Definition .0.1 (Hardy-Littlewood Maximal Function)

For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ define the Hardy-Littlewood maximal function for f

$$Hf(x) := \sup\{A_r(x) \mid r > 0\}$$
$$A_r(x) := \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| \, \mathrm{d}y.$$

Lemma .0.1

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then

(a) $A_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.

(b) $Hf(x)$ is Borel measurable.

Proof. WTS that if $(x_n, r_n) \to (x, r)$ then $A_{r_n}(x_n) \to A_r(x)$. Well

$$A_{r_n}(x_n) = \int |f(y)| \, 1_{B(x_n, r_n)} \to \int |f(y)| \, 1_{B(x,r)} = A_r(x)$$

because $1_{B(x_n,r_n)} \to 1_{B(x,r)}$ and we can use the dominated convergence theorem (bound r_n by R, x_n within R' of x, then $B(x_n, r_n) \subseteq B(x, R + R')$).

Now note that $(Hf)^{-1}((a, \infty)) = \bigcup_{r>0} A_r^{-1}((a, \infty))$ for any $a \in \mathbb{R}$. The right hand side is open, and so the preimage of (a, ∞) under Hf is always open, so Hf is Borel.

Recall .0.1

Markov ??,

$$m(\{x \in \mathbb{R}^d \mid |f(x)| \geq \alpha\}) \leq \frac{1}{\alpha} \int |f(x)| \, \mathrm{d}x.$$

Theorem .0.2 (Hardy-Littlewood maximal inequality)

There exists $C_d > 0$ depending only on d such that for all $f \in L^1(\mathbb{R}^d)$, for all $\alpha > 0$

$$m(\{x \in \mathbb{R}^d \mid Hf(x) > \alpha\}) \leq \frac{C_d}{\alpha} \int |f(x)| \, \mathrm{d}x.$$

We show $C_d = 3^d$ suffices.

Proof. Let $f \in L^1(\mathbb{R}^d)$ and let $\alpha > 0$. let $E = \{x \mid (Hf)(x) > \alpha\}$, which is Borel measurable by the lemma.

Then if $x \in E$, then there exists an $r_x > 0$ so that $A_{r_x}(x) > \alpha$. That is

$$m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x,r_x)} |f(y)| \, \mathrm{d}y.$$

Inner regularity gives that $m(E) = \sup\{m(K) \mid K \subseteq E, K \text{ compact}\}$. Let $K \subseteq E$ be compact. Then

$$K \subseteq \bigcup_{x \in K} B_{r_x}(x)$$

Thus $K \subseteq \bigcup_{i=1}^{N} B_i$. By Vitali (**??**) we may take $K \subseteq \bigcup_{j=1}^{m}(3B_j')$ where the B_j' are disjoint. Then

$$\begin{aligned} m(K) &\leq \sum_{j=1}^{m} m(3B_j') = 3^d \sum_{j=1}^{m} m(B_j') \\ &\leq \frac{3^d}{\alpha} \sum_{j=1}^{m} \int_{B_j'} |f(y)| \, \mathrm{d}y \\ &\leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \, \mathrm{d}y \end{aligned}$$

where the last line uses the disjointness. Now taking $\sup_K$ preserves the bound!