Announcements

- HW 12 posted, due next Saturday (4/16), last HW to collect.
- Exam will be April 27th, 1:30-3:30pm.

.1. Absolutely Continuous Functions

Definition .1.1

We say that $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous ($F \in AC$) means for all $\varepsilon > 0$, there exists a $\delta > 0$, such that if $(a_1, b_1), \dots, (a_N, b_N)$ are finitely many disjoint open intervals satisfying $\sum_{n=1}^N (b_n - a_n) < \delta$, then $\sum_{n=1}^N |F(b_n) - F(a_n)| < \varepsilon$.

Lemma .1.1

We have that

(1) If F is absolutely continuous, then it is uniformly continuous (take $N = 1$)

(2) If F is Lipschitz then F is absolutely continuous (easy).

(3) $F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$, $f \in L^1$, is absolutely continuous.

Proof of (3). We write this out as

$$\begin{aligned}\sum_{n=1}^N |F(b_n) - F(a_n)| &= \sum_{n=1}^N \left| \int_{a_n}^{b_n} f(t)\,\mathrm{d}t \right| \\ &\leq \sum_{n=1}^N \int_{a_n}^{b_n} |f(t)|\,\mathrm{d}t \\ &= \int_E |f(t)|\,\mathrm{d}t\end{aligned}$$

where $E = \bigcup_{n=1}^N (a_n, b_n)$, so $m(E) = \sum_{n=1}^N (b_n - a_n)$. By Midterm Q1, if $f \in L^1(X, \mu)$, for all $\varepsilon > 0$, there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |f| < \varepsilon$.

This directly implies that this function is absolutely continuous.

Example .1.1

The cantor function F is uniformly continuous. However, we will see that it is not absolutely continuous.

Proposition .1.2

Suppose $F \in NBV$, then F is absolutely continuous if and only if $\mu_F \ll m$.

Corollary .1.3

$F \in NBV \cap AC$ if and only if $F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$ for some $f \in L^1(\mathbb{R}, m)$. If this holds, we have $f = F'$ Lebesgue almost everywhere.

Lemma .1.4

If $F \in AC([a, b])$, then $F \in NBV([a, b])$

Proof. DIY

Theorem .1.5 (Fundamental Theorem of Calculus)

For $F \in [a,b] \to \mathbb{R}$, the following are equivalent

(1) $F \in AC([a,b])$.

(2) $F(x) - F(a) = \int_a^x f(t)\,\mathrm{d}t$ for some $f \in L^1([a,b],m)$.

(3) F is differentiable almost everywhere on $[a,b]$ and $F(x) - F(a) = \int_a^b F'(t)\,\mathrm{d}t$.

This follows directly from the above.

Proof of Proposition .1.2. Suppose $\mu_F \ll m$. Then $F(x) = \int_{-\infty}^x F'(t)\,\mathrm{d}t$, and $F' \in L^1(\mathbb{R},m)$, by **??**. Therefore $F \in AC$.

Now suppose $F \in AC$. Note that since F is continuous,

$$\mu_F((a,b)) = \lim_{n\to\infty} \mu_F((a, b-1/n]) = \lim_{n\to\infty} F(b-1/n) - F(a) = F(b) - F(a).$$

Now let E be a Borel set with $m(E) = 0$. Fix $\varepsilon > 0$, we will show $|\mu_F(E)| \le \varepsilon$. Let $\delta > 0$ be the constant from $F \in AC$.

Now there exist open $U_1 \supseteq U_2 \supseteq \cdots \supseteq E$ such that $\lim_{n\to\infty} m(U_n) = m(E) = 0$, and open $V_1 \supseteq V_2 \supseteq \cdots \supseteq E$ such that $\lim_{n\to\infty} \mu_F(V_n) = \mu_F(E)$ by regularity.

Let $O_n = U_n \cap V_n$, then $O_1 \supseteq O_2 \supseteq \cdots \supseteq E$, and by monotonicity (for μ_F decomposing into pos/neg first)

$$\lim_{n\to\infty} m(O_n) = m(E) = 0 \qquad \lim_{n\to\infty} \mu_F(O_n) = \mu_F(E).$$

Thus without loss of generality, we may assume $m(O_1) < \delta$. Each O_n is a countable union of disjoint intervals

$$O_n = \bigcup_{k=1}^{\infty} (a_k^n, b_k^n).$$

For any N we also have

$$\sum_{k=1}^{N} (b_k^n - a_k^n) \le m(O_n) \le m(O_1) < \delta.$$

Therefore

$$\begin{aligned} \left|\mu_F\left(\bigcup_{k=1}^{N} (a_k^n, b_k^n)\right)\right| &= \left|\sum_{k=1}^{N} \mu_F((a_k^n, b_k^n))\right| \\ &\le \sum_{k=1}^{N} |\mu_F((a_k^n, b_k^n))| \\ &\le \sum_{k=1}^{N} |F(b_k^n) - F(a_k^n)| < \varepsilon. \end{aligned}$$

Therefore

$$\begin{aligned} |\mu_F(O_n)| &= \lim_{N\to\infty} \left|\mu_F\left(\bigcup_{k=1}^{N} (a_k^n, b_k^n\right)\right| \le \varepsilon \\ |\mu_F(E)| &= \lim_{n\to\infty} |\mu_F(O_n)| \le \varepsilon. \end{aligned}$$

Therefore, taking $\varepsilon \to 0$, yields $\mu_F(E) = 0$. Therefore $\mu_F \ll m$.

Definition .1.2

Let μ be a finite signed Borel measure on $\mathbb{R}$.

- μ is called a discrete measure if there is a countable set $\{x_n\}$ and $c_n \neq 0$ such that $\sum_{n=1}^{\infty} |c_n| < \infty$ and $\mu = \sum_n c_n \delta_{x_n}$ (where δ_{x_n} is the Dirac delta at x_n).
- μ is called continuous if $\mu(\{a\}) = 0$ for all $a \in \mathbb{R}$.

Lemma .1.6

Given a finite signed Borel measure μ

(1) Any $\mu = \mu_d + \mu_c$, where μ_d is discrete, μ_c is continuous, uniquely.

(2) μ discrete implies $\mu \perp m$.

(3) $\mu \ll m$ implies μ is continuous.

Corollary .1.7

For μ a finite signed Borel measure on $\mathbb{R}$, we have that

$$\mu = \mu_d + \mu_{ac} + \mu_{sc}$$

where μ_d is discrete, μ_{ac} is absolutely continuous, and μ_{sc} is singularly continuous (to m).