I. Eisenstein Series

For now, define $\mathcal{E}_k(\Gamma) \coloneqq \mathcal{M}_k(\Gamma)/S_k(\Gamma)$ as the eisenstein space, eventually we'll make $\mathcal{E}_k(\Gamma)$ as a subspace of $\mathcal{M}_k(\Gamma)$, but that comes later.

Goal is to study $\mathcal{E}_k(\Gamma(N)), \mathcal{E}_k(\Gamma_1(N)), \mathcal{E}_k(\Gamma_0(N))$.

Recall I.0.1

For $k \geq 4$ even we defined $G_k(\tau) = \sum'_{(c,d)\in\mathbb{Z}^2} \frac{1}{(c\tau+d)^k}$. We then define $E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)}$.

Then

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k}.$$

The book defines

$$P_+ \coloneqq \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z} \right\} \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

Recall the structure of SL_2 given as

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \qquad U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

We will also have

$$E_k(\tau) = \frac{1}{2} \sum_{\gamma \in P_+ \setminus \mathrm{SL}_2(\mathbb{Z})} j(\gamma,\tau)^{-k}.$$

Exercise I.0.2

Check the above!

The adeles $\mathbb{A}$ are $\prod'_p \mathbb{Q}_p \times \mathbb{R}$ where for almost all p we have the p-coordinate lies in $\mathbb{Z}_p$ (what $\prod'$ means). Modular forms will later be related to automorphic representations $\mathrm{SL}_2(\mathbb{A})$.

We have by combining our earlier dimension formulas for $M_k(\Gamma(N))$ and $S_k(\Gamma(N))$.

$$\mathcal{E}_k(\gamma) = \begin{cases} \mathcal{E}_\infty & \text{if } k \geq 4 \text{ even} \\ \mathcal{E}_\infty^{reg} & \text{if } k \geq 3 \text{ odd } -I \notin \Gamma \\ \mathcal{E}_\infty - 1 & \text{if } k = 2 \\ \mathcal{E}_\infty^{reg}/2 & \text{if } k = 1, -I \notin \Gamma 1 \qquad\qquad \text{if } k = 0 \\ 0 & \text{if } k < 0, k > 0 \text{ odd } -I \in \Gamma. \end{cases}$$

Consider $\overline{v} \in (\mathbb{Z}/N\mathbb{Z})^2$ of order N. Let

$$\delta = \begin{pmatrix} a & b \\ c_v & d_v \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

where $(c_v, d_v) = \overline{v}$, then $\epsilon_N = 1/2$ if $N = 1, 2$ and 1 otherwise. Then we can consider

$$E_k^{\overline{v}}(\tau) = \epsilon_N \sum_{\gamma \in (P_+ \cap \Gamma(N)) \setminus \Gamma(N)\delta} j(\gamma,\tau)^{-k}$$

Proposition I.0.1

For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have $E_k^{\overline{v}}[\alpha]_k(\tau) = E_k^{\overline{v}\gamma}(\tau)$

Corollary I.0.2

$E_k^{\overline{v}}(\tau)$ is weight k $\Gamma(N)$ invariant, since then we essentially have $\overline{v} = \overline{v}\gamma$.

Proof. Ignore ϵ_N for convenience

$$E_k^{\overline{v}}[\gamma]_k(\tau) = j(\gamma,\tau)^{-k} \sum_{\gamma'} j(\gamma', \gamma(\tau))^{-k}.$$

Recall that $j(\gamma,\tau)j(\gamma',\gamma(\tau)) = j(\gamma'\gamma,\tau)$. Then

$$E_k^{\overline{v}}[\gamma]_k(\tau) = \sum_{\gamma'} j(\gamma'\gamma,\tau) = \sum_{\gamma'' \in (P_+ \cap \Gamma(N)) \backslash \Gamma(N)\delta\gamma} j(\gamma'',\tau) = E_k^{\overline{v}\gamma}(\tau).$$

One can prove holomorphicity of these things. But doing so is painful.

FACT: $E_k^{\overline{v}}(\tau)$ is weight k, $\Gamma(N)$ modular form for $k \geq 3$. We may also define for $\Gamma(N) \subseteq \Gamma$ the form

$$E_{k,\Gamma}^{\overline{v}}(\tau) = \sum_{\gamma_j \in \Gamma(N) \backslash \Gamma} E_k^{\overline{v}}[\gamma_j](\tau) \in \mathcal{M}_k(\Gamma).$$

For $N > 2$ and k even, one may calculate that $E_k\overline{v}$ is nonvanishing at $-d_v/c_v$ and vanishes at all other cusps.

Hence if we pick $\overline{v}$ which represents each cusp of $\Gamma(N)$, then $\{E_k^{\overline{v}}\}$ are linearly independent. The size of this is exactly the number of cusps $\mathcal{E}_\infty$!!! Wait this means it's a basis.

I.1. Dirichlet Characters

Definition I.1.1

A dirichlet character is a group homomorphism $\chi : G_N \coloneqq (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

The dirichlet characters themselves form a group $\chi_1\chi_2(m) = \chi_1(m)\chi_2(m)$. We'll call this $\widehat{G}_N$. Then $\widehat{G}_N \cong G_N$ in a non-canonical way.

We have

$$(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_{\substack{p^k \mid N \\ p^{k+1} \nmid N}} \left(\mathbb{Z}/p^k\mathbb{Z}\right)^\times$$

and the right hand side is cyclic for $p \neq 2$, and

$$(\mathbb{Z}/2^k\mathbb{Z})^\times = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{k-2}.$$

Lifting: If $d \mid N$, then there is a map $G_N \twoheadrightarrow G_d$, and so there is a map $\widehat{G}_d \hookrightarrow \widehat{G}_N$.

Definition I.1.2

We define the conductor of $\chi \in \widehat{G}_N$ to be the smallest d so that χ comes from $\widehat{G}_d$. We denote this by $\mathrm{Cond}(\chi)$. If $\mathrm{Cond}(\chi) = N$, then χ is called primitive.

Given $\chi \in \widehat{G}_N$, we may extend to $\chi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by sending everything not in $(\mathbb{Z}/N\mathbb{Z})^\times$ to zero. Likewise we get a map $\chi : \mathbb{Z} \to \mathbb{C}$ by sending $\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$.

Something that shows up a lot is a sum of the form $g(\chi) = \sum_{n=0}^{N-1} \chi(n)\mu_N^n$ where $\mu_N = e^{2\pi i N}$. One can thinkof this as a Fourier transform if we squint our eyes a bit (sums to integrals and such).

Application: We remember how $\Gamma_1(N)$ lies in $\Gamma_0(N)$ as

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}_{\bmod N} \qquad\qquad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}_{\bmod N}$$

and so $\Gamma_0(N)/\Gamma_1(N)$ is in fact $(\mathbb{Z}/N\mathbb{Z})^\times$. We define

$$M_k(N, \chi) \coloneqq \{f \in M_k(\Gamma_1(N)) \mid f[\gamma]_k = \chi(d_\gamma)f, \gamma \in \Gamma_0(N)\}.$$

We call these <u>modular forms of weight k of level N with Nebentypus character χ</u>. Then $M_k(N, 1) = M_k(\Gamma_0(N))$.

Fact:

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(N, \chi).$$

Why? Finite dimensional representation theory! Look at the action of $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ on the left hand side, which is a finite dimensional complex vector space. The irreducible representations are those things acting by χ, and so we take the "eigenspaces" of these to get the break up. Note the eigenspaces will often have multiplicity and not be irreducible themselves.

Recall orthogonality from representation theory as well, that is for fixed χ we have

$$\sum_{n \in G_N} \chi(n) = \begin{cases} \phi(N) & \text{if } \chi = 1 \\ 0 & \text{otherwise} \end{cases}$$

where ϕ is Euler's totient function, and for fixed n we have

$$\sum_{\chi \in \widehat{G}_N} \chi(n) = \begin{cases} \phi(N) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} .$$

This is the general fact from group theory that

$$\frac{1}{\phi(N)} \sum_{n \in G_N} \chi_1(n)\overline{\chi_2(n)} = \begin{cases} 1 & \text{if } \chi_1 = \chi_2 \\ 0 & \text{otherwise} \end{cases} .$$