Remark .0.1

There are no odd weight modular forms over $\mathrm{SL}_2(\mathbb{Z})$. Namely, $-I \in \mathrm{SL}_2(\mathbb{Z})$ gives $f(\tau) = f(\tau)(-1)^k$, thus k must be even.

Last time we used the example of the Eisenstein series $G_k(\tau)$ for $k > 2$ even. The q-expansion is

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

Definition .0.1

A modular form $f : \mathcal{H} \to \mathbb{C}$ is called a cuspform if $a_0(f) = 0$ in $\sum a_n(f)q^n$. We collect these as

$$S(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_k S_k(\mathrm{SL}_2(\mathbb{Z})).$$

Example .0.1

$(60G_4)^3 - 27(140G_6)^2 =: \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is a cuspform (using that we're a graded ring). In fact, it is nonzero! Check the degree 1 term of the q-expansion.

.1. Congruence subgroups

Definition .1.1

Define

$$\Gamma(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})).$$

In fact $\Gamma(N)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$. We say $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup if there exists $\Gamma(N) \subseteq \Gamma$.

Example .1.1

We will often consider the congruence subgroups

$$\Gamma_0(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$
$$\Gamma_0(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$

Exercise .1.2

$[\Gamma_1(N) : \Gamma(N)] = N$ and $[\Gamma_0(N) : \Gamma_1(N)] = \varphi(N)$, using the first isomorphism theorem to translate into $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Notation: For $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$, $f : \mathcal{H} \to \mathbb{C}$, we define

$$[\gamma]_k : f \mapsto f[\gamma]_k$$

via

$$(f[\gamma]_k)(\tau) \coloneqq (c\tau + d)^{-k} f(\gamma(\tau)).$$

For $f : \mathcal{H} \to \mathbb{C}$ we want to factor it through a map $\mathcal{H} \to D'$.

Note that $\Gamma(N) \subseteq \Gamma$ for some N, so $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$.If $h \in \mathbb{Z}_{>0}$ be the minimal so that $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$. This implies that $f(\tau + h) = f(\tau)$.

Now define $\mathcal{H} \to D' : \tau \mapsto e^{2\pi i \tau / h}$, so that f factors through $\mathcal{H}$. We get $g : D' \to \mathbb{C}$. This allows us to define f being holomorphic at ∞.

Pick $\alpha \in \mathrm{SL}_2(\mathbb{Z}), s \in \mathbb{Q}, \alpha(\infty)$. Given f, Γ-weakly modular, f is holomorphic at s if $f[\alpha]_k$ is holomorphic at ∞. Crunching the numbers gives that $f[\alpha]_k$ is weakly $\alpha^{-1}\Gamma\alpha$-modular (which is also a congruence subgroup) to make this work.

Definition .1.2

We call $f : \mathcal{H} \to \mathbb{C}$ modular of weight k with level Γ

(1) f is holomorphic on $\mathcal{H}$.
(2) f is weight k, Γ-invariant, so $f[\gamma]_k = f$ for $\gamma \in \Gamma$.
(3) $f[\alpha]_k$ is holomorphic at ∞, for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ (suffices to take finitely many α because $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ is finite, the q-series changes by a root of unity).
(4) f is called a cuspform if $a_0 = 0$ for $f[\alpha]_k$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

.2. Elliptic Curves as Complex Tori

Definition .2.1

$\Gamma = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \subseteq \mathbb{C}$ such that ω_1, ω_2 are a basis of $\mathbb{C}$ over $\mathbb{R}$.

We can assume $\gamma_1/\omega_2 \in \mathcal{H}$.

Exercise .2.1

Lattices $\Lambda = \Lambda'$ if and only if there exist matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that

$$\begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}.$$

Definition .2.2

A complex torus is $\mathbb{C}/\Lambda$ as a complex manifold. The complex structure depends on Λ. There is an inherited group structure via addition.

Observation: If $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is non-constant and holomorphic, then it is surjective.

Proof. Look at im f, which is closed (compactness),connected, and open (by the open mapping theorem).

Definition .2.3

An isogeny is a holomorphic homomorphism $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ which is nonconstant.

Example .2.2

$[N] : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, where $z \mapsto Nz$.

Exercise .2.3

$\mathbb{C}/\Lambda =: E$, and $E[N] := \ker[N]$. Describe $E[N]$ as a group.

It is fairly clear that $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$, by subdividing the lattice points in $\Lambda = \langle \omega_1, \omega_2 \rangle$.

Fact: Any isogeny $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is of the form $z + \Lambda \mapsto mz + \Lambda'$, $m \in \mathbb{C} \setminus \{0\}$.

Proposition .2.1

Isogeny is an equivalence relation on complex tori.

Proof. The only nontrivial portion is showing symmetry. Take an isogeny $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$, take $\varphi(z+\Lambda) = mz + \Lambda'$. This implies $m\Lambda \subseteq \Lambda'$. There exist naturals n_1, n_2 such that $\{n_1\omega_1', n_2\omega_2'\}$ is a basis of $m\Lambda$, where ω_1', ω_2' is a basis of Λ'.

Then $n_1 n_2 \Lambda' \subseteq m\Lambda$. Thus $n_1 n_2/m\Lambda' \subseteq \Lambda$.

We then define $\widehat{\varphi} : \mathbb{C}/\Lambda' \to \mathbb{C}/\Lambda$ by $\widehat{\varphi}(z + \Lambda') = n_1 n_2 z/m + \Lambda$.

Also $\widehat{\varphi} \circ \varphi = [n_1 n_2] = [\deg \varphi]$. Note $\deg[N] = N^2$.