Announcements

- Solutions for HW 3 posted
- HW 4 due tomorrow
- HW 5 will be posted.

Corollary .0.1 (Tonelli's for series and integrals)

For $g_n \geq 0$ and all measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Proof. Let $f_N = \sum_{n=1}^{N} g_n$. Then because $g_n \geq 0$ we have $0 \leq f_1 \leq f_2 \leq \cdots$. Furthermore

$$\lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

Thus **??** implies that

$$\begin{aligned} \lim_{N\to\infty} \int \sum_{n=1}^{N} g_n &= \int \sum_{n=1}^{\infty} g_n \\ \lim_{N\to\infty} \sum_{n=1}^{N} \int g_n &= \int \sum_{n=1}^{\infty} g_n \\ \sum_{n=1}^{\infty} \int g_n &= \int \sum_{n=1}^{\infty} g_n \end{aligned}$$

Theorem .0.2 (Fatou's Lemma)

Suppose $f_n \geq 0$ are measurable functions. Then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Recall .0.1

$\liminf$ obeys the following

$$\begin{aligned} \liminf_{n\to\infty} f_n &:= \lim_{k\to\infty} \inf_{n\geq k} f_n \\ &= \sup_{k\in\mathbb{N}} \inf_{n\geq k} f_n. \end{aligned}$$

Furhtermore we have that

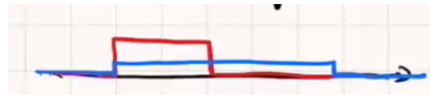
$$\lim_{n\to\infty} a_n \text{ exists} \iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$$

Proof. Let $g_k = \inf_{n\geq k} f_n$. Then each g_k is measurable and $0 \leq g_1 \leq g_2 \leq \cdots$.

Therefore by **??** we have

$$\int \liminf_{n\to\infty} f_n = \int \lim_{k\to\infty} g_k = \lim_{k\to\infty} \int g_k = \lim_{k\to\infty} \int \inf_{n\geq k} f_n.$$

(A) Escape to Horizontal ∞



(B) Escape to Width ∞

(C) Escape to Vertical ∞

We now know that $\inf_{n\geq k} f_n \leq f_m$ for all $m \geq k$. Therefore by monotonicity

$$\int \inf_{n\geq k} f_n \leq \int f_m \qquad (\forall\ m \geq k)$$
$$\int \inf_{n\geq k} f_n \leq \inf_{m\geq k} \int f_m.$$

Therefore

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \lim_{k\to\infty} \inf_{m\geq k} \int f_m = \liminf_{m\to\infty} \int f_m$$

This is exactly the result we wish to show!

Example .0.2

We'll use $(\mathbb{R}, \mathcal{L}, m)$.

(a) Escape to horizontal infinity: Take $f_n = 1_{(n,n+1)}$. Then $\int f_n = 1$ for all n, but $f_n \to 0$ pointwise. Thus Fatou's Lemma give us a strict inequality

$$0 = \int \liminf_{n\to\infty} f_n < \liminf_{n\to\infty} \int f_n = 1.$$

See Figure 1a.

(b) Escape to width infinity: Take $f_n = 1/n \cdot 1_{(0,n)}$. Then $\int f_n = 1$ for all n, but $f_n \to 0$ pointwise as well. See Figure 1b

(c) Escape to vertical infinity: Take $f_n = n1_{(0,1/n)}$. Then $\int f_n = 1$ for all n, but $f_n \to 0$ pointwise. See Figure 1c

Lemma .0.3 (Markov's Inequality)

Let $f \geq 0$ be measurable. Then for all $c \in [0, \infty]$ we have that

$$\mu(\underbrace{\{x \mid f(x) \geq c\}}_{E}) \leq \frac{1}{c} \int f.$$

Proof. We have that $f(x) \geq c1_E(x)$, and so by monotonicity

$$\int f \geq c \int 1_E = c\mu(E).$$

Proposition .0.4

If $f \geq 0$ is measurable, then

$$\int f = 0 \iff f = 0 \text{ almost everywhere.}$$

Namely if we let $A = \{x \mid f(x) > 0\}$ then

$$\int f \, \mathrm{d}\mu = 0 \iff \mu(A) = 0.$$

Recall that

$$\int f = \sup \left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}$$

Proof. We do this in steps

(1) Assume $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^{N} c_i 1_{E_i}$$

where E_i are disjoint and $c \in (0, \infty]$. Then saying that

$$\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0$$

if and only if$\mu(E_i) = 0$ for all i. Then this holds if and only if $\mu(A) = 0$ because $A = \bigcup_{i=1}^{N} E_i$.

(2) For general $f \geq 0$, we have

(a) Assume $\mu(A) = 0$. That is $f = 0$ almost everywhere. Now let $0 \leq \phi \leq f$ for ϕ simple. Then for all $x \in A^c$ we have $\phi(x) = 0$. Thus $\phi = 0$ almost everywhere, and $\int \phi = 0$.
Thus $\int f = 0$ by the definition of $\int f$.

(b) Now assume $\int f = 0$. Let $A_n = f^{-1}([-1/n, \infty])$. Then $A_1 \subseteq A_2 \subseteq \cdots$. We then know that

$$\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty \right] \right) = f^{-1}((0, \infty)) = A.$$

By continuity of the measure we know that

$$\mu(A) = \lim_{n \to \infty} \mu(A_n).$$

By Lemma .0.3, we see that

$$0 \leq \mu(A_n) \leq n \int f = 0.$$

Great! This shows that $\mu(A) = 0$.

Corollary .0.5

If $f, g \geq 0$ are measurable, and $f = g$ almost everywhere, then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}$. By assumption, $\mu(A) = 0$. Then

$$\begin{aligned} \int f &= \int f 1_A + \int f 1_{A^c} = 0 + \int g 1_{A^c} \\ &= \int g 1_A + \int g 1_{A^c} = \int g. \end{aligned}$$

Note: Almost all the theorems we've proved can be replaced by theorems dealing with almost everywhere conditions ☺