Remark .0.1

If we know that $H^*\mathbb{Q}[x] \otimes \bigwedge[\mathrm{d}x] = \mathbb{Q}$ in degree zero for x of even degree then

$$H^*\mathbb{Q}[x_1, \dots, x_n] \otimes \bigwedge[\mathrm{d}x_1, \dots, \mathrm{d}x_n] = \bigotimes_i H^*(\mathbb{Q}[x_i] \otimes \bigwedge[\mathrm{d}x_i]) = \mathbb{Q}$$

in degree zero. By the Kunneth Theorem

.1. Rational Homotopy Theory

If X is a simply connected CW complex of finite type (finitely many cells in each dimension). Equivalently (homotopically) the realization of a simplicial set with finitely many non-degenerate simplices in every dimension.

We calculate $\Omega^* X$ by taking limit over non-degenerate simplices. Use the fact that

$$\Omega^*\Delta^m = \mathbb{Q}[x_0, \dots, x_m]/\sum x_i = 1 \otimes \bigwedge[\mathrm{d}x_0, \dots, \mathrm{d}x_m]/\sum \mathrm{d}x_i = 0$$

where the degree of x_i is zero and the degree of $\mathrm{d}x_i$ is 1.

Then ΩX is a graded commutative DGA. We can of course talk about

Definition .1.1

A cell graded-commutative DGA is a DGA Q expressible as

$$\mathbb{Q} = A_{(0)} \subseteq A_{(1)} \subseteq \cdots \qquad A = \bigcup A_{(m)}$$

where we have that

$$A_{(m+1)} = A_{(m)} \otimes F[s_{mi} \mid i \in I_m] \otimes \bigwedge(\text{odd degree generators}) \otimes \mathbb{Q}[\text{even degree generators}].$$

Where $F(s_1, \dots, s_m)$ is the free graded commutative algebra on $s_1, \dots, s_m$. We say it has generators in degrees d_{mi} (even or odd).

Forces: cochain degrees ≥ 2. Furthermore we require $\mathrm{d}s_{mi} \subseteq A_{(m)}$. As an algebra $A_{(m)} = F(Q_m)$. We have an augmentation ideal $J_m = \{q \mid q \in Q_m\}$ of $A_{(m)}$.

We say A is minimal if $d_{si} \in J_m^2$ (the decomposable elements).

"Whitehead Theorem": Equivalences = quasiismorphisms = morphisms of graded-commutative DGAs inducing $\cong$ in cochain cohomology.

Theorem .1.1

There exists a derived category $\mathcal{D}$ of graded commutative DGAs, with respect to quasiisomorphism. Each isomorphism class in $\mathcal{D}$ contains a unique minimal DGA up to DGA isomorphism (this is called a minimal model).

Therefore we have for compact generated CW-complexes of finite type, a unique minimal model $A \to \Omega^* X$. Moreover $A = F[S]$ where $\mathbb{Q}S$ is the dual of rational homotopy groups of X.

If $S_m \subseteq S$ is the subset of generators of degree m then

$$\pi_m X \otimes \mathbb{Q} \cong \mathrm{Hom}(\mathbb{Q}S_m, \mathbb{Q}) = \mathrm{Map}(S_m, \mathbb{Q})$$

Example .1.1

Take $X = \mathbb{C}\mathrm{P}^m$. How is $\Omega^*\mathbb{C}\mathrm{P}^m$ represented?

Well $H^*(\mathbb{C}\mathrm{P}^m; \mathbb{Q}) = \mathbb{Q}[x]/(x^{m+1})$ with the cochain degree of x is 2.

Then we have a map of DGAs

$$\mathbb{Q}[u] \to \Omega^* \mathbb{C}\mathrm{P}^m$$
$$u \mapsto u, [u] = x$$

Impose the relation $x^{m+1} = 0$, so $u^{m+1} = \mathrm{d}v$. This gives a map

$$\mathbb{Q}[u] \otimes \bigwedge[v] \to \Omega^* \mathbb{C}\mathrm{P}^m$$

Where we have $\mathrm{d}v = u^{m+1}$, so the degree of v is $2m+1$. This is a quasiisomorphism with a bit of work.

Therefore

$$\pi_i \mathbb{C}\mathrm{P}^m \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 2, 2m+1 \\ 0 & \text{otherwise} \end{cases}$$

Homework #10

(2) Find the rational minimal model of S^m $(m > 1)$ and use it to calculate $\pi_k S^m \otimes \mathbb{Q}$ for all k.

Deligne-Morgan: A simply connected CW complex of finite type X is called formal if $\Omega^* X$ is quasiisomorphic to $H^*(X; \mathbb{Q})$ with zero differential

Theorem .1.2

Every simply connected smooth projective variety over $\mathbb{C}$ is formal.

Griffiths-Harris: Principles of Algebraic Geometry.

What does an alegbraic topologist make of this? "$\pi_m \otimes \mathbb{Q}$ are not interesting"

Or, perhaps, better point: The torsion is more interesting to algebraic topology.

Another thing worth mentioning: What if we replace $\mathbb{Q}$ with another field?

characteristic $0 \to$ same story

characteristic $> 0 \to$ doesn't work.

By complicated, we mean we get stuck on the first step. We are not able to make a model of $C^*(X; \mathbb{F}_p)$ which would be a graded-commutative DGA. (if you do HW problem 1, it does not work in characteristic > 0).

The fact that this fails in characteristic > 0 is related to something known as Steenrod operations.

I. Steenrod Operations

For X a CW complex of finite type then with the actions of swapping from $\mathbb{Z}/2$

$$C^*(X) \otimes C^*(X) \xrightarrow{\simeq} C^*(X \times X) \xrightarrow{C^*\Delta} C^*(X)$$

but this cannot be done $\mathbb{Z}/2$-equivariantly. The steenrod operations measure how much this fails using group homology.