If a cohomology theory is complex oriented and u is the Thom class, then

$$E^*\mathbb{CP}^\infty = E_*[[u]]$$

where we allow infinite sums which are homogeneous, with u having cohomological degree 2. One can compute this with AHSS

Similarly $E^*(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty) = E^*[[u_1, \ldots, u_m]]$.

Then $\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \to BU(m)$. This then gives a map

$$E^*BU(m) \to E^*(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty) = E_*[[u_1, \ldots, u_m]]$$

But note $\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty = B(S^1 \times \cdots \times S^1)$, with an action $\Sigma_m \subseteq U(m)$. On Homework, we proved inner automorphisms of G induce $\simeq \mathrm{Id}$ on BG.

This means it factors through as

$$E^*BU(m) \to E^*[[u_1, \ldots, u_m]]^{\Sigma_m} \to E^*[[u_1, \ldots, u_m]]$$

If $c_i = \sigma_i(u_1, \ldots, u_m)$ is the elementary symmetric polynomial this is $E^*[[c_1, \ldots, c_m]]$.

AHSS injects on E_2-terms so target collapses. Thus this is an isomorphism and

$$E^*BU(m) = E^*[[c_1, \ldots, c_m]]$$

We can then induct $BU(m-1)_+ \to BU(m)_+ \to BU((m)^{\gamma_{\mathbb{C}}^m}$ to show $\gamma_{\mathbb{C}}^m$ is E-oriented. Every complex bundle is then E-oriented. The symmetric polynomials $c_1, \ldots, c_m$ are called Chern classes.

Let ξ be an m-bundle on X, then $c_i \in E^{2i}X$. What is the classification of $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, the line bundle $\gamma_{\mathbb{C}}^1 \otimes \gamma_{\mathbb{C}}^1$.

Also $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, addition in $H^2(?; \mathbb{Z})$. Well this is a map

$$\begin{aligned} E^*[[u]] &\to E^*[[u_1, u_2]] \\ u &\mapsto F(u_1, u_2) = u_1 +_F u_2. \end{aligned}$$

We get properties like

$$x +_F 0 = x = 0 +_F x$$
$$x +_F y = y +_F x$$
$$(u +_F v) +_F w = u +_F (v +_F w)$$

a power series $F \in R[[u, v]$ which satisfies these two properties is called a formal group law (FGL).

Example .0.1

K-theory, $K_* = K_*(*) = \mathbb{Z}[\beta, \beta^{-1}]$ where β is a Bott class in degree 2. If we omit β from the notation, then

$$u = \gamma_{\mathbb{C}}^1 - 1 \in \widetilde{K}^0\mathbb{CP}^\infty$$

that is a virtual bundle of dimension zero. The tensor product of $u+1, v+1$ is $(u+1)(v+1)$. Subtracting 1, the formal group law is

$$u +_F v = u + v + uv.$$

This is called a multiplicative FGL.

For $H\mathbb{Z}$ (ordinary cohomology) we have $u +_F v = u + v$. This is called an additive FGL.

Example .0.2

$E^*\mathbb{RP}^\infty$. We then have a cofiber sequence

$$\mathbb{RP}^\infty_+ \to \mathbb{CP}^\infty_+ \xrightarrow{(\gamma^1_{\mathbb{C}})^2} \to (\mathbb{CP}^\infty)^{(\gamma^1_{\mathbb{C}})^2}.$$

If E is complex-oriented, then we have

$$E^*\mathbb{RP}^\infty \longleftarrow E^*[[u]] \xleftarrow{u+_F u} E^*[[u]][2].$$

In principle this is a long exact sequence, but if the right map is injective, it's a short exact sequence and

$$E^*\mathbb{RP}^\infty = E^*[[u]]/(u +_F u).$$

Example .0.3

K-theory (ignore the Bott class. Then

$$[2]_F u = (1+u)^2 - 1 = 2u + u^2.$$

This is injective on $\mathbb{Z}[u]$. Thus

$$\begin{aligned} K^0\mathbb{RP}^\infty &= \mathbb{Z}[[u]]/(1+u)^2 - 1 \\ K^1\mathbb{RP}^\infty &= 0 \end{aligned}$$

This is isomorphic for $t = 1 + u$ to

$$(\mathbb{Z}[t]/(t^2-1))^\wedge_{t-1} = \mathbb{Z}_2 \oplus \mathbb{Z}.$$

Where does this come from? Well t is essentially the tautological bundle. When restricted to $\mathbb{Z}/2$, $\mathbb{Z}[t]/(t^2-1)$ is the complex representation ring of $\mathbb{Z}/2$. That is

$$R(G) = K(\text{comm. monoid of f.d. complex representations of } G).$$

This is an example of the below theorem.

Theorem .0.1 (Atiyah-Segal Completion)

If G is a compact Lie group (including finite groups) then $K^0BG = (R(G))^\wedge_I$, where I is the augmentation ideal (virtual representations of dimension zero) and $K^1BG = 0$.

They considered <u>G-equivariant K-theory</u>. For compact CW-complexes, you want to take G-equivariant complex bunbldes. Then this comes from

$$K^0_G(*) = R(G) \qquad\qquad K^1_G(*) = 0.$$

This motivated the idea of equivariant generalized cohomology theory.

.1. More Formal Group Laws

Can we classify Formal Group Laws? Well let

$$F(x,y) = \sum_{i,j\geq 0} a_{i,j} x^i y^j$$

and consider the Lasard ring

$$L = \mathbb{Z}[a_{ij}]/(\text{relations from requiring that } F \text{ be an FGL}).$$

For example

$$a_{i,j} = a_{j,i} \qquad a_{io} = 0, i > 1 \ldots$$

So now F is an FGL on L. Then we have that

$$\{\text{FGLs on } R\} = \text{Mor}_{\text{Ring}}(L, R)$$

Theorem .1.1 (Lasard's)

We have that $L = \mathbb{Z}[x_1, x_2, \ldots]$.

We now notice the complex cobordism spectrum MU is complex-oriented ($(\mathbb{C}\text{P}^\infty)^{\gamma^1}$ is a term in the prespectrum).

Theorem .1.2 (Milnor-Novikov)

The FGL on $MU^*[[u]]$ (from complex orientation) is the Lasard universal formal group law. This is somehow familiar because ($MU_* = \mathbb{Z}[x_1, \ldots,]$)

Why are FGLs important? They come up in number theory. Natural question: can we make a complex-oriented spectrum from MU say by "killing generators", inverting others, and so on.

Answer: Not in the derived category! We need some coherence and some replacement for $\wedge$ not being strictly commutative, associative.

This becomes the general theory of

Brave new algebra, Spectral algebra, Higher algebra.

This subject is focused on how to introduce higher coherence.

How do FGLs come up in number theory. Well if K is a finite field extension of $\mathbb{Q}$, that is called a number field. We can describe Galois extensions of $K \subseteq L$ with abelian Galois group by their number theoretical properties. This area of mathematics is called class field theory.

Formal Group Laws cannot do Class Field Theory, but they *can* do it locally. The local question is to instead consider $\mathbb{Q}_p$ = fractions of $\mathbb{Z}_p$, where $\mathbb{Z}_p = \lim \mathbb{Z}/(p^n)$. A finite field extension $\mathbb{Q}_p \subseteq K$ is called a local number field.

The finite exntesions of $\mathbb{F}_p$ are $\mathbb{F}_{p^n}$, with $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathbb{Z}/n$.

We can lift to $\mathbb{Z}_p \subseteq W$. The field of fractions K (unramified degree in extension of $\mathbb{Q}_p$). Then $W = \mathcal{O}_K$.

Theorem .1.3 (Lubin-Tate)

if we have $x^{p_{i_1}} + \cdots + px = f(x)$ where the middle bit is divisible mod p, then there exists an FGL such that $[p]_F x = f(x)$ on $\mathcal{O}_K$.

Then

$$K[x]/f^{\circ n}(x)/f^{\circ(n-1)}(x)$$

this is a totally ramified (Eistenstein's polynomial) extension of K, which is both abelian + Galois. Furthermore F is an $\mathcal{O}_K$ module Then $[\alpha]x$ makes sense for $\alpha \in \mathcal{O}_k$. Furthermore the Galois action is

$$\alpha(x) \mapsto [\alpha]x.$$

Therefore we have that

$$\text{Gal}(L/K) = (\mathcal{O}_K/p^m\mathcal{O}_K)^\times.$$