Let $\mathcal{C}$ be a small category. Similarly as for monoids, we can generalize the Bar construction (aka the nerve). We have a simplicial set $B\mathcal{C}_\bullet$ where

$$B\mathcal{C}_n = \{\text{composable } n- \text{ tuples}\}$$
$$B\mathcal{C}_0 = \operatorname{Ob} C$$

The faces are compose γ_i, γ_{i+1}, and the degeneracies insert a unit.

Theorem .0.1

If $\mathcal{C}$ is a small symmetric monoidal category then $B\mathcal{C}$ is an E_∞-space.

Proof Sketch. The Street Construction: $\mathcal{C}$ is equivalent to a "permutative category"

Definition .0.1

A permutative category has an operation $\otimes$ which is strictly unital and associative. We have $\sigma : X \otimes Y \to Y \otimes X$ such that $\sigma^2 = \mathrm{Id}$ and also

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "X\otimes\sigma"] \arrow[dr, "\sigma"'] & & X \otimes Z \otimes Y \arrow[dl, "\sigma\otimes Y"] \\ & Z \otimes X \otimes Y & \end{tikzcd}$$

Then the operad $\mathcal{M}$ defined by $\mathcal{M}(n) = \Sigma_n$ acts on $\mathcal{C}$ in the sense that we have

$$\mathcal{M}(n) \times \mathcal{C} \times \cdots \times \mathcal{C} \to \mathcal{C}$$

and this is given by

$$g, X_1, \ldots, X_n \mapsto g^{-1} X_1 \otimes \cdots \otimes X_n g$$

$E\mathcal{M}_\bullet$ acts on $B\mathcal{C}_\bullet$. We then take simplicial realization and $E\mathcal{M}$ acts on $B\mathcal{C}$.

Comments:

(1) If a category $\mathcal{C}$ has an initial object (or terminal object) in particualr, or if it as zero, then $B\mathcal{C} \simeq *$ (same proof as $EX \simeq *$).

Often, what does give interesting examples is to take the subcategory of isomorphisms in some category.

(2) Let R be a commutative ring. Take $\operatorname{Ob} \mathcal{C} = \mathbb{N}_0$, and $\operatorname{Mor}(m,n) = 0$ for $m \neq n$ and $\operatorname{Mor}(m,m) = \mathrm{GL}_m(R)$.

The symmetric monoidal structure is just the block sum of matrices. This is a permutative category, and it's fairly clear that

$$B\mathcal{C} = \coprod_{m \geq 0} B \,\mathrm{GL}_m R.$$

This is an E_∞-space. Then $\pi_0 B\mathcal{C} = \mathbb{N}_0$. Thus we must apply the group completion $\overline{B\mathcal{C}} = \Omega B(B\mathcal{C})$ by viewing $B\mathcal{C}$ as a topological monoid.

This is an infinite loop space Z_m giving a cohomology theory $KR = Z_0 \times K_0 R$ (the algebraic K-theory, $K_0 R$ is discrete).

Theorem .0.2 (Quillen)
The group completion $\Omega B\left(\coprod_{m\geq 0} B\operatorname{GL}_m R\right) \simeq B\operatorname{GL}_\infty R^+ \times \mathbb{Z}$, where $+$ denotes the Quillen plus construction.

Proof Sketch. First construct a map

$$B\operatorname{GL}_\infty R^+ \to X := \left(\Omega B\left(\coprod_{m\geq 0} B\operatorname{GL}_m R\right)\right)_0$$

using the fact that X is an E_∞-space, thus a topological monoid, and so $\pi_1 X$ is abelian. Also $H_*X = H_*B\operatorname{GL}_\infty R$ because

$$H_*X = [\pi_0]X^{-1}\left(H_*\coprod_{m\geq 0} B\operatorname{GL}_m R\right) = H_*B\operatorname{GL}_\infty[t,t^{-1}]$$

where $t = (1)$. Thus by the universal property of the plus construction there is the desired map. This map is both an isomorphism in π_1 and an isomorphism in homology.

To finish the proof, one needs to show that $B\operatorname{GL}_\infty R^+$ is a simple space (to get weak equivalence). That is we need to show that π_1 acts trivially on π_n for $n > 1$ (we already know π_1 is abelian). A key step of this is in homework.

$G = \operatorname{GL}_\infty R, E = [\operatorname{GL}_\infty R, \operatorname{GL}_\infty R]$ and $[E,E] = E$.

The universal coer of BG^+ is BE^+ by construction. Why does $\pi_1 BG^+ = G/E$ act trivially on $\pi_n BE^+$. It is <u>not true</u> that the action of $g \in G$ on E by $h \in E \mapsto ghg^{-1}$ is by conjugation of an element of E.

However, for any m elements $h_1, \ldots, h_m$ we can find an element $q \in E$ such that

$$gh_ig^{-1} = qh_iq^{-1}.$$

An element of $\pi_m BE$ only meets finitely many simplices, and therefore finitely many h_i.

The whitehead lemma says that $\begin{bmatrix} g & 0 \\ 0 & g^{-1}\end{bmatrix} \in E$.

Homework #12

(1) Let G be a (discrete) group, $g \in G$. Then g acts on G by conjugation. Therefore g acts on BG by conjugation. Prove that the map $\gamma_g : BG \to BG$ is homotopic to the identity.

Better to think of $BG \cong EG/G$, where EG is the Čech resolution on which G acts on the left.

Then find a G-equivariant map

$$\begin{tikzcd} EG \arrow[r, "\varphi"] \arrow[d] & EG \arrow[d] \\ BG \arrow[r, "\gamma_g"'] & BG \end{tikzcd}$$

You may use the fact that EG is a free G-CW-complex (non-degenerate simplices are the cells). Then prove that any two self-maps of a contractible free G-CW-complex are G-equivariantly homotopic.

Recall that a G-CW-complex has the cells as G-sets and the attaching maps are G-equivariant. A free G-set is the same as a disjoint union of copies of G.