Announcements

- HW6 due tomorrow.
- HW6–Extra (Do not hand in)–about lsat 2 lectures.
- Midterm next Wednesday 6-8pm Chem 1400 (content up to last class–lecture 17).
  - Will be two classes taking the exam–take your own exam!
  - Bring computer + phone to scan exam to upload to gradescope afterwards.

Theorem .0.1

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces

(a) There is a measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

for every $A \in \mathcal{A}, B \in \mathcal{B}$.

(b) If μ, ν are σ-finite, $\mu \times \nu$ is unique.

Proof. Define $\pi : \mathcal{R} \to [0, \infty]$ by $\pi(A \times B) = \mu(A)\nu(B)$, and extending linearly

$$\pi(A \times B) = \mu(A)\nu(B)$$

$$\pi\left(\bigsqcup_{i=1}^{N} A_i \times B_i\right) = \sum_{i=1}^{n} \pi(A_i \times B_i).$$

Claim

π is a pre-measure

It is enough to check $\pi(A \times B) = \sum_{n=1}^{\infty} \pi(A_n \times B_n)$ if $A \times B = \bigsqcup_n A_n \times B_n$.

Since $A_n \times B_n$ are disjoint

$$1_{A \times B}(x, y) = \sum_{n=1}^{\infty} 1_{A_n \times B_n}(x, y)$$

Thus

$$1_A(x) 1_B(y) = \sum_{n=1}^{\infty} 1_{A_n}(x) 1_{B_n}(y).$$

Integrating with respect to x, and applying Tonelli's theorem for series and integrals:

$$\int_X 1_A(x) 1_B(y)\, \mathrm{d}\mu(x) = \sum_{n=1}^{\infty} \int_X 1_{A_n}(x) 1_{B_n}(y)\, \mathrm{d}\mu(x)$$

$$\mu(A) 1_B(y) = \sum_{n=1}^{\infty} \mu(A_n) 1_{B_n}(y).$$

For every y. Integrating again with respect to y and applying Tonelli's

$$\int_Y \mu(A) 1_B(y)\, \mathrm{d}\nu(y) = \sum_{n=1}^{\infty} \int_Y \mu(A_n) 1_{B_n}(y)\, \mathrm{d}\nu(y)$$

$$\mu(A)\nu(B) = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n).$$

Then π is a pre-measure, and so **??** gives $\mu \times \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

For (b), if μ, ν are σ-finite, then π is σ-finite on $\mathcal{R}$, then **??** applies.

Furthermore, we have that

$$(\mu \times \nu)(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \mid E \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

.1. Monotone Class Lemma

Definition .1.1 (Monotone Class)

If X is a set, and $C \subseteq \mathcal{P}(X)$, we say C is a monotone class on X if

- C is closed under countable increasing unions.
- C is closed under countable decreasing intersections.

Example .1.1

Of course, every σ-algebra is a monotone class.

If C_α are (arbitrarily many) monotone classes on a set X, then $\bigcap_\alpha C_\alpha$ is a monotone class. Then if $\mathcal{E} \subseteq \mathcal{P}(X)$, there is a unique smallest monotone class containing $\mathcal{E}$.

Theorem .1.1 (Monotone Class Lemma)

Suppose $\mathcal{A}_0$ is an algebra on X. Then

$$\langle \mathcal{A}_0 \rangle = \text{the monotone class generated by } \mathcal{A}_0$$

(the left hand side being the σ-algebra generated by $\mathcal{A}_0$).

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$ and let $\mathcal{C}$ be the monotone class generated by $\mathcal{A}_0$. Since $\mathcal{A}$ is a σ-algebra, it is a monotone class. It contains $\mathcal{A}_0$, and so $\mathcal{A} \supseteq \mathcal{C}$.

To show $\mathcal{C} \supseteq \mathcal{A}$, it is enough to show that $\mathcal{C}$ is a σ-algebra.

(1) $\emptyset \in \mathcal{A}_0 \subseteq \mathcal{C}$.

(2) Let $\mathcal{C}' = \{E \subseteq X \mid E^c \in \mathcal{C}\}$.
  - $\mathcal{C}'$ is a monotone class (easy)
  - $\mathcal{A}_0 \subseteq \mathcal{C}'$ because if $E \in \mathcal{A}_0$, then $E^c \in \mathcal{A}_0$, so $E^c \in \mathcal{C}$. Thus $E \in \mathcal{C}'$.

  Thus $\mathcal{C} \subseteq \mathcal{C}'$, and so $\mathcal{C}$ is closed under complements.

(3) For $E \subseteq X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.
  - $\mathcal{D}(E) \subseteq \mathcal{C}$
  - $\mathcal{D}(E)$ is a monotone class.
  - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subseteq \mathcal{D}(E)$ Why? Pick $F \in \mathcal{A}_0$. Well then $E \cup F \in \mathcal{A}_0 \subseteq \mathcal{C}$.

  Thus $C = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

(4) Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\}$. That is

$$\mathcal{D} = \{E \in \mathcal{C} \mid E \cup F, \forall F \in \mathcal{C}\}.$$

Then we see that

- $\mathcal{A}_0 \subseteq \mathcal{D}$ by Item (3)
- $\mathcal{D}$ is a monotone class (easy)
- $\mathcal{D} \subseteq \mathcal{C}$ by definition.

Thus $\mathcal{D} = \mathcal{C}$. Thus if $E, F \in \mathcal{C}$, then $E \cup F \in \mathcal{C}$. This shows that $\mathcal{C}$ is closed under finite unions.

(5) Now to show $\mathcal{C}$ is closed under countable unions, let $E_1, E_2, \ldots \in \mathcal{C}$. We may then define

$$F_N = \bigcup_{n=1}^{N} E_n \in \mathcal{C}$$

Then $F_1 \subseteq F_2 \subseteq \cdots$. Thus $\bigcup_N F_N \in \mathcal{C}$, but we see that $\bigcup_N F_N = \bigcup_n E_n$, and so we're done.

Proof.