Recall **??**, specifically the statement that every space X is m-equivalent to a CW-complex Z_m of dimension $\leq m$, and weakly equivalent to a CW-complex Z. Z_m is sometimes called a formal m-skeleton of X.

Remark .0.1

If $f : X \to Y$ is a weak equivalence (X, Y are any spaces) then f induces an $\cong$ in singular homology. $H_n f : H_n X \to H_n Y$.

Proof Sketch. Express singular homology in terms of maps from CW-complexes. Consider a singular cycle $c = \sum_k a_k \sigma_k$, $\sigma_k : \Delta^m \to X$. We can construct a CW-complex Z by taking $\coprod_K \Delta^m / \sim$, which is the minimal equivalence relation making it into a cycle (identify $(m-1)$-faces on which σ_k, σ_ℓ restrict to the same singular $(m-1)$-simplex).

c lifts to a singular cycle on Z. To show $H_n f$ is onto, let $c \in C_m Y$ be a cycle representing a class in $H_m Y$. We constructed a CW-complex (of dimension m) Z, $Z \to Y$ so that $c' \mapsto c$.

We can then lift $Z \to Y$ up to homotopy to a map $Z \to X$ using Whitehead's theorem. Thus we constructed a lift of $[c] \in H_m Y$ to $H_m X$ under f.

The argument for boundaries to show injectivity is analogous.

We add an axiom to generalized cohomology: $E_m f$ (resp. $E^m f$) is an isomorphism when f is a weak equivalence.

From the point of view of representing generalized cohomology by homotopy classes of maps into some based spaces: We need a sequence of based spaces Z_m with a based weak equivalence $Z_m \to \Omega Z_{m+1}$.

For a CW-complex X, $E^m(X) = [X, Z_m]$ (unbased).

For a general space X, find a weak equivalence $\gamma : X' \to X$ and define $E^m(X) := [X', Z_m]$. Then $E^m f$ is an isomoprhism when f is a weak equivalence.

How to prove the approximation statement from **??** from the first statement?

Proof. We do this by induction. The base case is to take $Z_0 \to X$, where Z_0 is the discrete set of path-components of X. This is of course onto in π_0.

Suppose we have an n-dimensional CW-complex Z_n and an n-equivalence $\gamma^m : Z_m \to X$. This is an isomorphism on π_i, $i < n$, and onto on π_n. γ^n may not be $\cong$ on π_n. There may be classes $\alpha_i : S^n \to Z_n$ so that $\gamma^n \circ \alpha_i \simeq *$.

We can just glue disks along each of these to fix the issue. Also γ^n may not be onto on π_{n+1}. To fix this if $\beta_j : S^{n+1} \to X$ is not represneted then

$$Z_{n+1} = Z_n \sqcup \coprod_i D^{n+1} \sqcup \coprod_j S^{n+1} / \sim$$

Where $\sim$ attaches D^{n+1} via α_i and S^{n+1} via their base point in Z_0.

By definition we get a map $\gamma^{n+1} : Z_{n+1} \to X$. This satisfies the inductive step because

- $\cong$ in π_i for $i < n$ comes from cellular approximation of maps, because we can approximate $S^n \to Z_{n+1}$ via maps $S^n \to Z_n$.
- For the same reason, it is onto on π_n. It is then injective on π_n by the gluings made above, as we killed all the relations.
- It is onto on π_{n+1} by construction.

We're done! For the infinite case set $Z = \bigcup_i Z_i$.

Notice: Say X is path-connected. Say $\pi_i(X) = 0$ for $i < m$ (we say X is $(m-1)$-connected). 1-connected means $\pi_1(X) = 0$, that is X is simply connected.

Then we can set $Z_{m-1} = *$. Furthermore, Z_m is a bouquet of spheres over generators of $\pi_m X$. Z_{m+1} is a bouqeut of spheres over generators of $\pi_m X$, and $\pi_{m+1}X$, and then we attach m-disks along relations in $\pi_m X$.

Definition .0.1 (Hurewicz Homomorphism)

$\pi_k X \to H_k(X;\mathbb{Z})$. This is given by taking some $\alpha : S^k \to X$ and mapping

$$H_k(S^k;\mathbb{Z}) \xrightarrow{H_k\alpha} H_k(X;\mathbb{Z})$$
$$1 \mapsto h(\alpha)$$

Computing cell homology, we get

Theorem .0.1 (The Hurewicz Theorem)

If X is $(m-1)$-connected, then the Hurewicz homomorphism $h : \pi_m X \to H_m(X;\mathbb{Z})$ is the abelianization if $m = 1$, and an isomorphism if $m > 1$.

Proof. Our construction of Z_{m+1} above makes this clear.

Homework # 8

(1) Compute $\pi_2(S^1 \vee S^2)$. Use universal cover and Hurewicz theorem.

It is also easy to construct by the methods above, a Cw-complex $K(\Pi, m)$, Π a group (abelian if $m > 1$) such that

$$\pi_i K(\Pi, m) = \begin{cases} \Pi & \text{if } i = m \\ 0 & \text{otherwise} \end{cases}.$$

We can construct Z_{m+1} by the above method (generators and defining relations of π). Then just keep attaching cells to kill all higher homotopy groups.

Same method implies that any two such CW-complexes $K(\Pi, m)$ are homotopy equivalent (use Whitehead Theorem).

We even get a weak equivalence $K(\Pi, m-1) \xrightarrow{\sim} \Omega K(\Pi, m)$. This way we can construct singular cohomology out of the Whitehead theorem. Namely this gives $[X, K(\Pi, m)] \to H^m(X;\Pi)$.

How do we do this for homology? Duality! We'll get there soon.