Definition .0.1

A simplicial set X which satisfies the inner Kan condition, which says any

$$\begin{tikzcd} \underline{V}_k^n \arrow[r] \arrow[d, hook] & X \\ \underline{\Delta}^n \arrow[ur, dashed] & \end{tikzcd}$$

for $0 < k < m$, then X is called a quasicategory.

Lurie uses quasicategory interchangeably with ∞-category, which is a vague term meaning many different but roughly equivalent things.

The homotopical information in an ∞-category is the same as a category where the sets of morphisms are given a topology and composition is continuous (topological category).

Back to Serre and Hurewicz!

Definition .0.2

X is a simple space provided that X is path-connected which means that $\pi_1(X)$ is abelian and $\pi_1(X)$ acts trivially on every $\pi_m(X)$.

Past homework showed that $S^1 \vee S^2$ is not simple.

Theorem .0.1 (The Relative Hurewicz Theorem)

If X is a simple space, then if $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i < m$, then the Hurewicz map $\pi_m(X) \to H_m(X; \mathbb{Z})$ becomes an isomorphism upon tensoring with $\mathbb{Q}$ to get $\pi_m(X) \otimes \mathbb{Q} to H_m(X; \mathbb{Q})$.

Proof Sketch. By induction, using the Serre spectral sequence of the fiber sequence $\Omega X \to * \to X$.

Theorem .0.2

If $f : X \to Y$ induces an isomorphism in $H_*(?, \mathbb{Q}) = H_*(?, \mathbb{Z}) \otimes \mathbb{Q}$ and X, Y are simply connected, then f induces and isomorphism in $\pi_*(?) \otimes \mathbb{Q}$.

Proof sketch. Following the method of homework, we get that $H_*(F(f)) \otimes \mathbb{Q} = 0$.

We would like to deduce $\pi_* F(f) \otimes \mathbb{Q} = 0$, which one does through the Relative Hurewicz Theorem

Remark .0.1

Let X be a path-connected space. Create a map $X \xrightarrow{f_m} X_m$ which induces an isomorphism on π_i for $i \leq m$. Then take $\pi_j(X_m) = 0$ for $j > m$ (attach cells to X to kill $\pi_{>m}$).

Then we have $X^m := F(f_m) \to X \xrightarrow{f_m} X_m$.

We get a tower $X_m \to X_{m-1} \to \cdots \to X_0$. We then get a fiber sequence

$K(\pi_{m+1}X, m+1) \to X_{m+1} \to X_m$. If X is a simple space, then this fiber sequence extends by 1 to the right

$$K(\pi_{m+1}X, m+1) \longrightarrow X_{m+1} \qquad X_m \longrightarrow K(\pi_{m+1}X, m+2)$$

This is called a Postnikov tower.

Rational homotopy theory: Localization of the full subcategory of $D\,\mathrm{Top}$ on simply connected (simple) spaces of finite type at equivalences being those maps inducing $\cong$ in $H_*(?; \mathbb{Q})$ (that is on $\pi_*(?) \otimes \mathbb{Q}$).

For a space X, there exists a model of $C^*(X; \mathbb{Q})$ which is a graded-commutative differential graded algebra.

Definition .0.3

A is a graded commutative DGA provided that if $x \in A_n, y \in A_m$ then

$$xy = (-1)^{nm} yx$$

and the differential satisfies

$$\mathrm{d}(xy) = \mathrm{d}x \cdot y + (-1)^m x \cdot \mathrm{d}y$$

For a simplex Δ^n, take differential forms on the affine envelope of Δ^n. Algebraically

$$\mathbb{Q}[x_0, \ldots, x_n]/(x_0 + \cdots + x_n = 1) \otimes \bigwedge[\mathrm{d}x_0, \ldots, \mathrm{d}x_m]/(\mathrm{d}x_0 + \cdots + \mathrm{d}x_n = 0)$$

If X is a simplicial complex (or a simplicial set), take the limit of these DGAs over its simplices. There's a paper by Sullivan in 1979 which is relevant.

For a space X, we construct in this way a graded commutative DGA over $\mathbb{Q}$. There is an appropriate notion of chain homotopy of graded commutative DGAs. Then this homotopy category satisfies colocalization with respect to cell DGAs.

A cell DGA is defined as $A_{(m)} \subseteq A_{(m+1)} \subseteq \cdots$ with $A = \bigcup_m A_{(m)}$ where

$$A_{(m+1)} = A_{(m)} \otimes F(x_i \mid i \in I_m)$$

where F is the free graded commutative algebra and x_i are homogeneous generators with $\mathrm{d}x_i \in A_{(m)}$. Recall that the free graded commutative algebra is $F(x) = \mathbb{Q}[x]$ for x in even degree and $\bigwedge[x] = \mathbb{Q}[x]/x^2$ if x is in odd degree. Take the tensor product for more than one generator.

In fact, A is called <u>minimal</u> if it is cell and $\mathrm{d}x_i$ is decomposable (sum of monomials and generators, each of which has monomial degree at least two).

Theorem .0.3

There exists a unique (up to DGA-isomorphism) minimal graded commutative DGA in each isomorphism class in the derived category.

<u>The upshot</u>: There is a <u>unique</u> model of a simple space of finite type up to $H_*(?; \mathbb{Q})$-equivalence by a minimal DGA (minimal model).

Homework #10

(1) Consider the commutative DGA

$$A = \mathbb{Q}[x] \otimes_{\mathbb{Q}} \bigwedge_{\mathbb{Q}} \bigwedge[\mathrm{d}x]$$

with $\deg(x)$ even. This is the tensor algebra, modulo $(\mathrm{d}x)^2 = 0$.

Prove that $H^i A = \mathbb{Q}$ if $i = 0$ and 0 if $i > 0$.

[Hint: Write down a basis of A