We may also consider that, via the decomposition introduced before, we have

$$\mathcal{E}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{E}_k(N, \chi).$$

Also, we may consider the unnormalized Eistenstein series with $\overline{v} \in (\mathbb{Z}/N\mathbb{Z})^2$ of order $N$, for $k \geq 3$,

$$G_k^{\overline{v}}(\tau) = \sum_{(c,d)\equiv\overline{v}}{}' \frac{1}{(c\tau + d)^k}.$$

Idea: $G_k^{\overline{v}} \in \mathcal{E}_k(\Gamma_0(N))$, so we can get something in $\mathcal{E}_k(\Gamma_1(N))$ by averaging over a finite set of coset representatives. This may be zero, you have to be careful! But thankfully it's not super hard to compute the Fourier expansions with some effort.

Take $u, v$ with $uv = N$, $\psi \in \widehat{G}_u, \varphi \in \widehat{G}_v$, with $\varphi$ primitive and $\varphi\psi(-1) = (-1)^k$. Then we may define

$$G_k^{\psi\varphi}(\tau) \coloneqq \sum_{c=0}^{u-1} \sum_{d=0}^{v-1} \sum_{e=0}^{u-1} \psi(d)\overline{\varphi(d)} G_k^{\overline{(cv, d+ev)}}(\tau).$$

For $\gamma \in \Gamma_1(N)$ we have

$$G_k^{\psi\varphi}[\gamma]_k = \psi\varphi(d_\gamma) G_k^{\psi\varphi}.$$

Thus $G_k^{\psi\varphi} \in M_k(N, \psi\varphi)$.

We can normalize this to $E_k^{\psi\varphi}(\tau)$. The idea then is to define for $t \in \mathbb{N}$

$$E_k^{\psi,\varphi,t} \coloneqq E_k^{\psi\varphi}(t\tau).$$

This won't always yield a modular form, but if $tuv \mid N$ then it is.

**Theorem .0.1**

$\{E_k^{\psi,\varphi,t}\}$ is a basis for $\mathcal{E}_k(\Gamma_1(N))$. If we impose $\psi\varphi = \chi$, then this is a basis for $\mathcal{E}_k(N, \chi)$.

The steps to proving something like this

- Prove everything converges (not much harder than standard Eisenstein series)
- Prove everything transforms properly (by construction essentially)
- Prove things are holomorphic (get weird zeta functions when writing down Fourier Expansion Hard!)
- Prove things are linearly independent by looking at Fourier series.

Suppose $N > 0$, $\overline{v}$ as before, $k$ is any integer, $\epsilon_N = 1/2$ if $N = 1, 2$ and 1 otherwise, then

$$E_k^{\overline{v}}(\tau, s) = \epsilon_N \sum_{\substack{(c,d)\equiv v \\ \gcd(c,d)=1}}{}' \frac{\operatorname{Im}(\tau)^s}{(c\tau + d)^k \left|c\tau + d\right|^{2s}}.$$

Fact, this converges absolutely and uniformly for

$$\{s \mid \operatorname{Re}(k + 2s) > 2\}.$$

If $k \geq 3$, this converges for $s = 0$. We can check this has the right transformation properties, and then there is at most one meromorphic continuation to the complex plane!!! One can find it, and $s = 0$ is not a pole for $N = 1, 2$.

## .1. Interlude on $L$-functions/$\zeta$-functions

**Definition .1.1**

We say $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ lies in the Selberg-class of functions if it converges absolutely for $\mathrm{Re}(s) > 1$ and

(1) Analyticity: there is a meromorphic continuation, and the only possible pole is at $s = 1$.

(2) Ramanujan: $a_1 = 1$, $a_n \ll_\varepsilon n^\varepsilon$ for all $\varepsilon > 0$.

(3) Functional Equation: There should be a $\gamma$ factor so that if $\Phi(s) \coloneqq \gamma(s)f(s)$ then

$$\Phi(s) = \overline{\Phi(1-\overline{s})}.$$

(4) Euler Product: We should be able to write $f$ as

$$f(s) = \prod_{p \text{ prime}} f_p(s)$$

where $f_p(s) = \exp\left(\sum_{n=1}^{\infty} \frac{b_{p^n}}{p^{ns}}\right)$

The primary example is the Riemann $\zeta$-function. Here we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

and

$$\Phi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Natural Constructions:

$$\begin{tikzcd}
\text{Galois Reps} \arrow[r] & L\text{-functions} & \text{Automorphic Forms} \arrow[l] \\
& \text{Algebraic Varieties} \arrow[u] &
\end{tikzcd}$$

For modular forms–namely eigenforms in a later sense–we have for $f = \sum_{n=0}^{\infty} a_n q^n$ then

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

will lie in the Selberg class.

Another important example is Artin $L$-functions. Take $\rho \in \mathrm{Rep}(\mathrm{Gal}(K/\mathbb{Q}))$ where $K/\mathbb{Q}$ is a finite Galois extension. Then there is an $L$-function

$$L(\rho, s) = \text{ramified primes} \times \prod_{\mathfrak{p}} \left(\mathrm{char}(\rho(\mathrm{Frob}(\mathfrak{p})))(N(\mathfrak{p})^{-1})\right)^{-1},$$

where char is the characteristic polynomial, and $N(\mathfrak{p})$ is the norm.

If $L = \mathbb{Q}$ and $\rho$ is trivial, then this is just the Riemann zeta function. Then for $\rho_{\mathrm{reg}}$ the canonical representation for $K/\mathbb{Q}$ we have

$$L(\rho_{\mathrm{reg}}, s) = \prod_{\mathfrak{p}} \frac{1}{1 - N_{K/\mathbb{Q}}(p)^{-s}}.$$

For $K = \mathbb{Q}(\mu_N)$ with $\mu_N = e^{2\pi i/N}$, we have $\mathrm{Gal}(K/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times$, and the Galois representations ar e Dirichlet characters $\chi$, and it turns out you get

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

These are called Dirichlet $L$-functions.

**Conjecture .1.1** (Artin)

$L(\rho, s)$ is analytic if $\rho \neq 1$.

The abelian case is ok. If the group is solvable it's also ok.

For going from algebraic varieties to $L$-functions, it has to do with counting the number of points of a variety $X$ over $\mathbb{F}_p$.

Meromorphic Continuation and the Functional Equation

Warmup: The $\Gamma$ function is defined as

$$\Gamma(s) \coloneqq \int_{t=0}^{\infty} e^{-t} t^s \frac{\mathrm{d}t}{t}$$

for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$. One may check that $\Gamma(s+1) = s\Gamma(s)$. This allows us to extend $\Gamma$ to $\mathrm{Re}(s) \leq 0$. Because

$$\Gamma(s) = \frac{\Gamma(s+1)}{s},$$

so this is defined for $\mathrm{Re}(s) > -1$ besides when $s = 0$, and then keep playing the game.

There is a generalization of this idea

**Definition .1.2**

Let $f : \mathbb{R}^+ \to \mathbb{C}$. We define the Mellin transform of $f$ to be

$$\mathfrak{M}f(s) = \int_{t=0}^{\infty} f(t) t^s \frac{\mathrm{d}t}{t}$$

Then if $f(t) = e^{-t}$ we have $\mathfrak{M}f(s) = \Gamma(s)$. We can define

$$\Theta(it) \coloneqq \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$$

$$\sum_{n=1}^{\infty} e^{-\pi n^2 t} = \frac{1}{2}(\Theta(it) - 1).$$

Taking the Mellin transform

$$\int_{t=0}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^s \frac{\mathrm{d}t}{t} = \frac{1}{2} \int_{t=0}^{\infty} (\theta(it) - 1) t^s \frac{\mathrm{d}t}{t}.$$

The left hand side has excellent convergence properties, so we may exchange the integral and the sum, which gives us for $\mathfrak{M}f$ on the left hand side

$$\mathfrak{M}f(s) = \sum_{n=1}^{\infty} (\pi n^2)^{-s} \Gamma(s) = \pi^{-s} \Gamma(s) \zeta(2s).$$

Then we may define $\mathfrak{M}f(s/2) =: \Phi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Splitting off the $0, 1$ portion of this

$$\frac{1}{2}\int_{t=0}^{1} (\Theta(it) - 1)t^{s/2}\frac{\mathrm{d}t}{t} = \frac{1}{2}\int_{0}^{1} \Theta(it)t^{s/2}\frac{\mathrm{d}t}{t}.$$

We also have a formula $\Theta(i/t) = t^{1/2}\Theta(it)$. Thus via a change of variables

$$\begin{aligned}
&\left[\frac{1}{2}\int_{t=1}^{\infty} \Theta(i/t)t^{-s/2}\frac{\mathrm{d}t}{t}\right] - \frac{1}{s} \\
&= \left[\frac{1}{2}\int_{t=1}^{\infty} \Theta(it)t^{1-s/2}\frac{\mathrm{d}t}{t}\right] - \frac{1}{s} \\
&= \left[\frac{1}{2}\int_{t=1}^{0} (\Theta(it) - 1)t^{1-s/2}\frac{\mathrm{d}t}{t}\right] - \frac{1}{s} - \frac{1}{1-s}.
\end{aligned}$$

One should then recombine things and show things are invariant under $s \mapsto 1 - s$.