Note that $\|Tv\|'' \leq \|T\|\|v\|'$ for $T : (V, \|\cdot\|') \to (W, \|\cdot\|'')$

**Theorem .0.1**

If $W$ is complete, then $L(V, W)$ is complete.

*Proof.* Suppose $T_n$ is a Cauchy sequence in $L(V, W)$. Fix $v \in V$. Then, let $w_n = T_n v \in W$. Also

$$\|w_n - w_m\| = \|T_n v - T_m v\| = \|(T_n - T_m)v\| \leq \|T_n - T_m\|\|v\|.$$

Thus $w_n$ is Cauchy, so it converges since $W$ is complete. We call its unique limit $Tv$. This makes $T : V \to W$ into a function. We must show it is a BLT and that $\|T_n - T\| \to 0$.

Finish the proof! See book or DIY

## .1. Dual of $L^p$ spaces

**Example .1.1**

Let $w \in \mathbb{R}^d$. Then we can consider

$$\max\{v \cdot w \mid \|v\|_2 = 1\} = \|w\|_2.$$

If $w \in \mathbb{C}^d$, this is similar we just do

$$\max\{|v \cdot w| \mid \|v\|_2 = 1\} = \|w\|_2.$$

These maxes are achieved by $v = \frac{\overline{w}}{\|w\|_w}$ if $w \neq 0$.

**Proposition .1.1**

Let $1/p + 1/q = 1$ with $1 \leq q < \infty$. For every $g \in L^q$,

$$\|g\|_q = \sup\left\{\left|\int fg\right| \mid \|f\|_p = 1\right\}.$$

Suppose $\mu$ is $\sigma$-finite. Then the result also holds for $q = \infty$, $p = 1$.

**Recall .1.2**

For $\alpha \in \mathbb{C}$, $\operatorname{sgn}\alpha := e^{i\theta}$ where $\alpha = |\alpha| e^{i\theta}$.

*Proof.* By Hölder's inequality we know that

$$\left|\int fg\right| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p\|g\|_q = \|g\|_q.$$

Thus the supremum is $\leq \|g\|_q$.

(1) Let

$$f(x) = \frac{\left|g(x)\overline{\operatorname{sgn}(g(x))}\right|^{q-1}}{\|g\|_q^{q-1}}$$

Then $\int |f|^p = 1$, and $\int fg = \|g\|_q$. Check this!

(2) DIY for handling the case when $\mu$ is $\sigma$-finite and $q = \infty, p = 1$.

**Remark .1.1**

One could use the above to prove Minkowski's inequality (as it only uses Hölder)

**Definition .1.1**

For a normed spave $(V, \|\cdot\|)$ its dual space is $V^* = L(V, \mathbb{R})$ or $V^* = L(V, \mathbb{C})$ (aka BLTs with codomain the scalar field).

$\ell \in V^*$ is called a linear functional on $V$. This means exactly that

- $\ell : V \to \mathbb{R}$ (or $\mathbb{C}$)
- $\ell$ linear
- There exists a $c \geq 0$ such that $|\ell(v)| = c\|v\|$.

Note: $V^*$ is always a Banach space (even if $V$ is not complete).

**Corollary .1.2**

We have the following:

(1) Let $1/p + 1/q = 1, 1 \leq q < \infty$. For $g \in L^q$ define $\ell_g \in L^p \to \mathbb{C}$ by

$$\ell_g(f) = \int fg.$$

Then $\ell_g \in (L^p)^*$. Furthermore, $\|\ell_g\| = \|g\|_q$.

(2) If $\mu$ is $\sigma$-finite then this also holds for $q = \infty, p = 1$.

*Proof.* $\ell_g$ is clearly linear in $f$ because the integral is linear. Then Proposition .1.1 gives in both (1) and (2) that

$$\|g\|_q = \sup\{|\ell_g(f)| \mid \|f\|_p = 1\} = \|\ell_g\|$$

and so $\ell_g$ is a BLT with the desired properties.