Recall .0.1

We have that

$$\mathrm{Jac}(X) = \Omega^1_{\mathrm{hol}}(X)^*/H_1(X,\mathbb{Z})$$
$$\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\,\mathrm{Div}^\ell(X).$$

and a theorem of Abel says that

$$\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X)$$
$$\sum n_x x \mapsto \sum_x n_x \int_{x_0}^x .$$

We had defined pushforwards

$$h_P : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$$
$$h_J : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$$

where $h : X \to Y$ is a map of compact Riemann Surfaces. The first was a norm map, and the second was pullback of differentials.

We now want the pullbacks. Let $h : X \to Y$, and let $X' = X - \mathcal{E}$, $Y' = Y - h(\mathcal{E})$ where we've cut out the ramified points (those with multiplicity). Then $h : X' \to Y'$ is a d-fold cover for some d.

To define the pullbacks we define the pushforwards of differentials

$$\mathrm{tr}_h : \Omega^1_{\mathrm{hol}}(X) \to \Omega^1_{\mathrm{hol}}(Y).$$

Let $y \in Y'$. Take a small $y \in U \subseteq Y'$ so that $h_i^{-1} : U \to U_i$ is defined (since this is a covering map). Then we define for $\omega \in \Omega^1_{\mathrm{hol}}(X)$ to be

$$(\mathrm{tr}_h\,\omega)\big|_U = \sum_{i=1}^d (h_i^{-1})^*(\omega\big|_U).$$

One must check this is well-defined on Y' and that it extends holomorphically to Y.

Dually, we get

$$\mathrm{tr}_h^* : \Omega^1_{\mathrm{hol}}(Y)^* \to \Omega^1_{\mathrm{hol}}(X)^*.$$

We need to pullback loops as well. Given a path δ in Y' and a basepoint $x \in h^{-1}(\delta(0)) \subseteq X'$, there is a unique path γ lying in X' which lifts δ and satisfies $\gamma(0) = x$. This gives d lifts total.

What if δ is in Y but only endpoints can be in $h(\mathcal{E})$? Then for each x, there are e_x many lifts γ which begin at x. There are then d lifts total.

If δ is a loop in Y', then $\gamma(1) \in h^{-1}(\delta(0))$ for any lift γ. Thus we can take the concatenation of all the lifts of δ. This will give us some collection of loops in cycles!

In other words, fixing $y_0 \in Y'$, then $\pi_1(y_0, Y')$ acts on $h^{-1}(y_0)$, and this is called the monodromy action.

Reverse change of variables

$$\int_{\delta \in Y'} (h^{-1})^*\omega = \int_{h^{-1}\circ\delta} \omega$$

for $\omega \in \Omega^1_{\mathrm{hol}}(X)$. Hence

$$\int_\delta \operatorname{tr}_h \omega = \sum_{\text{all lifts } \gamma} \int_\gamma \omega,$$

for δ lying in Y'. One can extend this formula to δ in Y, not just Y'.

Thus tr_h^* descends to

$$h^J : \operatorname{Jac}(Y) \to \operatorname{Jac}(X).$$

In fact, for $\lambda \in \Omega_{\mathrm{hol}}(Y)$ we have

$$(\operatorname{tr}_h \circ h^*)(\lambda) = \deg(h)\lambda.$$

As a consequence we have the fact that

$$h_J \circ h^J = [\deg h].$$

This is similar to the fact that we had for elliptic curves and isogenies!

Corollary .0.1

We have that

$$h_*(H_1(X, \mathbb{Z})) \subseteq H_1(Y, \mathbb{Z})$$

is of finite index.

What about for Picard Groups? For $h : X \to Y$, we have

$$h^* : \mathbb{C}(Y) \to \mathbb{C}(X)$$

and

$$\operatorname{div}(h^* g) = \sum_x e_x v_{h(x)}(g)[x] = \sum_y v_y(g) \sum_{x \in h^{-1}(y)} e_x[x].$$

This suggests we should define

$$h^D\left(\sum_y n_y[y]\right) = \sum_y n_y \sum_{x \in h^{-1}(y)} e_x[x].$$

This in fact gives you

$$h^P : \operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(X).$$

These maps actually commute with the Abel-Jacobi isomorphism $\operatorname{Pic}^0(-) \to \operatorname{Jac}(-)$!

.1. Jacobians and Hecke Operators

Suppose $\Gamma_1, \Gamma_2 \subseteq \mathrm{SL}_2(\mathbb{Z})$ are congruence subgroups. Then suppose $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then we can define

$$\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2 \qquad\qquad \Gamma_3' = \Gamma_1 \cap \alpha\Gamma_2\alpha^{-1}.$$

Then for the modular curves, we have a picture

$$\begin{tikzcd} X_3 \arrow[r, "\alpha", "\sim"'] \arrow[d, "\pi_2"'] & X_3' \arrow[d, "\pi_1"] \\ X_2 & X_1. \end{tikzcd}$$

We'll then define

$$[\Gamma_1 \alpha \Gamma_2]_2 : \mathrm{Div}(X_2) \to \mathrm{Div}(X_1)$$

which is given by $(\pi_1)_D \circ \alpha_D \circ (\pi_2)^D$. Now let $\gamma_{2,j}$ be representatives of the quotient $\Gamma_3 \backslash \Gamma_2$. Then recall that with $\beta_j := \alpha\gamma_{2,j}$ we have

$$\Gamma_1 \alpha \Gamma_2 = \bigsqcup_j \Gamma_1 \beta_j.$$

Upshot:

$$[\Gamma_1 \alpha \Gamma_2]^2 : \mathrm{Pic}^0(X_2) \to \mathrm{Pic}^0(X_1).$$

We can compute for $\Gamma_2 \tau \in X_2$, that we get

$$\begin{tikzcd} \{\Gamma_3 \gamma_{2,j}\tau\} \arrow[r, mapsto, "\alpha"] & \{\Gamma_3' \beta_j \tau\} \arrow[d, mapsto, "\pi_1"] \\ \Gamma_2 \tau \arrow[u, mapsto, "\pi_2^{-1}"] & \{\Gamma_1 \beta_j \tau\}. \end{tikzcd}$$

Explicitly, then the map is given by

$$[\Gamma_1 \alpha \Gamma_2]^2 \left(\sum_\tau n_\tau \Gamma_2 \tau \right) = \sum_\tau n_\tau \sum_j \Gamma_1 \beta_j \tau.$$

Remember that we had an isomorphism

$$w : S_2(\Gamma) \xrightarrow{\sim} \Omega^1_{\mathrm{hol}}(X(\Gamma)).$$

Then we must have

$$\mathrm{Jac}(X(\Gamma)) = S_2(\Gamma)^* / H_1(X(\Gamma), \mathbb{Z}).$$

We have defined a double coset operator

$$[\Gamma_1 \alpha \Gamma_2]_2 : S_2(\Gamma_1) \to S_2(\Gamma_2),$$

which induces a map

$$[\Gamma_1 \alpha \Gamma_2]_2^* : S_2(\Gamma_2)^* \to S_2(\Gamma_1)^*.$$

A priori this is not the same as $[\Gamma_1 \alpha \Gamma_2]^2$. But in fact these maps are the same!!!

Claim

Maps are the same. Essentially tr_{π_2} is defined on local patches which will be given by $\gamma_{2,j} \ldots$

Looking at $J_1(N) = \mathrm{Jac}(X(\Gamma_1(N)))$,

Proposition .1.1

Let $T = T_p, \langle d \rangle$. Then T acts on $J_1(N)$ by definition.

Easy consequence$T_p : S_2(\Gamma_1(N))^* \to S_2(\Gamma_1(N))^*$ descends to $J_1(N)$, and hence acts on $H_1(X_1, \mathbb{Z})$.

Then if $f = \operatorname{char} T_p$ has integer coefficients, then $f(T_p) = 0$ on $H_1(X_1, \mathbb{Z})$. Then $f(T_p) = 0$ on $S_2(\Gamma_1(N))^*$ hence $S_2(\Gamma_1(N))$.

Therefore the eigenvalues of T_p are algebraic integers. Then $a_p(f)$ are algebraic integers, so $a_n(f)$ is algebraic integer.