.1. **Oldforms and Newforms**

Last time we defined the Peterson inner product on $S_k(\Gamma)$. We then showed $S_k(\Gamma_1(N))$ has an orthonormal eigenbasis under $\{T_n, \langle n\rangle \mid (n, N) = 1\}$.

We'll work on the non-coprime case as well! We want to talk about modular forms "coming from lower level."

- If $M \mid N$ we have a trivial inclusion $S_k(\Gamma_1(M)) \hookrightarrow S_k(\Gamma_1(N))$ because $\Gamma_1(M) \supseteq \Gamma_1(N)$.
- Now suppose $d \mid N/M$, and let $\alpha_d = \left[\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right]$ (the action is $\alpha_d \tau = d\tau$). Then if $f \in S_k(\Gamma_1(M))$ then $f[\alpha_d]_k \in S_k(\Gamma_1(dM)) \subseteq S_k(\Gamma_1(N))$.

Proof. Fix $\gamma \in \Gamma_1(\delta M)$. Then we compute that

$$\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b\delta \\ c\delta^{-1} & d \end{pmatrix}.$$

Thus since c contains a factor of δ we have this conjugate lies in $\Gamma_1(M)$.

Thus for each $d \mid N$ we may define

$$\begin{aligned} \iota_d : S_k(\Gamma_1(Nd^{-1}))^2 &\to S_k(\Gamma_1(N)) \\ (f, g) &\mapsto f + g[\alpha_d]_k. \end{aligned}$$

Definition .1.1

We call the oldforms

$$S_k(\Gamma_1(N))^{\text{old}} := \operatorname{span}(\operatorname{im}(\iota_p) : p \mid N \text{ prime}).$$

We define the newforms $S_k(\Gamma_1(N))^{\text{old}}$ as the orthogonal complement of the oldforms under the Peterson inner product.

Proposition .1.1

For all $n \in \mathbb{Z}_{>0}$, these spaces are stable under $\{T_n, \langle n\rangle\}$.

Proof. Let $p \mid N$. Case 1 is to take $(d, N) = 1$. Let $T = \langle d\rangle$ or $T = T_{p'}$ for $p' \neq p$. Then we can consider the diagram

$$\begin{tikzcd} S_k(\Gamma_1(Np^{-1}))^2 \arrow[r, "\left[\begin{smallmatrix} T & 0 \\ 0 & T \end{smallmatrix}\right]"] \arrow[d, "\iota_p"'] & S_k(\Gamma_1(Np^{-1}))^2 \arrow[d, "\iota_p"] \\ S_k(\Gamma_1(N)) \arrow[r, "T"'] & S_k(\Gamma_1(N)) \end{tikzcd}$$

Showing this commutes is shows that the oldforms remain oldforms. For $T = \langle d\rangle_N$, we can show $\langle d\rangle_N = \langle d\rangle_{Np^{-1}} = [\alpha_p]\langle d\rangle_N[\alpha_p]^{-1}$. For the other case one must check $T_{p',Np^{-1}} = T_{p',N}$. Checking the compatibility with $[\alpha_p]$ is frankly awful. We check Dirichlet character by Dirichlet character. That is we check for $g \in S_k(Np^{-1}, \chi)$ that we have

$$(T_{p',Np^{-1}}g)[\alpha_p] = T_{p',N}(g[\alpha_p]).$$

We can check this at the level of Fourier series.

The one thing we haven't checked is T_p, as all other operators are zero or combinations of these via multiplication (and recursion for say T_{p^2}. We do the same thing with a different matrix. Namely

$$\begin{tikzcd} S_k(\Gamma_1(Np^{-1}))^2 \arrow[r, "{\left[\begin{smallmatrix} T_p & p^{k-1} \\ \langle p \rangle & 0 \end{smallmatrix}\right] I}"] \arrow[d, "\iota_p"'] & S_k(\Gamma_1(Np^{-1}))^2 \arrow[d, "\iota_p"] \\ S_k(\Gamma_1(N)) \arrow[r, "T_p"'] & S_k(\Gamma_1(N)) \end{tikzcd}$$

Proof for newforms is to show oldforms are invariant under $\langle n \rangle^*, T_n^*$. The only interesting case is $T_n, (n, N) > 1$¿ THen we have $T_n^* = \omega T_n \omega^{-1}$, where $\omega = \left[\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right]_k$.

We then need to suffer through the computation that

$$\iota_p \circ \begin{bmatrix} 0 & p^{k-2}\omega \\ \omega & 0 \end{bmatrix} = \omega \circ \iota_p.$$

Corollary .1.2

$S_k(\Gamma_1(N))^{\text{old,new}}$ each have an orthonormal basis under $\{T_n, \langle n \rangle \mid (n, N) = 1\}$.

Consider $L_d : d^{1-k}[\alpha_d]_k$. Then on Fourier series this acts very simply

$$\sum_{n=1}^{\infty} a_n q^n \mapsto \sum_{n=1}^{\infty} a_n q^{dn}.$$

Thus if $f \in L_d$, then $a_n(f) = 0$ for $d \nmid n$. Then to have $f \in \operatorname{span}(\operatorname{im} L_p \mid p \mid N)$ we must have $a_n(f) = 0$ for all $(n, N) = 1$.

Theorem .1.3 (Main lemma, Atkin-Lehmer)

The converse is true. That is if $a_n(f) = 0$ for all $(n, N) = 1$ then $f \in \operatorname{span}(\operatorname{im} L_p \mid p \mid N)$.

Proof of 1st Reduction. Define

$$\Gamma^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \mod N \right\}$$

Fact: $\alpha_N \Gamma_1(N) \alpha_N^{-1} = \Gamma^1(N)$.

We can consider a map

$$L^M = M^{k-1}[\alpha_M^{-1}] : S_k(\Gamma_1(M)) \to S_k(\Gamma^1(N))$$

which sends $\sum a_n q^n$ to $\sum a_n q_m^n$ where $q_M = e^{2\pi i \tau / M}$. Then in fact the following diagram commutes where $N = dM$,

$$S_k(\Gamma_1(M)) \xrightarrow{L_d} S_k(\Gamma_1(N))$$

$$S_k(\Gamma^1(M)) \xhookrightarrow[\text{Incl}]{} S_k(\Gamma^1(N))$$

by computing via Fourier series

$$\sum a_n q^n \xrightarrow{L_d} \sum a_n q^{dn}$$

$$\sum a_n q^{n/M} \underset{\text{Incl}}{\hookrightarrow} \sum a_n q^{n/M} = \sum a_n q^{dn/N}.$$

Thus the main lemma amounts to saying that if $f \in S_k(\Gamma^1(N))$, $f = \sum_n a_n(f) q_N^n$ with $a_n(f) = 0$ for all $(n, N) = 1$ then

$$f \in \sum_p S_k(\Gamma^1(Np^{-1})) \subseteq S_k(\Gamma^1(N)).$$

Proof of Second Reduction, projections. We work in $\Gamma(N)$. For $d \mid N$ define

$$\Gamma_d = \Gamma_1(N) \cap \Gamma^0(Nd^{-1}).$$

Fact: $\Gamma(N)\backslash\Gamma_d$ has representatives

$$\left\{ \begin{pmatrix} 1 & bN/d \\ 0 & 1 \end{pmatrix} \mid 0 \leq b \leq d \right\}.$$

We'll define the following

$$\begin{aligned} \pi_d : S_k(\Gamma(N)) &\to S_k(\Gamma_d) \subseteq S_k(\Gamma(N)) \\ f &\mapsto \frac{1}{d} \sum_{b=0}^{d-1} f \left[\begin{smallmatrix} 1 & bN/d \\ 0 & 1 \end{smallmatrix} \right]_k \\ \sum_{n=1}^{\infty} a_n q_N^n &\mapsto \sum_{n, d \mid n} a_n q_N^n. \end{aligned}$$

We then can define

$$\pi = \prod_{p \mid N} (\mathrm{Id} - \pi_p).$$

This kills everything that's not coprime to N. Thus the condition for the Main Lemma is that $f \in S_k(\Gamma^1(N)) \cap \ker(\pi)$. We can then apply some linear algebra

$$\ker \pi = \ker \left(\prod_{p \mid N} (\mathrm{Id} - \pi_p) \right) = \sum_{p \mid N} \ker(\mathrm{Id} - \pi_p) = \sum_{p \mid N} \mathrm{im}(\pi_p).$$

But wait we know that $\mathrm{im}(\pi_p) = S_k(\Gamma_p)$. Thus for our reduction we need to show that

$$S_k(\Gamma^1(N)) \cap \sum_{p \mid N} S_k(\Gamma_1(N)) \cap \Gamma^0(Np^{-1}) = \sum_{p \mid N} S_k(\Gamma^1(Np^{-1})).$$

The $\supseteq$ inclusion is true from previous discussion.

Proof. We know $G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on $S_k(\Gamma(N))$. We want to think of the spaces above as various fixed points of G. Write $G = \prod_i G_i = \prod_i \mathrm{SL}_2(\mathbb{Z}/p_i^{e_i})$ where $N = \prod_i p_i^{e_i}$. We then define H_i as

$$H_i \coloneqq \Gamma^1(p_i^{e_i})/\Gamma(p_i^{e_i})$$

and $H = \prod H_i$. Define

$$K_i = \frac{\Gamma_1(p_i^{e_i}) \cap \Gamma^0(p_i^{e_i-1})}{\Gamma(p_i^{e_i}}$$

Fact:

$$\langle \Gamma^1(p^e), \Gamma_1(p^e) \cap \Gamma^0(p^{e-1}) \rangle = \Gamma^1(p^{e-1}).$$

The third reduction becomes

$$S_k(\Gamma(N)) \cap \sum_{i=1}^n S_k(\Gamma(N))^{K_i} = \sum_{i=1}^n S_k(\Gamma(N)).$$

Now were looking at G acting on $S_k(\Gamma(N))$, we know that