Look at Elliptic points. Suppose $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ fixes $\tau \in \mathcal{H}$ and $c \neq 0$, this implies

$$c\tau^2 + (d-a)\tau - b = 0$$

and $ad - bc = 1$, so this implies $(d-a)^2 + 4bc < 0$, so $(d+a)^2 < 4$ which holds if and only if $|a+d| < 2$. Thus

$$\mathrm{char}(\gamma) = x^2 - (a+d)x + 1 = x^2 + 1 \text{ or } x^2 \pm x + 1.$$

Thus if $\gamma \neq \pm I$ and γ fixes some τ then one of

$$\mathrm{ord}(\gamma) = 3 \qquad \mathrm{ord}(\gamma) = 4 \qquad \mathrm{ord}(\gamma) = 6.$$

In these cases respectively we have

$$\gamma \sim \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^{\pm 1} \qquad \gamma \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\pm 1} \qquad \gamma \sim \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\pm 1}.$$

In the case $\mathrm{ord}(\gamma) = 6$, we can take the action of $\mathbb{Z}[\gamma]$ on $\mathbb{Z}^2$ making it into a $\mathbb{Z}[\gamma]$-module. We see that $\mathbb{Z}[\gamma]$ is a PID, so

$$\mathbb{Z}^2 = (\mathbb{Z}[\gamma])^r \oplus \bigoplus_I \mathbb{Z}[\gamma]/I$$

But there's no torsion, and $\mathbb{Z}[\gamma]$ has $\mathbb{Z}$-dimension two, since γ is a 6-th root of unity, and so its minimal polynomial has degree two, and $\mathbb{Z}[\gamma] \cong \mathbb{Z}[X]/\text{minpoly}$. This gives a map $\varphi : \mathbb{Z}[\gamma] \to \mathbb{Z}^2$ which is an isomorphism. Call $u = \varphi(1), v = \varphi(\gamma)$.

Then

$$\gamma[u,v] = [v, -u+v] = [u,v]\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\gamma[v,u] = [v,u]\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

One of $[u,v]$ or $[v,u]$ has determinant one, and move it over.

Proposition .0.1
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \Gamma_i$, $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \in \Gamma_{\mu_3}$ and nothing else. That is the elliptic points $Y(1) = Y(\mathrm{SL}_2(\mathbb{Z}))$ are $\{\pi(i), \pi(\mu_3)\}$ where μ_3 is a third root of unity.

Corollary .0.2
Elliptic points of $Y(\Gamma)$ are Γ-orbits in $\mathrm{SL}_2(\mathbb{Z})i$, $\mathrm{SL}_2(\mathbb{Z})\mu_3$.

.1. Cusps

Fact: $\mathrm{Stab}_\infty = \pm[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}]$, for $m \in \mathbb{Z}$.

Define $\mathcal{H}^\star = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. We'll define $X(\Gamma) = \mathcal{H}^\star/\Gamma$.

Exercise .1.1

There are finitely many images of $\mathbb{Q} \cup \{\infty\}$. There is only one orbit for $\mathrm{SL}_2(\mathbb{Z})$, as the action is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{m}{n} = \frac{am + bn}{cm + dn}.$$

But then $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] < \infty$, and so we can only split this orbit into finitely many pieces.

Definition .1.1

We call the finitely points in $X(\Gamma) \setminus Y(\Gamma)$ the cusps

We can take a topology on $\mathcal{H}^\star$ coming from the Riemann sphere, but then all of our cusps will be close together!!! This is awful! Instead, take a topology generated by

- Opens in $\mathcal{H}$
- $N_m \cup \{\infty\}$ where $N_m = \{\tau \in \mathcal{H} \mid \operatorname{im}(\tau) > m\}$.
- All $\mathrm{SL}_2(\mathbb{Z})$ orbits of $N_m \cup \{\infty\}$.

We then give $X(\Gamma)$ the quotient topology

Proposition .1.1

$X(\Gamma)$ is Hausdorff, compact, and connected.

Proof. For Hausdorff, there's three cases, two points in $\mathcal{H}$, a cusp and a point in $\mathcal{H}$, and then two cusps. For the first case, it's a simple proof using the properties of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. For s, τ a cusp and a point, prove $\mathrm{Im}(\gamma(\tau)) \leq \max(\mathrm{Im}(\tau), \mathrm{Im}(1/\tau))$.

Consider s_1, s_2 and $\alpha_i(\infty) = s_i$. Then $U_i = \alpha_i(N_2 \cup \{\infty\})$. If $\pi(U_1) \cap \pi(U_2) \neq \emptyset$, then

$$\gamma\alpha_1(\tau_1) = \alpha_2(\tau_2).$$

This will imply $\alpha_2^{-1}\gamma\alpha_1 : \tau_1 \mapsto \tau_2$. Claim: τ_1, τ_2 are translates of each other. This follows since they lie in the same $\mathrm{SL}_2(\mathbb{Z})$ orbit and they have "large" imaginary part. A messy computation yields that

$$\mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\mathrm{Im}(\tau)}{(d + c\,\mathrm{Re}(\tau))^2 + c^2(\mathrm{Im}(\tau))^2}$$

which is clearly less than 2 if $c \neq 0$, since $c \in \mathbb{Z}$. Thus τ_1, τ_2 are translates.

This will show $\alpha_2^{-1}\gamma\alpha_1$ fixes infinity, showing $s_1 \sim s_2$ in $X(\Gamma)$.

To show compactness it suffices to show this for a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. Namely

$$D^\star = D \cup \infty \qquad\qquad D = \{\tau \in \mathcal{H} \mid |\Re\tau| \leq 1/2, |\tau| \geq 1\}$$

as $X(\Gamma)$ will be a finite union of these with some gluings. Well if we have an open cover, we can assume one contains one of the $N_m \cup \{\infty\}$, but then $D \setminus N_m$ is clearly compact.

It turns out that $X(\Gamma)$ is a compact manifold. We must understand charts of the cusps. We now consider

$$h_{s,\Gamma} = |\mathrm{SL}_2(\mathbb{Z})_s / \{\pm I\}\Gamma_s| < \infty$$

Choose $\delta(s) = \infty, \delta \in \mathrm{SL}_2(\mathbb{Z})$ We then define $U_s = \delta^{-1}(N_2 \cup \{\infty\}), \psi : \rho \circ \delta$ where $\rho : z \mapsto e^{2\pi i z/h_s}$. One must check that the map ψ factors through the projection $U_s \xrightarrow{\pi} \pi(U_s)$.