Announcements

- HW7 due Thursday
- Avg/Median of Exam will move to an A-

.1. Lebesgue Differentiation Theorem

We should compare the Hardy-Littlewood inequality (**??**) to Markov's inequality (**??**). Namely there exists $C_d > 0$ (can take 3^d) such that for all $f \in L^1(\mathbb{R}^d)$, $\alpha > 0$ we have

$$m(\{x \mid (Hf)(x) > \alpha\}) \leq \frac{C_d}{\alpha} \int |f|$$
$$m(\{x \mid |f(x)| > \alpha\}) \leq \frac{1}{\alpha} \int |f|$$

Theorem .1.1

Let $f \in L^1$. Then

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0$$

for almost every x.

Proof. The result holds for $f \in C_c(\mathbb{R}^d)$, continuous with compact support (check). Why? Well then for any $\varepsilon > 0$ if r is small $|f(y) - f(x)| < \varepsilon$, so then the quantity

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y < \varepsilon.$$

Great!

Now let $f \in L^1(\mathbb{R}^d)$. Fix $\varepsilon > 0$. By density there exists $g \in C_c(\mathbb{R}^d)$ with $\|f - g\|_1 < \varepsilon$. We have

$$\int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y \leq \int_{B(x,r)} |f(y) - g(y)| \, \mathrm{d}y + \int_{B(x,r)} |g(y) - g(x)| \, \mathrm{d}y + \int_{B(x,r)} |g(x) - f(x)| \, \mathrm{d}y$$

Dividing all of these by $m(B(x,r))$, and taking $\limsup$ as $r \to 0$, we need to understand the error terms

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - g(x)| \, \mathrm{d}y = |g(x) - f(x)|$$
$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g(y)| \, \mathrm{d}y \leq (H(f - g))(x).$$

Define

$$Q(x) = \limsup_{r \to 0} \frac{1}{m(B(x,r))} \int B(x,r) \, |f(y) - f(x)| \, \mathrm{d}y.$$

We want to show $m(\{x \mid Q(x) > 0\}) = 0$. Let $E_\alpha = \{x \mid Q(x) > \alpha\}$. It is enough to show $m(E_\alpha) = 0$ for all $\alpha > 0$, because $\{x \mid Q(x) > 0\} = \bigcup_n E_{1/n}$. We know by the above that

$$Q(x) \leq (H(f - g))(x) + 0 + |g(x) - f(x)|.$$

Therefore

$$E_\alpha \subseteq \{x \mid (H(f - g))(x) > \alpha/2\} \cup \{x \mid |g(x) - f(x)| > \alpha/2\}.$$

By the Hardy-Littlewood maximal inequality and Markov

$$m(\{x \mid (H(f-g))(x) > \alpha/2\}) \leq \frac{2C_d}{\alpha}\int |f-g|$$
$$m(\{x \mid |g(x)-f(x)| > \alpha/2\}) \leq \frac{2}{\alpha}\int |f-g|$$

Thus

$$0 \leq m(E_\alpha) \leq \frac{2C_d}{\alpha}\|f-g\|_1 + \frac{2}{\alpha}\|f-g\|_1 \leq \frac{2(C_d+1)}{\alpha}\varepsilon$$

Taking $\varepsilon \to 0$, $m(E_\alpha)$ does not depend on ε, g so $m(E_\alpha) = 0$.

Corollary .1.2

This also holds for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$

Proof. DIY, partition $\mathbb{R}^d$ into countably many compact sets K_i then apply the theorem to $f1_{K_i}$ for each i.

Corollary .1.3

For $f \in L^1_{\text{loc}}$ for almost every x, we have

$$\lim_{r\to 0} \frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)\,\mathrm{d}y = f(x)$$

Proof. DIY, use that $f(x) = \frac{1}{m(B(x,r))}\int_{B(x,r)} f(x)\,\mathrm{d}y$ and the triangle inequality.

Definition .1.1

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The point $x \in \mathbb{R}^d$ is called a Lebesgue point of f if

$$\lim_{r\to 0} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\,\mathrm{d}y = 0$$

Corollary .1.2 tells us that almost all points in $\mathbb{R}^d$ are Lebesgue points for f.

Definition .1.2

We say measurable sets $\{E_r\}_{r>0}$ shrink nicely to x as $r \to 0$ if and only if $E_r \subseteq B(x,r)$ and there exists $c > 0$ such that $c \cdot m(B(x,r) \leq m(E_r)$.

Corollary .1.4 (Lebesgue Differentiation Theorem)

Suppose E_r shrink nicely to 0, $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, x a Lebesgue point of f. Then

$$\lim_{r\to 0} \frac{1}{m(E_r)}\int_{E_r+x} |f(y)-f(x)|\,\mathrm{d}y = 0$$
$$\lim_{r\to 0} \frac{1}{m(E_r)}\int_{E_r+x} f(y)\,\mathrm{d}y = f(x)$$

Corollary .1.5

If $f \in L^1_{\text{loc}}(\mathbb{R})$ then $F(x) = \int_0^x f(y)\,\mathrm{d}y$ is differentiable and $F'(x) = f(x)$ almost everywhere.

The rest of Chapter 3 of [Fol99] we will cover later (in 2-3 weeks).

I. Normed Vector Spaces

Folland sections 5.1,6.1,6.2 [Fol99].

I.1. Metric Spaces and Normed Spaces

Definition I.1.1

Let Y be a set, a function $\rho : Y \times Y \to [0, \infty)$ is a metric on Y provided that

(1) $\rho(x, y) = \rho(y, x)$

(2) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

(3) $\rho(x, y) = 0$ if and only if $x = y$.

The following make sense in a metric space

- Open/closed balls.
- Open/closed sets.
- Convergence sequences ($x_n \to x$ with respect to ρ if and only if $\lim_{x\to\infty} \rho(x_n, x) = 0$).
- Continuous functions.

Example I.1.1

We have the following metric spaces

(1) $\mathbb{Q}$, $\rho(x, y) = |x - y|$.

(2) $\mathbb{R}$, $\rho(x, y) = |x - y|$.

(3) $\mathbb{R}_+, \rho(x, y), |\ln(y/x)|$.

(4) $\mathbb{R}^d$, with

$$
\begin{aligned}
d_p(x, y) &= \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p} \\
d_\infty(x, y) &= \max_{1\leq i\leq d} |x_i - y_i| .
\end{aligned}
$$

These all give the same open sets (topologically equivalent)

(5) $C([0, 1])$, with

$$
\begin{aligned}
d_p(f, g) &= \left(\int_0^1 |f - g|^p\right)^{1/p} \\
d_\infty(f, g) &= \max_{x\in[0,1]} |f(x) - g(x)|
\end{aligned}
$$

(6) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Let Y be the set of measurable functions on X. Then

$$\rho(f, g) = \int \min(|f(x) - g(x)| , 1) \,\mathrm{d}\mu(x)$$

is a metric and $f_n \to f$ in ρ if and only if $f_n \to f$ in measure.