Recall .0.1

We have from last time that for $0 \leq r < 1$

$$e_n(x) := \frac{1}{\sqrt{2\pi}} e^{inx}$$
$$P_r(t) := \sum_{n=-\infty}^{\infty} e_n(t) r^{|n|} f \in L^1([-\pi,\pi]) \qquad \implies \sum_{n=-\infty}^{\infty} \hat{f}_n e_n(x) r^{|n|} = \int_{-\pi}^{\pi} P_r(x-y) f(y)\, \mathrm{d}y$$

uniformly in $x \in [-\pi,\pi]$. Furthermore, if $f \in C([-\pi,\pi])$ and $f(\pi) = f(-\pi)$ then

$$\lim_{r \to 1} \int_{-\pi}^{\pi} P_r(x-y) f(y)\, \mathrm{d}y = f(x)$$

uniformly in $x \in [-\pi,\pi]$.

Finally we also have

$$\|f\|_1 \leq \sqrt{2\pi}\|f\|_2 \leq 2\pi\|f\|_\infty$$

Theorem .0.1 (Fourier Series)

The set of $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ for $n \in \mathbb{Z}$ is an orthonormal basis of $L^2([-\pi,\pi])$.

We have from previous work that they are an orthonormal set. Thus we just need to show that for all $f \in L^2([-\pi,\pi])$ that there exists $h = \sum_{n=-M}^{N} \beta_n e_n(x)$ such that $\|f-h\|_2 < \varepsilon$. That is we need to show that the span of the e_n is dense.

Proof. Let $f \in L^2([-\pi,\pi])$. Fix $\varepsilon > 0$. Then there is a function $g \in C([-\pi,\pi])$ with $g(\pi) = g(-\pi)$ such that $\|f-g\|_2 < \frac{\varepsilon}{3}$. Why? Simple density argument.

Let $g_r(x) = \int_{-\pi}^{\pi} P_r(x-y) g(y)\, \mathrm{d}y$. By the above, there exists an $r \in [0,1)$ such that $\|g_r - g\|_\infty < \frac{\varepsilon}{3\sqrt{2\pi}}$.

Therefore $\|g_r - g\|_2 < \frac{\varepsilon}{3}$. Consider $g_{r,N}(x) = \sum_{n=-N}^{N} \hat{g}_n e_n(x) r^{|n|}$. By the above there exists an $N \in \mathbb{N}$ such that $\|g_{r,N} - g_r\|_\infty < \frac{\varepsilon}{3\sqrt{2\pi}}$, thus $\|g_{r,N} - g_r\|_2 < \frac{\varepsilon}{3}$. Therefore

$$\|f - g_{r,N}\| < \varepsilon$$

$g_{r,N}$ is a finite linear combination of e_n's, so these form an orthonormal basis as desired.

Example .0.2

Plancherel identity $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left|\hat{f}_n\right|^2$.

For $f(x) = x$, we have

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-inx}\, \mathrm{d}x = \begin{cases} 0 & \text{if } n = 0 \\ \frac{(-1)^n i\sqrt{2\pi}}{n} & \text{if } n \neq 0 \end{cases}.$$

Together these imply that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example .0.3

Isoperimetric inequality. Consider a parametric curve $(x(t), y(t)) \in \mathbb{R}^2$ with $t \in [-\pi,\pi]$. Assume that

(1) This is a closed curve, so $(x(-\pi), y(-\pi)) = (x(\pi), y(\pi))$.
(2) We assume these are smooth, but in fact we just need x, y are C^1 functions.
(3) The curve is simple.

Suppose that

$$L = \int_{-\pi}^{\pi} \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t = 2\pi.$$

What is the largest area A enclosed?

By Green's Theorem

$$\begin{aligned} A &= \frac{1}{2}\oint_C (x\,\mathrm{d}y - y\,\mathrm{d}x) \\ &= \frac{1}{2}\int_{-\pi}^{\pi} (x(t)y'(t) - x'(t)y(t))\,\mathrm{d}t. \end{aligned}$$

Arc length parametrization, so that $x'(t)^2+y'(t)^2 = 1$ for all t. Then the condition $L = 2\pi$ is automatically satisified and can be written as

$$\int_{-\pi}^{\pi} (x'(t)^2 + y'(t)^2)\,\mathrm{d}t = 2\pi.$$

For ease of computation, rewrite using complex numbers, i.e., $z(t) = x(t) + iy(t)$. This is then subject to

$$\int_{-\pi}^{\pi} \left|z'(t)\right|^2 \mathrm{d}t = 2\pi i.$$

Rewriting our above formula for area, we need to find the maximum of

$$A = \frac{1}{4i}\int_{-\pi}^{\pi} (\overline{z(t)}z'(t) - z(t)\overline{z'(t)})\,\mathrm{d}t.$$

Note z is C^1 and $z(\pi) = z(-\pi)$. Denote $\hat{z}_n = \alpha_n$. Now

$$\begin{aligned} \widehat{z'}_n &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} z'(t)e^{-int}\,\mathrm{d}t \\ &= \left(z(t)e^{-int}\right]_{\pi}^{\pi} + in\int_{-\pi}^{\pi} z(t)e^{-int}\,\mathrm{d}t \\ &= in\alpha_n. \end{aligned}$$

By Plancherel's identity, we have that

$$\int_{-\pi}^{\pi} \left|z'(t)\right|^2 \mathrm{d}t = \sum_{n=-\infty}^{\infty} \left|in\alpha_n\right|^2 = \sum_{n=-\infty}^{\infty} n^2\left|\alpha_n\right|^2 = 2\pi.$$

We know Parseval's identity says $\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \hat{f}_n\overline{\hat{g}_n}$.

Therefore

$$A = \frac{1}{4i}\sum_{n=-\infty} \infty\overline{\alpha_n}(in\alpha_n) - \alpha_n\overline{(in\alpha_n)} = \frac{1}{2}\sum_{n=-\infty}^{\infty} n\left|\alpha_n\right|^2.$$

Great! Then the question becomes what is the maximum of $\frac{1}{2}\sum_{n=-\infty}^{\infty} n\left|\alpha_n\right|^2$ subject to $\sum_{n=-\infty}^{\infty} n^2\left|\alpha_n\right|^2 = 2\pi$.

We will show that

$$2\pi - \sum_{-\infty}^{\infty} n\left|\alpha_n\right|^2 \geq 0$$

as we guess the maximum should be π, by virtue of the circle being theoretically optimal. Then we have

$$2\pi - \sum_{-\infty}^{\infty} n\left|\alpha_n\right|^2 = \sum_{n=-\infty}^{\infty}\left(n^2 - n\right)\left|\alpha_n\right|^2 \geq 0$$

because term by term every term is non-negative.

Thus the area cannot be more than π. We have $A = \pi$ if and only if equality holds above, that is $\alpha_n = 0$ for $n \neq 0, 1$. This means $z(t) = \alpha_0 + \alpha_1 e_1(x)$, that is

$$z(t) = \alpha_0 + Ce^{it}.$$

This means that $z(t)$ is a circle about α_0.

Professor Baik likes Dym and McKean's Fourier Series & Integrals [DM85].