Recall .0.1

Consider the following

- We have
$$B_{n,k} := \left\{x \in X \mid |f_n(x) - f(x)| < \frac{1}{k}\right\}$$
$$\{x \in X \mid \lim_{n\to\infty} f_n(x) = f(x)\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$
- $f_n \to f$ almost everywhere does not imply (and is not implied by) $f_n \to f$ in L^1.
- Markov's Inequality says for all $c > 0$ we have
$$\mu(\{x \in X \mid |g(x)| \geq c\}) \leq \frac{1}{c} \int |g|.$$
for all $c > 0$.

Proposition .0.1 (Fast L^1 convergence $\implies$ a.e. convergence)

Let $(X, \mathcal{A}, \mu)$ be a measure space with f_n, f measurable functions on X.

Assume that $\sum_{n=1}^{\infty} \|f_n - f\|_1 < \infty$. Then $f_n \to f$ almost everywhere.

Proof. Let $E = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c$. That is the set of points x where $f_n(x) \not\to f(x)$. It suffices to show for every fixed k that
$$\bigcap_{N=1}^{\infty} \bigcup_{n=N} B_{n,k}^c.$$
has measure zero. We do this using continuity of the measure. We see for every k, and for every n
$$\mu(B_{n,k}^c) \leq k \int |f_n - f|.$$
But then for each N
$$\mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) \leq \sum_{n=N}^{\infty} k\|f_n - f\|.$$
as $N \to \infty$, this goes to zero by convergence. By using the continuity of the measure, for every k we have
$$\mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N} B_{n,k}^c\right) = \lim_{N\to\infty} \mu\left(\bigcup_{n=N} B_{n,k}^c\right) = 0.$$
Because this happens for every k, we see that $\mu(E) = 0$. This finishes the proof.

Corollary .0.2

If $f_n \to f$ in L^1, there exists a subsequence $f_{n_j} \to f$ almost everywhere.

Proof. For every $j \in \mathbb{N}$, there exists $n_j \in \mathbb{N}$ such that $\|f_{n_j} - f\|_1 \leq 1/j^2$. Then $\sum_j \|f_{n_j} - f\|_1 < \infty$, and so the proposition can be applied.

Definition .0.1

Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. We say that $f_n \to f$ in measure provided thta for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu\left(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\right) = 0.$$

Example .0.2

Let $f_n = n1_{(0,1/n)}$ and $f = 0$. Then for every $\varepsilon > 0$, the set $\{x \mid |f_n(x) - f(x)| > \varepsilon\} \subseteq (0, 1/n)$. Thus $f_n \to 0$ in measure. But $f_n \not\to 0$ in L^1.

For typewriter functions $g_n \to 0$ in measure, and recall that $g_n \not\to 0$ almost everywhere.

$$\begin{tikzcd}
f_n \to f \text{ fast } L^1 \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & f_n \to f \text{ in } L^1 \arrow[r, shift left, Rightarrow] \arrow[d, shift left, Rightarrow, "/" marking] & f_n \to f \text{ in measure} \arrow[l, shift left, Rightarrow, "/" marking] \arrow[dl, shift left, Rightarrow, "/" marking] \arrow[d, Rightarrow] \\
& f_n \to f \text{ a.e.} \arrow[u, shift left, Rightarrow, "/" marking] \arrow[ur, shift left, Rightarrow, "/" marking] & \exists f_{n,j} \to f \text{a.e.}
\end{tikzcd}$$

Definition .0.2

f_n, f measurable on $(X, \mathcal{A}, \mu)$

(a) $f_n \to f$ uniformly almost everywhere if ther exists a null set F such that $f_n \to f$ uniformly on F^c

(b) $f_n \to f$ almost uniformly means for all $\varepsilon > 0$ there exists $F \in \mathcal{A}$ such that $\mu(F) < \varepsilon$, $f_n \to f$ uniformly on F^c.

Lemma .0.3

$f_n \to f$ uniformly on S if and only there exists $N_1, N_2, \ldots$ such that

$$S \subseteq \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Theorem .0.4 (Egoroff)

Let f_n, f be measurable on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$. Then, $f_n \to f$ almost everywhere if and only if $f_n \to f$ almost uniformly.

Proof. DIY the converse $\Longleftarrow$.

For $\Longrightarrow$, fix $\varepsilon > 0$. We know because $f_n \to f$ almost everywhere that

$$\mu\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0.$$

This implies for every k that

$$\mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0.$$

Then this implies the following using continuity of the measure and that $\mu(X) < \infty$,

$$\forall k \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall k\, \exists N_k \in \mathbb{N}\, \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c\right) < \frac{\varepsilon}{2^k}.$$

Now let $F = \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c$. We see that

$$\mu(F) < \varepsilon \qquad\qquad f_n \to f \text{ unif on } F^c$$

I. Product Measures

In the book this is pages 22-23 (section 1.2), and section 2.5, 2.6.

I.1. Product σ-algebras

Consider a product space $X = \prod_{\alpha \in I} X_\alpha$. That is $x = (x_\alpha)_{\alpha \in I}$. Of course formally we have $x : I \to \bigcup_\beta X_\beta$ such that $x(\alpha) \in X_\alpha$.

We have a coordinate map $\pi_\alpha : X \to X_\alpha$.

Definition I.1.1

Suppose $(X_\alpha, \mathcal{A}_\alpha)$ are measurable spaces for all $\alpha \in I$.

We define the product σ-algebra on $X = \prod_{\alpha \in I} X_\alpha$ to be

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1}(\mathcal{A}_\alpha) \right\rangle$$

where

$$\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \{\pi_\alpha^{-1}(E) \mid E \in \mathcal{A}_\alpha\}$$

Notation If $I = \{1, \ldots, d\}$ then $X = \prod_{i=1}^d X_i$ and $x = (x_1, \ldots, x_d)$ and $\bigotimes_{i=1}^d \mathcal{A}_i = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_d$ (allowing $d = \infty$ for countable sets)

Lemma I.1.1

If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \prod_{i=1}^{\infty} E_i \mid E_i \in \mathcal{A}_i \right\rangle$$

Proof. DIY.