I. Jacobians and Abelian Varieties

Let X be a Compact Riemann Surface

If the genus $g = 1$ (warmup), then $X = \mathbb{C}/\Lambda$ for some lattice Λ. Pick a differential $\mathrm{d}x$ on X. Then we can look at

$$X \to \{\text{path integrals on } X \text{ starting at } 0\}/\{\text{integrals over loops}\}$$
$$z + \Lambda \mapsto \int_0^{z+\Lambda} \mathrm{d}x/\text{integrals over loops}.$$

Any loop will be a combination of the two fundamental loops $0 \to \omega_1$ and $0 \to \omega_2$, where $\Lambda = \mathbb{Z}\langle \omega_1, \omega_2 \rangle$.

This is an isomorphism of groups so long as the differential is translation invariant. We want to generalize this to $g > 1$.

Let $\gamma : [0,1] \to X$ be some path. Fix $\omega \in \Omega^1_{\text{hol}}(X)$, that is a 1-form on charts which agrees on intersections. We can check $\int_\gamma \omega$ makes sense.

Let γ, γ' have the same endpoints. Then $\int_\gamma \omega$ and $\int_{\gamma'} \omega$ differ by an integral over a loop. If X is genus g then it looks like a sphere with g-many handles coming off of it.

Let $A_1, \ldots, A_g$ be the longitudinal loops at 0 and $B_1, \ldots, B_g$ be the latitudinal loops about these g handles. Fact: for any loop α at 0, there exists unique integers m_i, n_i so that

$$\int_\alpha \omega = \sum_i \left(m_i \int_{A_i} \omega + n_i \int_{B_i} \omega \right).$$

Definition I.0.1

Let $H_1(X, \mathbb{Z})$ be the $\mathbb{Z}$-linear combinations of A_i, B_i (this is the integral homology of X. This gives us a map

$$H_1(X,\mathbb{Z}) \hookrightarrow \Omega^1_{\text{hol}}(X)^* = \mathrm{Hom}_{\mathbb{C}}(\Omega^1_{\text{hol}}(X), \mathbb{C}).$$

Fact: $\Omega^1_{\text{hol}}(X)^* \cong H_1(X,\mathbb{Z}) \otimes \mathbb{R} =: H_1(X, \mathbb{R})$.

We define

Definition I.0.2

The Jacobian of X, denoted $\mathrm{Jac}(X)$ is

$$\Omega^1_{\text{hol}}(X)/H_1(X,\mathbb{Z}).$$

There is a map $X \hookrightarrow \mathrm{Jac}(X)$.

I.1. Connection to Divisors

Now we're gonna look at the connection to divisors

$$\mathrm{Div}^0(X) = \left\{ \sum_{x \in X} n_x[x] \mid n_x = 0 \text{ for almost all } x, \sum_x n_x = 0 \right\}$$
$$\mathrm{Div}^\ell(X) = \left\{ \delta \in \mathrm{Div}^0(X) \mid \delta = \mathrm{div}(f), f \in \mathbb{C}(X) \right\}.$$

Definition I.1.1

We call the Picard group of X

$$\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\,\mathrm{Div}^\ell(X).$$

In genus $g = 0$, we have $\mathrm{Pic}^0(X) = \{0\}$, because we can just manufacture a rational function for any divisor.

If $g > 0$, fix a basepoint x_0. The map

$$X \hookrightarrow \mathrm{Pic}^0(X)$$
$$x \mapsto [x] - [x_0].$$

This is in fact an embedding!

Theorem I.1.1 (Abel)

We have that $\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X)$, the map here is given by $\sum_x n_x[x] = \sum_x n_x \int_{x_0}^x$.

Theorem I.1.2 (Modularity)

Let E be a complex elliptic curve with $j(E) \in \mathbb{Q}$. Then there exists an N such that there is a map

$$J_0(N) \twoheadrightarrow E,$$

which is a holomorphic group homomorphism of complex tori, where $J_0(N) \coloneqq \mathrm{Jac}(X_0(N))$.

This automatically gives a map $X_0(N) \to E$, and one can argue it is still surjective. This version of Modularity turns out to be equivalent to the old one.

We also have a nice description of $\Omega^1_{\mathrm{hol}}(X(\Gamma))$, namely

$$\Omega^1_{\mathrm{hol}}(X(\Gamma)) = S_2(\Gamma).$$

We want to look at maps of Jacobians. Namely given X, Y compact riemann surfaces and a map $h : X \to Y$ we want to produce maps

$$h_J : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$$
$$h^J : \mathrm{Jac}(X) \leftarrow \mathrm{Jac}(Y)$$

We obviously have a map

$$h^* : \mathbb{C}(Y) \to \mathbb{C}(X)$$
$$g \mapsto g \circ h.$$

Then in fact $v_x(h^* g) = e_x v_{h(x)} g$ where e_x is the ramification number of h at x.

Recall that $\omega \in \Omega^1_{\mathrm{hol}}(Y)$. Then in charts this is $\omega = (\omega_i)$, where $\omega_i = f_i(q)\,\mathrm{d}q$. This induces a pushforward map as we know how to act on $f_i(q) \in \mathbb{C}(Y)$.

$$h^* : \Omega^1_{\mathrm{hol}}(Y) \to \Omega^1_{\mathrm{hol}}(X)$$

$$h_* : \Omega^1_{\mathrm{hol}}(X)^* \to \Omega^1_{\mathrm{hol}}(Y)^*$$

Let γ be a path in X, then $h(\gamma)$ is a path in Y, and it turns out for $\lambda \in \Omega^1_{\mathrm{hol}}(Y)$ we have

$$\int_\gamma h^*\lambda = \int_{h\circ\gamma} \lambda,$$

which is just a change of variables.

This then induces a map

$$h_J : \mathrm{Jac}(X) \to \mathrm{Jac}(Y).$$

Explicitly, we have

$$h_J\left(\sum_x n_x \int_{x_0}^x\right) \bullet = \sum_x n_x \int_{h(x_0)}^{h(x)} \bullet = \sum_x n_x \int_{x_0}^x h^*(\bullet).$$

where $\bullet \in \Omega^1_{\mathrm{hol}}(Y)$. We now turn to the Picard group. We can define a norm map

$$\begin{aligned} \mathrm{norm}_h &: \mathbb{C}(X) \to \mathbb{C}(Y) \\ (\mathrm{norm}_h f)(y) &= \prod_{x\in h^{-1}(y)} f(x)^{e_x}. \end{aligned}$$

Then we have

$$v_y(\mathrm{norm}_h f) = \sum_{x\in h^{-1}(y)} v_x(f).$$

Then we can look at

$$\mathrm{div}(\mathrm{norm}_h f) = \sum_y \sum_{x\in h^{-1}(y)} v_x(f)[y] = \sum_x v_x(f)[h(x)].$$

This lets us predict that

$$\begin{aligned} h_D &: \mathrm{Div}(X) \to \mathrm{Div}(Y) \\ h_D\left(\sum_x n_x[x]\right) &\mapsto \sum_x n_x[h(x)] \end{aligned}$$

so then

$$h_D(\mathrm{div}(f)) = \mathrm{div}(\mathrm{norm}_h f).$$

We then get a map

$$\begin{aligned} h_P &: \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y) \\ h_P([d]) &= [h_D(d)]. \end{aligned}$$

There is then a diagram of the form

$$\begin{tikzcd} \mathrm{Pic}^0(X) \arrow[r, "h_P"] \arrow[d, "\sim"'] & \mathrm{Pic}^0(Y) \arrow[d, "\sim"] \\ \mathrm{Jac}(X) \arrow[r, "h_J"'] & \mathrm{Jac}(Y). \end{tikzcd}$$