Announcements

- Final on 4/27 Wednesday 1:30-3:30
- Bring your phone/computer to scan/upload to Gradescope
- Content: Up to Lecture 36, i.e. all but Hilbert spaces & Fourier Analysis

Parseval's Identity says that if e_i is an orthonormal basis then

$$\langle x, y\rangle = \sum_{i=1}^{\infty} \alpha_n \overline{\beta_n}$$

where $\alpha_n = \langle x, e_n\rangle, \beta_n = \langle y, e_n\rangle$.

I. Intro to Fourier Analysis

I.1. Fourier Series

We will be considering the Hilbert space $L^2([-\pi, \pi])$ (which by scaling is equivalent to any finite interval).

Lemma I.1.1

The set

$$\left\{ e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} = \frac{1}{\sqrt{2\pi}}(\cos(nx) + i\sin(nx)) \right\}$$

is an orthonormal set in $L^2([-\pi, \pi])$.

Proof. We must evaluate

$$\langle e_n, e_m\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\, \mathrm{d}x = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Question: Is $\{e_n\}$ an orthonormal basis?

By Hölder's Inequality on $L^2([-\pi, \pi])$

$$\|f\|_1 \leq \|1\|_2 \|f\|_2 = \sqrt{2\pi} \cdot \|f\|_2 < \infty.$$

Likewise

$$\|f\|_2 = \sqrt{\int_{-\pi}^{\pi} |f(t)|^2 \,\mathrm{d}t} \leq \sqrt{\|f\|_\infty^2 2\pi}.$$

Therefore

$$\|f\|_1 \leq \sqrt{2\pi}\|f\|_2 \leq 2\pi\|f\|_\infty.$$

Definition I.1.1

For $f \in L^1([-\pi, \pi])$, its Fourier coefficients are

$$\hat{f}_n := \langle f, e_n\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(y) e^{-iny}\, \mathrm{d}y.$$

To show these form an orthonormal basis we must show that

$$\sum_{n=-M}^{N} \hat{f}_n e_n(x)$$

converges strongly to $f(x)$ as $M, N \to \infty$. Explicitly this is

$$\begin{aligned}\sum_{n=-m}^{N} \hat{f}_n e_n(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-M}^{N} \left[\int_{-\pi}^{\pi} f(y) e^{-iny}\,\mathrm{d}y\right] e^{inx} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-M}^{N} e^{in(x-y)}\right) \mathrm{d}y.\end{aligned}$$

Definition I.1.2

For $0 \le r < 1$, the Poisson kernel is

$$P_r(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{int} r^{|n|}$$

Explicitly, doing the sum as a geometric series

$$P_r(t) = \frac{1 - r^2}{2\pi(1 - 2r\cos t + r^2)}$$

Lemma I.1.2

For $f \in L^1([-\pi, \pi])$ and $0 \le r < 1$, $\sum_{n=-\infty}^{\infty} \hat{f}_n e_n(x) r^{|n|}$. This converges absolutely and uniformly for $x \in [-\pi, \pi]$ to $\int_{-\pi} P_r(x-y) f(y)\,\mathrm{d}y$.

Proof. We have that

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}_n e_n(x) r^{|n|}\right| \le \frac{1}{2\pi} \sum_{n=-\infty}^{m=\infty} \left(\int_{-\pi}^{\pi} \left|f(y) e^{-iny}\right| \mathrm{d}y\right) |e_n(x)|\, r^{|n|} = \frac{\|f\|_1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} < \infty.$$

Therefore Fubini's Theorem applies, and

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-\pi}^{\pi} f(y) e^{-iny}\,\mathrm{d}y\,\mathrm{d}y\right) e^{inx} r^{|n|} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(y) \sum_{n=-\infty} e^{in(x-y)} r^{|n|}\right) = \int_{-\pi}^{\pi} P_r(x-y) f(y)\,\mathrm{d}y.$$

Uniform convergence takes a small bit more work.

Note that $P_r(0) = \frac{1-r^2}{2\pi(1-r)^2} = \frac{1+r}{2\pi(1-r)} \to \infty$ as $r \to 1$. For any $t \ne 0$, we have

$$\frac{1 - r^2}{2\pi(1 - 2r\cos t + r^2)} \to 0$$

as the bottom is always nonzero and finite.

Lemma I.1.3

$P_r(t)$ form a "family of good kernels", i.e.

(1) $P_r(t) \ge 0$

(2) $\int_{-\pi}^{\pi} P_r(t)\,\mathrm{d}t = 1$.

(3) For every $\delta > 0$,

$$\int_{[-\pi,\pi]\setminus[-\delta,\delta]} P_r(t)\,\mathrm{d}t$$

Proof. For (2) use the first formula with Fubini. For (1),(3) use the second formula. Namely we have

$$\int_{[-\pi,\pi]\setminus[-\delta,\delta]} P_r(t)\,\mathrm{d}t \le \frac{1-r^2}{2\pi(1-2r\cos\delta+r^2)}2\pi \to 0$$

as $r \to 1$

Lemma I.1.4

For $f \in C([-\pi,\pi])$ satisfying $f(-\pi) = f(\pi)$, then

$$\lim_{r\to 1}\int_{-\pi}^{\pi} P_r(x-y)f(y)\,\mathrm{d}y = f(x)$$

uniformly for $x \in [-\pi,\pi]$.

Proof. Extend f to $f : \mathbb{R} \to \mathbb{C}$ setting $f(x+2\pi) = f(x)$, then f is uniformly continuous and bounded.

$$\begin{aligned}\int_{-\pi}^{\pi} P_r(x-y)f(y)\,\mathrm{d}y - f(x) &= \int_{-\pi}^{\pi} P_r(y)f(x-y)\,\mathrm{d}y - f(x)\\ &= \int_{-\pi}^{\pi} P_r(y)f(x-y)\,\mathrm{d}y - f(x)\int_{-\pi}^{\pi} P_r(y)\,\mathrm{d}y\\ &= \int_{-\delta}^{\delta} P_r(y)(f(x-y)-f(x))\,\mathrm{d}y + \int_{[-\pi,\pi]\setminus\ [-\delta,\delta]} P_r(y)(f(x-y)-f(x))\,\mathrm{d}y.\end{aligned}$$

Fix $\varepsilon > 0$, then f is uniformly continuous, so we can choose a $\delta > 0$ so that $|f(x-y)-f(x)| < \varepsilon$ for any choice of x. Then the left hand term is bounded by ε. For the right hand side, note that f is bounded, so for some M

$$\left|\int_{[-\pi,\pi]\setminus\ [-\delta,\delta]} P_r(y)(f(x-y)-f(x))\right|\mathrm{d}y \le M\int_{[-\pi,\pi]\setminus\ [-\delta,\delta]} P_r(y)\,\mathrm{d}y$$

Therefore, sending $r \to 1$ will send

$$\int_{-\pi}^{\pi} P_r(x-y)f(y)\,\mathrm{d}y - f(x) \to 0.$$