The map $\mathcal{A}_{k=2n}(\Gamma) \to \Omega^{\otimes n}(X(\Gamma))$ gives us a way to define the order of vanishing of a differential $\omega \in \Omega^{\otimes n}(X(\Gamma))$. On a cusp we write this as

$$v_0(\omega_j) = v_0\left(\frac{g_j(q)}{(hq)^{k/2}}\right)$$

where $z^n f[\alpha]_{2n}(z) = g_j(z^h)$. This is precisely

$$v_{\pi(\tau)}(f) - \frac{k}{2}\left(1 - \frac{1}{h}\right).$$

If we're at a cusp, we have a different type of function g_j with

$$v_0(\omega_j) = v_0\left(\frac{g_j(q)}{\left(\frac{2\pi i q}{h}\right)^{k/2}}\right) = v_{\pi(\rho)}(f) - \frac{k}{2}.$$

Unlike the order of vanishing of f (which can be non-integral), the order of vanishing of ω_j is always integer (as it's just the order of vanishing of some function.

Exercise .0.1

Show that

$$S_2(\Gamma) \leftrightarrow \Omega^{\otimes 1}_{\text{hol}}(X(\Gamma)).$$

.1. Computing Dimensions

What we want from this is the dimensions of $\mathcal{M}_{-k}(\Gamma), S_k(\Gamma) \subseteq \mathcal{A}_k(\Gamma)$. We will use the Riemann-Roch formula.

Recall .1.1

For X a compact Riemann surface we defined

$$\operatorname{Div}(X) = \{\sum_{x \in X} n_x[x] \mid n_x = 0, \text{ all but finitely many } x, n_x \in \mathbb{Z}\}$$

and

$$\deg(D) = \sum n_x \qquad\qquad D \geq D', n_x \geq n'_x.$$

We also define $\operatorname{Div}^0(X) = \deg^{-1}(\{0\})$. Then we have a map

$$\operatorname{div} : \mathbb{C}(X) \to \operatorname{Div}^0(X) \subseteq \operatorname{Div}(X),$$

whose image is called the principal divisors. Abel's Theorem says that

$$\operatorname{Div}^0(X)/\operatorname{div}(\mathbb{C}(X)) \cong \mathbb{C}^g/\Gamma_g$$

We also have

$$L(D) = \{f \in \mathbb{C}(X) \mid f = 0 \text{ or } \operatorname{div}(f) + D \geq 0\}.$$

And here we have

- $L(D)$ is a vector space.

- $\dim L(D) =: \ell(D)$.
- $\mathrm{div} : \Omega(X) \to \mathrm{Div}(X)$ is given by $\omega \mapsto v_0(f_x)$ where locally at x, $\omega = f_x(q)(\mathrm{d}q)^n$.
- If $\lambda \in \Omega^1(X)$, then $\mathrm{div}(\lambda)$ is a canonical divisor, since everything in $\Omega^1(X)$ is equivalent up to principal divisors.

Theorem .1.1 (Riemann-Roch)

Let X be a compact Riemann surface, then

$$\ell(D) = \deg D - g + 1 + \ell(\mathrm{div}(\lambda) - D)$$

where λ is the canonical divisor.

Corollary .1.2

We have that

(1) $\ell(\mathrm{div}(\lambda)) = g$.
(2) $\deg(\mathrm{div}(\lambda)) = 2g - 2$.
(3) $\deg(D) < 0$ implies $\ell(D) = 0$.
(4) $\deg(D) > 2g - 2$ implies $\ell(D) = \deg(D) - g + 1$.

We know that

$$\Omega^1(X(\Gamma)) \cong \mathbb{C}(X(\Gamma))\lambda$$
$$\Omega^1_{\mathrm{hol}}(X(\Gamma)) \to L(\lambda)$$
$$f_0\lambda \mapsto f_0$$

as the left and right hand sides both correspond to $\mathrm{div}(f_0) + \mathrm{div}(\lambda) \geq 0$. the upshot of this by the corollary above is $\dim S_2(\Gamma) = g$.

Now we'll derive dimensions for k even. Our orders of vanishing for forms have rationals in them, and we can get around this with flooring and previous work...

Namely, recall that for $f \in \mathcal{A}_k(\Gamma)$, $f \neq 0$, we know $\mathcal{A}_k(\Gamma) = \mathbb{C}\left(X(\Gamma)\right) f$. Then we see that

$$\mathcal{M}_k(\Gamma) = \{f_0 f \mid f_0 f = 0 \text{ or } \mathrm{div}(f_0 f) \geq 0\} \cong L(\lfloor \mathrm{div}(f) \rfloor).$$

We should now study $\lfloor \mathrm{div}(f) \rfloor$. Well, f corresponds to some $\omega(f) \in \Omega^{\otimes k/2}(X(\Gamma))$. Well we know that

$$\deg \omega(f) = \mathrm{div}(\lambda) \cdot \frac{k}{2} = (2g-2)\frac{k}{2} = k(g-1).$$

We may then compute that

$$\lfloor \mathrm{div}(f) \rfloor = \mathrm{div}(\omega) + \sum_i \left\lfloor \frac{k}{4} \right\rfloor x_{2,i} + \sum_i \left\lfloor \frac{k}{3} \right\rfloor x_{3,i} + \sum_i \frac{k}{2} x_i,$$

where $x_{2,i}, x_{3,i}$ are elliptic points and x_i are cusps. We then know that $\deg \lfloor \mathrm{div}(f) \rfloor > 2g-2$ for $k \geq 2$. Thus for $k \geq 2$ we see that

$$\dim(\mathcal{M}_k(\Gamma)) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \mathcal{E}_2 + \left\lfloor \frac{k}{3} \right\rfloor \mathcal{E}_3 + \frac{k}{2} \cdot \mathcal{E}_\infty.$$

For cusp forms we have a similar argument yielding for $k \geq 4$ that

$$S_k(\Gamma) = L\left(\left\lfloor \operatorname{div}(f) - \sum_i x_i \right\rfloor\right)$$
$$\dim S_k(\Gamma) = \dim(\mathcal{M}_k(\Gamma)) - \mathcal{E}_\infty.$$

We also know from previous work that

$$\dim S_2(\Gamma) = g.$$

We know that $\mathcal{M}_0(\Gamma) = \mathbb{C}$, and $S_0(\Gamma) = 0$. The book shows $\mathcal{M}_k(\Gamma) = 0$ for $k < 0$.

Proof Idea. If $f \in \mathcal{M}_k(\Gamma)$, then we'd have $\frac{f^{12}}{\Delta^k} \in S_0(\Gamma)\ldots$

Application: For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, let k be even, then

$$\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) = \{0\}\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) \qquad = S_k(\mathrm{SL}_2(\mathbb{Z})) \oplus \mathbb{C}E_k \qquad (k < 4)$$
$$\dim S_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor - 1 & \text{if } k \equiv 2 \pmod 12 \\ \left\lfloor \frac{k}{12} \right\rfloor & \text{otherwise} \end{cases}.$$

In fact this implies that $\mathcal{M}(\mathrm{SL}_2(\mathbb{C})) = \mathbb{C}[E_4, E_6]$ and $S(\mathrm{SL}_2(\mathbb{Z})) = \Delta \cdot \mathbb{C}[E_4, E_6]$.

How should we run this for k odd? When $-I \notin \Gamma$, it is in fact still true that

$$\dim(\mathcal{M}_k(\Gamma)) = \ell(\lfloor \operatorname{div}(f) \rfloor)$$

since this doesn't use differentials (since there will still be a nonzero f, need to check). There exists an $\omega \in \Omega^k(X(\Gamma))$ that pulls back to $f(\tau)^2 (\mathrm{d}\tau)^k$. In fact we can compute $\lfloor \operatorname{div}(f) \rfloor$ in terms of ω, to give the formula

$$\ell(\lfloor \operatorname{div}(f) \rfloor) = (k-1)(g-1) + \left\lfloor \frac{k}{3} \right\rfloor \mathcal{E}_{3^+} + \frac{k}{2}\mathcal{E}_\infty^{reg} + \frac{k-1}{2}\mathcal{E}_\infty^{irr}. \qquad (k \geq 3)$$