Example .0.1

For an example of Jordan decomposition, let $(X, \mathcal{A}, \mu)$ be a measure space, $f : X \to \overline{\mathbb{R}}$, and $\nu(E) = \int_E f \, d\mu$. Then

$$\nu^+(E) = \int_E f^+ \, d\mu \qquad \nu^-(E) = \int_E f^- \, d\mu.$$

Definition .0.1

Let ν be a signed measure on $(X, \mathcal{A})$. The total variation measure of ν is $|\nu| := \nu^+ + \nu^-$. This is a positive measure on X.

Example .0.2

In the above example, $|\nu|(E) = \int_E |f| \, d\mu$.

Lemma .0.1

We have the following

(1) $|\nu(E)| \leq |\nu|(E)$.

(2) E is ν-null if and only if E is $|\nu|$-null

(3) If κ is another signed measure then

$$\kappa \perp \nu \text{ if and only if } \kappa \perp |\nu| \text{ if and only if } (\kappa \perp \nu^+ \text{ and } \kappa \perp \nu^-).$$

Proof. DIY.

Definition .0.2

ν is finite if $|\nu|$ is a finite measure, and similarly for σ-finite.

This holds if and only if ν^+, ν^- are both finite (resp. σ-finite) measures.

.1. Absolutely Continuous Measures

Definition .1.1

Let μ be a positive measure, ν be a signed measure, both on $(X, \mathcal{A})$. We say that ν if absolutely continuous with respect to μ (written $\nu \ll \mu$) provided that for all $E \in \mathcal{A}, \mu(E) = 0$ implies $\nu(E) = 0$.

This is equivalent to every μ-null set being ν-null.

Example .1.1

If $(X, \mathcal{A}, \mu)$, $f : X \to \overline{\mathbb{R}}$, $\nu(E) = \int_E f \, d\mu$, then $\nu \ll \mu$.

Notation: $d\nu = f \, d\mu$ means ν is a signed measure defined by $\nu(E) = \int_E f \, d\mu$.

Lemma .1.1

If μ is a positive measure, ν is a signed measure on $(X, \mathcal{A})$, then

(1) $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $(\nu^+ \ll \mu, \nu^- \ll< \mu)$.

(2) $(\nu \ll \mu$ and $\nu \perp \mu)$ implies $\nu = 0$ (zeno measure)

Proof. DIY (1). For (2), write $X = A \cup B, A \cap B = \emptyset, A$ μ-null, B ν-null. Then

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A).$$

Then $E \cap A \subseteq A$, so $\nu(E \cap A) = 0$. By absolute continuity, $\nu(E \cap A) = 0$. Thus $\nu(E) = 0$.

Theorem .1.2 (Radon-Nikodym)

Suppose μ is a σ-finite positive measure, ν is a σ-finite signed measure, and suppose $\nu \ll \mu$. Then there exists $f : X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\nu = f\,\mathrm{d}\mu$, in other words $\nu(E) = \int_E f\,\mathrm{d}\mu$.

If g is another such function with $\mathrm{d}\nu = g\,\mathrm{d}\mu$ then $f = g$ μ-a.e.

Proof. Next class, we'll prove a more general Lebesgue-Radon-Nikodym theorem (**??**).

Definition .1.2

Suppose $\nu \ll \mu$. The Radon-Nikodym derivative of ν with respect to μ is a function $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} : X \to \overline{\mathbb{R}}$ such that $\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu$ for all $E \in \mathcal{A}$.

i.e. we have $\mathrm{d}\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu$.

Note: By Theorem .1.2, such a function exists and is unique up to equivalence μ-a.e. in the σ-finite case.

Example .1.2

Say $F(X) = e^{2x} : \mathbb{R} \to \mathbb{R}$. This is continuous and strictly increasing, so we may define a Lebesgue-Stieltjes measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

This is defined to be the unique locally finit emeasure satisfying $\mu_F([a,b]) = F(b) - F(a) = e^{2b} - e^{2a}$. Then one can check that

$$\mu_F(E) = \int_E 2e^{2x}\,\mathrm{d}x$$

by uniqueness and the classical FTC, since the RHS is a locally finite borel measure, and $\kappa([a,b]) = e^{2b} - e^{2a}$. Thus $\mu_F = \kappa$.

Therefore $\mu_F \ll m$ and $\frac{\mathrm{d}\mu_F}{\mathrm{d}m} = 2e^{2x} = \frac{\mathrm{d}F}{\mathrm{d}x}$.

Example .1.3

Let $C(X) : \mathbb{R} \to \mathbb{R}$ be the cantor function. Then $C'(x) = 0$ outside the cantor set. But we don't always have

$$\mu_C(E) \neq \int_E 0\,\mathrm{d}x$$

So the candidate derivative is 0, but this fails. In particular

$$C(b) - C(a) \neq \int_a^b C'(x)\,\mathrm{d}x.$$

In fact, $\mu_C \not\ll m$ because $\mu_C \perp m$ and $\mu_C \neq 0$.

Thus the existence of a derivative almost everywhere and continuity is not enough to guarantee a version of the FTC holds.