Announcements

- Solutions to HW 2 posted
- HW 3 due tomorrow
- HW 4 posted.

Recall .0.1

Let $X \subseteq [0, \infty]$. Then

(a) $\alpha = \sup X < \infty$ if and only if for all $x \in X, \alpha \geq x$, and for all $\varepsilon > 0$ there exists $x \in X$ such that $x + \varepsilon \geq \alpha$.

(b) $\alpha = \sup X = \infty$ if and only if for all $L > 0$ there exists $x \in X$ such that $x \geq L$.

Theorem .0.1

Let μ be an LS measure. Then, for all $A \in \mathcal{A}_\mu$,

(a) Outer regularity: $\mu(A) = \inf\{\mu(U) \mid \text{open } U \supseteq A\}$.

(b) Inner regularity: $\mu(A) = \sup\{\mu(K) \mid \text{compact } K \subseteq A\}$

Proof. Check part (a).

For part (b), let $S = \sup\{\cdots\}$. Montonicity then implies that $\mu(A) \geq s$. We must establish that $s \geq \mu(A)$.

(i) Assume A is a bounded set. Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{A}_\mu$. Then because $\overline{A}$ is bounded, $\mu(\overline{A}) < \infty$.

Fix $\varepsilon > 0$. We will show $s + \varepsilon \geq \mu(A)$. By (a), there exists an open set $U \supseteq \overline{A} \setminus A$ such that $\mu(U) - \mu(\overline{A} \setminus A) \leq \varepsilon$. But then

$$\mu(U) - \mu(\overline{A} \setminus A) \leq \varepsilon\mu(U \setminus (\overline{A} \setminus A)) \qquad \leq \varepsilon$$

Now let $K = A \setminus U$. Note that $K = \overline{A} \setminus U$. This tells us that K is compact, since it is a compact set cut an open set. Furthermore $K \subseteq A$.

It remains to show that $\mu(K) \geq \mu(A) - \varepsilon$. (DIY).

This shows that $s \geq \mu(K) \geq \mu(A) - \varepsilon$, and so we have the result by taking $\varepsilon \to 0$.

(ii) Suppose A is unbounded but $\mu(A) < \infty$. Write $A = \bigcup_{n=1}^\infty A_n$ where $A_n = A \cap [-n, n]$.

Then of course $A_1 \subseteq A_2 \subseteq \cdot$ we have that

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) < \infty.$$

Then for any $\varepsilon > 0$ we can find an A_N so that $\mu(A) - \mu(A_N) < \varepsilon$. We can also find a compact K such that $K \subseteq A_N \subseteq A$ so that $\mu(A_N) - \mu(K) < \varepsilon$.

Then the total error is $\mu(A) - \mu(K) < 2\varepsilon$. This shows the result.

(iii) Suppose $\mu(A) = \infty$. We still have $\lim_{n\to\infty} \mu(A_n) = \mu(A) = \infty$. This means for any $L > 0$, we can find some N such that $\mu(A_N) \geq L$.

But then we can find a compact set K so that $K \subseteq A_N \subseteq A$ so that $\mu(A_N) - \mu(K) < 1$, so $\mu(K) \geq L - 1$. Since L was arbitrary we see $s = \infty$ as desired.

Definition .0.1

Let X be a topological space.

A G_δ-set is $G = \bigcap_{i=1}^\infty U_i$ for U_i open.

An F_σ-set is $F = \bigcup_{i=1}^\infty F_i$, for F_i closed.

Theorem .0.2

Suppose μ is an LS measure. Then, the following are equivalent

(a) $A \in \mathcal{A}_\mu$

(b) $A = G \setminus M$, for G a G_δ-set, M μ-null.

(c) $A = F \cup N$, for F a F_σ-set, N μ-nul.

Proof. Clearly (b) $\implies$ (a) and (c) $\implies$ (a).

(a) $\implies$ (c) We do this in cases

(i) Assume $\mu(A) < \infty$. Inner regularity gives for every $n \in \mathbb{N}$ there exists a compact set $K_n \subseteq A$ such that $\mu(K_n) + 1/n \geq \mu(A)$.

Let $F = \bigcup_{n=1}^\infty K_n$. We must show $N = A \setminus F$ is μ-null. This is not too difficult.

(ii) Assume $\mu(A) = \infty$. Write $A = \bigcup_{k\in\mathbb{Z}} A_k$ such that $A_k = A \cap (k, k+1]$.

By (i), for every $k \in \mathbb{Z}$, $A_k = F_k \cup N_k$. Then $A = \bigcup_k F_k \cup \bigcup_k N_k$. These satisfy the necessary conditions.

(a) $\implies$ (b) Apply (c) to A^c to get $A^c = F \cup N$. But then

$$A = F^c \cap N^c = F^c \setminus N.$$

This means we are done.

Proposition .0.3

Let μ be an LS measure, $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$.

Then for all $\varepsilon > 0$, there exists an $I = \bigcup_{i=1}^{N(\varepsilon)} I_i$ disjoint open intervals such that $\mu(A \triangle I) \leq \varepsilon$.

Proof. DIY. Use outer regularity and the fact that every open set can be written as a countable union of disjoint open intervals.

.0.1. Properties of Lebesgue measure

Theorem .0.4

If $A \in \mathcal{L}$ then $A + s \in \mathcal{L}, rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$ and $m(A+s) = m(A)$, $m(rA) = |r|\, m(A)$.

Proof. DIY, should propogate from the fact that it's true for the h-intervals.

Example .0.2

We have the following strange behavior

(a) Let $\mathbb{Q} = \{r_i\}_{i=1}^\infty$ (which is dense in $\mathbb{R}$).

Let $\varepsilon > 0$. Now let $U = \bigcup_{i=1}^{\infty}(r - \varepsilon/2^i, r_i + \varepsilon/2^i)$. We know that U is an open subset which is dense in $\mathbb{R}$, and necessarily

$$m(U) \leq \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon.$$

But then $\partial U = \overline{U} \setminus U = \mathbb{R} \setminus U$ has measure $m(\partial U) = \infty$.

(b) There exists an uncountable set A with $m(A) = 0$.

(c) There exists A with $m(A) > 0$, but A contains no non-empty open interval.

(d) $A \notin \mathcal{L}$ Vitali set