Lets deal with examples of the cup product. That is when R is a commutative ring we have a map

$$\smile : H^*(X;R) \otimes H^*(X;R) \to H^*(X;R)$$

which is given by the Eilenberg-Zilbur theorem from $\Delta : X \to X \times X$, and gives $H^*(X;R)$ the structure of a graded commutative ring.

Lets cover the case when $X = BG$, for G a discrete group. $\pi_1 X = G$ and the universal cover $\widetilde{X}$ of X is contractible.

Then

$$H^*(X;R) = H^*(G;R) = \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z};\mathbb{R}).$$

Translating the sotry to algebra: Let C be a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$. Then $C \otimes_{\mathbb{Z}} C$ is a $\mathbb{Z}[G] \otimes \mathbb{Z}[G] = \mathbb{Z}[G \times G]$-free resolution of $\mathbb{Z}$ (by the Kunneth theorem)

We have the diagonal homomorphism $g \mapsto (g,g)$. Via the diagonal morphism, $C \otimes_{\mathbb{Z}} C$ is also a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$. This is free because G-action on $G \times G$ is a free action. By the functoriality of resolutions, there exists some map of $\mathbb{Z}[G]$-module chain complexes

$$C \to C \otimes_{\mathbb{Z}} C$$

which induces $1 : \mathbb{Z} \to \mathbb{Z}$ on H_0 (unique up to chain homotopy). Once we have this, we obtain a map

$$\mathrm{Hom}(C,R) \otimes_R \mathrm{Hom}(C,R) \to \mathrm{Hom}(C \otimes_{\mathbb{Z}} C, R) \to \mathrm{Hom}(C,R)$$

Example .0.1

Lets go with $G = \{1, \alpha\}$ with $\alpha^2 = 1$. Then the free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$ is

$$C : \cdots \xrightarrow{1-\alpha} \mathbb{Z}[G] \xrightarrow{1+\alpha} \mathbb{Z}[G] \xrightarrow{1-\alpha} \mathbb{Z}[G]$$

Then $G \times G = \{1, \alpha, \beta, \gamma\}$ with $\alpha^2 = \beta^2 = \gamma^2 = 1$ and $\alpha\beta = \gamma$. The double chain complex is

$$\begin{tikzcd}
 & \vdots & \vdots \\
\cdots \arrow[r] & \mathbb{Z}[G \times G] \arrow[r, "1-\alpha"] \arrow[d, "1-\beta"] & \mathbb{Z}[G \times G] \arrow[d, "1-\beta"] \\
\cdots \arrow[r, "1+\alpha"] & \mathbb{Z}[G \times G] \arrow[r, "1-\alpha"] & \mathbb{Z}[G \times G]
\end{tikzcd}$$

Now lets look at $C \to C \otimes_{\mathbb{Z}} C$, thinking of C with the maps γ. On each term, where do I send 1.

$$\begin{tikzcd}
 & \alpha \arrow[d, mapsto] & \\
1 \arrow[r, mapsto] & 1 - \gamma = 1 - \alpha + \alpha(1-\beta) & 1
\end{tikzcd}$$

And then we do this again

$$\begin{tikzcd}
 & & 1 \arrow[d, mapsto] \\
 & \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \alpha + \beta = \alpha - 1 + 1 - \beta \\
1 \arrow[r, mapsto] & 1 + \gamma = 1 + \alpha + \alpha(\beta - 1) &
\end{tikzcd}$$

Homework #5

(3) Denoting by e_n the $1 \in \mathbb{Z}[\{1,\gamma\}] \in C$, prove that

$$e_n \mapsto \sum_{\substack{\ell \text{ even} \\ k+\ell=n}} e_k \otimes e_\ell + \alpha \sum_{\substack{\ell \text{ odd} \\ k+\ell=n}} e_k \otimes e_\ell$$

Prove that this gives a ($\mathbb{Z}\{1,\gamma\}$-equivariant) chain map $C \to C \otimes_{\mathbb{Z}} C$.

Every $\mathbb{Z}/2$ in bidegree k,ℓ goes to $\mathbb{Z}/2$ in bidegree $k+\ell$. This tells us that

$$H^m(\mathbb{Z}/2;\mathbb{Z}/2) = H^m(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2$$

That is the cup product

$$\smile : H^k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^\ell(\mathbb{R}\mathrm{P}^\infty,\mathbb{Z}/2) \to H^{k+\ell}(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$$

is an isomorphism, since the left and right hand sides are both $\mathbb{Z}/2$. We conclude that $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[x]$. REcall that $H^n(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ is $\mathbb{Z}$ when $n=0$, $\mathbb{Z}/2$ in even degrees, and zero in odd degrees.

The cup product is functorial in the ring. Thus

$$H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}) \to H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$$

This is given as

$$\begin{array}{ccc}
\vdots & & \vdots \\
0 & \longrightarrow & \mathbb{Z}/2 \\
b^2, \mathbb{Z}/2 & \longrightarrow & a^4, \mathbb{Z}/2 \\
0 & \longrightarrow & \mathbb{Z}/2 \\
b, \mathbb{Z}/2 & \longrightarrow & a^2, \mathbb{Z}/2 \\
0 & \longrightarrow & \mathbb{Z}/2 \\
1, \mathbb{Z} & \longrightarrow & a^0, \mathbb{Z}/2
\end{array}$$

Thus $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}) = \mathbb{Z}[b]/(2b)$. If ℓ is any number then $H^*(\mathbb{Z}/\ell;\mathbb{Z}) = \mathbb{Z}[b]/(\ell b)$, where b is in degree 2. Note that if p and $a \in \mathbb{Z}/p = H^1(\mathbb{Z}/p,\mathbb{Z}/p)$ in degree one then

$$a \smile a = (-1)^{1\cdot 1} a \smile a = -a \smile a$$
$$a \smile a = 0$$

Thus $H^*(\mathbb{Z}/p;\mathbb{Z}/p) = \mathbb{Z}/p[b] \otimes_{\mathbb{Z}/p} \wedge \mathbb{Z}/p[a]$

Back to topology. The unit sphere S^∞ in $\mathbb{C}^\infty = \bigoplus_\infty \mathbb{C}$. Then S^1 acts on S^∞ by multiplying in every coordinate.

Thus $\mathbb{Z}/\ell < S^1$ acts on S^∞. $B\mathbb{Z}/\ell = S^\infty/\mathbb{Z}/\ell$.Also $\mathbb{C}\mathrm{P}^\infty = S^\infty/S^1$, which one can call BS^1, but S^1 is a topological group (not discrete).

$$
\begin{array}{ccc}
H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) & \longrightarrow & H^*(BZ/\ell;\mathbb{Z}) \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell \\
0 & & 0 \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell \\
0 & & 0 \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}
\end{array}
$$

One can deduce that $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) = \mathbb{Z}[b]$ (in degree 2). Functoriality gives $H^*(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) = \mathbb{Z}[b]/(b^{m+1})$.