Goal: Define $\operatorname{div}(f)$ for $f \in \mathcal{A}_k(\Gamma)$. We'll do this in cases

- Suppose $\tau \in \mathcal{H}$ with $\pi(\tau) \in X(\Gamma)$ is not a cusp. Note that $\tau \mapsto (c\tau + d)^k$ has no 0s or poles on $\mathcal{H}$ and

$$f(\gamma\tau) = \underbrace{j(\gamma, \tau)}_{(c\tau+d)^k} f(\tau).$$

The local coordinates at τ are of the form $q = (t - \tau)^h$ fr some h.

For $f(t) = a_m(t-\tau)^m$, then define $v_{\pi(\tau)}(f) = m/h$. In particular $v_{\pi(\tau)}(f) \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{2}\mathbb{Z}$. When $k = 0$, we have that f is an actual function on $X(\Gamma)$ so $m/h \in \mathbb{Z}$.

Suppose $\pi(\tau)$ is a cusp. We can focus on $\tau = \infty$ because it's similar elsewhere (transform to ∞)

Local coordinates are $q_h = e^{2\pi i\tau/h}$, where h is defined as the smallest positive integer satisfying

$$\{\pm I\}\Gamma_\infty = \{\pm I\}\langle\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)\rangle$$

To define "f meromorphic at ∞," we leveraged periodicity of f, we have $f(\tau + h) = (\pm 1)^k f(\tau)$. When it's $f(\tau + h) = f(\tau)$ we call the cusp regular, and otherwise it's irregular. Define $\hbar = h$ in the first case and $\hbar = 2h$ in the second case (aka the period).

Example .0.1

1/2 is irregular for $\Gamma = \Gamma_1(4)$, which is the only example for $\Gamma_0(N), \Gamma_1(N), \Gamma(N)$.

Let $h' = 2h$, then f is h'-periodic, and $f(\tau) = g(q_{h'})$ close to ∞. We define

$$v_{\pi(\infty)}(f) = \frac{m}{2}$$

where

$$g(q_{h'}) = \sum_{n=m}^{\infty} a_n q_{h'}^n.$$

In the regular case, $v_{\pi(\infty)}(f) = v_\infty(f)$ and in the irregular case $v_{\pi(\infty)}(f) = \frac{v_\infty(f)}{2}$.

I. Differentials

Intuition: If f is weight k Γ-invariant, k is even, then $f(\tau)(\mathrm{d}\tau)^{k/2}$ is honest to god Γ-invariant. Thus we should think of f as sort of differentials on the modular curve.

The Next Goal: Define these differentials appropriately

Definition I.0.1

For $V \subseteq \mathbb{C}$ open we define

$$\Omega^{\otimes n}(V) := \{f(q)(\mathrm{d}q)^n \mid f \text{ is meromorphic on } V\}$$

with $(\mathrm{d}q)^{n+m} := (\mathrm{d}q)^n(\mathrm{d}q)^m$. Then

$$\Omega(V) := \bigoplus_{n \in \mathbb{N}_0} \Omega^{\otimes n}(V)$$

is a graded ring of differentials

Suppose we have a holomorphic map $\varphi : V_1 \to V_2$, then we define the pullback

$$\varphi^* : \Omega^{\otimes n}(V_2) \to \Omega^{\otimes n}(V_1)$$
$$\varphi^*(f(q_2)(\mathrm{d}q_2)^n) \coloneqq f(\varphi(q_1))(\varphi'(q_1))^n(\mathrm{d}q_1)^n.$$

Exercise I.0.1

$(\varphi^*)^{-1} = (\varphi^{-1})^*$.

Definition I.0.2

For $U \subseteq X$ open, where X is a Riemann Surface, $\Omega^{\otimes}(U)$ is defined via the charts $\varphi_j : U_j \to V_j \subseteq \mathbb{C}$. Namely, we have $\omega \in \Omega^{\otimes n}(U)$ is a $(\omega_j) \in \prod_J \Omega^{\otimes n} V_j$ such that for

$$V_{j,k} \coloneqq \varphi_j(U_j \cap U_k) \qquad \varphi_{j,k} = \varphi_k \circ \varphi_j^{-1} : V_{j,k} \to V_{k,j}$$

such that

$$\omega_j\big|_{V_{j,k}} = \varphi_{j,k}^*\left(\omega\big|_{V_{k,j}}\right).$$

It is fairly simple then to define pullback everywhere.

We then have $\pi : \mathcal{H} \to X(\Gamma)$ then $\pi^* : \Omega^{\otimes n}(X(\Gamma)) \to \Omega^{\otimes n}(\mathcal{H})$.

But wait! The differential that is pulled back must then be Γ invariant. This will give us

$$\begin{aligned} \pi^*\omega = f(\tau)(\mathrm{d}\tau)^n &= \gamma^*(f(\tau)(\mathrm{d}\tau)^n) \\ &= f(\gamma\tau)(j(\gamma,\tau))^{-2n}(\mathrm{d}\tau)^n. \end{aligned}$$

Thus $f(\gamma\tau) = j(\gamma,\tau)^{2n} f(\tau)$, so $f \in \mathcal{A}_{2n}(\Gamma)$. This gives us an honest to god map

$$\Omega^{\otimes n}(X(\Gamma)) \to \mathcal{A}_{2n}(\Gamma).$$

Theorem I.0.1

This is a bijection.

Proof. Map in the other direction is an absolute shitshow. Take $f \in \mathcal{A}_{2n}(\Gamma)$, and call $k = 2n$. Work locally to construct $\omega(f) \in \Omega^{\otimes n}(X(\Gamma))$. We'll do this for the non-cusp points, but we won't check the gluing condition. Oops!

For $\tau \in U \subseteq H$ we constructed a map $\psi : U \xrightarrow{\rho\circ\delta} V$, and we showed this factors through as $\varphi : \pi(U) \to V$.

We'll instead construct "$\omega(f)$" in V so that it pulls back to the right thing in U, and then we'll pull it back to $\pi(U)$ via φ. We have $\delta \in \mathrm{GL}_2(\mathbb{C}), \alpha \coloneqq \delta^{-1}$. So the first step is to take $\lambda \coloneqq \alpha^*(f(\tau)(\mathrm{d}\tau)^n)$.

We define an extension of the $f[\gamma]_k$ formula as

$$f[\alpha]_k = (\det\alpha)^{k/2} j(\alpha,\tau)^{-k} f(\alpha(\tau)).$$

We in fact have $\alpha'(\tau) = \frac{\det\alpha}{(j(\alpha,\tau))^2}$. One may then check that

$$\lambda = (f[\alpha]_k)(z)(\mathrm{d}z)^n.$$

In contrast, ρ is not invertible, so the same trick does not work. Instead, we just have to think hard... If we have a non-elliptic point though, $\rho = \mathrm{Id}$ and we're done. Otherwise we should consider that λ is $\delta\Gamma\delta^{-1}$-invariant.

Lets defdine $\rho_h : z \mapsto \mu_h z$ where $\mu_h = e^{2\pi i/h}$. We have that $\rho_h^*(\lambda) = \lambda$ by invariance. But then this implies

$$\mu_h^n z^n (f[\alpha]_k)(\mu_h z) = z^n (f[\alpha]_k)(z).$$

Then $z^n f[\alpha]_k(z)$ is invariant under rotation by h, so it is equal to $g(z^h)$. We may then consider

$$\omega = \frac{g(q)(\mathrm{d}q)^n}{(h\,\mathrm{d}q)^n}.$$

In fact $\rho^*(\omega) = \lambda$ as desired.