Announcements

- Next week office hours: M 12:30-1:30, 3:05-3:50, T 1:30-2:30, No Thursday Office Hour.
- Exam: Wednesday 6-7:50, 7:50-8:00, upload to gradescope (birng your computer / phone)

Recall .0.1

$E \in \mathcal{A} \otimes \mathcal{B}$ implies $E_x \in \mathcal{B}, E^y \in \mathcal{A}$ for all $x \in X, y \in Y$.

The converse does not hold (see HW7).

.1. Fubini-Tonelli Theorem

Theorem .1.1 (Tonelli for characteristic functions)

Let $(X, \mathcal{A}, \mu)$, $(\mathcal{Y}, \mathcal{B}, \nu)$ be σ-finite measure spaces. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$. Then

(a) $\alpha(x) \coloneqq \nu(E_x) : X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.

(b) $\beta(y) \coloneqq \mu(E^y) : Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.

(c) We have that

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\,\mathrm{d}\mu(x) = \int_Y \mu(E^y)\,\mathrm{d}\nu(y)$$

Proof. This requires a few steps

(1) Assume μ, ν are <u>finite</u> measures. Let $\mathcal{C} = \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{(a),(b),(c) hold}\}$.

It is enough to prove $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{C}$. To do this, note that $\mathcal{R}$ is an algebra, so by **??** it is enough to show

- $\mathcal{R} \subseteq \mathcal{C}$
- $\mathcal{C}$ is a monotone class

Well! Let's do it!

- If $A \times B$ is a (measurable) rectangle, then

$$\alpha(x) = \nu((A \times B)_x) = \nu(B) 1_A(x).$$

This is clearly measurable, so (a) holds for $A \times B$. Similarly (b) holds. For part (c)

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$
$$\int_X \nu((A \times B)_x)\,\mathrm{d}\mu(x) = \int_X \nu(B) 1_A \,\mathrm{d}\mu = \nu(B)\mu(A).$$

Similarly for the other part of part (c). Extending to finite disjoint unions of rectangles is easy. Thus $\mathcal{R} \subseteq \mathcal{C}$

- Now let $E_n \in \mathcal{C}, E_1 \subseteq E_2 \subseteq \cdots$. We need to show $E = \bigcup_n E_n \in \mathcal{C}$. Then we see

$$E_x = \bigcup_{n=1}^{\infty} (E_n)_x, (E_1)_x \subseteq (E_2)_x \subseteq \cdots$$

Therefore by continuity from below

$$\alpha(x) = \nu(E_x) = \lim_{n \to \infty} \nu((E_n)_x) = \lim_{n \to \infty} \alpha_n(x).$$

Therefore α is a $\mathcal{A}$-measurable function. This shows (a) holds, and (b) is similar. For (c), we compute

$$\begin{aligned}(\mu \times \nu)(E) &= \lim_{n\to\infty} (\mu \times \nu)(E_n)\\ &= \lim_{n\to\infty} \int_X \nu((E_n)_x)\,\mathrm{d}\mu(x)\\ &= \int_X \nu(E_x)\,\mathrm{d}\mu(x)\end{aligned}$$

where the last equality holds by the monotone convergence theorem. Thus $E \in \mathcal{C}$. Great!

- Thus far we have not used the assumption that μ, ν are finite. Let $F_n \in \mathcal{C}$, $F_1 \supseteq F_2 \supseteq \cdots$. Need to show $F = \bigcap_n F_n \in \mathcal{C}$.
  Using that μ, ν are finite, we can use continity from above

  $$\alpha(x) = \nu(F_x) = \lim_{n\to\infty} \nu((F_n)_x) = \lim_{n\to\infty} \alpha_n(x).$$

  Great! We compute

  $$\begin{aligned}(\mu \times \nu)(E) &= \lim_{n\to\infty} (\mu \times \nu)(E_n)\\ &= \lim_{n\to\infty} \int_X \nu((E_n)_x)\,\mathrm{d}\mu(x)\\ &= \int_X \nu(E_x)\,\mathrm{d}\mu(x)\end{aligned}$$

  where we use the Dominated Convergence Theorem with $\nu(Y)$ as our dominating function, whose integral is $\mu(X)\nu(Y)$.

This proves the result when μ, ν are finite measures.

(2) Assume μ, ν are σ-finite. Write

$$\begin{gathered}X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n)\\ X_1 \subseteq X_2 \subseteq \cdots\\ Y_1 \subseteq Y_2 \subseteq \cdots\end{gathered}$$

where $\mu(X_k) < \infty, \nu(Y_k) < \infty$. DIY, for a hint let $E_n = (X_n \times Y_n) \cap E$. Note that E_n satisfies (a),(b),(c), and the argument from before showing an increasing countable union preserves these properties will hand us the result.

Theorem .1.2 (Fubini-Tonelli)

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.

(1) (Tonelli). If $f : X \times Y \to [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

(a) We have that

$$g(x) := \int_Y f(x, y)\,\mathrm{d}\nu(y) : X \to [0, \infty]$$

is a $\mathcal{A}$-measurable function.

(b) We have that

$$h(y) \coloneqq \int_X f(x,y)\,\mathrm{d}\mu(x) : Y \to [0,\infty]$$

is a $\mathcal{B}$-measurable function.

(c) We have that

$$\int_{X\times Y} f\,\mathrm{d}(\mu\times\nu) = \int_X \int_Y f(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

(2) (Fubini). If $f \in L^1(X\times Y, \mu\times\nu)$, then

(a) $f_x \in L^1(Y,\nu)$ for μ-almost every $x \in X$. Then the function

$$g(x) \coloneqq \int_Y f(x,y)\,\mathrm{d}\nu(y).$$

is defined μ-almost everywhere, and we claim $g(x) \in L^1(X,\mu)$.

(b) $f^y \in L^1(X,\mu)$ for ν-almost every $y \in Y$. Then the function

$$h(y) \coloneqq \int_X f(x,y)\,\mathrm{d}\mu(x).$$

is defined ν-almost everywhere, and we claim $h(x) \in L^1(Y,\nu)$.

(c) The iterated integral formula holds

$$\int_{X\times Y} f\,\mathrm{d}(\mu\times\nu) = \int_X \int_Y f(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

Proof. Read the textbook. Most of the work is done in Theorem .1.1, the rest is approximation.