I. **Homotopy Theory of** Based

We will be doing homotopy theory in Based because it is a category with zero, which means we can talk about kernels and cokernels!!!

Definition I.0.1

A homotopy $f_t : X \times [0,1] \to Y$ is called based provided that f_t preserves the basepoint for all $t \in [0,1]$.

In other words it's a based map of $f_t : X \wedge [0,1]_+ \to Y$ where $X \wedge [0,1]_+ = (X \times [0,1])/(* \times [0,1])$.

Definition I.0.2

The based mapping cone $C_{\text{based}}f$ associated to a given based map $f : Y \to X$ is given by

$$C_{\text{based}} = Cf/(* \times [0,1]) = (X \coprod (Y \times [0,1]))/(*,t) \sim *, (y,1) \sim *, (y,0) \sim f(y)$$

This is sometimes denoted by Cf despite the conflicting notation.

The based suspension $\Sigma Y = Y \wedge S^1 = C_{\text{based}}(Y \to *)$ is a special case.

If Y, X are CW complexes and f is a cellular map,

$$C_{\text{based}}f \simeq Cf$$

(same proof as for the suspension).

Denote by $[X,Y] = \text{Mor}_{\text{hBased}}(X,Y)$ (based homotopy classes of based maps $X \to Y$).

Proposition I.0.1

Let $f : Y \to X$ be a based map. Let Z be a based space. Then the induced sequence of

$$Y \xrightarrow{f} X \xrightarrow{i} Cf$$

given by applying $[-,Z]$

$$[Y,Z] \xleftarrow{[f,Z]} [X,Z] \xleftarrow{[i,Z]} [Cf,Z]$$

is exact. That is

$$\ker[f,Z] = \Im[i,Z]$$

Proof. To prove that $\text{im}[i,Z] \subseteq \ker[f,Z]$. This follows if $if \simeq 0$. But the mapping cone is almost rigged this way

$$h_t(y) = (y,t)$$

then $h_0 = if$ and $h_1 = 0$.

Now we need to prove that $\ker[f,Z] \subseteq \text{im}[i,Z]$. Well let $g \in \ker[f,Z]$ this means that we have a diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, dashed, "0"'] & X \arrow[r, "i"] \arrow[d, "g"] & Cf \\ & Z & \end{tikzcd}$$

Note that we have the following

$$Y \xrightarrow{f} X \xrightarrow{i} Cf \xrightarrow{j} Ci \simeq \Sigma X$$

Theorem I.0.2

Let $f : Y \to X$ be a based map. Let Z be a based space. Then we hve a long exact sequence

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
{[\Sigma^2 Cf, Z]} \arrow[r] & {[\Sigma^2 X, Z]} \arrow[r] & {[\Sigma^2 Y, Z]} \arrow[dll] \\
{[\Sigma Cf, Z]} \arrow[r] & {[\Sigma X, Z]} \arrow[r, "{[-\Sigma f, Z]}"] & {[\Sigma Y, Z]} \arrow[dll] \\
{[Cf, Z]} \arrow[r, "{[i, Z]}"] & {[X, Z} \arrow[r, "{[f, Z]}"] & {[Y, Z]}
\end{tikzcd}$$

(and

Observation: $[\Sigma X, Y]$ is a group, $[\Sigma^2 X, Y]$ is an abelian group (as is $[\Sigma^m X, Y]$) (same proof as for π_n).

Dualizing, Ω is right adjoint to $\Sigma : \text{Based} \to o\,\text{Based}$. And this also works in hBased.

The mapping cone also has a dual construction.

Definition I.0.3

Let $f : X \to Y$ be a map of based spaces. The homotopy fiber Ff is defined by

$$Ff := \{(x, \omega) \mid x \in X, \omega : [0, 1] \to Y, \omega(0) = f(x), \omega(1) = *\}$$

with the compact-open topology. And there's a canonical projection $Ff \xrightarrow{p} X$.

Lemma I.0.3

Let Z be a based space and let $f : X \to Y$ be a based map. Then the sequence

$$[Z, Ff] \xrightarrow{[Z,p]} [Z, X] \xrightarrow{[Z,f]} [Z, Y]$$

is exact.

Homework #6

(2) Prove Lemma I.0.3.

Notice also that

$$\Omega X \simeq Fq \xrightarrow{-\Omega f} \Omega Y \simeq Fp \xrightarrow{q} Ff \xrightarrow{p} X \xrightarrow{f} Y$$

A great reference for this is [1] (it's the best part of the book!).

Theorem I.0.4

Let $f : X \to Y$ be a based map and let Z be a based space. Then we have a long exact sequence

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
{[Z, \Omega^2 Ff]} \arrow[r] & {[Z, \Omega^2 X]} \arrow[r] & {[Z, \Omega^2 Y]} \arrow[dll] \\
{[Z, \Omega Ff]} \arrow[r] & {[Z, \Omega X]} \arrow[r, "{[Z, -\Omega f]}"] & {[Z, \Omega Y]} \arrow[dll] \\
{[Z, Ff]} \arrow[r, "{[Z, p]}"] & {[Z, X]} \arrow[r, "{[Z, f]}"] & {[Z, Y]}
\end{tikzcd}$$

Again $[Z, \Omega^n X] \cong [\Sigma^n Z, X]$ are groups for $n \geq 1$ and abelian groups for $n \geq 2$.

If we take $Z = S^0 = \{*, \infty\}$ then because $\Sigma^n S^0 = S^n$

$$[S^0, \Omega^n X] \cong [S^n, X] = \pi_n(x).$$

Thus there is a long exact sequence in homotopy groups:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners] \\
\pi_2(Ff) \arrow[r] & \pi_2(X) \arrow[r] & \pi_2(Y) \arrow[dll, rounded corners] \\
\pi_1(Ff) \arrow[r] & \pi_1(X) \arrow[r] & \pi_1(Y) \arrow[dll, rounded corners] \\
\pi_0(Ff) \arrow[r] & \pi_0(X) \arrow[r] & \pi_0(Y)
\end{tikzcd}$$

If $A \overset{i}{\hookrightarrow} X$ is an inclusion, this suggests defining

$$\pi_m(X, A) \coloneqq \pi_{m-1}(Fi)$$