In Yoneda Lemma:

$$
\begin{tikzcd}
 & \mathrm{Mor}_C(Y,Y) \arrow[d, "G(f)"] & Y \arrow[d, "f"] \\
* \arrow[ur, "\mathrm{const}(\mathrm{Id}_Y)"] \arrow[r, "\mathrm{const}(f)"'] & \mathrm{Mor}_C(Y,X) & X
\end{tikzcd}
$$

A functor is representable when $G(X) = \mathrm{Mor}_C(Y,X)$ for some Y.

This is, more generally, called a universal element

Definition .0.1

Let $G : C \to D$ be a functor and let $X \in D$. A universal element for X, G is a D-morphism $\mu : X \to G(Y)$ for some $Y \in C$ with

$$
\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[dr, "h"'] & G(Y) \arrow[d, "G(g)"] & Y \arrow[d, dashed, "g"', "\exists!"] \\
 & G(Z) & Z
\end{tikzcd}
$$

such that for every D-morphism $h : X \to G(Z)$ there exists a unique C-morphism $g : Y \to Z$ such that $h = G(g) \circ \mu$.

Example .0.1

$* \to \mathrm{Id}$ is a universal element for the representable functor $G : C \to \mathrm{Set}$ Where $G(Z) = \mathrm{Mor}_C(Y,Z)$. This is the statement of the Yoneda Lemma

The universal element for X, G if it exists, is unique up to isomorphism.

If the universal element exists for every object $x \in D$ for a fucntor $G : C \to D$, then Y_x is functorial in X. We have a functor $F : D \to C$ where $F(x)$ is universal for x, G.

Then we have

$$\mathrm{Mor}_D(X, G(Z)) \cong \mathrm{Mor}_C(F(X), Z)$$

naturally. In this case we say that $F : D \to C$ is left adjoint to G.

$\eta : X \to GF(X)$ is given by universality, and is called the unit of the adjunction. Symmetrically, we have a natural transformation $\varepsilon : FG(Y) \to y$ called the counit.

One can prove that F is left adjoint to G if and only if we have natural transformations $\eta : \mathrm{Id} \to GF$ and $\varepsilon : FG \to \mathrm{Id}$ such that

$$
\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[rr, bend right, "\mathrm{Id}"'] & FGF \arrow[r, "\varepsilon F"] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\eta G"] \arrow[rr, bend right, "\mathrm{Id}"'] & GFG \arrow[r, "G\varepsilon"] & G
\end{tikzcd}
$$

commute. These are called the triangle identities.

Example .0.2

Let R be a commutative ring. Then $M \otimes_R ?$ is left adjoin to $\mathrm{Hom}_R(M, ?)$.

Duals: $M^* : \mathrm{Hom}_R(M, R)$.

This also extends to R-Chain. There is a notion of a closed tensor category, an abelian category with $\otimes$ satisfying the "obvious axioms."

I. **Products in (co)homology**

Natural product in $H^*(X;R)$ where X is a space and R is a commutative ring. Start with $\Delta : X \to X \times X$. This gives

$$CX \to C(X \times X) \to CX \otimes CX$$

via the Eilenberg-Zilber theorem. Tensoring by R gives

$$C(X;R) \to C(X;R) \otimes_R C(X;R)$$

:wq Dualize the above example to get

$$C^*(X;R) \otimes_R C^*(X;R) \xrightarrow{\mu} C^*(X;R)$$
$$H^*(C^*(X;R)) \otimes_R H^*(C^*(X;R)) \xrightarrow{\mu} H^*(C^*(X;R))$$

This is called the cup product $\smile$.

Properties, it is associative, unital, and graded-commutative. Aka for $x \in H^k(X;R)$ and $y \in H^\ell(X;R)$ we have

$$x \smile y = (-1)^{k\ell}(y \smile x)$$

We actually get good rings (e.g. polynomial rings).

Homework #5

(2) Consider the functor $F : \mathrm{Grp} \to \mathrm{Ring}$ given by $G \mapsto \mathbb{Z}[G]$.

Prove that the right adjoint to F is the group of units

$$R \to \{g \in R \mid \exists k, gk = 1\}$$

(use universality).