.1. L^1 Spaces

Definition .1.1

Let V be a vector space over a field $\mathbb{R}$ or $\mathbb{C}$. A seminorm on V is $\|\cdot\| : V \to [0, \infty)$ satisfying

- $\|cv\| = |c| \, \|v\|$ for all $v \in V$ and c a scalar
- $\|v + w\| \leq \|v\| + \|w\|$.

A norm is a seminorm such that $\|v\| = 0 \iff v = 0$.

Lemma .1.1

A normed vector space is a metric space with metric $\rho(v, w) = \|v - w\|$.

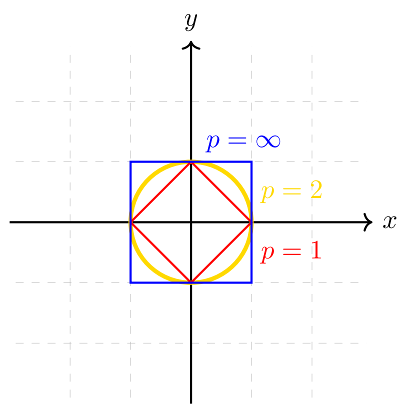
Proof. DIY.

Example .1.1

In $\mathbb{R}^d$ we have norms

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} & \text{if } p \in [1, \infty) \\ \max_{1 \leq i \leq d} |x_i| & \text{if } p = \infty \end{cases}$$

We call the unit ball $\{x \mid \|x\| \leq 1\}$. We see that we have the following



All $\|\cdot\}_p$ norms induce the same topology. i.e., if U is open in p-norm, it is open in p'-norm as well.

Recall that f is integrable means $\int |f| < \infty$ and $f = g$ almost everywhere implies $\int f = \int g$.

Definition .1.2

If $(X, \mathcal{A}, \mu)$ is a measure space, we say that

$L^1(X, \mathcal{A}, \mu) = L^1(X, \mu) = L^1(X) = L^1(\mu)$ is the set of integrable functions on X. This is a vector space.

Lemma .1.2

$L^1(X, \mathcal{A}, \mu)$ is a vector space with seminorm $\|f\|_1 = \int |f|$.

Definition .1.3

Define $f \sim g$ if $f = g$ almost everywhere. Then $L^1(X, \mathcal{A}, \mu)/\sim = L^1(X, \mathcal{A}, \mu)$.

This notation is confusing.

With this definition, $L^1(X, \mathcal{A}, \mu)$ is a normed vector space. Note then that

$$\rho(f, g) = \int |f - g| .$$

The dense subsets of L^1 are given by

Theorem .1.3

We have that

(a) {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1-metric).

(b) For $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$ and μ a Lebesgue-Stieltjes measure, we have that {integrable step functions} is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu$.

(c) $C_c^\infty(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

A step function on $\mathbb{R}$ is $\psi = \sum_{i=1}^N c_i 1_{I_i}$ where I_i is an interval.

And $C_c^\infty(\mathbb{R})$ is the collection of smooth functions with compact support.

Proof. Lets go!

(a) For any $f \in L^1(X, \mathcal{A}, \mu)$, we see there exist simple functions $0 \leq |\phi|_1 \leq |\phi_2| \leq \cdots \leq |f|$ such that $\phi_n \to f$ pointwise. But then

$$\lim_{n \to \infty} \int |\phi_n - f| = 0$$

by the Dominated Convergence Theorem, we win because $|\phi_n - f| \leq |\phi_n| + |f| \leq 2|f|$.

(b) It suffices to approximate 1_E by $\sum_{i=1}^N c_i 1_{I_i}$ for E a measurable set. Well we see that

$$\int |1_A - 1_B| = \mu(A \triangle B).$$

By regularity theorem for LS measure we see that for all $\varepsilon > 0$ there exists an $I = \bigcup_{i=1}^N I_i$ for I_i disjoint such that $\mu(E \triangle I) < \varepsilon$.

(c) It suffices to approximate $1_{(a,b)}$ by $g \in C_c^\infty(\mathbb{R})$ for $a, b \in \mathbb{R}$. Simply for $\varepsilon > 0$ glue together 0 on $(-\infty, a - \varepsilon/2)$ with 1 on (a, b) and 0 on $(b + \varepsilon/2, \infty)$.

Then we see that

$$\int \left|1_{(a,b)} - g\right| \mathrm{d}m \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$