We now prove **??**, completing the proof of **??** (which relies on this result).

Proof. Lets go!

($\Longrightarrow$) Suppose V is complete, and fix an absolutely convergent series $\sum_n v_n$. Define $s_N = \sum_{n=1}^N v_n$. It suffices to show the partial sums are a Cauchy Sequence.

Fix $\varepsilon > 0$, then because $\sum_{n=1}^\infty \|v_n\| < \infty$, there is an $K \in \mathbb{N}$ so that

$$\sum_{n=K}^{\infty} \|v_n\| < \varepsilon.$$

Now let $M > N > K$, we see that

$$\begin{aligned} \|s_M - s_N\| = \left\| \sum_{n=N+1}^{M} v_n \right\| &\leq \sum_{n=N+1}^{M} \|v_n\| \\ &\leq \sum_{n=N}^{\infty} \|v_n\| < \varepsilon. \end{aligned}$$

So this is Cauchy.

($\Longleftarrow$) Now suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. For all $j \in \mathbb{N}$, there exists an $N_j \in \mathbb{N}$ such that

$$\|v_n - v_m\| < \frac{1}{2^j}$$

for all $n, m \geq N_j$. WLOG, may assume $N_1 < N_2 < \cdots$.

Let $w_1 = v_{N_1}$, $w_j = v_{N_j} - v_{N_j - 1}$ for $j \geq 2$. Therefore

$$\sum_{j=1}^{\infty} \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} < \infty.$$

Thus $\sum_{j=1}^k w_j \to s \in V$ as $k \to \infty$. But by telescoping

$$v_{N_k} = \sum_{j=1}^{k} w_j \to s.$$

Now we claim that since v_n is Cauchy that $v_n \to s$.

Explicitly, take $\varepsilon > 0$, and let k be large enough so that $\|v_{N_k} - s\| < \varepsilon$ and $1/2^k < \varepsilon$. Then if $n > N_k$ then

$$\|v_n - s\| \leq \|v_n - v_{N_k}\| + \|v_{N_k} - s\| < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $v_n \to s$.

.1. Bounded Linear Transformations (BLTs)

Definition .1.1

Let $(V, \|\cdot\|), (W, \|\cdot\|')$ be normed spaces. A linear map $T : V \to W$ is called a bounded map if there exists $c \geq 0$ such that

$$\|Tv\|' \leq c\|v\|$$

for all $v \in V$.

Proposition .1.1

Suppose $T : (V, \|\cdot\|) \to (W, \|\cdot\|')$ is a linear map. Then the following are equivalent

(a) T is continuous

(b) T is continuous at 0

(c) T is a bounded map

Proof. (a) $\implies$ (b) is clear. For (b) $\implies$ (c) take $\varepsilon = 1$, then there exists a $\delta > 0$ such that $\|Tu\|' < 1$ if $\|u\| < \delta$.

Now take an arbitrary $v \in V$, $v \neq 0$. Let $u = \frac{\delta}{2\|v\|}v$. Then $\|u\| < \delta$. Therefore

$$\begin{aligned} \|Tu\|' &< 1 \\ \frac{\delta}{2\|v\|}\|Tv\|' &< 1 \\ \|Tv\|' &< \frac{2}{\delta}\|v\|. \end{aligned}$$

Then $2/\delta$ is our constant.

For (c) $\implies$ (a). Fix $v_0 \in V$. Then for some constant c

$$\|Tv - Tv_0\|' = \|T(v - v_0)\|' \leq c\|v - v_0\|.$$

Thus T is continuous, as when $v \to v_0$ the right hand side goes to zero, and so $Tv \to Tv_0$.

Example .1.1

Example time!

- We can look at

$$\begin{aligned} T : \ell^1 &\to \ell^1 \\ (a_1, a_2, \ldots) &\mapsto (a_2, a_3, \ldots). \end{aligned}$$

Then clearly $\|Ta\|_1 \leq \|a\|_1$, so T is a BLT.

- We can also look at $S : (C([-1, 1]), \|\cdot\|_1) \to \mathbb{C}$, where $Sf = f(0)$. S is not a BLT, because we can make

$$\begin{aligned} \|Tf\| &= |f(0)| = n \\ \|f\|_1 &= 1 \end{aligned}$$

for every $n \in \mathbb{N}$ (take f's graph to be a skinny triangle shooting up to n at 0).

- But $U : (C([-1, 1]), \|\cdot\|_\infty) \to \mathbb{C}$ defined by $Uf = f(0)$ is a BLT, because $|f(0)| \leq \|f\|_\infty$.
- Let A be an $n \times m$ matrix. Then $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $v \mapsto Av$ is a BLT.

Explicitly this is

$$(Tv)_i = (Av)_i = \sum_{j=1}^{m} A_{ij} v_j$$

- Let $K(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$. We'll define

$$T : (C[0, 1], \|\cdot\|_\infty) \to (C[0, 1], \|\cdot\|_\infty)$$

by

$$(Tf)(x) = \int_0^1 K(x, y) f(y)\,\mathrm{d}y.$$

This is an analogue of matrix multiplication (K is like a continuous matrix). This is a BLT.

- Lets look at $T : L^1(\mathbb{R}) \to (C(\mathbb{R}), \|\cdot\|_\infty)$ defined by

$$(Tf)(t) = \int_{-\infty}^{\infty} e^{-itx} f(x)\,\mathrm{d}x$$

that is the Fourier transform of f.

- $T : (C^\infty[0, 1], \|\cdot\|_\infty) \to (C^\infty[0, 1], \|\cdot\|_\infty)$. Define

$$(Tf)(x) = f'(x).$$

This is not a BLT. In contrast S, defined on the same spaces

$$(Sf)(x) = \int_0^x f(t)\,\mathrm{d}t$$

is bounded.

Definition .1.2

Let $L(V, W) = \{T : V \to W \mid T \text{ is a BLT}\}$, which is a vector space. For $T \in L(V, W)$, the operator norm of T is

$$\begin{aligned}
\|T\| &:= \inf\{c \geq 0 \mid \|Tv\| \leq c\|v\| \text{ for all } v \in V\} \\
&= \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \neq 0, v \in V\right\} \\
&= \sup\{\|Tv\| \mid \|v\| = 1, v \in V\}
\end{aligned}$$

check the equalities above.

Lemma .1.2

We have that

(a) The three definitions of $\|T\|$ above are all equal.
(b) $(L(V, W), \|\cdot\|)$ is indeed a normed space.

Proof. DIY.