Lemma .0.1

Let ν be a signed measure on $(X, \mathcal{A})$. Then

(1) If E is positive, $G \subseteq E$ is measurable, then G is positive.

(2) If E is negative, $G \subseteq E$ is measurable, then G is negative.

(3) If E is null, $G \subseteq E$ is measurable, then G is positive.

(4) $E_1, E_2, \ldots$ positive sets then $\bigcup_{i=1}^{\infty} E_n$ positive

Proof. DIY.

Lemma .0.2

Suppose that ν is a signed measure with $\nu : \mathcal{A} \to [-\infty, \infty)$. Suppose $E \in \mathcal{A}$ and $0 < \nu(E) < \infty$. Then there exists a measurable $A \subseteq E$ such A is a positive set and $\nu(A) > 0$.

Assuming this lemma we prove

Theorem .0.3 (Hahn Decomposition)

If ν is a signed measure on $(X, \mathcal{A})$, then there exist $P, N \in \mathcal{A}$ such that

$$P \cap N = \emptyset$$
$$P \cup N = X.$$

P is positive for ν, N is negative for ν.

If P', N' are another such pair, then $P \triangle P' = N \triangle N'$ is null for ν.

Proof of Uniqueness. We see that $P \setminus P' \subseteq P, P \setminus P' \subseteq N'$. Thus $P \setminus P' \subseteq P \cap N'$ is both positive and negative, hence $P \setminus P'$ is null.

Similarly for $P' \setminus P$, and then their union $P \triangle P'$ is null as well.

Proof of Existence. Without loss of generality suppose $\nu : \mathcal{A} \to [-\infty, \infty)$. If not, consider $-\nu$.

Let

$$s := \sup\{\nu(E) \mid E \in \mathcal{A} \text{ is a positive set}\}$$

which is a nonempty supremum because $\emptyset$ is positive. Then there exist $P_1, P_2, \ldots$ positive sets such that $\lim_{n\to\infty} \nu(P_n) = S$.

Then we have that $P = \bigcup_n P_n$ is positive by Lemma .0.1. Then $\nu(P) \le S$, and $\nu(P) = \nu(P_n) + \nu(P \setminus P_n) \ge \nu(P_n)$. Thus

$$\nu(P) \ge \lim_{n\to\infty} \nu(P_n) = s.$$

Hence $\nu(P) = s$ and the supremum is in fact a max. We then know that $s = \nu(P) < \infty$ because ν does not attain the value infinity.

Now let $N = X \setminus P$. We claim that N is negative. If not then there exists a measurable $E \subseteq N$ with $\nu(E) > 0$. By assumption, $\nu(E) < \infty$. Then $0 < \nu(E) < \infty$, so by Lemma .0.2 there exists a measurable $A \subseteq E$ such that A is positive and $\nu(A) > 0$.

But wait! We then know that

$$\nu(P \cup A) = \nu(P) + \nu(A) > \nu(P)$$

which is a contradiction since $P \cup A$ is a positive set, and $\nu(P)$ is maximal.

Therefore N is negative, and the theorem holds.

Proof of Lemma .0.2. If E is positive, we're done. Otherwise, there exist measurable subsets with negative measure. Let $n_1 \in \mathbb{N}$ be the least such n_1 such that there exists $E_1 \subseteq E$ with $\nu(E_1) < -1/n_1$.

If $E \setminus E_1$ is positive, we're done. Else we can inductively define $n_2, n_3, \ldots$ as well as $E_2, E_3, \ldots$.

Explicitly, if $E \setminus \bigcup_{i=1}^{k-1} E_i$ is not positive, let n_k be the least such that there exists $E_k \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ with $\nu(E_k) < -1/n_k$.

Note then that if $n_k \geq 2$, for all $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ we have that $\nu(B) \geq -\frac{1}{n_k - 1}$.

Now let $A = E \setminus \bigcup_{i=1}^{\infty} E_i$. Since $E = A \cup \bigcup_i E_i$ we have by countable additivity that

$$0 < \nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k) < \nu(A).$$

Furthermore, $\nu(E), \nu(A)$ are both in $(0, \infty)$, and we see that

$$0 < \nu(E) \leq \nu(A) - \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Therefore the sum on the RHS must converge, meaning that $1/n_k \to 0$ as $k \to \infty$. That is $\lim_{k\to\infty} n_k = \infty$.

Now if $B \subseteq A$, then $B \subseteq E \setminus \bigcup_{i=1}^{\infty} E_i$. Therefore $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$. By the note above, for large enough k such that $n_k \geq 2$ we have

$$\nu(B) \geq \frac{-1}{n_k - 1}$$

taking $k \to \infty$ we have $\nu(B) \geq 0$, and so A is a positive set as desired.

Definition .0.1

If μ, ν are signed measures on $(X, \mathcal{A})$, then we say $\mu \perp \nu$ (singular to each other) means there exists $E, F \in \mathcal{A}$ such that $E \cap F = \emptyset, E \cup F = X$, F is null for μ, E is null for ν.

Example .0.1

Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

(1) The Lebesgue measure m

(2) The Cantor measure μ_C defined by the Cantor function.

(3) The discrete measure $\mu_D = \delta_1 + 2\delta_{-1}$.

We can take $E = \mathbb{R} \setminus \{-1, 1\}, F = \{1, -1\}$ to see that $m \perp \mu_D$.

We can take $E = \mathbb{R} \setminus K$ and $F = K$ where K is the cantor set to see that $m \perp \mu_C$.

We can also see that $\mu_C \perp \mu_D$.

Theorem .0.4 (Jordan Decomposition Theorem)

Let ν be a signed measure on $(X, \mathcal{A})$. Then there exists unique positive measures ν^+, ν^- on $(X, \mathcal{A})$

such that for all $E \in \mathcal{A}$ we have

$$\nu(E) = \nu^+(E) - \nu^-(E) \qquad \nu^+ \perp \nu^-.$$

Proof. For existence take $\nu^+(E) \coloneqq \nu(E \cap P), \nu^-(E) \coloneqq -\nu(E \cap N)$. Uniqueness DIY.