Announcements

- No class on Monday (MLK day)
- HW 2 posted
- Another assignment
  - Get to Know Video review
  - Solution consent form.

Lemma .0.1

Improved version of **??**

Let μ^* be an outer measure on X. Suppose $B_1, B_2, \dots$ are <u>disjoint</u> C-measurable sets. Then for all $E \subseteq X$,

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i).$$

This also implies that μ^* is countably additive on C-measurable sets by setting $E = X$.

Proof. By countable subadditivity of μ^* we have that

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_n) \geq \mu^*\left(E \cap \cup_{n=1}^{\infty} B_n\right).$$

Now monotonicity and **??** implies that

$$\begin{aligned} \mu^*\left(E \cap \cup_{n=1}^{\infty} B_n\right) &\geq \mu^*\left(E \cap \cup_{n=1}^{N} B_n\right) \\ &\geq \sum_{n=1}^{N} \mu^*(E \cap B_n) \end{aligned}$$

by taking $N \to \infty$ we see that

$$\mu^*\left(E \cap \cup_{n=1}^{\infty} B_n\right) \geq \sum_{n=1}^{\infty} \mu^*(E \cap B_n)$$

These two inequalities imply the result.

Theorem .0.2 (Carathéodory Extension Theorem)

Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets (with respect to μ^*). Then

(a) $\mathcal{A}$ is a σ-algebra

(b) $\mu \coloneqq \mu^*\big|_{\mathcal{A}} : \mathcal{A} \to [0, \infty]$ is a measure on $(X, \mathcal{A})$.

(c) $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We do this by parts, (a) is hardest, (b) is easy-ish, and (c) is easy

(a) We break this down into five steps

  (a1) $\emptyset \in \mathcal{A}$, DIY.

  (a2) $\mathcal{A}$ is closed under complements, DIY.

  (a3) $\mathcal{A}$ is closed under finite unions (a4) $\mathcal{A}$ is closed under countable disjoint unions.

(a5) $\mathcal{A}$is closed under countable unions.

Lets go!

(a3) By induction, it is enough to show that if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. Fix $E \subseteq X$. We need to show that

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)).$$

We make the following convenient definitions

$$E_1 := E \cap A \cap B \qquad E_2 := (E \cap A) \setminus B \qquad E_3 := (E \cap B) \setminus A \qquad E_4 := E \setminus (A \cup B).$$

We need to show that

$$\mu^*(E_1 \cup E_2 \cup E_3 \cup E_4) = \mu^*(E_1 \cup E_2 \cup E_3) + \mu^*(E_4).$$

We know that

$$\mu^*(E_1 \cup E_2 \cup E_3 \cup E_4) = \mu^*(E_1 \cup E_2) + \mu^*(E_3 \cup E_4)$$
$$\mu^*(E_1 \cup E_2 \cup E_3) = \mu^*(E_1 \cup E_2) + \mu^*(E_3)$$

by testing against A, which is C-measurable.

By testing against B which is C-measurable that

$$\mu^*(E_3 \cup E_4) = \mu^*(3) + \mu^*(E_4).$$

The right hand side then becomes

$$\begin{aligned}\mu^*(E_1 \cup E_2 \cup E_3) + \mu^*(E_4) &= \mu^*(E_1 \cup E_2) + \mu^*(E_3) + \mu^*(E_4) \\ &= \mu^*(E_1 \cup E_2) + \mu^*(E_3 \cup E_4) \\ &= \mu^*(E_1 \cup E_2 \cup E_3 \cup E_4).\end{aligned}$$

(a4) We show $\mathcal{A}$ is closed under countable disjoint unions. Let $A_1, A_2, \ldots \in \mathcal{A}$ be disjoint. Fix $E \subseteq X$. We need to show that

$$\mu^*(E) = \mu^*\left(E \cap \bigcup_{n==1}^{\infty} A_n\right) + \mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right).$$

Because μ^* is countable subadditive we know that

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{n==1}^{\infty} A_n\right) + \mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right).$$

We then just need to show the other direction of the inequality. So fix $N \in \mathbb{N}$. We know by Item (a3) that $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$, and so by **??**, monotonicity, and countable subadditivity

$$\mu^*(E) = \mu^*\left(E \cap \bigcup_{n=1}^{N} A_n\right) + \mu^*\left(E \setminus \bigcup_{n=1}^{N}\right)$$

$$\geq \sum_{n=1}^{N} \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right).$$

By taking $N \to \infty$ and applying the result of countable subadditivity.

(a5) We claim that being closed under complement (a2), closed under finite unions (a3), and closed under countable disjoint unions (a4) suffices to show that $\mathcal{A}$ is closed under countable unions. To do this, fix $A_1, A_2, \ldots \in \mathcal{A}$. Now let

$$B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i.$$

Then $\bigcup_n A_n = \bigcup_n B_n$, but the B_n are disjoint, and all in $\mathcal{A}$ because of (a2),(a3).

(b) We know that $\mu(\emptyset) = \mu^*(\emptyset) = 0$, and countable additivity on $\mathcal{A}$ follows from Lemma .0.1 with $E = X$.

(c) On HW2!

Recall .0.1

Recall ??. That is let $\mathcal{E} \subseteq P(X)$ such that $\emptyset, X \in \mathcal{E}$.

Now let $\rho : \mathcal{E} \to [0, \infty]$ such that $\rho(\emptyset) = 0$. Then

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \mid E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supseteq A \right\}$$

is an outer measure on X.

Then we have the following

$$(\mathcal{E}, \rho) \xrightarrow{??} (P(X), \mu^*) \xrightarrow{\text{Theorem .0.2}} (\text{C-measurable sets}, \mu)$$

Question: Do we have $\mathcal{E} \subseteq \mathcal{A}$ and $\mu\big|_{\mathcal{E}} = \rho$? No!

Definition .0.1

Let $\mathcal{A}_0$ be an algebra on X (that is contains $\emptyset$, closed under complement, and closed under finite union).

We say $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a pre-measure if

(a) $\mu_0(\emptyset) = 0$

(b) Finite additivity: If $A_1, \ldots, A_n \in \mathcal{A}$ are disjoint then

$$\mu_0\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \mu_0(A_i)$$

(c) Countable additivity within $\mathcal{A}_0$: If $A \in \mathcal{A}_0$ and $A = \bigcup_{i=1}^{\infty} A_i$ for disjoint $A_i \in \mathcal{A}$, then

$$\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i)$$

In fact (a) + (c) imply (b) by taking empty sets.

Notation: [Fol99] uses $\mathcal{M}$ for σ-algebra and $\mathcal{A}$ for algebra. We use $\mathcal{A}$ for σ-algebra and $\mathcal{A}_0$ for algebra.

Example .0.2

By next Wednesday we will consider $\mathcal{A}_0$ as finite disjoint unions of $(a, b]$ and

$$\mu_0\left(\sum_{i=1}^{N}(a_i, b_i]\right) = \sum_{i=1}^{N}(b_i - a_i).$$

This will generate the Lebesgue measure on $\mathbb{R}$.

Lemma .0.3

μ_0 is monotone

Proof. DIY

Theorem .0.4 (Hahn-Kolmogorov Theorem)

Let μ_0 be a premeasure on the algebra $\mathcal{A}_0$ on X.

Let μ^* be the induced outer measure from $(\mathcal{A}_0, \mu_0)$ via **??**. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*.

Then (A, μ) extends $(\mathcal{A}_0, \mu_0)$. In other words, $\mathcal{A} \supseteq \mathcal{A}_0$ and $\mu\big|_{\mathcal{A}_0} = \mu_0$.

Proof. Let's go!

(a) We wish to show $\mathcal{A} \supseteq \mathcal{A}_0$. Let $A \in \mathcal{A}_0$. We need to show $A \in \mathcal{A}$, that is we need to show A is C-measurable. Concretely, for $E \subseteq X$ we need

$$\mu^*(E) \stackrel{?}{=} \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Now fix $E \subseteq X$. Countable subadditivity of μ^* implies that

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

For the other inequality, if $\mu^*(E) = \infty$, then we're clearly done. Thus we assume $\mu^*(E) < \infty$.

We use the room of $\varepsilon > 0$ trick. Fix $\varepsilon > 0$, then we will show that

$$\mu^*(E) + \varepsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

By the definition of μ^*, there are $B_1, B_2, \ldots \in \mathcal{A}_0$ so that $E \subseteq \bigcup_{n=1}^{\infty} B_n$ and

$$\begin{aligned}\mu^*(E) + \varepsilon &\geq \sum_{n=1}^{\infty}\mu_0(B_n) = \sum_{n=1}^{\infty}(\mu_0(B_n \cap A) + \mu_0(B_n \cap A^c)) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)\end{aligned}$$

because $B_n \cap A, B_n \cap A^c \in \mathcal{A}_0$ and their union contains $E \cap A$ and $E \cap A^c$ respectively. Perfect! Taking $\varepsilon \to 0$ yields the result.

(b) We need to show that $\mu\big|_{\mathcal{A}_0} = \mu_0$. Pick $A \in \mathcal{A}_0$. We want to show $\mu(A) = \mu_0(A)$, that is $\mu^*(A) = \mu_0(A)$.

To show $\mu^*(A) \leq \mu_0(A)$, just let

$$B_i = \begin{cases} A & \text{if } i = 1 \\ \emptyset & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0 \qquad\qquad \bigcup_{i=1}^{\infty} B_i \supseteq A.$$

Therefore

$$\mu^*(A) \leq \mu_0(A) + \sum_{i=2}^{\infty} \mu_0(\emptyset) = \mu_0(A).$$

Now we show that $\mu_0(A)$ is a lower bound on the sums $\sum_{i=1}^{\infty} \mu_0(B_i)$, so that $\mu^*(A) \geq \mu_0(A)$. Let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supseteq A$. Then define

$$C_1 = B_1 \cap A \qquad\qquad C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right).$$

Now note that each $C_i \in \mathcal{A}_0$, as we have only finitely many set operations. But then we know that

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

is a disjoint countable union.

Therefore because μ_0 is a premeasure, we know that

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i).$$

Taking the inf, we get $\mu_0(A) \leq \mu^*(A)$. Perfect! This finishes the proof!