Announcements

- Exam on Wednesday at Chem 1400
  - 6:00-7:50 (110 min)
  - 7:50-8:00 scan + upload to Gradescope (bring computer / phone)
- Office hour this week
  - Today 12:30-1:30, 3:05-3:50
  - Tomorrow 1:30-2:30
- No class Wednesday → office hour 11-12 instead.

.1. Lebesgue measure on $\mathbb{R}^d$

$\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}^d)$

Example .1.1

$(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is not complete.

Let $A \in \mathcal{L}$, $A \neq \emptyset$, $m(A) = 0$. Let $B \subseteq [0,1]$, $B \notin \mathcal{L}$ (Vitali set).

Let $E = A \times B$, $F = A \times [0,1]$. Then $E \subseteq F \in \mathcal{L} \otimes \mathcal{L}$ and $(m \times m)(F) = m(A)m([0,1]) = 0$.

If E were measurable, then every section of E would be measurable. One section is B, so E is not $\mathcal{L} \otimes \mathcal{L}$-measurable.

Definition .1.1 (Lebesgue Measure on $\mathbb{R}^d$)

Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the completion of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \cdots \times m)$, which is the same as the completion of $(\mathbb{R}^\lceil, \mathcal{L} \otimes \cdots \otimes \mathcal{L}, m \times \cdots \times m)$.

Recall .1.2

Recall from the Caratheodory extension theorem that calling a rectangle in $\mathbb{R}^d$ the product $R = \prod_{i=1}^d E_i$, $E_i \in \mathcal{B}(\mathbb{R})$, we have

$$m^d(E) = \inf \left\{ \sum_{k=1}^\infty m^d(R_k) : E \subseteq \bigcup_{k=1}^\infty R_k, R_k \text{ rectangle} \right\}$$

for all $E \in \mathcal{L}^d$.

Theorem .1.1 (Regularity of $\mathcal{L}^d$)

Let $E \in \mathcal{L}^d$, then we have the following:

(a) We have a formula from outer/inner regularity

$$m^d(E) = \inf\{m^d(O) \mid \text{ open } O \supseteq E\} = \sup\{m^d(K) \mid \text{ compact } K \subseteq E\} \tag{(a)}$$

(b) We also have for some A_1 a F_δ set, A_2 a G_δ set, N_1, N_2 null sets, that

$$E = A_1 \cup N_1 = A_2 \setminus N_2.$$

(c) If $m^d(E) < \infty$, for all $\varepsilon > 0$ there exists $R_1, \dots, R_m$ rectangles whose sides are intervals such that
$$m^d\left(E \triangle \left(\bigcup_{i=1}^{m} R_i\right)\right) < \varepsilon.$$

Theorem .1.2

Integrable "step functions" and $C_c(\mathbb{R}^d)$ (compactly supported continuous functions) are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$.

Theorem .1.3

Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.

Theorem .1.4 (Effect of linear transformation on Lebesgue measure)

If $T \in \mathrm{GL}(\mathbb{R}^d)$, $E \in \mathcal{L}^d$, then $T(E)$ is measurable and $m(T(E)) = |\det T| \cdot m(E)$.

Proof. See pages 71-81 of [Fol99], can skip every part except Theorem .1.4.

I. Differentiation on Euclidean Space

If we have $f : [a, b] \to \mathbb{R}$, there are two versions of the fundamental theorem of calculus

- We have
$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$
when f is sufficiently differentiable (later chapter).
- When f is continuous we have
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

The second statement is the same (by the definition of the derivative) as
$$\lim_{r\to 0^+} \frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t = f(x) = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} f(t)\,\mathrm{d}t.$$
We then see that
$$f(x) = \frac{1}{r}\int_x^{x+r} f(x)\,\mathrm{d}t.$$
Thus the above is equivalent to saying that
$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r}(f(t) - f(x))\,\mathrm{d}t = 0 = \lim_{r\to 0^+}\int_{x-r}^{x}(f(t) - f(x))\,\mathrm{d}t.$$
Now if $f : \mathbb{R}^d \to \mathbb{R}$, we can instead consider
$$\lim_{r\to 0^+}\frac{1}{\mathrm{vol}(B(x,r))}\int_{B(x,r)}(f(t) - f(x))\,\mathrm{d}t \stackrel{?}{=} 0$$
where $B(x, r)$ is a ball of radius r about x.

This question led to new and interesting techniques, our reference will be [Fol99] section 3.4.

I.1. **Hardy-Littlewood maximal function**

For an open ball in $\mathbb{R}^d$, $B = B(a, r)$, denote $cB = B(a, cr)$ for $c > 0$.

Lemma I.1.1 (Vitali-type Covering Lemma)

Let $B_1, \ldots, B_k$ be a finite collection of open balls in $\mathbb{R}^d$.

Then there exists a subcollection $B_1', \ldots, B_m'$ of disjoint open balls such that

$$\bigcup_{j=1}^{m} (3B_j') \supseteq \bigcup_{i=1}^{k} B_i$$

Proof. Greedy algorithm, take largest ball in the collection, and then next largest ball not intersecting the first one, etc.