I. Bott Periodicity

Let M be a Riemannian manifold, that is a smooth manifold M equipped with an inner product TM_x for each $x \in M$ which is smoothly varying in x.

Variational problem in a Riemannian manifold: Find the shortest path connecting two points (length of a path $\int \mathrm{d}s$)

Local Solution: second order differential equation = the geodesic equation. The solutions are called geodesics.

They are straight lines in $\mathbb{R}^m$, great circles on S^m (embedded as a unit sphere in $\mathbb{R}^{m+1}$).

Note: a geodesic may not actually be the shortest path between two points. For example walking around the "bad" portion of a great circle is still a geodesic.

Definition I.0.1

A (compact) symmetric space is a compact connected Riemannian manifold such that for each $P \in M$ there exists an isometry $\iota_P : M \to M$ such that $\iota_P(P) = P$ and $D\iota_P$ is -1 on TM_p.

Example I.0.1

S^m is a symmetric space. Take $(1, 0, \dots, 0) \in S^m$ then map

$$(x_0, x_1, \dots, x_m) \mapsto (x_0, -x_1, \dots, -x_m)$$

It turns out that this implies isometries act transitively on M and so M is a homogeneous space G/H (for G a compact Lie group and H a closed subgroup). And in fact, H is the fixed points for a certain kind of involution (Cartan involutions).

In fact a connected compact Lie group is a symmetric space. $\iota_1 = (g \mapsto g^{-1})$.

This is classified by Cartan

Definition I.0.2

Let M be a compact connected Riemannian manifold. Let $P, Q \in M$ and h be a homotopy class of paths between P, Q (rel boundary). Call $\nu = (P, Q, h)$.

The space of all paths from P to Q homotopic to h is homotopy equivalent to $(\Omega M)_0$ (a connected component of ΩM).

Consider $M^\nu \subseteq (\Omega M)_0$ be the subspace of the shortest geodesics (parameterized by scaled arc length).

Theorem I.0.1 (Bott [1])

If M is a symmetric space and $\nu = (P, Q, h)$, then M^ν is also a compact connected space. Furthermore $M^\nu \hookrightarrow (\Omega M)_0$ induces an isomorphism in π_i for $0 \le i < \alpha$, an onto map on π_α, where α is a certain number called the index.

The index α is defined as follows. If I have a geodesic h connecting P, Q then we can slightly deform this geodesic into a "nearby geodesic" satisfying the Jacobi equations. These can then cross at some point along h. We then say

$$\alpha_h = \sum_{R \in (P,Q)} \dim(\text{space of nearby geodesics crossing at } R)$$

$$\alpha = \min_h \alpha_h$$

For more information about this geometry see [2].

This implies

Theorem I.0.2 (Freudenthal Suspension)

$S^{m-1} \hookrightarrow \Omega S^m$ for $m > 1$ induces an isomorphism in homotopy groups π_k for $k \leq 2m-2$ and it is onto in degree $k = 2m-1$.

This shows that $\pi_k S^m \to \pi_{k+1} S^{m+1} \to \pi_{k+2} S^{m+2} \to \cdots$, then eventually these are isomorphisms. These are stable homotopy groups of spheres. Namely this is $\pi_{k+m} S^m$ for $m \gg 0$ and k fixed.

Homework #7

(2) Prove that for $m \geq 1$, $\pi_1(U(m)) \cong \mathbb{Z}$. Recall that $U(m)$ is unitary (can think of as complex linear maps which are also isometries) $m \times m$ (complex) matrices. Note that $U(m)$ acts on the unit sphere S^{2m-1} in $\mathbb{C}^m$ transitively.

The isotropy group is $U(m-1)$. Therefore we have a fibration sequence

$$U(m-1) \to U(m) \to S^{2m-1}.$$

use LES in homotopy groups for induction. $U(1) = S^1$, so we know the statement for that case.