Recall .0.1

Tonelli's theorem for series. If $a_{ij} \in [0, \infty]$ then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Read [Tao11], specifically Thm 0.0.2.

Proof of ??: Countable subadditivity. Let $A_1, A_2, \ldots \subseteq X$. We wish to show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

If one of the $\mu^*(A_n) = \infty$, the result holds. Thus it suffices to consider the case when all $\mu^*(A_n) < \infty$.

We will instead prove that for every $\varepsilon > 0$ we have that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

We can call this trick

Give yourself a room of $\varepsilon > 0$

For each $n \in \mathbb{N}$, there exists $E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supseteq A_n \qquad\qquad \mu^*(A_n) \leq \sum_{k=1}^{\infty} \rho(E_{n,k}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$$

Useful because $\mu^*(A_n) < \infty$. Here we have used the $\varepsilon/2^n$**-trick so that we don't accumulate infinite error**.

Then

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_{n,k} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{n,k}$$

$$\begin{aligned}\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{(n,k)\in\mathbb{N}^2} \rho(E_{k,n}) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \rho(E_{k,n}) \\ &\leq \sum_{n=1}^{\infty} \mu^*(E_n) + \frac{\varepsilon}{2^n} = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.\end{aligned}$$

Here we have used Tonelli's theorem, because each $\rho(E_{n,k})$ satisfies $0 \leq \rho(E_{n,k}) < \infty$. Perfect! This proves the result by taking $\varepsilon \to 0$.

Definition .0.1

[Carathéodory measurable] Let μ^* be an outer measure on X. We say that $A \subseteq X$ is Carathéodory measurable (abbrev. C-measurable) with respect to μ^* provided that for every $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$$

Lemma .0.1

Let μ^* be an outer measure on X. Suppose $B_1, \ldots, B_N$ are disjoint C-measurable sets. Then for all $E \subseteq X$,

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^N B_i\right)\right) = \sum_{i=1}^N \mu^*(E \cap B_i).$$

This also implies that μ^* is finitely additive on C-measurable sets by setting $E = X$.

Proof. We see that

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^N B_i\right)\right) = \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_{i=2}^N B_i\right)\right) = \cdots = \sum_{i=1}^N \mu^*(E \cap B_i).$$