**Remark .0.1**

If $f$ is as in **??** then $f/\mathbb{Q}$. Why? Well $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then

$$a_p(f^\sigma) = a_p(f)^\sigma$$

for almost all primes $a_p(f) = a_p(E) = \in \mathbb{Z}$ so $a_p(f) = a_p(f^\sigma)$. Strong Multiplicity one would then imply that $f = f^\sigma$

How do we relate the versions of modularity. Well we look for a map

$$X_{\mathbb{Q}}\text{-Mod} \to a_p - \text{-Mod}.$$

Well recall we had $\dim A'_f = [K_f : \mathbb{Q}]$ but since $f/\mathbb{Q}$ in the situation above, $A'_f$ is an elliptic curve (abelian variety of dimension one).

Have: $X_0(N) \to \mathrm{Pic}^0(X_0(N)) \to A'_f$. Then we can apply **??** to this setup. Then there's a $g$ with $a_p(g) = a_p(A'_f)$ (except at divisions), and you end up with $g = f$ in the proof. Why? Well the idea is the $a_p(f) - a_p(E)$ portion above, and applying strong multiplicity one.

Thus $a_p(f) = a_p(A'_f)$ for almost all $p$, when $f/\mathbb{Q}$.

**Theorem .0.1** (Carayol)

*$a_p(f) = a_p(A'_f)$ for all $p$.*

We then have $A'_f \twoheadrightarrow E$. Then it turns out $A'_f \cong E$ and $a_p(f) = a_p(A'_f) = a_p(E)$ for all $p$.

## .1. Some $L$-function stuff

Recall for a newform $f$ we defined $L(s, f) := \sum_{n=1}^{\infty} a_n(f)n^{-s}$. We were able to show that

$$L(s, f) = \prod_p \left(1 - a_p(f)p^{-s} + 1_N(p)p^{1-2s}\right)^{-1}$$

where $1_N(p)$ detects if $p \mid N$ where $f$ has level $N$. We can also define

$$t_{p^e} = p^e + 1 - \left|\widetilde{E}(\mathbb{F}_{p^e})\right|$$

Then we can define a local zeta function

$$Z_p(X, E) = \prod_{e=1}^{\infty} \exp\left(\frac{t_{p^e}(E)}{e} x^e\right).$$

One can show

$$Z_p(p^{-s}, E) = (1 - a_p(E)p^{-s} + 1_E(p)p^{1-2s})^{-1},$$

where $1_E$ is 1 if good reduction and 0 if bad reduction. This clearly depends on reduction type, and,

$$Z_p(p^{-s}, E) = \begin{cases} (1 - a_p(E)p^{-s} + p^{1-2s})^{-1} & \text{if good} \\ (1 - p^{-s})^{-1} & \text{if split} \\ (1 + p^{-s})^{-1} & \text{if non-split} \\ 1 & \text{if additive} \end{cases}.$$

Define

$$L(s,E) = \prod_p (1 - a_p(E)p^{-s} + 1_E(p)p^{1-2s}) = \sum_{n=1}^{\infty} \frac{a_n(E)}{n^s}.$$

Formally defined,$a_1(E), A_p(E) = p + 1 - |E(\mathbb{F}_p)|$. Furthermore

$$a_{p^e}(f) = a_p(E)a_{p^{e-1}}(E) - 1_E(p)pa_{p^{e-2}}(E).$$

Furthermore if $(m,n) = 1$ then $a_{mn}(E) = a_m(E)a_n(E)$.

**Theorem .1.1** (Modularity)

$L(s,f) = L(s,E)$. As a consequence $L(s,E)$ has a functional equation and an analytic continuation.

**Conjecture .1.2** (Birch-Swinnerton-Dyer)

$\operatorname{ord}_{s=1} L(s,E) = \operatorname{rank}(E/\mathbb{Q}) = r$ which is determined by $E(\mathbb{Q}) = \mathbb{Z}^r \oplus T$.

Then $L(s,E)$ converges when $\operatorname{Re}(s) > 2$. The functional equation determines $\operatorname{Re}(s) < 0$.

**Definition .1.1**

Let $K/\mathbb{Q}$ be an imaginary quadratic extension. An order $\mathcal{O} \subseteq \mathcal{O}_K$, $\operatorname{rank}_{\mathbb{Z}}(\mathcal{O}) = [K : \mathbb{Q}] = 2$.

In this simple case the orders are $\mathcal{O}_n = \mathbb{Z} + n\mathcal{O}_K$ where $n \in \mathbb{Z}_{\geq 1}$.

**Definition .1.2** (Heegner Point)

A Heegner point in $X_0(N)$ relative to $K$ is a pair $(E, C)$ such that $E, E/C$ have complex multiplication by the same order $\mathcal{O}$.

Then these will look like

$$\begin{aligned} x_n &\coloneqq (E = \mathbb{C}/\mathcal{O}_n, E' = E/C = \mathbb{C}/\mathcal{N}_n^{-1}) \\ &\mathcal{N} \subseteq \mathcal{O}_K, \mathcal{N}_n = \mathcal{N} \cap \mathcal{O}_n. \end{aligned}$$

with $\mathcal{O}_K/N \cong \mathbb{Z}/N\mathbb{Z}$, where inverse is taken with respect to the notion of fractional ideal.

The Heegner Hypothesis is that each $p \mid N$ splits in $K$, which implies there exist Heegner points in $X_0(N)$ for all $\mathcal{O}_N$. It turns out $x_n \in X_0(N)(H_n)$ where $H_n$ is a ring class field of $\mathcal{O}_n$.

This is a generalization of the Hilbertclass field, with Galois group $(\mathcal{O}_n/n\mathcal{O}_K)^\times / (\mathbb{Z}/N\mathbb{Z})^\times$.

Consider $E/\mathbb{Q}$ by modularity $X_0(N) \xrightarrow{\alpha} E$. Then we can consider the image this Heegner point $x_n \mapsto y_n \in E(H_n)$. We can then consider

$$\begin{aligned} \operatorname{tr}_n : E(H_{np}) &\to E(H_n) \\ z &\mapsto \sum_{\sigma \in \operatorname{Gal}(H_{np}/H_n)} \sigma(z). \end{aligned}$$

**Theorem .1.3**

$\operatorname{tr}_n(y_{np}) = a_p(E)y_n$.

*Proof.* We'll use Eichler-Shimura. We'll need the version where the composition

$$\operatorname{Pic}^0(X_0(N)) \xrightarrow{T_p - a_p(E)} \operatorname{Pic}^0(X_0(N)) \xrightarrow{\alpha} E$$

is zero. We might as well work in the picard group then! So we can look at

$$\begin{aligned} \mathrm{tr}_n(y_{np}) &= \mathrm{tr}(\alpha(x_{np})) = \alpha(\mathrm{tr}(x_{np})) \\ &= \alpha(T_p(x_n) = a_p(E)\alpha(x_n) = a_p(E)y_n. \end{aligned}$$

**Exercise .1.1**

Why is $\mathrm{tr}(x_{np}) = T_p(x_n)$? Idea: look at what we did for Hecke operators and Galois actions in the $X_1(N)$ moduli problem, and adapt a similar formula for $X_0(N)$.

Also probably understand $H_n$ better than I do (can't wait to learn class field theory one day).

Define

$$y_K \coloneqq \mathrm{tr}_{H_1/K}(y_1) \in E(K).$$

We need to say something about its height.

**Definition .1.3**

If $p \in E(K)$, we define the naive height as

$$h(p) \coloneqq \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} [K_v, \mathbb{Q}_v] \cdot \log \max(|x|_v, |y|_v, |z|_v),$$

where $M_K$ is all the places (absolute values in $K$)

We can also define the Neron-Tate Height

$$\widehat{h}_n(p) = \lim_{n \to \infty} \frac{h([2^n]p)}{4^n}.$$

This allows us to define a height pairing

$$\begin{aligned} \langle , \rangle &: E(K) \times E(K) \to \mathbb{R} \\ \langle P, Q \rangle &\coloneqq \frac{1}{2}\widehat{h}(P+Q) - \widehat{h}(P) - \widehat{h}(Q). \end{aligned}$$

It turns out that $\langle P, P \rangle = 0$ if and only if $P$ is torsion.

**Theorem .1.4** (Gross-Zagier)

If $E/\mathbb{Q}$ is an elliptic curve, $K$ is an imaginary quadratic field satisfying the Heegner Hypothesis. Then

$$L'(1, E_K) = c_{E,K} \cdot \langle y_K, y_K \rangle,$$

for some special number $c_{E,K}$ which is not terrible to write down.

Now write the analytic rank as $\mathrm{rk}_{an} = \mathrm{ord}_{s=1} L(s, E)$. The algebraic rank as $\mathrm{rk}_{alg} = \mathrm{rk}(E)$.

**Corollary .1.5**

$\mathrm{rk}_{an}(E_K) = 1$ then $\mathrm{rk}_{alg}(E_K) \geq 1$.

**Theorem .1.6** (Kolyvagin)

If $\operatorname{ord}(y_k) = \infty$, then $\operatorname{rk}_{alg}(E_K) = 1$.

This actually tells us that if $\operatorname{rk}_{an}(E_K) = 1$ implies $\operatorname{rk}_{alg}(E_K) = 1$.