I. Introduction & Syllabus

- Location: 269 Weiser
- Professor: Jinho Baik
- Syllabus – tentative
- Primary Text: Real analysis 1-3 (5,6,8) by Folland [Fol99].
- Supplementary Texts: Axler's measure theory and Tao's introduction to measure theory [Axl20; Tao11].

Broad strokes of the course

(1) Abstract measure Section II
(2) Integration
(3) Product measures
(4) Differentiation
(5) Signed and complex measures
(6) Banach and Hilbert Spaces
(7) L^p spaces
(8) Introduction to Fourier analysis.

Read [Fol99] Chapter 0, the following sections

0.1 Set theory
0.5 Extended real number system
0.6 Metric spaces (may skip Theorem 0.25).

II. Abstract Measure Theory

II.1. Motivation

For a set X, let $P(X)$ be the collection of all subsets of X (the power set).

Note that $P(X)$ has a lot more elements than X, for example if X is finite then $\#P(X) = 2^{\#X}$. Furthermore, $\mathbb{N}$ is obviously countable, but $P(\mathbb{N})$ is uncountable.

We can see this because there is a surjective function $\phi : P(\mathbb{N}) \to [0,1]$ given by

$$\phi(A) = 0.a_1a_2\cdots \qquad \text{(base 2)}$$

where

$$a_k = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{if } k \notin A \end{cases}.$$

Therefore $\#P(\mathbb{N}) \geq \#[0,1]$ (see 0.3 Cardinality if you like, not necessary). In general we always have $\#P(X) > \#X$.

We like to "measure" the "size" of subsets of X

Example II.1.1

Here are some good examples!

- Let $X = \{0, 1, 2, 3\}$. Then there are measures $\mu : P(X) \to [0, \infty]$ as below

$$\mu(A) = \#A \qquad \mu(\{0,1\}) = 2$$
$$\mu(A) = \sum_{i \in A} 2^i \qquad \mu(\{0,1\}) = 3$$

- For $X = \mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$ we have $\mu : P(\mathbb{N}_0) \to [0, \infty]$ via

$$\mu(A) = \#A \qquad \mu(\{2,4,6,\ldots\}) = \infty$$
$$\mu(A) = e^{-1} \sum_{i \in A} \frac{1}{i!} \qquad \mu(X) = 1$$

  Using $\mu(\{i\}) = a_i$ and $\mu(A) = \sum_{i \in A} a_i$ is a good measurement of A. This works because the sums are always countable.

- $X = \mathbb{R}$, want $\mu : P(\mathbb{R}) \to [0, \infty]$

$$\mu(A) = \#A \qquad \text{not interesting}$$
$$\mu(A) = \text{length of A?}.$$

  Here we'd have $\mu((a,b)) = b - a$. Can we extend μ reasonably to all subsets of $\mathbb{R}$? What if instead we take $\mu((a,b)) = e^b - e^a$? We also can't extend this to all subsets!

Theorem II.1.1 (Banach-Tarski)

In $\mathbb{R}^d$, for $d \geq 3$, one can divide a ball into finitely many subsets and put back into two balls of same radius.

We will try to define a "measure" on X, that is $\mu : \mathcal{A} \to [0, \infty]$ for a "suitable" $\mathcal{A} \subseteq P(X)$.

II.2. σ-algebras

Definition II.2.1

Let X be a set. A collection $\mathcal{A}$ of subsets of X is called a σ-algebra on X provided that

- $\emptyset \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements, aka if $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$.
- $\mathcal{A}$ is closed under countable unions. that is if $E_1, E_2, \ldots \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

These have some simple properties

- $X = \emptyset^c \in \mathcal{A}$.
- It's closed under countable intersections because

$$\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{\infty} E_i^c \right)^c$$

- $\bigcup_{i=1}^{N} E_i = E_1 \cup \cdots \cup E_N \cup \emptyset \cup \cdots$. So it is closed under finite unions (+ intersections).
- Closed under $E \setminus F = E \cap F^c$ and $E \triangle F = (E \cup F) \setminus (E \cap F)$.