Last time, we defined for a CW-complex X [more generally a CW-pair (X, Z)], a chain complex $C_m^{\text{cell}}(X, Z) := \mathbb{Z}[I_m]$, where I_m is the set of m-cells.

We also observed that $\mathbb{Z}[I_m] = \widetilde{H}_m(X_m/X_{m-1}) = H_m(X_m, X_{m-1})$.

This allows us to build a chain complex with coefficients or a cochain cell complex via $? \otimes A$ and $\operatorname{Hom}(?, A)$.

Furthermore, the differential $\mathrm{d}_m^{\text{cell}} : \mathbb{Z}[I_m] \to \mathbb{Z}[I_{m-1}]$ is obtained as a connecting map composed with an inclusion:

$$H_m(X_m, X_{m-1}) \xrightarrow{\partial} H_{m-1}(X_{m-1}) \longrightarrow H_{m-1}(X_{m-1}, X_{m-2})$$

This can be shown to give a chain complex as desired (see 592 Notes).

How do we actually compute $\mathrm{d}_m^{\text{cell}}$? Well it's 0 if $m = 0$. Then if $m = 1$, the 1-cells are oriented line segments, and:

$$\mathrm{d}_1^{\text{cell}}(e) = \text{beginning point} \ - \ \text{end point}$$

Now for $e \in I_m$ with $m > 1$ we compute $\mathrm{d}_m^{\text{cell}}(e)$ differently. Namely we have a map $f_m : S^{m-1} \times I_m \to X_{m-1}$. We can then write:

$$S^{m-1} \xrightarrow{f|_{S^{m-1}\times\{e\}}} X_{m-1} \qquad X_{m-1}/X_{m-2} = \bigvee_{I_{m-1}} S^{m-1}$$

We take this map in homology (apply $H_{m-1}(?; \mathbb{Z})$). It gives a map:

$$\mathbb{Z} \to \mathbb{Z}[I_{m-1}]1 \qquad\qquad \mapsto \mathrm{d}_m^{\text{cell}}(e)$$

We are using the fact that:

$$\widetilde{H}_{m-1}\left(\bigvee_{I_{m-1}} S^{m-1}\right) \cong \bigoplus_{i \in I_{m-1}} \widetilde{H}_{m-1}(S^{m-1}).$$

However, we could also just project this map down, sending every cell except c to the basepoint and mapping c by the identity:

$$\bigvee_{I_{m-1}} S^{m-1} \to S^{m-1}$$

And then take homology.

We are then given another problem! Given a continuous map $f : S^k \to S^k$ for $k = m - 1 \geq 1$, what does it induce in homology?

$$\begin{gathered} H_k(S^k) \longrightarrow H_k(S^k) \\ \mathbb{Z} \longrightarrow \mathbb{Z} \\ 1 \longmapsto \deg(f) \end{gathered}$$

We may homotope f to a smooth map, so let us assume $x \in S^k$ and there exists an open neighborhood U of x so that:

$$f^{-1}(U) = \coprod_{i=1}^{\ell} V_i$$

Such that $f : V_i \to U$ is a diffeomorphism (C^1), and we let $f : y_i \in V_i \mapsto x$.

Theorem .0.1

$\deg(f) = \sum_{i=1}^{\ell} \sigma_i$, where $\sigma_i = 1$ if $f\big|_{V_i}$ preserves orientation and $\sigma_i = -1$ if $f\big|_{V_i}$ reverses orientation. A good book for this material is Milnor's Topology from a differential viewpoint [1].

Example .0.1

Consider $\mathbb{R}\mathrm{P}^m$, which is the space of all lines through the origin in $\mathbb{R}^{m+1}$, or $S^m/x \sim -x$. This has a CW-complex structure. We

$$\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \cdots \subseteq \mathbb{R}^{m+1}$$
$$\mathbb{R}\mathrm{P}^0 \subseteq \mathbb{R}\mathrm{P}^1 \subseteq \cdots \subseteq \mathbb{R}\mathrm{P}^m$$

This is a CW-filtration, and $\mathbb{R}\mathrm{P}^m$ is an m-dimensional CW-complex (meaning it only has cells up to dimension m).

For $\mathbb{R}\mathrm{P}^2$ we have the 2-cell v_0 as the top hemisphere, in general the m-cell is $\{(x_0, \dots, x_m) \in S^m \mid x_m \geq 0\}$. the boundary is exactly when $x_m = 0$, whihc is S^{m-1}. The attaching map is then the quotient $S^{m-1} \to \mathbb{R}\mathrm{P}^{m-1}$.

So then we have that:

$$\begin{array}{cccccc} C^{\text{cell}}(\mathbb{R}\mathrm{P}^m) & \mathbb{Z} \longrightarrow & \mathbb{Z} \longrightarrow & \cdots \to & \mathbb{Z} \to & \mathbb{Z} \\ \text{degrees} & m & m-1 & \cdots & 1 & 0 \end{array}$$

The attaching map $S^{m-1} \to S^{m-1}_+/S^{m-2}$ sends the northern hemisphere to a point and the southern hemisphere to its antipode. After some work one works out that these maps are zero or two in homology:

$$\begin{array}{cccccc} C^{\text{cell}}(\mathbb{R}\mathrm{P}^m) & \mathbb{Z} \xrightarrow{1+(-1)^m} & \mathbb{Z} \xrightarrow{1+(-1)^{m-1}} & \cdots \xrightarrow{2} & \mathbb{Z} \xrightarrow{0} & \mathbb{Z} \\ \text{degrees} & m & m-1 & \cdots & 1 & 0 \end{array}$$

We can then compute that if m is even:

$$H_k(\mathbb{R}\mathrm{P}^m) \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < k < m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

And if m is odd we have:

$$H_k(\mathbb{R}\mathrm{P}^m) \begin{cases} \mathbb{Z} & \text{if } k = 0, m \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < k < m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

The cellular chain complex C^{cell} is not functorial in continuous maps, but it is functorial in cell maps $f : X \to Y$ where $f(X_k) \subseteq Y_k$. Because then there is an induced map $X_k/X_{k-1} \to Y_k/Y_{k-1}$. Then we can just take reduced homology to get wedges of spheres:

$$X_k/X_{k-1} \to Y_k/Y_{k-1}$$
$$\bigvee_{I_k^X} S^k = \widetilde{H}_k(X_k/X_{k-1}) \to \widetilde{H}_k(Y_k/Y_{k-1} = \bigvee_{I_k^Y} S^k$$

This can again be computed using the degree of maps $S^k \to S^k$.

Homework #3

1a) Calculate $H_k(\mathbb{RP}^m; \mathbb{Z}/2\mathbb{Z})$ by definition using cellular homology. You may use $C^{\text{cell}}(\mathbb{RP}^m)$ from class.
1b) Prove that the quotient $\varphi : \mathbb{RP}^m \to \mathbb{RP}^m/\mathbb{RP}^{m-1}$ (embedded as in class) is not homotopic to a constant map (use homology with suitable coefficients $H_m(\varphi; \mathbb{Z}/2\mathbb{Z})$).
1c) For which values of $m > 0$ is $H_m(\varphi; \mathbb{Z})$ non-zero?
1d) Construct an m-dimensional CW-complex X with only one m-cell such that the projection $\varphi : X \to X/X_{m-1}$ is homotopic to a constant map. [Think simple].

These are 5pts each and due next Monday (9/20).

Example .0.2

We can also look at $\mathbb{CP}^m$, which is the space of all lines through the origin in $\mathbb{C}^{m+1}$. That is, it is:

$$\left\{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} \mid \sum |z_j|^2 = 1\right\} / (z \sim z' \iff |z| = |z'|)$$

We also have a CW-filtration:

$$\mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \cdots \subseteq \mathbb{CP}^m$$

We have a $2m$-cell given by $\left\{(z_0, \ldots, z_m) \mid \sum |z_j|^2, z_m \in \mathbb{R}, z_m \geq 0\right\}$. We have a pushout:

$$\begin{array}{ccccc} S^{2m-1} & \longrightarrow & \mathbb{CP}^{m-1} & & \\ & & & & \\ D^{2m} & \longrightarrow & P & \longrightarrow & \mathbb{CP}^m \end{array}$$

To know the induced pushout map is a homeomorphism, one uses that it is bijective, P is compact, and $\mathbb{CP}^m$ is Hausdorff.

We can also compute $C^{\text{cell}}(\mathbb{CP}^m)$:

$$\begin{array}{ccccccc} C^{\text{cell}}(\mathbb{CP}^m) & \mathbb{Z} \longrightarrow & 0 \longrightarrow & \mathbb{Z} \longrightarrow & \cdots \to & 0 \to & \mathbb{Z} \\ \text{degrees} & 2m & 2m-1 & 2m-2 & \cdots & 1 & 0 \end{array}$$

So every map is the zero map. This allows us to say that:

$$H_k(\mathbb{CP}^m) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq m, \text{ even} \\ 0 & \text{otherwise} \end{cases}$$