Example .0.1 (Universal)

$BO(m) = \{m\text{-dimensional real vector subspaces of } \mathbb{R}^\infty\}$. That is $EO(m)/O(m)$.

If we then consider

$$\gamma_{\mathbb{R}}^m = \{(V, x) \mid V \subseteq \mathbb{R}^\infty, \dim V = m, x \in V\}.$$

Then there's a map $\gamma_{\mathbb{R}}^m \to BO(m)$.

Similarly for $\mathbb{C}$, with $BU(m)$ and $\gamma_{\mathbb{C}}^m \to BU(m)$.

Theorem .0.1

If X is paracompact, then

$$\{\cong \text{ clsses of real vector } m\text{-bundles on } X\} \cong [X, BO(m)]$$

And also

$$\{\cong \text{ clsses of complex vector } m\text{-bundles on } X\} \cong [X, BU(m)]$$

The map is given by $f : X \to BO(m)$ to $f^*\gamma_{\mathbb{R}}^m$ (the pullback), and likewise for complex vector bundles.

References: [2, 1]

This gives a geometric interpretation of K-theory (cohomology). A permutative category $\mathscr{C}$ with objects $\mathbb{N}_0$ and morphisms $m \to m$ given by $U(m)$.

We then set

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

So then

$$B\mathscr{C} = \coprod_{m \geq 0} BU(m)$$

is an E_∞-space, and by group completion

$$\Omega B(B\mathscr{C}) = BU^+ \times \mathbb{Z}$$

where BU^+ is the Quillen $+$-construction. Then

$$U = \bigcup_m U(m)$$

So we have that

$$\pi_0 U(m) = 0 \qquad \pi_0 BU(m) = 0 \pi_1 U(m) = \mathbb{Z} \qquad \pi_1 BU(m) = 0$$

By the fibration sequence $U(m) \to EU(m) = * \to BU(m)$. Thus $BU^+ = BU$.

Thus we have proved $BU \times \mathbb{Z}$ is an infinite loop space without using Bott periodicity.

Note that

$$BU \times |ZZ = \operatorname{colim}\left(\coprod_{m \in \mathbb{N}_0} \xrightarrow{\oplus 1} \coprod_{m \in \mathbb{N}_0} BU(m) \to \cdots\right)$$

If X is compact Hausdorff, then

$$\left[X, \bigcup Z_m\right] = \operatorname{colim}_m [X, Z_m]$$

where $Z_0 \subseteq Z_1 \subseteq \cdots$.

If X is compact then by definition

$$K^0 X = [X, BU \times \mathbb{Z}]$$

is the group completion of $\{\cong$ classes of complex vector bundles on $X\}$ which is a commutative monoid with Whitney sum.

Grothendieck construction K is left adjoint to the forgetful functor Ab $\to$ commutative monoids.

For example $K(\mathbb{N}_0) = \mathbb{Z}$.

Elements of $K^0(X)$ are virtual bundles. Namely they look like

$$(\xi, \mu)/\sim$$
$$(\xi, \mu) \sim (\xi', \mu') \iff \exists \nu \;\; \xi \oplus \mu' \oplus \nu \cong \xi' \oplus \mu \oplus \nu$$

We think of the pair (ξ, μ) as "$\xi - \mu$." For X compact, any virtual bundle is of the form $\xi - N$ for N trivial, should refer to Atiyah's K-theory.

This definition of $K^0 X$ as the group completion of isomorphism classes of vector bundles on X is not invariant under weak equivalence. For X CW we have $K^0(X) = [X, BU \times \mathbb{Z}]$.

Vector bundles also have a tensor product. This introduces a commutative ring structure on $K^0 X$. In fact, K is a commutative ring spectrum(for now, commutative monoid in *DSpectra* w.r.t $\wedge$).

Can we say something more rigit (= coherent) in *Spectra*, where $\wedge$ is not a symmetric monoidal product. Answer: yes. We will call this a multiplicative ∞ loop space.

Another example of a constructon of spectra:

.1. Cobordism

The geometric problem of cobordism: compact smooth manifolds without boundy M of dimension m, with equivalence defined by cobordism

Definition .1.1

We say two compact manifolds M, N of dimension m are cobordant if there is a compact manifold W of dimension $m + 1$so that $\partial W = M \coprod N$ with the normal data preserved.

That is

$$\nu_W^{\mathbb{R}^N} \oplus \tau_W \cong N \qquad \text{(trivial)}$$

so that

$$\tau\big|_M = \tau_M \oplus 1 \qquad\qquad \tau\big|_N = \tau_N \oplus 1.$$

For refernece see [2]

to avoid the problems, we prescribe some requirement on the normal bundle $\nu_M^{\mathbb{R}^N}$ (well-defined if whatever structure we require on $\nu_M^{\mathbb{R}^N}$ for $N \gg 0$ must be preserved by enlarging N, aka it is "stable").

Example .1.1

There are many examples

- $\mathbb{N}_0$ structure, then unoriented MO.
- Oriented (w.r.t $H\mathbb{Z}$), gives oritented MSO

- Complex gives complex MU.
- Trivial gives framed $\leq$.

Equivalence classes of each type of manifolds under cobordism are called cobordism groups MO_m, MSO_m, MU_m.

Why groups? The group operation is $\coprod$, and the inverse is to add the trivial bundle, reverse sign of 1-dimensional subspace in the isomoprhism class.

What