Definition .0.1

Consider the Hecke algebra over $\mathbb{Z}$ is defined as

$$\mathbb{T}_{\mathbb{Z}} = \mathbb{Z}[\{T_n, \langle n \rangle \mid n \in \mathbb{Z}^+\}],$$

as operators on $S_2(\Gamma_1(N))$ (so there will be relations, ex. T_{p^3} is related to T_{p^2}, T_p).

There is an evaluation map (and it is a homomorphism) for each normalized eigenform $f \in S_2(\Gamma_1(N))$ given by

$$\lambda_f : \mathbb{T}_{\mathbb{Z}} \to \mathbb{C}$$
$$Tf = \lambda_f(T) f.$$

Call $H_1 = H_1(X(\Gamma), \mathbb{Z})$, which is a finitely generated $\mathbb{Z}$-module. Then $\mathrm{End}(H_1)$ is a finitely generated $\mathbb{Z}$-module, and we know

$$\mathbb{T}_{\mathbb{Z}} \hookrightarrow \mathrm{End}(H_1)$$

from last time.

Likewise $\mathrm{im}(\lambda_f) = \mathbb{Z}[\{a_n(f)\}] \subseteq \mathbb{C}$. We may define $K_f = \mathbb{Q}(\{a_n(f)\})$. THen

$$|\mathrm{Hom}(K_f, \mathbb{C})| = [K_f : \mathbb{Q}].$$

If we have $\sigma \in \mathrm{Hom}(K_f, \mathbb{C})$ then we can take f to f^σ by mapping each coefficient in the Fourier series. Why the hell is this still a modular form?

Theorem .0.1

If $f \in S_2(N, \chi)$ and $\sigma \in \mathrm{Hom}(K_f, \mathbb{C})$, then $f \in S_2(N, \chi^\sigma)$. Furthermore, if f is a newform, then so is f^σ.

The rest of the class will be spent on proving this.

Recall .0.1 (Nakayama's Lemma, Commutative Algebra)

Suppose A is a commutative ring, $J \subseteq A$ is an ideal contained in all maximal ideals, and M is a finitely generated A-module. Then, if $M = JM$, we have that $M = \{0\}$.

Fix a basis $\varphi_1, \ldots, \varphi_{2g}$ of $H_1(X_1(N), \mathbb{Z})$ over $\mathbb{Z}$. Let $V = H_1(X_1(N), \mathbb{Z})_{\mathbb{C}}$. Now $\mathbb{T}_{\mathbb{Z}}$ acts on V, which is a complex vector space by its action on the basis (i.e., formally weirdly enough). Suppose $v \in V$ is a λ-eigenvector of $\mathbb{T}_{\mathbb{Z}}$, where $\lambda : \mathbb{T}_{\mathbb{Z}} \to \mathbb{C}$ is a homomorphism. Then if $\sigma \in \mathrm{Aut}(\mathbb{C})$ then v^σ is a λ^σ-eigenvector.

To proceed, we need to show the space of eigenvalues for V is the same as the space of eigenvalues for S_2. We'll construct a complement of $S_2^* \subseteq V$. We'll call the complement $\overline{S_2^*}$, and we'll study the eigenvalues of each piece of $V = S_2^* \oplus \overline{S_2^*}$.

Recall .0.2

Consider the operator $W_N = \left[\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right]_2$, and recall that $W_N T = T^* W_N$ for any Hecke operator T (where T^* is the adjoint for the Peterson inner product).

Define for each $g \in S_2$ a map

$$\psi_g : S_2 \to \mathbb{C}$$

$$h \mapsto \langle W_N g, h \rangle.$$

If we collect these into $\{\psi_g\} =: \overline{S_2^*}$, then $\overline{S_2^*}$ is a vector space and $g \mapsto \psi_g$ provides an isomorphism of vector spaces $S_2 \to \overline{S_2^*}$.

We actually need that they're isomorphic as a $\mathbb{T}_{\mathbb{Z}}$-module. This is fairly easy, and comes from the W_N factor.

Exercise .0.3

Verify that $S_2 \xrightarrow{\sim} \overline{S_2^*}$ as $\mathbb{T}_{\mathbb{Z}}$-modules.

Claim

$\mathbb{T}_{\mathbb{Z}}$-eigenvalues on S_2 and S_2^* are the same.

Proof. Let f be a normalized eigenform. Then take $\lambda_f : \mathbb{T}_{\mathbb{Z}} \to \mathbb{C}$, and let $J_f := \ker(\lambda_f)$. We will show $J_f S_2 \neq S_2$ using Nakayama. We know that J_f is a prime ideal (being a kernel), but we don't know J_f is contained in every maximal ideal. The idea is to localize $\mathbb{T}_{\mathbb{Z}}$ at J_f, and then show we didn't kill everything by localizing.

Now we can look at

$$S_2^*[J_f] := \{\varphi \in S_2^* \mid \varphi \circ T = 0, \forall T \in J_f\}.$$

Then we have a short exact sequence

$$0 \longrightarrow J_f S_2 \longrightarrow S_2 \longrightarrow S_2/J_f S_2 \longrightarrow 0,$$

which upon dualizing gives

$$0 \longleftarrow (J_f S_2)^* \longleftarrow S_2^* \longleftarrow (S_2/J_f S_2)^* \longleftarrow 0,$$

This implies that

$$S_2^* \supseteq (S_2/J_f S_2)^* \cong S_2^*[J_f].$$

We should show that the eigenvalue on the right hand side coming from f is the same as that on S_2.

Let $T \in \mathbb{T}_{\mathbb{Z}}$. Then for $\varphi \in S_2^*[J_f]$ we have

$$T \cdot \varphi = \varphi \cdot T = \varphi \circ [T - \lambda_f(T)\,\mathrm{Id}] + \lambda_f(T)\varphi.$$

The left hand side lies in J_f, so this becomes $T \cdot \varphi = \lambda_f(T)\varphi$. Perfect! This shows that if λ_f is an eigenvalue of S_2 then it is also an eigenvalue of S_2^* (and dualizing yields the converse).

Thus S_2 and $S_2^* \oplus \overline{S_2^*}$ have the same eigenvalues. Now we want to show that V and $S_2^* \oplus \overline{S_2^*}$ are isomorphic as $\mathbb{T}_{\mathbb{Z}}$-modules via

$$(z_1\varphi_1, \ldots, z_{2g}\varphi_{2g}) \mapsto \left(\sum_j z_j \varphi_j, \sum_j z_j \overline{\varphi}_j \right).$$

There is a short claim that this is well-defined, i.e. that the RHS lies in $\overline{S_2^*}$... this is an exercise.

It's injective as if $\sum_j z_j\varphi_j = 0$ and $\sum_j z_j\overline{\varphi}_j = 0$, then conjugating we get $\sum_j \overline{z}_j\varphi_j = 0$. This allows us to say $\sum \mathrm{Re}(z_j)\varphi_j = 0, \sum \mathrm{Im}(z_j)\varphi_j = 0$. But wait! As a real vector space the φ_j are all linearly independent, so $\mathrm{Re}(z_j) = 0, \mathrm{Im}(z_j) = 0$. Perfect! Then the $z_j = 0$.

Then they're complex vector spaces of the same dimension so they are isomorphic.

Why does this matter? Well take some $f \in S_2$ which is a normalized eigenform. So $\lambda_f : \mathbb{T}_{\mathbb{Z}} \to \mathbb{C}$ is an eigenvalue for S_2, so it is for V, and then λ_f^σ is an eigenvalue for V, but then it is an eigenvalue for S_2 by the above. So there is a $g \in S_2$ with eigenvalue λ_f^σ. Normalizing, we see the Fourier coefficients of g must be $\sigma(a_f(n))$ as Hecke operators can extract the Fourier coefficients.

This can similarly show $f \in S_2(N, \chi)$ maps to $f^\sigma \in S_2(N, \chi^\sigma)$, since diamond operators give the eigenvalue depending on χ for these. Showing f^σ is a newform if f is... should not be too hard

Corollary .0.2

$S_2(\Gamma_1)$ has a basis with $\mathbb{Q}$ Fourier coefficients.

Proof. Suppose f is a newform of level $m \mid N$ with field K. Let $\{\alpha_1, \ldots, \alpha_d\}$ be a basis of $\mathcal{O}_K$ as a $\mathbb{Z}$-module. Let $\sigma_1, \ldots, \sigma_d$ be embeddings $K_f \hookrightarrow \mathbb{C}$. Then consider the matrix $A = \left(\alpha_i^{\sigma_j}\right)$. Now we can look at

$$F = \begin{pmatrix} f^{\sigma_1} \\ \vdots \\ f^{\sigma_d} \end{pmatrix}$$

$$g = Af$$

$$g_i = \sum_j \alpha_i^{\sigma_j} f^{\sigma_j}$$

Notice thenthat $g_i^\sigma = g_i$ for any σ. Then we need A is invertible (fact from algebraic number theory). Then

$$\mathrm{span}(g_i) = \mathrm{span}(f^{\sigma_i}).$$

The proof then proceeds by some basic induction, working newform by newform.