I. Hecke Operators

I.1. Definitions and Computations

Call $\mathrm{GL}_2(\mathbb{Q})^+ \subseteq \mathrm{GL}_2(\mathbb{Q})$ the subgroup of positive determinant matrices. If we have $\Gamma_1, \Gamma_2 \in \mathrm{SL}_2(\mathbb{Z}), \alpha \in \mathrm{GL}_2(\mathbb{Q})^+$ we'll define an operator

$$[\Gamma_1 \alpha \Gamma_2]_k : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2).$$

Reminder: Double cosets are a little weird.

Exercise I.1.1

Suppose G is finite, H_1, H_2 are subgroups. Compute $|H_1 \alpha H_2|$ in temrs of cardinalities of subgroups of G. We have

$$|H_1 \alpha H_2| = \frac{|H_1| \cdot |H_2|}{|H_1 \cap \alpha H_2 \alpha^{-1}|}.$$

For $\beta \in \mathrm{GL}_2(\mathbb{Q})^+$ we define

$$f[\beta]_k(\tau) \coloneqq (\det \beta)^{k-1} j(\beta, \tau)^{-k} f(\beta(\tau)).$$

Now we see that

$$\Gamma_1 \alpha \Gamma_2 = \coprod_j \Gamma_1 \beta_j$$

for some $\beta_j \in \alpha \Gamma_2$, and we then define

$$f[\Gamma_1 \alpha \Gamma_2]_k = \sum_j f[\beta_j]_k.$$

We need to know: there are finitely many β_j, this doesn't depend on β_j, and this actually takes modular forms of weight k level Γ_1 to weight k level Γ_2 forms.

Fact: $\alpha^{-1} \Gamma \alpha \cap \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup.

Lemma I.1.1

We have that

$$\alpha^{-1} \Gamma_1 \alpha \cap \Gamma_2 \backslash \Gamma_2 \to \Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2,$$

and the left hand side is finite, so we only need finitely many β_j.

To show this gives a map as claimed, first check it's well-defined (does not depend on choice of β_j), then we take

$$f[\Gamma_1 \alpha \Gamma_2]_k[\gamma_2]_k = \sum_j f[\beta_j]_k[\gamma_2]_k = \sum_j f[\beta_j']_k = f[\Gamma_1 \alpha \Gamma_2]_k.$$

None of this effects holomorphicity on $\mathcal{H}$, but we need to check holomorphicity at the cusps. Recall this was $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The necessary lemma is

Lemma I.1.2

If $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$ and $\alpha\gamma = \gamma'$ then

$$\alpha = r \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

for $r \in \mathbb{Q}^+$. And this will not change holomorphicity at ∞.

This same proof will also show that if f is a cuspform then $f[\Gamma_1 \alpha \Gamma_2]_k$ is a cuspform.

Example I.1.2

If $\Gamma_1 \supseteq \Gamma_2$ and $\alpha = 1$ then we get the embedding $M_k(\Gamma_1) \hookrightarrow M_k(\Gamma_2)$.

If $\alpha^{-1}\Gamma_1\alpha = \Gamma_2$ then

$$f[\Gamma_1\alpha\Gamma_2]_k = f[\alpha]_k$$

and gives an isomorphism $\mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$.

If $\Gamma_1 \subseteq \Gamma_2$, $\{\gamma_{2,j}\}$ represents $\Gamma_1 \backslash \Gamma_3$

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\gamma_{2,j}]_k$$

Then

$$\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$$
$$\Gamma_3' = \Gamma_1 \cap \alpha\Gamma_2\alpha^{-1}$$

gives

$$\begin{tikzcd} \Gamma_3 \arrow[r, "\sim"] \arrow[d] & \Gamma_3' \arrow[d] \\ \Gamma_2 & \Gamma_1 \arrow[l, dashed] \end{tikzcd}$$

Then as moduli spaces

$$\begin{tikzcd} X_3 \arrow[r, "\sim"] \arrow[d] & X_3' \arrow[d] \\ X_2 & X_1 \arrow[l, dashed] \end{tikzcd}$$

Then we have

$$[\Gamma_1\alpha\Gamma_2]_k : \mathrm{Div}(X_2) \to \mathrm{Div}(X_1)$$

given by

$$x \mapsto \sum_{y \in \pi_2^{-1}(x)} e_y y \mapsto e_y \alpha y \alpha^{-1}$$
$$\mapsto \sum_y e_y \pi_1(\alpha y \alpha^{-1}).$$

Special Cases: $\Gamma_0(N), \Gamma_1(N)$, that is

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \quad \mathrm{mod}\ N$$

Given $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, we have a Diamond operator

$$\langle d \rangle = [\Gamma_1(N)\alpha\Gamma_1(N)]_k$$

where

$$\alpha \mapsto \begin{pmatrix} * & * \\ 0 & d \end{pmatrix}$$

where $\mapsto$ here is the reduction mod N. In particular since $\Gamma_1(N)$ is a normal subgroup of $\Gamma_1(N)$ we have

$$\langle d \rangle f = f[\alpha]_k$$

and in fact

$$\mathcal{M}_k(N, \chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) \mid \langle d \rangle f = \chi(d) f \text{ for all } d\}.$$

The next is $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right]$ where p is prime with

$$T_p \coloneqq [\Gamma_1(N)\alpha\Gamma_1(N)]_k.$$

Exercise I.1.3

$T_p, \langle d \rangle$ commute.

Proof. Note first that

$$\begin{aligned} \alpha^{-1}\Gamma_1 \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \Gamma_1 \alpha &= \Gamma_1 \alpha^{-1} \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \alpha \Gamma_1 \\ &= \Gamma_1 \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \Gamma_1. \end{aligned}$$

The second equality above requires a check. Then we know this is

$$\alpha^{-1} \coprod_j \Gamma_1 \beta_j \alpha = \coprod_j \Gamma_1 \alpha^{-1} \beta_j \alpha =: \coprod_j \Gamma_1 \beta_j'.$$

Then we can compute

$$T_p \langle d \rangle f = \sum_j f[\alpha]_k [\beta_j]_k = \sum_j f[\beta_j']_k [\alpha]_k = \langle d \rangle T_p f.$$