Proposition .0.1

Suppose $\mu(X) < \infty$, then for every $0 < p < q \leq \infty$, $L^q \subseteq L^p$.

Proof. You should check the $q = \infty$ case.

Suppose $q < \infty$. We see that

$$\int |f|^p = \int |f|^p \cdot 1 \leq \left(\int (|f|^p)^{q/p} \right)^{p/q} \left(\int 1^{q/(q-p)} \right)^{1-p/q}$$
$$= \left(\int |f|^q \right)^{p/q} \mu(X)^{1-p/q} < \infty.$$

Using Hölder's inequality with $q/p > 1$. Thus

$$\|f\|_p \leq \|f\|_q \mu(X)^{1/p - 1/q} < \infty.$$

Proposition .0.2

If $0 < p < q \leq \infty$ then $\ell^p \subseteq \ell^q$.

Proof. When $q = \infty$ we have

$$\|a\|_\infty^p = \left(\sup_i |a_i| \right)^p = \sup_i |a_i|^p \leq \sum_{i=1}^\infty |a_i|^p .$$

Thus $\|a\|_\infty \leq \|a\|_p$.

When $q < \infty$, we see that

$$\sum_{i=1}^\infty |a_i|^q = \sum_i |a_i|^p \cdot |a_i|^{q-p}$$
$$\leq \|a\|_\infty^{q-p} \sum_i |a_i|^p$$
$$j \leq \|a\|_p^{q-p} \|a\|_p^p = \|a\|_p^q.$$

Therefore

$$\|a\|_q \leq \|a\|_p.$$

Proposition .0.3

For all $0 < p < q < r \leq \infty$ we have $L^p \cap L^r \subseteq L^q$.

Proof. DIY.

.1. Banach Spaces

Definition .1.1

Let (Y, ρ) be a metric space. We call x_n a Cauchy sequence provided that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that for $n, m \geq N$ we have $\rho(x_n, x_m) < \varepsilon$.

Easy check: convergent sequences are Cauchy.

Definition .1.2

A metric space (Y, ρ) is called complete if every Cauchy sequence in Y converges.

Example .1.1

$\mathbb{Q}$ with $|x - y|$ is not complete, but $\mathbb{R}$ with the same metric is complete.

$C([0,1])$, with $\rho(f,g) = \|f-g\|_\infty$ is complete, but with $\rho(f,g) = \int |f-g|$ it is not complete.

Definition .1.3 (Banach Space)

A Banach Space is a complete normed vector space (i.e, a vector space equipped with a norm whose metric induced by the norm is complete).

Theorem .1.1

Let $(V, \|\cdot\|)$ be a normed space. Then,

$$V \text{ is complete} \iff \text{every absolutely convergent series is convergent}$$

i.e., if $\sum_{i=1}^\infty \|v_i\| < \infty$ then $\{\sum_{i=1}^N v_i\}_{N\in\mathbb{N}}$ converges to some $s \in V$.

Theorem .1.2 (Riesz-Fisher)

For every $1 \leq p \leq \infty, L^p(X, \mathcal{A}, \mu)$ is complete (hence a Banach space).

Proof. Lets go in pieces

(1) We handle the case where $1 \leq p < \infty$ first. Suppose $f_n \in L^p$ and $\sum_{n=1}^\infty \|f_n\|_p < \infty$.

We need to show that there is an $F \in L^p$ such that $\|\sum_{n=1}^N f_n - F\|_p \to 0$ as $N \to \infty$. We must show that

(i) $\sum_{n=1}^\infty f_n(x)$ is convergent almost everywhere. In fact we can show $\int \sum_{n=1}^\infty |f_n(x)| < \infty$.

(ii) $F \in L^p$, where $F(x) \coloneqq \sum_{n=1}^\infty f_n(x)$ almost everywhere and say is zero elsewhere.

(iii) $\|\sum_{n=1}^N f_n - F\|_p \to 0$ as $N \to \infty$.

Lets go!

(i) Let $G(x) = \sum_{n=1}^\infty |f_n(x)| = \sup_N \sum_{n=1}^N |f_n(x)|$, $G : X \to [0, \infty]$.

Let $G_N(x) = \sum_{n=1}^N |f_n(x)|$. Then $0 \leq G_1 \leq G_2 \leq \cdots \leq G$, $G_N \to G$. Furthermore $0 \leq G_1^p \leq G_2^p \leq \cdots \leq G^p$, $G_N^p \to G^p$.

Therefore by the Monotone Convergence Theorem (**??**)

$$\int G^p = \lim_{N\to\infty} \int G_N^p.$$

Now, by Minkowski

$$\|G_N\|_p \leq \sum_{n=1}^N \|f_n\|_p \leq \sum_{n=1}^\infty \|f_n\|_p \coloneqq B < \infty.$$

Thus

$$\int G(x)^p = \lim_{N\to\infty} \int G_N^p = \lim_{N\to\infty} \|G_N\|_p^p \le B^p < \infty.$$

Therefore G is finite almost everywhere as desired. This implies that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ almost everywhere so $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere.

Let $F(x) \coloneqq \sum_{n=1}^{\infty} f_n(x)$ if it converges, and otherwise $F(x) = 0$.

(ii) Now we see that

$$|F(x)| \le G(x)$$
$$\int |F|^p \le \int G^p < \infty.$$

So $F \in L^p$.

(iii) Now we see that

$$\left|\sum_{n=1}^{N} f_n(x) - F(x)\right|^p \le \left(\sum_{n=1}^{\infty} |f_n(x)| + |F(x)|\right)^p \le (2G(x))^p.$$

Well $2G \in L^p$, so $2G^p \in L^1$. Thus by the Dominated Convergence Theorem

$$\lim_{N\to\infty} \int \left|\sum n = 1^N f_n(x) - F(x)\right|^p \mathrm{d}x = 0.$$

And thus $\|\sum_{n=1}^{N} f_n - F\|_p \to 0$ as $N \to \infty$.