Given a small symmetric monoidal category $\mathscr{C}$ we can build $B\mathscr{C}$ an E_∞-space, and then $\Omega B(B\mathscr{C})$ is an infinite loop space (being a group completion). This then is a spectrum

Example .0.1

Let $\mathscr{C}$ be the category of finite sets with bijections (symmetric monoidal operation is $\coprod$), permutatively we have $\mathscr{C} = \coprod_{m\geq 0}\Sigma_m$. Then we have that

$$B\left(\coprod_{m\geq 0}\Sigma_m\right) = \coprod_{m\geq 0} B\Sigma_m$$

is an E_∞-space. What spectrum corresponds to the group completion?

$$\Omega B\left(\coprod_{m\geq 0} B\Sigma_m\right) \simeq B\Sigma_\infty^+ \times \mathbb{Z}.$$

We also have

$$\Omega B\left(\coprod_{m\geq 0} B\Sigma_m\right) = \operatorname{colim}_{m\to\infty}\Omega^m S^m.$$

The map $\Omega^m S^m \to \Omega^{m+1}S^{m+1}$ can come from a map $S^m \to \Omega S^{m+1}$ adjoint to $\Sigma S^m \xrightarrow{\cong} S^{m+1}$.

This spectrum is the spectrification of the pre-spectrum $D_m = S^m$ via $\Sigma S^m \xrightarrow{\cong} S^{m+1}$ giving a map $D_m \to \Omega D_{m+1}$. This is a special case of a general construction. Let X be a based space. Let

$$D_m = \Sigma^m X \qquad \Sigma\Sigma^m X \xrightarrow{\cong} \Sigma^{m+1}X \qquad \Sigma^m \xrightarrow{\subseteq} \Omega\Sigma^{m+1}X.$$

(D_m) is then an inclusion spectrum and $\Sigma^\infty X$ is the spectrification of this.

Definition .0.1

$\Sigma^\infty X$ is called the suspension spectrum of X.

So then we have a situation like

$$(\text{Finite sets}, \cong) \xrightarrow{\infty \text{ loop space machine}} \Sigma^\infty S^0$$

In some sense the suspension spectrum if free. Then we have

$$\text{Spectra} \to \text{Spaces}$$

$$E = (Z_m) \mapsto \Omega^\infty E \coloneqq Z_0.$$

The left adjoint of Ω^∞ is Σ^∞ (the suspension spectrum). The verification is quick.

The category of finite sets is a "free symmetric monoidal category on one point" so plugging it into our infinite loop space machine and getting back a free infinite loop space on "one point" is good.

That is if $(\mathscr{C}, \oplus)$ is a symmetric monoidal category and $X \in \mathrm{Ob}(C)$, then $* \to X$ necessarily requires that

$$S \mapsto \bigoplus_S X$$

If $\mathscr{D}$ is an operad and X is a space, the free $\mathscr{D}$-algebra on X is

$$\mathscr{D}X = \coprod_{m\geq 0}\mathscr{D}(m)\times_{\Sigma_m} X^m. \qquad (\times_{\Sigma_m} = \text{space of orbits})$$

(left adjoint to the forgetful functor).

If $\mathscr{D}$ is an E_∞-operad, $X = *$ then

$$\mathscr{D}X \simeq \coprod_{m\geq 0} E\Sigma_m \times_{\Sigma_m} * = \coprod_{m \geq 0} B\Sigma_m.$$

If $\mathscr{D} = E\mathcal{M}$ then these are all equalities.

You can then ask if the category (finite sets, $\cong, \coprod$) is symmetric monoidal equivalent to a strictly commutative associative unital category. It is not

Proof Sketch. If so, then $\coprod_{m\geq 0} B\Sigma_m$ as an E_∞-space would be equivalent to a topological commutative monoid. We can then look at the chains

$$C_*\left(\coprod_{m\geq 0} B\Sigma_m; \mathbb{F}_2\right)$$

is an E_∞-algebra in $\mathbb{F}_p$-chain. But it is not quasiisomorphic to a graded-commutative DGA because of Dyer-Lashof operations.

For example, $\alpha \in H_0 B\Sigma_1$, the Dyer-Lashof operations (which we defined) on α are the basis of $H_* B\Sigma_p$. That's how they were defined!

For $p = 2$ we have $H_i(B\Sigma_2; \mathbb{F}_2) = H_i(B\mathbb{Z}/2; \mathbb{F}_2) = \mathbb{Z}/2$. The generator is then equal by definition to $Q^i \alpha$.

Spectra: $\Sigma^\infty S^0$ is a spectrum, and so it gives a generalized cohomology theory. But here we see a generalized homology theory more naturally. This is called stable homotopy groups. Say X is a based CW-complex,

$$\begin{aligned}
(\Sigma^\infty X)_0 &= \operatorname{colim} \Omega^m \Sigma^m X \\
\pi_k(\Sigma^\infty X)_0 &= \pi_k \operatorname{colim} \Omega^m \Sigma^m X \\
&= \operatorname{colim} \pi_k \Omega^m \Sigma^m X \\
&= \operatorname{colim} \pi_{k+m} \Sigma^m X
\end{aligned}$$

This is called $\pi_k^{\text{stable}} X$.

Maybe every spectrum gives rise to a generalized homology theory. Maybe we could do homotopy theory of spectra?

For a spectrum E and a based space (compactly generated, weakly Hausdorff) X, and notationally $E = (Z_m)$ with structure maps $\rho_m : Z_m \to \Omega Z_{m+1}$.

$$F(X, E) = T_m \qquad T_m = F(X, Z_m) \qquad \text{(based maps)}$$

We can also define $X \wedge E$. Remember for based spaces this is

$$X \wedge Y = (X \times Y)/((X \times *) \cup (* \times Y)).$$

Then $X \wedge E = L(U_m)$ (spectrification) where $U_m := X \wedge Z_m$.

Homotopy of spectra $p, q : E \to F$ is $h : [0,1]_+ \wedge E \to F$ which is f on $\{0\}_+ \wedge E$ and g on $\{1\}_+ \wedge E = g$.

To define homotopy groups, I need to define spheres $\mathbb{S}^m, m \in \mathbb{Z}$. For $m \geq 0$ just take $\mathbb{S}^m := \Sigma^\infty S^m$.

Spectra have a shift functor

$$\begin{aligned}
[k] : \text{Spectra} &\to \text{Spectra} \\
E = (Z_m) &\mapsto E[k] = (Z_{m+k}).
\end{aligned}$$

To get negative spheres, take $\mathbb{S}^{\ell-k} = (\Sigma^\infty S^\ell)[-k]$. It turns out that this only depends on $\ell - k$ and not both variables, as we should hope.

Homework #12

(2) Prove that $\Sigma^\infty(\Sigma X)[-1] = \Sigma^\infty X$. (realize that spectrification only depends on tail of prespectrum).