Cobordism, compact smooth closed m-manifolds M with some normal data on $\nu_M^{\mathbb{R}^N}$ which can be

- no data (unoriented)
- oriented
- complex
- trivial bundle (this is called framed cobordism)

modding out by cobordism, that is $M_1 \sim M_2$ when

$$M_1 \coprod M_2 = \partial M$$

Where M has the same type of normal data, which restricts to M_i (usually with signs).

This is a group, whose operation is $\coprod$. How is this related to spectra?

We're now going to follow the Pontrjagin-Thom construction

Embed M as $M \subseteq \mathbb{R}^N \subseteq S^N$. Then there is a tubular neighborhood U of M, which is homeomorphic to $\nu_M^{\mathbb{R}^N}$ via some ι. This gives a map $S^N \to M^{\nu_M^{\mathbb{R}^N}}$ (which is the Thom space, or the 1-point compactification of $\nu_M^{\mathbb{R}^N}$). How? Well

$$S^N \xrightarrow{\varphi} M^{\nu_M^{\mathbb{R}^N}}$$
$$U \overset{\iota}{\mapsto} \nu_M^{\mathbb{R}^N}$$
$$S^N \setminus U \mapsto *$$

This still uses the manifold. But! We have classification of bundles: $\cong$ classes of k-real bundles on M via $[M, BO(k)]$. For oriented k-real bundles we have $[M, BSO(k)]$ (where $BSO(k)$ is the universal cover of $BO(k)$). For complex k-bundles we have $[M, BU(k)]$. And there is only one trivial k-bundle $[M, *] = *$.

We apply this classification to the normal bundle $\nu_M^{\mathbb{R}^N}$. We have $k = N - m$ expect in the complex case where $k = \frac{N-m}{2}$.

The classification map, say in the unoriented real case:

$$M \to BO(k) \qquad (k = N - m)$$
$$\nu_M^{\mathbb{R}^N} \to \gamma_R^k$$
$$M^{\nu_M^{\mathbb{R}^N}} \to BO(k)^{\gamma_{\mathbb{R}}^k}.$$

From the data of $\nu_M^{\mathbb{R}^N}$ we get a map

$$S^N \to BO(N-m)^{\gamma_{\mathbb{R}}^{N-m}}$$

with $N \gg 0$. A cobordism, by an analogous construction, on the manifold representing the cobordism, gives a homotopy. Thus by starting with a cobordism class we obtain a homotopy class

$$S^N \to BO(N-m)^{\gamma_{\mathbb{R}}^{N-m}}$$

In the oriented case, we have $S^N \to BSO(N-m)^{\gamma_{\mathbb{R}}^{N-m}}$. And in complex case we have $N - m = 2k$ and $S^N \to BU(k)^{\gamma_{\mathbb{C}}^k}$. In the trivial case we get $S^N \to S^{N-m}$.

Thom observed that there is an inverse to this procedure. Say we have $S^N \xrightarrow{f} BO(N-m)^{\gamma_{\mathbb{R}}^{N-m}}$. Because the Thom Space is locally nice, one can talk about transversality with respect to fibers. If f is transverse to the 0-section embedding $BO(N-m)$ in the Thom Space, then f^{-1}(0-section) is an m-manifold. In the cases with structure, it automatically gains the desired structure on $\nu_M^{\mathbb{R}^N}$.

Theorem .0.1 (Thom)

These two procedures are inverse to each other. For details see [1, 2].

What about this $N \gg 0$? Well then we have

$$MO_m = \operatorname{colim}_k \pi_{m+k} BO(k)^{\gamma_{\mathbb{R}}^k}$$
$$MSO_m = \operatorname{colim}_k \pi_{m+k} BSO(k)^{\gamma_{\mathbb{R}}^k}$$
$$MU_m = \operatorname{colim}_k \pi_{m+2k} BSO(k)^{\gamma_{\mathbb{C}}^k}$$
$$M_{\text{framed}}(m) = \operatorname{colim}_k \pi_{m+k} S^k = \pi_m \mathbb{S} = \pi_m^S.$$

The first three can be thought of as homotopy groups of twisted suspension spectra, which are now called Thom spectra. This means cobordism is intricately linked with stable homotopy theory.

Consider the complex case. We have a prespectrum $D_{2k} = BU(k)^{\gamma_{\mathbb{C}}^k}$. This is given by

$$\Sigma^2 D_{2k} \to D_{2k+2}$$
$$BU(k)^{\gamma_{\mathbb{C}}^k \oplus 1_{\mathbb{C}}} \to BU(k+1)^{\gamma_{\mathbb{C}}^{k+1}}$$

via the classsification of $(k+1)$-bundles. We could then just set $D_{2k+1} = \Sigma D_{2k}$.

We spectrify to get MU (in the other csaes $MO, MSO, \mathbb{S}$). This tells us that framed cobordism groups are stable homotopy groups of spheres, and we can get the first few stable homotopy groups this way before it becomes intractable.

Exercise .0.1

Show that $\pi_1 \mathbb{S} = \mathbb{Z}/2$ using this method.

Amazingly, in the other cases we listed, the cobordism groups (π_* of the Thom spectra MO, MSO, MU) can be completely calculated. This can be calculated by general methods of calculating homotopy groups of spectra. Namely, this uses the Adams spectral sequence.

Strategy: Look at $F(H\mathbb{Z}/p, H\mathbb{Z}/p)_* = A^*$ (the Steenrod Algebra), these are stable operations (that is natural transfromations) in mod p cohomology of spaces. We work modulo p because $\mathbb{F}_p$ is a field. We have that

$$F(H\mathbb{Z}, H\mathbb{Z}) = \mathbb{Z} \oplus (p\text{-torsion, all } p \text{ together}).$$

The Adams Spectral Sequence:

$$\operatorname{Ext}_{A^*}(H^* X, \mathbb{Z}/p) \Rightarrow (\pi_*^S X)_p^\wedge$$

That is $\pi_*^S X$ completed at p, where X is a CW-complex of finite type. This comes from

$$X \to X \wedge H\mathbb{Z}/p \to X_1$$
$$X_1 \to X_1 \wedge H\mathbb{Z}/p \to X_2$$

all of these cofibration sequences (mapping cones then) working entirely in the category of spectra. This leads to an exact couple, giving the adams spectral sequence. A great book for this is Ravenel's Complex Cobordism and Stable Homotopy Groups of Spheres [3]

Hard for $\mathbb{S}$, but for MO, MSO, MU it is relatively easy. For example

$$\pi_* MO = \mathbb{F}_2[y_i \mid i \neq 2^k - 1].$$

Furthermore

$$MO = \bigvee \Sigma^2 H\mathbb{Z}/2$$

This is just a sum of copies of $H\mathbb{Z}/2$, sometimes called a (GEM, a Generalized Eilenberg-Maclane spectrum aka nothing new).

However $\pi_* MU$ is more interesting

$$\pi_* MU = \mathbb{Z}[x_1, x_2, x_3, \ldots]$$

where $\deg(x_i) = 2i$. This is an interesting new spectrum (not a GEM). HOW?

.1. Complex Oriented Spectra

A commutative ring spectrum (commutative monoid in $DSpectra$) E is called complex-oriented when the universal complex line bundle $\gamma^1_{\mathbb{C}}$ on $\mathbb{C}\mathrm{P}^\infty = BU(1)$ is E-oriented.

What does the Thom Space $(\mathbb{C}\mathrm{P}^\infty)^{\gamma^1_{\mathbb{C}}}$ look like? For any bundle ξ on a space X, we have a cofiber sequence

$$S(\xi)_+ \to X_+ \to X^\xi$$

where $S(\xi)$ is the unit sphere bundle (given a Euclidean metric), equivalently $S(\xi) \simeq \xi \setminus$ (0-section). We then know that

$$S(\gamma^1_{\mathbb{C}} \simeq \gamma^1_0 \setminus \text{(0-section)} = \mathbb{C}^\infty \setminus 0 \simeq *.$$

Therefore $(\mathbb{C}\mathrm{P}^\infty)^{\gamma^1_{\mathbb{C}}} \simeq \mathbb{C}\mathrm{P}^\infty$. This comes from a cofiber sequence

$$S^0 \to \mathbb{C}\mathrm{P}^\infty_+ \to (\mathbb{C}\mathrm{P}^\infty)^{\gamma^1_{\mathbb{C}}}$$

Next time: talk more about complex oriented theories and formal group laws, why you might care about equivariant topology and structured / coherent topology.