Given a spectrum E and a CW-complex X, we can define the generalized homology and cohomology theory on X corresponding to E by

$$\widetilde{E}_m X = \pi_m(E \wedge X) \tag{$\star$}$$
$$\widetilde{E}^m X = \pi_{-m} F(X, E) = \pi_0 F(X, E[m]) \tag{1}$$
$$= [X, E_m]. \tag{2}$$

The first is motivated by the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$ (the corresponding generalized homology theory is $\pi_m^S X = \pi_m \Sigma^\infty X = \pi_m(X \wedge \mathbb{S})$). It turns out that for a CW-complex X, $? \wedge X : \text{Spectra} \to \text{Spectra}$ preserves weak equivalences.

Comment: With the (co)limit axioms on generalized homology and cohomology, and preservation by weak equivalence axiom, every generalized homology and cohomology theory is represented by some spectrum.

Example .0.1

K-theory cohomology comes from geometry (discuss later). There is still no known geometric interpretation of K-theory homology.

.1. Spectral Sequences: Revisited

Definition .1.1

An exact couple is a long exact sequence of the form

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Philosophy: from information about E, can we gain info about D.

We should want D, E to be $\mathbb{Z}$-graded, with i, j having degree 0 and k having degree -1.

Massey: Observed that $\mathrm{d}_1 = jk$ is a differential on E. We can then define $E' = H(E, \mathrm{d}_1)$. We can also define $D' = \operatorname{im}(i : D \to D)$. There is a derived exact couple

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

Intuitively, i' induced by i, k' induced by k, j' induced by $j \circ i^{-1}$. But this is not immediately seen to be well-defined.

We need another characterization of E'

$$E' = \frac{\ker jk}{\operatorname{im} jk} = \frac{k^{-1} \ker j}{j \operatorname{im} k} = \frac{k^{-1} \operatorname{im} i}{j \ker i}$$

Homework #13, Due: Monday Nov 29th

(1) Prove that
 (a) $\ker i' = \operatorname{im} k'$
 (b) $\ker j' = \operatorname{im} i'$
 (c) $\ker k' = \operatorname{im} j'$.

The spectral sequence arises by iterating this process.

Lemma .1.1

$$D^{(m)} = \operatorname{im}(i^m)$$
$$E^{(m)} = k^{-1}(\operatorname{im} i^m)/j(\ker i^m)$$

Often, we have an additional grading, making D, E bigraded. Then we understand $D_{p,q}, E_{p,q}$ are in total degree $n = p + q$. The traditional bidegrees are then

$$\begin{tikzcd} D \arrow[rr, "{(1,-1)}", "i"'] & & D \arrow[dl, "j", "{(0,0)}"'] \\ & E \arrow[ul, "k", "{(-1,0)}"'] & \end{tikzcd}$$

Then we have in the derived case

$$\begin{tikzcd} D^r \arrow[rr, "{(1,-1)}", "i"'] & & D^r \arrow[dl, "j", "{(1-r,r-1)}"'] \\ & E^r \arrow[ul, "k", "{(-1,0)}"'] & \end{tikzcd}$$

We then see d_r has degree $(-r, r-1)$ as usual.

A cohomological spectral sequence is the same, just reverse signs of p, q.

Example .1.1

AHSS (Atiyah-Hirzebruch Spectral Sequence) for a generalized homology theory L.

Here we have $D_{p,q} = L_{p+q}(X_p)$ where X is a CW complex and $E_{p,q} = L_{p+q}(X_p, X_{p-1})$.

The exact couple is

$$\begin{tikzcd} L_{p+q}(X_{p-1}) \arrow[r] & L_{p+q}(X_p) \arrow[r] & L_{p+q}(X_p, X_{p-1}) \arrow[r] & L_{p+q-1}(X_{p-1}) \\ D_{p-1,q+1} \arrow[r, "i"] & D_{p,q} \arrow[r, "j"] & E_{p,q} \arrow[r, "k"] & D_{p-1,q} \end{tikzcd}$$

If $r \gg 0$ implies $0 = \mathrm{d}^r : E^r_{p,q} \to \cdots$.

Define then $E^\infty_{p,q} = \operatorname{colim}_r E^r_{p,q}$. This happens here because i^{r-1} is the inclusion of a lower dimensional cell, and so its image is eventually zero. This then gives

$$\begin{aligned} E^\infty_{p,q} &= \frac{\ker(L_{p+q}(X_p, X_{p-1}) \to L_{p+q-1}(X_{p-1}))}{j(\ker L_{p+q}(X_p) \to L_{p+q}(X))} \\ &= \frac{\operatorname{im}(L_{p+q}(X_p) \to L_{p+q})(X_p, X_{p-1})}{\operatorname{im}(\ker(L_{p+q}(X_p) \to L_{p+q}(X)) \to L_{p+q}(X_p, X_{p-1}))} \\ &= L_{p+q}(X_p)/(\operatorname{im} L_{p+q}(X_{p-1}) + \ker(L_{p+q}(X_p) \to L_{p+q}X). \end{aligned}$$

Then

$$F_p L_{p+q} X \coloneqq \operatorname{im} L_{p+q} X_p \to L_{p+q} X).$$

Then $E^\infty_{p,q} = F_p L_{p+q} X / F_{p-1} L_{p+q} X$

<u>Note</u>: Cohomological AHSS similarly converges to $L^{p+q}X$ with

$$F^p L^{p+q} X = \ker(L^{p+q} X \to L^{p+q} X^{p-1}).$$

.2. Back to Spectra

In Equation ($\star$), we would really like X to also be a spectrum. Then we have

$$E_m X = L\pi_m(X \wedge E)$$
$$E^m X = \pi_{-m} F(X, E).$$

What do these mean? Well defining $X \wedge E$ by spectrifying $X_m \wedge E_m$ is wrong. A good way to see this is it doesn't satisfy $\Sigma^\infty Z \wedge E \sim Z \wedge E$ even for E cell.

Select two non-decreasing sequences α_m, β_m in $\mathbb{N}_0$ such that $\alpha_m + \beta_m = m$. With $\alpha_m, \beta_m \to \infty$ as $m \to \infty$.

Then

$$E \wedge F = LD \qquad D_m = E_{\alpha_m} \wedge F_{\beta_m}.$$

Then $F(Z, ?)$ is right adjoint to $Z \wedge ?$.

This gives a closed symmetric monoidal structure on $D\,$Spectra. But not on Spectra because of the choice of (α_m, β_m).

But you might want that structure on spectra! (rigid rings, modules in Spectra!)

Next Semester: Math 697 (introduction to current methods).