Last time we defined the Hecke operators $\langle d \rangle$, T_p. For convenience let N be fixed and write Γ_1 for $\Gamma_1(N)$.

Proposition .0.1

For $f \in \mathcal{M}_k(\Gamma_1)$, write the Fourier expansion as $f(\tau) = \sum a_n(f) q^n$ where $q = e^{2\pi i \tau}$. Then we may write the Fourier expansion of $T_p f$ explicitly

$$(T_p f)(\tau) = a_{np}(f) q^n + 1_N(p) p^{k-1} a_n(\langle p \rangle f) q^{np}.$$

where 1_N is the trivial character of N evaluated at p. In particular if $f \in \mathcal{M}_k(N, \chi)$ we have

$$(T_p f)(\tau) = a_{np}(f) q^n + 1_N(p) p^{k-1} \chi(f) a_n(f) q^{np}.$$

Proof. A group theory exercise yields if $p \nmid N$ then

$$\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1 = \coprod_{j=0}^{p-1} \Gamma_1 \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$$

and if $p \mid N$ then

$$\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1 = \Gamma_1 \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \coprod \coprod_{j=0}^{p-1} \Gamma_1 \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix},$$

where $mp - nN = 1$.

We'll only do the $p \mid N$ cosets first. Here we have

$$\begin{aligned}
(T_p f)(\tau) &= \sum_{j=0}^{p-1} f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \\
&= \sum_{j=0}^{p-1} p^{k-1} p^{-k} f\left(\frac{\tau + j}{p} \right) \\
&= \sum_{j=0}^{p-1} \sum_{n=0}^{\infty} \frac{a_n(f)}{p} e^{2\pi i n(\tau + j)/p} \\
&= \sum_{j=0}^{p-1} \sum_{n=0}^{\infty} \frac{a_n(f)}{p} \mu_p^{nj} q_p^n
\end{aligned}$$

where $\mu_p = e^{2\pi i/p}, q_p = e^{2\pi i \tau/p}$. We have that

$$\sum_{j=0}^{p-1} \mu_p^{nj} = \begin{cases} p & \text{if } p \mid n \\ 0 & \text{if } p \nmid n \end{cases}.$$

Thus this becomes

$$(T_p f)(\tau) = \sum_n a_{pn} q^n.$$

For the $p \nmid N$ case we take

$$f \left[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 1 \\ 0 & 1 \end{pmatrix} \right]_k = (\langle p \rangle f) \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}_k (\tau).$$

This is

$$\sum_n p^{k-1} a_n(\langle p\rangle f) e^{2\pi i n p \tau} = \sum_n p^{k-1} a_n(\langle p\rangle f) q^{np}.$$

Proposition .0.2

If $d, r \in (\mathbb{Z}/N\mathbb{Z})^\times$ and p, q are prime the

- $\langle d\rangle T_p = T_p \langle d\rangle$.
- $\langle d\rangle\langle r\rangle = \langle r\rangle\langle d\rangle$.
- $T_p T_q = T_q T_p$.

Now we may define $\langle n\rangle, T_n$ by

$$\langle n\rangle = \begin{cases} \langle n\rangle & \text{if } (n, N) = 1 \\ 0 & \text{if } (n, N) \neq 1 \end{cases}$$

$$T_n = \sum_{\substack{ad=n \\ a\mid d}} \langle a\rangle \left[\Gamma_1 \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma_1\right]_k$$

$$T_{p^2} = \left[\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \Gamma_1\right]_k + \langle p\rangle T_1.$$

One may check this satisfies the recursion

$$T_{p^r} = T_p T_p^{r-1} - p^{k-1}\langle p\rangle T_{p^{r-2}}.$$

Then we can define $T_n = \prod_i T_{p_i^{r_i}}$ where $n = \prod p_i^{r_i}$. Then

$$(T_n f)(\tau) = \sum_n a_m(T_n f) q^m$$

$$a_m(T_n f) = \sum_{d\mid(m,n)} d^{k-1} a_{mn/d^2}(\langle d\rangle f).$$

.1. Peterson Inner Product

Let $\tau = x + iy$, and write $\mathrm{d}\nu = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$, which is the "hyperbolic measure" on $\mathcal{H}$. One can prove that $\mathrm{d}\nu$ is actually $\mathrm{GL}_2^+(\mathbb{R})$-invariant. This lets us integrate over $\mathcal{H}^*$.

Recall we have the fundamental domain

$$D^* = \{\tau \in \mathcal{H} \mid |\mathrm{Re}(\tau)| \leq 1/2, |\tau| \geq 1\} \cup \{\infty\}.$$

We want to integrate on D^*. One may check that if $\varphi : \mathcal{H} \to \mathbb{C}$ is bounded and continuous then

$$\int_{D^*} \varphi(\alpha(\tau))\,\mathrm{d}\nu(\tau)$$

converges, where $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Take $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and write $\mathrm{SL}_2(\mathbb{Z}) = \coprod_j \{\pm I\}\Gamma\alpha_j$. If φ is Γ-invariant then the following will not depend on the choice of α_j,

$$\sum_j \int_{D^*} \varphi(\alpha_j(\tau))\,\mathrm{d}\nu(\tau) =: \int_{X(\Gamma)} \varphi(\tau)\,\mathrm{d}\nu(\tau).$$

We may then define

$$V_\Gamma \coloneqq \int_{X(\Gamma)} \mathrm{d}\nu(\tau)$$

Definition .1.1

We define the Peterson inner product of $f, g \in S_k(\Gamma)$ to be

$$\langle f, g\rangle \coloneqq \frac{1}{V_\Gamma}\int_{X(\Gamma)} f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k\,\mathrm{d}\nu(\tau).$$

We normalize by the volume so that the inner product remains the same over $\Gamma_1 \subseteq \Gamma_2$. It takes some work but we must check $\varphi(\tau) \coloneqq f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k$ is Γ-invariant.

Remark .1.1

We only need 1 of $f, g \in S_k(\Gamma)$ to be bounded.

Exercise .1.1

We can see that

$$\begin{aligned}
\mathrm{Im}(\gamma\tau) &= \frac{\mathrm{Im}(\tau)}{j(\gamma,\tau)\overline{j(\gamma,\tau)}} \\
\varphi(\gamma(\tau)) &= f(\gamma(\tau))\overline{g(\gamma(\tau))}\,\mathrm{Im}(\gamma(\tau))^k\,\mathrm{Im}(\gamma(\tau))^k \\
&= f[\gamma]_k j(\gamma,\tau)^k\overline{g[\gamma]_k j(\gamma,\tau)}^k\,\mathrm{Im}(\gamma(\tau))^k \\
&= f(\tau)g(\tau)\,\mathrm{Im}(\tau)^k = \varphi(\tau).
\end{aligned}$$

Want: $M_k(\Gamma_1(N))$ has an orthonormal basis of eigenvectors under $\{T_n, \langle n\rangle \mid (n, N) = 1\}$. We want to apply the spectral theorem, and we need $T_n, \langle n\rangle$ are normal.

Recall .1.2

The adjoint is defined by $\langle Tf, g\rangle = \langle f, T^*g\rangle$ we take T is normal provided that $TT^* = T^*T$.

One can get a simultaneous orthonormal basis of eigenvectors using that these operators commute and some linear algebra.

Here's a fact: For any Γ, let $\alpha' = \det(\alpha)\alpha^{-1}$. Then

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle = \langle f, g[\Gamma\alpha'\Gamma]_k\rangle.$$

This implies that

$$\langle p\rangle^* = \langle p^{-1}\rangle.$$

As the relevant matrix is represented as $\left(\begin{smallmatrix} n & s \\ N & 0 \; p \end{smallmatrix}\right)$ of determinant 1 and its inverse can be represented by a similar matrix with p^{-1} in the bottom right. Then for T_p^* w ehave

$$\begin{aligned}
\alpha &= \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \\
\alpha' &= p \begin{pmatrix} 1 & 0 \\ 0p^{-1} & \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & n \\ N & mp \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} p & n \\ N & m \end{pmatrix}.
\end{aligned}$$

The left hand side is in $\Gamma_1(m)$ and the right hand side is in Γ_0. Thus we have something like

$$\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1 \begin{pmatrix} p & n \\ N & m \end{pmatrix}.$$

Thus $T_p^* = \langle p \rangle^{-1} T p$.