Announcements

- Canvas/Modules
  - Lecture summary after each class
  - Suggested reading
- HW1 will be posted today: due next Thursday 1/13. 9pm.

Now lets look at examples of σ-algebras

Example .0.1

We have the followoing basic σ-algebras

- $\mathcal{A} = P(X)$, the power σ-algebra
- $\mathcal{A} = \{\emptyset, X\}$, the trivial σ-algebra
- Let $B \subseteq X$, $B \neq \emptyset, B \neq X$. Then

$$\mathcal{A} = \{\emptyset, B, B^c, X\}$$

  is a σ-algebra.

Lemma .0.1

Let $\mathcal{A}_i$ where $i \in I$ be a family of σ-algebras over a fixed set X.

Then $\bigcap_{i \in I} \mathcal{A}_i$ is a σ-algebra over X.

Proof. Clearly $\emptyset \in \bigcap_{i\in I} \mathcal{A}_i$ because $\emptyset \in \mathcal{A}_i$ for all i. Now if $E \in \bigcap_{i\in I} \mathcal{A}_i$, then $E^c \in \mathcal{A}_i$ for each i, so $E^c \in \bigcap_{i=1}^\infty \mathcal{A}_i$ as desired.

Now if $E_1, E_2, \ldots \in \bigcap_{i\in I} \mathcal{A}_i$, then of course $\bigcup_{j=1}^\infty E_j \in \mathcal{A}_i$ for each i, so $\bigcup_{j=1}^\infty E_j \in \bigcap_{i\in I} \mathcal{A}_i$. Great!

Definition .0.1

For $\mathcal{E} \subseteq P(X)$, let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X containing $\mathcal{E}$. We call $\langle \mathcal{E} \rangle$ the σ-algebra generated by $\mathcal{E}$.

Example .0.2

$\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{\emptyset, B^c\} \rangle$.

Remark .0.1

$\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$ (under the subset relation), and this uniquely characterizes $\mathcal{E}$.

Lemma .0.2

We have the following

(a) Suppose $\mathcal{E} \subseteq P(X)$ and $\mathcal{A}$ is a σ-algebra containing $\mathcal{A}$. Then $\langle \mathcal{E} \rangle \subseteq \mathcal{A}$.

(b) Suppose $\mathcal{E} \subseteq \mathcal{F} \subseteq P(X)$. Then $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{F} \rangle$ because $\mathcal{E} \subseteq \langle \mathcal{F} \rangle$.

Proof. DIY

Definition .0.2

For a topological space X, the Borel σ-algebra, $\mathcal{B}(X)$, is the σ-algebra generated by the collection of open sets in X.

Example .0.3

$\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a,b) \mid a < b, a, b \in \mathbb{R}\}$.
- $\mathcal{E}_2 = \{[a,b] \mid a < b\}$ because $[a,b] = ((-\infty, a) \cup (b, \infty))^c$.
- $\mathcal{E}_3 = \{(a,b] \mid a < b\}$ because $(a,b] = (a,b) \cup \{b\}$, and closed sets are in the Borel σ-algebra.
- All the open and closed rays. $(a,\infty), [a,\infty), (-\infty,b), (-\infty,b]$. Call these collections $\mathcal{E}_5$,$\mathcal{E}_6$,$\mathcal{E}_6$,$\mathcal{E}_7$, and $\mathcal{E}_8$.

Proposition .0.3

$\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \ldots, 8$.

Proof. We know that $\mathcal{E}_i \subseteq \mathcal{B}(\mathbb{R})$ by the arguments in the example. Thus $\langle \mathcal{E}_i \rangle \subseteq \mathcal{B}(\mathbb{R})$ by Lemma .0.2.

By definition $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$, where $\mathcal{E}$ is the collection of open sets. It is then enough to show $\mathcal{E} \subseteq \langle \mathcal{E}_i \rangle$. (if so $\langle \mathcal{E}_i \rangle \subseteq \mathcal{E}_i$).

Exercise .0.4

Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals (see Prop

Thus $\mathcal{B}(\mathbb{R}) \subseteq \langle I \rangle$, where I consists of the open intervals.

It is straightfoward to check open intervals are in $\langle \mathcal{E}_i \rangle$.

.1. Measures

Definition .1.1

A set X equipped with a σ-algebra $\mathcal{A}$ on X is called a measurable space (referred to as $(X, \mathcal{A})$).

If we equip this with a function $\mu : \mathcal{A} \to [0, \infty]$ satisfying

- $\mu(\emptyset) = 0$
- Countable additivity. That is if $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint then
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

We then call $(X, \mathcal{A}, \mu)$ a measure space and μ a measure on the space $(X, \mathcal{A})$.

We should only insist on countable additivity. Because

$$(0,1] = \bigcup_{i=0}^{\infty} \left(\frac{1}{2^{i+1}}, \frac{1}{2^i}\right]$$

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$(0,1] = \bigcup_{x \in (0,1]} \{x\}$$

$$(0,1] \neq \sum_{x \in (0,1]} 0.$$

A measure is also necessarily finite additive

Example .1.1 (a) For any $(X, \mathcal{A})$, $\mu(A) = \#A$ is called the counting measure.

(b) Let $x_0 \in X$. For any (X, A), the Dirac measure at x_0 is denoted by δ_{x_0} and takes the values

$$\delta_{x_0} = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A \end{cases}$$

(c) Note that measures are closed under pointwise scalar multiplication and pointwise addition. Thus for $(\mathbb{N}, P(\mathbb{N}))$ w eknow that

$$\mu(A) = \sum_{i \in A} a_i$$

is a measure where $a_i \in [0, \infty)$ for $i \in \mathbb{N}$.