There is no natural map filling the diagram below (commuting up to homotopy)

$$\begin{tikzcd} C(X) \arrow[r, "?"] \arrow[dr, "C(\Delta)"'] & C(X) \otimes C(X) \arrow[d, "\simeq"'] \\ & C(X \times X) \end{tikzcd}$$

where $C(X) = C(X; \mathbb{Z}/2)$ which is $\mathbb{Z}/2$-equivariant, using the action $x \otimes y \mapsto y \otimes x$.

This would bean that the chains in the image would be $\mathbb{Z}/2$-fixed. This turns out to be impossible!

What is posible? Consider a $\mathbb{Z}[\mathbb{Z}/2]$-free resolution of $\mathbb{Z}$, e.g.

$$\cdots \longrightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\alpha} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\alpha} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\alpha} \mathbb{Z}[\mathbb{Z}/2]$$

which we call R. It is possible to construct a $\mathbb{Z}/2$ equivariant

$$\begin{tikzcd} R \otimes C(X) \arrow[r, "?"] \arrow[dr, "R\otimes C(\Delta)"'] & C(X) \otimes C(X) \arrow[d, "\simeq"'] \\ & C(X \times X) \end{tikzcd}$$

R makes it a free $\mathbb{F}_2[\mathbb{Z}/2]$-modules. Universal element $\mathrm{Id} : \Delta^m \to \Delta^m$ (like in the Eilenberg-Zilber element), but we can $\otimes$ it with a free generator in R $(R \otimes C(X; \mathbb{Z}/2))_k$ is also representable.

Like in the cup product tratement. Dualize to cohomology

$$R \otimes C^*(X) \otimes C^*(X) \to C^*(X).$$

R remains homologically graded. So $C^*(X)$ is put in homological degree $-$.

We can also write

$$R \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C^*(X) \otimes C^*(X)) \to C^*(X).$$

$\mathbb{Z}/2$ acts on $H^*(X) \otimes H^*(X)$ by a permutation representation. We know $H^*(X)$ has basis $\alpha_i, i \in I$ and so we can map $\alpha_i \otimes \alpha_j \to \alpha_j \otimes \alpha_i$.

We then get a non-canonical map

$$R \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (H^*(X) \otimes H^*(X)) \to R \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C^*(X) \otimes C^*(X)) \to C^*(X).$$

However we do get a canonical map

$$H_*(\mathbb{Z}/2; H^*(X) \otimes H^*(X)) \to H^*(X)$$

with coefficients in $\mathbb{Z}/2$. If we order I then

$$H^*(X) \otimes H^*(X) = \bigoplus_{i<j} \mathbb{F}_2[\mathbb{Z}/2] \cdot \alpha_i \otimes \alpha_j \oplus \bigoplus_{i=j} \mathbb{F}_2 \cdot \alpha_i \otimes \alpha_i.$$

We know that $H_k(\mathbb{Z}/2; \mathbb{Z}/2) = H_k(\mathbb{R}\mathrm{P}^\infty, \mathbb{Z}/2) = \mathbb{Z}/2$ for all $k \geq 0$. Call this generator e_k.

If $\alpha_i \in H^m X$ (m depending on i), then

$$H_*(\mathbb{Z}/2; H^*(X) \otimes H^*(X)) \to H^*(X)$$

$$e_k \otimes \alpha_i \otimes \alpha_i \xmapsto{D_k} \; ? \in H^{2m-k} X$$

We may then define a Steenrod Operation

$$Sq^i = D_k : H^m X \to H^{m+i} X$$

by taking $k = m - i$.

Facts: $Sq^{>m} : H^m X \to ?$ (undefined, sometimes set to zero). The map $Sq^m : H^m X \to H^{2m} X$ sends x to x^2. And then $Sq^0(x) = x$, which is very nontrivial from this point of view.

Also $Sq^{<0}(x) = 0$, which is also nontrivial. The operations between 0 and m are completely mysterious.

We also have that

$$Sq^m(xy) = \sum_{i=0}^{m} Sq^i(x) Sq^{m-i}(y)$$

where juxtaposition denotes the cup product. The coproduct in $H_*(\mathbb{Z}/2, \mathbb{Z}/2)$ is $e_m \mapsto \sum e_i \otimes e_{m-i}$ (the dual to $H^*(\mathbb{Z}/2, \mathbb{Z}/2)$ being polynomial).

We can think of these as axioms

(1) $Sq^m(x) = x^2$

(2) $Sq^0(x) = x$
(3) $Sq^m(xy) = \sum_{i=0}^{m} Sq^i(x) Sq^{m-i}(y)$

Homework #10

(3) Calculate $Sq^m x^k$ with respect to $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$ with the degree of x being one. Use the axioms above.

There are compositions $Sq^i Sq^j = ?$ called Adam Relations. They're not deep but require prowess in combinatorics.

For $p > 2$ being the characteristic, we run into the fact that $\mathbb{Z}/p \subsetneq \Sigma_p$ (the symmetric group). So we really need to talk about $H_*(\Sigma_p; ?)$. Also because of signs, we can encounter either the trivial $\mathbb{Z}/p$-module or the sign representation.

We do have maps from functoriality.

$$H_*(\mathbb{Z}/p; ?) \underset{\text{res}}{\overset{\text{transfer}}{\rightleftarrows}} H_*(\Sigma_p; ?)$$

This maps $H_*(\Sigma_p; \mathbb{Z}/p)$ into direct summands of $H_*(\mathbb{Z}/p; \mathbb{Z}/p)$.

With this we'll get maps

$$p^i : H^m X \to H^{m+2i(p-1)} X \beta p^i : H^m X \qquad \to H^{m+2i(p-1)+1} X$$

β is the Bochstein from the short exact sequence in coefficients

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

which gives a long exact sequence in homology with connecting map

$$H^m(X; \mathbb{Z}/p) \xrightarrow{\beta} H^{m+1}(X; \mathbb{Z}/p)$$

For $p = 2$, $\beta = Sq^1$.

Example .0.1

$H^*(B\mathbb{Z}/p; \mathbb{Z}) = \mathbb{Z}[y]/(py)$ where $\deg y = 2$.

And also $H^*(B\mathbb{Z}/p; \mathbb{Z}/p) = \mathbb{Z}/p[y] \otimes \bigwedge[x]$ where $\deg x = 1$. There is also the integral Bochstein, which is the connecting map of

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0$$

Which will give $H^m(B\mathbb{Z}/p;\mathbb{Z}/p) \xrightarrow{\beta} H^{m+1}(B\mathbb{Z}/p;\mathbb{Z})$. For $m = 1$ you can derive from exactness that the integral bochstein is an isomorphism.

To get the mod p Bochstein, just compose with the map induced by $\mathbb{Z} \to \mathbb{Z}/p \to 0$.

We also have that Sq^i, β, p^i commute with the isomorphism

$$H^m(X) \to H^{m+1}(\Sigma X)$$

Once we get to stable homotopy theory we can use what's called the Adams Spectral Sequence, which takes these as data, to compute stable homotopy groups