Announcements

- Piazza.
- HW3 Q3 typo, $\mathcal{B} \subseteq \mathcal{A}_c$ should be $\mathcal{B} \subseteq \mathcal{A}$.
- Office hour reminder
  - M 12:30-1:30pm in-person.
  - T 1:30-2:30pm in-person.
  - Th 1-2pm online.

Theorem .0.1 (Locally finite Borel measures on $\mathbb{R}$)

Our work last time in fact classifies locally finite Borel measures on $\mathbb{R}$

(a) If $F : \mathbb{R} \to \mathbb{R}$ is a distribution function, then there exists a unique locally finite Borel measure μ_F on $\mathbb{R}$ satisfying $\mu_F((a, b]) = F(b) - F(a)$ for every $a < b$ in $\mathbb{R}$.

This essentially follows from **??**.

(b) Suppose $F, G : \mathbb{R} \to \mathbb{R}$ are distribution functions. Then, $\mu_F = \mu_G$ on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function. **(HW)**

(c) **??** implies that these are all of the locally finite Borel measures on $\mathbb{R}$.

.1. Lebesgue-Stieltjes measures on $\mathbb{R}$

The general sketch of what is going on

$$F \text{ dist. fn} \xrightarrow{HK} \mu_F \text{ on Carathéory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supseteq \mathcal{B}(\mathbb{R}).$$

Then HW3 implies that $(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}$ (the completion).

Definition .1.1 (Lebesgue-Stieltjes measure)

For a distribution function F, we call μ_F on $\mathcal{A}_{\mu_F}$ the Lebesgue-Stieltjes measure corresponding to F. A special case, when $F(x) = x$, is called the Lebesgue measure m on the Lebesgue σ-algebra $\mathcal{L}$.

Example .1.1 (Discrete Measures) (a) Write

$$F(x-) = \lim_{a \to x^-} F(a) \qquad\qquad F(x+) = \lim_{a \to x^+} F(a).$$

Then we have because F is right-continuous and increasing that

$$F(x-) \leq F(x) = F(x^+).$$

(HW)then gives us that $\mu_F(\{a\}) = F(a) - F(a-)$. As well we have

$$\mu_F([a, b]) = F(b) - F(a-)$$
$$\mu_F((a, b)) = F(b-) - F(a).$$

(b) Consider the following function

$$F(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Then $\mu_F(\{0\}) = 1$, $\mu_F(\mathbb{R}) = 1$, and $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. Generally

$$\mu_F(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$$

μ_F is called the Dirac measure at 0.

(c) Write $\mathbb{Q} = \{r_1, r_2, \ldots\}$. Then define

$$F_n(x) = \begin{cases} 1 & \text{if } x \geq r_n \\ 0 & \text{if } x < r_n \end{cases}$$

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$

Then we have $\mu_F(\{r\}) > 0$ for all $r \in \mathbb{Q}$, whereas $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$ **(HW)**.

(d) If F is continuous at a, then $\mu_F(\{a\}) = 0$.

(e) For $F(x) = x$ we have

$$m((a, b]) = m((a, b)) = m([a, b]) = b - a$$

(f) For $F(x) = e^x$ we have

$$\mu_F((a, b]) = \mu_F((a, b)) = \mu_F([a, b]) = e^b - e^a$$

Cases (b), (c) are examples of discrete measures

We make some quick definitions

Definition .1.2 (Dirac Measure)

Let $a \in \mathbb{R}$. The Dirac measure at a, denoted by δ_a, is the measure corresponding to the distribution function

$$F(x) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}.$$

We call a measure μ on X discrete if there is a countable set A such that $\mu(\{a\}) > 0$ for all $a \in A$ and $\mu(X \setminus A) = 0$.

Example .1.2 (Middle Thirds Cantor Set)

We define the Middle Thirds Cantor Set. We start with $K_0 = [0, 1]$. We remove the middle open interval. That is

$$K_1 = K_0 \setminus (1/3, 2/3).$$

$$K_2 = K_1 \setminus [(1/9, 2/9) \cup (7/9, 8/9)]$$

$$K_n = K_{n-1} \setminus \left[\bigcup_{k=0}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right) \right]$$

$$C = \bigcap_{n=1}^{\infty} K_n$$

We will show that C is uncountable, and that $m(C) = 0$.

We see that $x \in C$ if and only if $x = \sum_{n=1}^{\infty} a_n/3$ for some $a_n \in \{0, 2\}$

Keep in mind that $1/3 = 0.1 = 0.022222\ldots \in C$.

Example .1.3 (Cantor Function)

We define a function F to be 0 to the left of 0, 1 to the right of 1. Then we define F to be $1/2$ on $(1/3, 2/3)$. Then to be $1/4$ on $(1/9, 2/9)$ and $3/4$ on $(7/9, 8/9)$ and so on. This is called the Cantor Function.

We will show that F is continuous and increasing on **(HW)**, making it a distribution function.

We then have the following comparison

Cantor Measure	Lebesgue Measure
$\mu_F(\mathbb{R} \setminus C) = 0$	$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	$m(C) = 0$
$\mu_F(\{a\}) = 0$	$m(\{a\}) = 0.$

μ_F and m are said to be "singular to each other" which will be defined formally sometime later **??**.

.2. Regularity properties of Lebesgue-Stieltjes measures

Lemma .2.1

Let μ be a LS measure on $\mathbb{R}$. Then for all μ-measurable A we have

$$\begin{aligned}\mu(A) &= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supseteq A \right\} \\ &= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supseteq A \right\}.\end{aligned}$$

These come from the definition of outer measure (for the first), and continuity of the measure for the seocnd. Things like

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n] \qquad\qquad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n]$$

This is left as **(HW)**.