I. Hilbert Spaces

This is in [Fol99] section 5.5.

I.1. Inner Product Spaces

Definition I.1.1

Let V be a (complex) vector space. An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying

(1) We have linearity in the first argument

$$\langle \alpha x + \beta y, z \rangle$$

for all $x, y, z \in V$, and $\alpha, \beta \in \mathbb{C}$.

(2) We have that $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for every $x, y \in V$.

(3) $\langle x, x \rangle \in [0, \infty)$.

(4) $\langle x, x \rangle = 0$ if and only if $x = 0$.

Note, we have conjugate linearity in the second argument

$$\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$$

for any $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.

Example I.1.1

We have the following examples

- $\mathbb{R}^d$ with $\langle x, y \rangle = x \cdot y = \sum_{i=1}^d x_i y_i$.
- $\mathbb{C}^d$ with $\langle x, y \rangle = \sum_{i=1}^d x_i \overline{y_i}$.
- $L^2(X, \mu)$ with $\langle f, g \rangle = \int_X f \overline{g} \, d\mu$. Note by Hölder that

$$\left| \int_X f \overline{g} \right| \leq \| f \overline{g} \|_1 \leq \| f \|_2 \| g \|_2 < \infty.$$

  because $1/2 + 1/2 = 1$.
- A special case is ℓ^2, where we have

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

Definition I.1.2

Given an inner product space V, let $\| x \| = \sqrt{\langle x, x \rangle}$. We claim this is a norm, called the norm induced from the inner product.

We prove this is a norm below, after proving Theorem I.1.1.

Note that

$$\begin{aligned} \| x + y \|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \| x \|^2 + 2 \operatorname{Re} \langle x, y \rangle + \| y \|^2 \end{aligned}$$

Theorem I.1.1 (Cauchy-Schwarz Inequality)

We have that $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Proof. This is clear if $\langle x, y\rangle = 0$. Assume $\langle x, y\rangle \neq 0$. For every $\alpha \in \mathbb{C}$, we know that

$$0 \leq \|\alpha x - y\|^2 = |\alpha|^2 \|x\|^2 - 2\operatorname{Re}\alpha\langle x, y\rangle + \|y\|^2.$$

Write $\langle x, y\rangle = |\langle x, y\rangle| e^{i\theta}$, and take $\alpha = e^{-i\theta}t$ for arbitrary $t \in \mathbb{R}$. Then, the RHS gives

$$0 \leq \|x\|^2 t^2 - 2|\langle x, y\rangle| t + \|y\|^2.$$

Note this is a real quadratic function of t, with at most one real root. Thus the discriminant is ≤ 0. The discriminant is in fact

$$\begin{aligned} 4|\langle x, y\rangle|^2 - 4\|x\|^2\|y\|^2 &\leq 0 \\ |\langle x, y\rangle|^2 &\leq \|x\|^2\|y\|^2 \\ |\langle x, y\rangle| &\leq \|x\|\|y\|. \end{aligned}$$

Proof that Definition I.1.2 is a norm. We have that $\|x\| = 0 \iff x = 0$ from the definition of an inner product. We also have that

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\overline{\alpha}\langle x, x\rangle} = |\alpha| \|x\|.$$

The triangle inequality is less obvious, and comes from Theorem I.1.1. Namely

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \\ \|x+y\| &\leq \|x\| + \|y\|. \end{aligned}$$

Perfect!

Theorem I.1.2 (Parallelogram law)

Let V be a normed space. Then, $\|\cdot\|$ is induced by an inner product if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

for all $x, y \in V$.

Proof. The forward direction follows from

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\operatorname{Re}\langle x, y\rangle + \|y\|^2.$$

$$\|x \pm iy\|^2 = \|x\|^2 \pm 2\operatorname{Im}\langle x, y\rangle + \|y\|^2.$$

For the backwards direction, define

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

as motivated by the above relationship.

Check this is an inner product inducing the desired norm.

Example I.1.2

Consider $L^p(\mathbb{R}, m)$, $f = 1_{(0,1)}$, $g = 1_{(1,2)}$. We see the parallelogram law is satisfied only when $p = 2$. Thus $L^p(\mathbb{R}, m)$ is only an inner product space when $p = 2$.

Definition I.1.3 (Weak convergnece)

We say that $x_n \in V$ converges to $x \in V$ weakly provided that for any fixed $y \in V$, $\langle x_n - x, y\rangle \to 0$.

Lemma I.1.3 (Strong convergence $\implies$ Weak convergence)

Suppose V is an inner product space. If $x_n \to x$ strongly (i.e. $\|x_n - x\| \to 0$), then $x_n \to x$ weakly in the sense that for any fixed $y \in V$, we have $\langle x_n - x, y\rangle \to 0$.

Proof. Using the Cauchy-Schwarz inequality

$$0 \leq |\langle x_n - x, y\rangle| \leq \|x_n - x\| \cdot \|y\|.$$

Since $\|x_n - x\| \to 0$ and $\|y\|$ is constant in n, we have by the squeeze theorem that $\langle x_n - x, y\rangle \to 0$.

Example I.1.3

Consider ℓ^2, $x_n = (0, \ldots, 0, 1, 0, \ldots)$ and $x = 0$. Then x_n does not converge strongly to any vector. But, if we fix $y \in \ell^2$, then

$$\langle x_n - x, y\rangle = \overline{y_n}$$

which goes to 0 as $n \to \infty$ because $\sum_n |y_n|^2 < \infty$. Therefore $x_n \to 0$ weakly, but we see that

$$\|x_n - 0\| = \|x_n\| = 1.$$

Thus $x_n \not\to 0$ strongly.

I.2. Orthonormal Bases

Definition I.2.1

We say x, y are orthogonal if $\langle x, y\rangle = 0$, denoted $x \perp y$.

Lemma I.2.1 (Pythagorean Theorem)

If $x_1, \ldots, x_n \in V$, $\langle x_i, x_j\rangle = 0$ for all $I \neq j$, then

$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2 \tag{1}$$

Proof. Use that $\|x+y\| = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$ and induct.

Definition I.2.2

We call $\{e_i\}_{i \in I}$ an orthonormal set if

$$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$