Recall that $X(\Gamma)$ is a compact manifold

Theorem .0.1 (Modularity)

For E an elliptic curve such that $j(E) \in \mathbb{Q}$ there exists an N and a surjective map $X_0(N) \twoheadrightarrow E$ of compact Riemann surfaces.

Goal: Compute the genus of $X(\Gamma)$. Recall from the theory of compact riemann surfaces that

- If $f : X \to Y$ is a nonconstant map of compact riemann surfaces, then it is surjective.
- For all $y \in Y$, $f^{-1}(\{y\})$ is discrete, which implies $\left|f^{-1}(y)\right| < \infty$.
- Away from finitely many points of Y, $\left|f^{-1}(y)\right| = d$ is constant and we call this constant d the degree of f. We call those points where $\left|f^{-1}(y)\right| \neq d$ the ramification points. Consider $z \mapsto z^n$ as an example.
- For all $x \in X$, there exists some number e_x such that $\sum_{x \in f^{-1}(y)} e_x = d$, and we should think of e_x as the multiplicity or ramification number.

Important formula in this setting

Theorem .0.2 (Riemann-Hurwitz Formula)

If $f : X \to Y$ is a nonconstant map of compact connected Riemann surfaces then

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X} (e_x - 1)$$

where d is the degree of f, g_X, g_Y are the genuses of X, Y, and e_x ramification number at $x \in X$.

Proof Idea. Triangulate Y and generically you have d triangles in X for each triangle you start with, but we have to accoutn for ramification points.

In our case, we have $f : X(\Gamma) \to X(1)$, and $X(1)$ is a sphere, and so it is zero. Thus our formula simplifies to

$$2g - 2 = -2d + \sum_{x \in X} (e_x - 1).$$

The ramification points will be elliptic points and cusps.

Elliptic Points: If $\langle \gamma \rangle = \mathrm{SL}_2(\mathbb{Z})_\tau$ fixing τ, then $|\langle \gamma \rangle| = 4, 6$ and we have to worry about $i, \mu_3 = e^{2\pi i/3}$. Then

$$h_\tau = [\{\pm I\}\Gamma_T : \{\pm I\}] \in \{2, 3\}.$$

Let $\tau \in U \subseteq \mathcal{H}$ which is a coordinate chart and $\pi : \mathcal{H}^\star \to X(1), \pi_\Gamma : \mathcal{H}^\star \to X(\Gamma)$. Then we're looking at

$$\begin{tikzcd}
U \arrow[r, "\mathrm{Id}"] \arrow[d, "\pi_\Gamma"] \arrow[dd, bend right=60, dashed, "q \mapsto q^{h_\Gamma}"'] & U \arrow[d, "\pi"] \arrow[dd, bend left=60, dashed, "q \mapsto q^h"] \\
\pi_\Gamma(U) \arrow[r, "f"] \arrow[d, "\varphi_\Gamma"] & \pi(U) \arrow[d, "\varphi"] \\
V_\Gamma \arrow[r, "f_{\mathrm{loc}}"] & V
\end{tikzcd}$$

$$q \dashrightarrow q^{h/h_\Gamma}$$

We know $h/h_\Gamma \in \{1,2,3\}$. The interesting case is when τ is elliptic for $\mathrm{SL}_2(\mathbb{Z})$ but NOT Γ. Then this determines the ramification number.

Cusps: We have $z \mapsto e^{2\pi i z/h_\Gamma}$ where $h_\Gamma = |\mathrm{SL}_2(\mathbb{Z})_\infty| \,/\, |\{\pm I\}\Gamma_s|$. Then the ramificatio number is

$$e_x = \frac{h_\Gamma}{h} = h_\Gamma.$$

Say τ is elliptic, and consider $F_\tau = f^{-1}(\tau)$, and $\mathcal{E}_h$ is the number of elliptic points in F_τ for Γ, and n is the number of other points. Then

$$|F_\tau| = \mathcal{E}_h + n \qquad\qquad d = \sum_{x \in F_\tau} e_x = hn + \mathcal{E}_h.$$

We then see that

$$\sum_{x \in F_\tau} e_x - 1 = (h-1)n = \frac{h-1}{h}(d - \mathcal{E}_h).$$

For cusps, notice that

$$\sum_{x \in F_\infty} e_x - 1 = d - \mathcal{E}_\infty.$$

Therefore

$$\begin{aligned} 2g - 2 &= -2d + d - \mathcal{E}_\infty + \frac{1}{2}(d - \mathcal{E}_i) + \frac{2}{3}(d - \mathcal{E}_{\mu_3}) \\ &= \frac{1}{6}d - \mathcal{E}_\infty - \frac{1}{2}\mathcal{E}_i - \frac{2}{3}\mathcal{E}_{\mu_3} \\ g &= 1 + \frac{d}{12} - \frac{\mathcal{E}_\infty}{2} - \frac{\mathcal{E}_i}{4} - \frac{\mathcal{E}_{\mu_3}}{6}. \end{aligned}$$

Generally this computation is hard. Why is it important?

Idea:

Modular forms of weight $k \rightsquigarrow$ meromorphic Γ-invariant differentials on $\mathcal{H}$, $H^0(X(\Gamma), \Omega^{\otimes k})$.

The right hand side is computable using the Riemann-Roch theorem if you have seen it.

Definition .0.1

A function $f : \mathcal{H} \to \widehat{\mathbb{C}}$ is an automorphic function of weight k and level Γ if

(1) f is meromorphic on $\mathcal{H}$.

(2) f is weight k, Γ-invariant

(3) $f[\alpha]_k$ is meromorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

This is not an automorphic form if you have heard of that! We call these $\mathcal{A}_k(\Gamma)$, and we note that $\mathcal{A}_0(\Gamma)$ consists of the meromorphic functions on $X(\Gamma)$, as the function must descend. Let $\mathbb{C}(X)$ denote the meromorphic functions to $\mathbb{C}$ from X.

Definition .0.2

For X a compact riemann surface, $f \in \mathbb{C}(X)$,

$$\operatorname{div} f = \sum_X n_x [x].$$

We define the degree of $D \in \mathrm{Div}(X) = \mathbb{Z}X$ as

$$\deg D = \deg \sum_X n_x[x] = \sum_X n_x.$$

Fact: If X if a compact Riemann surface then

- If $f : X \to \mathbb{C}$ is holomorphic on X, then f is constant.
- $\mathbb{C}(\widehat{\mathbb{C}}) = \mathbb{C}(t)$.
- For f on the Riemann sphere, $\deg \operatorname{div} f = 0$.

Proposition .0.3

$\mathcal{A}_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}(j)$.

Recall that $j : \mathcal{H} \to \widehat{\mathbb{C}}$ is given by $j \coloneqq \frac{1728 g_2^3}{\Delta}$

Proof. Suppose $f \in \mathcal{A}_0(\mathrm{SL}_2(\mathbb{Z}))$. Then f has zeroes $z_1, \dots, z_n$ and poles $p_1, \dots, p_m$ in a fundamnetal domain for $\mathcal{H}$ (which we can think of as $X(1) \setminus \{\infty\}$). We can define

$$g(\tau) = \frac{\prod_i j(\tau) - j(z_i)}{\prod_j j(\tau) - j(p_i)}.$$

Then g has the same zeroes and poles as f in $\mathcal{H}$, because j is holomorphic on $\mathcal{H}$ with a pole at ∞. This implies f/g is holomorphic on $\mathcal{H}$, so it must be holomorphic on $X(1)$ as it will have the same behavior at ∞. Thus it will be constant!

Exercise .0.1

If $\mathcal{A}_k(\Gamma)$ is nonempty containing some f, then

$$\mathcal{A}_k(\Gamma) = \mathbb{C}(X(\Gamma))f.$$

Furthermore $j' \in \mathcal{A}_2(\Gamma)$, hence $\mathcal{A}_k(\Gamma)$ for k even is nonempty.