Example .0.1 (Lebesgue-Radon-Nikodym)

Let $\mu = m$, $\nu = \mu_F$ (Lebesgue-Stieltjes measure for F). We'll define $F(x)$ by

$$F(x) = \begin{cases} e^{3x} & \text{if } x \le 0 \\ 1 & \text{if } 0 < x < 1 \\ 5 & \text{if } x \ge 1 \end{cases}.$$

Then we will have that

$$\mu_F(E) = \int_{E \cap \mathbb{R}_{<0}} 3e^{3x} \,\mathrm{d}x + 4\delta_1(E).$$

It is enough to check on $(-\infty, x]$ because these are locally finite Borel measures on $\mathbb{R}$.

Then we have $\mu_F = \mathrm{d}\rho + \mathrm{d}\lambda = f\,\mathrm{d}m + \mathrm{d}\lambda$ where $f = 1_{\mathbb{R}_{<0}} 3e^{3x}$ and $\lambda = 4\delta_1$, $\lambda \perp m$.

Read: Theorem 3.5, Proposition 3.9, Corollary 3.10 of section 3.2 of [Fol99]

Skip: Complex measures (section 3.3).

Recall .0.2

If $\nu = \nu^+ - \nu^-$, we defined the total variation $|\nu| = \nu^+ + \nu^-$, see **??**.

Then we have $|\nu(E)| \le |\nu|(E)$.

.1. Lebesgue Differentiation Theorem for regular Borel measures on $\mathbb{R}^d$

See page 99 of [Fol99].

Definition .1.1

A Borel signed measure ν on $\mathbb{R}^d$ is called regular if

(1) $|\nu|(K) < \infty$ for all compact K.

(2) We have outer regularity

$$|\nu|(E) = \inf\{|\nu|(U) \mid \text{ open } U \supseteq E\}$$

for every Borel set E.

Example .1.1

Any Lebesgue-Stieltjes measure on $\mathbb{R}$ has this property (see **????**).

In fact, so is the difference of two of them (at least if one of them is finite).

The Lebesgue measure on $\mathbb{R}^d$ is regular.

Note: From Item (1), if ν is regular then ν is σ-finite. Also if $\mathrm{d}\nu = f\,\mathrm{d}m$ is regular, then

$$|\nu|(K) = \int_K |f| \,\mathrm{d}m < \infty$$

for all compact K. Thus $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Lemma .1.1

$f \in L^1_{\text{loc}}(\mathbb{R}^d)$ if and only if $\mathrm{d}\nu = f\,\mathrm{d}m$ is regular

Proof. Skip–read the book.

Recall .1.2

Remember the Lebesgue Differentiation theorem (????).

Here we had that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ implies that for Lebesgue almost every x,

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, \mathrm{d}y = f(x)$$

for any E_r shrinking nicely to x (??, think of $B_r(x)$).

Corollary .1.2

Let ρ be a regular signed Borel measure on $\mathbb{R}^d$. Suppose $\rho \ll m$. Then $\mathrm{d}\rho = f \, \mathrm{d}m$ for some $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, So then for Lebesgue almost every x we have

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, \mathrm{d}y = f(x).$$

Writing this in a nice way, using established notation, this is

$$\lim_{r \to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for every E_r shrinking nicely to x.

Proposition .1.3

Let λ be a regular positive Borel measure on $\mathbb{R}^d$. Suppose $\lambda \perp m$.

For Lebesgue almost every x, we have

$$\lim_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for every E_r shrinking to x nicely (equivalently shrinking to 0 nicely).

Proof. It is enough to consider $E_r = B(x, r)$. We wish to prove that

$$\begin{aligned} G \coloneqq G &\coloneqq \left\{ x \mid \limsup_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} \neq 0 \right\} \\ &= \bigcup_{n=1}^{\infty} G_n \\ G_n &\coloneqq \left\{ x \mid \limsup_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} > \frac{1}{n} \right\}. \end{aligned}$$

It is enough to show $m(G_n) = 0$ for all n.

$\lambda \perp m$, so we know $\mathbb{R}^d = A \cup B$ disjoint, $\lambda(A) = 0$, $m(B) = 0$. Thus it suffices to show $m(G_n \cap A) = 0$.

Fix $\varepsilon > 0$, since λ is regular, there exists an open set $U \supseteq A$ such that $\lambda(U) \leq \lambda(A) + \varepsilon = \varepsilon$.

For every $x \in G_n \cap A$, there is an $r_x > 0$ such that $\lambda(B(x, r_x))/m(B(x, r_x)) > 1/n$ and $B(x, r_x) \subseteq U$.

Let $K \subseteq G_n \cap A$, compact. Then $K \subseteq \bigcup_{x \in K} B(x, r_x)$. By compactness, we can take a finite subcover, and then use Vitali (??) to find $B_1, B_2, \ldots, B_N$ disjoint each of type $B(x, r_x)$ such that $K \subseteq \bigcup_i 3B_i$.

Therefore

$$m(K) \leq 3^d \sum_{i=1}^{N} m(B_i) \leq 3^d n \sum_{i=1}^{N} \lambda(B_i)$$

$$= 3^d n\lambda\left(\bigcup_i B_i\right) \leq 3^d n\lambda(U) = 3^d n\varepsilon.$$

Thus by inner regularity, $m(G_n \cap A) \leq 3^d n\varepsilon$ for any $\varepsilon > 0$. Taking $\varepsilon \to 0$ yields $m(G_n \cap A) = 0$, so then $m(G_n) = 0$ as desired.