Homework #8

(2) The Quillen + construction. Let X be a connected CW-complex. Let $\pi_1(X,x) = G$, $x \in X_0$. Let $H \subseteq G$ be a subgroup such that $[H,H] = H$, that is $H^{ab} = 0$.

Attach a 2-cell e_k to each element $h \in H$ to form a CW-complex Y. Note (Y,X) is a CW-pair, and Y is connected. Let $p : \widetilde{Y} \to Y$ be the universal cover. Let $\widetilde{X} = p^{-1}(X)$. Choose a lift $\widetilde{e}_k$ of each cell e_k.

Then $\widetilde{e}_k$ represents a class of $\alpha_h \in H_2(\widetilde{Y}, \widetilde{X})$.

(a) Prove that α_h lifts to a class $\overline{\alpha}_h \in H_2(\widetilde{Y})$ (use the fact that the abelianization of H is zero, the long exact sequence in homology, and Hurewicz).

(b) Observe that $\overline{\alpha}_h$ is in the image by the Hurewicz map of an element $u_h \in \pi_2(\widetilde{Y}, \widetilde{x})$ where $\widetilde{x} \in \widetilde{X}$. Form a CW-complex X^+ by attaching a 3-cell to Y along each $p \circ u_h : S^2 \to Y$.

(c) Prove that the inclusion $X \to X^+$ induces an isomorphism in homology (use cellular homology, the additional attached cells cancel out).

(d) $\pi_1(X^+, x) = G/\overline{H}$ where $\overline{H}$ is generated by all $g^{-1}hg$, $g \in G$, $h \in H$ (aka the smallest normal subgroup of G containing H).

Example .0.1

Say $G = H = A_n$ for $n \geq 5$. Then $[A_n, A_n] = A_n$ so we can form the plus construction.

Then $BA_n \xrightarrow{i} BA_n^+$, and by this $H_k BA_n \xrightarrow{i_*} H_k BA_n^+$. Then $\pi_1 BA_n^+ = 0$.

Thus homology is not an adequate measure of equivalence of spaces (does not imply weak equivalence).

The reason Quillen invented this was to define higher algebraic K-theory of commutative rings. If R is a commutative ring, put $\mathrm{GL}_\infty R = \bigcup_{n \geq 0} \mathrm{GL}_n R$. The analogy with K-theory (imperfect), note $U(m) \subseteq \mathrm{GL}_m \mathbb{C}$ is a homotopy equivalence by the Gram-Schmidt process. Then $\mathrm{GL}_\infty \mathbb{C} \simeq U$.

But we're cheating, $\mathrm{GL}_\infty \mathbb{C}$ has topology from $\mathbb{C}$. For R general it is considered discrete. If we consider $H = [\mathrm{GL}_\infty R, \mathrm{GL}_\infty R] \subseteq \mathrm{GL}_\infty R = G$ then

Theorem .0.1 (Steinberg)

$[H,H] = H$.

Quillen: Take $K_m R = \pi_m(B\,\mathrm{GL}_\infty R^+)$ (with respect to H), for $m > 0$. Then set $K_0 R$ to be the Grothendieck group of isomorphism classes of finitely generated projective R-modules. People knew earlier that $K_1 R = \mathrm{GL}_\infty R / [\mathrm{GL}_\infty R, \mathrm{GL}_\infty R]$, and there was a natural geometry to this. But people could not do it purely algebraically, and instead were able to do it with homotopy groups.

Definition .0.1

HELP (Homotopy Extension and Lifting Property). A map $f : X \to Y$ satisfies HELP with respect to a pair (Z, A) if the following diagram completes

$$\begin{tikzcd}
A \arrow[rr, "0"] \arrow[dd, hook] & & A \times [0,1] \arrow[dl, "h"'] \arrow[dd, hook] & & A \arrow[ll, "0"'] \arrow[dl, "g"'] \arrow[dd, hook] \\
& Y & & X \arrow[ll, "e"] & \\
Z \arrow[rr, "0"'] \arrow[ur, "f"] & & Z \times [0,1] \arrow[ul, dashed, "\widetilde{h}"] & & Z \arrow[ll, "1"] \arrow[ul, dashed, "\widetilde{g}"']
\end{tikzcd}$$

Include garbage can intuition

Lemma .0.2 (The HELP Lemma)
If $\pi_{m-1}(e)$ is injective and $\pi_m(e)$ is onto, then e satisfies HELP for the pair $(Z, A) = (D^m, S^{m-1})$.

Proof Sketch. Put a lid on first (injectivity property). If the garbage can does not fill, move the lid (onto property). Be careful when $m = 1$. More detail is in [**may**].

Lemma .0.3 (The HELP Lemma 2)
If $e : X \to Y$ is an m-equivalence (resp. weak equivalence), then it satisfies HELP with respect to CW-pairs of dimension $\leq n$ (resp. all CW-pairs)
(Induction on cells).

Proof of Whitehead's Theorem. We now prove **??**. We will use HELP. Let $e : X \to Y$ (changed notation, permuted) be an m-equivalence (or weak equivalence). We wish to study the map $[Z, e] : [Z, X] \to [Z, Y]$ for a CW-complex Z.

For surjectivity, apply HELP to the pair $(Z, \emptyset)$. For injectivity apply HELP to $(Z \times [0, 1], Z \times \{0, 1\})$.

Next Time: This applies in more general settings than spaces. In particular, we can use it on chain complexes for derived functors and derived categories. We also really want to use it for spectra, which will allow us to understand duality.