Definition .0.1

We say $f \in \mathcal{M}_k(\Gamma_1(N))$ is an eigenform if it is an eigenvector for all $\langle n \rangle, T_n$.

We say it is normalized if $a_1(f)$. A newform is an eigenform in $S_k(\Gamma_1(N))^{\text{new}}$.

The eigenvalues for diamond operators $\langle n \rangle$ will just be $\chi(n)$ where $f \in \mathcal{M}_k(N, \chi)$. What about for T_n? Recall the formula is

$$a_m(T_n f) = \sum_{d \mid (m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f).$$

Setting $m = 1$ yields

$$a_1(T_n f) = \chi(1) a_n(f) = a_n(f).$$

thus the eigenvalue is $a_n(f)/a_1(f)$.

Proposition .0.1

For $f \in S_k(\Gamma_1(N))^{\text{new}}$, an eigenvector for $\{\langle n \rangle, T_n \mid (n, N) = 1\}$ is an eigenform.

Proof. All we have to check are the T_n.

Claim

For $f \in S_k(\Gamma_1(N))^{\text{new}}$, we have $a_1(f) \neq 0$.

If not, then we know for $(n, N) = 1$, we have

$$a_n(f) = a_1(T_n f) = c_n a_1(f) = 0.$$

The main lemma then would tell us $f \in S_k(\Gamma_1(N))^{\text{old}}$ because $a_n(f) \neq 0$ whenever f is a newform and $(n, N) = 1$.

Without loss of generality, assume $a_1(f) = 1$. Let $m \in \mathbb{Z}^+$, and consider $g_m = T_m f - a_m(f) f$. Then g_m is still an eigenform away from N (that is for $T_n, (n, N) = 1$). Furthermore $a_1(g_m) = 0$. Thus g_m is an oldform and a newform. Thus $g_m = 0$, so $T_m f = a_m(f) f$.

Corollary .0.2 (Multiplicity 1)

If f, f' have the same T_m eigenvalues then $f' = cf$.

Proof. The eigenvalues are the coefficients upon normalization!

Theorem .0.3

We have

$$B_k(N) := \{f(n\tau) \mid f \text{ is a newform of level } M, nM \mid N\}$$

is a basis for $S_k(\Gamma_1(N))$.

Proof. We look at

$$S_k(\Gamma_1(N)) = S_k(\Gamma_1)^{\text{new}} \oplus \sum_{p \mid N} \iota_p(S_k(\Gamma_1(Np^{-1})))^2.$$

Spanning happens via induction.

Linear independence. Choose minimal linear combination

$$\sum_{i,j} c_{i,j} f_i(n_{i,j}\tau) = 0$$

where $f_i \in S_k(M_i, \chi_j)$. We can in fact require that all the χ_i lift to the same χ. Namely we can do this by applying $\langle d\rangle - \widetilde{\chi}_i(d)$ for some d with $\widetilde{\chi}_i(d) \neq \widetilde{\chi}_j(d)$ to get a nontrivial relation with fewer terms.

By applying $T_p - a_p(f_i)$ we can require all fourier coefficients away from N to agree, as otherwise we'd have a nontrivial relation with fewer terms.

Strong Multiplicity One implies the f_i must be the same, and then we're actually done.

Proposition .0.4

Let $f \in \mathcal{M}_k(N, \chi)$. Then f is a normalized eigenform if and only if the Fourier coefficients satisfy

(1) $a_1(f) = 1$
(2) $a_{p^r}(f) = a_p(f)a_{p^{r-1}}(f) - \chi(p)p^{k-1}a_{p^{r-2}}(f)$.
(3) $a_{mn}(f) = a_m(f)a_n(f)$ for m, n coprime.

Proof. The forward direction is a bunch of computation. For the converse, we need to show

$$a_m(T_p f) = a_p(f)a_m(f)$$

for all p, m. If $p \nmid M$ then

$$a_m(T_p f) = a_{pm}(f) = a_p(f)a_m(f).$$

If $p \mid m$, write $m = p^r m'$ for $p \nmid m'$, then

$$a_m(T_p f) = a_{p^{r+1}m'}(f) + \chi(p)p^{k-1}a_{m'p^{r-1}}(f)$$

via the formula. Then

$$\begin{aligned} a_m(T_p f) &= a_{m'}(f)\left[a_{p^{r+1}}(f) + \chi(p)p^{k-1}a_{p^{r-1}}(f)\right] \\ &= a_{m'}(f)a_p(f)a_{p^r}(f). \\ &= a_p(f)a_m(f). \end{aligned}$$

Fact: $E_k^{\psi,\varphi}$ satisfy this. You just write down the Fourier coefficients...

.1. Connection with L-functions

Let $f \in \mathcal{M}_k(\Gamma_1(\mathbb{N}))$. We may define for a complex variable $s \in \mathbb{C}$

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}.$$

The convergence of $L(s, f)$ in a half place will be given by estimating the Fourier coefficients. Namely it converges if $\mathrm{Re}(s) > k$, and if it is a cuspform then it converges if $\mathrm{Re}(s) > \frac{k}{2} + 1$.

Theorem .1.1

The following are equivalent

- f is a normalized eigenform
- We have a product as

$$L(s,f) = \prod_p \left(1 - a_p p^{-s} + \chi(p) p^{k-1-2s}\right)^{-1}.$$

Proof. Being a normalized eigenform is equivalent to conditions (1),(2),(3) from before.

Exercise .1.1

Let

$$X = \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}},$$

then X is the p-part of the Euler product.

Idea: Plug in condition (2) for $r \geq 2$, and find an equation X must satisfy. Doing this in reverse shows relation (2) if we have the Euler product.

Taking $s \to +\infty$ yields $L(s,f) = 1$ if and only if $a_1(f) = 1$.

Fact: Let g be a function on prime powers. Then

$$\prod_p \sum_{r=0} g(p^r) = \sum_{n=1}^{\infty} \prod_{p^r || n} g(p^r).$$

Assuming (1),(2),(3) We then write

$$\begin{aligned}
L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} &= \sum_{n=1}^{\infty} \left(\prod_{p^r | a_{p^n}} n^{-s} \right) \\
&= \sum_{n=1}^{\infty} \prod_{p^r | n} \frac{a_{p^r}}{p^{rs}} \\
&= \prod_p \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}} \\
&= \prod_p (1 - a_p p^{-s} + \chi(p) p^{k-1-2s})^{-1}.
\end{aligned}$$

Running the equalities backwards gives essentially the converse.

Now we'll look at functional equations. Let $f = \sum a_n q^n \in S_k(\Gamma_1(\mathbb{N}))$. Recall that the Mellin transform of some function ϕ is defined to be

$$\psi(s) = \int_{t=0}^{\infty} \phi(it) t^s \frac{\mathrm{d}t}{t}.$$

Proposition .1.2

The Mellin transform of $f = \sum a_n q^n$ is $\frac{1}{(2\pi)^s} \Gamma(s) L(s,f)$.

Well we see that

$$\begin{aligned} g(s) &= \int_{t=0}^{\infty} \sum_n a_n e^{-2\pi n t} t^s \frac{\mathrm{d}t}{t} \\ &= \sum_n a_n \int_{t=0}^{\infty} e^{-2\pi n t} t^s \frac{\mathrm{d}t}{t} \\ &= \sum_n \frac{a_n}{(2\pi)^s n^s} \Gamma(s) \\ &= \frac{1}{(2\pi)^s} \Gamma(s) L(s, f). \end{aligned}$$

via change of variables.

Definition .1.1

Let $\Gamma_N = \frac{N^{s/2}}{(2\pi)^s}\Gamma(s)L(s,f)$. Then define the operator W_N as

$$f \mapsto i^k N^{1-k/2} f \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix} \right]_k .$$

This is in fact an involution and so has eigenvalues ± 1.

Theorem .1.3

If $f \in S_k(\Gamma_1(N))^{\pm}$ (eigenspaces for W_N) then $\Gamma_N(s) = \pm \Gamma_N(k-s)$.

This implies that $L(s, f)$ has an analytic continuation just as for the Riemann zeta function.