Let $f : X \to Y$ be a based map. We would like to understand the actual fiber $F \coloneqq f^{-1}(*)$ in terms of our understanding of the homotopy fiber Ff.

Definition .0.1

The map $f : X \to Y$ is called a fibration provided that it satisfies the homotopy lifting property. Namely if Z is some space and we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\iota_0"'] & X \arrow[d, "f"] \\ Z \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & Y \end{tikzcd}$$

Given a map $g : Z \to X$ and a homotopy $h : Z \times [0,1] \to Y$ so that $h(z,0) = fg(z)$ then there exists a homotopy $\widetilde{h} : Z \times [0,1] \to X$ such that $H(z,0) = g(z)$ and $fH(z,t) = h(z,t)$.

Proposition .0.1

If $f : X \to Y$ is a based fibration then $F(f) \simeq f^{-1}(*)$.

Homework # 6

(3) Prove Proposition .0.1. For hints, consider that

$$f^{-1}(*) \xrightarrow{\alpha} F(f)$$
$$x \mapsto (x, \mathrm{const}_*).$$

If f is a fibration, how do we go $F(f) \to f^{-1}(*)$. Well consider that we have a commutative diagram

$$\begin{tikzcd} F(f) \arrow[r, "p"] \arrow[d, hook, "\iota_0"'] & X \arrow[d, "f"] \\ F(f) \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & Y \end{tikzcd}$$

$$((x,\omega),t) \longmapsto \omega(t)$$

Thus there is a lift $\widetilde{h} : F(f) \times [0,1] \to X$. Take $\gamma \coloneqq \widetilde{h}_1$.

To show $\alpha\gamma \simeq : F(f) \to F(f)$, use the path only part of the way.

For $\gamma\alpha \simeq \mathrm{Id}$ we need a map $f^{-1}(*) \to f^{-1}(*)$. $\widetilde{h}$ will preserve the const_* downstairs. . . should allow us to construct the homotopy.

Definition .0.2

A map $f : X \to Y$ is called a fiber bundle provided that for all $y \in Y$ there is some open neighborhood U of y such that there is a diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "\cong"] \arrow[d, "f"'] & U \times F \arrow[d] \\ U \arrow[r, "\mathrm{Id}"'] & U \end{tikzcd}$$

Definition .0.3

Let Y be a space. A refinement of an open cover $\{U_i\}_{i \in I}$ is another open cover $\{V_j\}_{j \in J}$ such that for all$j$ there exists and i with $V_j \subseteq U_i$.

An open cover $\{U_i\}_{i\in I}$ is called locally finite if for all $x \in Y$ there is an open neighborhood W of x and a finite set $F \subseteq I$ such that if $i \in I \setminus F$ then $U_i \cap W = \emptyset$.

A space is paracompact if every open cover has a locally finite refinement.

Example .0.1

Almost all nice spaces are paracompact

- All manifolds are paracompact.
- All CW-complexes are paracompact.

Theorem .0.2

If $f : X \to Y$ is a fiber bundle and Y is paracompact then f is a fibration.

See May's book for a proof [**may**].

Example .0.2

A covering is a fiber bundle whose fiber F is discrete. This implies that it satisfies the homotopy lifting property *with uniqueness.*

Let $f : X \to Y$ be a based covering with Y paracompact. Then $F(f) \simeq \underbrace{f^{-1}(*)}_{\text{discrete}}$. Thus $\pi_m f^{-1}(* = 0$ for $m > 0$.

Now the long exact sequence gives us that for $m \geq 2$

$$0 = \pi_m(F) \longrightarrow \pi_m(X) \xrightarrow{\cong} \pi_m(Y) \longrightarrow \pi_{m-1}(F) = 0$$

Therefore $\pi_m f$ is an isomorphism for $m \geq 2$.

This shows that $\pi_m(S^1) = 0$ for $m \geq 2$ because the universal cover of S^1 is $\mathbb{R} \to S^1$ and $\mathbb{R} \simeq *$. Thus $S^1 \simeq B\mathbb{Z}$.

More generally, if the universal covering of a nice space X is contractible, then $\pi_m(X) = 0$ for $m \geq 2$. We would then call X "hyperbolic" in the most general sense.

All surfaces are hyperbolic except $S^2, \mathbb{RP}^2$. Caution: geometers would not consider the torus hyperbolic, but it does satisfy this property. This implies that all surfaces except $S^2, \mathbb{RP}^2$ have that $X \simeq B\pi_1 X$.

We also know that $\pi_m(\mathbb{RP}^\infty) = 0$ for $m > 1$ because the universal cover of $\mathbb{RP}^\infty$ is $S^\infty \simeq *$. This shows that $\mathbb{RP}^\infty$ is a $B\mathbb{Z}/2$.

Definition .0.4

Notice that $S^1 \to S^{2n+1} \xrightarrow{f} \mathbb{CP}^n$ is an action, where we view S^1 as the unit sphere in $\mathbb{C}$ acting by multiplication on the unit sphere in $\mathbb{C}^{m+1}$

$$\lambda \cdot (z_0, \dots, z_m) = (\lambda z_0, \dots, \lambda z_m).$$

Thus this is a fiber bundle, and hence a fibration.

The most striking case is $m = 1$, because then $\mathbb{CP}^1 \cong S^2$. as then we have $S^1 \to S^3 \xrightarrow{f} S^2$, which is called the Hopf fibration

Proposition .0.3

$\pi_m(S^3) \cong \pi_m(S^2)$ for $m \geq 3$.

Proof. We have the following long exact sequence for $m > 2$ from the Hopf fibration

$$0 = \pi_m(S^1) \longrightarrow \pi_m(S^3) \xrightarrow{\cong} \pi_m(S^2) \longrightarrow \pi_{m-1}(S^1) = 0$$

And so we're done.

This is most striking when $m = 3$, because then $\pi_3(S^3) \cong \mathbb{Z}$, so $\pi_3(S^2) = \mathbb{Z}$.

Actually, $\pi_{4m-1}S^{2m}$ is infinite, $\pi_m S^m = \mathbb{Z}$, and all other homotopy groups of spheres are finite.

This really clues us in to how complex homotopy groups are, as $\pi_m(S^n)$ can be nonzero even when $m > n$.

Now lets think about how to construct generalized cohomology theories. In the based version this is given by the axioms

$$X \longrightarrow Y \longrightarrow Cf$$

$$\widetilde{E}^m Cf \longrightarrow \widetilde{E}^m Y \longrightarrow \widetilde{E}^m X$$

is exact. And also $\widetilde{E}^{m+1}\Sigma X \cong \widetilde{E}^m X$ naturally. We could also require the axiom that

$$\widetilde{E}^m \bigvee_i X_i \cong \prod_{i \in I} \widetilde{E}^m X_i$$

Suppose I give you based spaces Z_m, $m \in \mathbb{Z}$, such that $Z_m \simeq \Omega Z_{m+1}$. Then we can define $\widetilde{E}^m X = [X, Z_m]$. Then of course

$$\widetilde{E}^{m+1}\Sigma X = [\Sigma X, Z_{m+1}] = [X, \Omega Z_{m+1}] = [X, Z_m] = \widetilde{E}^m X.$$

We already proved exactness, and the product formula also holds. It turns out that <u>every</u> generalized cohomology theory is obtained this way.