Consider the self-isogenies, we know that $\mathbb{Z} \subseteq \mathrm{Isog}(E, E)$.

Complex Multiplication curves are those such that $\mathbb{Z} \subsetneq \mathrm{Isog}(E, E)$, and in this case we will have that $\mathrm{Isog}(E, E) \subseteq \mathcal{O}_K$, where $K$ is a quadratic imaginary number field.

## .1. Modular Curves

These are Moduli spaces of elliptic curves.

**Definition .1.1**

If $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, then $Y(\Gamma) = \mathcal{H}/\Gamma$, which we call the modular curve for $\Gamma$.

**Exercise .1.1**

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then elliptic curves up to isomorphism are in bijection with $Y(\Gamma)$.

Namely $\tau \in \mathcal{H} \mapsto \mathbb{C}/(\tau\mathbb{Z} + \mathbb{Z})$

**Example .1.2**

$\Gamma_0(N), \Gamma_1(N), \Gamma(N)$ are congruence subgroups, and

$$Y_0(N) \cong \{(E, C) \mid E \text{ is an elliptic curve}, C \subseteq E[N], E \text{ cyclic of order } N\}/\sim$$
$$Y_1(N) \cong \{(E, Q) \mid Q \text{ is a point of order } N\}/\sim .Y(N) \qquad \cong \{(E, (P, Q)) \mid P, Q \text{ generate } E[N], \langle P, Q \rangle$$

where $\langle P, Q \rangle$ is the Weil pairing (see book/homework). There are of course maps $Y_1(N) \to Y_0(N) \to Y(N) \to Y(\mathrm{SL}_2(\mathbb{Z}))$.

We have a map $\Delta : \mathcal{H} \to \mathbb{C}$ called the modular discriminant defined by $\Delta = g_2^3 - 27g_3^3, g_2 = 60G_4, g_3 = 140G_6$. We also may consider

$$j : \mathcal{H} \to \mathbb{C}$$
$$j = \frac{1728g_2^3}{\Delta}$$

which is weight zero and holomorphic on $\mathcal{H}$ but not at $\infty$. We can actually think of $j$ as $j : \{E\}/\sim \to \mathbb{C}$ which is an invariant on elliptic curves, called the $j$-invariant. The modularity theorem will concern

- Elliptic curves $E$ where $j(E) \in \mathbb{Q}$
- CM elliptic curves imply $j(E)$ is algebraic.

As some examples, $j(i) = 1728, j(\mu_3) = 0$, $\mu_N := e^{2\pi i/N}$. We also can consider moonshine theory–concerning the coefficients of $j$ and the monster group.

Modular curves can be viewed as Riemann surfaces

- Give $Y(\gamma)$ a manifold structure
- Compactify $Y(\Gamma) \subseteq X(\Gamma)$.

We have a map $\pi : \mathcal{H} \to Y(\Gamma)$, and we give $Y(\Gamma)$ the quotient topology. How do we show $Y(\Gamma)$ is Hausdorff?

**Proposition .1.1**

If $\tau_1, \tau_2 \in \mathcal{H}$, then there exists neighborhoods $U_i$ containing $\tau_i$ such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\gamma(U_1) \cap U_2 \neq \emptyset$ implies $\gamma(\tau_1) = \tau_2$.

*Proof.* Choose any $U_1', U_2'$ containing $\tau_1, \tau_2$ with compact closure. First we need a claim.

**Claim**

$\gamma(U_1') \cap U_2' \neq \emptyset$ for finitely many $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Well we know $\mathcal{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})$. Take a section $S : x + yi \mapsto \frac{1}{\sqrt{y}}\left[\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right]$. Then

$$e_1, e_2 \in \mathcal{H}, \gamma(e_1) = e_2 \iff \gamma \in S(e_1)\,\mathrm{SO}_2(\mathbb{R}) S(e_2)^{-1}.$$

If we let $e_1, e_2$ range over $\overline{U}_1, \overline{U}_2$, then $\gamma$ lies in a compact subset of $\mathrm{SL}_2(\mathbb{R})$

Thus the number of such $\gamma$ is finite since $\mathrm{SL}_2(\mathbb{Z})$ is discrete. $F$ is the finite set of such $\gamma$, for each $\gamma \in F$, choose disjoint $U_{1,\gamma}, U_{2,\gamma}$ containing $\gamma(\tau_1), \tau_2$ respectively. Then

$$U_1 = U_1' \cap \bigcap_\gamma \gamma^{-1}(U_{1,\gamma}) \qquad U_2 = U_2' \cap \bigcap_\gamma U_{2,\gamma}$$

**Corollary .1.2**

$Y(\Gamma)$ is Hausdorff

We now want to construct charts, tha tis for each $\pi(\tau) \in Y(\Gamma)$, we want $\widetilde{U} \subseteq Y(\Gamma)$, a homeomorphism $\varphi : \widetilde{U} \to V \subseteq \mathbb{C}$ onto $V$ open, and we want holomorphic transition maps.

The $Y(\Gamma)$ are in fact "ramified covers." If $\tau$ is only fided by $\Gamma \cap \{\pm I\}$ then take a small neighborhood $U$ of $\tau$, then $\pi : U \to \widetilde{U}$ is a homeomorphism.

**Definition .1.2**

Let $\Gamma$ be a congruence subgroup. We say $\tau$ is <u>elliptic</u> in $\Gamma$ if $\mathrm{Stab}_\Gamma(\tau) \supsetneq \{\pm I\}$.

Fact: For each $\tau$, $\Gamma_\tau$ is finite cyclic (of order 1,2,3,4,6).

**Definition .1.3**

$h_\tau = |\Gamma_\tau / (\Gamma \cap \{\pm I\})|$

We may then choose $U \subseteq \mathcal{H}$ such that $\gamma(U) \cap U \neq \emptyset$ implies $\gamma \in \Gamma_\tau$. We also know elliptic points are discrete. Then $U \xrightarrow{\psi = \rho \circ \delta} \widehat{\mathbb{C}}$ where $\rho(z) = z^{h_\tau}$, and

$$\delta : z \mapsto \begin{bmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{bmatrix} z$$

where $\delta(\tau) = 0, \delta(\overline{\tau}) = \infty$. This will induce a map $\varphi : \pi(U) \to \widehat{\mathbb{C}}$ giving us a chart.