In notation, we quite often identify a CW-complex X with the spectrum $\Sigma^\infty X$.

Spanier-Whitehead Duality: For $X \subseteq S^N$ (say a simplicial subcomplex, then $D\Sigma^\infty X = \Sigma^\infty(S^N \setminus X)[-N+1]$.

Recall .0.1

when Z in $DSpectra$ is strongly dualizable

$$DZ = F(Z, S^0)$$

What about X_+? Well we have a cofiber sequence

$$X_+ \to S^0 \to \Sigma X$$

And after applying Σ^∞ we have

$$DS^0 \to DX_+ \to DS^0.$$

That is

$$S^N \setminus X[-N] \to S^N[-N] = S^0 \to DX_+ \to S^N \setminus X[1-N] = DX.$$

last term is the mapping cone.

Answer:

$$D(\Sigma^\infty X_+) = D(X_+) = C(S^N \setminus X \to S^N)[-N]$$

if U is an open neighborhood of X in S^N, this is the same as

$$C(U \setminus X \to U)[-N]$$

A particularly interesting case is $X = M$ being a compact smooth N-manifold. Then we can embed

$$M \subseteq \mathbb{R}^N \subseteq S^N = \mathbb{R}^N \cup \{\infty\}$$

I. Vector Bundles

General structure: A family of finite-dimensional vector spaces indexed by X.

Form a category of space over X, whose objects are continuous maps $Y \to X$ and whose morphisms are diagrams

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

Definition I.0.1 (Vector Bundle)

A topological vector space over X can then be defined in this category. Explicitly for a total space E with a map $p : E \to X$ there are addition, multiplication, negation, and zero maps as below. For $\lambda \in \mathbb{R}$,

$$\begin{tikzcd} E \times E \arrow[rr, "+"] \arrow[dr, "p\times p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "\lambda\cdot -"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[dr, "\mathrm{Id}_X"'] & E \arrow[d, "p"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "-"] \arrow[dr, "p"'] & E \arrow[d, "p"] \\ & X \end{tikzcd}$$

satisfying the obvious commutative diagrams.

which is locally isomorphic to a product with $\mathbb{R}^n$. That is there exists an open cover U_i of X such that bulling back to each U_i, $p^{-1}(U_i) \to U_i$ is isomorphic, as a vector space over U_i, to $U_i \times \mathbb{R}^n \to U_i$.

Example I.0.1

Möbius Band $\to S^1$.

Tubular Neighborhood Theorem: If $M \subseteq M'$ is a smooth embedding of compact manifolds, then there exists an open neighborhood U of M in M' which is homeomorphic to the total space of a vector bundle (with M embedded as the 0-section).

The normal bundle of M in M' for example.

For $M \subseteq \mathbb{R}^N \subseteq S^N$ we have

$$\begin{aligned} DM_+ &= C(S^N \setminus M \to S^N)[-N] \\ &= C(U \setminus M \to U)[-N] \\ &= C(E \setminus M \to E)[-N] \end{aligned}$$

where U is a tubular neighborhood with bundle E, and $E \setminus M \to E$ is the inclusion by embedding M into E via the 0-section. Call $E_0 \coloneqq E \setminus M$. We also have $E = \nu_M S^N$, where $\nu_M S^N$ is the normal bundle of M in S^N

For a vector bundle E, what does $C(E_0 \to E)$ look like?

Say, the bundle $E \xrightarrow{p} X$ over X has X compact.

Claim

$C(E_0 \to E)$ is homootpy equivalent to the 1-point compactification of E, which is equivalent to $D(E)/S(E)$ (the 1-point compactification of the open disk bundle)

Where $S(E) \to D(E)$ is the inclusion of the unit sphere bundle into the unit disk bundle.

The 1-point compactification of E (where $p : E \to X$ is a vector bundle, X compact) is called the Thom space of E, someitmes denoted by X^E or $T(E)$.

If X is not compact, $D(E)/S(E)$ does not compactify E, so we need to define the Thom space by

$$X^E \coloneqq \operatorname{colim}_{\substack{Z \subseteq X \\ \text{compact}}} Z^E$$

The conclusion: If M is a connected compact m-manifold embedded in $\mathbb{R}^N$ then

$$DM_+ = M^{\nu_M \mathbb{R}^N}[-N]$$

What can we say about $\nu_M \mathbb{R}^N$. Well

$$\nu_M \mathbb{R}^N \oplus T(M) = N$$

where $\oplus = \times_M$ is the whitney sum, $T(M)$ is the tangent bundle of M, and N is the trivial bundle of dimension N.

One can prove that (since M is compact), if $N \gg 0$ then the Whitney sum component of a bundle ξ in N (a bundle μ such that $\xi \oplus \mu \cong N$) is uniquely determined after isomorphism.

So in fact, selecting $N \gg 0$, $\nu_M \mathbb{R}^N$ is determined (we say: the normal bundle is stably determined).

Next: E-orientability for a commutative ring spectrum E (commutative monoid in $DSpectra$), leading us to E-Poincaré duality.