.1. Abelian Varieties and Modularity

Fix $f \in S_2(\Gamma_1(N))$ a newform of level N, then $\lambda_f : \mathbb{T}_{\mathbb{Z}} \to \mathbb{C}$ was defined last time as an evaluation map (for the eigenvalue), $I_f = \ker \lambda_f$, and we now define

$$A_f \coloneqq J_1(N)/I_f J_1(N).$$

Note I_f, A_f only depend on the Galois orbit of f (in the sense discussed last time).

Well we know $\mathbb{T}_{\mathbb{Z}}/I_f$ acts on A_f, and we can look at this as a diagram

$$\begin{tikzcd} J_1(N) \arrow[r, "T_p"] \arrow[d] & J_1(N) \arrow[d] \\ A_f \arrow[r, "a_p"'] & A_f. \end{tikzcd}$$

Namely we have that

$$(a_p \cdot \varphi)(f^\sigma) = \varphi(a_p(f^\sigma) f^\sigma)$$

for $\varphi \in A_f$, and $a_p(f^\sigma)$ is the p-th Fourier coefficient of f^σ. We wish to study A_f. We say $f_1 \sim f_2$ if there is a Galois action σ for which $f_1 = f_2^\sigma$. The equivalence class of f is denoted $[f]$. We now define

$$V_f \coloneqq \operatorname{span}(\{g \in [f]\}) \subseteq S_2(N).$$

We have since the galois orbits are linearly independent (easy check) that

$$\dim V_f = [K_f : \mathbb{Q}].$$

Now define

$$\Lambda_f = H_1(X(N), \mathbb{Z})\big|_{V_f} \subseteq V_f^*.$$

Proposition .1.1

$A_f \cong V_f^*/\Lambda_f$. Furthermore, this right hand side is a complex torus of dimension $[K_f : \mathbb{C}]$.

Proof. Condense notation as $S_2 = S_2(\Gamma_1(N)), H_1 = H_1(X_1(N), \mathbb{Z})$. Then by definition

$$\begin{aligned} A_f &= \frac{J_1(N)}{I_f J_1(N)} = \frac{S_2^*/H_1}{I_f(S_2^*/H_1)} \\ &\cong \frac{S_2^*}{I_f S_2^* + H_1} \cong \frac{S_2^*/I_f S_2^*}{\text{image of } H_1 \text{ in } S_2^*/I_f S_2^*}. \end{aligned}$$

Last time we had that this (on top) is the dual of the annihilator, $S_2[I_f]^*$

$$A_f \cong \frac{S_2[I_f]^*}{H_1\big|_{S_2[I_f]}}.$$

We will show that $V_f = S_2[I_f]$, and then the result follows. We will also show Λ_f is actually a lattice.

(1) We know $V_f \subseteq S_2[I_f]$. We need to know this is an equality. The strategy is just to compute the dimension of $S_2[I_f]$. Well

$$\dim(S_2[I_f]) = \dim(S_2[I_f]^*) = \dim(S_2^*/I_f S_2^*).$$

Then we have a pairing

$$\begin{aligned} \mathbb{T}_{\mathbb{C}} \times S_2 &\to \mathbb{C} \\ (T, g) &\mapsto a_1(Tg). \end{aligned}$$

Then we get $\mathbb{T}_{\mathbb{C}} \to S_2^*$. We claim the pairing is bilinear, non-degenerate.

- Bilinearity is easy.
- If $g \in S_2$, and $(T, g) = 0$ for all $\mathbb{T}_{\mathbb{C}}$, then $(T_n, g) = a_1(T_n g) = a_n(g)$, so $g = 0$.
- If $T \in \mathbb{T}_{\mathbb{C}}$ and $(T, g) = 0$ for all $g \in S_2$. But then we see that

$$a_n(Tg) = a_1(T_n T g) = a_1(T T_n g) = 0.$$

  Thus $Tg = 0$ for all g, so $T = 0$.

This shows an isomorphism $\mathbb{T}_{\mathbb{C}} \to S_2^*$. Thus

$$\dim(S_2[I_f]) = \dim(S_2^*/I_f S_2^*) = \dim(\mathbb{T}_{\mathbb{C}}/I_f \mathbb{T}_{\mathbb{C}}).$$

And in fact, since $\mathbb{T}_{\mathbb{Z}} \otimes \mathbb{C}$ surjects onto $\mathbb{T}_{\mathbb{C}}$ we have

$$\dim(\mathbb{T}_{\mathbb{C}}/I_f \mathbb{T}_{\mathbb{C}}) \leq \dim\left(\frac{\mathbb{T}_{\mathbb{Z}} \otimes \mathbb{C}}{I_f \otimes \mathbb{C}}\right) = \dim\left(\frac{\mathbb{T}_{\mathbb{Z}}}{I_f} \otimes \mathbb{C}\right) = \operatorname{rank}(\mathbb{T}_{\mathbb{Z}}/I_f)$$

The second to last equality follows because $\mathbb{C}$ is free over $\mathbb{Z}$, and $\mathbb{Z}$ is a PID, so tensor product by $\mathbb{C}$ is exact. We finally claim

$$\operatorname{rank}(\mathbb{T}_{\mathbb{Z}}/I_f) = [K_f : \mathbb{Q}].$$

because $\lambda_f : \mathbb{T}_{\mathbb{Z}} \to \mathbb{C}$ provides an isomorphism of $\mathbb{T}_{\mathbb{Z}}/I_f$ with the $\mathbb{Z}$-module generated by the coefficients of f in $\mathbb{C}$.

This in fact gives equality of the dimensions so $V_f = S_2[I_f]$. Further we get a nice fact that

$$\frac{\mathbb{T}_{\mathbb{Z}} \otimes \mathbb{C}}{I_f \otimes \mathbb{C}} \to \frac{\mathbb{T}_{\mathbb{C}}}{I_f \mathbb{T}_{\mathbb{C}}}$$

is an isomorphism!

(2) Showing that Λ_f is a lattice is a big computation like this that we will not do.

Clarification for people

$$\begin{aligned} \mathbb{T}_{\mathbb{Z}} &= \mathbb{Z}\{T_n, \langle n \rangle\} \subseteq \operatorname{End}(S_2(\Gamma_1(N))) \\ \mathbb{T}_{\mathbb{C}} &= \mathbb{C}\{T_n, \langle n \rangle\} \subseteq \operatorname{End}(S_2(\Gamma_1(N))), \end{aligned}$$

but in fact $\mathbb{T}_{\mathbb{Z}} \otimes \mathbb{C} \neq \mathbb{T}_{\mathbb{C}}$. Not actually true... but one can imagine T_2 scales by 3, and T_3 scales by $\sqrt{2}$ and everything else is zero. Then we would have $\mathbb{T}_{\mathbb{Z}} \otimes \mathbb{C} \cong \mathbb{Z}^2 \otimes \mathbb{C} = \mathbb{C}^2$, and $\mathbb{T}_{\mathbb{C}} \cong \mathbb{C}$.

We do have a surjection

$$\mathbb{T}_{\mathbb{Z}} \otimes \mathbb{C} \twoheadrightarrow \mathbb{T}_{\mathbb{C}}$$

as mentioned in the proof above.

Then $J_1(N)/I_f J_1(N) = A_f \cong V_f^*/\Lambda_f$ is a torus as desired.

Theorem .1.2

There is an isogeny (surjective homomorphism with finite kernel)

$$J_1(N) \to \bigoplus_{f,\ \text{level}\ N_f} A_f^{m_f}$$

where m_f is the number of divisors of N/N_f.

Proof. Must use the basis for $S_2(\Gamma_1(N))$. These were $f(n\tau)$ were f is a newform of some level and $n \mid N/N_f$. We rewrite the basis of $S_2(\Gamma_1(N))$ as

$$B_2(N) = \coprod_{[f]} \coprod_{n \mid N/N_f} \coprod_{\sigma} \{f^\sigma(n\tau)\}$$

We then define a map

$$\begin{aligned} \Psi_{f,n} : S_2(\Gamma_1(N))^* &\to V_f^* \\ \varphi &\mapsto \psi \end{aligned}$$

such that

$$\psi\left(\sum_{j=1}^d z_j f^{\sigma_j}(\tau)\right) = \sum_{j=1}^d z_j n \varphi(f^{\sigma_j}(n\tau)).$$

We then claim that

Claim

$\Psi_{f,n}$ takes $H_1(X_1(N), \mathbb{Z})$ to $\Lambda_f = H_1(X_1(N_f), \mathbb{Z})\big|_{V_f}$.

Let $\varphi = \int_\alpha$, where α is a loop. Then

$$\psi(f^\sigma(\tau)) = n \int_\alpha f^\sigma(n\tau)\,\mathrm{d}\tau = \int_{\widetilde{\alpha}} f^\sigma(\tau)\,\mathrm{d}\tau.$$

where $\widetilde{\alpha} = n\alpha$. One can show that $\widetilde{\alpha}$ is a lift of a loop in $X_1(N_f)$.

We then obtain

$$\Psi = \prod_{f,n} \psi_{f,n} : S_2(\Gamma_1(N))^* \to \bigoplus_{f,n} V_f^* = \bigoplus_f (V_f^*)^{m_f}.$$

By the claim, this descends to a map

$$\overline{\Psi} : J_1(N) \to \bigoplus_f A_f^{m_f}.$$

We now must show $\overline{\psi}$ is an isogeny. We'll start with surjectivity. If φ is the dual vector of $f^\sigma(n\tau)$ then $\psi_{f,n}(\varphi)$ sends $f^\sigma(\tau)$ to n and everything else to 0, and $\psi_{g,n}(\varphi)$ is zero.

This makes up the basis that we'd like to have! To prove the finite kernel property, we need to show the image of H_1 in Λ_f under $\psi_{f,n}$ has the same rank as Λ_f.

This is a computation that is not too difficult.

This will allow us to state the modularity theorem in better terms, namely the surjection $J_1(N) \twoheadrightarrow E$ of the modularity theorem will be a specific map $AF_{\twoheadrightarrow}E$ for a specific newform!

Note: We've done everything for $\Gamma_1(N)$, we could do everything for $\Gamma_0(N)$. Note $X_1(N)$ surjects onto $X_0(N)$, and so indeed what we've done is precisely stronger. If we define

$$A'_f = J_0(N_f)/I_f J_0(N_f) \cong (V'_f)^*/\Lambda'_f,$$

and we get a map

$$J_0(N) \to \bigoplus_f (A'_f)^{m_f}.$$

The modularity theorem is then stated as

Theorem .1.3 (Modularity Theorem)

If E is an elliptic curve with $j(E) \in \mathbb{Q}$ then there exists an N and $f \in S_2(\Gamma_0(N))$ with a surjection $A'_f \twoheadrightarrow E$.