longrightarrow \mathbb{Z} & \longrightarrow & \cdots \to & \mathbb{Z} & \to & \mathbb{Z} \\ \text{degrees} & m & m-1 & & \cdots & 1 & & 0 \end{array}$$

The attaching map $S^{m-1} \to S^{m-1}_+/S^{m-2}$ sends the northern hemisphere to a point and the southern hemisphere to its antipode. After some work one works out that these maps are zero or two in homology:

$$\begin{array}{cccccc} C^{\text{cell}}(\mathbb{R}\mathrm{P}^m) & \mathbb{Z} \xrightarrow{1+(-1)^m} & \mathbb{Z} \xrightarrow{1+(-1)^{m-1}} & \cdots \xrightarrow{2} & \mathbb{Z} \xrightarrow{0} & \mathbb{Z} \\ \text{degrees} & m & m-1 & \cdots & 1 & 0 \end{array}$$

We can then compute that if m is even:

$$H_k(\mathbb{R}\mathrm{P}^m) \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < k < m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

And if m is odd we have:

$$H_k(\mathbb{R}\mathrm{P}^m) \begin{cases} \mathbb{Z} & \text{if } k = 0, m \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < k < m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

The cellular chain complex C^{cell} is not functorial in continuous maps, but it is functorial in cell maps $f : X \to Y$ where $f(X_k) \subseteq Y_k$. Because then there is an induced map $X_k/X_{k-1} \to Y_k/Y_{k-1}$. Then we can just take reduced homology to get wedges of spheres:

$$X_k/X_{k-1} \to Y_k/Y_{k-1}$$
$$\bigvee_{I_k^X} S^k = \widetilde{H}_k(X_k/X_{k-1}) \to \widetilde{H}_k(Y_k/Y_{k-1} = \bigvee_{I_k^Y} S^k$$

This can again be computed using the degree of maps $S^k \to S^k$.

Homework #3

1a) Calculate $H_k(\mathbb{R}\mathrm{P}^m; \mathbb{Z}/2\mathbb{Z})$ by definition using cellular homology. You may use $C^{\text{cell}}(\mathbb{R}\mathrm{P}^m)$ from class.

1b) Prove that the quotient $\varphi : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-1}$ (embedded as in class) is not homotopic to a constant map (use homology with suitable coefficients $H_m(\varphi; \mathbb{Z}/2\mathbb{Z})$).

1c) For which values of $m > 0$ is $H_m(\varphi; \mathbb{Z})$ non-zero?

1d) Construct an m-dimensional CW-complex X with only one m-cell such that the projection $\varphi : X \to X/X_{m-1}$ is homotopic to a constant map. [Think simple].

These are 5pts each and due next Monday (9/20).

Example II.3.2

We can also look at $\mathbb{C}\mathrm{P}^m$, which is the space of all lines through the origin in $\mathbb{C}^{m+1}$. That is, it is:

$$\left\{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} \mid \sum |z_j|^2 = 1\right\} / (z \sim z' \iff |z| = |z'|)$$

We also have a CW-filtration:

$$\mathbb{C}\mathrm{P}^0 \subseteq \mathbb{C}\mathrm{P}^1 \subseteq \cdots \subseteq \mathbb{C}\mathrm{P}^m$$

We have a $2m$-cell given by $\left\{(z_0, \ldots, z_m) \mid \sum |z_j|^2, z_m \in \mathbb{R}, z_m \geq 0\right\}$. We have a pushout:

$$\begin{tikzcd} S^{2m-1} \arrow[r] & \mathbb{C}\mathrm{P}^{m-1} & \\ D^{2m} \arrow[r] & P \arrow[r] & \mathbb{C}\mathrm{P}^m \end{tikzcd}$$

To know the induced pushout map is a homeomorphism, one uses that it is bijective, P is compact, and $\mathbb{C}\mathrm{P}^m$ is Hausdorff.

We can also compute $C^{\text{cell}}(\mathbb{C}\mathrm{P}^m)$:

$C^{\text{cell}}(\mathbb{C}\mathrm{P}^m)$	$\mathbb{Z} \longrightarrow$	$0 \longrightarrow$	$\mathbb{Z} \longrightarrow$	$\cdots \to$	$0 \to$	$\mathbb{Z}$
degrees	$2m$	$2m-1$	$2m-2$	$\cdots$	1	0

So every map is the zero map. This allows us to say that:

$$H_k(\mathbb{C}\mathrm{P}^m) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq m, \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

II.4. **The Generalized Homology of CW-complexes**

Let us try to define $C^{\text{cell}}(X)$ for a generalized homology theory E and see what goes wrong. We let $E_m \coloneqq E_m(*)$ be the coefficients E_*.

We know that:

$$\widetilde{E}_m(S^0) = E_m$$
$$\widetilde{E}_{m+k}(S^k) = \widetilde{E}_m(S^0) = E_m$$

For a CW-complex X, we have that:

$$\widetilde{E}_{p+q}(X_p/X_{p-1}) = E_q[I_p]$$
$$E_{p+q}(X_p, X_{p-1}) = E_q[I_p]$$

We do get a differential:

$$\mathrm{d} : E_{p+q}(X_p, X_{p-1}) \xrightarrow{\partial} E_{p+q-1}(X_{p-1}) \to E_{p+q-1}(X_{p-1}, X_{p-2})$$

However, we now have a chain complex for each choice of $q \in \mathbb{Z}$. We draw this for p increasing to the right and q increasing on the upper side:

$$\begin{array}{ccccccccc}
q = 2 & & E_2[I_0] & \longleftarrow & E_2[I_1] & \longleftarrow & E_2[I_2] & \longleftarrow & \cdots \\
q = 1 & & E_1[I_0] & \longleftarrow & E_1[I_1] & \longleftarrow & E_1[I_2] & \longleftarrow & \cdots \\
q = 0 & & E_0[I_0] & \longleftarrow & E_0[I_1] & \longleftarrow & E_0[I_2] & \longleftarrow & \cdots \\
q = -1 & & E_{-1}[I_0] & \longleftarrow & E_{-1}[I_1] & \longleftarrow & E_{-1}[I_2] & \longleftarrow & \cdots \\
 & & p = 0 & & p = 1 & & p = 2 & & \cdots
\end{array}$$

We have a "total dimension" $p+q$. This is called the E^1-page of a spectral sequence. And this is in fact the Atiyah-Hirzebruch spectral sequence. In general we can have:

$$\begin{tikzcd}
q=2 & E^1_{0,2} & E^1_{1,2} \arrow[l] & E^1_{2,2} \arrow[l] & \cdots \arrow[l] \\
q=1 & E^1_{0,1} & E^1_{1,1} \arrow[l] & E^1_{2,1} \arrow[l] & \cdots \arrow[l] \\
q=0 & E^1_{0,0} & E^1_{1,0} \arrow[l] & E^1_{2,0} \arrow[l] & \cdots \arrow[l] \\
q=-1 & E^1_{0,-1} & E^1_{1,-1} \arrow[l] & E^1_{2,-1} \arrow[l] & \cdots \arrow[l] \\
 & p=0 & p=1 & p=2 & \cdots
\end{tikzcd}$$

The homology of each sequence is called the E^2-page.

$$\begin{tikzcd}
q=2 & E^1_{0,2} & E^1_{1,2} \arrow[l] & E^1_{2,2} \arrow[l] & \cdots \arrow[l] \\
q=1 & E^1_{0,1} & E^1_{1,1} \arrow[l] & E^1_{2,1} \arrow[l] & \cdots \arrow[l] \\
q=0 & E^1_{0,0} & E^1_{1,0} \arrow[l] & E^1_{2,0} \arrow[l] & \cdots \arrow[l] \\
q=-1 & E^1_{0,-1} & E^1_{1,-1} \arrow[l] & E^1_{2,-1} \arrow[l] & \cdots \arrow[l] \\
 & p=0 & p=1 & p=2 & \cdots
\end{tikzcd}$$

We get a differential $\mathrm{d}_2 : E^2_{p,q} \to E^2_{p-1,q+1}$:

$$\begin{tikzcd}
E^1_{0,2} & E^1_{1,2} & E^1_{2,2} & \cdots \\
E^1_{0,1} & E^1_{1,1} \arrow[ul] & E^1_{2,1} \arrow[ul] & \cdots \arrow[ul] \\
E^1_{0,0} & E^1_{1,0} \arrow[ul] & E^1_{2,0} \arrow[ul] & \cdots \arrow[ul] \\
E^1_{0,-1} & E^1_{1,-1} \arrow[ul] & E^1_{2,-1} \arrow[ul] & \cdots \arrow[ul]
\end{tikzcd}$$

In general we get a differential on the r-th page $\mathrm{d}_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

And we take algebraic homology:

$$E_{p,q}^{r+1} = H(E_{p,q}^r)$$

We can then define:

$$E_{p,q}^{\infty} = \operatorname{colim}_r E_{p,q}^r 2$$

Still a whole plane full of groups. Have we calculated $E_{p+q}(X)$? Well we can consider:

$$F_p E_{p+q} X \coloneqq \Im(E_{p+q} X_p \to E_{p+q} X)$$

Then $F_{-1} = 0 \subseteq F_0 \subseteq F_1 \subseteq \cdots$. This is an increasing filtration (complete), and we have:

$$\bigcup F_p E_{p+q} X = E_{p+q} X$$

Which follows by the limit axiom of homology:

Theorem II.4.1 (Atiyah-Hirzebruch)

$E_{p,q}^{\infty} = F_p E_{p+q} X / F_{p-1} E_{p+q} X$. This is called the associated graded object.

We have $E_{p,q}^2 = H_p(X, E_q)$.

For an abelian group A, a complete filtration on A is:

$$0 = F_{-1} A \subseteq F_0 A \subseteq F_1 A \subseteq \cdots$$

Where $A = \bigcup_i F_i A$. Then the associated graded object is:

$$E^0 A_i = (F_i A / F_{i-1} A)_{i \geq 0}$$

Example II.4.1

Suppose that $(F_i A)$ is a complete filtration of an abelian group A and suppose $(F_i A / F_{i-1} A) \cong \mathbb{Z}[S_i]$. Then:

$$A = \bigoplus_i \mathbb{Z}[S_i] = \mathbb{Z}\left[\coprod_i S_i\right]$$

Proof. There is a short exact sequence:

$$0 \longrightarrow F_{i-1} A \longrightarrow F_i A \longrightarrow F_i A / F_{i-1} A \longrightarrow 0$$

A free abelian group is projective so this splits. For splitting just lifts the free generators $\in S_i$ to $F_i A$ and extends by the universal property.

We can conclude $F_i A \cong F_{i-1} A \oplus F_i A / F_{i-1} A$. Induction finishes the proof.

This is called an extension in a spectral sequence. What good is a spectral sequence (for example the AHSS, Atiyah-Hirzebruch spectral sequence) when we only know $E_{p,q}^2$ and not the higher differentials?

The simplest scenario: sometimes they are ruled out.

Example II.4.2

When $E = H(?; A)$ is the ordinary homology. In that case AHSS looks like the following on the first page:

$$\begin{array}{lllll} q=1 & 0 & 0 & 0 & \cdots \\ \\ q=0 & A[I_0] \longleftarrow & A[I_1] \longleftarrow & A[I_2] \longleftarrow & \cdots \end{array}$$

$$\begin{array}{lllll} q=-1 & 0 & 0 & 0 & \cdots \end{array}$$

Thus no higher differentials are possible, and no extensions are possible because the associated graded object only has one term.

So if we prove that the AHSS works, it implies the theorem about $H^{\text{cell}} = H^{\text{singular}}$.

Someitmes the differential can also be ruled out in a more subtle way. For example when $E^2_{p,q} = 0$ for $p+q$ odd (any higher differential will decrease the total dimension by one). In this case, we can still have extensions.

Homework #3

(2) Suppose a generalized homology theory K has coefficients $K_{2n} = \mathbb{Z}$, $K_{2m-1} = 0$, $m \in \mathbb{Z}$. Calculate $K_\ell \mathbb{C}\mathrm{P}^m$ for all $\ell \in \mathbb{Z}$. Use AHSS, and put together the information mentioned in class.

Remark II.4.1

In most (homological) spectral sequences, the pages are denoted by E^r_{pq}. This has nothing to do with the generalized homology theory E in last class. On Homework, generalized homology theory is called K, the spectral sequence terms should still be denoted by E^r_{pq}.

Today we show that the groups $H_m(X,Y;A), H^m(X,Y;A)$ are completely determined (algebraically) by $H_m(X,Y) = H_m(X,Y;\mathbb{Z})$ and $H^m(X,Y) = H^m(X,Y;\mathbb{Z})$.

However, the way they are determined is not completely functorial. The key point: $C(X,Y)$ (singular chain complex) are chain complexes of free abelian groups (terms are free abelian groups).

Theorem II.4.2

Any chain complex of free abelian groups can be written as follows:

$$C \cong \bigoplus_{m\in\mathbb{Z}} \mathscr{H}_m[m]$$

The brackets denote a "shift" of a chain complex by m, and $\mathscr{H}_m$ is a complex of the form:

$$\begin{array}{cccccc} \cdots \longrightarrow & 0 \longrightarrow & B_m \longrightarrow & Z_m \longrightarrow & 0 \longrightarrow & \cdots \\ \cdots & 2 & 1 & 0 & -1 & \cdots \end{array}$$

Where the map $B_m \to Z_m$ is injective and $H_0\mathscr{H}_m = H_m(C)$ (note that $H_k\mathscr{H}_m = 0$ for $k \neq 0$).

Proof. $C : \cdots \to C_{m+1} \to C_m \to C_{m-1} \to \cdots$. We let $Z_m \coloneqq \ker \mathrm{d}_m$ and $B_m \coloneqq \operatorname{im} \mathrm{d}_{m+1}$. Then $H_0\mathscr{H}_m = H_m C \coloneqq Z_m/B_m$.

Note that we have a short exact sequence:

$$0 \longrightarrow Z_m \xrightarrow{\subseteq} C_m \longrightarrow B_{m-1} \longrightarrow 0$$

Now $B_{m-1} \subseteq Z_{m-1} \subseteq C_{m-1}$. C_{m-1} is free abelian, so B_{m-1} is free abelian as well (see [5] for the algebra).

Thus this splits (say by s_{m-1}), and we have that:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{m+1} \arrow[r] & C_m \arrow[r] & C_{m-1} \arrow[r] & C_{m-2} \arrow[r] & \cdots \\
\cdots \arrow[r] & Z_{m+1} \oplus B_m \arrow[r] \arrow[u, "\subseteq \oplus s_m"] & Z_m \oplus B_{m-1} \arrow[r] \arrow[u, "\subseteq \oplus s_{m-1}"] & Z_{m-1} \oplus B_{m-2} \arrow[r] \arrow[u, "\subseteq \oplus s_{m-2}"] & Z_{m-2} \oplus B_{m-3} \arrow[r] \arrow[u, "\subseteq \oplus s_{m-3}"] & \cdots
\end{tikzcd}$$

Given the simple complex $\mathscr{H} : B \subseteq Z$, we have $H_0\mathscr{H} = H$. What is $H_*(\mathscr{H} \otimes A)$? Well $H_0(\mathscr{H} \otimes A) = H \otimes A$. Why? Well $\otimes$ is right exact so:

$$0 \longrightarrow B \longrightarrow Z \longrightarrow H \longrightarrow 0$$

$$B \otimes A \longrightarrow Z \otimes A \longrightarrow H \otimes A \longrightarrow 0$$

And then we set $\operatorname{Tor}_1^{\mathbb{Z}}(H, A) \coloneqq H_1(\mathscr{H} \otimes A)$. Is this well-defined? For this to be well-defined, the answer needs to depend only on H, not on $\mathscr{H}$. We'll postpone this for now, and we'll prove it later in greater generality.

Similarly, what is the cohomology of $\operatorname{Hom}(\mathscr{H}, A)$? Well we have left exactness so:

$$0 \longrightarrow B \longrightarrow Z \longrightarrow H \longrightarrow 0$$

$$\operatorname{Hom}(B, A) \longleftarrow \operatorname{Hom}(Z, A) \longleftarrow \operatorname{Hom}(H, A) \longleftarrow 0$$

Then $H^0 \operatorname{Hom}(X, A) = \operatorname{Hom}(H, A)$. We then set $\operatorname{Ext}^1_{\mathbb{Z}}(H, A) \coloneqq H^1 \operatorname{Hom}(X, A)$.

Example II.4.3

Let $H = \mathbb{Z}/2\mathbb{Z}$ and $A = \mathbb{Z}$. Then $\mathscr{H} : \mathbb{Z} \xrightarrow{2} \mathbb{Z}$. Homming into $\mathbb{Z}$ we have:

$$\operatorname{Hom}(\mathscr{H}, A) : \mathbb{Z} \xleftarrow{2} \mathbb{Z}$$

Then $\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0 = \operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$. And $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

From the structure theorem of chain complexes of free abelian groups, commutation of homology with shifts, and direct sum, we get the following wonderful result

Theorem II.4.3 (The Universal Coefficient Theorem)

We have that

$$H_m(X, Y; A) \cong (H_m(X, Y) \otimes A) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{m-1}(X, Y), A)$$
$$H^m(X, Y; A) \cong \operatorname{Hom}(H_m(X, Y), A) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{m-1}(X, Y), A)$$

Thus we've reduced the problem to figuring out how to calculate $\operatorname{Tor}_1^{\mathbb{Z}}$ and $\operatorname{Ext}^1_{\mathbb{Z}}$.

A slight catch: This is not completely functorial, namely the splittings are not natural transformations. Functorially, we only have short exact sequences:

$$0 \longrightarrow H_m(X, Y) \otimes A \longrightarrow H_m(X, Y; A) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H_{m-1}(X, Y), A) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_{m-1}(X, Y), A) \longrightarrow H^m(X, Y; A) \longrightarrow \operatorname{Hom}(H_m(X, Y), A) \longrightarrow 0$$

These split, but not naturally.

This actually works for any chain complex of free abelian groups.

Homework #3

3a) Calculate $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, where $m, n \in \mathbb{Z}$.

3b) Calculate $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ where $m, n \in \mathbb{Z}$.

The cases where one of them is 0 may need special care.

A headstart on next class–the general theory of all this. Namely, resolutions.

Definition II.4.1

Let R be any commutative ring, and let M be an R-module. A **free R-resolution of M** is a chain complex of free R-modules

$$\mathscr{C} \qquad \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0$$

Then $H_0\mathscr{C} = M$, $H_k\mathscr{C} = 0$ for $k \neq 0$.

Now let N be any other R-module. We define:

$$\mathrm{Tor}_m^R(M, N) := H_m(\mathscr{C} \otimes_R N)$$
$$\mathrm{Ext}_R^m(M, N) := H^m \operatorname{Hom}_R(\mathscr{C}, N)$$

It is still true that:

$$\mathrm{Tor}_0^R(M, N) = M \otimes_R N$$
$$\mathrm{Ext}_R^0(M, N) = \operatorname{Hom}_R(M, N)$$

Before we defined the R-modules $\mathrm{Ext}_R^m, \mathrm{Tor}_m^R$ for a commutative ring R. If R is not commutative, then $\mathrm{Ext}_R^m(M, N)$ is defined if M, N are both left R-modules. In general, then, $\mathrm{Ext}_R^m(M, N)$ is just an abelian group. Then Tor_R^m is defined when M is a right R-module and N is a left R-module, and it is only an abelian group.

Example II.4.4

Let G be a group. The group ring $\mathbb{Z}[G]$ is the free abelian group on G wiht multiplication given by the multiplication in G (and extended by distributivity).

For example, if $G = \mathbb{Z}/2\mathbb{Z}$. Then we can consider $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$. Let $G = \{1, \alpha\}$ be the particular representative, with $\alpha \cdot \alpha = 1$. Then:

$$(k + \ell\alpha)(n + m\alpha) = (kn + \ell m) + (km + \ell n)\alpha$$

Bad habit (in general G): A $\mathbb{Z}[G]$ module is called a "G-module." This clashes with other terminology. This really means that G acts on M by linear maps. And of course a left G-action and a right action are equivalent by $gm \leftrightarrow mg^{-1}$.

Definition II.4.2

Let M be a G-module. We define the homology and cohomology of G with coefficients in M by:

$$H_m(G; M) := \mathrm{Tor}_m^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$
$$H^m(G; M) := \mathrm{Ext}_{\mathbb{Z}[G]}^m(\mathbb{Z}, M)$$

Back to topology.

Definition II.4.3

Let G be a group, A G-CW-complex is a G-equivariant space (space with a G-action) X where $X = \bigcup_{n \in \mathbb{N}_0}$ where $X_{-1} = \emptyset$ (indeed we can take X_{-1} to be a G-space to get a G-CW-pair).

I_m (the set of m-cells) is a G-set (set with a G-action). Furthermore, $f_m : I_m \times S^{m-1} \to X_{m-1}$ is a G-equivariant map (when tking the G-action to be trivial on the sphere). Then X_m is a pushout:

$$\begin{tikzcd} I_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d, hook] & X_{m-1} \arrow[d, hook] \\ I_n \times D^{n-1} \arrow[r] & X_m \end{tikzcd}$$

Suppose we have a G-space X which is both free (all the I_m are free G-sets, aka $gx = x \implies g = 1$) and $X \simeq *$ non-equivariantly..

Then $C^{\text{cell}}X$ is a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$. Why? Well because $\mathbb{Z}I_m$ is a free $\mathbb{Z}[G]$-module, and we have an exact sequence (because $X \simeq *$) given by:

$$\cdots \longrightarrow \mathbb{Z}I_m \longrightarrow \mathbb{Z}I_{m-1} \longrightarrow \cdots \longrightarrow \mathbb{Z}I_0 \longrightarrow \mathbb{Z}$$

So we have that X/G is CW-complex (with set of m-cells I_m/G). Furthermore $C^{\text{cell}}(X/G) = C^{\text{cell}}(X) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. More generally, using that I_m is G-free:

$$\mathbb{Z}[I_m] \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}[I_m/G]$$

Likewise,

$$C_{\text{cell}}(X/G) = \operatorname{Hom}_{\mathbb{Z}[G]}(C^{\text{cell}}(X), \mathbb{Z})$$
$$\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}I_m, \mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}[I_m/G], \mathbb{Z})$$

We can conclude that:

$$H_m(G; \mathbb{Z}) = H_m(X/G)$$
$$H^m(G; \mathbb{Z}) = H^m(X/G)$$

We sometimes write $BG = X/G$, where $X* \simeq$ (non-equivariantly) is a free G-CW-complex. This is also sometimes called the classifying space of G.

We call $X = EG$, and it is the universal cover of BG via the quotient map. Therefore $\pi_1(BG) = G$ and $\pi_k(EG) = \pi_k(BG) = 0$ for $k > 1$. We will come back to this in more detail.

Example II.4.5

Let $G = \{1, \alpha\} \cong \mathbb{Z}/2\mathbb{Z}$. Consider $\mathbb{RP}^\infty := \bigcup_n \mathbb{RP}^n$ (a CW-complex with the union topology). Then $\mathbb{RP}^\infty$ is ain fact a BG. Why?

Well the universal cover of $\mathbb{RP}^\infty$ is S^∞, which we know to be contractible (and will be a EG). To see this think of a homotopy:

$$h_t(x_0, x_1, \ldots) = \frac{t(x_0, x_1, \ldots) + (1-t)(0, x_0, 0, x_1, 0, x_2, \ldots)}{\|t(x_0, x_1, \ldots) + (1-t)(0, x_0, 0, x_1, 0, x_2, \ldots)\|}$$

Thus the identity is homotopic to $(x_0, x_1, \ldots) \mapsto (0, x_0, 0, x, 0, x_2, \ldots)$. Then we can use the straight line homotopy to $(1, 0, 0, \ldots)$. This gives a homotopy from the identity to the constant map.

The map $S^\infty \to \mathbb{RP}^\infty$ is the quotient map identifying antipodal points. Thus letting α act on S^∞ by the antipodal map we have the desired structure. We then see that:

$$H_k(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = H_k\mathbb{RP}^\infty = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k > 0 \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = H^k\mathbb{RP}^\infty = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k > 0 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Homework #4

1a) Prove that the following is a free $\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$-resolution of $\mathbb{Z}$ (with the trivial action), where $k \in \mathbb{Z}\setminus\{0\}$:

$$\cdots \longrightarrow \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}] \xrightarrow{T} \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}] \xrightarrow{T} \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$$

Let $\mathbb{Z}/k\mathbb{Z} = \{1, \alpha, \alpha^2, \ldots, \alpha^{k-1}\}$ with $\alpha^k = 1$. Then we can set $T(1) = 1 - \alpha$, Then $N(1) = 1 + \alpha + \alpha^2 + \cdots + \alpha^{k-1}$.

1b) Calculate $H_n(\mathbb{Z}/k\mathbb{Z};\mathbb{Z})$ and $H^n(\mathbb{Z}/k\mathbb{Z};\mathbb{Z})$ when $k \neq 0$.

Note: S^∞ can be considered as the unit sphere in $\mathbb{C}$, namely

$$\{(z_0, z_1, \ldots) \mid z_m \in \mathbb{C}, \text{finitely many nonzero}, \sum |z_m|^2 = 1\}.$$

Then $\mathbb{Z}/k\mathbb{Z} \hookrightarrow S^1$ via the k-th roots of unity. One can then make S^∞ a $\mathbb{Z}/k\mathbb{Z}$-CW-complex via a bit of work. Then $B\mathbb{Z}/k\mathbb{Z} = S^\infty/(\mathbb{Z}/k\mathbb{Z})$. This is sometimes known as an infinite lens space.

One can make it in such a way that $C^{\text{cell}}E\mathbb{Z}/k\mathbb{Z}$ can be chosen to be precisely the resolution given in Homework #4 1a.

Next time: Correctness of definition of Tor, Ext.

Today we start the proof that the definitions of Tor and Ext are correct. For simplicity, we assume R is a commutatative rign.

We start with a key lemma.

Lemma II.4.4

Let $C_\bullet$ and $D_\bullet$ be free R-resolutions of R-modules M, N, respectively. Let $\varphi : M \to N$ be a homomorphism of R-modules. Then there exists a, unique up to chain homotopy, R-chain map $\widetilde{\varphi} : C_\bullet \to D_\bullet$ such that $H_0\widetilde{\varphi} = \varphi$.

Comments: Recall $H_0C = M$, $H_0D = N$, $H_kC = H_kD = 0$ for $k > 0$.

Definition II.4.4

Let $f, g : C_\bullet \to D_\bullet$ be chain maps. a chain homotopy $h : f \simeq g$ has

$$\mathrm{d}h + h\,\mathrm{d} = f - g$$

Notice that $h : C_n \to D_{n+1}$

Proof of Existence. Let $M_1 = \operatorname{im} \mathrm{d}_1^C = C_1 / \ker \mathrm{d}_1^C$ and $N_1 = \operatorname{im} \mathrm{d}_1^D = C_1 / \ker \mathrm{d}_1^D$. We have short exact sequences, and because C_0, D_0 are free we can lift in the generators:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, dashed] & C_0 \arrow[r] \arrow[d, dashed, "\widetilde{\varphi}_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & D_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Homological algebra then guarantees a map $M_1 \to N_1$. Note that $\cdots \to C_1$ and $\cdots \to D_1$ are free resolutions of M_1, N_1, because $\ker \mathrm{d}_1^C = \operatorname{im} \mathrm{d}_2^C$ and $\ker \mathrm{d}_1^D = \operatorname{im} \mathrm{d}_2^D$. Thus we can construct $\widetilde{\varphi}_k$ inductively from the map $M_1 \to N_1$.

Proof of Uniqueness. Suppose $\widetilde{\varphi}_1, \widetilde{\varphi}_2 : C_\bullet \to D_\bullet$ both induce φ in H_0. Then $\widetilde{\varphi}_1 - \widetilde{\varphi}_2$ induces 0. Thus it suffices to show that if $\widetilde{\varphi}$ induces zero, then it is chain homotopic to zero.

Consider the augmented resolutions $C_m \to \cdots \to C_1 \to C_0 \to M \to 0$ and likewise for D. We construct h_0 by noting that for $x \in C_0$ we have $\mathrm{d}_0 \widetilde{\varphi}(x) = \varphi(\mathrm{d}_0(x)) = 0$. Thus $\widetilde{\varphi}(x) = \mathrm{d}_1 y$ for some $y \in D_1$. We can then lift on free generators.

Now suppose we have constructed the homotopy $h : \widetilde{\varphi} \simeq 0$ in degrees $< m$.

Then we know that:

$$\begin{aligned}
\mathrm{d}_m(\widetilde{\varphi}_m - h\,\mathrm{d}_m) &= \mathrm{d}_m \widetilde{\varphi}_m - \mathrm{d}_m h\,\mathrm{d}_m \\
&= \widetilde{\varphi}_{m-1}\,\mathrm{d}_m - \mathrm{d} h\,\mathrm{d} \\
&= h\,\mathrm{d}\,\mathrm{d} + \mathrm{d} h\,\mathrm{d} - \mathrm{d} h\,\mathrm{d} = 0
\end{aligned}$$

Thus by exactness $(\widetilde{\varphi}_m - h\,\mathrm{d}_m)(x) = \mathrm{d}y$ for free generators $x \in C_m$, and so we can lift on the free generators.

Corollary II.4.5

Free resolution is a functor R-Mod $\to$ h-R-Chain.

Both $\operatorname{Hom}(?, N)$ and $? \otimes_R N$ preserve chain homotopy, and chain homotopy preserves homology. Thus Tor and Ext are well-defined.

Recipe: If F is any additive functor R-Mod $\to \mathscr{C}$ to an additive category which preserves chain homotopy (aka extends to a functor on the homotopy category of R-chains), then we can define left derived functors $L_m F$ by applying F to a free resolution and then taking H_m.

Instead of a free resolution, it suffices to require that C_m be projective (projective resolution).

Definition II.4.5

P is projective if for every $g : M \twoheadrightarrow N$ and $f : P \to N$ there exists a $\overline{f} : P \to M$ such that $g\overline{f} = f$.

$$\begin{tikzcd}
 & M \arrow[d, "g"] \\
P \arrow[ur, dashed, "\overline{f}"] \arrow[r, "f"'] & N \arrow[d] \\
 & 0
\end{tikzcd}$$

We turn around the arrows and say Q is injcetive if for every $g : N \hookrightarrow M$ and $f : N \to Q$ there exists a $\overline{f} : M \to Q$ such that $\overline{f}g = f$.

$$\begin{tikzcd} 0 \arrow[d] & \\ N \arrow[r, "f"] \arrow[d, "g"'] & Q \\ M \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

We also have injective resolutions, where we have $Q_0 \to Q_1 \to Q_2 \to \cdots$ with $H^0 Q = M$ and $H^m Q = 0$ for $m > 0$.

Lemma II.4.6

Injective resolutions ar ea functor R-Mod to h-R-Chain.

Existence of "enough injectives". In Ab, injective is exactly being divisible, that is for all $x \in G$ there exists $m \in \mathbb{N}$ such that there exists $y \in G$ with $my = x$.

Injective R-modules are given as $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ with G a divisible abelian group. This will give that injective resolutions exist. The rest of the proof is the same as for projectives.

Right derived functors $R^m F$ are then defined by applying F to an injective resolution and taking m-th cohomology.

The case of Tor and Ext is:

$$\begin{aligned} \mathrm{Tor}_m^R(?, N) &= L_m(? \otimes_R N) \\ \mathrm{Tor}_m^R(M, ?) &= L_m(M \otimes_R ?) \\ \mathrm{Ext}_R^m(?, N) &= L_m(\mathrm{Hom}_R(?, N)) \\ \mathrm{Ext}_R^m(M, ?) &= R^m(\mathrm{Hom}_R(M, ?)) \end{aligned}$$

Note if F is right exact then $L_0 F = F$ and if F is left exact then $R^0 F = F$.

Next Time: Abelian Categories.

Homework #4

(2) Prove that if P, T are projective resolutions of R-modules M, N and $\varphi : M \to N$ is a homomorphism of R-modules then there exists a (unique up to R-chain homotopy) $\widetilde{\varphi} : P \to T$ such that $H_0 \widetilde{\varphi} = \varphi$.

II.5. Abelian Categories

We take a quick digression to define limits/colimits in a category.

Definition II.5.1

A diagram is a functor $D : I \to \mathscr{C}$.

Definition II.5.2

A cone over a diagram $D : I \to \mathscr{C}$ is an object X in $\mathscr{C}$ so that for each $i \in I$ there is a map

$\eta_i : X \to D(i)$. Furthermore for every map $i \xrightarrow{f} j$ in I we have the following commuting triangle:

$$\begin{tikzcd} & X \arrow[dl, "\eta_i"'] \arrow[dr, "\eta_j"] & \\ D(i) \arrow[rr, "D(f)"] & & D(j) \end{tikzcd}$$

Example II.5.1

Product is the limit of a diagram with index category $I = \{*_1, *_2\}$. Pullback is also a limit over a diagram $\bullet \to \bullet \leftarrow \bullet$. Equalizers are also limits.

Definition II.5.3

A limit $\lim D$ over a diagram $D : I \to \mathscr{C}$ is a "universal cone"

That is $\lim D$ is a cone over D, with maps $\eta_i : \lim D \to D(i)$ such that for any other cone T over D with maps $\mu_i : T \to D(i)$ we have that there is a unique arrow $f : T \to \lim D$ such that the following diagram commutes for all i:

$$\begin{tikzcd} T \arrow[d, "f"] \arrow[dd, bend right=60, "\mu_i"'] \\ \lim D \arrow[d, "\eta_i"] \\ D(i) \end{tikzcd}$$

Dually wqe have the notion of a colimit.

Example II.5.2

The coproduct, pushouts, and coequalizers are all colimits.

Definition II.5.4

$\operatorname{colim} \emptyset$ is called an initial object, as there is a unique arrow $\operatorname{colim} \emptyset \to T$ for every T lying in $\mathscr{C}$. Likewise $\lim \emptyset$ is called a terminal object, as there is a unique arrow $T \to \lim \emptyset$ for every T lying in $\mathscr{C}$

Note: Limits and colimits are only defined up to isomorphism (given by the universal property). However there is only one such isomorphism at the level of cones (i.e. respecting the limiting cones over the diagram).

Definition II.5.5

A functor is called right exact when it preserves finite colimits. It is called left exact when it preserves finite limits.

Now we'll think a bit about abelian categories.

Definition II.5.6

A category with zero has an initial object and a final object such that the unique morphism $I \to T$ from the initial object to the terminal object is an isomorphism.

In such a category for any X, Y we have an arrow $0 : X \to Y$ given by the unique composition $X \to 0 \to Y$.

Example II.5.3

$\text{Ab}, R\text{-Mod}, \text{BasedSpaces}, \text{BasedSets}$.

In a category with zero, we can define kernels and cokernels.

Definition II.5.7

In a category with zero, $\ker f$ is the equalizer of $f : X \to Y$ and 0. That is we take the limit over the diagram:

$$X \underset{0}{\overset{f}{\rightrightarrows}} Y$$

Likewise a cokernel $\operatorname{coker} f$ is the coequalizer of $f : X \to Y$ and 0 (that is the colimit of the above diagram.

Definition II.5.8

In any category, $f : X \to Y$ is a monomorphism when for any $g, h : Z \to X$ such that $fg = fh$ we have $g = h$.

Likewise $f : X \to Y$ is an epimorphism when for any $u, v : Y \to Z$ such that $uf = vf$ we have $u = v$.

Definition II.5.9

An abelian category is an Ab-enriched category with zero and with finite limits and colimits such that every epimorphism is a cokernel and every monomorphism is a kernel.

You can prove a lot of nice properties about abelian categories. Including all the additive properties of abelian groups.

- $X \oplus Y \cong X \prod Y \cong X \coprod Y$.
- $\operatorname{Mor}_{\mathscr{C}}(X, Y)$ is an abelian group and composition is bilinear.

Definition II.5.10

Enough projectives in an abelian category $\mathscr{C}$ provided that for all X in $\mathscr{C}$ there exists a projective P and an epimorphism $P \twoheadrightarrow X$.

This gives us projective resolutions and left derived functors. If you have enough injectives (that is for every object X you have a monomorphism $X \hookrightarrow Q$ into an injective) that gives you injective resolutions and right derived functors.

We have enough projectives and injectives in R-Mod. You also have enough injectives in abelian sheaves.

II.6. **Commutativity of** Tor

We want to show that $\operatorname{Tor}_m^R(M, N) \cong \operatorname{Tor}_m^R(N, M)$, and that this isomorphism is canonical.

Idea: Resolve both M, N. Call C a free resolution of M and D a free resolution of N. Redefine $\operatorname{Tor}_m^R(M, N) = H_m(C \otimes_R D)$. We need to define $C \otimes_R D$, and prove that we get the same thing.

Given two chain complexes C and D, we must define their tensor product $C \otimes_R D$.

If D is just an R-module N, then we want $C \otimes_R N$. It's certainly then incorrect to take the componentwise tensor product.

We can take a two-dimensional grid of tensor products:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_m \otimes D_m \arrow[r] & C_{m-1} \otimes D_m \arrow[r] & \cdots \\
\cdots \arrow[r] & C_m \otimes D_{m-1} \arrow[r] & C_{m-1} \otimes D_{m-1} \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Definition II.6.1

This is a double chain complex $A_{n,m}$. We have two differentials $\partial : A_{n,m} \to A_{n-1,m}$ and $\delta : A_{n,m} \to A_{n,m-1}$ such that:

$$\partial\partial = 0 \qquad \delta\delta = 0 \qquad \partial\delta = \delta\partial$$

Definition II.6.2

We define $(C \otimes_R D)_{n,m} = C_n \otimes_R D_m$, the tensor product of two chain complexes, to be the double chain complex with differentials given by $\mathrm{d}^C \otimes \mathrm{Id}_D$ and $\mathrm{Id}_C \otimes \mathrm{d}^D$.

Definition II.6.3

Given a double chain complex A, the totalization $|A|$ is a chain complex given by

$$|A|_m = \bigoplus_{k+\ell=m} A_{k,\ell}$$
$$\mathrm{d}x = \partial x + (-1)^k \delta x$$

Reversing roles of k, ℓ gives an isomorphic chain complex. Apply the sign $(-1)^{k\ell}$ to $x \in A_{k,\ell}$.

We can then redefine $\mathrm{Tor}_m^R(M, N) = H_m(|C \otimes_R D|)$ where C, D are projective resolutions of M, N. Nobody writes the totalization, so $H_m(|C \otimes_R D|) = H_m(C \otimes_R D)$.

Homework #4

(3) Suppose we have a double chain complex C such that
- $H_m(C_{k,*}, \partial) = 0$
- $C_{k,\ell} = 0$ if $\ell < 0$ (equiv. $\ell < N$ fixed)

So cut off in the bottom, rows exact. Then $H_m(|C|) = 0$. (Hint: First prove it when there exists a L such that for all k $C_{k,\ell} = 0$ if $\ell > L$. Can induct on L using short exact sequences of chain complexes, which leads to a long exact sequence in homology. Then express C as a colimit of such sequences with L increasing, use commutation of homology with colimits of sequences).

Using this, it's fairly easy to prove that Tor is commutative using the program outlined below. You consider the augmented resolution $\widetilde{C}$ of M given by $\cdots \to C_m \to C_{m-1} \to \cdots \to C_0 \to M \to 0$ for a free resolution C of M. This is exact, and there is actually a short exact sequence $0 \to M[-1] \to \widetilde{C} \to C \to 0$.

By the homework $H_*(\widetilde{C} \otimes_R D) = 0$. By long exact sequence, we then have that:

$$H_m(\widetilde{C} \otimes_R D) = 0 \longrightarrow H_m(C \otimes_R D) \longrightarrow H_{m-1}(M[-1] \otimes_R D) \longrightarrow H_{m-1}(\widetilde{C} \otimes_R D) = 0$$

If C and D are free R-resolutions of R-modules M, N (for R a commutative ring) then

$$H_m(C \otimes_R D) = H_m(C \otimes_R N) = \mathrm{Tor}_m^R(M, N)$$

Therefore $\mathrm{Tor}_m^R(M, N) \cong \mathrm{Tor}_m^R(N, M)$, because $C \otimes_R D \cong D \otimes_R C$.

Let $R = \mathbb{Z}$. A corollary is

Corollary II.6.1 (Kunneth Theorem)

Let C, D be chain complexes of free abelian groups. Then

$$H_m(C \otimes D) \cong \bigoplus_{k+\ell=m} H_k(C) \otimes H_\ell(D) \oplus \bigoplus_{k+\ell=m-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_k(C), H_\ell(D))$$

naturally we have a exact sequence

$$0 \longrightarrow \textstyle\bigoplus_{k+\ell=m} H_k(C) \otimes H_\ell(D) \longrightarrow H_m(C \otimes D) \longrightarrow \bigoplus_{k+\ell=m-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_k(C), H_\ell(D)) \longrightarrow 0$$

but the splitting is not natural.

Proof. Although C and D are not free resolutions, recall that they are direct sums of shifted free resolutions

$$C = \bigoplus_{m \in \mathbb{Z}} \mathscr{H}_m[m]$$

where $\mathscr{H}_m$ is a $\mathbb{Z}$-free resolution of $H_m C$. Similarly for D.

What about an arbitrary commutative ring R, C and D are chain complexes of free R-modules?

We have a Kunneth spectral sequence:

$$E^2_{pq} = \bigoplus_{k+\ell=q} \mathrm{Tor}_p^R(H_k C, H_\ell D) \Rightarrow H_{p+q}(C \otimes_R D)$$

For $R = \mathbb{Z}$ (more generally a principal ideal domain) we only have $\mathrm{Tor}_0^R = \otimes_R$, Tor_1^R, $E^2_{pq} = 0$ for $p > 1$.

No room for d^r for $r > 1$, so $E^2 = E^\infty$, and the spectral sequence collapses.

<u>Comment</u>: If R is a field, every module is free. Thus

$$H_m(M \otimes_R N) = \bigoplus_{k+\ell=m} H_k(M) \otimes_R H_\ell(N)$$

<u>Comment:</u> For any commutatative ring R, we have a natural homomorphism of R-modules

$$H_k(C) \otimes_R H_\ell(D) \to H_{k+\ell}(C \otimes_R D)[c] \otimes [c'] \qquad \mapsto [c \otimes c']$$

if $c = \mathrm{d}a$ then $c \otimes c' = \mathrm{d}(a \otimes c')$.

Can this be used to calculate $H_m(X \times Y)$ for X, Y spaces? Well we're looking at $C(X \times Y)$ versus $C(X) \otimes C(Y)$. That is maps $\Delta^m \to X \times Y$ versus $(\Delta^k \to X) \otimes (\Delta^\ell \to Y)$.

This is different except in degree zero, because

$$C_0(X \times Y) \cong C_0(X) \otimes C_0(Y)$$
$$\mathbb{Z}[X \times Y]\mathbb{Z}X \otimes \mathbb{Z}Y$$

Theorem II.6.2 (Eilenberg-Zilber)

There exist natural chain maps

$$\varphi : C(X) \otimes C(Y) \to C(X \times Y)$$
$$\psi : C(X \times Y) \to C(X) \otimes C(Y)$$

which are Id in degree zero and moreover $\varphi\psi$ is naturally homotopic to Id and $\psi\varphi$ is naturally homotopic to Id.

In other words: $C(X \times Y)$ is naturally homotopic to $C(X) \otimes X(Y)$.

Note: Also similarly for coefficients in a commutative ring R. Especially useful when R is a field via the Kunneth Theorems.

Over $\mathbb{Z}$ this gives a kunneth theorme for spaces

$$H_m(X \times Y) \cong \bigoplus_{k+\ell=m} H_k(X) \otimes H_\ell(Y) \oplus \bigoplus_{k+\ell=m-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_k(X), H_\ell(Y))$$

In Eilenberg-Zilbur theorem, natural homotopy h means each h_m is natural.

Strategy for proving Eileberg-Zilbur theorem. We are trying to construct natural homomorphisms

$$C_m(X \times Y) \to \Phi$$
$$C_k(X) \otimes C_\ell(Y) \to \Phi$$

where $\Phi : \text{Top} \times \text{Top} \to \text{Ab}$. Because $C_m(X \times Y) = \mathbb{Z}[S_m(X \times Y)]$ and $C_k(X) \otimes C_\ell(Y) = \mathbb{Z}(S_k X \times S_\ell Y)$, our problem is equivalent to constructing a natural map of sets

$$S_m(X \times Y) \to \Phi$$
$$S_k(X) \times S_\ell(Y) \to \Phi$$

Lemma II.6.3 (Yoneda Lemma)

Natural transformations $\operatorname{Mor}_C(T, ?) \to \Psi$ where $\Psi : C \to \text{Set}$ are in bijection with elements of $\Psi(T)$.

To prove the Eilenberg-Zilbur theorem, proceed by induction on m (resp. $k + \ell$).

Suppose there is a natural transformation $C_k X \otimes C_\ell Y \to C_{k+\ell} X \times Y$ is constructed for $k + \ell < m$. Let $k + \ell = m$.

We need to construct a natural transformation

$$\varphi : C_k X \otimes C_\ell Y \to C_{k+\ell} X \times Y$$

By Yoneda lemma, we only need to construct

$$\varphi(\underbrace{(\mathrm{Id} : \Delta^k \to \Delta^k) \otimes (\mathrm{Id} : \Delta^\ell \to \Delta^\ell)}_{z}) = u \in C_{k+\ell}(\Delta^k \times \Delta^\ell)$$

We must have that $\mathrm{d}u = \varphi(\mathrm{d}z)$, but we already know $\varphi(\mathrm{d}z)$ by inductive hypothesis. Butthen

$$\mathrm{d}\varphi(\mathrm{d}z) = \varphi(\mathrm{d}\,\mathrm{d}z) = 0$$

$\varphi(\mathrm{d}z)$ is then a cycle, so it is a boundary

$$H_{m-1}(\Delta^k \times \Delta^\ell) = 0$$

($m = 1$ needs a special case). Thus $\varphi(\mathrm{d}z) = \mathrm{d}u$ for some u. We then just can lift on these free generators.

Other direction $\psi : C_m(X \times Y) \to \bigoplus_{k+\ell=m} C_k(X) \otimes C_\ell(Y)$.

We need to map this on the identity $\Delta^m \to \Delta^m$, we special case $m = 1$. For $m > 1$ by Kunneth theorem

$$H_m(C\Delta^m \otimes C\Delta^m) = 0.$$

Homework #5

(1) Prove that for EZ maps φ, ψ we have $\psi\varphi \simeq \mathrm{Id}_{CX \otimes CY}$ naturally.

Use induction. For $k + \ell = m$ we need to construct a natural map

$$h : C_k X \otimes C_\ell Y \to \bigoplus_{p+q=m+1} C_p X \otimes C_q Y$$

where both sides are considered as functors $\mathrm{Top} \times \mathrm{Top} \to \mathrm{Ab}$.

The funtor $C_k X \otimes C_\ell Y$ is again the free abelian group on $S_k X \times S_\ell Y$.

By Yoneda lemma, we need to construct $h(z) \in (C(\Delta^k) \otimes C(\Delta^\ell))_{m+1}$ where

$$z = (\mathrm{Id}_{\Delta^k}, \mathrm{Id}_{\Delta^\ell}) \in S_k(\Delta^k) \times S_\ell(\Delta^\ell).$$

We want that

$$\mathrm{d}h(z) + h\,\mathrm{d}z = \psi\varphi(z) - z$$

Thus

$$\mathrm{d}h(z) = \psi\varphi(z) - z - h\,\mathrm{d}z$$

We verify that $\mathrm{d}(\psi\varphi(z) - z - h\,\mathrm{d}z) = 0$. Then the target has zero homology in degree $m \geq 1$.

In Yoneda Lemma:

$$\begin{tikzcd} & \mathrm{Mor}_C(Y,Y) \arrow[d, "G(f)"] & Y \arrow[d, "f"] \\ * \arrow[ur, "\mathrm{const}(\mathrm{Id}_Y)"] \arrow[r, "\mathrm{const}(f)"'] & \mathrm{Mor}_C(Y,X) & X \end{tikzcd}$$

A functor is representable when $G(X) = \mathrm{Mor}_C(Y, X)$ for some Y.

This is, more generally, called a universal element

Definition II.6.4

Let $G : C \to D$ be a functor and let $X \in D$. A universal element for X, G is a D-morphism $\mu : X \to G(Y)$ for some $Y \in C$ with

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[dr, "h"'] & G(Y) \arrow[d, "G(g)"] & Y \arrow[d, dashed, "g", "\exists!"'] \\ & G(Z) & Z \end{tikzcd}$$

such that for every D-morphism $h : X \to G(Z)$ there exists a unique C-morphism $g : Y \to Z$ such that $h = G(g) \circ \mu$.

Example II.6.1

$* \to \mathrm{Id}$ is a universal element for the representable functor $G : C \to \mathrm{Set}$ Where $G(Z) = \mathrm{Mor}_C(Y, Z)$. This is the statement of the Yoneda Lemma

The universal element for X, G if it exists, is unique up to isomorphism.

If the universal element exists for every object $x \in D$ for a fucntor $G : C \to D$, then Y_x is functorial in X. We have a functor $F : D \to C$ where $F(x)$ is universal for x, G.

Then we have

$$\mathrm{Mor}_D(X, G(Z)) \cong \mathrm{Mor}_C(F(X), Z)$$

naturally. In this case we say that $F : D \to C$ is left adjoint to G.

$\eta : X \to GF(X)$ is given by universality, and is called the unit of the adjunction. Symmetrically, we have a natural transformation $\varepsilon : FG(Y) \to y$ called the counit.

One can prove that F is left adjoint to G if and only if we have natural transformations $\eta : \mathrm{Id} \to GF$ and $\varepsilon : FG \to \mathrm{Id}$ such that

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[rr, bend right, "\mathrm{Id}"'] & FGF \arrow[r, "\varepsilon F"] & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[rr, bend right, "\mathrm{Id}"'] & GFG \arrow[r, "G\varepsilon"] & G \end{tikzcd}$$

commute. These are called the triangle identities.

Example II.6.2

Let R be a commutative ring. Then $M \otimes_R ?$ is left adjoin to $\mathrm{Hom}_R(M, ?)$.

Duals: $M^* : \mathrm{Hom}_R(M, R)$.

This also extends to R-Chain. There is a notion of a closed tensor category, an abelian category with $\otimes$ satisfying the "obvious axioms."

III. Products in (co)homology

Natural product in $H^*(X; R)$ where X is a space and R is a commutative ring. Start with $\Delta : X \to X \times X$. This gives

$$CX \to C(X \times X) \to CX \otimes CX$$

via the Eilenberg-Zilber theorem. Tensoring by R gives

$$C(X; R) \to C(X; R) \otimes_R C(X; R)$$

:wq Dualize the above example to get

$$C^*(X; R) \otimes_R C^*(X; R) \xrightarrow{\mu} C^*(X; R)$$

$$H^*(C^*(X; R)) \otimes_R H^*(C^*(X; R)) \xrightarrow{\mu} H^*(C^*(X; R))$$

This is called the cup product $\smile$.

Properties, it is associative, unital, and graded-commutative. Aka for $x \in H^k(X; R)$ and $y \in H^\ell(X; R)$ we have

$$x \smile y = (-1)^{k\ell}(y \smile x)$$

We actually get good rings (e.g. polynomial rings).

Homework #5

(2) Consider the functor $F : \mathrm{Grp} \to \mathrm{Ring}$ given by $G \mapsto \mathbb{Z}[G]$.

Prove that the right adjoint to F is the group of units

$$R \to \{g \in R \mid \exists k, gk = 1\}$$

(use universality).

Lets deal with examples of the cup product. That is when R is a commutative ring we have a map

$$\smile : H^*(X;R) \otimes H^*(X;R) \to H^*(X;R)$$

which is given by the Eilenberg-Zilbur theorem from $\Delta : X \to X \times X$, and gives $H^*(X;R)$ the structure of a graded commutative ring.

Lets cover the case when $X = BG$, for G a discrete group. $\pi_1 X = G$ and the universal cover $\widetilde{X}$ of X is contractible.

Then

$$H^*(X;R) = H^*(G;R) = \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z};\mathbb{R}).$$

Translating the sotry to algebra: Let C be a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$. Then $C \otimes_{\mathbb{Z}} C$ is a $\mathbb{Z}[G] \otimes \mathbb{Z}[G] = \mathbb{Z}[G \times G]$-free resolution of $\mathbb{Z}$ (by the Kunneth theorem)

We have the diagonal homomorphism $g \mapsto (g,g)$. Via the diagonal morphism, $C \otimes_{\mathbb{Z}} C$ is also a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$. This is free because G-action on $G \times G$ is a free action. By the functoriality of resolutions, there exists some map of $\mathbb{Z}[G]$-module chain complexes

$$C \to C \otimes_{\mathbb{Z}} C$$

which induces $1 : \mathbb{Z} \to \mathbb{Z}$ on H_0 (unique up to chain homotopy). Once we have this, we obtain a map

$$\mathrm{Hom}(C,R) \otimes_R \mathrm{Hom}(C,R) \to \mathrm{Hom}(C \otimes_{\mathbb{Z}} C, R) \to \mathrm{Hom}(C,R)$$

Example III.0.1

Lets go with $G = \{1, \alpha\}$ with $\alpha^2 = 1$. Then the free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$ is

$$C : \cdots \xrightarrow{1-\alpha} \mathbb{Z}[G] \xrightarrow{1+\alpha} \mathbb{Z}[G] \xrightarrow{1-\alpha} \mathbb{Z}[G]$$

Then $G \times G = \{1, \alpha, \beta, \gamma\}$ with $\alpha^2 = \beta^2 = \gamma^2 = 1$ and $\alpha\beta = \gamma$. The double chain complex is

$$\begin{tikzcd}
 & \vdots & \vdots \\
\cdots \arrow[r] & \mathbb{Z}[G \times G] \arrow[r, "1-\alpha"] \arrow[d, "1-\beta"] & \mathbb{Z}[G \times G] \arrow[d, "1-\beta"] \\
\cdots \arrow[r, "1+\alpha"] & \mathbb{Z}[G \times G] \arrow[r, "1-\alpha"] & \mathbb{Z}[G \times G]
\end{tikzcd}$$

Now lets look at $C \to C \otimes_{\mathbb{Z}} C$, thinking of C with the maps γ. On each term, where do I send 1.

$$\begin{tikzcd}
 & \alpha \arrow[d, mapsto] & \\
1 \arrow[r, mapsto] & 1 - \gamma = 1 - \alpha + \alpha(1-\beta) & 1
\end{tikzcd}$$

And then we do this again

$$\begin{tikzcd}
 & & 1 \arrow[d, mapsto] \\
 & \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \alpha+\beta = \alpha - 1 + 1 - \beta \\
1 \arrow[r, mapsto] & 1+\gamma = 1 + \alpha + \alpha(\beta - 1) &
\end{tikzcd}$$

Homework #5

(3) Denoting by e_n the $1 \in \mathbb{Z}[\{1,\gamma\}] \in C$, prove that

$$e_n \mapsto \sum_{\substack{\ell \text{ even} \\ k+\ell=n}} e_k \otimes e_\ell + \alpha \sum_{\substack{\ell \text{ odd} \\ k+\ell=n}} e_k \otimes e_\ell$$

Prove that this gives a ($\mathbb{Z}\{1,\gamma\}$-equivariant) chain map $C \to C \otimes_{\mathbb{Z}} C$.

Every $\mathbb{Z}/2$ in bidegree k,ℓ goes to $\mathbb{Z}/2$ in bidegree $k+\ell$. This tells us that

$$H^m(\mathbb{Z}/2;\mathbb{Z}/2) = H^m(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2$$

That is the cup product

$$\smile\, : H^k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^\ell(\mathbb{R}\mathrm{P}^\infty,\mathbb{Z}/2) \to H^{k+\ell}(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$$

is an isomorphism, since the left and right hand sides are both $\mathbb{Z}/2$. We conclude that $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[x]$. REcall that $H^n(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ is $\mathbb{Z}$ when $n=0$, $\mathbb{Z}/2$ in even degrees, and zero in odd degrees.

The cup product is functorial in the ring. Thus

$$H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}) \to H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$$

This is given as

$$\begin{array}{ccc}
\vdots & & \vdots\, 0 \longrightarrow \mathbb{Z}/2 \\
b^2, \mathbb{Z}/2 & \longrightarrow & a^4, \mathbb{Z}/2 \\
0 & \longrightarrow & \mathbb{Z}/2 \\
b, \mathbb{Z}/2 & \longrightarrow & a^2, \mathbb{Z}/2 \\
0 & \longrightarrow & \mathbb{Z}/2 \\
1, \mathbb{Z} & \longrightarrow & a^0, \mathbb{Z}/2
\end{array}$$

Thus $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}) = \mathbb{Z}[b]/(2b)$. If ℓ is any number then $H^*(\mathbb{Z}/\ell;\mathbb{Z}) = \mathbb{Z}[b]/(\ell b)$, where b is in degree 2. Note that if p and $a \in \mathbb{Z}/p = H^1(\mathbb{Z}/p,\mathbb{Z}/p)$ in degree one then

$$a \smile a = (-1)^{1\cdot 1} a \smile a = -a \smile a$$
$$a \smile a = 0$$

Thus $H^*(\mathbb{Z}/p;\mathbb{Z}/p) = \mathbb{Z}/p[b] \otimes_{\mathbb{Z}/p} \wedge \mathbb{Z}/p[a]$

Back to topology. The unit sphere S^∞ in $\mathbb{C}^\infty = \bigoplus_\infty \mathbb{C}$. Then S^1 acts on S^∞ by multiplying in every coordinate.

Thus $\mathbb{Z}/\ell < S^1$ acts on S^∞. $B\mathbb{Z}/\ell = S^\infty/\mathbb{Z}/\ell$.Also $\mathbb{C}\mathrm{P}^\infty = S^\infty/S^1$, which one can call BS^1, but S^1 is a topological group (not discrete).

$$H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \longrightarrow H^*(BZ/\ell;\mathbb{Z})$$

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell \\ & & \\ 0 & & 0 \\ & & \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell \\ & & \\ 0 & & 0 \\ & & \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

One can deduce that $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) = \mathbb{Z}[b]$ (in degree 2). Functoriality gives $H^*(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) = \mathbb{Z}[b]/(b^{m+1})$.

IV. **Homotopy Theory of** Based

We will be doing homotopy theory in Based because it is a category with zero, which means we can talk about kernels and cokernels!!!

Definition IV.0.1

A homotopy $f_t : X \times [0,1] \to Y$ is called based provided that f_t preserves the basepoint for all $t \in [0,1]$.

In other words it's a based map of $f_t : X \wedge [0,1]_+ \to Y$ where $X \wedge [0,1]_+ = (X \times [0,1])/(* \times [0,1])$.

Definition IV.0.2

The based mapping cone $C_{\text{based}}f$ associated to a given based map $f : Y \to X$ is given by

$$C_{\text{based}} = Cf/(* \times [0,1]) = (X \coprod (Y \times [0,1]))/(*,t) \sim *, (y,1) \sim *, (y,0) \sim f(y)$$

This is sometimes denoted by Cf despite the conflicting notation.

The based suspension $\Sigma Y = Y \wedge S^1 = C_{\text{based}}(Y \to *)$ is a special case.

If Y, X are CW complexes and f is a cellular map,

$$C_{\text{based}}f \simeq Cf$$

(same proof as for the suspension).

Denote by $[X,Y] = \mathrm{Mor}_{\text{hBased}}(X,Y)$ (based homotopy classes of based maps $X \to Y$).

Proposition IV.0.1

Let $f : Y \to X$ be a based map. Let Z be a based space. Then the induced sequence of

$$Y \xrightarrow{f} X \xrightarrow{i} Cf$$

given by applying $[-,Z]$

$$[Y, Z] \xleftarrow{[f,Z]} [X, Z] \xleftarrow{[i,Z]} [Cf, Z]$$

is exact. That is

$$\ker[f, Z] = \Im[i, Z]$$

Proof. To prove that $\mathrm{im}[i, Z] \subseteq \ker[f, Z]$. This follows if $if \simeq 0$. But the mapping cone is almost rigged this way

$$h_t(y) = (y, t)$$

then $h_0 = if$ and $h_1 = 0$.

Now we need to prove that $\ker[f, Z] \subseteq \mathrm{im}[i, Z]$. Well let $g \in \ker[f, Z]$ this means that we have a diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, dashed, "0"'] & X \arrow[r, "i"] \arrow[d, "g"] & Cf \\ & Z & \end{tikzcd}$$

Note that we have the following

$$Y \xrightarrow{f} X \xrightarrow{i} Cf \xrightarrow{j} Ci \simeq \Sigma X$$

Theorem IV.0.2

Let $f : Y \to X$ be a based map. Let Z be a based space. Then we hve a long exact sequence

$$\begin{tikzcd} & & \cdots \\ {[\Sigma^2 Cf, Z]} \arrow[r] & {[\Sigma^2 X, Z]} \arrow[r] & {[\Sigma^2 Y, Z]} \\ {[\Sigma Cf, Z]} \arrow[r] & {[\Sigma X, Z]} \arrow[r, "{[-\Sigma f, Z]}"] & {[\Sigma Y, Z]} \\ {[Cf, Z]} \arrow[r, "{[i,Z]}"] & {[X, Z} \arrow[r, "{[f,Z]}"] & {[Y, Z]} \end{tikzcd}$$

(and

Observation: $[\Sigma X, Y]$ is a group, $[\Sigma^2 X, Y]$ is an abelian group (as is $[\Sigma^m X, Y]$) (same proof as for π_n).

Dualizing, Ω is right adjoint to $\Sigma : \mathrm{Based} \to o\,\mathrm{Based}$. And this also works in hBased.

The mapping cone also has a dual construction.

Definition IV.0.3

Let $f : X \to Y$ be a map of based spaces. The homotopy fiber Ff is defined by

$$Ff \coloneqq \{(x, \omega) \mid x \in X, \omega : [0, 1] \to Y, \omega(0) = f(x), \omega(1) = *\}$$

with the compact-open topology. And there's a canonical projection $Ff \xrightarrow{p} X$.

Lemma IV.0.3

Let Z be a based space and let $f : X \to Y$ be a based map. Then the sequence

$$[Z, Ff] \xrightarrow{[Z,p]} [Z, X] \xrightarrow{[Z,f]} [Z, Y]$$

is exact.

Homework #6

(2) Prove Lemma IV.0.3.

Notice also that

$$\Omega X \simeq Fq \xrightarrow{-\Omega f} \Omega Y \simeq Fp \xrightarrow{q} Ff \xrightarrow{p} X \xrightarrow{f} Y$$

A great reference for this is [6] (it's the best part of the book!).

Theorem IV.0.4

Let $f : X \to Y$ be a based map and let Z be a based space. Then we have a long exact sequence

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (\tikztotarget)}] \\
{[Z, \Omega^2 Ff]} \arrow[r] & {[Z, \Omega^2 X]} \arrow[r] & {[Z, \Omega^2 Y]} \arrow[dll] \\
{[Z, \Omega Ff]} \arrow[r] & {[Z, \Omega X]} \arrow[r, "{[Z, -\Omega f]}"] & {[Z, \Omega Y]} \arrow[dll] \\
{[Z, Ff]} \arrow[r, "{[Z, p]}"] & {[Z, X]} \arrow[r, "{[Z, f]}"] & {[Z, Y]}
\end{tikzcd}$$

Again $[Z, \Omega^n X] \cong [\Sigma^n Z, X]$ are groups for $n \geq 1$ and abelian groups for $n \geq 2$.
If we take $Z = S^0 = \{*, \infty\}$ then because $\Sigma^n S^0 = S^n$

$$[S^0, \Omega^n X] \cong [S^n, X] = \pi_n(x).$$

Thus there is a long exact sequence in homotopy groups:

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
\pi_2(Ff) \arrow[r] & \pi_2(X) \arrow[r] & \pi_2(Y) \arrow[dll] \\
\pi_1(Ff) \arrow[r] & \pi_1(X) \arrow[r] & \pi_1(Y) \arrow[dll] \\
\pi_0(Ff) \arrow[r] & \pi_0(X) \arrow[r] & \pi_0(Y)
\end{tikzcd}$$

If $A \overset{i}{\hookrightarrow} X$ is an inclusion, this suggests defining

$$\pi_m(X, A) := \pi_{m-1}(Fi)$$

Let $f : X \to Y$ be a based map. We would like to understand the actual fiber $F := f^{-1}(*)$ in terms of our understanding of the homotopy fiber Ff.

Definition IV.0.4

The map $f : X \to Y$ is called a <u>fibration</u> provided that it satisfies the <u>homotopy lifting property</u>. Namely if Z is some space and we have a commutative diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "\iota_0"'] & X \arrow[d, "f"] \\
Z \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & Y
\end{tikzcd}$$

Given a map $g : Z \to X$ and a homotopy $h : Z \times [0,1] \to Y$ so that $h(z, 0) = fg(z)$ then there exists a homotopy $\widetilde{h} : Z \times [0,1] \to X$ such that $H(z, 0) = g(z)$ and $fH(z,t) = h(z,t)$.

Proposition IV.0.5

If $f : X \to Y$ is a based fibration then $F(f) \simeq f^{-1}(*)$.

Homework # 6

(3) Prove Proposition IV.0.5. For hints, consider that

$$f^{-1}(*) \xrightarrow{\alpha} F(f)$$
$$x \mapsto (x, \mathrm{const}_*).$$

If f is a fibration, how do we go $F(f) \to f^{-1}(*)$. Well consider that we have a commutative diagram

$$\begin{tikzcd} F(f) \arrow[r, "p"] \arrow[d, hook, "\iota_0"'] & X \arrow[d, "f"] \\ F(f) \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & Y \end{tikzcd}$$

$$((x,\omega),t) \longmapsto \omega(t)$$

Thus there is a lift $\widetilde{h} : F(f) \times [0,1] \to X$. Take $\gamma := \widetilde{h}_1$.

To show $\alpha\gamma \simeq: F(f) \to F(f)$, use the path only part of the way.

For $\gamma\alpha \simeq \mathrm{Id}$ we need a map $f^{-1}(*) \to f^{-1}(*)$. $\widetilde{h}$ will preserve the const_* downstairs. . . should allow us to construct the homotopy.

Definition IV.0.5

A map $f : X \to Y$ is called a fiber bundle provided that for all $y \in Y$ there is some open neighborhood U of y such that there is a diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "\cong"] \arrow[d, "f"'] & U \times F \arrow[d] \\ U \arrow[r, "\mathrm{Id}"'] & U \end{tikzcd}$$

Definition IV.0.6

Let Y be a space. A refinement of an open cover $\{U_i\}_{i\in I}$ is another open cover $\{V_j\}_{j\in J}$ such that for all$j$ there exists and i with $V_j \subseteq U_i$.

An open cover $\{U_i\}_{i\in I}$ is called locally finite if for all $x \in Y$ there is an open neighborhood W of x and a finite set $F \subseteq I$ such that if $i \in I \setminus F$ then $U_i \cap W = \emptyset$.

A space is paracompact if every open cover has a locally finite refinement.

Example IV.0.1

Almost all nice spaces are paracompact

- All manifolds are paracompact.
- All CW-complexes are paracompact.

Theorem IV.0.6

If $f : X \to Y$ is a fiber bundle and Y is paracompact then f is a fibration.

See May's book for a proof [6].

Example IV.0.2

A covering is a fiber bundle whose fiber F is discrete. This implies that it satisfies the homotopy lifting property *with uniqueness.*

Let $f : X \to Y$ be a based covering with Y paracompact. Then $F(f) \simeq \underbrace{f^{-1}(*)}_{\text{discrete}}$. Thus $\pi_m f^{-1}(* = 0$ for $m > 0$.

Now the long exact sequence gives us that for $m \geq 2$

$$0 = \pi_m(F) \longrightarrow \pi_m(X) \xrightarrow{\cong} \pi_m(Y) \longrightarrow \pi_{m-1}(F) = 0$$

Therefore $\pi_m f$ is an isomorphism for $m \geq 2$.

This shows that $\pi_m(S^1) = 0$ for $m \geq 2$ because the universal cover of S^1 is $\mathbb{R} \to S^1$ and $\mathbb{R} \simeq *$. Thus $S^1 \simeq B\mathbb{Z}$.

More generally, if the universal covering of a nice space X is contractible, then $\pi_m(X) = 0$ for $m \geq 2$. We would then call X "hyperbolic" in the most general sense.

All surfaces are hyperbolic except $S^2, \mathbb{RP}^2$. Caution: geometers would not consider the torus hyperbolic, but it does satisfy this property. This implies that all surfaces except $S^2, \mathbb{RP}^2$ have that $X \simeq B\pi_1 X$.

We also know that $\pi_m(\mathbb{RP}^\infty) = 0$ for $m > 1$ because the universal cover of $\mathbb{RP}^\infty$ is $S^\infty \simeq *$. This shows that $\mathbb{RP}^\infty$ is a $B\mathbb{Z}/2$.

Definition IV.0.7

Notice that $S^1 \to S^{2n+1} \xrightarrow{f} \mathbb{CP}^n$ is an action, where we view S^1 as the unit sphere in $\mathbb{C}$ acting by multiplication on the unit sphere in $\mathbb{C}^{m+1}$

$$\lambda \cdot (z_0, \ldots, z_m) = (\lambda z_0, \ldots, \lambda z_m).$$

Thus this is a fiber bundle, and hence a fibration.

The most striking case is $m = 1$, because then $\mathbb{CP}^1 \cong S^2$. as then we have $S^1 \to S^3 \xrightarrow{f} S^2$, which is called the <u>Hopf fibration</u>

Proposition IV.0.7

$\pi_m(S^3) \cong \pi_m(S^2)$ for $m \geq 3$.

Proof. We have the following long exact sequence for $m > 2$ from the Hopf fibration

$$0 = \pi_m(S^1) \longrightarrow \pi_m(S^3) \xrightarrow{\cong} \pi_m(S^2) \longrightarrow \pi_{m-1}(S^1) = 0$$

And so we're done.

This is most striking when $m = 3$, because then $\pi_3(S^3) \cong \mathbb{Z}$, so $\pi_3(S^2) = \mathbb{Z}$.

Actually, $\pi_{4m-1} S^{2m}$ is infinite, $\pi_m S^m = \mathbb{Z}$, and all other homotopy groups of spheres are finite.

This really clues us in to how complex homotopy groups are, as $\pi_m(S^n)$ can be nonzero even when $m > n$.

Now lets think about how to construct generalized cohomology theories. In the based version this is given by the axioms

$$X \longrightarrow Y \longrightarrow Cf$$

$$\widetilde{E}^m Cf \longrightarrow \widetilde{E}^m Y \longrightarrow \widetilde{E}^m X$$

is exact. And also $\widetilde{E}^{m+1}\Sigma X \cong \widetilde{E}^m X$ naturally. We could also require the axiom that

$$\widetilde{E}^m \bigvee_i X_i \cong \prod_{i\in I} \widetilde{E}^m X_i$$

Suppose I give you based spaces Z_m, $m \in \mathbb{Z}$, such that $Z_m \simeq \Omega Z_{m+1}$. Then we can define $\widetilde{E}^m X = [X, Z_m]$. Then of course

$$\widetilde{E}^{m+1}\Sigma X = [\Sigma X, Z_{m+1}] = [X, \Omega Z_{m+1}] = [X, Z_m] = \widetilde{E}^m X.$$

We already proved exactness, and the product formula also holds. It turns out that every generalized cohomology theory is obtained this way.

Definition IV.0.8

The mapping cocylinder of a map $f : X \to Y$ is

$$Nf := \{(x, \omega) \mid x \in X,\ \omega : [0,1] \to Y,\ \omega(0) = f(x)\}$$

The projection $(x, \omega) \mapsto x$ is a homotopy equivalence $N(f) \simeq X$. Furthermore $(x,\omega) \mapsto \omega(1)$ is a fibration. This leads to a way to replace maps by fibrations

$$\begin{tikzcd} Cf \arrow[r] & Nf \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & Y \arrow[d, "\mathrm{Id}"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

The dual version with $f : Y \to X$ and the mapping cylinder $Mf = (Y \times [0,1]) \coprod X/(y,0) \sim f(y)$ we have $y \mapsto (y, 1)$ is a cofibration.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\simeq"] & Mf \arrow[r] \arrow[d] & Cf \\ Y \arrow[r, "f"'] & X & \end{tikzcd}$$

Theorem IV.0.8 (Serre Spectral Sequence)

Let $F \to X \xrightarrow{f} Y$ be a fibration with $\pi_0 Y, \pi_1 Y = 0$. Then there is a spectral sequence

$$E^2_{pq} = H_p(Y, H_q(F; A)) \Rightarrow H_{p+q}(X; A)$$

(Y general also works. Have to use homology with local coefficients).

Recall that

$$\mathrm{d}^r : E^r_{pq} \to E^r_{p-r,q+r-1} E^{r+1} \qquad = H(E^r, \mathrm{d}^r)$$

$$E^\infty_{pq} = \operatorname{colim} E^r_{pq}$$

There is an exhaustive filtration $F_{-1} = 0$, $F_p H_m(X; A)$ with $\bigcup F_p H_m(X; A) = H_m(X; A)$.

We don't have to worry about convergence because the serre spectral sequence exists entirely in the first quadrant. Recall also that

$$F_p H_{p+q} X / F_{p-1} H_{p+q}(X) = E^\infty_{pq}$$

Also there is a cohomological spectral sequence:

$$E_2^{pq} = H^p(Y; H^q(F; A)) \Rightarrow H^{p+q}(X; A)$$

In this case we have

$$\mathrm{d}_r : E_r^{pq} \to E_r^{p+r,q-r+1}$$
$$E_{r+1} = H(E_r, \mathrm{d}_r)$$
$$F_0 H^m(X; A) = H^m(X; A)$$
$$F_0 \supseteq F_1 \supseteq \cdots$$

And for $N \gg 0$ we have $F_N H^m(X; A) = 0$. We then also have

$$E_\infty^{pq} = \operatorname{colim} E_r^{pq} = F^p H^{p+q}(X; A)/F^{p+1} H^{p+q}(X; A).$$

If A is a commutative ring, then E_r is a spectral sequence of rings. That is E_r are bigraded rings, graded-commutative with respect to the total degree $p+q$ and d_r satisfies

$$\mathrm{d}_r(xy) = (\mathrm{d}_r x) \cdot y + (-1)^{|x|} x \cdot \mathrm{d}_r(y)$$

Where $|x|$ is the total degree of x.

This is in Serre's thesis (paraphrased less rigorously in Spanier's book).

Homework #7

(1) Calculate completely the homological and cohomological Serre spectral sequence of the Hopf fibration $S^1 \to S^3 \to S^2$ with coefficients in $\mathbb{Z}$. (Note this is unreduced, so we have to worry about degree zero).

…back to generalized cohomology. In the based version for $X \xrightarrow{f} \xrightarrow{i} Cf$ we have

(1) $\widetilde{E}^m Cf \xrightarrow{i^*} \widetilde{E}^m Y \xrightarrow{f^*} \widetilde{E}^m X$ is exact

(2) $\widetilde{E}^m X \cong \widetilde{E}^{m+1} \Sigma X$.

$*$ $\widetilde{E}^m \bigvee_i X_i \cong \prod_i \widetilde{E}^m(X_i)$

If I have a sequence of based spaces Z_m, $m \in \mathbb{Z}$ and $Z_m \xrightarrow{\simeq} \Omega$ then I can put $\widetilde{E}^m X := [X, Z_m]$. It turns out (in a proper setting) to be an if and only if, every generalized cohomology (satisfying $3*$) is given like this.

…Something is being swept under the rug. Namely $C_{\text{based}} f$ may not be homotopy equivalent to $C_{\text{unbased}} f$ and $\Sigma X \simeq SX$ might not hold. We should require that $* \hookrightarrow X, Y$ are cofibrations for the unbased and based constructions to agree.

Example IV.0.3

"Ordinary" cohomology = singular cohomology with coefficients in A (abelian group). What are the spaces Z_m? Well test it for $X = S^k$. Then

$$\pi_k Z_m = [S^k, Z_m] = \widetilde{H}^m(S^k; A) = \begin{cases} A & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$

This is called an <u>Eilenberg-MacLane space</u>, $K(A, m)$ has $\pi_m(K(A, m)) = A$ and $\pi_n(K(A, m))$ whenever $n \neq m$. (for $m = 1$, $K(A, 1) \simeq BA$)

There are loose ends to tie up

(1) Are the spaces $K(A, m)$ unique up to homotopy equivalence?

(2) Do we automatically have $K(A, m-1) \simeq \Omega K(A, m)$?

(3) For what (based) spaces X is $\widetilde{H}^m(X; A)$ determined just by the fact that $Z_m = K(A, m)$?

An answer: For X a CW-complex $\widetilde{H}^m(X;A)$ is determined. Namely $X_m/X_{m-1} = \bigvee_{I_m} S^m$, and we know the cohomology of this. Functoriality hands us $C^*_{\text{cell}}(X;A)$, telling us $H^m_{\text{cell}}(X;A) = H^m(X;A)$.

The moral of the story:

- Develop a notion of "equivalence of spaces" using homotopy groups.
- Approximate spaces by CW-complexes with respect to this equivalence
- An equivalence of CW-complexes is a homotopy equivalence.

Next time: Another (first "nontrivial") example of generalized cohomology, K-theory. Based on

$$U(m) = m \times n \text{ complex matrices whose columns form an orthonormal basis of } \mathbb{C}^m.$$

Then $U(m) \subseteq U(m+1)$ with

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

Then $U := \bigcup_m U(m)$ with the union topology.

Theorem IV.0.9 (Bott)

Bott periodicity tells us that $\Omega^2 U \simeq U$.

The corresponding generalized cohomology theory for $Z_{2m+1} := U$, $Z_{2m} := \Omega U$ is called (topological complex) K-theory.

V. Bott Periodicity

Let M be a Riemannian manifold, that is a smooth manifold M equipped with an inner product TM_x for each $x \in M$ which is smoothly varying in x.

Variational problem in a Riemannian manifold: Find the shortest path connecting two points (length of a path $\int \mathrm{d}s$)

Local Solution: second order differential equation = the geodesic equation. The solutions are called geodesics.

They are straight lines in $\mathbb{R}^m$, great circles on S^m (embedded as a unit sphere in $\mathbb{R}^{m+1}$).

Note: a geodesic may not actually be the shortest path between two points. For example walking around the "bad" portion of a great circle is still a geodesic.

Definition V.0.1

A (compact) symmetric space is a compact connected Riemannian manifold such that for each $P \in M$ there exists an isometry $\iota_P : M \to M$ such that $\iota_P(P) = P$ and $D\iota_P$ is -1 on TM_p.

Example V.0.1

S^m is a symmetric space. Take $(1, 0, \dots, 0) \in S^m$ then map

$$(x_0, x_1, \dots, x_m) \mapsto (x_0, -x_1, \dots, -x_m)$$

It turns out that this implies isometries act transitively on M and so M is a homogeneous space G/H (for G a compact Lie group and H a closed subgroup). And in fact, H is the fixed points for a certain kind of involution (Cartan involutions).

In fact a connected compact Lie group is a symmetric space. $\iota_1 = (g \mapsto g^{-1})$.

This is classified by Cartan

Definition V.0.2

Let M be a compact connected Riemannian manifold. Let $P, Q \in M$ and h be a homotopy class of paths between P, Q (rel boundary). Call $\nu = (P, Q, h)$.

The space of all paths from P to Q homotopic to h is homotopy equivalent to $(\Omega M)_0$ (a connected component of ΩM).

Consider $M^\nu \subseteq (\Omega M)_0$ be the subspace of the shortest geodesics (parameterized by scaled arc length).

Theorem V.0.1 (Bott [2])

If M is a symmetric space and $\nu = (P, Q, h)$, then M^ν is also a compact connected space. Furthermore $M^\nu \hookrightarrow (\Omega M)_0$ induces an isomorphism in π_i for $0 \le i < \alpha$, an onto map on π_α, where α is a certain number called the index.

The index α is defined as follows. If I have a geodesic h connecting P, Q then we can slightly deform this geodesic into a "nearby geodesic" satisfying the Jacobi equations. These can then cross at some point along h. We then say

$$\alpha_h = \sum_{R \in (P,Q)} \dim(\text{space of nearby geodesics crossing at } R)$$
$$\alpha = \min_h \alpha_h$$

For more information about this geometry see [4].

This implies

Theorem V.0.2 (Freudenthal Suspension)

$S^{m-1} \hookrightarrow \Omega S^m$ for $m > 1$ induces an isomorphism in homotopy groups π_k for $k \le 2m-2$ and it is onto in degree $k = 2m-1$.

This shows that $\pi_k S^m \to \pi_{k+1} S^{m+1} \to \pi_{k+2} S^{m+2} \to \cdots$, then eventually these are isomorphisms. These are stable homotopy groups of spheres. Namely this is $\pi_{k+m} S^m$ for $m \gg 0$ and k fixed.

Homework #7

(2) Prove that for $m \ge 1$, $\pi_1(U(m)) \cong \mathbb{Z}$. Recall that $U(m)$ is unitary (can think of as complex linear maps which are also isometries) $m \times m$ (complex) matrices. Note that $U(m)$ acts on the unit sphere S^{2m-1} in $\mathbb{C}^m$ transitively.

The isotropy group is $U(m-1)$. Therefore we have a fibration sequence

$$U(m-1) \to U(m) \to S^{2m-1}.$$

use LES in homotopy groups for induction. $U(1) = S^1$, so we know the statement for that case.

VI. Whitehead's Theorem and CW approximation

Definition VI.0.1

A map $f : X \to Y$ is called an m-equivalence if $\pi_0 f : \pi_0 X \to \pi_0 Y$ is onto and for all $x \in X$, $\pi_k f : \pi_k(X, x) \to \pi_k(Y, f(x))$ is

(a) An isomorphism for $k < m$

(b) Onto for $k = m$.

A weak equivalence (or equivalence) is a map $f : X \to Y$ which is an m-equivalence for all m.

From now on $[Z, X] = \mathrm{Mor}_{\mathrm{hTop}}(Z, X)$ (unbased).

Theorem VI.0.1 (Whitehead's Theorem)

This is a two-parter!

(1) Let $f : X \to Y$ be an m-equivalence (resp. weak equivalence). Then $[Z, f] : [Z, X] \to [Z, Y]$ is a bijection when Z is a CW-complex with dimension $< m$ and onto when Z is a CW-complex of dimension m (resp. bijective for every CW-complex Z).

(2) For every space X there exists an m-equivalence $\gamma_X^m : Z_m \to X$ where Z_m is a CW-complex of dimension $\leq m$ (resp. a weak equivalence $\gamma : Z \to X$ where Z is a CW complex).

Back to Bott's Theorem

Let M be a compact Riemannian manifold (connected). $P, Q \in M$ points, h a homotopy class of paths from P to Q, $\nu = (P, Q, h)$.

$$M^\nu = \{\text{all shortest (by arc length) geodesics from } P \text{ to } Q \text{ path-homotopic to } h\}$$

The index α: The minimum index α_k of geodesics path-homotopic to h where $\alpha_k > 0$.

The index of a geodesic α_k is the sum over points R interior to h of the dimension of the space of nearby geodesics beginning at the same starting point and also coinciding in R.

M^ν includes into the space of paths from P to Q in M, which is homotopy equivalent to $(\Omega M)_0$.

Theorem VI.0.2 (Bott)

If M is a compact symmetric space and $\nu = (P, Q, h)$ as above then M^ν is a compact symmetric space and

$$\iota_\nu : M^\nu \to (\Omega M)_0$$

is an $(\alpha - 1)$-equivalence.

Example VI.0.1

$M = S^n$, then $M^\nu \simeq S^{n-1}$ as the shortest geodesics are the meridians. Then $\alpha_k = 2(m-1)$. Thus $S^{m-1} \to \Omega S^m$ is a $[2(m-1) - 1]$-equivalence.

General principle of compact symmetric spaces M (contains connected compact Lie groups). Take a closed geodesic at P. Then $M = G/H$ where G is the group of isometries and $H = \mathrm{Iso}(P)$. Then $M^\nu = H/\mathrm{Iso}(P, Q)$, where Q is the opposite point along the geodesic.

Example VI.0.2

Complex Bott periodicity. Take $M = U(2m)$, P to be the identity, and $Q = -P$. Then take h to be e^{ix} along the diagonal. Then

$$M^\nu = U(2m)/(U(m) \times U(m))$$

Index is $2m + 2 \to \infty$.

Thus $U/(U \times U) \to (\Omega U)_0$ is a weak equivalence.

$U \to U/(U \times \{e\}) \to U/(U \times U)$ is a fibration sequence. Therefore $U \simeq \Omega(U/(U \times U))$.

We can write $BU := U/(U \times U)$. Bott's theorems say $BU \xrightarrow{\sim} (\Omega U)_0$.

Likewise $BU \times \mathbb{Z} \xrightarrow{\sim} \Omega U$. This means that $\Omega(BU \times \mathbb{Z}) \simeq U$.

Thus $\Omega^2 U \simeq U$.

Complex K-theory is then

$$K^m(X) = [X, Z_m]$$

Where Z_m is defined by

$$Z_m := \begin{cases} BU \times \mathbb{Z} & \text{if } m \text{ even} \\ U & \text{if } m \text{ odd} \end{cases}$$

What happens if we replace $U(m)$ by $O(m)$? Well then it becomes a mit more complicated. Let $BO = O/(O \times O)$, and note that $U(m) \subseteq O(2m)$, and also that if $\mathrm{Sp}(m)$ is the $m \times m$ unitary quaternion matrices then $\mathrm{Sp}(m) \subseteq U(2m)$.

n	$\Omega^n(BO \times \mathbb{Z})$	Z_{-}
$8m$	$BO \times \mathbb{Z}$	Z_{8m}
$8m+1$	O	Z_{8m-1}
$8m+2$	O/U	Z_{8m-2}
$8m+3$	U/Sp	Z_{8m-3}
$8m+4$	$B\,\mathrm{Sp} \times \mathbb{Z}$	Z_{8m-4}
$8m+5$	Sp	Z_{8m-5}
$8m+6$	Sp/U	Z_{8m-6}
$8m+7$	U/O	Z_{8m-7}
$8(m+1)$	$BO \times \mathbb{Z}$	$Z_{8(m-1)}$.

For X a CW-comples, $\mathrm{KO}^m X := [X, Z_m]$.

Homework #7

(3) Calculate $\mathrm{KO}^m(*)$, $\pi_m(BO \times \mathbb{Z})$, $m \in \mathbb{Z}$.
Use that
- $O(m)$ has two path-connected components
- $U(m), \mathrm{Sp}(m)$ are path-connected
- $O(m-1) \to O(m) \to S^{m-1}$ is an action (fibration sequence) in $\mathbb{R}^m$.
- $\mathrm{Sp}(m-1) \to \mathrm{Sp}(m) \to S^{4m-1}$ is an action (fibration sequence) in $\mathbb{H}^m$.

(4) Prove that if $f : X \to Y$ is a weak equivalence and X, Y are CW-complexes then f is a homotopy equivalence. (use Whitehead's Theorem)

Homework due Wednesday October 20th at 8PM.

Recall Theorem VI.0.1, specifically the statement that every space X is m-equivalent to a CW-complex Z_m of dimension $\leq m$, and weakly equivalent to a CW-complex Z. Z_m is sometimes called a formal m-skeleton of X.

Remark VI.0.1

If $f : X \to Y$ is a weak equivalence (X, Y are any spaces) then f induces an $\cong$ in singular homology. $H_n f : H_n X \to H_n Y$.

Proof Sketch. Express singular homology in terms of maps from CW-complexes. Consider a singular cycle $c = \sum_k a_k \sigma_k$, $\sigma_k : \Delta^m \to X$. We can construct a CW-complex Z by taking $\coprod_K \Delta^m / \sim$, which is the minimal equivalence relation making it into a cycle (identify $(m-1)$-faces on which σ_k, σ_ℓ restrict to the same singular $(m-1)$-simplex).

c lifts to a singular cycle on Z. To show $H_n f$ is onto, let $c \in C_m Y$ be a cycle representing a class in $H_m Y$. We constructed a CW-complex (of dimension m) Z, $Z \to Y$ so that $c' \mapsto c$.

We can then lift $Z \to Y$ up to homotopy to a map $Z \to X$ using Whitehead's theorem. Thus we constructed a lift of $[c] \in H_m Y$ to $H_m X$ under f.

The argument for boundaries to show injectivity is analogous.

We add an axiom to generalized cohomology: $E_m f$ (resp. $E^m f$) is an isomorphism when f is a weak equivalence.

From the point of view of representing generalized cohomology by homotopy classes of maps into some based spaces: We need a sequence of based spaces Z_m with a based weak equivalence $Z_m \to \Omega Z_{m+1}$.

For a CW-complex X, $E^m(X) = [X, Z_m]$ (unbased).

For a general space X, find a weak equivalence $\gamma : X' \to X$ and define $E^m(X) := [X', Z_m]$. Then $E^m f$ is an isomoprhism when f is a weak equivalence.

How to prove the approximation statement from Theorem VI.0.1 from the first statement?

Proof. We do this by induction. The base case is to take $Z_0 \to X$, where Z_0 is the discrete set of path-components of X. This is of course onto in π_0.

Suppose we have an n-dimensional CW-complex Z_n and an n-equivalence $\gamma^m : Z_m \to X$. This is an isomorphism on π_i, $i < n$, and onto on π_n. γ^n may not be $\cong$ on π_n. There may be classes $\alpha_i : S^n \to Z_n$ so that $\gamma^n \circ \alpha_i \simeq *$.

We can just glue disks along each of these to fix the issue. Also γ^n may not be onto on π_{n+1}. To fix this if $\beta_j : S^{n+1} \to X$ is not represneted then

$$Z_{n+1} = Z_n \sqcup \coprod_i D^{n+1} \sqcup \coprod_j S^{n+1} / \sim$$

Where $\sim$ attaches D^{n+1} via α_i and S^{n+1} via their base point in Z_0.

By definition we get a map $\gamma^{n+1} : Z_{n+1} \to X$. This satisfies the inductive step because

- $\cong$ in π_i for $i < n$ comes from cellular approximation of maps, because we can approximate $S^n \to Z_{n+1}$ via maps $S^n \to Z_n$.
- For the same reason, it is onto on π_n. It is then injective on π_n by the gluings made above, as we killed all the relations.
- It is onto on π_{n+1} by construction.

We're done! For the infinite case set $Z = \bigcup_i Z_i$.

Notice: Say X is path-connected. Say $\pi_i(X) = 0$ for $i < m$ (we say X is $(m-1)$-connected). 1-connected means $\pi_1(X) = 0$, that is X is simply connected.

Then we can set $Z_{m-1} = *$. Furthermore, Z_m is a bouquet of spheres over generators of $\pi_m X$. Z_{m+1} is a bouqeut of spheres over generators of $\pi_m X$, and $\pi_{m+1} X$, and then we attach m-disks along relations in $\pi_m X$.

Definition VI.0.2 (Hurewicz Homomorphism)

$\pi_k X \to H_k(X; \mathbb{Z})$. This is given by taking some $\alpha : S^k \to X$ and mapping

$$H_k(S^k; \mathbb{Z}) \xrightarrow{H_k \alpha} H_k(X; \mathbb{Z})$$
$$1 \mapsto h(\alpha)$$

Computing cell homology, we get

Theorem VI.0.3 (The Hurewicz Theorem)
If X is $(m-1)$-connected, then the Hurewicz homomorphism $h : \pi_m X \to H_m(X;\mathbb{Z})$ is the abelianization if $m = 1$, and an isomorphism if $m > 1$.

Proof. Our construction of Z_{m+1} above makes this clear.

Homework # 8

(1) Compute $\pi_2(S^1 \vee S^2)$. Use universal cover and Hurewicz theorem.

It is also easy to construct by the methods above, a Cw-complex $K(\Pi, m)$, Π a group (abelian if $m > 1$) such that

$$\pi_i K(\Pi, m) = \begin{cases} \Pi & \text{if } i = m \\ 0 & \text{otherwise} \end{cases} .$$

We can construct Z_{m+1} by the above method (generators and defining relations of π). Then just keep attaching cells to kill all higher homotopy groups.

Same method implies that any two such CW-complexes $K(\Pi, m)$ are homotopy equivalent (use Whitehead Theorem).

We even get a weak equivalence $K(\Pi, m-1) \xrightarrow{\sim} \Omega K(\Pi, m)$. This way we can construct singular cohomology out of the Whitehead theorem. Namely this gives $[X, K(\Pi, m)] \to H^m(X; \Pi)$.

How do we do this for homology? Duality! We'll get there soon.

Homework #8

(2) The Quillen + construction. Let X be a connected CW-complex. Let $\pi_1(X, x) = G$, $x \in X_0$. Let $H \subseteq G$ be a subgroup such that $[H, H] = H$, that is $H^{ab} = 0$.

Attach a 2-cell e_k to each element $h \in H$ to form a CW-complex Y. Note (Y, X) is a CW-pair, and Y is connected. Let $p : \widetilde{Y} \to Y$ be the universal cover. Let $\widetilde{X} = p^{-1}(X)$. Choose a lift $\widetilde{e}_k$ of each cell e_k.

Then $\widetilde{e}_k$ represents a class of $\alpha_h \in H_2(\widetilde{Y}, \widetilde{X})$.

(a) Prove that α_h lifts to a class $\overline{\alpha}_h \in H_2(\widetilde{Y})$ (use the fact that the abelianization of H is zero, the long exact sequence in homology, and Hurewicz).

(b) Observe that $\overline{\alpha}_h$ is in the image by the Hurewicz map of an element $u_h \in \pi_2(\widetilde{Y}, \widetilde{x})$ where $\widetilde{x} \in \widetilde{X}$. Form a CW-complex X^+ by attaching a 3-cell to Y along each $p \circ u_h : S^2 \to Y$.

(c) Prove that the inclusion $X \to X^+$ induces an isomorphism in homology (use cellular homology, the additional attached cells cancel out).

(d) $\pi_1(X^+, x) = G/\overline{H}$ where $\overline{H}$ is generated by all $g^{-1}hg$, $g \in G$, $h \in H$ (aka the smallest normal subgroup of G containing H).

Example VI.0.3
Say $G = H = A_n$ for $n \geq 5$. Then $[A_n, A_n] = A_n$ so we can form the plus construction.
Then $BA_n \xrightarrow{i} BA_n^+$, and by this $H_k BA_n \xrightarrow{i_*} H_k BA_n^+$. Then $\pi_1 BA_n^+ = 0$.
Thus homology is not an adequate measure of equivalence of spaces (does not imply weak equivalence).

The reason Quillen invented this was to define higher algebraic K-theory of commutative rings. If R is a commutative ring, put $\mathrm{GL}_\infty R = \bigcup_{n \geq 0} \mathrm{GL}_n R$. The analogy with K-theory (imperfect), note $U(m) \subsetneq \mathrm{GL}_m \mathbb{C}$ is a homotopy equivalence by the Gram-Schmidt process. Then $\mathrm{GL}_\infty \mathbb{C} \simeq U$.

But we're cheating, $\operatorname{GL}_\infty \mathbb{C}$ has topology from $\mathbb{C}$. For R general it is considered discrete. If we consider $H = [\operatorname{GL}_\infty R, \operatorname{GL}_\infty R] \subseteq \operatorname{GL}_\infty R = G$ then

Theorem VI.0.4 (Steinberg)

$[H, H] = H$.

Quillen: Take $K_m R = \pi_m(B \operatorname{GL}_\infty R^+)$ (with respect to H), for $m > 0$. Then set $K_0 R$ to be the Grothendieck group of isomorphism classes of finitely generated projective R-modules. People knew earlier that $K_1 R = \operatorname{GL}_\infty R / [\operatorname{GL}_\infty R, \operatorname{GL}_\infty R]$, and there was a natural geometry to this. But people could not do it purely algebraically, and instead were able to do it with homotopy groups.

Definition VI.0.3

HELP (Homotopy Extension and Lifting Property). A map $f : X \to Y$ satisfies HELP with respect to a pair (Z, A) if the following diagram completes

$$\begin{tikzcd}
A \arrow[rr, "0"] \arrow[dd, hook] & & A \times [0,1] \arrow[dl, "h"'] \arrow[dd, hook] & & A \arrow[ll, "0"'] \arrow[dl, "g"'] \arrow[dd, hook] \\
& Y & & X \arrow[ll, "e"] & \\
Z \arrow[ur, "f"] \arrow[rr, "0"'] & & Z \times [0,1] \arrow[ul, dashed, "\widetilde{h}"] & & Z \arrow[ll, "1"] \arrow[ul, dashed, "\widetilde{g}"']
\end{tikzcd}$$

Include garbage can intuition

Lemma VI.0.5 (The HELP Lemma)

If $\pi_{m-1}(e)$ is injective and $\pi_m(e)$ is onto, then e satisfies HELP for the pair $(Z, A) = (D^m, S^{m-1})$.

Proof Sketch. Put a lid on first (injectivity property). If the garbage can does not fill, move the lid (onto property). Be careful when $m = 1$. More detail is in [6].

Lemma VI.0.6 (The HELP Lemma 2)

If $e : X \to Y$ is an m-equivalence (resp. weak equivalence), then it satisfies HELP with respect to CW-pairs of dimension $\leq n$ (resp. all CW-pairs)

(Induction on cells).

Proof of Whitehead's Theorem. We now prove Theorem VI.0.1. We will use HELP. Let $e : X \to Y$ (changed notation, permuted) be an m-equivalence (or weak equivalence). We wish to study the map $[Z, e] : [Z, X] \to [Z, Y]$ for a CW-complex Z.

For surjectivity, apply HELP to the pair $(Z, \emptyset)$. For injectivity apply HELP to $(Z \times [0,1], Z \times \{0,1\})$.

Next Time: This applies in more general settings than spaces. In particular, we can use it on chain complexes for derived functors and derived categories. We also really want to use it for spectra, which will allow us to understand duality.

VI.1. Derived Categories

Definition VI.1.1

Setup: C is a category $E \subseteq C$ is a subcategory (we call the morphisms of E equivalences and write $f : X \xrightarrow{\sim} Y$). We assume "2 out of three property" that for $f : X \to Y$, $g : Y \to Z$ that if two out of f, g, gf are equivalences then so is the third. Also we assume that all isomorphisms belong to E.

A derived category (if one exists) with respect to this data is a category DC together with a functor $\Phi : C \to DC$ which is universal among functors $F : C \to Q$ which take all morphisms in E into isomorphisms.

Strict version: $\Phi(e)$ is an isomoprhism for all $e \in \mathrm{Mor}(E)$. For all $F : C \to Q$ satisfying $F(e)$ is an isomoprhism for all $e \in \mathrm{Mor}(E)$ there exists a uique $G : DC \to Q$ with $G\Phi = F$.

Lax version: For all $e \in \mathrm{Mor}(E)$, $\Phi(e)$ is an isomorphism. For all $F : C \to Q$ satisfying this same property there exists a $G : DC \to Q$ togehter with a natural isomorphism $\kappa : G \circ \Phi \to F$. For any other functor $H : DC \to Q$ together with a natural isomorphism $\lambda : H \circ \Phi \to F$ there exists a unique $\mu : H \to G$ with $\lambda = \kappa \circ (\mu\Phi)$.

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "\Phi"'] & {} & Q \\ & DC \arrow[u, phantom, "\Uparrow\kappa"] \arrow[ur, dashed, "G"'] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "\Phi"'] & {} & Q \\ & DC \arrow[u, phantom, "\kappa\Uparrow\ \Uparrow\lambda"] \arrow[ur, dashed, "G"] \arrow[ur, dashed, bend right, "H"'] \arrow[ur, phantom, "\Uparrow\mu"] & \end{tikzcd}$$

Observation: The two definitions are equivalent if $\Phi : C \to DC$ is bijective on objects.

Concrete construction of derived categories:

where we have an analog of the Whitehead Theorem (or its dual) (given above definition)

An object $Z \in \mathrm{Ob}(C)$ is called co-local if for every $e : X \xrightarrow{\sim} Y$ we have

$$\mathrm{Mor}_C(Z, X) \xrightarrow[\cong]{\mathrm{Mor}_C(Z,e)} \mathrm{Mor}_C(Z, Y)$$

is a bijection. For example, CW-complexes are co-local. We say that we have co-localization if for every $X \in \mathrm{Ob}(C)$ there exists X' which is co-local and an equivalence $X' \xrightarrow{\sim} X$. If always we have $X' \in B$ for some $B \subseteq \mathrm{Ob}(C)$ (with every object of B colocal), we say this is colocalization by B.

This situation is exactly the content of Whitehead's theorem in hTop.

Theorem VI.1.1

If C, E are as above and we have a colocalization by $B \subseteq \mathrm{Ob}(C)$ then the derived category DC exists and it is equivalence to the full subcategory of C on B.

Proof. $\Phi : C \to DC$ given by $X \mapsto X'$. This is functorial because for $f : X \to Y$ we can use colocality of X' to get a unique map $X' \to Y'$.

The other checks are similarly trivial.

Definition VI.1.2

A cell complex is a space $X = \bigcup X_{(m)}$

$\emptyset = X_{(-1)} \subseteq X_{(0)} \subseteq \cdots$.

$X_{(m)}$ is obtained from$X_{(m-1)}$ by attaching cell in any dimension $J_m, d_n : J_m \to \mathbb{N}_0$ with

$$f_m : \coprod_{j \in J_m} S^{d_j - 1} \to X_{(m)-1}$$

$X_{(m)}$ is the pushout

$$\begin{tikzcd} \coprod_{j \in J_m} S^{d_j-1} \arrow[r, "f_m"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X_{(m-1)} \arrow[d] \\ \coprod_{j \in J_m} D^{d_j} \arrow[r] & X_{(m)} \end{tikzcd}$$

Observe that cell complexes also satisfy the Whitehead Theorem.

Homework #9

(1) Prove that every cell complex is homotopically equivalent to a CW-complex.

Let A be an abelian category with coproducts and enough projectives

For all $X \in \mathrm{Ob}(A)$ there exists a projective P and an epimorphism $P \twoheadrightarrow X$.

We can look at h-A-Chain. We can define a cell chain complex

$$0 = X_{(-1)} \subseteq X_{(0)} \subseteq \cdots$$

We say $P_{(m)}$ is a projective chain complex with zero differentials.

We then can take

$$CP_{(m)} \qquad P_{(m)} \longrightarrow P_{(m)k} \oplus P_{(m)(k-1)} \longrightarrow P_{(m)(k-1)} \oplus P_{(m)(k-2)} \longrightarrow \cdots$$

$H_* CP_{(m)} = 0$.

We require $X_{(m)}$ is a pushout

$$\begin{tikzcd}
P_{(m)} \arrow[r, "f_m"] \arrow[d, hook] & X_{(m-1)} \arrow[d] \\
CP_{(m)} \arrow[r] & X_{(m)}
\end{tikzcd}$$

One can prove the Whitehead Theorem precisely analogously

The equivalences are quasiisomprhisms (chain maps which induce isomorphisms in homology),

Theorem VI.1.2

Cell chain complexes are colocal in h-A-Chain with respect to quasiisomorphisms and one has colocalization by cell chain complexes.

We define

$$DA := Dh\text{-A-Chain}$$

called the "derived category of the abelian category A."

If A is an abelian category, we denote

$$DA \simeq Dh\text{-A-Chain}$$

Equivalences are quasiisomorphisms (induce isomorphisms in homology).

Cell chain complexes are colocal and we have colocalization when there are coproducts and enough projectives. For objects $X, Y \in A$ we have

$$\mathrm{Ext}^m_A(X, Y) = \mathrm{Mor}_{DA}(X, Y[m])$$

Proof. If C is a projective resolution in degree 0, then C is cell (individual degrees = cells).

Then by definition

$$\mathrm{Ext}^m_A(X, Y) := \mathrm{Mor}_{h\text{-A-Chain}}(C, Y[m]) = H^m(\mathrm{Hom}(X, Y)) = \mathrm{Mor}_{DA}(X, Y[m]).$$

because C is colocal. ♥

What about Tor (not in every abelian category, must have $\otimes$ first).

VI.2. **Derived Functors**

The most general notion does not even involve derived categories, but instead just involves a functor $\Phi : C \to D$.

Given a functor $F : C \to Q$, may not factor through Φ. But is there a "universal" functor $D \to Q$ with respect to this data. Two ways

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\Phi"'] & Q \\ D \arrow[ur, "F'"'] \arrow[u, phantom, shift left=6, "\Leftarrow\eta"] & \end{tikzcd}$$

such that for every $G : D \to Q$ provided with a $\kappa : G\Phi \Rightarrow F$ we have a unique $\mu : G \to F'$ with

$$\kappa = \eta \circ \mu\Phi$$

that is

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\Phi"'] & Q \\ D \arrow[ur, "F'"'] \arrow[ur, bend right, "G"'] & \end{tikzcd}$$

This is called a right Kan extension, aka a left derived functor. Denoted by LF.

Example VI.2.1

Suppose $\Phi : C \to DC$, with $\gamma_X : X' \xrightarrow{\sim} X$ a colocalization. Then if $F : C \to Q$ then the (total) left derived functor exists and is defined by

$$LF(X) = F(X')$$

We have the morphism $F(X') \to F(X)$ via $F(\gamma_X)$. If $G : DC \to Q$ and $G\Phi \Rightarrow F$ then the map $G(X') \to F(X')$ is handed to us because $G(X') \cong G(X)$.

If I have a functor $? \otimes N$ then

$$\mathrm{Tor}^R_m(?, N) = L(H_m(? \otimes N)).$$

That is

$$\mathrm{Tor}^R_m(M, N) = H_m(C \otimes_R N)$$

where C is a projective resolution of M (colocal).

What about localization

Theorem VI.2.1

If an abelian category A has enough injectives and products then h-A-Chain has localization with respect to co-cell chain complexes. Turn around arrows in the definition of cell chain complexes, replace projective by injective.

Technical Issue: H_* does not commute with inverse limits of sequences. Say we have the following chain complexes

$$\cdots \to X_2 \to X_1 \to X_0$$

then $H_m \lim_k X_k$ is not in general isomorphic to $\lim_k H_m(X_k)$.

The symmetric statement for colimits holds. In general lim of a sequence has one right derived functor $\lim^1$. However, $\lim^1 = 0$ if we have the Mitag-Leffler condition in each

$$\cdots \longrightarrow X_1 \longrightarrow X_0$$

$$\cdots \longrightarrow H_m X_1 \longrightarrow H_m X_0$$

The Mitag-Leffler condition (in an abelian category) says that the composed images at each stage eventually become constant.

Important notes

- There are abelian categories which have enough injectives but not enough projectives
- If we have localization, we have right derived functors (defined symmetrically to left derived functors, instead a left Kan extension) e.g. (sheaf cohomology is to apply right derived functors to global sections).

Homework #9

(2) Prove that if $f : X \to Y$ induces an isomorphism in homology (coefficients in $\mathbb{Z}$) and X, Y are simply connected, then f is a weak equivalence.

(Consider the Serre spectral sequence in homology of the fiber sequence $Ff \to X \to Y$).

$$E^2_{pq} \coloneqq H_p(Y, H_q(Ff)) \Rightarrow H_{p+q}(X).$$

We have an increasing filtration F_p on $H_{p+q}(X)$, and $E^\infty_{pq} = F_p H_{p+q}(X)/F_{p-1}H_{p+q}(X)$.

If you think of this

$$\begin{tikzcd} H_p(X) = F_p H_p(X) \arrow[r] \arrow[rr, bend right, "H_p f"'] & E^\infty_{p,0} \arrow[r, hook] & E^2_{p,0} = H_p(Y) \end{tikzcd}$$

called the edge map. Deduce that $H_p Ff = 0$ for all $p > 0$. Then observe that $\pi_1(Ff)$ is abelian (long exact sequence on homotopy groups). Finally, deduce that $\pi_m(Ff) = 0$ for all m, and conclude that that f must be a weak equivalence via the long exact sequence on homotopy groups.

VI.3. **Localization in Topology**

There is an alternative approach to deriving $h\,\mathrm{Top}$. Essentially we mimic the formal structure on the singular set $S_m X = \{\Delta^m \to X\}$. These formal structures will be called simplicial sets.

To do this, we must determine what distinguished maps are there between the standard simplices Δ^m? Well, we have faces

$$\partial_i : \Delta^m \to \Delta^{m+1}$$
$$[t_0, \ldots, t_m] \mapsto [t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_m].$$

We can realize that this corresponds to

$$\{0, \ldots, m\} \to \{0, \ldots, m+1\}$$
$$j \mapsto j \qquad (j < i)$$
$$j \mapsto j + 1. \qquad (j \geq i)$$

Compositions correspond to order-preserving injections. There is also a map in the other direction.

$$\Delta^m \to \Delta^{m-1}$$
$$[t_0, \ldots, t_m] \mapsto [t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_m]$$

This corresponds to the map

$$\begin{aligned} \{0, \ldots, m\} &\to \{0, \ldots, m-1\} \\ j &\mapsto j && (j \le i) \\ j &\mapsto j - 1. && (j > i) \end{aligned}$$

These are called <u>degeneracies</u>, and are order-preserving surjections. Triangulation of objects with simplicial sets can be used to verify the homotopy axiom of homology.

> **Definition VI.3.1**
>
> We call Δ the simplicial category. The objects are $\mathbb{N}_0$, and we write $\underline{m} = \{0, \ldots, m\}$. The morphisms are non-strictly order preserving maps.

> **Definition VI.3.2**
>
> If C is a category, the category of simplicial objects in C ($\Delta^{\mathrm{op}} - C$) is the category of functors $\Delta^{\mathrm{op}} \to C$ and natural transformations.

We will talk about simplicial sets. Consider that $\overline{\Delta}(m) := \Delta^m$ is a functor $\Delta \to \mathrm{Top}$, called the topological realization.

The topological realization of a simplicial set $S : \Delta^{\mathrm{op}} \to \mathrm{Set}$ is left adjoint to the singular set functor.

$$|S| = \coprod_{m \in \mathbb{N}_0} S_m \times \Delta^n / (s, fx) \sim (Sf(s), x) \qquad (f : m \to_\Delta n)$$

Triangulation of prism says that

$$(S \times T)_m = S_m \times T_m$$
$$|S \times T| \cong |S| \times |S|$$

Simplicial sets generalize simplicial complexes, "are" CW-complexes (after calculation).

$\underline{\Delta^n} : k \in \mathrm{Ob}(\Delta) \to \mathrm{Mor}_\Delta(k, n)$ is a simplicial model of Δ^n. We then have that

$$\underline{\Delta}^0 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \underline{\Delta}^1$$

is a model for the unit interval.

> **Definition VI.3.3**
>
> A simplicial homotopy is a natural transformation $\underline{\Delta}^1 \times S \to T$.
>
> We call two morphisms $f, g : S \to T$ <u>simplicially homotopic</u> if they are equivalent in the smallest equivalence relation containing simplicial homotopy.

Then we have $h - \Delta^{\mathrm{op}}$-Set with objects simplcial sets and morphisms simplicial homotopy classes of Δ^{op}-morphisms.

This category has <u>localization</u> with respect to <u>Kan complexes</u>

Definition VI.3.4

A Kan complex S is a simplicial set satisfying the Kan condition. To phrase this, consider V_k^n in Δ^n, which is obtained by ommitting the open n-simplex and the k-th face.

In terms of simplicial sets this is

$$\underline{V}_k^n : j \mapsto \{f \in \mathrm{Hom}_\Delta(j, m) \mid \{0, \ldots, k-1, k+1, \ldots, m\} \not\subseteq \mathrm{im}\, f\}.$$

We have a natural injection $\underline{V}_k^m \hookrightarrow \underline{\Delta}^n$.

Then S satisfies the Kan condition provided that every morphism $f : \underline{V}_k^n \to S$ extends to a morphism $\overline{f} : \Delta^n \to S$

$$\begin{tikzcd} \underline{V}_k^n \arrow[r, "f"] \arrow[d, hook] & S \\ \Delta^n \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

We say that S is a minimal Kan complex if the extension $\overline{f}$ is unique.

Definition VI.3.5

An equivalence of simplicial sets is a morphism $f : S \to T$ such that $|f| : |S| \to |T|$ is a weak equivalence (homotopy equivalence since these are CW complexes).

Theorem VI.3.1

The simplicial realization induces an equivalence of categories

$$D - \Delta^{\mathrm{op}}\text{-Set} = Dh - \Delta^{\mathrm{op}}\text{-Set} \xrightarrow{\simeq} Dh\,\mathrm{Top} = D\,\mathrm{Top}\,.$$

The $=$ above comes from the fact that inverting equvalences identifies homotopic maps.

Addendum: Every Kan complex is simplicial homotopy equivalent to a unique (up to isomorphism) minimal Kan complex.

We conclude that

MINIMAL KAN COMPLEXES (up to $\cong$) ARE IN BIJECTION
WITH WEAK HOMOTOPY TYPES

where a weak homotopy type are homotopy classes of CW-complexes. Unfortunately, this is not a practical solution because minimal Kan complexes are extremely difficult to write down (try doing it for S^1, $\mathbb{RP}^\infty$, these are easier because they are $K(G, 1)$s).

Homework #9

(3) Prove that for every space X, the singular set $S_\bullet X$ satisfies the Kan condition.

Another topic is localization within $D\,\mathrm{Top}$. If E is some genrealized homology theory (preserving weak equivalence), then we can say that $f : X \to Y$ is an E-equivalence if $E_* f$ is an isomorphism.

Example VI.3.1

$E = H\mathbb{Z}$. Then

- $H\mathbb{Z}$-equivalence is not in general a weak equivalence.
- $H\mathbb{Z}$-equivalence is a weak equivalence if X, Y are simply connected.

Theorem VI.3.2 (Bousfield)

$D\,\mathrm{Top}$ has localization with respect to E-equivalence for any chosen generalized homology E.

Even if E is an ordinary homology theory, of interest is $E = H\mathbb{Q}$ localization is called rationalization. $H\mathbb{Z}/p$ is called p-completion, and $H\mathbb{Z}_{(p)}$ is called p-localization, where $\mathbb{Z}_{(p)} = \{k^{-1} \mid p \nmid k\}\mathbb{Z}$.

Say for simply connected spaces, this is fairly well understood. Lets consider $H\mathbb{Q}$.

Theorem VI.3.3

If $f : X \to Y$ induces an isomorphism in $H\mathbb{Q}$, and X, Y are simply connected. Then f induces an isomorphism $\pi_m X \otimes \mathbb{Q} \cong \pi_m Y \otimes \mathbb{Q}$.

Serre in 1953 provied this in the case $\pi_2 f$ is onto (Annals of Math). He then wrote

"Nous insisterons par la-desrus" $\leftrightarrow$ "We shall not insist on it"

Definition VI.3.6

A simplicial set X which satisfies the <u>inner Kan condition</u>, which says any

$$\begin{tikzcd} \underline{V}_k^n \arrow[r] \arrow[d, hook] & X \\ \underline{\Delta}^n \arrow[ur, dashed] & \end{tikzcd}$$

for $0 < k < m$, then X is called a <u>quasicategory</u>.

Lurie uses quasicategory interchangeably with ∞-category, which is a vague term meaning many different but roughly equivalent things.

The homotopical information in an ∞-category is the same as a category where the sets of morphisms are given a topology and composition is continuous (topological category).

Back to Serre and Hurewicz!

Definition VI.3.7

X is a <u>simple space</u> provided that X is path-connected which means that $\pi_1(X)$ is abelian and $\pi_1(X)$ acts trivially on every $\pi_m(X)$.

Past homework showed that $S^1 \vee S^2$ is not simple.

Theorem VI.3.4 (The Relative Hurewicz Theorem)

If X is a simple space, then if $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i < m$, then the Hurewicz map $\pi_m(X) \to H_m(X; \mathbb{Z})$ becomes an isomorphism upon tensoring with $\mathbb{Q}$ to get $\pi_m(X) \otimes \mathbb{Q} to H_m(X; \mathbb{Q})$.

Proof Sketch. By induction, using the Serre spectral sequence of the fiber sequence $\Omega X \to * \to X$.

Theorem VI.3.5

If $f : X \to Y$ induces an isomorphism in $H_*(?, \mathbb{Q}) = H_*(?, \mathbb{Z}) \otimes \mathbb{Q}$ and X, Y are simply connected, then f induces and isomorphism in $\pi_*(?) \otimes \mathbb{Q}$.

Proof sketch. Following the method of homework, we get that $H_*(F(f)) \otimes \mathbb{Q} = 0$.

We would like to deduce $\pi_* F(f) \otimes \mathbb{Q} = 0$, which one does through the Relative Hurewicz Theorem

Remark VI.3.1

Let X be a path-connected space. Create a map $X \xrightarrow{f_m} X_m$ which induces an isomorphism on π_i for $i \leq m$. Then take $\pi_j(X_m) = 0$ for $j > m$ (attach cells to X to kill $\pi_{>m}$).

Then we have $X^m := F(f_m) \to X \xrightarrow{f_m} X_m$.

We get a tower $X_m \to X_{m-1} \to \cdots \to X_0$. We then get a fiber sequence

$K(\pi_{m+1}X, m+1) \to X_{m+1} \to X_m$. If X is a simple space, then this fiber sequence extends by 1 to the right

$$K(\pi_{m+1}X, m+1) \longrightarrow X_{m+1} \qquad X_m \longrightarrow K(\pi_{m+1}X, m+2)$$

This is called a Postnikov tower.

Rational homotopy theory: Localization of the full subcategory of $D\,\mathrm{Top}$ on simply connected (simple) spaces of finite type at equivalences being those maps inducing $\cong$ in $H_*(?;\mathbb{Q})$ (that is on $\pi_*(?) \otimes \mathbb{Q}$).

For a space X, there exists a model of $C^*(X;\mathbb{Q})$ which is a graded-commutative differential graded algebra.

Definition VI.3.8

A is a graded commutative DGA provided that if $x \in A_n, y \in A_m$ then

$$xy = (-1)^{nm} yx$$

and the differential satisfies

$$\mathrm{d}(xy) = \mathrm{d}x \cdot y + (-1)^m x \cdot \mathrm{d}y$$

For a simplex Δ^n, take differential forms on the affine envelope of Δ^n. Algebraically

$$\mathbb{Q}[x_0, \ldots, x_n]/(x_0 + \cdots + x_n = 1) \otimes \bigwedge[\mathrm{d}x_0, \ldots, \mathrm{d}x_m]/(\mathrm{d}x_0 + \cdots + \mathrm{d}x_n = 0)$$

If X is a simplicial complex (or a simplicial set), take the limit of these DGAs over its simplices. There's a paper by Sullivan in 1979 which is relevant.

For a space X, we construct in this way a graded commutative DGA over $\mathbb{Q}$. There is an appropriate notion of chain homotopy of graded commutative DGAs. Then this homotopy category satisfies colocalization with respect to cell DGAs.

A cell DGA is defined as $A_{(m)} \subseteq A_{(m+1)} \subseteq \cdots$ with $A = \bigcup_m A_{(m)}$ where

$$A_{(m+1)} = A_{(m)} \otimes F(x_i \mid i \in I_m)$$

where F is the free graded commutative algebra and x_i are homogeneous generators with $\mathrm{d}x_i \in A_{(m)}$. Recall that the free graded commutative algebra is $F(x) = \mathbb{Q}[x]$ for x in even degree and $\bigwedge[x] = \mathbb{Q}[x]/x^2$ if x is in odd degree. Take the tensor product for more than one generator.

In fact, A is called minimal if it is cell and $\mathrm{d}x_i$ is decomposable (sum of monomials and generators, each of which has monomial degree at least two).

Theorem VI.3.6

There exists a unique (up to DGA-isomorphism) minimal graded commutative DGA in each isomorphism class in the derived category.

The upshot: There is a unique model of a simple space of finite type up to $H_*(?;\mathbb{Q})$-equivalence by a minimal DGA (minimal model).

Homework #10

(1) Consider the commutative DGA

$$A = \mathbb{Q}[x] \otimes_{\mathbb{Q}} \bigwedge_{\mathbb{Q}} [\mathrm{d}x]$$

with $\deg(x)$ even. This is the tensor algebra, modulo $(\mathrm{d}x)^2 = 0$.

Prove that $H^i A = \mathbb{Q}$ if $i = 0$ and 0 if $i > 0$.

[Hint: Write down a basis of A

Remark VI.3.2

If we know that $H^*\mathbb{Q}[x] \otimes \bigwedge[\mathrm{d}x] = \mathbb{Q}$ in degree zero for x of even degree then

$$H^*\mathbb{Q}[x_1, \dots, x_n] \otimes \bigwedge[\mathrm{d}x_1, \dots, \mathrm{d}x_n] = \bigotimes_i H^*(\mathbb{Q}[x_i] \otimes \bigwedge[\mathrm{d}x_i]) = \mathbb{Q}$$

in degree zero. By the Kunneth Theorem

VI.4. Rational Homotopy Theory

If X is a simply connected CW complex of finite type (finitely many cells in each dimension). Equivalently (homotopically) the realization of a simplicial set with finitely many non-degenerate simplices in every dimension.

We calculate $\Omega^* X$ by taking limit over non-degenerate simplices. Use the fact that

$$\Omega^*\Delta^m = \mathbb{Q}[x_0, \dots, x_m] / \sum x_i = 1 \otimes \bigwedge[\mathrm{d}x_0, \dots, \mathrm{d}x_m] / \sum \mathrm{d}x_i = 0$$

where the degree of x_i is zero and the degree of $\mathrm{d}x_i$ is 1.

Then ΩX is a graded commutative DGA. We can of course talk about

Definition VI.4.1

A cell graded-commutative DGA is a DGA Q expressible as

$$\mathbb{Q} = A_{(0)} \subseteq A_{(1)} \subseteq \cdots \qquad A = \bigcup A_{(m)}$$

where we have that

$$A_{(m+1)} = A_{(m)} \otimes F[s_{mi} \mid i \in I_m] \otimes \bigwedge(\text{odd degree generators}) \otimes \mathbb{Q}[\text{even degree generators}].$$

Where $F(s_1, \dots, s_m)$ is the free graded commutative algebra on $s_1, \dots, s_m$. We say it has generators in degrees d_{mi} (even or odd).

Forces: cochain degrees ≥ 2. Furthermore we require $\mathrm{d}s_{mi} \subseteq A_{(m)}$. As an algebra $A_{(m)} = F(Q_m)$. We have an augmentation ideal $J_m = \{q \mid q \in Q_m\}$ of $A_{(m)}$.

We say A is minimal if $d_{si} \in J_m^2$ (the decomposable elements).

"Whitehead Theorem": Equivalences = quasiismorphisms = morphisms of graded-commutative DGAs inducing $\cong$ in cochain cohomology.

Theorem VI.4.1

There exists a derived category $\mathcal{D}$ of graded commutative DGAs, with respect to quasiisomorphism. Each isomorphism class in $\mathcal{D}$ contains a unique minimal DGA up to DGA isomorphism (this is called a minimal model).

Therefore we have for compact generated CW-complexes of finite type, a unique minimal model $A \to \Omega^* X$. Moreover $A = F[S]$ where $\mathbb{Q}S$ is the dual of rational homotopy groups of X.

If $S_m \subseteq S$ is the subset of generators of degree m then

$$\pi_m X \otimes \mathbb{Q} \cong \mathrm{Hom}(\mathbb{Q}S_m, \mathbb{Q}) = \mathrm{Map}(S_m, \mathbb{Q})$$

Example VI.4.1

Take $X = \mathbb{C}\mathrm{P}^m$. How is $\Omega^*\mathbb{C}\mathrm{P}^m$ represented?

Well $H^*(\mathbb{C}\mathrm{P}^m; \mathbb{Q}) = \mathbb{Q}[x]/(x^{m+1})$ with the cochain degree of x is 2.

Then we have a map of DGAs

$$\mathbb{Q}[u] \to \Omega^*\mathbb{C}\mathrm{P}^m$$
$$u \mapsto u, [u] = x$$

Impose the relation $x^{m+1} = 0$, so $u^{m+1} = \mathrm{d}v$. This gives a map

$$\mathbb{Q}[u] \otimes \bigwedge[v] \to \Omega^*\mathbb{C}\mathrm{P}^m$$

Where we have $\mathrm{d}v = u^{m+1}$, so the degree of v is $2m+1$. This is a quasiisomorphism with a bit of work.

Therefore

$$\pi_i\mathbb{C}\mathrm{P}^m \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 2, 2m+1 \\ 0 & \text{otherwise} \end{cases}$$

Homework #10

(2) Find the rational minimal model of S^m $(m > 1)$ and use it to calculate $\pi_k S^m \otimes \mathbb{Q}$ for all k.

Deligne-Morgan: A simply connected CW complex of finite type X is called formal if Ω^*X is quasiisomorphic to $H^*(X; \mathbb{Q})$ with zero differential

Theorem VI.4.2

Every simply connected smooth projective variety over $\mathbb{C}$ is formal.

Griffiths-Harris: Principles of Algebraic Geometry.

What does an alegbraic topologist make of this? "$\pi_m \otimes \mathbb{Q}$ are not interesting"

Or, perhaps, better point: The torsion is more interesting to algebraic topology.

Another thing worth mentioning: What if we replace $\mathbb{Q}$ with another field?

$$\text{characteristic } 0 \to \text{same story}$$
$$\text{characteristic } > 0 \to \text{doesn't work.}$$

By complicated, we mean we get stuck on the first step. We are not able to make a model of $C^*(X; \mathbb{F}_p)$ which would be a graded-commutative DGA. (if you do HW problem 1, it does not work in characteristic > 0).

The fact that this fails in characteristic > 0 is related to something known as Steenrod operations.

VII. Steenrod Operations

For X a CW complex of finite type then with the actions of swapping from $\mathbb{Z}/2$

$$C^*(X) \otimes C^*(X) \xrightarrow{\simeq} C^*(X \times X) \xrightarrow{C^*\Delta} C^*(X)$$

but this cannot be done $\mathbb{Z}/2$-equivariantly. The steenrod operations measure how much this fails using group homology.

There is no natural map filling the diagram below (commuting up to homotopy)

$$\begin{tikzcd} C(X) \arrow[r, "?"] \arrow[dr, "C(\Delta)"'] & C(X) \otimes C(X) \arrow[d] \arrow[dl, phantom, "\simeq"] \\ & C(X \times X) \end{tikzcd}$$

where $C(X) = C(X; \mathbb{Z}/2)$ which is $\mathbb{Z}/2$-equivariant, using the action $x \otimes y \mapsto y \otimes x$.

This would bean that the chains in the image would be $\mathbb{Z}/2$-fixed. This turns out to be impossible!

What is posible? Consider a $\mathbb{Z}[\mathbb{Z}/2]$-free resolution of $\mathbb{Z}$, e.g.

$$\cdots \longrightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\alpha} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\alpha} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\alpha} \mathbb{Z}[\mathbb{Z}/2]$$

which we call R. It is possible to construct a $\mathbb{Z}/2$ equivariant

$$\begin{tikzcd} R \otimes C(X) \arrow[r, "?"] \arrow[dr, "R\otimes C(\Delta)"'] & C(X) \otimes C(X) \arrow[d] \arrow[dl, phantom, "\simeq"] \\ & C(X \times X) \end{tikzcd}$$

R makes it a free $\mathbb{F}_2[\mathbb{Z}/2]$-modules. Universal element $\mathrm{Id} : \Delta^m \to \Delta^m$ (like in the Eilenberg-Zilber element), but we can $\otimes$ it with a free generator in R $(R \otimes C(X; \mathbb{Z}/2))_k$ is also representable.

Like in the cup product tratement. Dualize to cohomology

$$R \otimes C^*(X) \otimes C^*(X) \to C^*(X).$$

R remains homologically graded. So $C^*(X)$ is put in homological degree $-$.

We can also write

$$R \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C^*(X) \otimes C^*(X)) \to C^*(X).$$

$\mathbb{Z}/2$ acts on $H^*(X) \otimes H^*(X)$ by a permutation representation. We know $H^*(X)$ has basis $\alpha_i, i \in I$ and so we can map $\alpha_i \otimes \alpha_j \to \alpha_j \otimes \alpha_i$.

We then get a non-canonical map

$$R \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (H^*(X) \otimes H^*(X)) \to R \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C^*(X) \otimes C^*(X)) \to C^*(X).$$

However we do get a canonical map

$$H_*(\mathbb{Z}/2; H^*(X) \otimes H^*(X)) \to H^*(X)$$

with coefficients in $\mathbb{Z}/2$. If we order I then

$$H^*(X) \otimes H^*(X) = \bigoplus_{i<j} \mathbb{F}_2[\mathbb{Z}/2] \cdot \alpha_i \otimes \alpha_j \oplus \bigoplus_{i=j} \mathbb{F}_2 \cdot \alpha_i \otimes \alpha_i.$$

We know that $H_k(\mathbb{Z}/2; \mathbb{Z}/2) = H_k(\mathbb{RP}^\infty, \mathbb{Z}/2) = \mathbb{Z}/2$ for all $k \geq 0$. Call this generator e_k.

If $\alpha_i \in H^m X$ (m depending on i), then

$$\begin{aligned} H_*(\mathbb{Z}/2; H^*(X) \otimes H^*(X)) &\to H^*(X) \\ e_k \otimes \alpha_i \otimes \alpha_i &\xmapsto{D_k} \; ? \in H^{2m-k} X \end{aligned}$$

We may then define a Steenrod Operation

$$Sq^i = D_k : H^m X \to H^{m+i} X$$

by taking $k = m - i$.

Facts: $Sq^{>m} : H^m X \to ?$ (undefined, sometimes set to zero). The map $Sq^m : H^m X \to H^{2m} X$ sends x to x^2. And then $Sq^0(x) = x$, which is very nontrivial from this point of view.

Also $Sq^{<0}(x) = 0$, which is also nontrivial. The operations between 0 and m are completely mysterious.

We also have that

$$Sq^m(xy) = \sum_{i=0}^{m} Sq^i(x) Sq^{m-i}(y)$$

where juxtaposition denotes the cup product. The coproduct in $H_*(\mathbb{Z}/2, \mathbb{Z}/2)$ is $e_m \mapsto \sum e_i \otimes e_{m-i}$ (the dual to $H^*(\mathbb{Z}/2, \mathbb{Z}/2)$ being polynomial).

We can think of these as axioms

(1) $Sq^m(x) = x^2$

(2) $Sq^0(x) = x$

(3) $Sq^m(xy) = \sum_{i=0}^{m} Sq^i(x) Sq^{m-i}(y)$

Homework #10

(3) Calculate $Sq^m x^k$ with respect to $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$ with the degree of x being one. Use the axioms above.

There are compositions $Sq^i Sq^j = ?$ called Adam Relations. They're not deep but require prowess in combinatorics.

For $p > 2$ being the characteristic, we run into the fact that $\mathbb{Z}/p \subsetneq \Sigma_p$ (the symmetric group). So we really need to talk about $H_*(\Sigma_p; ?)$. Also because of signs, we can encounter either the trivial $\mathbb{Z}/p$-module or the sign representation.

We do have maps from functoriality.

$$H_*(\mathbb{Z}/p; ?) \underset{\text{res}}{\overset{\text{transfer}}{\rightleftarrows}} H_*(\Sigma_p; ?)$$

This maps $H_*(\Sigma_p; \mathbb{Z}/p)$ into direct summands of $H_*(\mathbb{Z}/p; \mathbb{Z}/p)$.

With this we'll get maps

$$p^i : H^m X \to H^{m+2i(p-1)} X \beta p^i : H^m X \qquad \to H^{m+2i(p-1)+1} X$$

β is the Bochstein from the short exact sequence in coefficients

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

which gives a long exact sequence in homology with connecting map

$$H^m(X; \mathbb{Z}/p) \xrightarrow{\beta} H^{m+1}(X; \mathbb{Z}/p)$$

For $p = 2$, $\beta = Sq^1$.

Example VII.0.1

$H^*(B\mathbb{Z}/p; \mathbb{Z}) = \mathbb{Z}[y]/(py)$ where $\deg y = 2$.

And also $H^*(B\mathbb{Z}/p; \mathbb{Z}/p) = \mathbb{Z}/p[y] \otimes \bigwedge[x]$ where $\deg x = 1$. There is also the integral Bochstein, which is the connecting map of

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0$$

Which will give $H^m(B\mathbb{Z}/p;\mathbb{Z}/p) \xrightarrow{\beta} H^{m+1}(B\mathbb{Z}/p;\mathbb{Z})$. For $m = 1$ you can derive from exactness that the integral bochstein is an isomorphism.

To get the mod p Bochstein, just compose with the map induced by $\mathbb{Z} \to \mathbb{Z}/p \to 0$.

We also have that Sq^i, β, p^i commute with the isomorphism

$$H^m(X) \to H^{m+1}(\Sigma X)$$

Once we get to stable homotopy theory we can use what's called the Adams Spectral Sequence, which takes these as data, to compute stable homotopy groups

VIII. Operads

Definition VIII.0.1

A symmetric monoidal category $\mathcal{C}$ has a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

along with a unit $1 \in \mathrm{Ob}\,\mathcal{C}$. We wish for this to be commutative, associative, and unital. We need this in the 2-categorical sense. Namely we need natural isomorphisms

$$A \otimes B \cong_{\iota_{AB}} B \otimes A$$
$$A \otimes (B \otimes C) \cong_{\alpha_{ABC}} (A \otimes B) \otimes C$$
$$A \cong_{\mu_A} A \otimes 1.$$

We also need some axioms. These are called coherence diagrams. Consider a word in the operator an units in a commutative monoid such as

$$((a \cdot b) \cdot c) \cdot d \to (a \cdot b) \cdot (\gamma \cdot d) \to a \cdot (b \cdot (c \cdot d))$$

But we can also do it in a different way

$$((a \cdot b) \cdot c) \cdot d \to (a \cdot (b \cdot c)) \cdot d \to a \cdot ((b \cdot c) \cdot d) \to a \cdot (b \cdot (c \cdot d))$$

Any time I can do this in two different ways, I get a coherence diagram for symmetric monoidal categories. This example is known as the pentagram diagram.

The actual coherence diagrams may be found on wikipedia.

Definition VIII.0.2

A closed symmetric monoidal category is one where for every object X, $X \otimes ? : \mathcal{C} \to \mathcal{C}$ has a right adjoint $\mathrm{Hom}(X, ?)$.

Example VIII.0.1

Set, $\times$; compactly generated spaces, $\times$; R-Mod, $\otimes$; R-Chain, $\otimes$.

Definition VIII.0.3

In a symmetric monoidal category one can define an operad. This is a collection of objects $\mathcal{D}(m)$ for $m \in \mathbb{N}_0$ satisfying the same formal properties as $\mathrm{Hom}(X^{\otimes m}, X)$ for some object X.

What structure maps do we have?

(1) $1 \xrightarrow{\iota} \mathcal{D}(1)$

(2) Σ_m acts on $X^{\otimes m}$ by permutation. Thus we require Σ_m acts on $\mathcal{D}(m)$.
(3) There is a map $\mathrm{Hom}(X^{\otimes m}, X) \otimes \bigotimes_{i=1}^m \mathrm{Hom}(X^{\otimes k_i}, X) \to \mathrm{Hom}(X^{\otimes \sum_i k_i}, X)$. Thus we require a map $\mathcal{D}(m) \otimes \bigotimes_{i=1}^m \mathcal{D}(k_i) \xrightarrow{\gamma} \mathcal{D}(k_1 + \cdots + k_m)$.

Axioms: Associativity, permutations, two unitalities.

Recommend the book by May. Geometry of Iterated Loop Spaces. [7].

There is an obvious notion of homomorphism of operads $\mathcal{D}_1 \to \mathcal{D}_2$ given by maps $\mathcal{D}_1(m) \to \mathcal{D}_2(m)$ preserving the above operations.

The operad $\mathrm{Hom}(X^{\otimes m}, X)$ is called the endomorphism operad $\mathrm{End}(X)$.

Definition VIII.0.4

An object X is called a $\mathcal{D}$-algebra (for an operad $\mathcal{D}$) if we are given a homomorphism of operads $\mathcal{D} \to \mathrm{End}(X)$. Equivalently in terms of maps $\mathcal{D}(m) \otimes X^{\otimes m} \to X$ satisfying some diagrams.

Back to cochains (with coefficients in $\mathbb{F}_p$). There is an operad $\mathcal{E}$ in $\mathbb{F}_p$-chain such that

(1) $\mathcal{E}(m) \simeq \mathbb{F}_p$ by chain homotopy (chain contractible).
(2) $\mathcal{E}(m)$ is a chain complex of free $\mathbb{F}_p(\Sigma_m)$-modules. This is the same thing as a linear action of the group.

Such an operad is called an E_∞-operad. An algebra of such an operad is called an E_∞-algebra. Running these through a colocalization game, we get a unique derived category of E_∞ algebras.

Theorem VIII.0.1 (Hinich-Schectman)

For a space X, $C^*(X; \mathbb{F}_p)$ has a natural structure of an E_∞-algebra.

(Proof: a souped up version of Eilenberg-Zilber theorem).

Example VIII.0.2

Another example of an operad on $\mathbb{F}_p$-chain given by $\mathcal{N}(m) = \mathbb{F}_p$. Then an $\mathcal{N}$-algebra is the same thing as a graded commutative DGA.

The structure maps are $\mathbb{F}_p \otimes X \otimes \cdots \otimes X \to X$. The $\mathbb{F}_p$ is a unit so it gets killed, and the signs of a graded commutative DGA come from the signs in the chain complex tensor product.

Remark VIII.0.1

Just as we defined Steenrod operations, we can define operations in the homology of an E_∞-$\mathbb{F}_p$-algebra.

Caution: This time, $Sq^0 = 1$ and $Sq^i = 0$ for $i < 0$ do not hold.

A convention in the context of E_∞-algebras is to put $Q^i = Sq^{-i}$. These are called Dyer-Landof operations.

Homework #11

(1) Write down the axiom diagrams for an operad (Ok to use reference, but adapt it exactly to the concept covered in class).

Note: We have not constructed any example of an E_∞-operad yet!

One method is to construct an E_∞-operad in spaces, and apply chains.

E_∞-operad in Top would satisfy

(1) $\mathcal{E}(m) \simeq *$
(2) $\mathcal{E}(m)$ has the homotopy type of a CW-complex with Σ_m acting freely on cells.

Note: This requires constructing a map $C_*(X) \otimes C_*(Y) \xrightarrow{\varphi} C_*(X \times Y)$ which is commutative, associative, and unital strictly (on the nose).

There is such a map (not in the opposite direction becuase of Steenrod operations being nonzer) called the shuffle map (standard transformation of a product of two standard simplices).

E_∞-algebras in spaces were in fact discovered first, and have a very close conection with generalized cohomology. This is called Infinite Loop Space Theory.

VIII.1. **Constructing E_∞ Operads**

An E_∞ operad in spaces consists of the following

(1) $\mathcal{C}(m) \simeq$ CW-complex, Σ_m-equivariantly, and Σ_m acts freely on the cells (when G acts on the sets of cells of a CW-complex we call this a G-CW-complex).

(2) $\mathcal{C}(m) \simeq *$ (non-equivariantly).

Start with any operad $\mathcal{M}$ satisfying (1). For example $\mathcal{M}(m) = \Sigma_m$. Then a $\mathcal{M}$-algebra is a monoid (an associative, unital).

Čech resolution If X is an object of a category $\mathcal{G}$ (with product), then this builds a simplicial object EX in the same category $\mathcal{G}$, that is a functor $\Delta^{\text{op}} \to \mathcal{G}$.

Then we set $EX_m = \underbrace{X \times \cdots \times X}_{m+1 \text{ times}}$. Labeling these coordinates $0, \dots, m$ then the i-th face map $\{0, \dots, m-1\} \to \{0, \dots, m\}$ gets mapped to the projection away from the i-th coordinate.

The degeneracies are given by applying the diagonal $X \xrightarrow{\Delta} X \times X$ in the appropriate coordinate given by $\{0, \dots, m+1\} \to \{0, \dots, m\}$. Namely this sends $i, i+1$ to i, so apply the diagonal to the i-th coordinate. In some sense we have "$EX = X^\Delta$," or as a right Kan Extension along $\Delta \to *$.

In Set, Top (compactly generated weakly Hausdorff spaces see [6]). Here we have the geometric realization. If $Y_\bullet$ is a simplicial space (simplicial object in Top, then

$$|Y_\bullet| = \coprod Y_m xx \Delta^m / (y, \alpha t) \sim (Y_\bullet(\alpha) y, t) \qquad (\alpha \in \text{Mor}(\Delta))$$

It suffices to just take faces and degeneracies (the generators).

Proposition VIII.1.1

If $X \neq \emptyset$, then $|EX| \simeq *$.

Proof sketch. We have some basepoint $* \in X$. Then we have that

$$|EX| = \coprod_{m \geq 0} X^{\{0, \dots, m\}} \times \Delta^m / (y, \alpha t) \sim (EX(\alpha) y, t).$$

We have a map $h_s : |EX| \to |EX|$ given by

$$h_s((x_0, \dots, x_m), [t_0, \dots, t_m]) = ((x_0, \dots, x_m, *), [(1-s)t_0, \dots, (1-s)t_m, s]).$$

Homemwork #11

(2) Verify that this definition is compatible with face and degeneracy identification, proving that for a non-empty space X, $|EX| \simeq *$.

If $s = 0$, then $h_0 = \text{Id}$, and if $s = 1$¡ then h_1 is constant at $(*, 1)$ by face/degeneracy identifications.

Geometric realization preserves products (triangulation of $\Delta^m \times \Delta^n$ by shuffles). If $\mathcal{D}$ is a simplicial operad in spaces, then $|\mathcal{D}_\bullet|$ is also an operad. This shows us by definition then that $|E\mathcal{M}|$ is an E_∞ operad.

Definition VIII.1.1

An E_∞-space is an algebra over an E_∞-operad in spaces.

We can play the game to show that $\mathcal{D}$-algebras have colocalization, giving a derived category.

Theorem VIII.1.2

The derived category does not depend on the particular E_∞-operad chosen.

Proof sketch. If $\mathcal{D}$, ξ are E_∞-operads then there is a diagram

$$\begin{tikzcd} & \mathcal{D} \times \xi \arrow[dl, "\text{proj. } \pi_1"'] \arrow[dr, "\text{proj. } \pi_2"] & \\ \mathcal{D} & & \xi \end{tikzcd}$$

For a homomorphism of operads $f : \xi \to \mathcal{D}$ we have a pullback functor $f^* : \mathcal{D}\text{-algebra} \to \xi\text{-algebra}$, one proves that π_1^*, π_2^* induce equivalence of derived categories of algebra.s

[7] does this more concretely without derived categories.

VIII.2. Infinite Loop Space Theory

Recall that a generalized cohomology theory is determined by some based spaces Z_n where $n \in \mathbb{Z}$ equipped with weak equivalences

$$Z_n \xrightarrow{\sim} \Omega Z_{n+1}. \tag{$\star$}$$

In fact $\mathbb{N}_0$ would do. GivenZ_0, define $Z_{-m} = \Omega^m Z_0$.

The spaces Z_m of $(*)$ are called infinite loops spaces. Peter May notices that infinite loop spaces (up to $\simeq$) are E_∞-spaces, and connected E_∞-spacesm are infinite loop spaces.

Application: Construction of generalized cohomology theories. For example, we can consider alegbraic K-theory.

Why are infinite loop spaces E_∞-spaces. Consider that E_∞-spaec are commutative monoids up to homotopy and all reaonsonable higher homotopies.

What does this have to do with loops: π_m is commutative for $m \geq 2$. Consider a space of the form $\Omega^* X$, X is a based space, and $\Omega^m X$ is $\mathrm{Hom}(([0,1]^m, \partial[0,1]^m), (X, *))$.

Peter May invented an operad so that m-loop spaces are E_∞ algebras over this operad ξ_m.

The litte n-cubes operad $\mathcal{E}_m(k)$ is merely a configuration of k cubes in $[0,1]^m$ with disjoint images.

It is obvious then that $\Omega^m X$ (as defined above) is a $\mathcal{C}_m$-aglgebra (same as our proof of commutativity of π,).

Inclusions of operads

$$\mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \qquad \cdots .$$

Take a little cubes $\times [0,1]$ Then

$$\mathcal{C}_\infty = \bigcup \mathcal{C}_n.$$

May tells us that $\mathcal{C}_\infty$ is a C_∞-algebra, that is an E_∞-operad algebra.

$\mathcal{C}_m$ is the little m-cubes operad. It acts on the loop space $\Omega^m X$ so that it is a $\mathcal{C}_m$-algebra.

We can then take $\mathcal{C}_\infty = \bigcup_m \mathcal{C}_m$ We wish to see that $\mathcal{C}_\infty$ acts on an infinite loop space $Z_m \xrightarrow{\sim} \Omega Z_{m+1}$.

Definition VIII.2.1

A collection of based spaces (Z_m), $m \subseteq \mathbb{N}_0$ together with based homeomorphisms $\rho_m : Z_m \xrightarrow{\cong} \Omega Z_{m+1}$ is called a (May) spectrum.

A morphism of spectra $(Z_m) \to (T_m)$ is a collection of based maps $f_m : Z_m \to T_m$ with commutative diagrams

$$\begin{tikzcd} Z_m \arrow[r, "f_m"] \arrow[d] & T_m \arrow[d] \\ \Omega Z_{m+1} \arrow[r, "\Omega f_{m+1}"'] & \Omega T_{m+1} \end{tikzcd}$$

Therefore if (Z_m) is a May spectrum, then obviously $\mathcal{C}_\infty$ acts on each Z_m (i.e. each Z_m is an E_∞-space).

Can we make a spectrum out of $Z_m \xrightarrow{\sim} \Omega Z_{m=1}$ that would give the same generalized cohomology theory on CW-complexes?

Definition VIII.2.2

A prespectrum is defined the same way as a spectrum, except no condition is given on the continuous map ρ_m (besides being a based continous map)

Thus there is a forgetful functor Spectra $\to$ Prespectra. One can prove that there is a left adjoint (i.e., a free functor) L : Prespectra $\to$ Spectra, which we call spectrification. This was proved by Freyd-Kelly in a transfinite argument.

For the moment we should not we're working with the following convenient category of spaces (see [6]).

- Weakly Hausdorff, compactly generated spaces
- Closed symmetric monoidal category under $\times$.

$L(D_n)_k$ can be described explicitly if (D_n) is an inclusion prespectrum which means that $\rho_n : D_n \to \Omega D_{n+1}$.

Then we have that

$$(L(D_n))_k = \operatorname{colim} \Omega^m D_{k+m} = \operatorname{colim}(D_m \hookrightarrow \Omega D_{m+1} \hookrightarrow \Omega^2(D_{m+2}) \hookrightarrow \cdots).$$

The issue with non-inclusions: Ω commutes past colimit of a sequence of inclusions, but not an arbitrary sequence.

Then for general $Z_m \xrightarrow{\sim} \Omega Z_{m+1}$, we replace them by inclusions by looking at the based mapping cylinder $\Sigma Z_m \to Z_{m+1}$ recursively.

Theorem VIII.2.1 (May)

A connected E_∞-space ($\simeq$ CW-complex) is $\simeq$ to an infinite loop space (Z_0 for some spectrum (Z_m)). [7]

Note: since it doesn't matter which E_∞-operad we are using. We may as well use $E\mathcal{M}$ where $\mathcal{M}(k) = \Sigma_k$. By construction then we have a map of operads $\mathcal{M} \to E\mathcal{M}$. An $\mathcal{M}$-algebra is a topological monoid. Thus an $E\mathcal{M}$-space (E_∞-space) is a topological monoid.

Since $E\mathcal{M}_k \simeq *$, this topological monoid is commutative up to homotopy (and higher homotopies). In particular, π_0 for such a space is a commutative monoid. If X was an infinite loop space, $\pi_0 X$ would be forced to be an abelian group (associated generalized cohomology theory gives E has $E^0(*) = \pi_0 X$).

One can construct a "group completion" of X, say $X \to \overline{X}$ which satisfies

- On π_0 is the K-groupification (the universal abelian group on this commutative monoid, $K\pi_0 X$).
- $H_*(\overline{X};\mathbb{Z}) = [\pi_0 X]^{-1} H_*(X;\mathbb{Z})$. Where $H_*(X;\mathbb{Z})$, homology of a commutative monoid is a graded commutative ring, using the product $\mu : X \times X \to X$ and chain-approximation $CX \otimes CX \to CX$.

VIII.2.1. **Homework #11**

(3a) Prove that a path-connected topological monoid X is a simple space. Namely $\pi_1 X$ is commutative and acts trivially on $\pi_m X$ for $m > 1$.

> **Recall VIII.2.1**
>
> We should recall how $\pi_1 X$ acts on $\pi_m X$. 2

(3b) Prove that $S^2 \vee S^1$ is not homotopy equivalent to a topological monoid. (Consider how π_1 acts on π_2).

Let first G be a (discrete) group. Recall the Čech resolution $EG = |EG|$, where $EG_\bullet$ is the simplicial set $EG_m = G^{\{0,\dots,m\}}$.

G acts on EG as $g(g_0, \dots, g_m) = (gg_0, \dots, gg_m)$, and it acts freely, properly discontinuously, and all the nice things.

We then call $BG \coloneqq EG/G$. Thus we have $G \to EG \to BG$ as a fibration (where $EG \to BG$) is a universal covering. This shows another construction of BG because $\pi_1 BG = G$ and $\pi_m BG = 0$ for $m > 1$.

Another description of BG which can be generalized. Again we have $BG = |BG_\bullet|$. And we define

$$BG_m = G^m = \{(h_1, \dots, h_m)\}.$$

Then

$$(g_0, g_1, \dots, g_m) \mapsto (h_1, \dots, h_m)$$

Where we put $h_i = g_{i-1}^{-1} g_i$. This is called dehomogenization (and is an isomorphism).

$$g_0^{-1} g_1 = (gg_0)^{-1}(gg_1).$$

We wish to describe the faces and degeneracies directly in terms of $(h_1, \dots, h_m)$.

- For the 0-face, drop h_1 to get $(h_2, \dots, h_m)$
- For the i-th face for $1 \le i \le m-1$ we get $(h_1, \dots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \dots, h_m)$
- For the m-th face, drop h_m to get $(h_1, \dots, h_{m-1})$.
- For degeneracies, insert a unit. $(h_1, \dots, h_{i-1}, 1, h_i, \dots, h_n)$.

This is BG (Bar construction = classifying space = nerve). We can already see that we can do this for a monoid, getting us BM for a monoid M.

Now we see that an E_∞-space X is a topological monoid. The group completion is constructed by

$$\overline{X} = \Omega BX$$

Let $\mathcal{C}$ be a small category. Similarly as for monoids, we can generalize the Bar construction (aka the nerve). We have a simplicial set $B\mathcal{C}_\bullet$ where

$$B\mathcal{C}_n = \{\text{composable } n-\text{ tuples}\}$$
$$B\mathcal{C}_0 = \operatorname{Ob} C$$

The faces are compose γ_i, γ_{i+1}, and the degeneracies insert a unit.

> **Theorem VIII.2.2**
>
> If $\mathcal{C}$ is a small symmetric monoidal category then $B\mathcal{C}$ is an E_∞-space.

Proof Sketch. The Street Construction: $\mathcal{C}$ is equivalent to a "permutative category"

Definition VIII.2.3

A permutative category has an operation $\otimes$ which is strictly unital and associative. We have $\sigma : X \otimes Y \to Y \otimes X$ such that $\sigma^2 = \mathrm{Id}$ and also

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "X\otimes\sigma"] \arrow[dr, "\sigma"'] & & X \otimes Z \otimes Y \arrow[dl, "\sigma\otimes Y"] \\ & Z \otimes X \otimes Y & \end{tikzcd}$$

Then the operad $\mathcal{M}$ defined by $\mathcal{M}(n) = \Sigma_n$ acts on $\mathcal{C}$ in the sense that we have

$$\mathcal{M}(n) \times \mathcal{C} \times \cdots \times \mathcal{C} \to \mathcal{C}$$

and this is given by

$$g, X_1, \ldots, X_n \mapsto g^{-1} X_1 \otimes \cdots \otimes X_n g$$

$EM_\bullet$ acts on $B\mathcal{C}_\bullet$. We then take simplicial realization and $E\mathcal{M}$ acts on $B\mathcal{C}$.

Comments:

(1) If a category $\mathcal{C}$ has an initial object (or terminal object) in particualr, or if it as zero, then $B\mathcal{C} \simeq *$ (same proof as $EX \simeq *$).

Often, what does give interesting examples is to take the subcategory of isomorphisms in some category.

(2) Let R be a commutative ring. Take $\mathrm{Ob}\,\mathcal{C} = \mathbb{N}_0$, and $\mathrm{Mor}(m, n) = 0$ for $m \neq n$ and $\mathrm{Mor}(m, m) = \mathrm{GL}_m(R)$.

The symmetric monoidal structure is just the block sum of matrices. This is a permutative category, and it's fairly clear that

$$B\mathcal{C} = \coprod_{m \geq 0} B\,\mathrm{GL}_m\, R.$$

This is an E_∞-space. Then $\pi_0 B\mathcal{C} = \mathbb{N}_0$. Thus we must apply the group completion $\overline{B\mathcal{C}} = \Omega B(B\mathcal{C})$ by viewing $B\mathcal{C}$ as a topological monoid.

This is an infinite loop space Z_m giving a cohomology theory $KR = Z_0 \times K_0 R$ (the algebraic K-theory, $K_0 R$ is discrete).

Theorem VIII.2.3 (Quillen)

The group completion $\Omega B \left(\coprod_{m \geq 0} B\,\mathrm{GL}_m\, R \right) \simeq B\,\mathrm{GL}_\infty\, R^+ \times \mathbb{Z}$, where $+$ denotes the Quillen plus construction.

Proof Sketch. First construct a map

$$B\,\mathrm{GL}_\infty\, R^+ \to X := \left(\Omega B \left(\coprod_{m \geq 0} B\,\mathrm{GL}_m\, R \right) \right)_0$$

using the fact that X is an E_∞-space, thus a topological monoid, and so $\pi_1 X$ is abelian. Also $H_* X = H_* B\,\mathrm{GL}_\infty\, R$ because

$$H_* X = [\pi_0] X^{-1} \left(H_* \coprod_{m \geq 0} B\,\mathrm{GL}_m\, R \right) = H_* B\,\mathrm{GL}_\infty [t, t^{-1}]$$

where $t = (1)$. Thus by the universal property of the plus construction there is the desired map. This map is both an isomorphism in π_1 and an isomorphism in homology.

To finish the proof, one needs to show that $B\,\mathrm{GL}_\infty R^+$ is a simple space (to get weak equivalence). That is we need to show that π_1 acts trivially on π_n for $n > 1$ (we already know π_1 is abelian). A key step of this is in homework.

$G = \mathrm{GL}_\infty R, E = [\mathrm{GL}_\infty R, \mathrm{GL}_\infty R]$ and $[E, E] = E$.

The universal coer of BG^+ is BE^+ by construction. Why does $\pi_1 BG^+ = G/E$ act trivially on $\pi_n BE^+$. It is <u>not true</u> that the action of $g \in G$ on E by $h \in E \mapsto ghg^{-1}$ is by conjugation of an element of E.

However, for any m elements $h_1, \ldots, h_m$ we can find an element $q \in E$ such that

$$gh_ig^{-1} = qh_iq^{-1}.$$

An element of $\pi_m BE$ only meets finitely many simplices, and therefore finitely many h_i.

The whitehead lemma says that $\left[\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right] \in E$.

Homework #12

(1) Let G be a (discrete) group, $g \in G$. Then g acts on G by conjugation. Therefore g acts on BG by conjugation. Prove that the map $\gamma_g : BG \to BG$ is homotopic to the identity.

Better to think of $BG \cong EG/G$, where EG is the Čech resolution on which G acts on the left. Then find a G-equivariant map

$$\begin{tikzcd} EG \arrow[r, "\varphi"] \arrow[d] & EG \arrow[d] \\ BG \arrow[r, "\gamma_g"'] & BG \end{tikzcd}$$

You may use the fact that EG is a free G-CW-complex (non-degenerate simplices are the cells). Then prove that any two self-maps of a contractible free G-CW-complex are G-equivariantly homotopic.

Recall that a G-CW-complex has the cells as G-sets and the attaching maps are G-equivariant. A free G-set is the same as a disjoint union of copies of G.

Given a small symmetric monoidal category $\mathscr{C}$ we can build $B\mathscr{C}$ an E_∞-space, and then $\Omega B(B\mathscr{C})$ is an infinite loop space (being a group completion). This then is a spectrum

Example VIII.2.2

Let $\mathscr{C}$ be the category of finite sets with bijections (symmetric monoidal operation is $\coprod$), permutatively we have $\mathscr{C} = \coprod_{m \geq 0} \Sigma_m$. Then we have that

$$B\left(\coprod_{m \geq 0} \Sigma_m\right) = \coprod_{m \geq 0} B\Sigma_m$$

is an E_∞-space. What spectrum corresponds to the group completion?

$$\Omega B\left(\coprod_{m \geq 0} B\Sigma_m\right) \simeq B\Sigma_\infty^+ \times \mathbb{Z}.$$

We also have

$$\Omega B\left(\coprod_{m\geq 0} B\Sigma_m\right) = \mathrm{colim}_{m\to\infty}\,\Omega^m S^m.$$

The map $\Omega^m S^m \to \Omega^{m+1}S^{m+1}$ can come from a map $S^m \to \Omega S^{m+1}$ adjoint to $\Sigma S^m \xrightarrow{\cong} S^{m+1}$.

This spectrum is the spectrification of the pre-spectrum $D_m = S^m$ via $\Sigma S^m \xrightarrow{\cong} S^{m+1}$ giving a map $D_m \to \Omega D_{m+1}$. This is a special case of a general construction. Let X be a based space. Let

$$D_m = \Sigma^m X \qquad \Sigma\Sigma^m X \xrightarrow{\cong} \Sigma^{m+1}X \qquad \Sigma^m \xrightarrow{\subseteq} \Omega\Sigma^{m+1}X.$$

(D_m) is then an inclusion spectrum and $\Sigma^\infty X$ is the spectrification of this.

Definition VIII.2.4

$\Sigma^\infty X$ is called the suspension spectrum of X.

So then we have a situation like

$$(\text{Finite sets}, \cong) \xrightarrow{\infty\text{ loop space machine}} \Sigma^\infty S^0$$

In some sense the suspension spectrum if free. Then we have

$$\text{Spectra} \to \text{Spaces}$$
$$E = (Z_m) \mapsto \Omega^\infty E \coloneqq Z_0.$$

The left adjoint of Ω^∞ is Σ^∞ (the suspension spectrum). The verification is quick.

The category of finite sets is a "free symmetric monoidal category on one point" so plugging it into our infinite loop space machine and getting back a free infinite loop space on "one point" is good.

That is if $(\mathscr{C}, \oplus)$ is a symmetric monoidal category and $X \in \mathrm{Ob}(C)$, then $* \to X$ necessarily requires that

$$S \mapsto \bigoplus_S X$$

If $\mathscr{D}$ is an operad and X is a space, the free $\mathscr{D}$-algebra on X is

$$\mathscr{D}X = \coprod_{m\geq 0} \mathscr{D}(m) \times_{\Sigma_m} X^m. \qquad (\times_{\Sigma_m} = \text{space of orbits})$$

(left adjoint to the forgetful functor).

If $\mathscr{D}$ is an E_∞-operad, $X = *$ then

$$\mathscr{D}X \simeq \coprod_{m\geq 0} E\Sigma_m \times_{\Sigma_m} * = \coprod_{m\geq 0} B\Sigma_m.$$

If $\mathscr{D} = E\mathcal{M}$ then these are all equalities.

You can then ask if the category (finite sets, $\cong, \coprod$) is symmetric monoidal equivalent to a strictly commutative associative unital category. It is not

Proof Sketch. If so, then $\coprod_{m\geq 0} B\Sigma_m$ as an E_∞-space would be equivalent to a topological commutative monoid. We can then look at the chains

$$C_*\left(\coprod_{m\geq 0} B\Sigma_m; \mathbb{F}_2\right)$$

is an E_∞-algebra in $\mathbb{F}_p$-chain. But it is not quasiisomorphic to a graded-commutative DGA because of Dyer-Lashof operations.

For example, $\alpha \in H_0 B\Sigma_1$, the Dyer-Lashof operations (which we defined) on α are the basis of $H_* B\Sigma_p$. That's how they were defined!

For $p = 2$ we have $H_i(B\Sigma_2; \mathbb{F}_2) = H_i(B\mathbb{Z}/2; \mathbb{F}_2) = \mathbb{Z}/2$. The generator is then equal by definition to $Q^i\alpha$.

Spectra: $\Sigma^\infty S^0$ is a spectrum, and so it gives a generalized cohomology theory. But here we see a generalized homology theory more naturally. This is called stable homotopy groups. Say X is a based CW-complex,

$$\begin{aligned}
(\Sigma^\infty X)_0 &= \operatorname{colim} \Omega^m \Sigma^m X \\
\pi_k(\Sigma^\infty X)_0 &= \pi_k \operatorname{colim} \Omega^m \Sigma^m X \\
&= \operatorname{colim} \pi_k \Omega^m \Sigma^m X \\
&= \operatorname{colim} \pi_{k+m} \Sigma^m X
\end{aligned}$$

This is called $\pi_k^{\text{stable}} X$.

Maybe every spectrum gives rise to a generalized homology theory. Maybe we could do homotopy theory of spectra?

For a spectrum E and a based space (compactly generated, weakly Hausdorff) X, and notationally $E = (Z_m)$ with structure maps $\rho_m : Z_m \to \Omega Z_{m+1}$.

$$F(X, E) = T_m \qquad T_m = F(X, Z_m) \qquad \text{(based maps)}$$

We can also define $X \wedge E$. Remember for based spaces this is

$$X \wedge Y = (X \times Y)/((X \times *) \cup (* \times Y)).$$

Then $X \wedge E = L(U_m)$ (spectrification) where $U_m \coloneqq X \wedge Z_m$.

Homotopy of spectra $p, q : E \to F$ is $h : [0,1]_+ \wedge E \to F$ which is f on $\{0\}_+ \wedge E$ and g on $\{1\}_+ \wedge E = g$.

To define homotopy groups, I need to define spheres $\mathbb{S}^m, m \in \mathbb{Z}$. For $m \geq 0$ just take $\mathbb{S}^m \coloneqq \Sigma^\infty S^m$.

Spectra have a shift functor

$$\begin{aligned}
[k] &: \text{Spectra} \to \text{Spectra} \\
E = (Z_m) &\mapsto E[k] = (Z_{m+k}).
\end{aligned}$$

To get negative spheres, take $\mathbb{S}^{\ell-k} = (\Sigma^\infty S^\ell)[-k]$. It turns out that this only depends on $\ell - k$ and not both variables, as we should hope.

Homework #12

(2) Prove that $\Sigma^\infty(\Sigma X)[-1] = \Sigma^\infty X$. (realize that spectrification only depends on tail of prespectrum).

For a spectrum E, $\Sigma E = S^1 \wedge E$. Also, the category of spectra (Spectra) has all limits and colimits.

Take limits "space-wise," colimits are done space-wise to obtain a prespectrum, so then spectrify.

$\mathbb{S}^m \coloneqq \Sigma^\infty S^k[m-k]$. Per homework this does not depend on the choice of $k \geq 0$. We also have that

$$\Sigma^\infty(\Sigma X) = \Sigma(\Sigma^\infty X) = S^1 \wedge \Sigma^\infty X.$$

We can define hSpectra by using smash products to define homotopies. $[X, Y]$ denotes the set of homotopy classes of morphisms between X, Y. We can then of course define homotopy groups of a spectrum E via

$$\pi_m E = [\mathbb{S}^m, E]. \qquad (m \in \mathbb{Z})$$

These are always abelian groups because $\mathbb{S}^m = \Sigma^2 \mathbb{S}^{m-2}$, the proof is the same as for based spaces.

We can also define the mapping cone (homotopy cofibre) of a morphism of spectra $f : E \to F$

$$Cf := \operatorname{colim} \left(\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "0"'] & F \\ E \wedge [0,1]_+ & \\ E \arrow[u, "1"] \arrow[r] & * \end{tikzcd} \right)$$

$$Ff := \operatorname{colim} \left(\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "0"'] & F \\ E \wedge [0,1]_+ & \\ E \arrow[u, "1"] \arrow[r] & * \end{tikzcd} \right)$$

Note: ΣE is not the same as $E[1]$ for a general spectrum. On adjoints, equivalently, $\Omega E = F(S^1, E)$ is not the same as $E[-1]$.

If $E = (Z_m)_{m \in \mathbb{N}_0}$ with structure map $\rho_m : Z_m \to \Omega Z_{m+1}$. Then $(\Omega E)_m = \Omega Z_m$, with structure maps $\rho'_m : \Omega Z_m \to \Omega \Omega Z_{m+1}$. We need a switch of coordinates $T : \Omega^2 Z_{m+1} \to \Omega^2 Z_{m+1}$ then

$$\rho'_m = T \circ \Omega \rho_m.$$

Proposed isomorphism of spectra $E[-1] \to \Omega E$ given by $Z_{m-1} \xrightarrow{\rho_{m-1}} \Omega Z_m$. But when trying to check the compatibility we see

$$\begin{tikzcd} Z_{m-1} \arrow[r, "\rho_{m-1}"] \arrow[d, "\rho_{m-1}"] & \Omega Z_m \arrow[d, "T \circ \Omega \rho_m"] \\ \Omega Z_m \arrow[r, "\Omega \rho_m"'] & \Omega \Omega Z_{m+1} \end{tikzcd}$$

Does not commute! This is wrong!

Definition VIII.2.5

A spectrum E is called a cell spectrum provided that $E = \operatorname{colim} E_{(m)}$ with

$$\bigvee_{i \in I_m} \mathbb{S}^{d_i} \xrightarrow{f_m} E_{(m)}.$$

We should have that $Cf_m = E_{m+1}$.

Theorem VIII.2.4 (May,Lewis)

hSpectra has colocalization with respect to cell spectra and the class E of weak equivalences.

A weak equivalence is of course a morphism of spectra $f : E \to F$ which induces an isomorphism in all π_k for all $k \in \mathbb{Z}$.

The derived category $D\,\text{Spectra} = D\,\text{hSpectra}$ with respect to weak equivalences is called the stable homotopy category

Theorem VIII.2.5 (May,Lewis)

On $D\,\mathrm{Spectra}$, Ω and $L\Sigma$ are inverse equivalences of categories isomorphic to $[-1]$ and $[1]$ respectively (where L denotes the left derived functor, aka cell approximate first).

Proposition VIII.2.6

If $f : E \to F$ is a morphism of spectra then $Ff \sim LCf[-1]$ weakly.

L symbol is usually omitted because mathematicians are lazy.

Proof Sketch. We have analogously to based spaces for a map $f : E \to F$ a long exact sequence

$$[W, \Omega E] \longrightarrow [W, \Omega F] \longrightarrow [W, Ff] \longrightarrow [W, E] \longrightarrow [W, F].$$

We can prove in fact for E, F cell that

$$[W, E] \longrightarrow [W, F] \longrightarrow [W, Cf]$$

is also exact. The idea being that

$$\begin{tikzcd} E \arrow[r] & F \arrow[r] & Cf \arrow[r] & \Sigma E \arrow[r, "-\Sigma f"] & \Sigma F \\ W \arrow[r, "\mathrm{Id}"'] \arrow[u, dashed, "\ell"] & W \arrow[r] \arrow[u, "g"] & * \arrow[r] \arrow[u, "h"] & \Sigma W \arrow[r, "-\mathrm{Id}"'] \arrow[u, dashed, "k"] & \Sigma W \arrow[u, dashed, "\Sigma\ell"] \end{tikzcd}$$

using the theorem multiple times and then use the 5-lemma to show $Cf[-1] \xrightarrow{\sim} Ff$.

It follows that finite products are isomorphic to finite coproducts in $D\,\mathrm{Spectra}$.

It turns out that the stable homotopy category is triangulated, and has a lot of structure.

$D\,\mathrm{Spectra}, DA$ (for A an abelian category) has products / coproducts.

Homework #12

(3) In $D\,\mathrm{Ab}$ the map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ does not have a kernel (i.e. there is no equalizer between 2 and 0).

This is set up with cell chain complexes and chain homotopy classes of maps. Shifts $\mathbb{Z}[k]$ are cell complexes. If $[?, ?] = \mathrm{Mor}_{D\,\mathrm{Ab}}$. Then

$$[\mathbb{Z}[k], C] = H_k(C).$$

We also can see that for abelian groups A, B (considered as chain complexes in degree zero)

$$[A[-k], B] = [A, B[k]] = \mathrm{Ext}^k(A, B)$$

Lemma VIII.2.7

$\mathrm{Ab} \to D\,\mathrm{Ab}$ sending A to A is an inclusion of a full subcategory. This is sometimes called the heart of the derived category with respect to chain homology. Also sometimes called the t-structure.

To see this, note that free resolutions are cell approximations. We proved in class that morphisms between free resolutions are the same as morphisms between the abelian groups.

The proof then becomes

(a) If $K = \ker(2 : \mathbb{Z} \to \mathbb{Z})$ exists in $D\,\mathrm{Ab}$, then $H_i K = 0$ for $i \neq 0$. Non-zero would violate uniqueness of the limit. Therefore $K \in \mathrm{Ab}$.

(b) But then $K = \ker(2 : \mathbb{Z} \to \mathbb{Z})$ in Ab. But there exists a nonzero morphism $C \xrightarrow{f} \mathbb{Z}$ in $D\,\mathrm{Ab}$ so that $2f = 0$.
Hint, use that $[A[-k], B] = [A, B[k]] = \mathrm{Ext}^k(A, B)$.

Given a spectrum E and a CW-complex X, we can define the generalized homology and cohomology theory on X corresponding to E by

$$\widetilde{E}_m X = \pi_m(E \wedge X) \tag{$\star$}$$
$$\widetilde{E}^m X = \pi_{-m} F(X, E) = \pi_0 F(X, E[m]) \tag{1}$$
$$= [X, E_m]. \tag{2}$$

The first is motivated by the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$ (the corresponding generalized homology theory is $\pi_m^S X = \pi_m \Sigma^\infty X = \pi_m(X \wedge \mathbb{S})$). It turns out that for a CW-complex X, $? \wedge X : \text{Spectra} \to \text{Spectra}$ preserves weak equivalences.

Comment: With the (co)limit axioms on generalized homology and cohomology, and preservation by weak equivalence axiom, every generalized homology and cohomology theory is represented by some spectrum.

Example VIII.2.3

K-theory cohomology comes from geometry (discuss later). There is still no known geometric interpretation of K-theory homology.

VIII.3. Spectral Sequences: Revisited

Definition VIII.3.1

An exact couple is a long exact sequence of the form

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Philosophy: from information about E, can we gain info about D.

We should want D, E to be $\mathbb{Z}$-graded, with i, j having degree 0 and k having degree -1.

Massey: Observed that $\mathrm{d}_1 = jk$ is a differential on E. We can then define $E' = H(E, \mathrm{d}_1)$. We can also define $D' = \mathrm{im}(i : D \to D)$. There is a derived exact couple

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

Intuitively, i' induced by i, k' induced by k, j' induced by $j \circ i^{-1}$. But this is not immediately seen to be well-defined.

We need another characterization of E'

$$E' = \frac{\ker jk}{\mathrm{im}\, jk} = \frac{k^{-1} \ker j}{j \,\mathrm{im}\, k} = \frac{k^{-1} \mathrm{im}\, i}{j \ker i}$$

Homework #13, Due: Monday Nov 29th

(1) Prove that
   (a) $\ker i' = \mathrm{im}\, k'$
   (b) $\ker j' = \mathrm{im}\, i'$

(c) $\ker k' = \operatorname{im} j'$.

The spectral sequence arises by iterating this process.

Lemma VIII.3.1

$$D^{(m)} = \operatorname{im}(i^m)$$
$$E^{(m)} = k^{-1}(\operatorname{im} i^m)/j(\ker i^m)$$

Often, we have an additional grading, making D, E bigraded. Then we understand $D_{p,q}, E_{p,q}$ are in total degree $n = p + q$. The traditional bidegrees are then

$$\begin{tikzcd} D \arrow[rr, "{(1,-1)}", "i"'] & & D \arrow[dl, "j", "{(0,0)}"'] \\ & E \arrow[ul, "k", "{(-1,0)}"'] & \end{tikzcd}$$

Then we have in the derived case

$$\begin{tikzcd} D^r \arrow[rr, "{(1,-1)}", "i"'] & & D^r \arrow[dl, "j", "{(1-r,r-1)}"'] \\ & E^r \arrow[ul, "k", "{(-1,0)}"'] & \end{tikzcd}$$

We then see d_r has degree $(-r, r-1)$ as usual.

A cohomological spectral sequence is the same, just reverse signs of p, q.

Example VIII.3.1

AHSS (Atiyah-Hirzebruch Spectral Sequence) for a generalized homology theory L.

Here we have $D_{p,q} = L_{p+q}(X_p)$ where X is a CW complex and $E_{p,q} = L_{p+q}(X_p, X_{p-1})$.

The exact couple is

$$L_{p+q}(X_{p-1}) \longrightarrow L_{p+q}(X_p) \longrightarrow L_{p+q}(X_p, X_{p-1}) \longrightarrow L_{p+q-1}(X_{p-1})$$

$$D_{p-1,q+1} \xrightarrow{\quad i \quad} D_{p,q} \xrightarrow{\quad j \quad} E_{p,q} \xrightarrow{\quad k \quad} D_{p-1,q}$$

If $r \gg 0$ implies $0 = \mathrm{d}^r : E^r_{p,q} \to \cdots$.

Define then $E^\infty_{p,q} = \operatorname{colim}_r E^r_{p,q}$. This happens here because i^{r-1} is the inclusion of a lower dimensional cell, and so its image is eventually zero. This then gives

$$\begin{aligned} E^\infty_{p,q} &= \frac{\ker(L_{p+q}(X_p, X_{p-1}) \to L_{p+q-1}(X_{p-1}))}{j(\ker L_{p+q}(X_p) \to L_{p+q}(X))} \\ &= \frac{\operatorname{im}(L_{p+q}(X_p) \to L_{p+q})(X_p, X_{p-1})}{\operatorname{im}(\ker(L_{p+q}(X_p) \to L_{p+q}(X)) \to L_{p+q}(X_p, X_{p-1}))} \\ &= L_{p+q}(X_p)/(\operatorname{im} L_{p+q}(X_{p-1}) + \ker(L_{p+q}(X_p) \to L_{p+q}X). \end{aligned}$$

Then

$$F_p L_{p+q} X \coloneqq \operatorname{im} L_{p+q} X_p \to L_{p+q} X).$$

Then $E^\infty_{p,q} = F_p L_{p+q} X / F_{p-1} L_{p+q} X$

Note: Cohomological AHSS similarly converges to $L^{p+q}X$ with

$$F^p L^{p+q} X = \ker(L^{p+q} X \to L^{p+q} X^{p-1}).$$

VIII.4. **Back to Spectra**

In Equation ($\star$), we would really like X to also be a spectrum. Then we have

$$E_m X = L\pi_m(X \wedge E)$$
$$E^m X = \pi_{-m} F(X, E).$$

What do these mean? Well defining $X \wedge E$ by spectrifying $X_m \wedge E_m$ is wrong. A good way to see this is it doesn't satisfy $\Sigma^\infty Z \wedge E \sim Z \wedge E$ even for E cell.

Select two non-decreasing sequences α_m, β_m in $\mathbb{N}_0$ such that $\alpha_m + \beta_m = m$. With $\alpha_m, \beta_m \to \infty$ as $m \to \infty$. Then

$$E \wedge F = LD \qquad D_m = E_{\alpha_m} \wedge F_{\beta_m}.$$

Then $F(Z, ?)$ is right adjoint to $Z \wedge ?$.

This gives a closed symmetric monoidal structure on D Spectra. But not on Spectra because of the choice of (α_m, β_m).

But you might want that structure on spectra! (rigid rings, modules in Spectra!)

Next Semester: Math 697 (introduction to current methods).

The derived cateogry D Spectra is a closed symmetric monoidal category with $\wedge$ and $F(-,-)$.

Definition VIII.4.1

In a symmetric monoidal category $\mathscr{C}$ (operation $\otimes$). A strong dual of an object X is an object Y together with morphisms

$$\mu : 1 \to Y \otimes X \qquad \varepsilon : X \otimes Y \to 1$$

such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}\otimes\mu"] \arrow[rr, bend right, "\mathrm{Id}"'] & X \otimes Y \otimes X \arrow[r, "\varepsilon\otimes\mathrm{Id}"] & X \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "\mu\otimes\mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"'] & Y \otimes X \otimes Y \arrow[r, "\mathrm{Id}\otimes\varepsilon"] & X \end{tikzcd}$$

If this holds, we call X (and symmetrically Y) strongly dualizable and write $Y = DX$

Comments: We have the following

(1) If $Y = DX$ is a strong dual of X, then $DX \otimes ?$ is both right and left adjoint to $X \otimes ?$ (use definition of adjunction via triangle identities).

(2) If $\mathscr{C}$ is closed, $F(X, ?)$ is the right adjoint to $X \otimes ?$, and adjoints are unique, so if X is strongly dualizable then

$$DX \otimes Y \cong F(X, Y)$$
$$DX \cong F(X, 1)$$

(3) If X is strongly dualizable, then

$$X \otimes F(Z, T) \cong F(Z, X \otimes T)$$

(4) $DDX \cong X$.

Example VIII.4.1

If $\mathbb{F}$ is a field, then in $\mathbb{F}-\mathrm{Vect}$ the category of vector spaces over $\mathbb{F}$, then with $\otimes$ the tensor product we have the usual duals.

Definition VIII.4.2

E is a ring-spectrum if we have

$$\mu : E \wedge E \to E \qquad\qquad \varepsilon : \mathbb{S} \to E$$

and we have the following commutative diagram in $DSpectra$

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mathrm{Id} \wedge \mu"] \arrow[d, "\mu \wedge \mathrm{Id}"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"] & E \end{tikzcd}$$

and similarly an identity axiom

Example VIII.4.2

If $\mathscr{C} = D\,\mathrm{Spectra}$, $\otimes = \wedge$.

For a space X and a commutative ring spectrum E, E^*X is a graded commutative ring (working in $D\,\mathrm{Top}$). Why? Well consider

$$\begin{aligned} E^*X \otimes E^*X = F(X_+, E)_* \otimes F(X_+, E)_* &\to F(X_+ \wedge X_+, E \wedge E)_* \\ &\hookrightarrow F(X \times X_+, E \wedge E)_* \xrightarrow{\Delta^* \circ F(-,\mu)} F(X_+, E)_* = E^*X \end{aligned}$$

we can define morphisms of R-module spectra by commutativity with the operation. For R-modules $M \to N$

$$\begin{tikzcd} R \wedge M \arrow[r] \arrow[d] & M \arrow[d] \\ R \wedge N \arrow[r] & N \end{tikzcd}$$

But the mapping cone Cf is not in general an R-module.

Back to strong duality. Which objects are strongly dualizable in $D\,\mathrm{Spectra}$ and what are their strong duals?

Answer (Spanier): The best source is Adams stable homotopy + generalized cohomology [1]. Namely these are $\Sigma^\infty X[m]$ ($m \in \mathbb{Z}$) where X is a finite cell spectrum.

Note: In $D\,\mathrm{Spectra}$, we define for spectra E, X

$$E_m X = \pi_m(X \wedge E) \qquad\qquad E^m X = \pi_{-m}(F(X, E))$$

If X is strongly dualizable, then

$$E_m X = \pi_m(X \wedge E) \cong \pi_m(F(DX, E)) = E^{-m}(DX)$$

Spanier gave a geometric model of strong duality in $D\,\mathrm{Spectra}$ before all of this was understood. The model is entirely in spaces. Select some $N > 0$, and suppose $K, L \subseteq S^N$ with $K \cap L = \emptyset$. (The case ($\star$) we are interested in: K, L are simplicial subcomplexes of some triangulation of S^N. Further L is a deformation retract of $S^N - K$ and likewise K is a deformation retract of $S^N - L$). Then $\Sigma^\infty K \simeq DL[N-1]$

The way to construct the relevant maps is to select points $a \in K, b \in L$ and a simple path ω in S^N from a to b. Furthermore, require $\omega(t) \notin K, L$ if $t \neq 0, 1$. Select the basepoint ∞ to be $\omega(1/2)$. Then $S^N \setminus \{\infty\} = \mathbb{R}^N$. Thus we may define

$$\mu' : K \times L \to S^{N-1}$$
$$(x, y) \mapsto \frac{x - y}{\|x - y\|}$$

We then have that $K \times \{b\} \simeq$ const and likewise for L. This gives us a deformed map $\mu : K \wedge L \to S^{N-1}$. Taking suspension spectra

$$\Sigma^\infty \mu : \Sigma^\infty K \wedge \Sigma^\infty L \to \mathbb{S}^{N-1}$$
$$\Sigma^\infty K \wedge \Sigma^\infty L[1 - N] \to S^0$$

when $K = S^N \setminus L$ we can also get ε and verify triangular identities on space level by hand.

This is called Spanier-Whitehead Duality

In notation, we quite often identify a CW-complex X with the spectrum $\Sigma^\infty X$.

Spanier-Whitehead Duality: For $X \subseteq S^N$ (say a simplicial subcomplex, then $D\Sigma^\infty X = \Sigma^\infty(S^N \setminus X)[-N + 1]$.

> **Recall VIII.4.3**
>
> when Z in *DSpectra* is strongly dualizable
>
> $$DZ = F(Z, S^0)$$

What about X_+? Well we have a cofiber sequence

$$X_+ \to S^0 \to \Sigma X$$

And after applying Σ^∞ we have

$$DS^0 \to DX_+ \to DS^0.$$

That is

$$S^N \setminus X[-N] \to S^N[-N] = S^0 \to DX_+ \to S^N \setminus X[1 - N] = DX.$$

last term is the mapping cone.

Answer:

$$D(\Sigma^\infty X_+) = D(X_+) = C(S^N \setminus X \to S^N)[-N]$$

if U is an open neighborhood of X in S^N, this is the same as

$$C(U \setminus X \to U)[-N]$$

A particularly interesting case is $X = M$ being a compact smooth N-manifold. Then we can embed

$$M \subseteq \mathbb{R}^N \subseteq S^N = \mathbb{R}^N \cup \{\infty\}$$

IX. Vector Bundles

General structure: A family of finite-dimensional vector spaces indexed by X.

Form a category of space over X, whose objects are continuous maps $Y \to X$ and whose morphisms are diagrams

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

Definition IX.0.1 (Vector Bundle)

A topological vector space over X can then be defined in this category. Explicitly for a total space E with a map $p : E \to X$ there are addition, multiplication, negation, and zero maps as below. For $\lambda \in \mathbb{R}$,

$$\begin{tikzcd} E \times E \arrow[rr, "+"] \arrow[dr, "p \times p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "\lambda \cdot -"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr, "0"] \arrow[dr, "\mathrm{Id}_X"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "-"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

satisfying the obvious commutative diagrams.

which is locally isomorphic to a product with $\mathbb{R}^n$. That is there exists an open cover U_i of X such that bulling back to each U_i, $p^{-1}(U_i) \to U_i$ is isomorphic, as a vector space over U_i, to $U_i \times \mathbb{R}^n \to U_i$.

Example IX.0.1

Möbius Band $\to S^1$.

Tubular Neighborhood Theorem: If $M \subseteq M'$ is a smooth embedding of compact manifolds, then there exists an open neighborhood U of M in M' which is homeomorphic to the total space of a vector bundle (with M embedded as the 0-section).

The normal bundle of M in M' for example.

For $M \subseteq \mathbb{R}^N \subseteq S^N$ we have

$$\begin{aligned} DM_+ &= C(S^N \setminus M \to S^N)[-N] \\ &= C(U \setminus M \to U)[-N] \\ &= C(E \setminus M \to E)[-N] \end{aligned}$$

where U is a tubular neighborhood with bundle E, and $E \setminus M \to E$ is the inclusion by embedding M into E via the 0-section. Call $E_0 \coloneqq E \setminus M$. We also have $E = \nu_M S^N$, where $\nu_M S^N$ is the normal bundle of M in S^N

For a vector bundle E, what does $C(E_0 \to E)$ look like?

Say, the bundle $E \xrightarrow{p} X$ over X has X compact.

Claim

$C(E_0 \to E)$ is homootpy equivalent to the 1-point compactification of E, which is equivalent to $D(E)/S(E)$ (the 1-point compactification of the open disk bundle)

Where $S(E) \to D(E)$ is the inclusion of the unit sphere bundle into the unit disk bundle.

The 1-point compactification of E (where $p : E \to X$ is a vector bundle, X compact) is called the Thom space of E, someitmes denoted by X^E or $T(E)$.

If X is not compact, $D(E)/S(E)$ does not compactify E, so we need to define the Thom space by

$$X^E := \operatorname{colim}_{\substack{Z\subseteq X \\ \text{compact}}} Z^E$$

<u>The conclusion</u>: If M is a connected compact m-manifold embedded in $\mathbb{R}^N$ then

$$DM_+ = M^{\nu_M \mathbb{R}^N}[-N]$$

What can we say about $\nu_M\mathbb{R}^N$. Well

$$\nu_M\mathbb{R}^N \oplus T(M) = N$$

where $\oplus = \times_M$ is the whitney sum, $T(M)$ is the tangent bundle of M, and N is the trivial bundle of dimension N.

One can prove that (since M is compact), if $N \gg 0$ then the Whitney sum component of a bundle ξ in N (a bundle μ such that $\xi \oplus \mu \cong N$) is uniquely determined after isomorphism.

So in fact, selecting $N \gg 0$, $\nu_M\mathbb{R}^N$ is determined (we say: the normal bundle is <u>stably determined</u>).

<u>Next</u>: E-orientability for a commutative ring spectrum E (commutative monoid in *DSpectra*), leading us to E-Poincaré duality.

For a compact connected smoothly embedded m-manifold $M \subseteq \mathbb{R}^N$ we have by Spanier-Whitehead duality that

$$DM_+ = M^{\nu_M \mathbb{R}^N}[-N].$$

Recall that the Thom space of a vector bundle $\xi \to X$ for X compact is

$$X^\xi = \text{1-point compactification of } \xi$$

For general X, we have

$$X^\xi = \operatorname{colim}_{\substack{Z\subseteq X \\ \text{compact}}} Z^\xi.$$

Recall: If E is a spectrum and X is strongly dualizable, then

$$E_k X = E^{-k} DX.$$

We know $\Sigma^\infty M_+$ for a compact smooth connected manifold is strongly dualizable, so

$$E_k M = \widetilde{E}^{N-k} M^{\nu_M \mathbb{R}^N}$$

We also see that $\nu_M\mathbb{R}^N$ has dimension $N-m$. We can think of M^ξ as a "twisted suspension" of M by the dimension of the bundle. Indeed if $\xi = \ell$ was a trivial bundle, then $M^\xi = \Sigma^\ell M_+$.

Under what circumstances can we "untwist the Thom space to the eyes of the spectrum E"?

Suppose E is a commutative ring spectrum (a commutative monoid in *DSpectra*).

Thom realized that if ξ is an m-bundle on X, then there is a natural map

$$\theta : X^\xi \to X^\xi \wedge X_+$$
$$y \in \xi \mapsto (y, \operatorname{proj} y)$$
$$\infty \mapsto \infty$$

It is an exercise to check continuity at ∞.

If X is a CW-complex

$$\widetilde{E}^*(\theta) : \widetilde{E}^*(X^\xi \wedge X_+) \to \widetilde{E}^*(X^\xi)$$

and using that it is a ring theory, we have a map

$$\widetilde{E}^k(X^\xi) \otimes E^\ell(X) \to \widetilde{E}^{k_\ell}(X^\xi \wedge X_+) \to \widetilde{E}^{k+\ell}(X^\xi).$$

The m-bundle ξ is called E-orientable if there exists a class $u \in \widetilde{E}^m(X^\xi)$ (called the Thom class) which for each point $x \in X$ restricts to a unit.

That is

$$\widetilde{E}^m(X^\xi) \to \widetilde{E}^m(\{x\}^\xi) = \widetilde{E}^m(S^m) = E_0(*).$$

If E is a ring spectrum then $E_0(*)$ is a commutative ring, and so we can just trace u through this map and see if it becomes a unit.

Thom Isomorphism Theorem:

If an m-bundle ξ is E-orientable with Thom class u, then

$$\Xi : \widetilde{E}^m(X^\xi) \otimes E^\ell(X) \to \widetilde{E}^{m+\ell}(X^\xi)$$

restricts to an isomorphism

$$\Xi(u \otimes ?) : E^\ell(X) \to \widetilde{E}^{m+\ell}(X^\xi)$$

Proof sketch. Take open cover $\{U_i\}_{i\in I}$ of X where $\xi\big|_{U_i}$ is trivial for each i. Then use the Meyer-Vietoris sequence and the five lemma.

If I is infinite, a limit argument is needed.

Note that in fact

$$E^\ell(X) = \widetilde{E}^{m+\ell}(\Sigma^m X_+).$$

Thus if ξ is E-orientable, then X^ξ "untwists" to the eyes of E.

Definition IX.0.2 (First Version)

A compact connected m-manifold is E-orientable for a commutative ring spectrum E when the normal bundle $\nu_M\mathbb{R}^N$ is E-orientable.

Then we can conclude that

$$E_k(M) \cong \widetilde{E}^{N-k} M^{\nu_M\mathbb{R}^N} \cong E^{N-k-N+m}(M) = E^{m-k}(M)$$

because $\dim \nu_M\mathbb{R}^N = N - m$.

This is called E-Poincare duality.

How can we make the definition of orientability more elegant? The Thom class of $\nu_M\mathbb{R}^N$ (if there is one) is in

$$\widetilde{E}^{N-m} M^{\nu_M\mathbb{R}^N} \cong E_m(M)$$

Definition IX.0.3 (Final version)

An E-orientation of a compacted connected smooth m-manifold M is a class $[M] \in E_m(M)$ (sometimes called the "fundamental class") such that the embedding of pairs $(M, \emptyset) \xrightarrow{\iota_x} (M, M \setminus \{x\})$, for every $x \in M$ sends $[M]$ to a unit.

That is we see that

$$E_m(M) \to E_m(M, M \setminus \{x\}) = \widetilde{E}_m(C\iota_x) \cong E_m(U, U \setminus \{x\}) \cong E_m(S^m) = E_0(*)$$

where $x \in U \cong \mathbb{R}^m$ is open. Again $E_0(*)$ is a ring and we can define this correctly.

Theorem IX.0.1 (Poincaré duality)

If M is an E-orientable compact connected m-manifold then Spanier-Whitehead duality, using the Thom class corresponding to the fundamental class $[M]$, define an isomorphism

$$E_k M \cong E^{m-k} M$$

Remark IX.0.1

For $E = H\mathbb{Z}/2$ (ordinary cohomology with coefficients $\mathbb{Z}/2$) every (compact smooth connected) manifold is $H\mathbb{Z}/2$-orientable.

Note that $H\mathbb{Z}/2_0(*) = \mathbb{Z}/2$ has a unique non-zero element.

$H\mathbb{Z}$-orientability is equivalent to $H\mathbb{R}$-orientability which (at least for compact, smooth, connected manifolds) is equivalent to the existence of a nowhere vanishing differential m-form.

This is related to the statement that $\mathbb{Z}$ only has two units.

X. A Plethora of Examples

For the last week, we will talk about Examples of Spectra, that is of generalized homology/cohomology.

Example X.0.1 (Universal)

$BO(m) = \{m\text{-dimensional real vector subspaces of } \mathbb{R}^\infty\}$. That is $EO(m)/O(m)$.

If we then consider

$$\gamma_{\mathbb{R}}^m = \{(V, x) \mid V \subseteq \mathbb{R}^\infty, \dim V = m, x \in V\}.$$

Then there's a map $\gamma_{\mathbb{R}}^m \to BO(m)$.

Similarly for $\mathbb{C}$, with $BU(m)$ and $\gamma_{\mathbb{C}}^m \to BU(m)$.

Theorem X.0.1

If X is paracompact, then

$$\{\cong \text{ clsses of real vector } m\text{-bundles on } X\} \cong [X, BO(m)]$$

And also

$$\{\cong \text{ clsses of complex vector } m\text{-bundles on } X\} \cong [X, BU(m)]$$

The map is given by $f : X \to BO(m)$ to $f^*\gamma_{\mathbb{R}}^m$ (the pullback), and likewise for complex vector bundles.

References: [8, 3]

This gives a geometric interpretation of K-theory (cohomology). A permutative category $\mathscr{C}$ with objects $\mathbb{N}_0$ and morphisms $m \to m$ given by $U(m)$.

We then set

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

So then

$$B\mathscr{C} = \coprod_{m \geq 0} BU(m)$$

is an E_∞-space, and by group completion

$$\Omega B(B\mathscr{C}) = BU^+ \times \mathbb{Z}$$

where BU^+ is the Quillen $+$-construction. Then

$$U = \bigcup_m U(m)$$

So we have that

$$\pi_0 U(m) = 0 \qquad \pi_0 BU(m) = 0 \pi_1 U(m) = \mathbb{Z} \qquad \pi_1 BU(m) = 0$$

By the fibration sequence $U(m) \to EU(m) = * \to BU(m)$. Thus $BU^+ = BU$.

Thus we have proved $BU \times \mathbb{Z}$ is an infinite loop space without using Bott periodicity.

Note that

$$BU \times |ZZ = \operatorname{colim} \left(\coprod_{m \in \mathbb{N}_0} \xrightarrow{\oplus 1} \coprod_{m \in \mathbb{N}_0} BU(m) \to \cdots \right)$$

If X is compact Hausdorff, then

$$\left[X, \bigcup Z_m\right] = \operatorname{colim}_m [X, Z_m]$$

where $Z_0 \subseteq Z_1 \subseteq \cdots$.

If X is compact then by definition

$$K^0 X = [X, BU \times \mathbb{Z}]$$

is the group completion of $\{\cong$ classes of complex vector bundles on $X\}$ which is a commutative monoid with Whitney sum.

Grothendieck construction K is left adjoint to the forgetful functor Ab $\to$ commutative monoids.

For example $K(\mathbb{N}_0) = \mathbb{Z}$.

Elements of $K^0(X)$ are virtual bundles. Namely they look like

$$(\xi, \mu)/\sim$$
$$(\xi, \mu) \sim (\xi', \mu') \iff \exists \nu \;\; \xi \oplus \mu' \oplus \nu \cong \xi' \oplus \mu \oplus \nu$$

We think of the pair (ξ, μ) as "$\xi - \mu$." For X compact, any virtual bundle is of the form $\xi - N$ for N trivial, should refer to Atiyah's K-theory.

This definition of $K^0 X$ as the group completion of isomorphism classes of vector bundles on X is not invariant under weak equivalence. For X CW we have $K^0(X) = [X, BU \times \mathbb{Z}]$.

Vector bundles also have a tensor product. This introduces a commutative ring structure on $K^0 X$. In fact, K is a commutative ring spectrum(for now, commutative monoid in $DSpectra$ w.r.t $\wedge$).

Can we say something more rigit (= coherent) in *Spectra*, where $\wedge$ is not a symmetric monoidal product. Answer: yes. We will call this a multiplicative ∞ loop space.

Another example of a constructon of spectra:

X.1. **Cobordism**

The geometric problem of cobordism: compact smooth manifolds without boundy M of dimension m, with equivalence defined by cobordism

> **Definition X.1.1**
>
> We say two compact manifolds M, N of dimension m are cobordant if there is a compact manifold W of dimension $m + 1$so that $\partial W = M \coprod N$ with the normal data preserved.
>
> That is
>
> $$\nu_W^{\mathbb{R}^N} \oplus \tau_W \cong N \qquad \text{(trivial)}$$
>
> so that
>
> $$\tau\big|_M = \tau_M \oplus 1 \qquad\qquad \tau\big|_N = \tau_N \oplus 1.$$
>
> For refernece see [8]

to avoid the problems, we prescribe some requirement on the normal bundle $\nu_M^{\mathbb{R}^N}$ (well-defined if whatever structure we require on $\nu_M^{\mathbb{R}^N}$ for $N \gg 0$ must be preserved by enlarging N, aka it is “stable”).

> **Example X.1.1**
>
> There are many examples
>
> - $\mathbb{N}_0$ structure, then unoriented MO.
> - Oriented (w.r.t $H\mathbb{Z}$), gives oritented MSO
> - Complex gives complex MU.
> - Trivial gives framed $\leq$.

Equivalence classes of each type of manifolds under cobordism are called cobordism groups MO_m, MSO_m, MU_m.

Why groups? The group operation is $\coprod$, and the inverse is to add the trivial bundle, reverse sign of 1-dimensional subspace in the isomoprhism class.

What

Cobordism, compact smooth closed m-manifolds M with some normal data on $\nu_M^{\mathbb{R}^N}$ which can be

- no data (unoriented)
- oriented
- complex
- trivial bundle (this is called framed cobordism)

modding out by cobordism, that is $M_1 \sim M_2$ when

$$M_1 \coprod M_2 = \partial M$$

Where M has the same type of normal data, which restricts to M_i (usually with signs).

This is a group, whose operation is $\coprod$. How is this related to spectra?

We're now going to follow the Pontrjagin-Thom construction

Embed M as $M \subseteq \mathbb{R}^N \subseteq S^N$. Then there is a tubular neighborhood U of M, which is homeomorphic to $\nu_M^{\mathbb{R}^N}$ via some ι. This gives a map $S^N \to M^{\nu_M^{\mathbb{R}^N}}$ (which is the Thom space, or the 1-point compactification of

$\nu_M^{\mathbb{R}^N}$). How? Well

$$S^N \xrightarrow{\varphi} M^{\nu_M^{\mathbb{R}^N}}$$
$$U \overset{\iota}{\mapsto} \nu_M^{\mathbb{R}^N}$$
$$S^N \setminus U \mapsto *$$

This still uses the manifold. But! We have classification of bundles: $\cong$ classes of k-real bundles on M via $[M, BO(k)]$. For oriented k-real bundles we have $[M, BSO(k)]$ (where $BSO(k)$ is the universal cover of $BO(k)$). For complex k-bundles we have $[M, BU(k)]$. And there is only one trivial k-bundle $[M, *] = *$.

We apply this classification to the normal bundle $\nu_M^{\mathbb{R}^N}$. We have $k = N - m$ expect in the complex case where $k = \frac{N-m}{2}$.

The classification map, say in the unoriented real case:

$$M \to BO(k) \qquad (k = N - m)$$
$$\nu_M^{\mathbb{R}^N} \to \gamma_R^k$$
$$M^{\nu_M^{\mathbb{R}^N}} \to BO(k)^{\gamma_{\mathbb{R}}^k}.$$

From the data of $\nu_M^{\mathbb{R}^N}$ we get a map

$$S^N \to BO(N-m)^{\gamma_{\mathbb{R}}^{N-m}}$$

with $N \gg 0$. A cobordism, by an analogous construction, on the manifold representing the cobordism, gives a homotopy. Thus by starting with a cobordism class we obtain a homotopy class

$$S^N \to BO(N-m)^{\gamma_{\mathbb{R}}^{N-m}}$$

In the oriented case, we have $S^N \to BSO(N-m)^{\gamma_{\mathbb{R}}^{N-m}}$. And in complex case we have $N - m = 2k$ and $S^N \to BU(k)^{\gamma_{\mathbb{C}}^k}$. In the trivial case we get $S^N \to S^{N-m}$.

Thom observed that there is an inverse to this procedure. Say we have $S^N \xrightarrow{f} BO(N-m)^{\gamma_{\mathbb{R}}^{N-m}}$. Because the Thom Space is locally nice, one can talk about transversality with respect to fibers. If f is transverse to the 0-section embedding $BO(N-m)$ in the Thom Space, then f^{-1}(0-section) is an m-manifold. In the cases with structure, it automatically gains the desired structure on $\nu_M^{\mathbb{R}^N}$.

Theorem X.1.1 (Thom)

These two procedures are inverse to each other. For details see [8, 10].

What about this $N \gg 0$? Well then we have

$$MO_m = \operatorname{colim}_k \pi_{m+k} BO(k)^{\gamma_{\mathbb{R}}^k}$$
$$MSO_m = \operatorname{colim}_k \pi_{m+k} BSO(k)^{\gamma_{\mathbb{R}}^k}$$
$$MU_m = \operatorname{colim}_k \pi_{m+2k} BSO(k)^{\gamma_{\mathbb{C}}^k}$$
$$M_{\text{framed}}(m) = \operatorname{colim}_k \pi_{m+k} S^k = \pi_m \mathbb{S} = \pi_m^S.$$

The first three can be thought of as homotopy groups of twisted suspension spectra, which are now called Thom spectra. This means cobordism is intricately linked with stable homotopy theory.

Consider the complex case. We have a prespectrum $D_{2k} = BU(k)^{\gamma_{\mathbb{C}}^k}$. This is given by

$$\Sigma^2 D_{2k} \to D_{2k+2}$$

$$BU(k)^{\gamma_{\mathbb{C}}^k \oplus 1_{\mathbb{C}}} \to BU(k+1)^{\gamma_{\mathbb{C}}^{k+1}}$$

via the classsification of $(k+1)$-bundles. We could then just set $D_{2k+1} = \Sigma D_{2k}$.

We spectrify to get MU (in the other csaes $MO, MSO, \mathbb{S}$). This tells us that framed cobordism groups are stable homotopy groups of spheres, and we can get the first few stable homotopy groups this way before it becomes intractable.

Exercise X.1.2

Show that $\pi_1 \mathbb{S} = \mathbb{Z}/2$ using this method.

Amazingly, in the other cases we listed, the cobordism groups (π_* of the Thom spectra MO, MSO, MU) can be completely calculated. This can be calculated by general methods of calculating homotopy groups of spectra. Namely, this uses the Adams spectral sequence.

Strategy: Look at $F(H\mathbb{Z}/p, H\mathbb{Z}/p)_* = A^*$ (the Steenrod Algebra), these are stable operations (that is natural transfromations) in mod p cohomology of spaces. We work modulo p because $\mathbb{F}_p$ is a field. We have that

$$F(H\mathbb{Z}, H\mathbb{Z}) = \mathbb{Z} \oplus (p\text{-torsion, all } p \text{ together}).$$

The Adams Spectral Sequence:

$$\mathrm{Ext}_{A^*}(H^* X, \mathbb{Z}/p) \Rightarrow (\pi_*^S X)_p^\wedge$$

That is $\pi_*^S X$ completed at p, where X is a CW-complex of finite type. This comes from

$$X \to X \wedge H\mathbb{Z}/p \to X_1$$

$$X_1 \to X_1 \wedge H\mathbb{Z}/p \to X_2$$

all of these cofibration sequences (mapping cones then) working entirely in the category of spectra. This leads to an exact couple, giving the adams spectral sequence. A great book for this is Ravenel's Complex Cobordism and Stable Homotopy Groups of Spheres [12]

Hard for $\mathbb{S}$, but for MO, MSO, MU it is relatively easy. For example

$$\pi_* MO = \mathbb{F}_2[y_i \mid i \neq 2^k - 1].$$

Furthermore

$$MO = \bigvee \Sigma^2 H\mathbb{Z}/2$$

This is just a sum of copies of $H\mathbb{Z}/2$, sometimes called a (GEM, a Generalized Eilenberg-Maclane spectrum aka nothing new).

However $\pi_* MU$ is more interesting

$$\pi_* MU = \mathbb{Z}[x_1, x_2, x_3, \ldots]$$

where $\deg(x_i) = 2i$. This is an interesting new spectrum (not a GEM). HOW?

X.2. Complex Oriented Spectra

A commutative ring spectrum (commutative monoid in $DSpectra$) E is called complex-oriented when the universal complex line bundle $\gamma_{\mathbb{C}}^1$ on $\mathbb{C}\mathrm{P}^\infty = BU(1)$ is E-oriented.

What does the Thom Space $(\mathbb{C}\mathrm{P}^\infty)^{\gamma^1_\mathbb{C}}$ look like? For any bundle ξ on a space X, we have a cofiber sequence

$$S(\xi)_+ \to X_+ \to X^\xi$$

where $S(\xi)$ is the unit sphere bundle (given a Euclidean metric), equivalently $S(\xi) \simeq \xi \setminus (0\text{-section})$. We then know that

$$S(\gamma^1_\mathbb{C} \simeq \gamma^1_0 \setminus (0\text{-section}) = \mathbb{C}^\infty \setminus 0 \simeq *.$$

Therefore $(\mathbb{C}\mathrm{P}^\infty)^{\gamma^1_\mathbb{C}} \simeq \mathbb{C}\mathrm{P}^\infty$. This comes from a cofiber sequence

$$S^0 \to \mathbb{C}\mathrm{P}^\infty_+ \to (\mathbb{C}\mathrm{P}^\infty)^{\gamma^1_\mathbb{C}}$$

Next time: talk more about complex oriented theories and formal group laws, why you might care about equivariant topology and structured / coherent topology.

If a cohomology theory is complex oriented and u is the Thom class, then

$$E^*\mathbb{C}\mathrm{P}^\infty = E_*[[u]]$$

where we allow infinite sums which are homogeneous, with u having cohomological degree 2. One can compute this with AHSS

Similarly $E^*(\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty) = E^*[[u_1, \ldots, u_m]]$.

Then $\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty \to BU(m)$. This then gives a map

$$E^*BU(m) \to E^*(\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty) = E_*[[u_1, \ldots, u_m]]$$

But note $\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty = B(S^1 \times \cdots \times S^1)$, with an action $\Sigma_m \subseteq U(m)$. On Homework, we proved inner automorphisms of G induce $\simeq$ Id on BG.

This means it factors through as

$$E^*BU(m) \to E^*[[u_1, \ldots, u_m]]^{\Sigma_m} \to E^*[[u_1, \ldots, u_m]]$$

If $c_i = \sigma_i(u_1, \ldots, u_m)$ is the elementary symmetric polynomial this is $E^*[[c_1, \ldots, c_m]]$.

AHSS injects on E_2-terms so target collapses. Thus this is an isomorphism and

$$E^*BU(m) = E^*[[c_1, \ldots, c_m]]$$

We can then induct $BU(m-1)_+ \to BU(m)_+ \to BU((m)^{\gamma^m_\mathbb{C}}$ to show $\gamma^m_\mathbb{C}$ is E-oriented. Every complex bundle is then E-oriented. The symmetric polynomials $c_1, \ldots, c_m$ are called Chern classes.

Let ξ be an m-bundle on X, then $c_i \in E^{2i}X$. What is the classification of $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$, the line bundle $\gamma^1_\mathbb{C} \otimes \gamma^1_\mathbb{C}$.

Also $\mathbb{C}\mathrm{P}^\infty = K(\mathbb{Z}, 2)$, addition in $H^2(?; \mathbb{Z})$. Well this is a map

$$\begin{aligned} E^*[[u]] &\to E^*[[u_1, u_2]] \\ u &\mapsto F(u_1, u_2) = u_1 +_F u_2. \end{aligned}$$

We get properties like

$$x +_F 0 = x = 0 +_F x$$
$$x +_F y = y +_F x$$
$$(u +_F v) +_F w = u +_F (v +_F w)$$

a power series $F \in R[[u,v]$ which satisfies these two properties is called a formal group law (FGL).

Example X.2.1

K-theory, $K_* = K_*(*) = \mathbb{Z}[\beta, \beta^{-1}]$ where β is a Bott class in degree 2. If we omit β from the notation, then

$$u = \gamma^1_{\mathbb{C}} - 1 \in \widetilde{K}^0 \mathbb{C}\mathrm{P}^\infty$$

that is a virtual bundle of dimension zero. The tensor product of $u+1, v+1$ is $(u+1)(v+1)$. Subtracting 1, the formal group law is

$$u +_F v = u + v + uv.$$

This is called a multiplicative FGL.

For $H\mathbb{Z}$ (ordinary cohomology) we have $u +_F v = u + v$. This is called an additive FGL.

Example X.2.2

$E^*\mathbb{R}\mathrm{P}^\infty$. We then have a cofiber sequence

$$\mathbb{R}\mathrm{P}^\infty_+ \to \mathbb{C}\mathrm{P}^\infty_+ \xrightarrow{(\gamma^1_{\mathbb{C}})^2} (\mathbb{C}\mathrm{P}^\infty)^{(\gamma^1_{\mathbb{C}})^2}.$$

If E is complex-oriented, then we have

$$E^*\mathbb{R}\mathrm{P}^\infty \longleftarrow E^*[[u]] \xleftarrow{u +_F u} E^*[[u]][2].$$

In principle this is a long exact sequence, but if the right map is injective, it's a short exact sequence and

$$E^*\mathbb{R}\mathrm{P}^\infty = E^*[[u]]/(u +_F u).$$

Example X.2.3

K-theory (ignore the Bott class. Then

$$[2]_F u = (1+u)^2 - 1 = 2u + u^2.$$

This is injective on $\mathbb{Z}[u]$. Thus

$$K^0\mathbb{R}\mathrm{P}^\infty = \mathbb{Z}[[u]]/(1+u)^2 - 1$$
$$K^1\mathbb{R}\mathrm{P}^\infty = 0$$

This is isomorphic for $t = 1 + u$ to

$$(\mathbb{Z}[t]/(t^2-1))^\wedge_{t-1} = \mathbb{Z}_2 \oplus \mathbb{Z}.$$

Where does this come from? Well t is essentially the tautological bundle. When restricted to $\mathbb{Z}/2$, $\mathbb{Z}[t]/(t^2-1)$ is the complex representation ring of $\mathbb{Z}/2$. That is

$$R(G) = K(\text{comm. monoid of f.d. complex representations of } G).$$

This is an example of the below theorem.

Theorem X.2.1 (Atiyah-Segal Completion)

If G is a compact Lie group (including finite groups) then $K^0BG = (R(G))^\wedge_I$, where I is the augmentation ideal (virtual representations of dimension zero) and $K^1BG = 0$.

They considered G-equivariant K-theory. For compact CW-complexes, you want to take G-equivariant complex bunbldes. Then this comes from

$$K^0_G(*) = R(G) \qquad\qquad K^1_G(*) = 0.$$

This motivated the idea of equivariant generalized cohomology theory.

X.3. More Formal Group Laws

Can we classify Formal Group Laws? Well let

$$F(x, y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j$$

and consider the Lasard ring

$$L = \mathbb{Z}[a_{ij}]/(\text{relations from requiring that } F \text{ be an FGL}).$$

For example

$$a_{i,j} = a_{j,i} \qquad\qquad a_{io} = 0, i > 1 \ldots$$

So now F is an FGL on L. Then we have that

$$\{\text{FGLs on } R\} = \mathrm{Mor}_{\mathrm{Ring}}(L, R)$$

> **Theorem X.3.1** (Lasard's)
> We have that $L = \mathbb{Z}[x_1, x_2, \ldots]$.

We now notice the complex cobordism spectrum MU is complex-oriented ($(\mathbb{C}\mathrm{P}^\infty)^{\gamma^1}$ is a term in the prespectrum).

> **Theorem X.3.2** (Milnor-Novikov)
> The FGL on $MU^*[[u]]$ (from complex orientation) is the Lasard universal formal group law. This is somehow familiar because ($MU_* = \mathbb{Z}[x_1, \ldots,]$)

Why are FGLs important? They come up in number theory. Natural question: can we make a complex-oriented spectrum from MU say by "killing generators", inverting others, and so on.

Answer: Not in the derived category! We need some coherence and some replacement for $\wedge$ not being strictly commutative, associative.

This becomes the general theory of

Brave new algebra, Spectral algebra, Higher algebra.

This subject is focused on how to introduce higher coherence.

How do FGLs come up in number theory. Well if K is a finite field extension of $\mathbb{Q}$, that is called a number field. We can describe Galois extensions of $K \subseteq L$ with abelian Galois group by their number theoretical properties. This area of mathematics is called class field theory.

Formal Group Laws cannot do Class Field Theory, but they *can* do it locally. The local question is to instead consider $\mathbb{Q}_p$ = fractions of $\mathbb{Z}_p$, where $\mathbb{Z}_p = \lim \mathbb{Z}/(p^n)$. A finite field extension $\mathbb{Q}_p \subseteq K$ is called a local number field.

The finite exntesions of $\mathbb{F}_p$ are $\mathbb{F}_{p^n}$, with $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathbb{Z}/n$.

We can lift to $\mathbb{Z}_p \subseteq W$. The field of fractions K (unramified degree in extension of $\mathbb{Q}_p$). Then $W = \mathcal{O}_K$.

Theorem X.3.3 (Lubin-Tate)

if we have $x^{p_{i_1}} + \cdots + px = f(x)$ where the middle bit is divisible mod p, then there exists an FGL such that $[p]_F x = f(x)$ on $\mathcal{O}_K$.

Then

$$K[x]/f^{\circ n}(x)/f^{\circ(n-1)}(x)$$

this is a totally ramified (Eistenstein's polynomial) extension of K, which is both abelian + Galois. Furthermore F is an $\mathcal{O}_K$ module Then $[\alpha]x$ makes sense for $\alpha \in \mathcal{O}_k$. Furthermore the Galois action is

$$\alpha(x) \mapsto [\alpha]x.$$

Therefore we have that

$$\mathrm{Gal}(L/K) = (\mathcal{O}_K/p^m\mathcal{O}_K)^\times.$$

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TODOS: