If C and D are free R-resolutions of R-modules M, N (for R a commutative ring) then

$$H_m(C \otimes_R D) = H_m(C \otimes_R N) = \operatorname{Tor}_m^R(M, N)$$

Therefore $\operatorname{Tor}_m^R(M, N) \cong \operatorname{Tor}_m^R(N, M)$, because $C \otimes_R D \cong D \otimes_R C$.

Let $R = \mathbb{Z}$. A corollary is

Corollary .0.1 (Kunneth Theorem)

Let C, D be chain complexes of free abelian groups. Then

$$H_m(C \otimes D) \cong \bigoplus_{k+\ell=m} H_k(C) \otimes H_\ell(D) \oplus \bigoplus_{k+\ell=m-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_k(C), H_\ell(D))$$

naturally we have a exact sequence

$$0 \longrightarrow \bigoplus\nolimits_{k+\ell=m} H_k(C) \otimes H_\ell(D) \longrightarrow H_m(C \otimes D) \longrightarrow \bigoplus\nolimits_{k+\ell=m-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_k(C), H_\ell(D)) \longrightarrow 0$$

but the splitting is not natural.

Proof. Although C and D are not free resolutions, recall that they are direct sums of shifted free resolutions

$$C = \bigoplus_{m \in \mathbb{Z}} \mathscr{H}_m[m]$$

where $\mathscr{H}_m$ is a $\mathbb{Z}$-free resolution of $H_m C$. Similarly for D.

What about an arbitrary commutative ring R, C and D are chain complexes of free R-modules?

We have a Kunneth spectral sequence:

$$E_{pq}^2 = \bigoplus_{k+\ell=q} \operatorname{Tor}_p^R(H_k C, H_\ell D) \Rightarrow H_{p+q}(C \otimes_R D)$$

For $R = \mathbb{Z}$ (more generally a principal ideal domain) we only have $\operatorname{Tor}_0^R = \otimes_R$, Tor_1^R, $E_{pq}^2 = 0$ for $p > 1$.

No room for d^r for $r > 1$, so $E^2 = E^\infty$, and the spectral sequence collapses.

<u>Comment</u>: If R is a field, every module is free. Thus

$$H_m(M \otimes_R N) = \bigoplus_{k+\ell=m} H_k(M) \otimes_R H_\ell(N)$$

<u>Comment:</u> For any commutatative ring R, we have a natural homomorphism of R-modules

$$H_k(C) \otimes_R H_\ell(D) \to H_{k+\ell}(C \otimes_R D)[c] \otimes [c'] \qquad \mapsto [c \otimes c']$$

if $c = \mathrm{d}a$ then $c \otimes c' = \mathrm{d}(a \otimes c')$.

Can this be used to calculate $H_m(X \times Y)$ for X, Y spaces? Well we're looking at $C(X \times Y)$ versus $C(X) \otimes C(Y)$. That is maps $\Delta^m \to X \times Y$ versus $(\Delta^k \to X) \otimes (\Delta^\ell \to Y)$.

This is different except in degree zero, because

$$C_0(X \times Y) \cong C_0(X) \otimes C_0(Y)$$

$$\mathbb{Z}[X \times Y]\mathbb{Z}X \otimes \mathbb{Z}Y$$

Theorem .0.2 (Eilenberg-Zilber)

There exist natural chain maps

$$\varphi : C(X) \otimes C(Y) \to C(X \times Y)$$
$$\psi : C(X \times Y) \to C(X) \otimes C(Y)$$

which are Id in degree zero and moreover $\varphi\psi$ is naturally homotopic to Id and $\psi\varphi$ is naturally homotopic to Id.

In other words: $C(X \times Y)$ is naturally homotopic to $C(X) \otimes X(Y)$.

Note: Also similarly for coefficients in a commutative ring R. Especially useful when R is a field via the Kunneth Theorems.

Over $\mathbb{Z}$ this gives a kunneth theorme for spaces

$$H_m(X \times Y) \cong \bigoplus_{k+\ell=m} H_k(X) \otimes H_\ell(Y) \oplus \bigoplus_{k+\ell=m-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_k(X), H_\ell(Y))$$

In Eilenberg-Zilbur theorem, natural homotopy h means each h_m is natural.

Strategy for proving Eileberg-Zilbur theorem. We are trying to construct natural homomorphisms

$$C_m(X \times Y) \to \Phi$$
$$C_k(X) \otimes C_\ell(Y) \to \Phi$$

where $\Phi : \mathrm{Top} \times \mathrm{Top} \to \mathrm{Ab}$. Because $C_m(X \times Y) = \mathbb{Z}[S_m(X \times Y)]$ and $C_k(X) \otimes C_\ell(Y) = \mathbb{Z}(S_k X \times S_\ell Y)$, our problem is equivalent to constructing a natural map of sets

$$S_m(X \times Y) \to \Phi$$
$$S_k(X) \times S_\ell(Y) \to \Phi$$

Lemma .0.3 (Yoneda Lemma)

Natural transformations $\mathrm{Mor}_C(T, ?) \to \Psi$ where $\Psi : C \to \mathrm{Set}$ are in bijection with elements of $\Psi(T)$.

To prove the Eilenberg-Zilbur theorem, proceed by induction on m (resp. $k + \ell$).

Suppose there is a natural transformation $C_k X \otimes C_\ell Y \to C_{k+\ell} X \times Y$ is constructed for $k + \ell < m$. Let $k + \ell = m$.

We need to construct a natural transformation

$$\varphi : C_k X \otimes C_\ell Y \to C_{k+\ell} X \times Y$$

By Yoneda lemma, we only need to construct

$$\varphi(\underbrace{(\mathrm{Id} : \Delta^k \to \Delta^k) \otimes (\mathrm{Id} : \Delta^\ell \to \Delta^\ell)}_{z}) = u \in C_{k+\ell}(\Delta^k \times \Delta^\ell)$$

We must have that $\mathrm{d}u = \varphi(\mathrm{d}z)$, but we already know $\varphi(\mathrm{d}z)$ by inductive hypothesis. Butthen

$$\mathrm{d}\varphi(\mathrm{d}z) = \varphi(\mathrm{d}\,\mathrm{d}z) = 0$$

$\varphi(\mathrm{d}z)$ is then a cycle, so it is a boundary

$$H_{m-1}(\Delta^k \times \Delta^\ell) = 0$$

($m = 1$ needs a special case). Thus $\varphi(\mathrm{d}z) = \mathrm{d}u$ for some u. We then just can lift on these free generators.

Other direction $\psi : C_m(X \times Y) \to \bigoplus_{k+\ell=m} C_k(X) \otimes C_\ell(Y)$.

We need to map this on the identity $\Delta^m \to \Delta^m$, we special case $m = 1$. For $m > 1$ by Kunneth theorem

$$H_m(C\Delta^m \otimes C\Delta^m) = 0.$$

Homework #5

(1) Prove that for EZ maps φ, ψ we have $\psi\varphi \simeq \mathrm{Id}_{CX\otimes CY}$ naturally.

Use induction. For $k + \ell = m$ we need to construct a natural map

$$h : C_k X \otimes C_\ell Y \to \bigoplus_{p+q=m+1} C_p X \otimes C_q Y$$

where both sides are considered as functors $\mathrm{Top} \times \mathrm{Top} \to \mathrm{Ab}$.

The funtor $C_k X \otimes C_\ell Y$ is again the free abelian group on $S_k X \times S_\ell Y$.

By Yoneda lemma, we need to construct $h(z) \in (C(\Delta^k) \otimes C(\Delta^\ell))_{m+1}$ where

$$z = (\mathrm{Id}_{\Delta^k}, \mathrm{Id}_{\Delta^\ell}) \in S_k(\Delta^k) \times S_\ell(\Delta^\ell).$$

We want that

$$\mathrm{d}h(z) + h\,\mathrm{d}z = \psi\varphi(z) - z$$

Thus

$$\mathrm{d}h(z) = \psi\varphi(z) - z - h\,\mathrm{d}z$$

We verify that $\mathrm{d}(\psi\varphi(z) - z - h\,\mathrm{d}z) = 0$. Then the target has zero homology in degree $m \geq 1$.