The derived cateogry $D\,\mathrm{Spectra}$ is a closed symmetric monoidal category with $\wedge$ and $F(-,-)$.

Definition .0.1

In a symmetric monoidal category $\mathscr{C}$ (operation $\otimes$). A strong dual of an object X is an object Y together with morphisms

$$\mu : 1 \to Y \otimes X \qquad \varepsilon : X \otimes Y \to 1$$

such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}\otimes\mu"] \arrow[rr, bend right, "\mathrm{Id}"'] & X \otimes Y \otimes X \arrow[r, "\varepsilon\otimes\mathrm{Id}"] & X \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "\mu\otimes\mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"'] & Y \otimes X \otimes Y \arrow[r, "\mathrm{Id}\otimes\varepsilon"] & X \end{tikzcd}$$

If this holds, we call X (and symmetrically Y) strongly dualizable and write $Y = DX$

Comments: We have the following

(1) If $Y = DX$ is a strong dual of X, then $DX\otimes ?$ is both right and left adjoint to $X\otimes ?$ (use definition of adjunction via triangle identities).

(2) If $\mathscr{C}$ is closed, $F(X, ?)$ is the right adjoint to $X\otimes ?$, and adjoints are unique, so if X is strongly dualizable then

$$DX \otimes Y \cong F(X, Y)$$
$$DX \cong F(X, 1)$$

(3) If X is strongly dualizable, then

$$X \otimes F(Z, T) \cong F(Z, X \otimes T)$$

(4) $DDX \cong X$.

Example .0.1

If $\mathbb{F}$ is a field, then in $\mathbb{F} - \mathrm{Vect}$ the category of vector spaces over $\mathbb{F}$, then with $\otimes$ the tensor product we have the usual duals.

Definition .0.2

E is a ring-spectrum if we have

$$\mu : E \wedge E \to E \qquad \varepsilon : \mathbb{S} \to E$$

and we have the following commutative diagram in $DSpectra$

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mathrm{Id}\wedge\mu"] \arrow[d, "\mu\wedge\mathrm{Id}"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"] & E \end{tikzcd}$$

and similarly an identity axiom

Example .0.2

If $\mathscr{C} = D\,\mathrm{Spectra}$, $\otimes = \wedge$.

For a space X and a commutative ring spectrum E, E^*X is a graded commutative ring (working in $D\,\mathrm{Top}$). Why? Well consider

$$\begin{aligned} E^*X \otimes E^*X = F(X_+, E)_* \otimes F(X_+, E)_* &\to F(X_+ \wedge X_+, E \wedge E)_* \\ &\hookrightarrow F(X \times X_+, E \wedge E)_* \xrightarrow{\Delta^* \circ F(-,\mu)} F(X_+, E)_* = E^*X \end{aligned}$$

we can define morphisms of R-module spectra by commutativity with the operation. For R-modules $M \to N$

$$\begin{tikzcd} R \wedge M \arrow[r] \arrow[d] & M \arrow[d] \\ R \wedge N \arrow[r] & N \end{tikzcd}$$

But the mapping cone Cf is not in general an R-module.

Back to strong duality. Which objects are strongly dualizable in $D\,\mathrm{Spectra}$ and what are their strong duals?

Answer (Spanier): The best source is Adams stable homotopy + generalized cohomology [1]. Namely these are $\Sigma^\infty X[m]$ ($m \in \mathbb{Z}$) where X is a finite cell spectrum.

Note: In $D\,\mathrm{Spectra}$, we define for spectra E, X

$$E_m X = \pi_m(X \wedge E) \qquad\qquad E^m X = \pi_{-m}(F(X, E))$$

If X is strongly dualizable, then

$$E_m X = \pi_m(X \wedge E) \cong \pi_m(F(DX, E)) = E^{-m}(DX)$$

Spanier gave a geometric model of strong duality in $D\,\mathrm{Spectra}$ before all of this was understood. The model is entirely in spaces. Select some $N > 0$, and suppose $K, L \subseteq S^N$ with $K \cap L = \emptyset$. (The case ($\star$) we are interested in: K, L are simplicial subcomplexes of some triangulation of S^N. Further L is a deformation retract of $S^N - K$ and likewise K is a deformation retract of $S^N - L$). Then $\Sigma^\infty K \simeq DL[N-1]$

The way to construct the relevant maps is to select points $a \in K, b \in L$ and a simple path ω in S^N from a to b. Furthermore, require $\omega(t) \notin K, L$ if $t \neq 0, 1$. Select the basepoint ∞ to be $\omega(1/2)$. Then $S^N \setminus \{\infty\} = \mathbb{R}^N$. Thus we may define

$$\begin{aligned} \mu' : K \times L &\to S^{N-1} \\ (x, y) &\mapsto \frac{x - y}{\|x - y\|} \end{aligned}$$

We then have that $K \times \{b\} \simeq \text{const}$ and likewise for L. This gives us a deformed map $\mu : K \wedge L \to S^{N-1}$. Taking suspension spectra

$$\begin{gathered} \Sigma^\infty \mu : \Sigma^\infty K \wedge \Sigma^\infty L \to \mathbb{S}^{N-1} \\ \Sigma^\infty K \wedge \Sigma^\infty L[1 - N] \to S^0 \end{gathered}$$

when $K = S^N \setminus L$ we can also get ε and verify triangular identities on space level by hand.

This is called Spanier-Whitehead Duality