I. Introduction and Motivation

Goals:

- Goals of the book: To explain the statement of the modularity theorem.
  - The book introduces many things: modular forms, elliptic curves, modular curves. These are all relevant to modern mathematics, and so are their generalizations, that is: automorphic forms/representations, abelian varieties, Shimura varieties.
  - The first is the $\mathrm{SL}_2(\mathbb{R})$ version, and the rest are the general G versions.
- Our Goal: Be able to think about these things both in specific and in general.

II. The Basics

II.1. Modular Forms

Definition II.1.1

The modular group is $\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a, b, c, d \in \mathbb{Z} \right\}$.

Exercise II.1.1

This group is generated by

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

We'll also think often of the upper half-plane $\mathcal{H} \subseteq \widehat{\mathbb{C}}$, which is the set $\{a + bi \mid b > 0\}$.

We know $\mathrm{SL}_2(\mathbb{R})$ acts on $\widehat{\mathbb{C}}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

Then $\mathcal{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})$.

Definition II.1.2

Let $k \in \mathbb{Z}$. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is called weaklymodular of weight k provided that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Example II.1.2

If in weight zero, this is $\mathrm{SL}_2(\mathbb{Z})$-invariant. Then $f : (\mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})) / \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}$.

Example II.1.3

Consider $\mathrm{d}\tau$. Then for $f(\tau)\,\mathrm{d}\tau$ to be invariant we need f to be weight two, as $\mathrm{d}(\gamma(\tau)) = (c + d\tau)^{-2}\,\mathrm{d}\tau$.

Definition II.1.3

A modular form $f : \mathcal{C} \to \mathbb{C}$ of weight k is

- weakly modular of weight k.

- holomorphic on $\mathcal{H}$.
- holomorphic at ∞.

Let D be the complex unit disk, $D' = D \setminus \{0\}$. Then $\tau \mapsto e^{2\pi i\tau}$ takes $\mathcal{H} \to D'$ and is $\mathbb{Z}$-periodic. Because $f(\tau) = f(\tau+1)$ for any weakly modular form, we know f factors through the map $\mathcal{H} \to D'$ as some $g : D' \to \mathbb{C}$. Saying f is holomorphic at ∞ is equivalent to saying that it extends holomorphically to D.

We reserve the letter $q = e^{2\pi i\tau}$. We know $g(q) = \sum_{n\in\mathbb{Z}} a_n q^n$ for $q \in D'$. Holomorphic at ∞ can also be understood as $a_n = 0$ for $n < 0$. Thus we have a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n.$$

Set $M_k(\mathrm{SL}_2(\mathbb{Z}))$ to be the weight k modular forms, then

Exercise II.1.4

Try this:

$$M(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_k M_k(\mathrm{SL}_2(\mathbb{Z})).$$

Actual Example: "Weight k Eisenstein series" for $k > 2$ even.

$$G_k(\tau) = \sum_{(c,d)}{}' \frac{1}{(c\tau + d)^k}.$$

where $\sum'_{(c,d)}$ means

$$\sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}}.$$

Exercise II.1.5

G_k is weakly modular of weight k.

Strategy: Write it out and then use that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the index set.

For holomorphicity use the fact that

$$\sum_{d\in\mathbb{Z}} \frac{1}{\tau + d} = \pi\cot(\pi\tau) = \pi i - 2\pi i \sum_{m\geq 0} q^m.$$

differentiating $k-1$ times gives

$$\sum_{d\in\mathbb{Z}} \frac{1}{(\tau + d)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m\geq 1} m^{k-1} q^m.$$

Then we have

$$\begin{aligned}\sum_{(c,d)}{}' \frac{1}{(c\tau + d)^k} &= \sum_{d>0} \frac{1}{d^k} + 2\sum_{c=1}^{\infty}\left(\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau + d)^k}\right) \\ &= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\end{aligned}$$