.1. **The Generalized Homology of CW-complexes**

Let us try to define $C^{\text{cell}}(X)$ for a generalized homology theory E and see what goes wrong. We let $E_m \coloneqq E_m(*)$ be the coefficients E_*.

We know that:

$$\widetilde{E}_m(S^0) = E_m$$
$$\widetilde{E}_{m+k}(S^k) = \widetilde{E}_m(S^0) = E_m$$

For a CW-complex X, we have that:

$$\widetilde{E}_{p+q}(X_p/X_{p-1}) = E_q[I_p]$$
$$E_{p+q}(X_p, X_{p-1}) = E_q[I_p]$$

We do get a differential:

$$\mathrm{d} : E_{p+q}(X_p, X_{p-1}) \xrightarrow{\partial} E_{p+q-1}(X_{p-1}) \to E_{p+q-1}(X_{p-1}, X_{p-2})$$

However, we now have a chain complex for each choice of $q \in \mathbb{Z}$. We draw this for p increasing to the right and q increasing on the upper side:

$$\begin{array}{ccccccc}
q=2 & E_2[I_0] & \longleftarrow & E_2[I_1] & \longleftarrow & E_2[I_2] & \longleftarrow \cdots \\
\\
q=1 & E_1[I_0] & \longleftarrow & E_1[I_1] & \longleftarrow & E_1[I_2] & \longleftarrow \cdots \\
\\
q=0 & E_0[I_0] & \longleftarrow & E_0[I_1] & \longleftarrow & E_0[I_2] & \longleftarrow \cdots \\
\\
q=-1 & E_{-1}[I_0] & \longleftarrow & E_{-1}[I_1] & \longleftarrow & E_{-1}[I_2] & \longleftarrow \cdots \\
\\
 & p=0 & & p=1 & & p=2 & \quad\cdots
\end{array}$$

We have a "total dimension" $p+q$. This is called the E^1-page of a spectral sequence. And this is in fact the Atiyah-Hirzebruch spectral sequence. In general we can have:

$$\begin{tikzcd}
q=2 & E^1_{0,2} & E^1_{1,2} \arrow[l] & E^1_{2,2} \arrow[l] & \cdots \arrow[l] \\
q=1 & E^1_{0,1} & E^1_{1,1} \arrow[l] & E^1_{2,1} \arrow[l] & \cdots \arrow[l] \\
q=0 & E^1_{0,0} & E^1_{1,0} \arrow[l] & E^1_{2,0} \arrow[l] & \cdots \arrow[l] \\
q=-1 & E^1_{0,-1} & E^1_{1,-1} \arrow[l] & E^1_{2,-1} \arrow[l] & \cdots \arrow[l] \\
& p=0 & p=1 & p=2 & \cdots
\end{tikzcd}$$

The homology of each sequence is called the E^2-page.

$$\begin{tikzcd}
q=2 & E^1_{0,2} & E^1_{1,2} \arrow[l] & E^1_{2,2} \arrow[l] & \cdots \arrow[l] \\
q=1 & E^1_{0,1} & E^1_{1,1} \arrow[l] & E^1_{2,1} \arrow[l] & \cdots \arrow[l] \\
q=0 & E^1_{0,0} & E^1_{1,0} \arrow[l] & E^1_{2,0} \arrow[l] & \cdots \arrow[l] \\
q=-1 & E^1_{0,-1} & E^1_{1,-1} \arrow[l] & E^1_{2,-1} \arrow[l] & \cdots \arrow[l] \\
& p=0 & p=1 & p=2 & \cdots
\end{tikzcd}$$

We get a differential $\mathrm{d}_2 : E^2_{p,q} \to E^2_{p-1,q+1}$:

$$\begin{tikzcd}
E^1_{0,2} & E^1_{1,2} & E^1_{2,2} & \cdots \\
E^1_{0,1} & E^1_{1,1} \arrow[ul] & E^1_{2,1} \arrow[ul] & \cdots \arrow[ul] \\
E^1_{0,0} & E^1_{1,0} \arrow[ul] & E^1_{2,0} \arrow[ul] & \cdots \arrow[ul] \\
E^1_{0,-1} & E^1_{1,-1} \arrow[ul] & E^1_{2,-1} \arrow[ul] & \cdots \arrow[ul]
\end{tikzcd}$$

In general we get a differential on the r-th page $\mathrm{d}_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

And we take algebraic homology:

$$E^{r+1}_{p,q} = H(E^r_{p,q})$$

We can then define:

$$E^\infty_{p,q} = \operatorname{colim}_r E^r_{p,q} 2$$

Still a whole plane full of groups. Have we calculated $E_{p+q}(X)$? Well we can consider:

$$F_p E_{p+q} X \coloneqq \Im(E_{p+q} X_p \to E_{p+q} X)$$

Then $F_{-1} = 0 \subseteq F_0 \subseteq F_1 \subseteq \cdots$. This is an increasing filtration (complete), and we have:

$$\bigcup F_p E_{p+q} X = E_{p+q} X$$

Which follows by the limit axiom of homology:

Theorem .1.1 (Atiyah-Hirzebruch)

$E^\infty_{p,q} = F_p E_{p+q} X / F_{p-1} E_{p+q} X$. This is called the associated graded object.

We have $E^2_{p,q} = H_p(X, E_q)$.

For an abelian group A, a complete filtration on A is:

$$0 = F_{-1} A \subseteq F_0 A \subseteq F_1 A \subseteq \cdots$$

Where $A = \bigcup_i F_i A$. Then the associated graded object is:

$$E^0 A_i = (F_i A / F_{i-1} A)_{i \geq 0}$$

Example .1.1

Suppose that $(F_i A)$ is a complete filtration of an abelian group A and suppose $(F_i A / F_{i-1} A) \cong \mathbb{Z}[S_i]$. Then:

$$A = \bigoplus_i \mathbb{Z}[S_i] = \mathbb{Z}\left[\coprod_i S_i\right]$$

Proof. There is a short exact sequence:

$$0 \longrightarrow F_{i-1} A \longrightarrow F_i A \longrightarrow F_i A / F_{i-1} A \longrightarrow 0$$

A free abelian group is projective so this splits. For splitting just lifts the free generators $\in S_i$ to $F_i A$ and extends by the universal property.

We can conclude $F_i A \cong F_{i-1} A \oplus F_i A / F_{i-1} A$. Induction finishes the proof.

This is called an extension in a spectral sequence. What good is a spectral sequence (for example the AHSS, Atiyah-Hirzebruch spectral sequence) when we only know $E^2_{p,q}$ and not the higher differentials?

The simplest scenario: sometimes they are ruled out.

Example .1.2

When $E = H(?; A)$ is the ordinary homology. In that case AHSS looks like the following on the first page:

$$q = 1 \qquad 0 \qquad 0 \qquad 0 \qquad \cdots$$

$$q = 0 \qquad A[I_0] \longleftarrow A[I_1] \longleftarrow A[I_2] \longleftarrow \cdots q = -1 \qquad 0 \qquad 0 \qquad 0 \qquad \cdots$$

Thus no higher differentials are possible, and no extensions are possible because the associated graded object only has one term.

So if we prove that the AHSS works, it implies the theorem about $H^{\text{cell}} = H^{\text{singular}}$.

Someitmes the differential can also be ruled out in a more subtle way. For example when $E^2_{p,q} = 0$ for $p + q$ odd (any higher differential will decrease the total dimension by one). In this case, we can still have extensions.

Homework #3

(2) Suppose a generalized homology theory K has coefficients $K_{2n} = \mathbb{Z}$, $K_{2m-1} = 0$, $m \in \mathbb{Z}$. Calculate $K_\ell \mathbb{C}\mathrm{P}^m$ for all $\ell \in \mathbb{Z}$. Use AHSS, and put together the information mentioned in class.