For a spectrum E, $\Sigma E = S^1 \wedge E$. Also, the category of spectra (Spectra) has all limits and colimits. Take limits "space-wise," colimits are done space-wise to obtain a prespectrum, so then spectrify. $\mathbb{S}^m \coloneqq \Sigma^\infty S^k[m-k]$. Per homework this does not depend on the choice of $k \geq 0$. We also have that

$$\Sigma^\infty(\Sigma X) = \Sigma(\Sigma^\infty X) = S^1 \wedge \Sigma^\infty X.$$

We can define hSpectra by using smash products to define homotopies. $[X, Y]$ denotes the set of homotopy classes of morphisms between X, Y. We can then of course define homotopy groups of a spectrum E via

$$\pi_m E = [\mathbb{S}^m, E]. \qquad (m \in \mathbb{Z})$$

These are always abelian groups because $\mathbb{S}^m = \Sigma^2 \mathbb{S}^{m-2}$, the proof is the same as for based spaces.

We can also define the mapping cone (homotopy cofibre) of a morphism of spectra $f : E \to F$

$$Cf \coloneqq \operatorname{colim} \left(\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "0"'] & F \\ E \wedge [0,1]_+ & \\ E \arrow[u, "1"] \arrow[r] & * \end{tikzcd} \right)$$

$$Ff \coloneqq \operatorname{colim} \left(\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "0"'] & F \\ E \wedge [0,1]_+ & \\ E \arrow[u, "1"] \arrow[r] & * \end{tikzcd} \right)$$

<u>Note</u>: ΣE is not the same as $E[1]$ for a general spectrum. On adjoints, equivalently, $\Omega E = F(S^1, E)$ is not the same as $E[-1]$.

If $E = (Z_m)_{m \in \mathbb{N}_0}$ with structure map $\rho_m : Z_m \to \Omega Z_{m+1}$. Then $(\Omega E)_m = \Omega Z_m$, with structure maps $\rho'_m : \Omega Z_m \to \Omega\Omega Z_{m+1}$. We need a switch of coordinates $T : \Omega^2 Z_{m+1} \to \Omega^2 Z_{m+1}$ then

$$\rho'_m = T \circ \Omega \rho_m.$$

Proposed isomorphism of spectra $E[-1] \to \Omega E$ given by $Z_{m-1} \xrightarrow{\rho_{m-1}} \Omega Z_m$. But when trying to check the compatibility we see

$$\begin{tikzcd} Z_{m-1} \arrow[r, "\rho_{m-1}"] \arrow[d, "\rho_{m-1}"] & \Omega Z_m \arrow[d, "T \circ \Omega \rho_m"] \\ \Omega Z_m \arrow[r, "\Omega \rho_m"'] & \Omega\Omega Z_{m+1} \end{tikzcd}$$

Does not commute! This is wrong!

Definition .0.1

A spectrum E is called a <u>cell spectrum</u> provided that $E = \operatorname{colim} E_{(m)}$ with

$$\bigvee_{i \in I_m} \mathbb{S}^{d_i} \xrightarrow{f_m} E_{(m)}.$$

We should have that $Cf_m = E_{m+1}$.

Theorem .0.1 (May,Lewis)

hSpectra has colocalization with respect to cell spectra and the class E of weak equivalences.

A weak equivalence is of course a morphism of spectra $f : E \to F$ which induces an isomorphism in all π_k for all $k \in \mathbb{Z}$.

The derived category $D\,\mathrm{Spectra} = D\,\mathrm{hSpectra}$ with respect to weak equivalences is called the stable homotopy category

Theorem .0.2 (May,Lewis)

On $D\,\mathrm{Spectra}$, Ω and $L\Sigma$ are inverse equivalences of categories isomorphic to $[-1]$ and $[1]$ respectively (where L denotes the left derived functor, aka cell approximate first).

Proposition .0.3

If $f : E \to F$ is a morphism of spectra then $Ff \sim LCf[-1]$ weakly.

L symbol is usually omitted because mathematicians are lazy.

Proof Sketch. We have analogously to based spaces for a map $f : E \to F$ a long exact sequence

$$[W, \Omega E] \longrightarrow [W, \Omega F] \longrightarrow [W, Ff] \longrightarrow [W, E] \longrightarrow [W, F].$$

We can prove in fact for E, F cell that

$$[W, E] \longrightarrow [W, F] \longrightarrow [W, Cf]$$

is also exact. The idea being that

$$\begin{tikzcd}
E \arrow[r] & F \arrow[r] & Cf \arrow[r] & \Sigma E \arrow[r, "-\Sigma f"] & \Sigma F \\
W \arrow[u, dashed, "\ell"] \arrow[r, "\mathrm{Id}"'] & W \arrow[u, "g"] \arrow[r] & * \arrow[u, "h"] \arrow[r] & \Sigma W \arrow[u, dashed, "k"] \arrow[r, "-\mathrm{Id}"'] & \Sigma W \arrow[u, dashed, "\Sigma\ell"]
\end{tikzcd}$$

using the theorem multiple times and then use the 5-lemma to show $Cf[-1] \xrightarrow{\sim} Ff$.

It follows that finite products are isomorphic to finite coproducts in $D\,\mathrm{Spectra}$.

It turns out that the stable homotopy category is triangulated, and has a lot of structure.

$D\,\mathrm{Spectra}, DA$ (for A an abelian category) has products / coproducts.

Homework #12

(3) In $D\,\mathrm{Ab}$ the map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ does not have a kernel (i.e. there is no equalizer between 2 and 0).

This is set up with cell chain complexes and chain homotopy classes of maps. Shifts $\mathbb{Z}[k]$ are cell complexes. If $[?, ?] = \mathrm{Mor}_{D\,\mathrm{Ab}}$. Then

$$[\mathbb{Z}[k], C] = H_k(C).$$

We also can see that for abelian groups A, B (considered as chain complexes in degree zero)

$$[A[-k], B] = [A, B[k]] = \mathrm{Ext}^k(A, B)$$

Lemma .0.4

$\mathrm{Ab} \to D\,\mathrm{Ab}$ sending A to A is an inclusion of a full subcategory. This is sometimes called

> the heart of the derived category with respect to chain homology. Also sometimes called the t-structure.

To see this, note that free resolutions are cell approximations. We proved in class that morphisms between free resolutions are the same as morphisms between the abelian groups.

The proof then becomes

(a) If $K = \ker(2 : \mathbb{Z} \to \mathbb{Z})$ exists in $D\,\mathrm{Ab}$, then $H_i K = 0$ for $i \neq 0$. Non-zero would violate uniqueness of the limit. Therefore $K \in \mathrm{Ab}$.

(b) But then $K = \ker(2 : \mathbb{Z} \to \mathbb{Z})$ in Ab. But there exists a nonzero morphism $C \xrightarrow{f} \mathbb{Z}$ in $D\,\mathrm{Ab}$ so that $2f = 0$.

Hint, use that $[A[-k], B] = [A, B[k]] = \mathrm{Ext}^k(A, B)$.