I. Whitehead's Theorem and CW approximation

Definition I.0.1

A map $f : X \to Y$ is called an m-equivalence if $\pi_0 f : \pi_0 X \to \pi_0 Y$ is onto and for all $x \in X$, $\pi_k f : \pi_k(X, x) \to \pi_k(Y, f(x))$ is

(a) An isomorphism for $k < m$

(b) Onto for $k = m$.

A weak equivalence (or equivalence) is a map $f : X \to Y$ which is an m-equivalence for all m.

From now on $[Z, X] = \mathrm{Mor}_{\mathrm{hTop}}(Z, X)$ (unbased).

Theorem I.0.1 (Whitehead's Theorem)

This is a two-parter!

(1) Let $f : X \to Y$ be an m-equivalence (resp. weak equivalence). Then $[Z, f] : [Z, X] \to [Z, Y]$ is a bijection when Z is a CW-complex with dimension $< m$ and onto when Z is a CW-complex of dimension m (resp. bijective for every CW-complex Z).

(2) For every space X there exists an m-equivalence $\gamma_X^m : Z_m \to X$ where Z_m is a CW-complex of dimension $\leq m$ (resp. a weak equivalence $\gamma : Z \to X$ where Z is a CW complex).

Back to Bott's Theorem

Let M be a compact Riemannian manifold (connected). $P, Q \in M$ points, h a homotopy class of paths from P to Q, $\nu = (P, Q, h)$.

$$M^\nu = \{\text{all shortest (by arc length) geodesics from } P \text{ to } Q \text{ path-homotopic to } h\}$$

The index α: The minimum index α_k of geodesics path-homotopic to h where $\alpha_k > 0$.

The index of a geodesic α_k is the sum over points R interior to h of the dimension of the space of nearby geodesics beginning at the same starting point and also coinciding in R.

M^ν includes into the space of paths from P to Q in M, which is homotopy equivalent to $(\Omega M)_0$.

Theorem I.0.2 (Bott)

If M is a compact symmetric space and $\nu = (P, Q, h)$ as above then M^ν is a compact symmetric space and

$$\iota_\nu : M^\nu \to (\Omega M)_0$$

is an $(\alpha - 1)$-equivalence.

Example I.0.1

$M = S^n$, then $M^\nu \simeq S^{n-1}$ as the shortest geodesics are the meridians. Then $\alpha_k = 2(m-1)$. Thus $S^{m-1} \to \Omega S^m$ is a $[2(m-1)-1]$-equivalence.

General principle of compact symmetric spaces M (contains connected compact Lie groups). Take a closed geodesic at P. Then $M = G/H$ where G is the group of isometries and $H = \mathrm{Iso}(P)$. Then $M^\nu = H/\mathrm{Iso}(P, Q)$, where Q is the opposite point along the geodesic.

Example I.0.2

Complex Bott periodicity. Take $M = U(2m)$, P to be the identity, and $Q = -P$. Then take h to be

e^{ix} along the diagonal. Then

$$M^\nu = U(2m)/(U(m) \times U(m))$$

Index is $2m + 2 \to \infty$.

Thus $U/(U \times U) \to (\Omega U)_0$ is a weak equivalence.

$U \to U/(U \times \{e\}) \to U/(U \times U)$ is a fibration sequence. Therefore $U \simeq \Omega(U/(U \times U))$.

We can write $BU \coloneqq U/(U \times U)$. Bott's theorems say $BU \xrightarrow{\sim} (\Omega U)_0$.

Likewise $BU \times \mathbb{Z} \xrightarrow{\sim} \Omega U$. This means that $\Omega(BU \times \mathbb{Z}) \simeq U$.

Thus $\Omega^2 U \simeq U$.

Complex K-theory is then

$$K^m(X) = [X, Z_m]$$

Where Z_m is defined by

$$Z_m \coloneqq \begin{cases} BU \times \mathbb{Z} & \text{if } m \text{ even} \\ U & \text{if } m \text{ odd} \end{cases}$$

What happens if we replace $U(m)$ by $O(m)$? Well then it becomes a mit more complicated. Let $BO = O/(O \times O)$, and note that $U(m) \subseteq O(2m)$, and also that if $\mathrm{Sp}(m)$ is the $m \times m$ unitary quaternion matrices then $\mathrm{Sp}(m) \subseteq U(2m)$.

n	$\Omega^n(BO \times \mathbb{Z})$	Z_-
$8m$	$BO \times \mathbb{Z}$	Z_{8m}
$8m+1$	O	Z_{8m-1}
$8m+2$	O/U	Z_{8m-2}
$8m+3$	U/Sp	Z_{8m-3}
$8m+4$	$B\,\mathrm{Sp} \times \mathbb{Z}$	Z_{8m-4}
$8m+5$	Sp	Z_{8m-5}
$8m+6$	Sp/U	Z_{8m-6}
$8m+7$	U/O	Z_{8m-7}
$8(m+1)$	$BO \times \mathbb{Z}$	$Z_{8(m-1)}$.

For X a CW-comples, $\mathrm{KO}^m X \coloneqq [X, Z_m]$.

Homework #7

(3) Calculate $\mathrm{KO}^m(*)$, $\pi_m(BO \times \mathbb{Z})$, $m \in \mathbb{Z}$.

Use that

- $O(m)$ has two path-connected components
- $U(m), \mathrm{Sp}(m)$ are path-connected
- $O(m-1) \to O(m) \to S^{m-1}$ is an action (fibration sequence) in $\mathbb{R}^m$.
- $\mathrm{Sp}(m-1) \to \mathrm{Sp}(m) \to S^{4m-1}$ is an action (fibration sequence) in $\mathbb{H}^m$.

(4) Prove that if $f : X \to Y$ is a weak equivalence and X, Y are CW-complexes then f is a homotopy equivalence. (use Whitehead's Theorem)

Homework due Wednesday October 20th at 8PM.