Now we take a digression to quickly review some category theory which will help us in our definitions. For a reference see Category Theory in Context by Emily Riehl [1].

Definition .0.1

A category $\mathscr{C}$ has a class of objects $\operatorname{Ob}\mathscr{C}$ and of morphisms $\operatorname{Mor}\mathscr{C}$. There are maps $S, T : \operatorname{Mor}\mathscr{C} \to \operatorname{Ob}\mathscr{C}$ which stand for source and target as well as $\operatorname{Id} : \operatorname{Ob}\mathscr{C} \to \operatorname{Mor}\mathscr{C}$, and notably $S \circ \operatorname{Id}, T \circ \operatorname{Id}$ are both the identity on objects.

We call $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ the class of all $f \in \operatorname{Mor}\mathscr{C}$ such that $S(f) = X$ and $T(f) = Y$. This is sometimes also denoted by $\mathscr{C}(X, Y)$, and we usually assume that this is a set. We also sometimes write $f : X \to Y$ to mean that $f \in \operatorname{Hom}_{\mathscr{C}}(X, Y)$ when the ambient category is clear.

Furthermore if $f : X \to Y$ and $g : Y \to Z$ then we define $g \circ f : X \to Z$. This is associative when defined and for $f : X \to Y$:

$$\operatorname{Id}_Y \circ f = f = f \circ \operatorname{Id}_X$$

Example .0.1

There are a variety of examples:

Name	Objects	Morphisms
Set	sets	functions
Grp	groups	homomorphisms
Ab	abelian groups	homomorphisms
Top	spaces	continuous maps

Also given any category $\mathscr{C}$ there is a category $\mathscr{C}^{\mathrm{op}}$ which has the same objects as $\mathscr{C}$ and the morphisms point in the opposite direction with composition also reversed.

Definition .0.2

A functor $F : \mathscr{C} \to \mathscr{D}$ is a map of objects and of morphisms which preserves Id, S, T, and composition. One can of course compose functors.

Why are we concerned about this? Well we have functors in algebraic topology

$$C_m : \mathrm{Top} \to \mathrm{Ab}$$
$$C : \mathrm{Top} \to \mathrm{Chain}$$

This category Chain has objects chain complexes, and the maps are collections of group homomorphisms $f_n : A_n \to B_n$ satisfying the commutative diagram below

$$\begin{tikzcd}
A_n \arrow[r, "\mathrm{d}_n^A"] \arrow[d, "f_n"] & A_{n-1} \arrow[d, "f_{n-1}"'] \\
B_n \arrow[r, "\mathrm{d}_n^B"] & B_{n-1}
\end{tikzcd}$$

Furthermore $H_m : \mathrm{Chain} \to \mathrm{Ab}$ is a functor, and so we may define the composition $H_m : \mathrm{Top} \to \mathrm{Ab}$, overloading notation.

Even better, $? \otimes ? : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$ is a functor, where the product of categories is appropriately defined.

For a specified abelian group A, $? \otimes A : \mathrm{Ab} \to \mathrm{Ab}$ is a functor, which is defined on morphisms as

$$(f \otimes A)(x \otimes a) = f(x) \otimes a$$

This will allow us to construct a homology with coefficients functor via "abstract nonsense." Namely, if C is a chain complex then $C \otimes A$ is a chain complex given below:

$$\cdots \longrightarrow C_m \otimes A \xrightarrow{\mathrm{d}_m \otimes A} C_{m-1} \otimes A \xrightarrow{\mathrm{d}_{m-1} \otimes A} \cdots$$

Perfect! Thus with A an abelian group, $? \otimes A : \text{Chain} \to \text{Chain}$ is a functor. We then know that $C(X; A) :- (CX) \otimes A$, and this will be a functor.

Therefore $H_m(X; A) := H_m C(X; A)$ is a functor as well.

At first, this seems strange, as homology with coefficients is determined by homology, and so it cannot contain new information. However, it contains some new and interesting information.

For cohomology, recall that we defined $C^m(X; A) = \text{Hom}(C_m X, A)$. Notice that $\text{Hom}(?, A) : \text{Ab} \to \text{Ab}^{\text{op}}$ is a functor. It is defined on objects via pointwise addition, and it is defined on morphisms as follows.

Let $f : B \to D$ be a morphism of abelian groups. Then we define $\text{Hom}(f, A) : \text{Hom}(D, A) \to \text{Hom}(B, A)$ as foloows. If we have a morphism $h : D \to A$ then:

$$\begin{tikzcd} D \arrow[r, "h"] & A \\ B \arrow[u, "f"] \arrow[ur, dashed, bend right, "\text{Hom}(f,A)(h) = h \circ f"'] & \end{tikzcd}$$

Definition .0.3

A functor $F : C^{\text{op}} \to D$ is called a contravariant functor from C to D. A "normal" functor is called covariant. Hom is covariant in the first coordinate, aka if A is fixed then $\text{Hom}(A, ?)$ is a covariant functor from $\text{Ab} \to \text{Ab}$.

Now say we have a chain complex

$$\cdots \longrightarrow C_m \xrightarrow{\mathrm{d}_m} C_{m-1} \xrightarrow{\mathrm{d}_{m-1}} \cdots$$

Then we may apply $\text{Hom}(?, A)$ everywhere, and we get a chain complex in the "reverse" direction:

$$\cdots \longleftarrow \text{Hom}(C_m, A) \xleftarrow{\text{Hom}(\mathrm{d}_m, A)} \text{Hom}(C_{m-1}, A) \xleftarrow{\text{Hom}(\mathrm{d}_{m-1}, A)} \cdots$$

We say that Cochain the category of such "reversed" chains. If C^* is a cochain complex, then of course defining the chain complex $C_m := C^{-m}$ gives us an equivalence $\text{Chain} \cong \text{Cochain}$. We may also define cohomology of a cochain complex as $H^m(C) = \ker \mathrm{d}^m / \operatorname{im} \mathrm{d}^{m-1}$.

Great! This means we may define $C^*(X; A) = \text{Hom}(CX, A)$ as a cochain complex and then:

$$H^m(X; A) = H_{-m}(C^*(X; A)) = H_{-m}(\text{Hom}(CX, A))$$

Functoriality of cohomology then just follows by composing functors:

$$\text{Top} \xrightarrow{C} \text{Chain} \xrightarrow{\text{Hom}(?, A)} \text{Cochain}^{\text{op}} \xrightarrow{H^*} \text{Ab}$$

So cohomology is a contravariant functor. Why do we care about cohomology?

- We encounter it in geometry (de Rham)
- Cohomology has additional structure. if R is a commutative ring, then $H^*(?; R)$ is a functor from spaces into commutative rings.

Homework due 2021-09-07

(3a) Write down the differential in $C(X;A)$, $C^*(X;A)$ in elements.

(3b) Say in a few words why d_m needs (and preserves) the finiteness condition and why d^m does not.