Something that could go wrong when reducing maps. Look at

$$h : \mathbb{P}^1 \to \mathbb{P}^1$$
$$[x : y] \mapsto [px : y].$$

Then $\widetilde{h}$ doesn't quite make sense, as it maps $[1 : 0]$ to $[0 : 0]$ (which is not in $\mathbb{P}^1_p$

Corollary .0.1

Suppose C, C' are nonsingular and projective with good reduction at p and $g(C') > 0$.

(a) If h is surjective, then $\widetilde{h}$ is surjective.

(b) If $k : C' \to C''$ and $g(C'') > 0$ then $\widetilde{k \circ h} = \widetilde{k} \circ \widetilde{h}$.

(c) h is an isomorphism implies $\widetilde{h}$ is an isomorphism.

Theorem .0.2

The map $\mathrm{Div}^0(C) \to \mathrm{Div}^0(\widetilde{C})$ where $p \mapsto \widetilde{p}$ is well-defined, and furthermore

$$\mathrm{Div}^\ell(C) \to \mathrm{Div}^\ell(\widetilde{(C)}).$$

However, it is not necessasarily true that the reduction of the divisor of a function is the divisor of the reduction of the function.

This then induces a map

$$\mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\widetilde{C}).$$

Theorem .0.3

?? is true for $E/\overline{\mathbb{Q}}$, h an isogeny.

Fix ideals $p \subseteq \mathbb{Z}$ and $\mathfrak{p} \subseteq \overline{\mathbb{Z}}$, and $p \nmid N$.

Recall .0.1

$E/\overline{Q}$ has good reduction if and only if $j(E) \in \overline{\mathbb{Z}}_{(\mathfrak{p})}$.

Definition .0.1

Consider the set

$$S_1(N,)'_{\text{good}} = \{(E, Q) \in S_1(N) \mid E \text{ has good reduction at } \text{ and } j(\widetilde{E}) \neq 0, 1728\}.$$

We also define

$$\widetilde{S}_1(N) = \{(E, Q) \mid E/\overline{\mathbb{F}}_p, Q \in E[N]\}$$

We also define

$$\widetilde{S}_1(N)' = \{(E, Q) \in \widetilde{S}_1(N) \mid j(E) \neq 0, 1728\}.$$

We also have a surjection $S_1(N)'_{\text{good}} \twoheadrightarrow \widetilde{S}_1(N)'$.

Consider the modular curve $X_1(N)$. We had a universal elliptic curve E_j living over this. The function field was x-coordinates of torsion on this curve. We can also consider $\widetilde{E}_j$,

$$\widetilde{E}_j : y^2 + xy = x^3 - \left(\frac{36}{j-1728}\right)x - \frac{1}{j-1728}.$$

Fix $Q \in \widetilde{E}_j[N]$ of order N. Let $\varphi_{1,N} \in \mathbb{F}_p(j)[X]$ be the minimal polynomial of $x(Q)$.

We can then define

Definition .0.2

$\mathbb{K}_1^p(N) = \mathbb{F}_p(j)[X]/\varphi_{1,N}(X)$.

This is our candidate function field. It is easy to show this is a function field. Thus there exists a nonsingular projective curve corresponding to this, and we must ask if that is the same as $\widetilde{X}_1(N)$ (which as of now we don't even know if that has good reduction!).

Theorem .0.4 (Igusa)

For the modular curve $X_1(N)$,

- $X_1(N)$ has good reduction at p.
- $\mathbb{F}_p(\widetilde{X_1(N)}) \xrightarrow{\sim} \mathbb{K}_1^p(N)$.
- There is a commutative diagram

$$\begin{tikzcd} S_1(N)'_{\text{good}} \arrow[r, "\psi"] \arrow[d] & X_1(N) \arrow[d] \\ \widetilde{S}_1(N)' \arrow[r, "\widetilde{\psi}"] & \widetilde{X_1(N)} \end{tikzcd}$$

Corollary .0.5

There is a commutative diagram

$$\begin{tikzcd} \mathrm{Div}^0(S_1(N)'_{\text{good}}) \arrow[r] \arrow[d] & \mathrm{Pic}^0(X_1(N)) \arrow[d] \\ \mathrm{Div}^0(\widetilde{S}_1(N)') \arrow[r] & \mathrm{Pic}^0(\widetilde{X_1(N)}) \end{tikzcd}$$

.1. Eichler-Shimura Relation

Idea: Compute $\widetilde{T}_p : \mathrm{Pic}^0(\widetilde{X}_1(N)) \to \mathrm{Pic}^0(\widetilde{X}_1(N))$.

Warmup: Consider the diamond operator $\langle d\rangle$, We have $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$. The quotient is $(\mathbb{Z}/N\mathbb{Z})^\times$ and we pick a d here. We pick a matrix

$$\begin{bmatrix} a & 0 \\ c & \delta \end{bmatrix} \in \Gamma_0(N)$$

reducing to d. We can think of conjugation by this matrix acting on $\Gamma_0(N)$, and we can think of it as a double coset operator as well. We then get a map

$$\langle d\rangle : X_1(N) \to X_1(N)$$
$$\langle d\rangle_* : \mathrm{Pic}^0(X_1(N)) \to \mathrm{Pic}^0(X_1(N)).$$

Since this comes from an actual honest to god map of curves, we're actually fine.

General double coset operators. Let Γ_1, Γ_2 be congruence subgroups and

$$\Gamma_3 = \Gamma_1 \cap g^{-1}\Gamma_2 g$$
$$\Gamma_3' = g\Gamma_1 g^{-1} \cap \Gamma_2.$$

There are then maps

$$\begin{tikzcd} X_3 \arrow[r, leftrightarrow] \arrow[d] & X_3' \arrow[d] \\ X_1 & X_2. \end{tikzcd}$$

In the T_p case, $\Gamma_1, \Gamma_2 = \Gamma_1(N)$. Then

$$\Gamma_{1,0}(N,p) = \Gamma_1(N) \cap \Gamma_0(Np).$$

Then one gets maps

$$\begin{tikzcd} & X_{1,0}(N,p) \arrow[dl] \arrow[dr] & \\ X_1(N) & & X_1(N) \end{tikzcd}$$

The problem is $X_{1,0}(N,p)$ does not have good reduction at p. The reduction somehow looks like 2 copies of $\widetilde{X_1(N)}$ glued at the supersingular points.

The books says in fact we can sort of reduce this diagram, but we have to wrestle with $X_{1,0}(N,p)$ having singular reduction.

Assuming $\widetilde{T}_p$ is well-defined, we compute it.

Recall .1.1

Eigenvalues of T_p are coefficients of forms. We would like to do point counts for the reduced modular curves.

We have $a_p(f)$ is the coefficent in the modular curve, and we'd like to relate that to $a_p(\widetilde{E})$ (a point count of $\overline{\mathbb{F}_p}^2$ points on $\widetilde{E}$).

We should also recall what the Hecke operator does on the moduli problem

Recall .1.2

We have that

$$T_p : \mathrm{Div}^0(S_1(N)) \to \mathrm{Div}^0(S_1(N))$$
$$T_p[E,Q] = \sum_C [E/C, Q + C],$$

where the sum is over all $C \subseteq E$ of order p with $C \cap \langle Q \rangle = 0$. In our case this second condition is vacuous since $p \nmid N$, and Q has order N.

Also recall that if E has ordinary reduction at p, then so does E/C. Thus we can split this computation into an ordinary and supersingular computation.

Let $E/\overline{\mathbb{Q}}$ have ordinary reduction at , and let

$$C_0 = \ker(E[p] \twoheadrightarrow \widetilde{E}[p]).$$

And of course $|C_0| = p$.

Lemma .1.1

We need to know what the reduction looks like, well

$$[\widetilde{E/C}, \widetilde{Q+C}] = \begin{cases} [\widetilde{E}^{\sigma_p}, \widetilde{Q}^{\sigma_p}] & \text{if } C = C_0 \\ (\widetilde{E}^{\sigma_p^{-1}}, [p]\widetilde{Q}^{\sigma_p^{-1}}) & \text{if } C \neq C_0 \end{cases}.$$

Proof when $C = C_0$. Let $E' = E/C, Q' = Q + C = \varphi(Q)$, where $\varphi : E \to E'$. Let $\psi : E' \to E$ be the dual isogeny.

Consider the diagram

$$\begin{tikzcd} E'[p] \arrow[r, "\psi"] \arrow[d] & E[p] \arrow[d] \\ \widetilde{E'[p]} \arrow[r, "\widetilde{\psi}"'] & \widetilde{E}[p] \end{tikzcd}$$

We know this commutes, so then we have the following steps

- $\psi(E'[p]) \subseteq E[p]$ as order p.
- $\psi(E'[p]) \subseteq C$, and this implies $\psi(E'[p]) = C$.
- $\widetilde{E'[p]} \subseteq \ker \widetilde{\psi}$.
- $\ker(\widetilde{\psi}) = \widetilde{E'[p]}$

Upshot: compute the degrees of everything in sight.

$$\deg[p]_{\widetilde{E'}} = p^2 \qquad \deg(\widetilde{\varphi}) = p \qquad \deg(\widetilde{\psi}) = p.$$

Hence,

$$\begin{aligned} \deg_{\text{sep}}[p]_{\widetilde{E'}} &= p & \deg_{\text{insep}}[p]_{\widetilde{E'}} &= p \\ \deg_{\text{sep}} \widetilde{\psi} &= p & \deg_{\text{insep}} \widetilde{\psi} &= 1 \\ \deg_{\text{sep}} \widetilde{\varphi} &= 1 & \deg_{\text{insep}} \widetilde{\varphi} &= p. \end{aligned}$$

This implies that $\widetilde{\varphi} = \iota \circ \sigma_p$, where ι is an isomorphims and σ_p is the Frobenius map. With $\iota : \widetilde{E}^{\sigma_p} \to \widetilde{E}$.

This is a field extensions sort of argument (splitting into separable/inseparable). Then ι induces an equivalence

$$\iota : [\widetilde{E'}, \widetilde{Q'}] \leftrightarrow [\widetilde{E}^{\sigma_p}, \widetilde{Q}^{\sigma_p}].$$

The other computation is similar.