Today we start the proof that the definitions of Tor and Ext are correct. For simplicity, we assume R is a commutatative rign.

We start with a key lemma.

Lemma .0.1

Let $C_\bullet$ and $D_\bullet$ be free R-resolutions of R-modules M, N, respectively. Let $\varphi : M \to N$ be a homomorphism of R-modules. Then there exists a, unique up to chain homotopy, R-chain map $\widetilde{\varphi} : C_\bullet \to D_\bullet$ such that $H_0\widetilde{\varphi} = \varphi$.

Comments: Recall $H_0C = M$, $H_0D = N$, $H_kC = H_kD = 0$ for $k > 0$.

Definition .0.1

Let $f, g : C_\bullet \to D_\bullet$ be chain maps. a chain homotopy $h : f \simeq g$ has

$$\mathrm{d}h + h\,\mathrm{d} = f - g$$

Notice that $h : C_n \to D_{n+1}$

Proof of Existence. Let $M_1 = \operatorname{im} \mathrm{d}_1^C = C_1/\ker \mathrm{d}_1^C$ and $N_1 = \operatorname{im} \mathrm{d}_1^D = C_1/\ker \mathrm{d}_1^D$. We have short exact sequences, and because C_0, D_0 are free we can lift in the generators:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, dashed] & C_0 \arrow[r] \arrow[d, dashed, "\widetilde{\varphi}_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & D_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Homological algebra then guarantees a map $M_1 \to N_1$. Note that $\cdots \to C_1$ and $\cdots \to D_1$ are free resolutions of M_1, N_1, because $\ker \mathrm{d}_1^C = \operatorname{im} \mathrm{d}_2^C$ and $\ker \mathrm{d}_1^D = \operatorname{im} \mathrm{d}_2^D$. Thus we can construct $\widetilde{\varphi}_k$ inductively from the map $M_1 \to N_1$.

Proof of Uniqueness. Suppose $\widetilde{\varphi}_1, \widetilde{\varphi}_2 : C_\bullet \to D_\bullet$ both induce φ in H_0. Then $\widetilde{\varphi}_1 - \widetilde{\varphi}_2$ induces 0. Thus it suffices to show that if $\widetilde{\varphi}$ induces zero, then it is chain homotopic to zero.

Consider the augmented resolutions $C_m \to \cdots \to C_1 \to C_0 \to M \to 0$ and likewise for D. We construct h_0 by noting that for $x \in C_0$ we have $\mathrm{d}_0\widetilde{\varphi}(x) = \varphi(\mathrm{d}_0(x)) = 0$. Thus $\widetilde{\varphi}(x) = \mathrm{d}_1 y$ for some $y \in D_1$. We can then lift on free generators.

Now suppose we have constructed the homotopy $h : \widetilde{\varphi} \simeq 0$ in degrees $< m$.

Then we know that:

$$\begin{aligned}
\mathrm{d}_m(\widetilde{\varphi}_m - h\,\mathrm{d}_m) &= \mathrm{d}_m\widetilde{\varphi}_m - \mathrm{d}_m h\,\mathrm{d}_m \\
&= \widetilde{\varphi}_{m-1}\,\mathrm{d}_m - \mathrm{d}h\,\mathrm{d} \\
&= h\,\mathrm{d}\,\mathrm{d} + \mathrm{d}h\,\mathrm{d} - \mathrm{d}h\,\mathrm{d} = 0
\end{aligned}$$

Thus by exactness $(\widetilde{\varphi}_m - h\,\mathrm{d}_m)(x) = \mathrm{d}y$ for free generators $x \in C_m$, and so we can lift on the free generators.

Corollary .0.2

Free resolution is a functor R-Mod $\to$ h-R-Chain.

Both $\operatorname{Hom}(?, N)$ and $? \otimes_R N$ preserve chain homotopy, and chain homotopy preserves homology. Thus Tor and Ext are well-defined.

Recipe: If F is any additive functor $R\text{-Mod} \to \mathscr{C}$ to an additive category which preserves chain homotopy (aka extends to a functor on the homotopy category of R-chains), then we can define left derived functors $L_m F$ by applying F to a free resolution and then taking H_m.

Instead of a free resolution, it suffices to require that C_m be projective (projective resolution).

Definition .0.2

P is projective if for every $g : M \twoheadrightarrow N$ and $f : P \to N$ there exists a $\overline{f} : P \to M$ such that $g\overline{f} = f$.

$$\begin{tikzcd}
& M \arrow[d, "g"] \\
P \arrow[ur, dashed, "\overline{f}"] \arrow[r, "f"'] & N \arrow[d] \\
& 0
\end{tikzcd}$$

We turn around the arrows and say Q is injcetive if for every $g : N \hookrightarrow M$ and $f : N \to Q$ there exists a $\overline{f} : M \to Q$ such that $\overline{f}g = f$.

$$\begin{tikzcd}
0 \arrow[d] & \\
N \arrow[r, "f"] \arrow[d, "g"'] & Q \\
M \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}$$

We also have injective resolutions, where we have $Q_0 \to Q_1 \to Q_2 \to \cdots$ with $H^0 Q = M$ and $H^m Q = 0$ for $m > 0$.

Lemma .0.3

Injective resolutions ar ea functor R-Mod to h-R-Chain.

Existence of "enough injectives". In Ab, injective is exactly being divisible, that is for all $x \in G$ there exists $m \in \mathbb{N}$ such that there exists $y \in G$ with $my = x$.

Injective R-modules are given as $\operatorname{Hom}_{\mathbb{Z}}(R, G)$ with G a divisible abelian group. This will give that injective resolutions exist. The rest of the proof is the same as for projectives.

Right derived functors $R^m F$ are then defined by applying F to an injective resolution and taking m-th cohomology.

The case of Tor and Ext is:

$$\begin{aligned}
\operatorname{Tor}_m^R(?, N) &= L_m(? \otimes_R N) \\
\operatorname{Tor}_m^R(M, ?) &= L_m(M \otimes_R ?) \\
\operatorname{Ext}_R^m(?, N) &= L_m(\operatorname{Hom}_R(?, N)) \\
\operatorname{Ext}_R^m(M, ?) &= R^m(\operatorname{Hom}_R(M, ?))
\end{aligned}$$

Note if F is right exact then $L_0 F = F$ and if F is left exact then $R^0 F = F$.

Next Time: Abelian Categories.

Homework #4

(2) Prove that if P, T are projective resolutions of R-modules M, N and $\varphi : M \to N$ is a homomorphism of R-modules then there exists a (unique up to R-chain homotopy) $\widetilde{\varphi} : P \to T$ such that $H_0\widetilde{\varphi} = \varphi$.