If A is an abelian category, we denote

$$DA \simeq Dh\text{-A-Chain}$$

Equivalences are quasiisomorphisms (induce isomorphisms in homology).

Cell chain complexes are colocal and we have colocalization when there are coproducts and enough projectives. For objects $X, Y \in A$ we have

$$\mathrm{Ext}_A^m(X, Y) = \mathrm{Mor}_{DA}(X, Y[m])$$

Proof. If C is a projective resolution in degree 0, then C is cell (individual degrees = cells).

Then by definition

$$\mathrm{Ext}_A^m(X, Y) \coloneqq \mathrm{Mor}_{h\text{-A-Chain}}(C, Y[m]) = H^m(\mathrm{Hom}(X, Y)) = \mathrm{Mor}_{DA}(X, Y[m]).$$

because C is colocal.

What about Tor (not in every abelian category, must have $\otimes$ first).

.1. Derived Functors

The most general notion does not even involve derived categories, but instead just involves a functor $\Phi : C \to D$.

Given a functor $F : C \to Q$, may not factor through Φ. But is there a "universal" functor $D \to Q$ with respect to this data. Two ways

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\Phi"'] & Q \\ D \arrow[ur, "F'"'] \arrow[u, phantom, shift left=4, "\Nwarrow\eta"] & \end{tikzcd}$$

such that for every $G : D \to Q$ provided with a $\kappa : G\Phi \Rightarrow F$ we have a unique $\mu : G \to F'$ with

$$\kappa = \eta \circ \mu\Phi$$

that is

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\Phi"'] & Q \\ D \arrow[ur, "F'"] \arrow[ur, bend right, "G"'] \arrow[u, phantom, shift left=4, "\Nwarrow\eta"] \arrow[ur, phantom, shift right=2, "\Nwarrow\mu"] & \end{tikzcd}$$

This is called a right Kan extension, aka a left derived functor. Denoted by LF.

> **Example .1.1**
>
> Suppose $\Phi : C \to DC$, with $\gamma_X : X' \xrightarrow{\sim} X$ a colocalization. Then if $F : C \to Q$ then the (total) left derived functor exists and is defined by
>
> $$LF(X) = F(X')$$
>
> We have the morphism $F(X') \to F(X)$ via $F(\gamma_X)$. If $G : DC \to Q$ and $G\Phi \Rightarrow F$ then the map $G(X') \to F(X')$ is handed to us because $G(X') \cong G(X)$.

If I have a functor $? \otimes N$ then

$$\mathrm{Tor}_m^R(?, N) = L(H_m(? \otimes N)).$$

That is

$$\mathrm{Tor}^R_m(M,N) = H_m(C \otimes_R N)$$

where C is a projective resolution of M (colocal).

What about localization

Theorem .1.1

If an abelian category A has enough injectives and products then h-A-Chain has localization with respect to co-cell chain complexes. Turn around arrows in the definition of cell chain complexes, replace projective by injective.

Technical Issue: H_* does not commute with inverse limits of sequences. Say we have the following chain complexes

$$\cdots \to X_2 \to X_1 \to X_0$$

then $H_m \lim_k X_k$ is not in general isomorphic to $\lim_k H_m(X_k)$.

The symmetric statement for colimits holds. In general lim of a sequence has one right derived functor $\lim^1$. However, $\lim^1 = 0$ if we have the Mitag-Leffler condition in each

$$\cdots \longrightarrow X_1 \longrightarrow X_0$$

$$\cdots \longrightarrow H_m X_1 \longrightarrow H_m X_0$$

The Mitag-Leffler condition (in an abelian category) says that the composed images at each stage eventually become constant.

Important notes

- There are abelian categories which have enough injectives but not enough projectives
- If we have localization, we have right derived functors (defined symmetrically to left derived functors, instead a left Kan extension) e.g. (sheaf cohomology is to apply right derived functors to global sections).

Homework #9

(2) Prove that if $f : X \to Y$ induces an isomorphism in homology (coefficients in $\mathbb{Z}$) and X, Y are simply connected, then f is a weak equivalence.

(Consider the Serre spectral sequence in homology of the fiber sequence $Ff \to X \to Y$).

$$E^2_{pq} \coloneqq H_p(Y, H_q(Ff)) \Rightarrow H_{p+q}(X).$$

We have an increasing filtration F_p on $H_{p+q}(X)$, and $E^\infty_{pq} = F_p H_{p+q}(X)/F_{p-1}H_{p+q}(X)$.

If you think of this

$$\begin{tikzcd} H_p(X) = F_p H_p(X) \arrow[r] \arrow[rr, bend right, "H_p f"'] & E^\infty_{p,0} \arrow[r, hook] & E^2_{p,0} = H_p(Y) \end{tikzcd}$$

called the edge map. Deduce that $H_p Ff = 0$ for all $p > 0$. Then observe that $\pi_1(Ff)$ is abelian (long exact sequence on homotopy groups). Finally, deduce that $\pi_m(Ff) = 0$ for all m, and conclude that that f must be a weak equivalence via the long exact sequence on homotopy groups.