To summarize last class

- $\|f\|_p = \left(\int |f|^p\right)^{1/p}$ is a norm if $1 \le p < \infty$.
- Hölder's Inequality (**??**) says that $\|fg\|_1 \le \|f\|_p \|g\|_q$ for $1/p + 1/q = 1$. That is

$$\int |fg| \le \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

- Minkowski's Inequality (**??**) says that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ for $1 \le p < \infty$.

Definition .0.1

For a measurable function f on $(X, \mathcal{A}, \mu)$ we define

$$S = \{\alpha \ge 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\} = \{\alpha \ge 0 \mid |f(x)| \le \alpha \text{ almost everywhere}\}$$

Define the essential supremum of f to be $\|f\|_\infty = \inf S$ if $S \ne 0$ and $\|f\|_\infty = \infty$ if $S = \emptyset$.

Let $L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}$, and $\ell^\infty = L^\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ where ν is the counting measure.

Example .0.1

Consider $(\mathbb{R}, \mathcal{L}, m)$. Then

$$f(x) = \frac{1}{x} 1_{(0,\infty)}(x) \notin L^\infty$$
$$g(x) = x 1_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty.$$

If f is continuous on $(\mathbb{R}, \mathcal{L}, m)$ then $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$.

For $a \in \ell^\infty$ we have $\|a\|_\infty = \sup_{i \in \mathbb{N}} |a_i|$, and sequences in ℓ^∞ are exactly the bounded sequences.

Lemma .0.1

We have the following

(1) Suppose $f \in L^\infty(X, \mathcal{A}, \mu)$. For $\alpha \ge \|f\|_\infty$, we have $\mu(\{x \mid |f(x)| > \alpha\}) = 0$.

For $\alpha < \|f\|_\infty$ we have $\mu(\{x \mid |f(x)| > \alpha\}) > 0$.

(2) In particular, $|f(x)| \le \|f\|_\infty$ almost everywhere.

(3) $f \in L^\infty$ if and only if there exists a bounded measurable function g such that $f = g$ almost everywhere.

Proof. DIY.

Theorem .0.2

We have that

(1) $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$ (motivation: $1/1 + 1/\infty = 1$).

(2) $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

(3) $f_n \to f$ in L^∞ if and only if $f_n \to f$ uniformly almost everywhere (i.e., there is a null set A such that $f_n \to f$ uniformly on A^c).

Proof. DIY. We'll do (3) $\implies$

Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$. Let $A = \bigcup_n A_n$, we see that $\mu(A) = 0$.

For $x \in A^c$ and for every n, we have $|f_n(x) - f(x)| \leq \|f_n - f\|_\infty$. Given $\varepsilon > 0$, there is an N so that $\|f_n - f\|_\infty < \varepsilon$ for all $n \geq N$. But then for all $x \in A^c$, $|f_n(x) - f(x)| < \varepsilon$ as well.

Great! This proves the claim.

Proposition .0.3

We have that

(1) For $1 \leq p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.

(2) For $1 \leq p < \infty$, the collection of step functions with finite measure support is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$. So is $C_c(\mathbb{R})$.

(3) For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Note: $C_c(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

Proof. DIY.

.1. Embedding Properties of L^pspaces

Definition .1.1

Two norms $\|\cdot\|, \|\cdot\|'$ on V are equivalent if there exists $c_1, c_2 > 0$ such that

$$c_1\|v\| \leq \|v\|' \leq c_2\|v\|$$

for all $v \in V$. Note that these norms give the same topological properties (open sets, closed sets, convergence, etc.)

Note that this is an equivalence relation on norms.

Example .1.1

For $\mathbb{R}^d$, we have the norms $\|\cdot\|_p$ for $1 \leq p \leq \infty$. All of these are equivalent. We see that for $1 \leq p < \infty$

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} \leq (d\|x\|_\infty^p)^{1/p} = d^{1/p}\|x\|_\infty.$$

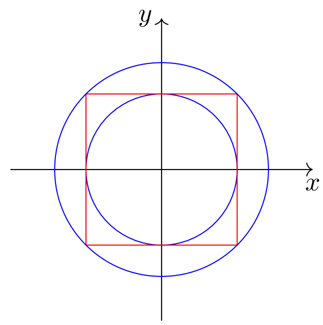
And also

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} \geq (\|x\|_\infty^p)^{1/p} = \|x\|_\infty.$$

Thus $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\infty$ for every $1 \leq p < \infty$, transitivity gives that they are all equivalent.

Another way of thinking of this, by assuming $v \neq 0$ and scaling by some t, we may assume v lies on the unit circle in one of the norms. Then we are squeezing a unit circle in $\|\cdot\|'$ between two circles of radius c_1, c_2 in $\|\cdot\|$.

In a picture we have to show that $\|\cdot\|_2, \|\cdot\|_\infty$ are equivalent, we have



Since $\|\cdot\|_\infty$ circles are squares.

Example .1.2

For $1 \le p, q \le \infty$, We have $L^p(\mathbb{R}, m)$-norm and $L^q(\mathbb{R}, m)$-norm are not equivalent, even worse, we have that

$$L^p(\mathbb{R}, m) \nsubseteq L^1(\mathbb{R}, m)$$
$$L^p(\mathbb{R}, m) \nsupseteq L^1(\mathbb{R}, m)$$