Theorem .0.1

Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to [0, \infty]$. Then the following are equivalent

(a) f is $\mathcal{A}$-measurable.

(b) There exists simple functions $0 \le \phi_1 \le \phi_2 \le \cdots \le f$ such that

$$\lim_{n\to\infty} \phi_n(x) = f(x)$$

for all $x \in X$. I.e., f is a pointwise convergence upward limit of simple functions.

Proof. (b) $\implies$ (a) is easy because $f(x) = \sup_{n\in\mathbb{N}} \phi_n(x)$, and so f is a supremum of measurable functions.

Now assume f is $\mathcal{A}$-measurable. Now fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty])$. For every $0 \le k \le 2^{2n} - 1$ let $E_{n,k} = f^{-1}([k/2^n, (k+1)/2^n))$.

Let

$$\phi_n \coloneqq \sum_{k=0}^{2^{2n}-1} \frac{k 1_{E_{n,k}}}{2^n} + 2^n 1_{F_n}.$$

This implies $0 \le \phi_1 \le \phi_2 \le \cdots f$. Now for all $x \in X \setminus F_n$ we have that

$$0 \le f(x) - \phi_n(x) \le \frac{1}{2^n}$$

Then $F_1 \supseteq F_2 \supseteq \cdots$, and $\bigcap_{n=1}^\infty F_n = f^{-1}(\infty)$. This shows that for $x \in f^{-1}([0,\infty)) = X \setminus \bigcap_n F_n$. Thus $\lim_{n\to\infty} \phi_n(x) = f(x)$.

Then for $x \in f^{-1}(\{\infty\}) = \bigcap_n F_n$, which implies $\phi_n(x) \ge 2^n$. Thus $\lim_{n\to\infty} \phi_n(x) = \infty = f(x)$.

Corollary .0.2

If f is bounded on a set $A \subseteq \mathbb{R}$, then $\phi_n \to f$ uniformly on A.

Corollary .0.3

$f : X \to \mathbb{C}$ is a measurable function if and only if there exist simple functions $\phi_n : X \to \mathbb{C}$ such that $0 \le |\phi_1| \le |\phi_2| \le \cdots |f|$ such that ϕ_n converges pointwise to f (if f is bounded the convergence can be made uniform).

.1. Integration of nonnegative functions

Definition .1.1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then let

$$\phi = \sum_{i=1}^N c_i 1_{E_i} : X \to [0, \infty]$$

be a simple function with each $c_i \in [0, \infty)$. We define

$$\int \phi \coloneqq \int \phi \, \mathrm{d}\mu \coloneqq \int_X \phi \, \mathrm{d}\mu \coloneqq \sum_{i=1}^N c_i \mu(E_i).$$

This is called the integral of ϕ

For $A \in \mathcal{A}$ we define the notation

$$\int_A \phi := \int_A \phi \,\mathrm{d}\mu := \int \phi 1_A \,\mathrm{d}\mu$$

Proposition .1.1

Let $\phi, \psi \geq 0$ be simple functions. Then,

- This definition is well-defined.
- $\int c\phi = c \int \phi$ for $c \in [0, \infty)$.
- $\int \phi + \psi = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x)$ for all x implies $\int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \,\mathrm{d}\mu$ is a measure on $(X, \mathcal{A})$.

Proof. DIY.

Definition .1.2

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a measurable function. Then we define

$$\int f := \int f \,\mathrm{d}\mu :== \sup \left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

We have a few properties

- If f is a simple function, then the two definitions of $\int f$ agree
- $\int cf = c \int f$ for all $c \in [0, \infty)$.
- If $f \geq g \geq 0$ then $\int f \geq \int g$.
- But $\int f + g \stackrel{?}{=} \int f + \int g$. This really uses that f, g are measurable.

Theorem .1.2 (Monotone Convergence Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then let $f_n : X \to [0, \infty]$ be monotonically increasing measurable functions (i.e., $0 \leq f_1 \leq f_2 \leq \cdots$).

Let $f(x) := \sup_n f_n(x) = \lim_{n\to\infty} f_n(x)$. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. Note that $\lim_{n\to\infty} f_n(x)$ converges and $\lim_{n\to\infty} f_n$ converges because they are both monotone.

We know $f_n \leq f$, and so

$$\int f_n \leq \int f \implies \lim_{n\to\infty} \int f_n \leq \int f.$$

Now fix a simple function $0 \leq \phi \leq f$. It is enough to show that

$$\lim_{n\to\infty} \int f_n \geq \int \phi.$$

Fix $\alpha \in (0, 1)$. It is enough to prove that

$$\lim_{n\to\infty} \int f_n \geq \alpha \int \phi.$$

Then we can take the supremum over α, and then take a supremum over ϕ! Now we know that $\alpha\phi < f$. Let $A_n = \{x \mid f_n(x) \geq \alpha\phi(x)\}$.

We know $A_n \in \mathcal{A}$ (using measurability). Furthermore $A_1 \subseteq A_2 \subseteq \cdots$. We see that $\bigcup_n A_n = X$ (check!). Therefore

$$\int f_n \geq \int f_n 1_{A_n} \geq \int \alpha\phi 1_{A_n} = \alpha\nu(A_n) \coloneqq \int_{A_n} \phi.$$

We know ν is a measure, and so using continuity

$$\lim_{n\to\infty} \int f_n \geq \alpha \lim_{n\to\infty} \nu(A_n) = \alpha\nu(X) = \alpha \int \phi.$$

Corollary .1.3

Let $f, g \geq 0$ be measurable. Then $\int f + g = \int f + \int g$.

Proof. There exist simple functions $0 \leq \phi_1 \leq \phi_2 \leq \cdots$ for $\phi_n \to f$ pointwise, and likewise $0 \leq \psi_1 \leq \psi_2 \leq \cdots$ for $\psi_n \to g$ pointwise. Then by Theorem .1.2 we have

$$\int f + g = \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \int \phi_n + \int \psi_n = \int f + \int g.$$