.1. Localization in Topology

There is an alternative approach to deriving $h\,\mathrm{Top}$. Essentially we mimic the formal structure on the singular set $S_m X = \{\Delta^m \to X\}$. These formal structures will be called simplicial sets.

To do this, we must determine what distinguished maps are there between the standard simplices Δ^m? Well, we have faces

$$\begin{aligned}\partial_i : \Delta^m &\to \Delta^{m+1}\\ [t_0, \dots, t_m] &\mapsto [t_0, \dots, t_{i-1}, 0, t_i, \dots, t_m].\end{aligned}$$

We can realize that this corresponds to

$$\begin{aligned}\{0, \dots, m\} &\to \{0, \dots, m+1\}\\ j &\mapsto j && (j < i)\\ j &\mapsto j+1. && (j \geq i)\end{aligned}$$

Compositions correspond to order-preserving injections. There is also a map in the other direction.

$$\begin{aligned}\Delta^m &\to \Delta^{m-1}\\ [t_0, \dots, t_m] &\mapsto [t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_m]\end{aligned}$$

This corresponds to the map

$$\begin{aligned}\{0, \dots, m\} &\to \{0, \dots, m-1\}\\ j &\mapsto j && (j \leq i)\\ j &\mapsto j-1. && (j > i)\end{aligned}$$

These are called degeneracies, and are order-preserving surjections. Triangulation of objects with simplicial sets can be used to verify the homotopy axiom of homology.

Definition .1.1

We call Δ the simplicial category. The objects are $\mathbb{N}_0$, and we write $\underline{m} = \{0, \dots, m\}$. The morphisms are non-strictly order preserving maps.

Definition .1.2

If C is a category, the category of simplicial objects in C ($\Delta^{\mathrm{op}} - C$) is the category of functors $\Delta^{\mathrm{op}} \to C$ and natural transformations.

We will talk about simplicial sets. Consider that $\overline{\Delta}(m) \coloneqq \Delta^m$ is a functor $\Delta \to \mathrm{Top}$, called the topological realization.

The topological realization of a simplicial set $S : \Delta^{\mathrm{op}} \to \mathrm{Set}$ is left adjoint to the singular set functor.

$$|S| = \coprod_{m \in \mathbb{N}_0} S_m \times \Delta^n / (s, fx) \sim (Sf(s), x) \qquad (f : m \to_\Delta n)$$

Triangulation of prism says that

$$\begin{aligned}(S \times T)_m &= S_m \times T_m\\ |S \times T| &\cong |S| \times |S|\end{aligned}$$

Simplicial sets generalize simplicial complexes, "are" CW-complexes (after calculation).

$\underline{\Delta^n} : k \in \mathrm{Ob}(\Delta) \to \mathrm{Mor}_\Delta(k, n)$ is a simplicial model of Δ^n. We then have that

$$\underline{\Delta}^0 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \underline{\Delta}^1$$

is a model for the unit interval.

Definition .1.3

A simplicial homotopy is a natural transformation $\underline{\Delta}^1 \times S \to T$.

We call two morphisms $f, g : S \to T$ simplicially homotopic if they are equivalent in the smallest equivalence relation containing simplicial homotopy.

Then we have $h - \Delta^{\mathrm{op}}$-Set with objects simplcial sets and morphisms simplicial homotopy classes of Δ^{op}-morphisms.

This category has localization with respect to Kan complexes

Definition .1.4

A Kan complex S is a simplicial set satisfying the Kan condition. To phrase this, consider V_k^n in Δ^n, which is obtained by ommitting the open n-simplex and the k-th face.

In terms of simplicial sets this is

$$\underline{V_k^n} : j \mapsto \{f \in \mathrm{Hom}_\Delta(j, m) \mid \{0, \dots, k-1, k+1, \dots, m\} \not\subseteq \mathrm{im}\, f\}.$$

We have a natural injection $\underline{V_k^m} \hookrightarrow \underline{\Delta}^n$.

Then S satisfies the Kan condition provided that every morphism $f : \underline{V_k^n} \to S$ extends to a morphism $\overline{f} : \Delta^n \to S$

$$\begin{tikzcd} \underline{V_k^n} \arrow[r, "f"] \arrow[d, hook] & S \\ \Delta^n \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

We say that S is a minimal Kan complex if the extension $\overline{f}$ is unique.

Definition .1.5

An equivalence of simplicial sets is a morphism $f : S \to T$ such that $|f| : |S| \to |T|$ is a weak equivalence (homotopy equivalence since these are CW complexes).

Theorem .1.1

The simplicial realization induces an equivalence of categories

$$D - \Delta^{\mathrm{op}}\text{-Set} = Dh - \Delta^{\mathrm{op}}\text{-Set} \xrightarrow{\simeq} Dh\,\mathrm{Top} = D\,\mathrm{Top}\,.$$

The $=$ above comes from the fact that inverting equvalences identifies homotopic maps.

Addendum: Every Kan complex is simplicial homotopy equivalent to a unique (up to isomorphism) minimal Kan complex.

We conclude that

MINIMAL KAN COMPLEXES (up to $\cong$) ARE IN BIJECTION
WITH WEAK HOMOTOPY TYPES

where a weak homotopy type are homotopy classes of CW-complexes. Unfortunately, this is not a practical solution because minimal Kan complexes are extremely difficult to write down (try doing it for S^1, $\mathbb{R}\mathrm{P}^\infty$, these are easier because they are $K(G, 1)$s).

Homework #9

(3) Prove that for every space X, the singular set $S_\bullet X$ satisfies the Kan condition.

Another topic is localization within $D\,\mathrm{Top}$. If E is some genrealized homology theory (preserving weak equivalence), then we can say that $f : X \to Y$ is an E-equivalence if $E_* f$ is an isomorphism.

Example .1.1

$E = H\mathbb{Z}$. Then

- $H\mathbb{Z}$-equivalence is not in general a weak equivalence.
- $H\mathbb{Z}$-equivalence is a weak equivalence if X, Y are simply connected.

Theorem .1.2 (Bousfield)

$D\,\mathrm{Top}$ has localization with respect to E-equivalence for any chosen generalized homology E.

Even if E is an ordinary homology theory, of interest is $E = H\mathbb{Q}$ localization is called rationalization. $H\mathbb{Z}/p$ is called p-completion, and $H\mathbb{Z}_{(p)}$ is called p-localization, where $\mathbb{Z}_{(p)} = \{k^{-1} \mid p \nmid k\}\mathbb{Z}$.

Say for simply connected spaces, this is fairly well understood. Lets consider $H\mathbb{Q}$.

Theorem .1.3

If $f : X \to Y$ induces an isomorphism in $H\mathbb{Q}$, and X, Y are simply connected. Then f induces an isomorphism $\pi_m X \otimes \mathbb{Q} \cong \pi_m Y \otimes \mathbb{Q}$.

Serre in 1953 provied this in the case $\pi_2 f$ is onto (Annals of Math). He then wrote

“Nous insisterons par la-desrus” $\leftrightarrow$ “We shall not insist on it”