For a compact connected smoothly embedded m-manifold $M \subseteq \mathbb{R}^N$ we have by Spanier-Whitehead duality that

$$DM_+ = M^{\nu_M \mathbb{R}^N}[-N].$$

Recall that the Thom space of a vector bundle $\xi \to X$ for X compact is

$$X^\xi = \text{1-point compactification of } \xi$$

For general X, we have

$$X^\xi = \operatorname{colim}_{\substack{Z \subseteq X \\ \text{compact}}} Z^\xi.$$

Recall: If E is a spectrum and X is strongly dualizable, then

$$E_k X = E^{-k} DX.$$

We know $\Sigma^\infty M_+$ for a compact smooth connected manifold is strongly dualizable, so

$$E_k M = \widetilde{E}^{N-k} M^{\nu_M \mathbb{R}^N}$$

We also see that $\nu_M \mathbb{R}^N$ has dimension $N - m$. We can think of M^ξ as a "twisted suspension" of M by the dimension of the bundle. Indeed if $\xi = \ell$ was a trivial bundle, then $M^\xi = \Sigma^\ell M_+$.

Under what circumstances can we "untwist the Thom space to the eyes of the spectrum E"?

Suppose E is a commutative ring spectrum (a commutative monoid in $DSpectra$).

Thom realized that if ξ is an m-bundle on X, then there is a natural map

$$\begin{aligned} \theta : X^\xi &\to X^\xi \wedge X_+ \\ y \in \xi &\mapsto (y, \operatorname{proj} y) \\ \infty &\mapsto \infty \end{aligned}$$

It is an exercise to check continuity at ∞.

If X is a CW-complex

$$\widetilde{E}^*(\theta) : \widetilde{E}^*(X^\xi \wedge X_+) \to \widetilde{E}^*(X^\xi)$$

and using that it is a ring theory, we have a map

$$\widetilde{E}^k(X^\xi) \otimes E^\ell(X) \to \widetilde{E}^{k_\ell}(X^\xi \wedge X_+) \to \widetilde{E}^{k+\ell}(X^\xi).$$

The m-bundle ξ is called E-orientable if there exists a class $u \in \widetilde{E}^m(X^\xi)$ (called the Thom class) which for each point $x \in X$ restricts to a unit.

That is

$$\widetilde{E}^m(X^\xi) \to \widetilde{E}^m(\{x\}^\xi) = \widetilde{E}^m(S^m) = E_0(*).$$

If E is a ring spectrum then $E_0(*)$ is a commutative ring, and so we can just trace u through this map and see if it becomes a unit.

Thom Isomorphism Theorem:

If an m-bundle ξ is E-orientable with Thom class u, then

$$\Xi : \widetilde{E}^m(X^\xi) \otimes E^\ell(X) \to \widetilde{E}^{m+\ell}(X^\xi)$$

restricts to an isomorphism

$$\Xi(u\otimes ?) : E^\ell(X) \to \widetilde{E}^{m+\ell}(X^\xi)$$

Proof sketch. Take open cover $\{U_i\}_{i\in I}$ of X where $\xi\big|_{U_i}$ is trivial for each i. Then use the Meyer-Vietoris sequence and the five lemma.

If I is infinite, a limit argument is needed.

Note that in fact

$$E^\ell(X) = \widetilde{E}^{m+\ell}(\Sigma^m X_+).$$

Thus if ξ is E-orientable, then X^ξ "untwists" to the eyes of E.

Definition .0.1 (First Version)

A compact connected m-manifold is E-orientable for a commutative ring spectrum E when the normal bundle $\nu_M\mathbb{R}^N$ is E-orientable.

Then we can conclude that

$$E_k(M) \cong \widetilde{E}^{N-k} M^{\nu_M\mathbb{R}^N} \cong E^{N-k-N+m}(M) = E^{m-k}(M)$$

because $\dim \nu_M\mathbb{R}^N = N - m$.

This is called E-Poincare duality.

How can we make the definition of orientability more elegant? The Thom class of $\nu_M\mathbb{R}^N$ (if there is one) is in

$$\widetilde{E}^{N-m} M^{\nu_M\mathbb{R}^N} \cong E_m(M)$$

Definition .0.2 (Final version)

An E-orientation of a compacted connected smooth m-manifold M is a class $[M] \in E_m(M)$ (sometimes called the "fundamental class") such that the embedding of pairs $(M,\emptyset) \xrightarrow{\iota_x} (M, M\setminus\{x\})$, for every $x \in M$ sends $[M]$ to a unit.

That is we see that

$$E_m(M) \to E_m(M, M\setminus\{x\}) = \widetilde{E}_m(C\iota_x) \cong E_m(U, U\setminus\{x\}) \cong E_m(S^m) = E_0(*)$$

where $x \in U \cong \mathbb{R}^m$ is open. Again $E_0(*)$ is a ring and we can define this correctly.

Theorem .0.1 (Poincaré duality)

If M is an E-orientable compact connected m-manifold then Spanier-Whitehead duality, using the Thom class corresponding to the fundamental class $[M]$, define an isomorphism

$$E_k M \cong E^{m-k} M$$

Remark .0.1

For $E = H\mathbb{Z}/2$ (ordinary cohomology with coefficients $\mathbb{Z}/2$) every (compact smooth connected) manifold is $H\mathbb{Z}/2$-orientable.

Note that $H\mathbb{Z}/2_0(*) = \mathbb{Z}/2$ has a unique non-zero element.

$H\mathbb{Z}$-orientability is equivalent to $H\mathbb{R}$-orientability which (at least for compact, smooth, connected manifolds) is equivalent to the existence of a nowhere vanishing differential m-form.

This is related to the statement that $\mathbb{Z}$ only has two units.

I. A Plethora of Examples

For the last week, we will talk about Examples of Spectra, that is of generalized homology/cohomology.