Definition .0.1

A **cell map** (cellular map, CW-map) between CW-pairs $f : (X, Z) \to (Y, T)$ is a continuous map which preserves skeleta. That is $f(X_n) \subseteq Y_n$.

Theorem .0.1

Every (continuous) map between CW-pairs is homotopic to a cell map.

Proof in Hatcher (Theorem 4.8 [1]). An elaboration of the proof. Why is every map $f : S^k \to S^m$ for $k < m$ homotopic to the constant map. It's clear if images misses a point $S^m \setminus \{*\} \simeq *$. But $f \simeq$ smooth map, which always misses a point.

Proposition .0.2

If (X, Z) is a CW-pair, then $X/Z \simeq C\iota$ where $\iota : Z \hookrightarrow X$. As a consequence

$$E_m(X, Z) \cong \widetilde{E}_m(C\iota) \cong \widetilde{E}_m(X/Z)$$

This works more generally when $Z \hookrightarrow X$ has the homotopy extension property (HEP), which holds for CW-pairs)

Definition .0.2

The mapping cylinder Mf of a map $f : Y \to X$ is given as

$$Mf : ((Y \times [0,1]) \coprod X)/(y, 0) \sim f(y)$$

Definition .0.3

A map $f : Z \to X$ is a **cofibration** (satisfies HEP) if there is a left inverse $r : X \times [0,1] \to Mf$ of the map

$$\overline{f} : Mf \to X \times [0,1]$$
$$(y, t) \mapsto (f(y), t)$$
$$x \mapsto (x, 0)$$

A more explicit definition is given by the commuting diagram below, which means that if we have g_0 and g_t commuting then there must exist a $\widetilde{g}_t$.

$$\begin{tikzcd}
Z \arrow[rr, "\iota_0"] \arrow[dd, "f"'] & & Z \times I \arrow[dl, "g_t"'] \arrow[dd, "f \times \mathrm{Id}"] \\
& Y & \\
X \arrow[ur, "g_0"] \arrow[rr, "\iota_0"'] & & X \times I \arrow[ul, dashed, "\widetilde{g}_t"']
\end{tikzcd}$$

See [2] Chapter 6 for details.

A CW-pair is a cofibration. Only need to observe that $S^{m-1} \subseteq D^m$ is a cofibration, because cofibrations do well with pushouts. This means we need a retract of

$$S^{m-1} \times [0,1] \cup D^m \times \{0\} \hookrightarrow D^m \times [0,1]$$

But this is homeomorphic to

$$D^m \times \{0\} \hookrightarrow D^m \times [0,1]$$

And this has a retract given by taking every (x,t) to $(x,0)$..

If $\iota : Z \hookrightarrow X$ is a cofibration then $C\iota \simeq X/Z$. We know that $M\iota \overset{j}{\subseteq} X \times [0,1]$ has a left inverse. We can perform $M\iota/(Z \times \{1\})$, and this gives $C\iota \overset{j'}{\subseteq} X \times [0,1]/Z \times \{1\}$ has a left inverse r'.

Restrict r' to $X/Z \cong X \times \{1\}/Z \times \{1\} \xrightarrow{\ell} C\iota$.

We claim that ℓ is a homotopy inverse to $c : C\iota \to X/Z$. The details of this will be on the homework

Calculating (Co)homology of CW-pairs

First we'll look at Ordinary homology with coefficients in $\mathbb{Z}$. Make a chain complex $C^{\text{cell}}(X,Z)$. Namely, look at the homology

$$H_k(X_m, X_{m-1}) \cong \widetilde{H}_k(X_m/X_{m-1}) \cong \widetilde{H}_k\left(\bigvee_{I_m} S^m\right) = \bigoplus_{I_m} \widetilde{H}_k(S^m) = \begin{cases} 0 & \text{if } k \neq m \\ \mathbb{Z}I_m & \text{if } k = m \end{cases}$$

We can calculate $\widetilde{H}_k(S^m)$ by noting it is the m-fold suspension of $S^0 = \{*, *'\}$.

That is $H_m(X_m, X_{m-1}) = \mathbb{Z}I_m$ is the free abelian group on the set of m-cells. We then have from the long exact sequence of a pair the map ∂_m below, which we can combine with the inclusion j_{m-1}:

$$H_m(X_m, X_{m-1}) \xrightarrow{\partial_m} H_{m-1}(X_{m-1}) \xrightarrow{j_{m-1}} H_{m-1}(X_{m-1}, X_{m-2})$$

We can set $\mathrm{d}^{\text{cell}}_m = j_{m-1} \circ \partial_m$. Some calculations with long exact sequences of pairs shows that this gives a chain complex.

This allows us to define $C^{\text{cell}}(X)$ as

$$\cdots \longrightarrow \mathbb{Z}I_m \xrightarrow{\mathrm{d}^{\text{cell}}} \mathbb{Z}I_m \longrightarrow \cdots$$

And we can of course define

$$C^{\text{cell}}(X;A) = C^{\text{cell}}(X) \otimes A$$
$$C_{\text{cell}}(X;A) = \mathrm{Hom}(C^{\text{cell}}(X), A)$$

Theorem .0.3

We in fact have

$$H_m(X;A) = H_m(C^{\text{cell}}(X;A))$$
$$H^m(X;A) = H^m(C_{\text{cell}}(X;A))$$

The proof will be later.

Next time: How to calculate d^{cell}.

Homework #2

(3) Prove that if $Z \overset{\iota}{\hookrightarrow} X$ is a cofibration then $X/Z \simeq C\iota$. (detailed hint in lecture).