Using reduced (co)homology, we can simplify to talking about based spaces instead of about pairs. However, $E_m(X, A) \not\cong \widetilde{E}_m(X/A)$, where X/A is the quotient space (even made into a Hausdorff space). Although this holds for special classes of pairs (X, A), we cannot use it to reduce.

We can get rid of this problem by defining some new constructions.

Definition .0.1

The **mapping cone** CY of a space Y is definde to be

$$CY \coloneqq (Y \times [0, 1])/(Y \times \{1\})$$

The **mapping cone** Cf of a map $f : Y \to X$ is defined to be

$$Cf \coloneqq (X \coprod CY)/(y, 0) \sim f(y)$$

The quotient topology here is universal. That is a map $Cf \to Z$ is in a natural bijection with maps $g : X \to Z$ such that $g \circ f$ is nullhomotopic.

Definition .0.2

Given a space Y, its **suspension** SY is defined by

$$SY = (Y \times [0, 1])/(y, 0) \sim (y', 0), (y, 1) \sim (y', 1)$$

The upshot of mapping cones?

Proposition .0.1

For an inclusion $f : Y \to X$, $\widetilde{E}_m(Cf) \cong E_m(X, Y)$, and likewise $\widetilde{E}^m(Cf) \cong E^m(X, Y)$.

Proof. This is just some simple arguments from the Ellenberg-Steenrod axioms

$$\begin{aligned} \widetilde{E}_m(Cf) &\cong E_m(Cf, *) \cong E_m(Cf, CY) \\ &\cong E_m(C_-f, C_-Y) \cong E_m(X, Y) \end{aligned}$$

Where we define:

$$C_-Y \coloneqq Y \times [0, 1/2]$$

$$C_-f \coloneqq (X \coprod C_-Y)/(y, 0) \sim f(y)$$

The third isomorphism above follows by excision on $CY \setminus C_-Y \subseteq CY \subseteq Cf$, and the others follow by homotopy equivalences between pairs $(Cf, *) \simeq (Cf, CY)$ and $(C_-f, C_-Y) \simeq (X, Y)$.

Similarly for cohomology.

We always have an inclusion $X \xrightarrow{\iota} Cf$. We can then ask what is Ci? Well

$$Ci \cong (CX \coprod CY)/(y, 0) \sim (f(y), 0)$$

This then allows us to see that $Ci \simeq SY$, where SY is the suspension (see Definition .0.2). This is visualized by Figure 1 Why is this? Well there are maps $SY \to C\iota \to SY = C\iota/CX$ which give a homotopy equivalence. Explicitly for $SY \to C\iota$, we map

$$(y, t) \mapsto (f(y), 1 - 2t) \qquad (0 \le t \le 1/2)$$

$$(y, t) \mapsto (y, 2t - 1) \qquad (1/2 \le t \le 1)$$

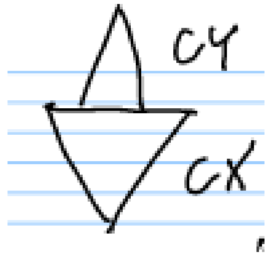

FIGURE 1. $C\iota$ for the inclusion $\iota : X \to Cf$

This suggests that $\widetilde{E}_m(X) \cong \widetilde{E}_{m+1}(SX)$ (which will be on homework).

It also suggests an alternative formulation of the Ellenberg-Steenrod axioms.

Definition .0.3

Functors $\widetilde{E}_m$: hBased $\to$ Ab are called a generalized based homology theory provided that:

(1) We have an exact sequence for every map of spaces $f : Y \to X$:

$$\widetilde{E}_m(Y) \xrightarrow{\widetilde{E}_m(f)} \widetilde{E}_m(X) \xrightarrow{\widetilde{E}_m(\iota)} \widetilde{E}_m(Cf)$$

where $\iota : X \hookrightarrow Cf$ is the inclusion.

(2) There is a natural isomorphism

$$\widetilde{E}_m(X) \cong \widetilde{E}_{m+1}(SX)$$

for all $m \in \mathbb{Z}$.

Similarly for cohomology. The product axiom involves the wedge sum.

Definition .0.4

Given based spaces X_i we define their **wedge sum** by:

$$\bigvee_{i \in I} X_i \coloneqq \coprod_i X_i / *_i \sim *_j$$

Definition .0.5

We call a generalized based homology theory $\widetilde{E}_m$ **additive** provided that the inclusions provide an isomorphism

$$\bigoplus_{i \in I} \widetilde{E}_m X_i \to \widetilde{E}_m \left(\bigvee_{i \in I} X_i \right)$$

Likewise, a generalized based cohomology theory $\widetilde{E}^m$ is called **additive** provided that the inclusions induce an isomorphism

$$\prod_{i \in I} \widetilde{E}_m X_i \leftarrow \widetilde{E}_m \left(\bigvee_{i \in I} X_i \right)$$

THe based and unbased sets of axioms are equivalence. Why? Well given an unbased theory E_m we may define $\widetilde{E}_m(X) \coloneqq E_m(X, *)$ and prove the suspension axiom as well as exactness.

Likewise, given a based theory $\widetilde{E}_m$ we may define $E_m(X) \coloneqq \widetilde{E}_m(X_+)$ where $X_+ \coloneqq X \coprod \{*\}$. For $f : Y \hookrightarrow X$ we define $E_m(X, Y) \coloneqq \widetilde{E}_m(Cf)$.

We then can prove a long exact sequnece from $C\iota \simeq SY$ for $\iota : X \to Cf$ and the suspension axiom.

Similarly for cohomology

.1. Computing ordinary (co)homology

How do we actually compute it? Well we need a nice category of spaces. The CW-complexes.

Definition .1.1

Let $X = \bigcup_{i \geq -1} X_i$, where

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$$

are given the subspace topology, and $Z \subseteq X$ is closed if and only if $Z \cap X_i$ is closed in X_i for each i. We say X is a **CW-complex**.

We mandate that X_m is built from X_{m-1} by adjoining m-cells along their boundaries to X_{m-1}. For clarity recall the definitions of an m-cell D^m and its boundary $S^{m-1} = \partial D^m$.

$$D^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid \sum x_i^2 \leq 1\} S^{m-1} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i^2 = 1\}$$

More formally, we are given a set I_m of m-cells, and there is a map $f_m : I_m \times S^{m-1} \to X_{m-1}$ called the **attaching map** so that the following is a pushout diagram

$$\begin{tikzcd} I_m \times S^{m-1} \arrow[r, "f_m"] \arrow[d, hook] & X_{m-1} \arrow[d, hook] \\ I_m \times D^m \arrow[r] & X_m \end{tikzcd}$$

This gives a formula for X_m as follows:

$$X_m = (X_{m-1} \coprod (I_m \times D^m))/(i, y) \sim f_m(i, y)$$

Often X_m is called the m**-skeleton.**

Definition .1.2

A **CW-pair** is defined the same way except $X_{-1} = Z$ instead of $\emptyset$.

Homework #2

(1) There is a long exact sequence in reduced homology for any based inclusion $i : Y \to X$

$$\cdots \longrightarrow \widetilde{E}_m(Y) \longrightarrow \widetilde{E}_m(X) \longrightarrow E_m(X, Y) \longrightarrow \widetilde{E}_{m-1}(Y) \longrightarrow \cdots$$

Hint: a long exact sequence is a chain complex with homology 0. Consider the LES of the inclusion $* \to *$ and map it into the unbased LES of i. Then consider the "quotient chain complex"

(2) Show that $\widetilde{E}_m(X) \cong \widetilde{E}_{m+1}(SX)$.

This essentially follows by the following, letting

$$S_+X := X \times [1/2, 1]/(x, 1) \sim (x', 1) \cong CX \simeq *$$
$$S_0X := X \times [1/2, 3/4]$$
$$S_-X := X \times [0, 3/4]/(x, 0) \sim (x', 0) \cong CX \simeq *$$

Apply the long exact sequence of a pair to show

$$E_{m+1}(S_-X, S_0X) \cong \widetilde{E}_m(S_0X) \cong \widetilde{E}_m(X).$$

Then apply excision and homotopy equivalence to show that

$$E_{m+1}(SX, *) \cong E_{m+1}(SX, S_+X) \cong E_{m+1}(S_-X, S_0X).$$