**Theorem .0.1**

We have the following

(1) Let $1/p + 1/q = 1$, $1 \leq q < \infty$. The map $T : L^q \to (L^p)^*$ given by $Tg = \ell_g$ is an isometric linear isomorphism. (isometric meaning $Tg$ has the same norm as $g$).

This means that

- $T$ is a BLT.
- $T$ is bijective.
- $T$ is norm-preserving.

(2) If $\mu$ is $\sigma$-finite then this also holds for $q = \infty, p = 1$.

Even if $\mu$ is $\sigma$-finite we might not have $L^1 \cong (L^\infty)^*$.

Also note that $L^2 \cong (L^2)^*$. Also for all $1 < p < \infty$ we have $(L^p)^{**} \cong L^p$.

*Proof.* We have already proved this is isometric in **??**, it is clearly linear, and isometry implies injectivity.

We will prove that it is surjective later. See

Prove surjectivity

# I. Signed and Complex Measures

See [Fol99] Chapter 3.

**Recall I.0.1**

Suppose $f : X \to [0, \infty]$ is a measurable function on $(X, \mathcal{A}, \mu)$.

We can define $\nu(E) = \int_E f \, \mathrm{d}\mu$ for $E \in \mathcal{A}$, and $\nu$ is a measure on $(X, \mathcal{A})$.

This gives a map from the set of non-negative measurable functions on $X$ to measures on $X$. This is injective if we identify functions which are equal almost everywhere. But it is not necessarily surjective. We can then think of measures as a generalization of functions.

For an example, think of a dirac delta measure on $\mathbb{R}$. This is not the Lebesgue integral of any non-negative measurable function.

What if instead we took $f : X \to \mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$. We could take the same construction to get $\nu(E) = \int_E f \, \mathrm{d}\mu$, but this is no longer a measure as it can take $\mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$ values.

## I.1. Signed Measures

**Definition I.1.1**

Let $(X, \mathcal{A})$ be a measurable space. A signed measure is $\nu : \mathcal{A} \to [-\infty, \infty)$ or $\nu : \mathcal{A} \to (-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$.
- If $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint then

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$$

where the series on the RHS converges absolutely if $\nu\left(\bigcup_{i=1}^{\infty}\right) \in (-\infty, \infty)$. This means the series does not depend on re-arrangment.

**Example I.1.1**

Consider

(a) $\nu$ is a positive measure, then $\nu$ is a signed measure.

(b) If we have positive measures $\mu_1, \mu_2$ such that either $\mu_1(X) < \infty$ or $\mu_2(X) < \infty$, then $\nu = \mu_1 - \mu_2$ is a signed measure.

(c) If $f : X \to \overline{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ such that $\int_X f^+ \,\mathrm{d}\mu < \infty$ or $\int_X f^- \,\mathrm{d}\mu < \infty$, we can define

$$\nu(E) = \int_E f \,\mathrm{d}\mu$$

and this will be a signed measure.

Note: The following weird things happen with signed measures

(1) $A \subseteq B$ does not imply $\nu(A) \leq \nu(B)$, as $\nu(B) = \nu(A) + \nu(B \setminus A)$, and $\nu(B \setminus A)$ may be negative.

(2) If $A \subseteq B$ and $\nu(A) = \infty$, then $\nu(B) = \infty$, because $\nu(B \setminus A) \in (-\infty, \infty]$.

(3) Similarly if $A \subseteq B$ and $\nu(A) = -\infty$ then $\nu(B) = -\infty$.

**Lemma I.1.1**

If $\nu$ is a signed measure on $(X, \mathcal{A})$, then:

(1) If $E_n \in \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \cdots$ then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \nu(E_N).$$

We call this continuity from below.

(2) If $E_n \in \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \cdots$, and $-\infty < \nu(E_1) < \infty$ then

$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \nu(E_N).$$

We call this continuity from above.

*Proof.* DIY, or read [Fol99].

**Definition I.1.2**

Let $\nu$ be a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$. We say that

(1) $E$ is positive for $\nu$ if for all $F \subseteq E$, $\nu(F) \geq 0$.

(2) $E$ is negative for $\nu$ if for all $F \subseteq E$, $\nu(F) \leq 0$.

(3) $E$ is null for $\nu$ if for all $F \subseteq E$, $\nu(F) = 0$.

Note:

(1) If $E$ is a positive set, $F \subseteq E$, then $\nu(F) \leq \nu(E)$.

(2) If $E$ is a negative set, $F \subseteq E$, then $\nu(F) \geq \nu(E)$.

Todo list