Notes on
MATH 597
(Intro to Graduate Real Analysis)

September 21, 2023

Faye Jackson

Contents

I. Introduction & Syllabus

- Location: 269 Weiser
- Professor: Jinho Baik
- Syllabus – tentative
- Primary Text: Real analysis 1-3 (5,6,8) by Folland [Fol99].
- Supplementary Texts: Axler's measure theory and Tao's introduction to measure theory [Axl20; Tao11].

Broad strokes of the course

(1) Abstract measure Section II
(2) Integration
(3) Product measures
(4) Differentiation
(5) Signed and complex measures
(6) Banach and Hilbert Spaces
(7) L^p spaces
(8) Introduction to Fourier analysis.

Read [Fol99] Chapter 0, the following sections

0.1 Set theory
0.5 Extended real number system
0.6 Metric spaces (may skip Theorem 0.25).

II. Abstract Measure Theory

II.1. Motivation

For a set X, let $P(X)$ be the collection of all subsets of X (the power set).

Note that $P(X)$ has a lot more elements than X, for example if X is finite then $\#P(X) = 2^{\#X}$. Furthermore, $\mathbb{N}$ is obviously countable, but $P(\mathbb{N})$ is uncountable.

We can see this because there is a surjective function $\phi : P(\mathbb{N}) \to [0,1]$ given by

$$\phi(A) = 0.a_1 a_2 \cdots \qquad \text{(base 2)}$$

where

$$a_k = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{if } k \notin A \end{cases}.$$

Therefore $\#P(\mathbb{N}) \geq \#[0,1]$ (see 0.3 Cardinality if you like, not necessary). In general we always have $\#P(X) > \#X$.

We like to "measure" the "size" of subsets of X

Example II.1.1

Here are some good examples!

- Let $X = \{0,1,2,3\}$. Then there are measures $\mu : P(X) \to [0,\infty]$ as below

$$\mu(A) = \#A \qquad \mu(\{0,1\}) = 2$$
$$\mu(A) = \sum_{i\in A} 2^i \qquad \mu(\{0,1\}) = 3$$

- For $X = \mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$ we have $\mu : P(\mathbb{N}_0) \to [0,\infty]$ via

$$\mu(A) = \#A \qquad \mu(\{2,4,6,\ldots\}) = \infty$$
$$\mu(A) = e^{-1}\sum_{i\in A} \frac{1}{i!} \qquad \mu(X) = 1$$

  Using $\mu(\{i\}) = a_i$ and $\mu(A) = \sum_{i\in A} a_i$ is a good measurement of A. This works because the sums are always countable.

- $X = \mathbb{R}$, want $\mu : P(\mathbb{R}) \to [0,\infty]$

$$\mu(A) = \#A \qquad \text{not interesting}$$
$$\mu(A) = \text{length of A?}.$$

  Here we'd have $\mu((a,b)) = b - a$. Can we extend μ reasonably to all subsets of $\mathbb{R}$? What if instead we take $\mu((a,b)) = e^b - e^a$? We also can't extend this to all subsets!

Theorem II.1.1 (Banach-Tarski)

In $\mathbb{R}^d$, for $d \geq 3$, one can divide a ball into finitely many subsets and put back into two balls of same radius.

We will try to define a "measure" on X, that is $\mu : \mathcal{A} \to [0,\infty]$ for a "suitable" $\mathcal{A} \subseteq P(X)$.

II.2. σ-algebras

Definition II.2.1

Let X be a set. A collection $\mathcal{A}$ of subsets of X is called a σ-algebra on X provided that

- $\emptyset \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements, aka if $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$.
- $\mathcal{A}$ is closed under countable unions. that is if $E_1, E_2, \ldots \in \mathcal{A}$ then $\bigcup_{i=1}^\infty E_i \in \mathcal{A}$.

These have some simple properties

- $X = \emptyset^c \in \mathcal{A}$.
- It's closed under countable intersections because

$$\bigcap_{i=1}^\infty E_i = \left(\bigcup_{i=1}^\infty E_i^c\right)^c$$

- $\bigcup_{i=1}^N E_i = E_1 \cup \cdots \cup E_N \cup \emptyset \cup \cdots$. So it is closed under finite unions (+ intersections).
- Closed under $E \setminus F = E \cap F^c$ and $E \triangle F = (E \cup F) \setminus (E \cap F)$.

Announcements

- Canvas/Modules
  - Lecture summary after each class
  - Suggested reading
- HW1 will be posted today: due next Thursday 1/13. 9pm.

Now lets look at examples of σ-algebras

Example II.2.1

We have the followoing basic σ-algebras

- $\mathcal{A} = P(X)$, the power σ-algebra
- $\mathcal{A} = \{\emptyset, X\}$, the trivial σ-algebra
- Let $B \subseteq X$, $B \neq \emptyset, B \neq X$. Then

  $$\mathcal{A} = \{\emptyset, B, B^c, X\}$$

  is a σ-algebra.

Lemma II.2.1

Let $\mathcal{A}_i$ where $i \in I$ be a family of σ-algebras over a fixed set X.

Then $\bigcap_{i \in I} \mathcal{A}_i$ is a σ-algebra over X.

Proof. Clearly $\emptyset \in \bigcap_{i \in I} \mathcal{A}_i$ because $\emptyset \in \mathcal{A}_i$ for all i. Now if $E \in \bigcap_{i \in I} \mathcal{A}_i$, then $E^c \in \mathcal{A}_i$ for each i, so $E^c \in \bigcap_{i=1}^{\infty} \mathcal{A}_i$ as desired.

Now if $E_1, E_2, \ldots \in \bigcap_{i \in I} \mathcal{A}_i$, then of course $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}_i$ for each i, so $\bigcup_{j=1}^{\infty} E_j \in \bigcap_{i \in I} \mathcal{A}_i$. Great!

Definition II.2.2

For $\mathcal{E} \subseteq P(X)$, let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X containing $\mathcal{E}$. We call $\langle \mathcal{E} \rangle$ the $\underline{\sigma\text{-algebra generated by } \mathcal{E}}$.

Example II.2.2

$\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{\emptyset, B^c\} \rangle$.

Remark II.2.1

$\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$ (under the subset relation), and this uniquely characterizes $\mathcal{E}$.

Lemma II.2.2

We have the following

(a) Suppose $\mathcal{E} \subseteq P(X)$ and $\mathcal{A}$ is a σ-algebra containing $\mathcal{A}$. Then $\langle \mathcal{E} \rangle \subseteq \mathcal{A}$.

(b) Suppose $\mathcal{E} \subseteq \mathcal{F} \subseteq P(X)$. Then $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{F} \rangle$ because $\mathcal{E} \subseteq \langle \mathcal{F} \rangle$.

Proof. DIY

Definition II.2.3

For a topological space X, the Borel σ-algebra, $\mathcal{B}(X)$, is the σ-algebra generated by the collection of open sets in X.

Example II.2.3

$\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a, b) \mid a < b, a, b \in \mathbb{R}\}$.
- $\mathcal{E}_2 = \{[a, b] \mid a < b\}$ because $[a, b] = ((-\infty, a) \cup (b, \infty))^c$.
- $\mathcal{E}_3 = \{(a, b] \mid a < b\}$ because $(a, b] = (a, b) \cup \{b\}$, and closed sets are in the Borel σ-algebra.
- All the open and closed rays. $(a, \infty), [a, \infty), (-\infty, b), (-\infty, b]$. Call these collections $\mathcal{E}_5$,$\mathcal{E}_6$,$\mathcal{E}_6$,$\mathcal{E}_7$, and $\mathcal{E}_8$.

Proposition II.2.3

$\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \ldots, 8$.

Proof. We know that $\mathcal{E}_i \subseteq \mathcal{B}(\mathbb{R})$ by the arguments in the example. Thus $\langle \mathcal{E}_i \rangle \subseteq \mathcal{B}(\mathbb{R})$ by Lemma II.2.2.

By definition $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$, where $\mathcal{E}$ is the collection of open sets. It is then enough to show $\mathcal{E} \subseteq \langle \mathcal{E}_i \rangle$. (if so $\langle \mathcal{E}_i \rangle \subseteq \mathcal{E}_i$).

Exercise II.2.4

Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals (see Prop

Thus $\mathcal{B}(\mathbb{R}) \subseteq \langle I \rangle$, where I consists of the open intervals.

It is straightfoward to check open intervals are in $\langle \mathcal{E}_i \rangle$.

II.3. Measures

Definition II.3.1

A set X equipped with a σ-algebra $\mathcal{A}$ on X is called a measurable space (referred to as $(X, \mathcal{A})$).

If we equip this with a function $\mu : \mathcal{A} \to [0, \infty]$ satisfying

- $\mu(\emptyset) = 0$
- Countable additivity. That is if $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

We then call $(X, \mathcal{A}, \mu)$ a measure space and μ a measure on the space $(X, \mathcal{A})$.

We should only insist on countable additivity. Because

$$(0, 1] = \bigcup_{i=0}^{\infty} \left(\frac{1}{2^{i+1}}, \frac{1}{2^i}\right]$$

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$(0, 1] = \bigcup_{x \in (0,1]} \{x\}$$

$$(0, 1] \neq \sum_{x \in (0,1]} 0.$$

A measure is also necessarily finite additive

Example II.3.1 (a) For any $(X, \mathcal{A})$, $\mu(A) = \#A$ is called the counting measure.

(b) Let $x_0 \in X$. For any (X, A), the Dirac measure at x_0 is denoted by δ_{x_0} and takes the values

$$\delta_{x_0} = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A \end{cases}$$

(c) Note that measures are closed under pointwise scalar multiplication and pointwise addition. Thus for $(\mathbb{N}, P(\mathbb{N}))$ w eknow that

$$\mu(A) = \sum_{i \in A} a_i$$

is a measure where $a_i \in [0, \infty)$ for $i \in \mathbb{N}$.

Announcements

- Get to know you Video
- HW1 Due Thursday 9pm
- Office Hours (not today)
  - M 12:30-1:30, T 1:30-2:30 in-person EH5838
  - Thursday 1-2, online

Recall: Definition II.3.1

Note: For $A, B \in \mathcal{A}$, $A \subseteq B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B).$$

And thus $\mu(A) \leq \mu(B)$ and $\mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$. We must always be careful with ∞ when we subtract, because $\infty - \infty$ is not well-defined.

Theorem II.3.1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then we have the following properties

(1) Monotonicity: $A \subseteq B \in \mathcal{A} \implies \mu(A) \leq \mu(B)$.

(2) Countable subadditivity: If $A_1, A_2, \ldots \in \mathcal{A}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

(3) Continuity from below / Montone Convergence Theorem (MCT) for sets: Given $A_1, A_2, \ldots \in \mathcal{A}$ satisfying $A_1 \subseteq A_2 \subseteq \cdots$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n)$$

(4) Continuity from above: Given $A_1, A_2, \ldots \in \mathcal{A}$satisfying $A_1 \supseteq A_2 \supseteq \cdots$ and $\mu(A_1) < \infty$ then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n)$$

Proof. (1) and (2) DIY.

For part (3), let $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$. Then we know that

$$\bigcup_{i=1}^{\infty} A_i = \bigsqcup_{i=1}^{\infty} B_i$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i)$$

$$= \lim_{n\to\infty} \mu\left(\bigsqcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n).$$

For part (4), let $E_i = A_1 \setminus A_i$. Then $E_1 \subseteq E_2 \subseteq \cdots$. Then

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_1 \setminus A_i = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right)$$

Now note that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty.$$

Therefore we have that

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= \mu(A_1) - \lim_{n\to\infty} \mu(E_n)$$

$$= \mu(A_1) - \lim_{n\to\infty} \mu(A_1) - \mu(A_n)$$

$$= \lim_{n\to\infty} \mu(A_n).$$

Example II.3.2

TAke $\mathbb{N}, \mathcal{P}(\mathbb{N})$ with the counting measure. Then let $A_n = \{n, n+1, n+2, \dots\}$. Then note that $A_1 \supseteq A_2 \supseteq \cdots$ and

$$\bigcap_{i=1}^{\infty} A_i = \emptyset \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$$

But $\mu(A_n) = \infty$ for each n. This shows that finiteness is necessary for part (4).

Definition II.3.2

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

- $A \subseteq X$ is a μ-null set if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subseteq X$ is a μ-subnull set if there exists a μ-null set B with $A \subseteq B$. Note: A is not necessarily $\mathcal{A}$-measurable.
- $(X, \mathcal{A}, \mu)$ is a complete measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Definition II.3.3 (Almost everywhere)

A statement $P(x)$ quantified over $x \in X$, holds μ-a.e. (almost everywhere) if the set $\{x \in X \mid P(x) \text{ does not hold}\}$ is μ-null.

Definition II.3.4

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

- μ is a finite measure if $\mu(X) < \infty$.
- μ is a σ-finite measure if $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in \mathcal{A}$ and $\mu(X_n) < \infty$.

HW: Every measure space can be "completed" by expanding the relevant σ-algebra and expanding the definition of the measure.

II.4. Building Measures

Definition II.4.1 (Outer measure)

An outer measure on X is $\mu^* : P(X) \to [0, \infty]$ such that

- $\mu^*(\emptyset) = 0$
- Monotonicity: If $A \subseteq B$ then $\mu^*(A) \le \mu^*(B)$.
- Countable subadditivity: That is

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu^*(A_i).$$

For every $A_1, A_2, \ldots \subseteq X$.

Example II.4.1

For $A \subseteq \mathbb{R}$,

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) \mid \bigcup_{i=1}^{\infty}(a_i, b_i) \supseteq A\right\}$$

is an outer measure due to Proposition II.4.1 by taking $\mathcal{E} = \{(a,b) \mid -\infty \le a \le b \le \infty\}$ and $\rho((a,b)) = b - a$.

This is called the Lebesgue outer measure on $\mathbb{R}$.

Proposition II.4.1

Let $\mathcal{E} \subseteq P(X)$ such that $\emptyset, X \in \mathcal{E}$. Then let $\rho : \mathcal{E} \to [0, \infty]$ such that $\rho(\emptyset) = 0$.

Then

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \mid E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supseteq A\right\}$$

is an outer measure on X. Note! It may fail that μ^* may not be ρ when restricted to $\mathcal{E}$. We need more conditions to guarantee that!

The proof of this proposition will introduce two very important tricks that we will use over and over.

Proof of Proposition II.4.1: The easy parts. We will not have time to do the proof today, but we will sketch out the easy steps

(1) μ^* is well-defined: This is easy, since inf is taken over a non-empty set bounded below by zero.
(2) $\mu^*(\emptyset) = 0$. Just take all the $E_i = \emptyset$ to get a minimum
(3) $A \subseteq B$ implies $\mu^*(A) \leq \mu^*(B)$ because every cover of B by elements of $\mathcal{E}$ also covers A.

Next class: we will prove countable subadditivity.

> **Recall II.4.2**
>
> Tonelli's theorem for series. If $a_{ij} \in [0, \infty]$ then
> $$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$
> Read [Tao11], specifically Thm 0.0.2.

Proof of Proposition II.4.1: Countable subadditivity. Let $A_1, A_2, \ldots \subseteq X$. We wish to show that
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$
If one of the $\mu^*(A_n) = \infty$, the result holds. Thus it suffices to consider the case when all $\mu^*(A_n) < \infty$.

We will instead prove that for every $\varepsilon > 0$ we have that
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$
We can call this trick

Give yourself a room of $\varepsilon > 0$

For each $n \in \mathbb{N}$, there exists $E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ such that
$$\bigcup_{k=1}^{\infty} E_{n,k} \supseteq A_n \qquad \mu^*(A_n) \leq \sum_{k=1}^{\infty} \rho(E_{n,k}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$$
Useful because $\mu^*(A_n) < \infty$. Here we have used the **$\varepsilon/2^n$-trick so that we don't accumulate infinite error**.

Then
$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_{n,k} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{n,k}$$
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{(n,k)\in\mathbb{N}^2} \rho(E_{k,n}) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \rho(E_{k,n})$$
$$\leq \sum_{n=1}^{\infty} \mu^*(E_n) + \frac{\varepsilon}{2^n} = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$
Here we have used Tonelli's theorem, because each $\rho(E_{n,k})$ satisfies $0 \leq \rho(E_{n,k}) < \infty$. Perfect! This proves the result by taking $\varepsilon \to 0$.

Definition II.4.2

[Carathéodory measurable] Let μ^* be an outer measure on X. We say that $A \subseteq X$ is Carathéodory measurable (abbrev. C-measurable) with respect to μ^* provided that for every $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$$

Lemma II.4.2

Let μ^* be an outer measure on X. Suppose $B_1, \ldots, B_N$ are disjoint C-measurable sets. Then for all $E \subseteq X$,

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = \sum_{i=1}^{N} \mu^*(E \cap B_i).$$

This also implies that μ^* is finitely additive on C-measurable sets by setting $E = X$.

Proof. We see that

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_{i=2}^{N} B_i\right)\right) = \cdots = \sum_{i=1}^{N} \mu^*(E \cap B_i).$$

Announcements

- No class on Monday (MLK day)
- HW 2 posted
- Another assignment
  - Get to Know Video review
  - Solution consent form.

Lemma II.4.3

Improved version of Lemma II.4.2

Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots$ are disjoint C-measurable sets. Then for all $E \subseteq X$,

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i).$$

This also implies that μ^* is countably additive on C-measurable sets by setting $E = X$.

Proof. By countable subadditivity of μ^* we have that

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_n) \geq \mu^*\left(E \cap \cup_{n=1}^{\infty} B_n\right).$$

Now monotonicity and Lemma II.4.2 implies that

$$\begin{aligned}\mu^*\left(E \cap \cup_{n=1}^{\infty} B_n\right) &\geq \mu^*\left(E \cap \cup_{n=1}^{N} B_n\right) \\ &\geq \sum_{n=1}^{N} \mu^*(E \cap B_n)\end{aligned}$$

by taking $N \to \infty$ we see that

$$\mu^*\left(E \cap \cup_{n=1}^\infty B_n\right) \geq \sum_{n=1}^\infty \mu^*(E \cap B_n)$$

These two inequalities imply the result.

Theorem II.4.4 (Carathéodory Extension Theorem)

Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets (with respect to μ^*). Then

(a) $\mathcal{A}$ is a σ-algebra

(b) $\mu := \mu^*\big|_{\mathcal{A}} : \mathcal{A} \to [0, \infty]$ is a measure on $(X, \mathcal{A})$.

(c) $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We do this by parts, (a) is hardest, (b) is easy-ish, and (c) is easy

(a) We break this down into five steps

(a1) $\emptyset \in \mathcal{A}$, DIY.

(a2) $\mathcal{A}$ is closed under complements, DIY.

(a3) $\mathcal{A}$ is closed under finite unions

(a4) $\mathcal{A}$ is closed under countable disjoint unions.

(a5) $\mathcal{A}$is closed under countable unions.

Lets go!

(a3) By induction, it is enough to show that if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. Fix $E \subseteq X$. We need to show that

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)).$$

We make the following convenient definitions

$$E_1 := E \cap A \cap B \qquad E_2 := (E \cap A) \setminus B \qquad E_3 := (E \cap B) \setminus A \qquad E_4 := E \setminus (A \cup B).$$

We need to show that

$$\mu^*(E_1 \cup E_2 \cup E_3 \cup E_4) = \mu^*(E_1 \cup E_2 \cup E_3) + \mu^*(E_4).$$

We know that

$$\mu^*(E_1 \cup E_2 \cup E_3 \cup E_4) = \mu^*(E_1 \cup E_2) + \mu^*(E_3 \cup E_4)$$
$$\mu^*(E_1 \cup E_2 \cup E_3) = \mu^*(E_1 \cup E_2) + \mu^*(E_3)$$

by testing against A, which is C-measurable.

By testing against B which is C-measurable that

$$\mu^*(E_3 \cup E_4) = \mu^*(3) + \mu^*(E_4).$$

The right hand side then becomes

$$\begin{aligned}\mu^*(E_1 \cup E_2 \cup E_3) + \mu^*(E_4) &= \mu^*(E_1 \cup E_2) + \mu^*(E_3) + \mu^*(E_4) \\ &= \mu^*(E_1 \cup E_2) + \mu^*(E_3 \cup E_4) \\ &= \mu^*(E_1 \cup E_2 \cup E_3 \cup E_4).\end{aligned}$$

(a4) We show $\mathcal{A}$ is closed under countable disjoint unions. Let $A_1, A_2, \ldots \in \mathcal{A}$ be disjoint. Fix $E \subseteq X$. We need to show that

$$\mu^*(E) = \mu^*\left(E \cap \bigcup_{n==1}^{\infty} A_n\right) + \mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right).$$

Because μ^* is countable subadditive we know that

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{n==1}^{\infty} A_n\right) + \mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right).$$

We then just need to show the other direction of the inequality. So fix $N \in \mathbb{N}$. We know by Item (a3) that $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$, and so by Lemma II.4.2, monotonicity, and countable subadditivity

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \bigcup_{n=1}^{N} A_n\right) + \mu^*\left(E \setminus \bigcup_{n=1}^{N}\right) \\ &\geq \sum_{n=1}^{N} \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right).\end{aligned}$$

By taking $N \to \infty$ and applying the result of countable subadditivity.

(a5) We claim that being closed under complement (a2), closed under finite unions (a3), and closed under countable disjoint unions (a4) suffices to show that $\mathcal{A}$ is closed under countable unions. To do this, fix $A_1, A_2, \ldots \in \mathcal{A}$. Now let

$$B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i.$$

Then $\bigcup_n A_n = \bigcup_n B_n$, but the B_n are disjoint, and all in $\mathcal{A}$ because of (a2),(a3).

(b) We know that $\mu(\emptyset) = \mu^*(\emptyset) = 0$, and countable additivity on $\mathcal{A}$ follows from Lemma II.4.3 with $E = X$.

(c) On HW2!

Recall II.4.3

Recall Proposition II.4.1. That is let $\mathcal{E} \subseteq P(X)$ such that $\emptyset, X \in \mathcal{E}$.

Now let $\rho : \mathcal{E} \to [0, \infty]$ such that $\rho(\emptyset) = 0$. Then

$$\mu^*(A) := \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \mid E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supseteq A\right\}$$

is an outer measure on X.

Then we have the following

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition II.4.1}} (P(X), \mu^*) \xrightarrow{\text{Theorem II.4.4}} (\text{C-measurable sets}, \mu)$$

Question: Do we have $\mathcal{E} \subseteq \mathcal{A}$ and $\mu\big|_{\mathcal{E}} = \rho$? No!

Definition II.4.3

Let $\mathcal{A}_0$ be an algebra on X (that is contains $\emptyset$, closed under complement, and closed under finite union).

We say $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a pre-measure if

(a) $\mu_0(\emptyset) = 0$

(b) Finite additivity: If $A_1, \ldots, A_n \in \mathcal{A}$ are disjoint then

$$\mu_0\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \mu_0(A_i)$$

(c) Countable additivity within $\mathcal{A}_0$: If $A \in \mathcal{A}_0$ and $A = \bigcup_{i=1}^{\infty} A_i$ for disjoint $A_i \in \mathcal{A}$, then

$$\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i)$$

In fact (a) + (c) imply (b) by taking empty sets.

Notation: [Fol99] uses $\mathcal{M}$ for σ-algebra and $\mathcal{A}$ for algebra. We use $\mathcal{A}$ for σ-algebra and $\mathcal{A}_0$ for algebra.

Example II.4.4

By next Wednesday we will consider $\mathcal{A}_0$ as finite disjoint unions of $(a, b]$ and

$$\mu_0\left(\sum_{i=1}^{N} (a_i, b_i]\right) = \sum_{i=1}^{N} (b_i - a_i).$$

This will generate the Lebesgue measure on $\mathbb{R}$.

Lemma II.4.5

μ_0 is monotone

Proof. DIY

Theorem II.4.6 (Hahn-Kolmogorov Theorem)

Let μ_0 be a premeasure on the algebra $\mathcal{A}_0$ on X.

Let μ^* be the induced outer measure from $(\mathcal{A}_0, \mu_0)$ via Proposition II.4.1. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*.

Then (A, μ) extends $(\mathcal{A}_0, \mu_0)$. In other words, $\mathcal{A} \supseteq \mathcal{A}_0$ and $\mu\big|_{\mathcal{A}_0} = \mu_0$.

Proof. Let's go!

(a) We wish to show $\mathcal{A} \supseteq \mathcal{A}_0$. Let $A \in \mathcal{A}_0$. We need to show $A \in \mathcal{A}$, that is we need to show A is C-measurable. Concretely, for $E \subseteq X$ we need

$$\mu^*(E) \stackrel{?}{=} \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Now fix $E \subseteq X$. Countable subadditivity of μ^* implies that

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

For the other inequality, if $\mu^*(E) = \infty$, then we're clearly done. Thus we assume $\mu^*(E) < \infty$.

We use the room of $\varepsilon > 0$ trick. Fix $\varepsilon > 0$, then we will show that

$$\mu^*(E) + \varepsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

By the definition of μ^*, there are $B_1, B_2, \ldots \in \mathcal{A}_0$ so that $E \subseteq \bigcup_{n=1}^\infty B_n$ and

$$\begin{aligned} \mu^*(E) + \varepsilon &\geq \sum_{n=1}^\infty \mu_0(B_n) = \sum_{n=1}^\infty (\mu_0(B_n \cap A) + \mu_0(B_n \cap A^c)) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \end{aligned}$$

because $B_n \cap A, B_n \cap A^c \in \mathcal{A}_0$ and their union contains $E \cap A$ and $E \cap A^c$ respectively. Perfect! Taking $\varepsilon \to 0$ yields the result.

(b) We need to show that $\mu\big|_{\mathcal{A}_0} = \mu_0$. Pick $A \in \mathcal{A}_0$. We want to show $\mu(A) = \mu_0(A)$, that is $\mu^*(A) = \mu_0(A)$.

To show $\mu^*(A) \leq \mu_0(A)$, just let

$$B_i = \begin{cases} A & \text{if } i = 1 \\ \emptyset & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0 \qquad\qquad \bigcup_{i=1}^\infty B_i \supseteq A.$$

Therefore

$$\mu^*(A) \leq \mu_0(A) + \sum_{i=2}^\infty \mu_0(\emptyset) = \mu_0(A).$$

Now we show that $\mu_0(A)$ is a lower bound on the sums $\sum_{i=1}^\infty \mu_0(B_i)$, so that $\mu^*(A) \geq \mu_0(A)$. Let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^\infty B_i \supseteq A$. Then define

$$C_1 = B_1 \cap A \qquad\qquad C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right).$$

Now note that each $C_i \in \mathcal{A}_0$, as we have only finitely many set operations. But then we know that

$$A = \bigcup_{i=1}^\infty C_i \in \mathcal{A}_0$$

is a disjoint countable union.

Therefore because μ_0 is a premeasure, we know that

$$\mu_0(A) = \sum_{i=1}^\infty \mu_0(C_i) \leq \sum_{i=1}^\infty \mu_0(B_i).$$

Taking the inf, we get $\mu_0(A) \leq \mu^*(A)$. Perfect! This finishes the proof!

Announcements

- HW 2 due tomorrow

- HW 1 solutions (by you) – Canvas/HW 1 page
- Piazza made for the class

Definition II.4.4

We call $(\mathcal{A}, \mu)$ the Hahn-Kolmogorov (HK) extension of $(\mathcal{A}_0, \mu_0)$ where $\mathcal{A}_0$ is an algebra and μ_0 is a premeasure.

Namely, we define

$$\begin{aligned}
\mu^* &: \mathcal{P}(X) \to [0, \infty] \\
\mu^*(E) &:= \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \bigcup_{i=1}^{\infty} B_i \supseteq E \right\} \\
\mathcal{A} &:= \{A \subseteq X \mid \forall E \subseteq X, \;\; \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)\} \\
\mu &:= \mu^*\big|_{\mathcal{A}}.
\end{aligned}$$

We have by Theorem II.4.6 that $\mathcal{A}_0 \subseteq \mathcal{A}$ and that $\mu\big|_{\mathcal{A}_0} = \mu_0$.

Theorem II.4.7 (Uniqueness of HK extension)

Let $\mathcal{A}_0$ be an algebra on X, μ_0 a pre-measure on $\mathcal{A}_0$.

Let $(\mathcal{A}, \mu)$ be the HK extension of $(\mathcal{A}_0, \mu_0)$.

Let $(\mathcal{A}', \mu')$ be some other extension of $(\mathcal{A}_0, \mu_0)$.

If μ_0 is σ-finite (recall Definition II.3.4), then $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Corollary II.4.8

Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite.

Then there exists a unique measure μ on $\langle \mathcal{A}_0 \rangle$ that extends μ_0.

Furthermore,

(i) the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$ **(HW)**

(ii) We have a formula for all $A \in \overline{\langle \mathcal{A} \rangle}$

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \bigcup_{i=1}^{\infty} B_i \supseteq A \right\}$$

Proof of Theorem II.4.7. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show that $\mu^*(A) = \mu(A) = \mu'(A)$. Again we prove two inequalities

(a) Show $\mu^*(A) \geq \mu'(A)$ **(HW)**

(b) We will show $\mu^*(A) \leq \mu'(A)$. First

   (i) Assume $\mu^*(A) < \infty$. Then fix $\varepsilon > 0$, then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup_{i=1}^{\infty} B_i \supseteq A$ so that

$$\begin{aligned}
\mu(A) + \varepsilon \geq \sum_{i=1} \mu_0(B_i) &= \sum_{i=1}^{\infty} \mu(B_i) \\
&\geq \mu(B)
\end{aligned}$$

Then since $A \subseteq B, \mu(A) < \infty$, we know that

$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon.$$

On the other hand using continuity from below

$$\begin{aligned}\mu(B) &= \lim_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) \\ &= \lim_{N\to\infty} \mu'\left(\bigcup_{i=1}^{N} B_i\right) = \mu'(B)\end{aligned}$$

Then we have by part (a) that

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \varepsilon$$

Perfect! We win by taking $\varepsilon \to 0$.

(ii) Assume $\mu(A) = \infty$. Because μ_0 is σ-finite we know $X = \bigcup_{i=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ satisfying $\mu_0(X_n) < \infty$.
Replacing X_n by $X_1 \cup \cdots \cup X_n \in \mathcal{A}_0$, we may assume $X_1 \subseteq X_2 \subseteq \cdots$.
Then note $\mu(A \cap X_n) < \infty$ so by part (i) we have

$$\mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

Now by continuity of the measure

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) \leq \lim_{n\to\infty} \mu'(A \cap X_n) = \mu'(A).$$

This finishes the proof!

Announcements

- HW 3 posted.
- Piazza.

II.5. Borel measures on $\mathbb{R}$

Definition II.5.1

A function $F : \mathbb{R} \to \mathbb{R}$ is an inreasing function provided that for $x \leq y$ we have $F(x) \leq F(y)$.

A function $F : \mathbb{R} \to \mathbb{R}$ which is increasing and right-continuous (that is $\lim_{x\to a^+} F(x) = F(a)$ for all a) is called a distribution function.

Example II.5.1

These functions are distributions

- $F(x) = x$
- $F(x) = e^x$
- $F(X) = 1$ for $x \geq 0$ and $F(x) = 0$ for $x < 0$.

- Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$. Then

$$F_n(x) = \begin{cases} 1 & \text{if } x \geq r_n \\ 0 & \text{if } x < r_n \end{cases}$$

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}$$

F is a distribution function **(HW)**

Note: If F is increasing, we have that

$$F(\infty) := \lim_{x\to\infty} F(x) \qquad\qquad F(-\infty) = \lim_{x\to-\infty} F(x)$$

exist in $[-\infty, \infty]$.

Definition II.5.2

In probability theory, cumulative distribution function is a distribution function with $F(\infty) = 1, F(-\infty) = 0$.

There are distributions [Fol99], but these are different from distribution functions.

Definition II.5.3

If X is a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called locally finite if $\mu(K) < \infty$ for all compact sets $K \subseteq X$.

Lemma II.5.1

Let μ be a locally finite Borel measure on $\mathbb{R}$. From this we can define

$$F_\mu(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}.$$

Definition II.5.4

The h-intervals are $\emptyset, (a, b), (a, \infty), (-\infty, b], (-\infty, \infty)$.

Lemma II.5.2

Let $\mathcal{H}$ be the collection of finite disjoint unions of h-intervals. Then $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. DIY

Proposition II.5.3 (Distribution function defines a Pre-measure)

Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. For all the h-intervals I define $\ell(I) := \ell_F(I)$

$$\ell_F(\emptyset) = 0$$

$$\ell_F((a, b]) = F(b) - F(a)$$

$$\ell_F((a, \infty)) = F(\infty) - F(a)$$

$$\ell_F((-\infty, b)) = F(b) - F(-\infty)$$

$$\ell_F((-\infty,\infty)) = F(\infty) - F(-\infty).$$

We now define $\mu_0 \coloneqq \mu_{0,F} : \mathcal{H} \to [0,\infty]$ by

$$\mu_0(A) = \sum_{k=1}^{N} \ell_F(I_k)$$

if A may be written as a finite disjoint union $\bigcup_{k=1}^{N} I_k$ of h-intervals.

Then μ_0 is well-defined and a pre-measure on $\mathcal{H}$.

Proof. There are a few conditions to verify

(a) μ_0 is well-defined. This can be shown by taking a common "refinement" of two expressions $I_1, \ldots, I_N$ and $J_1, \ldots, J_M$ which both union to $A \subseteq \mathcal{H}$.

(b) $\mu_0(\emptyset) = 0$ ✓.

(c) μ_0 is finitely additive ✓.

(d) μ_0 is countably additive within $\mathcal{H}$. That is suppose $A \in \mathcal{H}$ and $A = \bigcup_{i=1}^{\infty} A_i$, disjoint union, $A_i \in \mathcal{H}$. These cases look something like

$$(0,1] = \bigcup_{i=1}^{\infty} \left(\frac{1}{i+1}, \frac{1}{i} \right].$$

It is enough to consider the case where $A = I$, $A_k = I_k$ all h-intervals. (why?)

Furthermore the statement is easy to extend to the infinite cases, so we focus on $I = (a,b]$ **(HW)**

Suppose that $(a,b] = \bigcup_{n=1}^{\infty} (a_n, b_n]$, disjoint. We must check that

$$F(b) - F(a) \stackrel{?}{=} \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

We know that for all N

$$\begin{aligned}
(a,b] &\supseteq \bigcup_{n=1}^{N} (a_n, b_n] \\
F(b) - F(a) &\geq \sum_{n=1}^{N} (F(b_n) - F(a_n)) \\
F(b) - F(a) &\geq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).
\end{aligned}$$

Fix $\varepsilon > 0$. Since F is right-continuous, there exists $a' > a$ such that $F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, ther eis a point $b_n' > b_n$ such that $F(b_n') - F(b_n) < \frac{\varepsilon}{2^n}$. We then see that

$$\begin{aligned}
[a', b] &\subseteq \bigcup_{n=1}^{\infty} (a_n, b_n') \\
[a', b] &\subseteq \bigcup_{n=1}^{N} (a_n, b_n') \\
(a', b] &\subseteq \bigcup_{n=1}^{N} (a_n, b_n']
\end{aligned}$$

$$\begin{aligned}
F(b) - F(a') &\leq \sum_{n=1}^{N} F(b_n') - F(a_n) \\
F(b) - F(a) \leq F(b) &= F(a') + \varepsilon \\
&\leq \varepsilon + \sum_{n=1}^{\infty} F(b_n') - F(a_n) \\
&\leq \varepsilon + \sum_{n=1}^{\infty} F(b_n) - F(a_n) + \frac{\varepsilon}{2^n} \\
&= 2\varepsilon + \sum_{n=1}^{\infty} F(b_n) - F(a_n).
\end{aligned}$$

taking $\varepsilon \to 0$ yields the result.

Announcements

- Piazza.
- HW3 Q3 typo, $\mathcal{B} \subseteq \mathcal{A}_c$ should be $\mathcal{B} \subseteq \mathcal{A}$.
- Office hour reminder
  - M 12:30-1:30pm in-person.
  - T 1:30-2:30pm in-person.
  - Th 1-2pm online.

Theorem II.5.4 (Locally finite Borel measures on $\mathbb{R}$)

Our work last time in fact classifies locally finite Borel measures on $\mathbb{R}$

(a) If $F : \mathbb{R} \to \mathbb{R}$ is a distribution function, then there exists a unique locally finite Borel measure μ_F on $\mathbb{R}$ satisfying $\mu_F((a, b]) = F(b) - F(a)$ for every $a < b$ in $\mathbb{R}$.

This essentially follows from Theorem II.4.7.

(b) Suppose $F, G : \mathbb{R} \to \mathbb{R}$ are distribution functions. Then, $\mu_F = \mu_G$ on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function. **(HW)**

(c) Lemma II.5.1 implies that these are all of the locally finite Borel measures on $\mathbb{R}$.

II.6. Lebesgue-Stieltjes measures on $\mathbb{R}$

The general sketch of what is going on

$$F \text{ dist. fn} \xRightarrow{HK} \mu_F \text{ on Carathéory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supseteq \mathcal{B}(\mathbb{R}).$$

Then HW3 implies that $(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}$ (the completion).

Definition II.6.1 (Lebesgue-Stieltjes measure)

For a distribution function F, we call μ_F on $\mathcal{A}_{\mu_F}$ the <u>Lebesgue-Stieltjes measure</u> corresponding to F. A special case, when $F(x) = x$, is called the Lebesgue measure m on the Lebesgue σ-algebra $\mathcal{L}$.

Example II.6.1 (Discrete Measures) (a) Write

$$F(x-) = \lim_{a \to x^-} F(a) \qquad F(x+) = \lim_{a \to x^+} F(a).$$

Then we have because F is right-continuous and increasing that

$$F(x-) \leq F(x) = F(x^+).$$

(HW) then gives us that $\mu_F(\{a\}) = F(a) - F(a-)$. As well we have

$$\mu_F([a,b]) = F(b) - F(a-)$$
$$\mu_F((a,b)) = F(b-) - F(a).$$

(b) Consider the following function

$$F(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Then $\mu_F(\{0\}) = 1$, $\mu_F(\mathbb{R}) = 1$, and $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. Generally

$$\mu_F(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$$

μ_F is called the Dirac measure at 0.

(c) Write $\mathbb{Q} = \{r_1, r_2, \ldots\}$. Then define

$$F_n(x) = \begin{cases} 1 & \text{if } x \geq r_n \\ 0 & \text{if } x < r_n \end{cases}$$
$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$

Then we have $\mu_F(\{r\}) > 0$ for all $r \in \mathbb{Q}$, whereas $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$ **(HW)**.

(d) If F is continuous at a, then $\mu_F(\{a\}) = 0$.

(e) For $F(x) = x$ we have

$$m((a,b]) = m((a,b)) = m([a,b]) = b - a$$

(f) For $F(x) = e^x$ we have

$$\mu_F((a,b]) = \mu_F((a,b)) = \mu_F([a,b]) = e^b - e^a$$

Cases (b), (c) are examples of discrete measures

We make some quick definitions

Definition II.6.2 (Dirac Measure)

Let $a \in \mathbb{R}$. The Dirac measure at a, denoted by δ_a, is the measure corresponding to the distribution function

$$F(x) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}.$$

We call a measure μ on X discrete if there is a countable set A such that $\mu(\{a\}) > 0$ for all $a \in A$ and $\mu(X \setminus A) = 0$.

Example II.6.2 (Middle Thirds Cantor Set)

We define the Middle Thirds Cantor Set. We start with $K_0 = [0,1]$. We remove the middle open interval. That is

$$\begin{aligned} K_1 &= K_0 \setminus (1/3, 2/3). \\ K_2 &= K_1 \setminus [(1/9, 2/9) \cup (7/9, 8/9)] \\ K_n &= K_{n-1} \setminus \left[\bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right)\right] \\ C &= \bigcap_{n=1}^{\infty} K_n \end{aligned}$$

We will show that C is uncountable, and that $m(C) = 0$.

We see that $x \in C$ if and only if $x = \sum_{n=1}^{\infty} a_n/3$ for some $a_n \in \{0,2\}$

Keep in mind that $1/3 = 0.1 = 0.022222\ldots \in C$.

Example II.6.3 (Cantor Function)

We define a function F to be 0 to the left of 0, 1 to the right of 1. Then we define F to be 1/2 on $(1/3, 2/3)$. Then to be 1/4 on $(1/9, 2/9)$ and 3/4 on $(7/9, 8/9)$ and so on. This is called the Cantor Function.

We will show that F is continuous and increasing on **(HW)**, making it a distribution function.

We then have the following comparison

Cantor Measure	Lebesgue Measure
$\mu_F(\mathbb{R} \setminus C) = 0$	$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	$m(C) = 0$
$\mu_F(\{a\}) = 0$	$m(\{a\}) = 0.$

μ_F and m are said to be "singular to each other" which will be defined formally sometime later Definition VII.1.3.

II.7. Regularity properties of Lebesgue-Stieltjes measures

Lemma II.7.1

Let μ be a LS measure on $\mathbb{R}$. Then for all μ-measurable A we have

$$\begin{aligned} \mu(A) &= \inf\left\{\sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supseteq A\right\} \\ &= \inf\left\{\sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supseteq A\right\}. \end{aligned}$$

These come from the definition of outer measure (for the first), and continuity of the measure for the seocnd. Things like

$$(a,b) = \bigcup_{n=1}^{\infty} (a, b - 1/n] \qquad (a,b] = \bigcap_{n=1}^{\infty} (a, b + 1/n]$$

This is left as (HW).

Announcements

- Solutions to HW 2 posted
- HW 3 due tomorrow
- HW 4 posted.

Recall II.7.1

Let $X \subseteq [0, \infty]$. Then

(a) $\alpha = \sup X < \infty$ if and only if for all $x \in X, \alpha \geq x$, and for all $\varepsilon > 0$ there exists $x \in X$ such that $x + \varepsilon \geq \alpha$.

(b) $\alpha = \sup X = \infty$ if and only if for all $L > 0$ there exists $x \in X$ such that $x \geq L$.

Theorem II.7.2

Let μ be an LS measure. Then, for all $A \in \mathcal{A}_\mu$,

(a) Outer regularity: $\mu(A) = \inf\{\mu(U) \mid \text{open } U \supseteq A\}$.

(b) Inner regularity: $\mu(A) = \sup\{\mu(K) \mid \text{compact } K \subseteq A\}$

Proof. Check part (a).

For part (b), let $S = \sup\{\cdots\}$. Montonicity then implies that $\mu(A) \geq s$. We must establish that $s \geq \mu(A)$.

(i) Assume A is a bounded set. Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{A}_\mu$. Then because $\overline{A}$ is bounded, $\mu(\overline{A}) < \infty$.

Fix $\varepsilon > 0$. We will show $s + \varepsilon \geq \mu(A)$. By (a), there exists an open set $U \supseteq \overline{A} \setminus A$ such that $\mu(U) - \mu(\overline{A} \setminus A) \leq \varepsilon$. But then

$$\mu(U) - \mu(\overline{A} \setminus A) \leq \varepsilon\mu(U \setminus (\overline{A} \setminus A)) \qquad\qquad \leq \varepsilon$$

Now let $K = A \setminus U$. Note that $K = \overline{A} \setminus U$. This tells us that K is compact, since it is a compact set cut an open set. Furthermore $K \subseteq A$.

It remains to show that $\mu(K) \geq \mu(A) - \varepsilon$. (DIY).

This shows that $s \geq \mu(K) \geq \mu(A) - \varepsilon$, and so we have the result by taking $\varepsilon \to 0$.

(ii) Suppose A is unbounded but $\mu(A) < \infty$. Write $A = \bigcup_{n=1}^\infty A_n$ where $A_n = A \cap [-n, n]$.

Then of course $A_1 \subseteq A_2 \subseteq \cdot$ we have that

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) < \infty.$$

Then for any $\varepsilon > 0$ we can find an A_N so that $\mu(A) - \mu(A_N) < \varepsilon$. We can also find a compact K such that $K \subseteq A_N \subseteq A$ so that $\mu(A_N) - \mu(K) < \varepsilon$.

Then the total error is $\mu(A) - \mu(K) < 2\varepsilon$. This shows the result.

(iii) Suppose $\mu(A) = \infty$. We still have $\lim_{n\to\infty} \mu(A_n) = \mu(A) = \infty$. This means for any $L > 0$, we can find some N such that $\mu(A_N) \geq L$.

But then we can find a compact set K so that $K \subseteq A_N \subseteq A$ so that $\mu(A_N) - \mu(K) < 1$, so $\mu(K) \geq L - 1$. Since L was arbitrary we see $s = \infty$ as desired.

Definition II.7.1

Let X be a topological space.

A G_δ-set is $G = \bigcap_{i=1}^{\infty} U_i$ for U_i open.

An F_σ-set is $F = \bigcup_{i=1}^{\infty} F_i$, for F_i closed.

Theorem II.7.3

Suppose μ is an LS measure. Then, the following are equivalent

(a) $A \in \mathcal{A}_\mu$

(b) $A = G \setminus M$, for G a G_δ-set, M μ-null.

(c) $A = F \cup N$, for F a F_σ-set, N μ-nul.

Proof. Clearly (b) $\implies$ (a) and (c) $\implies$ (a).

(a) $\implies$ (c) We do this in cases

(i) Assume $\mu(A) < \infty$. Inner regularity gives for every $n \in \mathbb{N}$ there exists a compact set $K_n \subseteq A$ such that $\mu(K_n) + 1/n \geq \mu(A)$.

Let $F = \bigcup_{n=1}^{\infty} K_n$. We must show $N = A \setminus F$ is μ-null. This is not too difficult.

(ii) Assume $\mu(A) = \infty$. Write $A = \bigcup_{k \in \mathbb{Z}} A_k$ such that $A_k = A \cap (k, k+1]$.

By (i), for every $k \in \mathbb{Z}$, $A_k = F_k \cup N_k$. Then $A = \bigcup_k F_k \cup \bigcup_k N_k$. These satisfy the necessary conditions.

(a) $\implies$ (b) Apply (c) to A^c to get $A^c = F \cup N$. But then

$$A = F^c \cap N^c = F^c \setminus N.$$

This means we are done.

Proposition II.7.4

Let μ be an LS measure, $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$.

Then for all $\varepsilon > 0$, there exists an $I = \bigcup_{i=1}^{N(\varepsilon)} I_i$ disjoint open intervals such that $\mu(A \triangle I) \leq \varepsilon$.

Proof. DIY. Use outer regularity and the fact that every open set can be written as a countable union of disjoint open intervals.

II.7.1. **Properties of Lebesgue measure**

Theorem II.7.5

If $A \in \mathcal{L}$ then $A + s \in \mathcal{L}, rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$ and $m(A+s) = m(A)$, $m(rA) = |r|\, m(A)$.

Proof. DIY, should propogate from the fact that it's true for the h-intervals.

Example II.7.2

We have the following strange behavior

(a) Let $\mathbb{Q} = \{r_i\}_{i=1}^{\infty}$ (which is dense in $\mathbb{R}$).

Let $\varepsilon > 0$. Now let $U = \bigcup_{i=1}^{\infty}(r - \varepsilon/2^i, r_i + \varepsilon/2^i)$. We know that U is an open subset which is dense in $\mathbb{R}$, and necessarily

$$m(U) \leq \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon.$$

But then $\partial U = \overline{U} \setminus U = \mathbb{R} \setminus U$ has measure $m(\partial U) = \infty$.

(b) There exists an uncountable set A with $m(A) = 0$.

(c) There exists A with $m(A) > 0$, but A contains no non-empty open interval.

(d) $A \notin \mathcal{L}$ Vitali set

III. Integration

III.1. Measurable functions

Definition III.1.1 (Measurable Functions)

Suppose $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measure spaces. We call $f : X \to Y$ $(\mathcal{A}, \mathcal{B})$-measurable if for all $B \in \mathcal{B}$ we have $f^{-1}(\mathcal{B}) \in \mathcal{A}$.

Lemma III.1.1

Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$ measurable if and only if for all $E \in \mathcal{E}$ we have $f^{-1}(E) \in \mathcal{A}$.

Proof. The forward direction is clear.

For the converse, let $\mathcal{D} = \{E \subseteq Y \mid f^{-1}(E) \in \mathcal{A}\}$. We see that $\mathcal{E} \subseteq \mathcal{D}$ by assumption. It is not difficult to check that $\mathcal{D}$ is a σ-algebra.

Then $\mathcal{B} = \langle \mathcal{E} \rangle \subseteq \mathcal{D}$, proving the claim.

Definition III.1.2

Let $(X, \mathcal{A})$ be a measurable space

- $f : X \to \mathbb{R}$
- $f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$
- $f : X \to \mathbb{C}$

is $\mathcal{A}$-measurable if

- f is $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable
- f is $(\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable
- $\Re f, \Im f : X \to \mathbb{R}$ are $\mathcal{A}$-measurable.

where we have $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subseteq \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$.

Example III.1.1

For $\mathcal{A} = \mathcal{P}(X)$, every function is $\mathcal{A}$-measurable. if $\mathcal{A} = \{\emptyset, X\}$, the only $\mathcal{A}$-measurable functions are the constant functions.

Lemma III.1.2

$f : X \to \mathbb{R}$, then the following are are equivalent

(a) f is $\mathcal{A}$-measurable.
(b) For all $a \in \mathbb{R}$, $f^{-1}((a, \infty)) \in \mathcal{A}$.
(c) For all $a \in \mathbb{R}$, $f^{-1}([a, \infty)) \in \mathcal{A}$.
(d) For all $a \in \mathbb{R}$, $f^{-1}((-\infty, a)) \in \mathcal{A}$
(e) For all $a \in \mathbb{R}$, $f^{-1}((-\infty, a])) \in \mathcal{A}$.

As a consequence, continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ are $\mathcal{B}(\mathbb{R})$-measurable (Borel measurable).

Proof. Lemma III.1.1.

Properties Let $f, g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable

(a) Let $\phi : \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R})$-measurable (i.e., Borel measurable). Then $\phi \circ f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.
(b) As a consequence, $-f, 3f, f^2, |f|$ are $\mathcal{A}$-measurable, and $1/f$ is $\mathcal{A}$-measurable if $f(x) \neq 0$ for all $x \in X$.
(c) $f + g$ is $\mathcal{A}$-measurable, as given by the following equality of sets

$$(f+g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} (f^{-1}((r, \infty)) \cap g^{-1}((a-r, \infty))).$$

For a quick proof of this equality.

If $f(x) + g(x) > a$, then $f(x) > a - g(x)$. And thus there exists some $r \in \mathbb{Q}$ so that $f(x) > r > a - g(x)$.

Likewise if $f(x) > r$ and $g(x) > a - r$, then $f(x) + g(x) > a$.

(d) fg is $\mathcal{A}$-measurable because

$$f(x)g(x) = \frac{1}{2}\left((f(x) + g(x))^2 - f(x)^2 - g(x)^2\right)$$

(e) $(f \vee g)(x) = \max\{f(x), g(x)\}$, $(f \wedge g)(x) = \min\{f(x), g(x)\}$ are both $\mathcal{A}$-measurable because

$$(f \vee g)^{-1}((a, \infty)) = f^{-1}((a, \infty)) \cup g^{-1}((a, \infty))$$
$$(f \vee g)^{-1}((a, \infty)) = f^{-1}((a, \infty)) \cap g^{-1}((a, \infty)).$$

(f) If $f_n : X \to \overline{\mathbb{R}}$ are $\mathcal{A}$-measurable, then $\sup_n f_n, \inf_n f_n, \limsup_{n\to\infty} f_n, \liminf_{n\to\infty} f_n$ are all $\mathcal{A}$-measurable. We give a quick proof

Call $\sup_n f_n$ as g. We need to check that

$$g^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} f^{-1}((a, \infty]).$$

For $\subseteq$, $\sup_n f_n(x) = g(x) > a$, and so necessarily there is an $n \in \mathbb{N}$ such that $f_n(x) > a$. The other direction is easy. Also $\inf_n f_n$ is exactly the same with the opposite type of interval.

Now we note that $\limsup_n f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n$. Thus this also must be measurable.

(g) Let $f_n : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If $f(x) = \lim_{n\to\infty} f_n(x)$ converges (in $\mathbb{R}$) for every $x \in X$, then f is $\mathcal{A}$-measurable.

Definition III.1.3

For $f : X \to \overline{\mathbb{R}}$ define $\operatorname{supp} f = \{x \in X \mid f(x) \neq 0\}$. This is called the support of f.

Definition III.1.4

For $f : X \to \overline{\mathbb{R}}$, let $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

We call f^+, f^- the positive and negative part of f respectively. Note that $\operatorname{supp}(f^+) \cap \operatorname{supp} f^- = \emptyset$.

Furthermore, $f = f^+ - f^-$, and so f is $\mathcal{A}$-measurable if and only if f^+, f^- are $\mathcal{A}$-measurable.

Definition III.1.5 (Characteristic Function)

For $E \subseteq X$, the characteristic (indicator) function of E is given by

$$\chi_E(x) \coloneqq 1_E(x) \coloneqq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Note that 1_E is $\mathcal{A}$-measurable if and only if $E \in \mathcal{A}$.

Definition III.1.6 (Simple Function)

A simple function $\phi : X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes finitely many values.

If $\phi(X) = \{c_1, \ldots, c_N\}$, then $E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A}$, and $\phi = \sum_{i=1}^N c_i 1_{E_i}$.

Note that $c_i \neq \pm\infty$.

Theorem III.1.3

Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to [0, \infty]$. Then the following are equivalent

(a) f is $\mathcal{A}$-measurable.

(b) There exists simple functions $0 \le \phi_1 \le \phi_2 \le \cdots \le f$ such that

$$\lim_{n\to\infty} \phi_n(x) = f(x)$$

for all $x \in X$. I.e., f is a pointwise convergence upward limit of simple functions.

Proof. (b) $\implies$ (a) is easy because $f(x) = \sup_{n\in\mathbb{N}} \phi_n(x)$, and so f is a supremum of measurable functions.

Now assume f is $\mathcal{A}$-measurable. Now fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty])$. For every $0 \le k \le 2^{2n} - 1$ let $E_{n,k} = f^{-1}([k/2^n, (k+1)/2^n))$.

Let

$$\phi_n \coloneqq \sum_{k=0}^{2^{2n}-1} \frac{k 1_{E_{n,k}}}{2^n} + 2^n 1_{F_n}.$$

This implies $0 \le \phi_1 \le \phi_2 \le \cdots f$. Now for all $x \in X \setminus F_n$ we have that

$$0 \le f(x) - \phi_n(x) \le \frac{1}{2^n}$$

Then $F_1 \supseteq F_2 \supseteq \cdots$, and $\bigcap_{n=1}^\infty F_n = f^{-1}(\infty)$. This shows that for $x \in f^{-1}([0, \infty)) = X \setminus \bigcap_n F_n$. Thus $\lim_{n\to\infty} \phi_n(x) = f(x)$.

Then for $x \in f^{-1}(\{\infty\}) = \bigcap_n F_n$, which implies $\phi_n(x) \ge 2^n$. Thus $\lim_{n\to\infty} \phi_n(x) = \infty = f(x)$.

Corollary III.1.4

If f is bounded on a set $A \subseteq \mathbb{R}$, then $\phi_n \to f$ uniformly on A.

Corollary III.1.5

$f : X \to \mathbb{C}$ is a measurable function if and only if there exist simple functions $\phi_n : X \to \mathbb{C}$ such that $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots |f|$ such that ϕ_n converges pointwise to f (if f is bounded the convergence can be made uniform).

III.2. Integration of nonnegative functions

Definition III.2.1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then let

$$\phi = \sum_{i=1}^{N} c_i 1_{E_i} : X \to [0, \infty]$$

be a simple function with each $c_i \in [0, \infty)$. We define

$$\int \phi := \int \phi \, \mathrm{d}\mu := \int_X \phi \, \mathrm{d}\mu := \sum_{i=1}^{N} c_i \mu(E_i).$$

This is called the integral of ϕ

For $A \in \mathcal{A}$ we define the notation

$$\int_A \phi := \int_A \phi \, \mathrm{d}\mu := \int \phi 1_A \, \mathrm{d}\mu$$

Proposition III.2.1

Let $\phi, \psi \geq 0$ be simple functions. Then,

- This definition is well-defined.
- $\int c\phi = c \int \phi$ for $c \in [0, \infty)$.
- $\int \phi + \psi = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x)$ for all x implies $\int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \, \mathrm{d}\mu$ is a measure on $(X, \mathcal{A})$.

Proof. DIY.

Definition III.2.2

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a measurable function. Then we define

$$\int f := \int f \, \mathrm{d}\mu :== \sup \left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

We have a few properties

- If f is a simple function, then the two definitions of $\int f$ agree
- $\int cf = c \int f$ for all $c \in [0, \infty)$.
- If $f \geq g \geq 0$ then $\int f \geq \int g$.
- But $\int f + g \stackrel{?}{=} \int f + \int g$. This really uses that f, g are measurable.

Theorem III.2.2 (Monotone Convergence Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then let $f_n : X \to [0, \infty]$ be monotonically increasing measurable functions (i.e., $0 \le f_1 \le f_2 \le \cdots$).

Let $f(x) := \sup_n f_n(x) = \lim_{n\to\infty} f_n(x)$. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. Note that $\lim_{n\to\infty} f_n(x)$ converges and $\lim_{n\to\infty} f_n$ converges because they are both monotone.

We know $f_n \le f$, and so

$$\int f_n \le \int f \implies \lim_{n\to\infty} \int f_n \le \int f.$$

Now fix a simple function $0 \le \phi \le f$. It is enough to show that

$$\lim_{n\to\infty} \int f_n \ge \int \phi.$$

Fix $\alpha \in (0, 1)$. It is enough to prove that

$$\lim_{n\to\infty} \int f_n \ge \alpha \int \phi.$$

Then we can take the supremum over α, and then take a supremum over ϕ! Now we know that $\alpha\phi < f$. Let $A_n = \{x \mid f_n(x) \ge \alpha\phi(x)\}$.

We know $A_n \in \mathcal{A}$ (using measurability). Furthermore $A_1 \subseteq A_2 \subseteq \cdots$. We see that $\bigcup_n A_n = X$ (check!). Therefore

$$\int f_n \ge \int f_n 1_{A_n} \ge \int \alpha\phi 1_{A_n} = \alpha\nu(A_n) := \int_{A_n} \phi.$$

We know ν is a measure, and so using continuity

$$\lim_{n\to\infty} \int f_n \ge \alpha \lim_{n\to\infty} \nu(A_n) = \alpha\nu(X) = \alpha \int \phi.$$

Corollary III.2.3

Let $f, g \ge 0$ be measurable. Then $\int f + g = \int f + \int g$.

Proof. There exist simple functions $0 \le \phi_1 \le \phi_2 \le \cdots$ for $\phi_n \to f$ pointwise, and likewise $0 \le \psi_1 \le \psi_2 \le \cdots$ for $\psi_n \to g$ pointwise. Then by Theorem III.2.2 we have

$$\int f + g = \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \int \phi_n + \int \psi_n = \int f + \int g.$$

Announcements

- Solutions for HW 3 posted
- HW 4 due tomorrow
- HW 5 will be posted.

Corollary III.2.4 (Tonelli's for series and integrals)

For $g_n \geq 0$ and all measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Proof. Let $f_N = \sum_{n=1}^{N} g_n$. Then because $g_n \geq 0$ we have $0 \leq f_1 \leq f_2 \leq \cdots$. Furthermore

$$\lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

Thus Theorem III.2.2 implies that

$$\begin{aligned}
\lim_{N\to\infty} \int \sum_{n=1}^{N} g_n &= \int \sum_{n=1}^{\infty} g_n \\
\lim_{N\to\infty} \sum_{n=1}^{N} \int g_n &= \int \sum_{n=1}^{\infty} g_n \\
\sum_{n=1}^{\infty} \int g_n &= \int \sum_{n=1}^{\infty} g_n
\end{aligned}$$

Theorem III.2.5 (Fatou's Lemma)

Suppose $f_n \geq 0$ are measurable functions. Then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Recall III.2.1

$\liminf$ obeys the following

$$\begin{aligned}
\liminf_{n\to\infty} f_n &:= \lim_{k\to\infty} \inf_{n\geq k} f_n \\
&= \sup_{k\in\mathbb{N}} \inf_{n\geq k} f_n.
\end{aligned}$$

Furhtermore we have that

$$\lim_{n\to\infty} a_n \text{ exists} \iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$$

Proof. Let $g_k = \inf_{n\geq k} f_n$. Then each g_k is measurable and $0 \leq g_1 \leq g_2 \leq \cdots$.

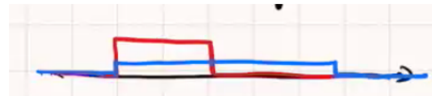
Therefore by Theorem III.2.2 we have

$$\int \liminf_{n\to\infty} f_n = \int \lim_{k\to\infty} g_k = \lim_{k\to\infty} \int g_k = \lim_{k\to\infty} \int \inf_{n\geq k} f_n.$$

We now know that $\inf_{n\geq k} f_n \leq f_m$ for all $m \geq k$. Therefore by monotonicity

$$\begin{aligned}
\int \inf_{n\geq k} f_n &\leq \int f_m && (\forall\ m \geq k) \\
\int \inf_{n\geq k} f_n &\leq \inf_{m\geq k} \int f_m.
\end{aligned}$$

(A) Escape to Horizontal ∞



(B) Escape to Width ∞

(C) Escape to Vertical ∞

Therefore

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \lim_{k\to\infty} \inf_{m\geq k} \int f_m = \liminf_{m\to\infty} \int f_m$$

This is exactly the result we wish to show!

Example III.2.2

We'll use $(\mathbb{R}, \mathcal{L}, m)$.

(a) Escape to horizontal infinity: Take $f_n = 1_{(n,n+1)}$. Then $\int f_n = 1$ for all n, but $f_n \to 0$ pointwise. Thus Fatou's Lemma give us a strict inequality

$$0 = \int \liminf_{n\to\infty} f_n < \liminf_{n\to\infty} \int f_n = 1.$$

See Figure 1a.

(b) Escape to width infinity: Take $f_n = 1/n \cdot 1_{(0,n)}$. Then $\int f_n = 1$ for all n, but $f_n \to 0$ pointwise as well. See Figure 1b

(c) Escape to vertical infinity: Take $f_n = n1_{(0,1/n)}$. Then $\int f_n = 1$ for all n, but $f_n \to 0$ pointwise. See Figure 1c

Lemma III.2.6 (Markov's Inequality)

Let $f \geq 0$ be measurable. Then for all $c \in [0,\infty]$ we have that

$$\mu(\underbrace{\{x \mid f(x) \geq c\}}_{E}) \leq \frac{1}{c}\int f.$$

Proof. We have that $f(x) \geq c1_E(x)$, and so by monotonicity

$$\int f \geq c \int 1_E = c\mu(E).$$

Proposition III.2.7

If $f \geq 0$ is measurable, then

$$\int f = 0 \iff f = 0 \text{ almost everywhere.}$$

Namely if we let $A = \{x \mid f(x) > 0\}$ then

$$\int f \, \mathrm{d}\mu = 0 \iff \mu(A) = 0.$$

Recall that

$$\int f = \sup \left\{ \int \phi \mid 0 \le \phi \le f, \phi \text{ simple} \right\}$$

Proof. We do this in steps

(1) Assume $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^{N} c_i 1_{E_i}$$

where E_i are disjoint and $c \in (0, \infty]$. Then saying that

$$\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0$$

if and only if$\mu(E_i) = 0$ for all i. Then this holds if and only if $\mu(A) = 0$ because $A = \bigcup_{i=1}^{N} E_i$.

(2) For general $f \ge 0$, we have

(a) Assume $\mu(A) = 0$. That is $f = 0$ almost everywhere. Now let $0 \le \phi \le f$ for ϕ simple. Then for all $x \in A^c$ we have $\phi(x) = 0$. Thus $\phi = 0$ almost everywhere, and $\int \phi = 0$.
Thus $\int f = 0$ by the definition of $\int f$.

(b) Now assume $\int f = 0$. Let $A_n = f^{-1}([-1/n, \infty])$. Then $A_1 \subseteq A_2 \subseteq \cdots$. We then know that

$$\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty \right] \right) = f^{-1}((0, \infty)) = A.$$

By continuity of the measure we know that

$$\mu(A) = \lim_{n \to \infty} \mu(A_n).$$

By Lemma III.2.6, we see that

$$0 \le \mu(A_n) \le n \int f = 0.$$

Great! This shows that $\mu(A) = 0$.

Corollary III.2.8

If $f, g \ge 0$ are measurable, and $f = g$ almost everywhere, then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \ne g(x)\}$. By assumption, $\mu(A) = 0$. Then

$$\begin{aligned}\int f &= \int f 1_A + \int f 1_{A^c} = 0 + \int g 1_{A^c} \\ &= \int g 1_A + \int g 1_{A^c} = \int g.\end{aligned}$$

Note: Almost all the theorems we've proved can be replaced by theorems dealing with almost everywhere conditions ☺

III.3. Integration of complex functions

Definition III.3.1

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to \overline{R}, g : X \to \mathbb{C}$ be measurable (for g, this means that both $\Re g, \Im g$ are measurable).

Then f, g are called integrable if $\int |f| < \infty$. Then we define

$$\int f = \int f^+ - \int f^- \qquad \int g = \int \Re g + i \int \Im g.$$

For $f : X \to \overline{\mathbb{R}}$ we can define

$$\int f = \begin{cases} \infty & \text{if } \int f^+ = \infty, \int f^- < \infty \\ -\infty & \text{if } \int f^+ < \infty, \int f^- = \infty \end{cases}$$

Lemma III.3.1

Let $f, g : X \to \overline{R}$ or $\mathbb{C}$ integrable. Assume that $f(x) + g(x)$ is well-defined for all $x \in X$. That is we never see $\infty + (-\infty)$ or $(-\infty) + \infty$.

Then we have that

(a) $f + g$, cf for all $c \in \mathbb{C}$ are integrable.

(b) $\int f + g = \int f + \int g$. This is non-trivial because $(f + g)^+ \neq f^+ + g^+$.

(c) $\left|\int f\right| \leq \int |f|$.

Proof. Check [Fol99] pg 53.

Lemma III.3.2

Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be an integrable function on X. Then

(a) f is finite almost everywhere (i.e. $\{x \in X \mid |f(x)| = \infty\}$ is a null set).

(b) The set $\{x \in X \mid f(x) \neq 0\}$ is σ-finite

Proof. HW5 Q8 (think Markov).

Proposition III.3.3

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

(a) If h is integrable on X, then

$$\int_E h = 0, \quad \forall E \in \mathcal{A} \iff \int |h| = 0 \iff h = 0 \text{ almost everywhere}$$

the second $\iff$ was done last class.

(b) If f, g are integrable on X, then

$$\int_E f = \int_E g, \quad \forall E \in \mathcal{A} \iff f = g \text{ almost everywhere}$$

Proof. Let's go!

(a) We have that

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0.$$

This handles one implication. Now assume $\int_E h = 0$ for all $E \in \mathcal{A}$. Then write

$$h = u + iv = u^+ - u^- + i(v^+ - v^-).$$

Then let $B = \{x \mid u^+(x) > 0\}$. By assumption

$$0 = \int_B h = \Re \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+.$$

Therefore $u^+ = 0$ almost everywhere. Similarly, u^-, v^+, v^- are zero almost everywhere. This gives us that h is zero almost everywhere as desired.

Theorem III.3.4 (Dominated Convergence Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Here is the setup!

(1) Let f_n be integrable on X.

(2) $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere.

(3) There is a $g : X \to [0, \infty]$ such that g is integrable and

$$|f_n(x)| \leq g(x) \text{ almost everywhere for all } n \in \mathbb{N}.$$

Then we have that

$$\lim_{n\to\infty} \int f_n = \int f = \int \lim_{n\to\infty} f_n$$

Proof. Let F be the countable union of null sets on which (1)-(3) may fail. Modifying the definition of f_n, f, g on F, we may assume (1)-(3) hold everywhere because modifying on a null set does not change the integral.

We consider the $\overline{\mathbb{R}}$-valued case only ($\mathbb{C}$-valued case, check yourself). Note that (2),(3) imply that f is integrable, because $|f| \leq g(x)$.

Then $g + f_n \geq 0$ and $g - f_n \geq 0$ because $-g \leq f_n \leq g$. Then Fatou's Lemma tells us that

$$\int g + f \leq \liminf_{n\to\infty} \int g + f_n$$
$$\int g - f \leq \liminf_{n\to\infty} \int g - f_n.$$

Using linearity and cancellation (because $\int g < \infty$) this shows that

$$\int f \leq \liminf_{n\to\infty} \int f_n \qquad\qquad -\int f \leq \liminf \int -f_n = -\limsup \int f_n.$$

Therefore

$$\int f \leq \liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f.$$

This shows the limit exists as well as the desired result!

Corollary III.3.5 (Fubini's Theorem for series and integrals)

Suppose f_n are integrable functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty.$$

Then we have that

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Take $G(x)$ to be $\sum_{n=1}^{\infty} |f_n(x)|$. Then $G(x) \geq |F_N(x)|$ where $F_N(x) = \sum_{n=1}^{N} f_n(x)$.

By Corollary III.2.4 we have that

$$\int G(x) = \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Then the Dominated Convergence Theorem hands us the result

III.4. L^1 Spaces

Definition III.4.1

Let V be a vector space over a field $\mathbb{R}$ or $\mathbb{C}$. A seminorm on V is $\|\cdot\| : V \to [0, \infty)$ satisfying

- $\|cv\| = |c|\,\|v\|$ for all $v \in V$ and c a scalar
- $\|v + w\| \leq \|v\| + \|w\|$.

A norm is a seminorm such that $\|v\| = 0 \iff v = 0$.

Lemma III.4.1

A normed vector space is a metric space with metric $\rho(v, w) = \|v - w\|$.

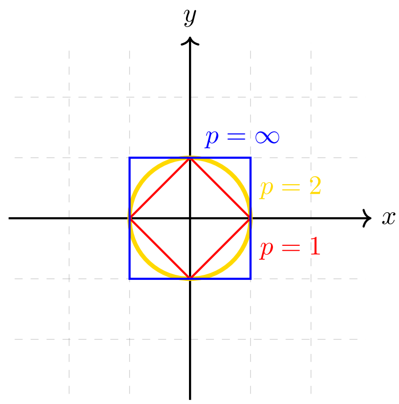
Proof. DIY.

Example III.4.1

In $\mathbb{R}^d$ we have norms

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} & \text{if } p \in [1, \infty) \\ \max_{1\leq i\leq d} |x_i| & \text{if } p = \infty \end{cases}$$

We call the unit ball $\{x \mid \|x\| \leq 1\}$. We see that we have the following



All $\|\cdot\}_p$ norms induce the same topology. i.e., if U is open in p-norm, it is open in p'-norm as well.

Recall that f is integrable means $\int |f| < \infty$ and $f = g$ almost everywhere implies $\int f = \int g$.

Definition III.4.2

If $(X, \mathcal{A}, \mu)$ is a measure space, we say that

$L^1(X, \mathcal{A}, \mu) = L^1(X, \mu) = L^1(X) = L^1(\mu)$ is the set of integrable functions on X. This is a vector space.

Lemma III.4.2

$L^1(X, \mathcal{A}, \mu)$ is a vector space with seminorm $\|f\|_1 = \int |f|$.

Definition III.4.3

Define $f \sim g$ if $f = g$ almost everywhere. Then $L^1(X, \mathcal{A}, \mu)/ \sim = L^1(X, \mathcal{A}, \mu)$.

This notation is confusing.

With this definition, $L^1(X, \mathcal{A}, \mu)$ is a normed vector space. Note then that

$$\rho(f, g) = \int |f - g| .$$

The dense subsets of L^1 are given by

Theorem III.4.3

We have that

(a) {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1-metric).

(b) For $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$ and μ a Lebesgue-Stieltjes measure, we have that {integrable step functions} is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu$.

(c) $C_c^\infty(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

A step function on $\mathbb{R}$ is $\psi = \sum_{i=1}^N c_i 1_{I_i}$ where I_i is an interval.

And $C_c^\infty(\mathbb{R})$ is the collection of smooth functions with compact support.

Proof. Lets go!

(a) For any $f \in L^1(X, \mathcal{A}, \mu)$, we see there exist simple functions $0 \le |\phi|_1 \le |\phi_2| \le \cdots \le |f|$ such that $\phi_n \to f$ pointwise. But then

$$\lim_{n\to\infty} \int |\phi_n - f| = 0$$

by the Dominated Convergence Theorem, we win because $|\phi_n - f| \le |\phi_n| + |f| \le 2|f|$.

(b) It suffices to approximate 1_E by $\sum_{i=1}^N c_i 1_{I_i}$ for E a measurable set. Well we see that

$$\int |1_A - 1_B| = \mu(A \triangle B).$$

By regularity theorem for LS measure we see that for all $\varepsilon > 0$ there exists an $I = \bigcup_{i=1}^N I_i$ for I_i disjoint such that $\mu(E \triangle I) < \varepsilon$.

(c) It suffices to approximate $1_{(a,b)}$ by $g \in C_c^\infty(\mathbb{R})$ for $a, b \in \mathbb{R}$. Simply for $\varepsilon > 0$ glue together 0 on $(-\infty, a - \varepsilon/2)$ with 1 on (a, b) and 0 on $(b + \varepsilon/2, \infty)$.

Then we see that

$$\int \left|1_{(a,b)} - g\right| \mathrm{d}m \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

III.5. Riemann Integrability

Definition III.5.1 (Riemann Integral)

Let f be a bounded function $f : [a, b] \to \mathbb{R}$. Now fix some partition $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$. We define the upper and lower Riemann sums

$$L(f, P) = \sum_{i=1}^k (t_i - t_{i-1}) \cdot \inf_{[t_{i-1}, t_i]} f$$
$$U(f, P) = \sum_{i=1}^k (t_i - t_{i-1}) \cdot \sup_{[t_{i-1}, t_i]} f.$$

Then note that if P' is a refinement of P then

$$L(f, P) \le L(f, P') \le U(f, P') \le U(f, P).$$

And if P, Q are any partitions with common refinement $P \cup Q$ then

$$L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q).$$

Thus we can define the lower/upper Riemann integrals as

$$\underline{I}(f) = \sup_P L(f, P) \qquad \overline{I}(f) = \inf_P U(f, P).$$

We say that f is Riemann integrable provided that

$$\underline{I}(f) = \overline{I}(f).$$

and we call this common value $\int_a^b f(x)\,\mathrm{d}x$ the Riemann integral.

Theorem III.5.1

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then we see that

(a) If f is Riemann integrable, then f is Lebesgue measurable (and so Lebesgue integrable because f is bounded). Furthermore the two integrals agree

$$\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}m$$

(b) f is Riemann integrable if and only if f is continuous almost everywhere.

Proof. Pick partitions P_n such that $L(f, P_n)$ converges to $\underline{I}$ upwards and $U(f, P_n)$ converges to $\overline{I}$ downwards (taking refinements if needed).

Define functions for $P_n = \{a = t_0 < \cdots < t_k\}$ by

$$\begin{aligned}
\phi_n &= \sum_{i=1} \left(\inf_{[t_{i-1},t_i]} f \right) 1_{(t_{i-1},t_i]} \\
\psi_n &= \sum_{i=1} \left(\inf_{[t_{i-1},t_i]} f \right) 1_{(t_{i-1},t_i]} \\
\phi &= \sup_n \phi_n \\
\psi &= \inf_n \psi_n .
\end{aligned}$$

THen ϕ, ψ are Lebesgue (Borel) measurable functions. Note there exists $M > 0$ such that $|f| < M1_{[a,b]}$ and so $|\phi_n|, |\psi_n| \leq M1_{[a,b]}$. Then

$$\int \phi_n \,\mathrm{d}m = L(f, P_n) \qquad\qquad \int \psi_n \,\mathrm{d}m = U(f, P_n).$$

Now by the dominated convergence theorem

$$\begin{aligned}
\underline{I} &= \lim_{n\to\infty} \int \phi_n \,\mathrm{d}m = \int \phi \,\mathrm{d}m \\
\overline{I} &= \lim_{n\to\infty} \int \psi_n \,\mathrm{d}m = \int \psi \,\mathrm{d}m.
\end{aligned}$$

Thus f is Riemann integrable if and only if $\int \psi = \int \psi$ which holds if and only if $\int(\psi - \phi) = 0$ which holds if and only if $\psi = \phi$ Lebesgue almost everywhere.

Recall that $\phi \leq f \leq \psi$, so this holds if and only if $f = \phi$ almost everywhere (which implies f is Lebesgue measurable because the Lebesgue measure is complete).

This proves part (a). Part (b) follows by similar arguments.

III.6. **Mode of Convergence**

Definition III.6.1

Say that $f_n, f : X \to \mathbb{C}$ and $S \subseteq X$. We can say that

- $f_n \to f$ pointwise on S provided that for all $x \in S$, and for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|f_n(x) - f(x)| < \varepsilon$.
- $f_n \to f$ uniformly on S provided that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in S$ we have $|f_n(x) - f(x)| < \varepsilon$.

Note: We can change for every $\varepsilon > 0$ to for every $k \in \mathbb{N}$ we have $|f_n(x) - f(x)| < 1/k$ by the Archimedean principle.

Lemma III.6.1

Let $B_{n,k} = \{x \in X \mid |f_n(x) - f(x)| < 1/k\}$. Then we have that

(a) $f_n \to f$ pointwise on S if and only if

$$S \subseteq \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}$$

(b) $f_n \to f$ uniformly on S if and only if there exist integers $N_1, N_2, \ldots \in \mathbb{N}$ such that

$$S \subseteq \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}$$

Definition III.6.2

Let $(X, \mathcal{A}, \mu)$ be a measure space

(a) $f_n \to f$ almost everywhere provided that there is a null set E such that $f_n \to f$ pointwise on E^c.

(b) $f_n \to f$ in L^1 provided that

$$\lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \int |f_n - f| = 0.$$

Example III.6.1

Consider $(\mathbb{R}, \mathcal{L}, m)$, we'll have $f = 0$.

(1) $f_n = 1_{(n,n+1)}$.

(2) $f_n = 1/n \cdot 1_{(0,n)}$.

(3) $f_n = n1_{(0,1/n)}$

(4) the typewriter functions. We define f_1 supported on $[0,1]$, f_2 supporterd on $[0,1/2]$, f_3 supported on $[1/2,1]$, f_4 supported on $[0,1/4]$, f_5 supported on $[1/4,1/2]$, f_5 supported on $[1/2,3/4]$...

Then (1)-(3) we have $f_n \to f$ pointwise, $f_n \not\to f$ in L^1.

For (4) we have $f_n \to f$ in L^1, but $f_n \not\to f$ almost everywhere. Note that (4) has a convergent subsequence to f almost everywhere.

Recall III.6.2

Consider the following

- We have

$$B_{n,k} := \left\{x \in X \mid |f_n(x) - f(x)| < \frac{1}{k}\right\}$$

$$\{x \in X \mid \lim_{n\to\infty} f_n(x) = f(x)\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$

- $f_n \to f$ almost everywhere does not imply (and is not implied by) $f_n \to f$ in L^1.
- Markov's Inequality says for all $c > 0$ we have

$$\mu(\{x \in X \mid |g(x)| \geq c\}) \leq \frac{1}{c} \int |g|.$$

for all $c > 0$.

Proposition III.6.2 (Fast L^1 convergence $\implies$ a.e. convergence)

Let $(X, \mathcal{A}, \mu)$ be a measure space with f_n, f measurable functions on X.

Assume that $\sum_{n=1}^{\infty} \|f_n - f\|_1 < \infty$. Then $f_n \to f$ almost everywhere.

Proof. Let $E = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c$. That is the set of points x where $f_n(x) \not\to f(x)$. It suffices to show for every fixed k that

$$\bigcap_{N=1}^{\infty} \bigcup_{n=N} B_{n,k}^c.$$

has measure zero. We do this using continuity of the measure. We see for every k, and for every n

$$\mu(B_{n,k}^c) \leq k \int |f_n - f| \,.$$

But then for each N

$$\mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) \leq \sum_{n=N}^{\infty} k\|f_n - f\|.$$

as $N \to \infty$, this goes to zero by convergence. By using the continuity of the measure, for every k we have

$$\mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N} B_{n,k}^c\right) = \lim_{N\to\infty} \mu\left(\bigcup_{n=N} B_{n,k}^c\right) = 0.$$

Because this happens for every k, we see that $\mu(E) = 0$. This finishes the proof.

Corollary III.6.3

If $f_n \to f$ in L^1, there exists a subsequence $f_{n_j} \to f$ almost everywhere.

Proof. For every $j \in \mathbb{N}$, there exists $n_j \in \mathbb{N}$ such that $\|f_{n_j} - f\|_1 \leq 1/j^2$. Then $\sum_j \|f_{n_j} - f\|_1 < \infty$, and so the proposition can be applied.

Definition III.6.3

Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. We say that $f_n \to f$ in measure provided thta for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu\left(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\right) = 0.$$

Example III.6.3

Let $f_n = n1_{(0,1/n)}$ and $f = 0$. Then for every $\varepsilon > 0$, the set $\{x \mid |f_n(x) - f(x)| > \varepsilon\} \subseteq (0, 1/n)$. Thus $f_n \to 0$ in measure. But $f_n \not\to 0$ in L^1.

For typewriter functions $g_n \to 0$ in measure, and recall that $g_n \not\to 0$ almost everywhere.

$$\begin{tikzcd}
f_n \to f \text{ fast } L^1 \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & f_n \to f \text{ in } L^1 \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift left, "/" marking] & f_n \to f \text{ in measure} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[d, Rightarrow] \arrow[dl, Rightarrow, shift left, "/" marking] \\
 & f_n \to f \text{ a.e.} \arrow[u, Rightarrow, shift left, "/" marking] \arrow[ur, Rightarrow, shift left, "/" marking] & \exists f_{n,j} \to f \text{a.e.}
\end{tikzcd}$$

Definition III.6.4

f_n, f measurable on $(X, \mathcal{A}, \mu)$

(a) $f_n \to f$ uniformly almost everywhere if ther exists a null set F such that $f_n \to f$ uniformly on F^c

(b) $f_n \to f$ almost uniformly means for all $\varepsilon > 0$ there exists $F \in \mathcal{A}$ such that $\mu(F) < \varepsilon$, $f_n \to f$ uniformly on F^c.

Lemma III.6.4

$f_n \to f$ uniformly on S if and only there exists $N_1, N_2, \ldots$ such that

$$S \subseteq \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Theorem III.6.5 (Egoroff)

Let f_n, f be measurable on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$. Then, $f_n \to f$ almost everywhere if and only if $f_n \to f$ almost uniformly.

Proof. DIY the converse $\Longleftarrow$.

For $\Longrightarrow$, fix $\varepsilon > 0$. We know because $f_n \to f$ almost everywhere that

$$\mu\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0.$$

This implies for every k that

$$\mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0.$$

Then this implies the following using continuity of the measure and that $\mu(X) < \infty$,

$$\forall k \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall k\ \exists N_k \in \mathbb{N}\ \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c\right) < \frac{\varepsilon}{2^k}.$$

Now let $F = \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c$. We see that

$$\mu(F) < \varepsilon \qquad\qquad f_n \to f \text{ unif on } F^c$$

IV. Product Measures

In the book this is pages 22-23 (section 1.2), and section 2.5, 2.6.

IV.1. Product σ-algebras

Consider a product space $X = \prod_{\alpha \in I} X_\alpha$. That is $x = (x_\alpha)_{\alpha \in I}$. Of course formally we have $x : I \to \bigcup_\beta X_\beta$ such that $x(\alpha) \in X_\alpha$.

We have a coordinate map $\pi_\alpha : X \to X_\alpha$.

Definition IV.1.1

Suppose $(X_\alpha, \mathcal{A}_\alpha)$ are measurable spaces for all $\alpha \in I$.

We define the product σ-algebra on $X = \prod_{\alpha \in I} X_\alpha$ to be

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1}(\mathcal{A}_\alpha) \right\rangle$$

where

$$\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \{\pi_\alpha^{-1}(E) \mid E \in \mathcal{A}_\alpha\}$$

Notation If $I = \{1, \ldots, d\}$ then $X = \prod_{i=1}^d X_i$ and $x = (x_1, \ldots, x_d)$ and $\bigotimes_{i=1}^d \mathcal{A}_i = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_d$ (allowing $d = \infty$ for countable sets)

Lemma IV.1.1

If I is countable, then

$$\bigotimes_{i=1}^\infty \mathcal{A}_i = \left\langle \prod_{i=1}^\infty E_i \mid E_i \in \mathcal{A}_i \right\rangle$$

Proof. DIY.

Lemma IV.1.2

Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle$ for all $\alpha \in I$.

(a) $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b) We have

$$\bigotimes_\alpha \mathcal{A}_\alpha = \left\langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle.$$

(c) If I is countable, then

$$\bigotimes_{i=1}^\infty \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^\infty E_i \mid E_i \in \mathcal{E}_i \right\} \right\rangle$$

Proof. DIY (b), (c), as they should not be difficult from (a).

Let's prove part (a). In general, if $f : Y \to Z$ and $\mathcal{B}$ is a σ-algebra on Z. Then $f^{-1}(\mathcal{B})$ is a σ-algebra Why is that?

- $f^{-1}(\emptyset) = \emptyset$.
- $f^{-1}(B)^c = f^{-1}(B^c)$
- $\bigcup_n f^{-1}(B_n) = f^{-1}\left(\bigcup_n B_n\right)$.

Hence $\pi_\alpha^{-1}(\mathcal{A}_\alpha)$ is a σ-algebra on X. Furthermore it is clear that $\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subseteq \pi_\alpha^{-1}(\mathcal{A}_\alpha)$. Therefore $\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subseteq \pi_\alpha^{-1}(\mathcal{A}_\alpha)$.

Now we prove the other direction. Consider

$$\mathcal{M} := \{B \subseteq X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle\}.$$

We must then simply prove that $\mathcal{A}_\alpha \subseteq \mathcal{M}$. Since $\langle \mathcal{E}_\alpha \rangle = \mathcal{A}_\alpha$, it suffices to show that

- $\mathcal{M}$ is a σ-algebra. This is easy, because we're taking preimages with set operations.
- $\mathcal{E}_\alpha \subseteq \mathcal{M}$. This is trivial by definition of $\mathcal{M}$.

Thus $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle \subseteq \mathcal{M}$.

Thus if $E \in \mathcal{A}_\alpha$, then $E \in M$, so $\pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha)\rangle$. Therefore $\pi_\alpha^{-1}(\mathcal{A}_\alpha) \subseteq \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha)\rangle$.

Theorem IV.1.3

Suppose $X_1, \dots, X_d$ are metric spaces. Let $X = \prod_{i=1}^\infty X_i$ with product metric. For concreteness say $\rho(x,y) = \sum_i \rho_i(x_i, y_i)$ where ρ_i is the metric on X_i.

Then,

(a) We have that

$$\bigotimes_{i=1}^d \mathcal{B}(X_i) \subseteq \mathcal{B}(X).$$

(b) If, in addition, each X_i has a countable dense subset, then

$$\bigotimes_{i=1}^d \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof. DIY (see Homework 1 while doing part (b)).

Example IV.1.1

We have that

$$\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}).$$

Consider $f = u + iv : X \to \mathbb{C}$, with $\mathcal{A}$ a σ-algebra on X. Then $u^{-1}(E), v^{-1}(E) \in \mathcal{A}$ for all $E \in \mathcal{B}(\mathbb{R})$ if and only if $f^{-1}(F) \in A$ for all F in $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

All of this so far was pages 22-23. Now we pick back up at page 65 of [Fol99].

Definition IV.1.2

Let X, Y are sets. Now

(a) For $E \subseteq X \times Y$, $E_x = \{y \in Y \mid (x,y) \in E$, $E^y = \{x \in X \mid (x,y) \in E\}$.

(b) For $f : X \times Y \to Z$, define $f_x : Y \to Z$ and $f^y : X \to Z$ by $f_x(y) = f(x,y) = f^y(x)$.

Example IV.1.2

$(1_E)_x = 1_{E_x}$, similarly $(1_E)^y = 1_{E^y}$.

Proposition IV.1.4

Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces. Then

(a) If $E \in \mathcal{A} \otimes \mathcal{B}$, then $E_x \in \mathcal{B}$, $E^y \in \mathcal{A}$ for all $x \in X, y \in Y$.

(b) If $f : X \times Y \to Z$ is measurable $(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$ for some measurable space $(Z, \mathcal{C})$. Then f_x is $(\mathcal{B}, \mathcal{C})$ measurable, f^y is $(\mathcal{A}, \mathcal{C}$-measurable for all $x \in X, y \in Y$.

Proof. (b) follows from (a). We prove (a). Let

$$\mathcal{F} = \{E \subseteq X \times Y \mid \forall\, x \in X, y \in Y\ E_x \in \mathcal{B}, E^y \in \mathcal{A}\}.$$

Then

- $\mathcal{F}$ is a σ-algebra. This works because $(E^c)_x = E_x^c$, and similar statements hold for unions.
- We recall that

$$\mathcal{R}_0 \coloneqq \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}\mathcal{A} \otimes \mathcal{B} \qquad = \langle \mathcal{R}_0 \rangle.$$

  It is not difficult to show $\mathcal{R}_0 \subseteq \mathcal{F}$.

Then $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{F}$, and we're done with part (a).

Midterm may use things up to this point

IV.2. Product Measures

Definition IV.2.1

Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces. A (measurable) rectangle is $R = A \times B$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let $\mathcal{R}_0$ be the set of all (measurable) rectangles. Then let

$$\mathcal{R} = \left\{ \bigcup_{i=1}^{N} R_i \mid N \in \mathbb{N}, R_1, \ldots, R_N \text{ are disjoint rectangles} \right\}$$

Lemma IV.2.1

$\mathcal{R}$ is an algebra, and $\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$.

Proof. DIY, noting that $(A \times B)^c = (A^c \times Y) \sqcup (A \times B^c)$.

Announcements

- HW6 due tomorrow.
- HW6–Extra (Do not hand in)–about lsat 2 lectures.
- Midterm next Wednesday 6-8pm Chem 1400 (content up to last class–lecture 17).
  - Will be two classes taking the exam–take your own exam!
  - Bring computer + phone to scan exam to upload to gradescope afterwards.

Theorem IV.2.2

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces

(a) There is a measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

for every $A \in \mathcal{A}, B \in \mathcal{B}$.

(b) If μ, ν are σ-finite, $\mu \times \nu$ is unique.

Proof. Define $\pi : \mathcal{R} \to [0, \infty]$ by $\pi(A \times B) = \mu(A)\nu(B)$, and extending linearly

$$\pi(A \times B) = \mu(A)\nu(B)$$

$$\pi\left(\bigsqcup_{i=1}^{N} A_i \times B_i\right) = \sum_{i=1}^{n} \pi(A_i \times B_i).$$

Claim

π is a pre-measure

It is enough to check $\pi(A \times B) = \sum_{n=1}^{\infty} \pi(A_n \times B_n)$ if $A \times B = \bigsqcup_n A_n \times B_n$.

Since $A_n \times B_n$ are disjoint

$$1_{A \times B}(x, y) = \sum_{n=1}^{\infty} 1_{A_n \times B_n}(x, y)$$

Thus

$$1_A(x)1_B(y) = \sum_{n=1}^{\infty} 1_{A_n}(x)1_{B_n}(y).$$

Integrating with respect to x, and applying Tonelli's theorem for series and integrals:

$$\int_X 1_A(x)1_B(y)\,\mathrm{d}\mu(x) = \sum_{n=1}^{\infty} \int_X 1_{A_n}(x)1_{B_n}(y)\,\mathrm{d}\mu(x)$$

$$\mu(A)1_B(y) = \sum_{n=1}^{\infty} \mu(A_n)1_{B_n}(y).$$

For every y. Integrating again with respect to y and applying Tonelli's

$$\int_Y \mu(A)1_B(y)\,\mathrm{d}\nu(y) = \sum_{n=1}^{\infty} \int_Y \mu(A_n)1_{B_n}(y)\,\mathrm{d}\nu(y)$$

$$\mu(A)\nu(B) = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n).$$

Then π is a pre-measure, and so Theorem II.4.6 gives $\mu \times \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

For (b), if μ, ν are σ-finite, then π is σ-finite on $\mathcal{R}$, then Theorem II.4.7 applies.

Furthermore, we have that

$$(\mu \times \nu)(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \mid E \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B}\right\}.$$

IV.3. Monotone Class Lemma

Definition IV.3.1 (Monotone Class)

If X is a set, and $C \subseteq \mathcal{P}(X)$, we say C is a monotone class on X if

- C is closed under countable increasing unions.
- C is closed under countable decreasing intersections.

Example IV.3.1

Of course, every σ-algebra is a monotone class.

If C_α are (arbitrarily many) monotone classes on a set X, then $\bigcap_\alpha C_\alpha$ is a monotone class. Then if $\mathcal{E} \subseteq \mathcal{P}(X)$, there is a unique smallest monotone class containing $\mathcal{E}$.

Theorem IV.3.1 (Monotone Class Lemma)

Suppose $\mathcal{A}_0$ is an algebra on X. Then

$$\langle \mathcal{A}_0 \rangle = \text{the monotone class generated by } \mathcal{A}_0$$

(the left hand side being the σ-algebra generated by $\mathcal{A}_0$).

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$ and let $\mathcal{C}$ be the monotone class generated by $\mathcal{A}_0$. Since $\mathcal{A}$ is a σ-algebra, it is a monotone class. It contains $\mathcal{A}_0$, and so $\mathcal{A} \supseteq \mathcal{C}$.

To show $\mathcal{C} \supseteq \mathcal{A}$, it is enough to show that $\mathcal{C}$ is a σ-algebra.

(1) $\emptyset \in \mathcal{A}_0 \subseteq \mathcal{C}$.

(2) Let $\mathcal{C}' = \{E \subseteq X \mid E^c \in \mathcal{C}\}$.
   - $\mathcal{C}'$ is a monotone class (easy)
   - $\mathcal{A}_0 \subseteq \mathcal{C}'$ because if $E \in \mathcal{A}_0$, then $E^c \in \mathcal{A}_0$, so $E^c \in \mathcal{C}$. Thus $E \in \mathcal{C}'$.

   Thus $\mathcal{C} \subseteq \mathcal{C}'$, and so $\mathcal{C}$ is closed under complements.

(3) For $E \subseteq X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.
   - $\mathcal{D}(E) \subseteq \mathcal{C}$
   - $\mathcal{D}(E)$ is a monotone class.
   - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subseteq \mathcal{D}(E)$ Why? Pick $F \in \mathcal{A}_0$. Well then $E \cup F \in \mathcal{A}_0 \subseteq \mathcal{C}$.

   Thus $C = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

(4) Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\}$. That is

$$\mathcal{D} = \{E \in \mathcal{C} \mid E \cup F, \forall F \in \mathcal{C}\}.$$

   Then we see that
   - $\mathcal{A}_0 \subseteq \mathcal{D}$ by Item (3)
   - $\mathcal{D}$ is a monotone class (easy)
   - $\mathcal{D} \subseteq \mathcal{C}$ by definition.

   Thus $\mathcal{D} = \mathcal{C}$. Thus if $E, F \in \mathcal{C}$, then $E \cup F \in \mathcal{C}$. This shows that $\mathcal{C}$ is closed under finite unions.

(5) Now to show $\mathcal{C}$ is closed under countable unions, let $E_1, E_2, \ldots \in \mathcal{C}$. We may then define

$$F_N = \bigcup_{n=1}^{N} E_n \in \mathcal{C}$$

   Then $F_1 \subseteq F_2 \subseteq \cdots$. Thus $\bigcup_N F_N \in \mathcal{C}$, but we see that $\bigcup_N F_N = \bigcup_n E_n$, and so we're done.

Proof.

Announcements

- Next week office hours: M 12:30-1:30, 3:05-3:50, T 1:30-2:30, No Thursday Office Hour.
- Exam: Wednesday 6-7:50, 7:50-8:00, upload to gradescope (birng your computer / phone)

Recall IV.3.2

$E \in \mathcal{A} \otimes \mathcal{B}$ implies $E_x \in \mathcal{B}, E^y \in \mathcal{A}$ for all $x \in X, y \in Y$.

The converse does not hold (see HW7).

IV.4. Fubini-Tonelli Theorem

Theorem IV.4.1 (Tonelli for characteristic functions)

Let $(X, \mathcal{A}, \mu)$, $(\mathcal{Y}, \mathcal{B}, \nu)$ be σ-finite measure spaces. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$. Then

(a) $\alpha(x) \coloneqq \nu(E_x) : X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.

(b) $\beta(y) \coloneqq \mu(E^y) : Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.

(c) We have that

$$(\mu \times \nu)(E) = \int_X \nu(E_x) \, \mathrm{d}\mu(x) = \int_Y \mu(E^y) \, \mathrm{d}\nu(y)$$

Proof. This requires a few steps

(1) Assume μ, ν are <u>finite</u> measures. Let $\mathcal{C} = \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{(a),(b),(c) hold}\}$.

It is enough to prove $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{C}$. To do this, note that $\mathcal{R}$ is an algebra, so by Theorem IV.3.1 it is enough to show

- $\mathcal{R} \subseteq \mathcal{C}$
- $\mathcal{C}$ is a monotone class

Well! Let's do it!

- If $A \times B$ is a (measurable) rectangle, then

$$\alpha(x) = \nu((A \times B)_x) = \nu(B) 1_A(x).$$

This is clearly measurable, so (a) holds for $A \times B$. Similarly (b) holds. For part (c)

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$
$$\int_X \nu((A \times B)_x) \, \mathrm{d}\mu(x) = \int_X \nu(B) 1_A \, \mathrm{d}\mu = \nu(B)\mu(A).$$

Similarly for the other part of part (c). Extending to finite disjoint unions of rectangles is easy. Thus $\mathcal{R} \subseteq \mathcal{C}$

- Now let $E_n \in \mathcal{C}, E_1 \subseteq E_2 \subseteq \cdots$. We need to show $E = \bigcup_n E_n \in \mathcal{C}$. Then we see

$$E_x = \bigcup_{n=1}^{\infty} (E_n)_x, (E_1)_x \subseteq (E_2)_x \subseteq \cdots$$

Therefore by continuity from below

$$\alpha(x) = \nu(E_x) = \lim_{n \to \infty} \nu((E_n)_x) = \lim_{n \to \infty} \alpha_n(x).$$

Therefore α is a $\mathcal{A}$-measurable function. This shows (a) holds, and (b) is similar. For (c), we compute

$$\begin{aligned}(\mu \times \nu)(E) &= \lim_{n\to\infty} (\mu \times \nu)(E_n) \\ &= \lim_{n\to\infty} \int_X \nu((E_n)_x)\, \mathrm{d}\mu(x) \\ &= \int_X \nu(E_x)\, \mathrm{d}\mu(x)\end{aligned}$$

where the last equality holds by the monotone convergence theorem. Thus $E \in \mathcal{C}$. Great!

- Thus far we have not used the assumption that μ, ν are finite. Let $F_n \in \mathcal{C}$, $F_1 \supseteq F_2 \supseteq \cdots$. Need to show $F = \bigcap_n F_n \in \mathcal{C}$.

  Using that μ, ν are finite, we can use continity from above

$$\alpha(x) = \nu(F_x) = \lim_{n\to\infty} \nu((F_n)_x) = \lim_{n\to\infty} \alpha_n(x).$$

  Great! We compute

$$\begin{aligned}(\mu \times \nu)(E) &= \lim_{n\to\infty} (\mu \times \nu)(E_n) \\ &= \lim_{n\to\infty} \int_X \nu((E_n)_x)\, \mathrm{d}\mu(x) \\ &= \int_X \nu(E_x)\, \mathrm{d}\mu(x)\end{aligned}$$

  where we use the Dominated Convergence Theorem with $\nu(Y)$ as our dominating function, whose integral is $\mu(X)\nu(Y)$.

This proves the result when μ, ν are finite measures.

(2) Assume μ, ν are σ-finite. Write

$$\begin{gathered}X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n) \\ X_1 \subseteq X_2 \subseteq \cdots \\ Y_1 \subseteq Y_2 \subseteq \cdots\end{gathered}$$

where $\mu(X_k) < \infty, \nu(Y_k) < \infty$. DIY, for a hint let $E_n = (X_n \times Y_n) \cap E$. Note that E_n satisfies (a),(b),(c), and the argument from before showing an increasing countable union preserves these properties will hand us the result.

Theorem IV.4.2 (Fubini-Tonelli)

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.

(1) (Tonelli). If $f : X \times Y \to [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

(a) We have that

$$g(x) := \int_Y f(x, y)\, \mathrm{d}\nu(y) : X \to [0, \infty]$$

is a $\mathcal{A}$-measurable function.

(b) We have that

$$h(y) \coloneqq \int_X f(x,y)\,\mathrm{d}\mu(x) : Y \to [0,\infty]$$

is a $\mathcal{B}$-measurable function.

(c) We have that

$$\int_{X\times Y} f\,\mathrm{d}(\mu\times\nu) = \int_X \int_Y f(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

(2) (Fubini). If $f \in L^1(X\times Y, \mu\times\nu)$, then

(a) $f_x \in L^1(Y,\nu)$ for μ-almost every $x \in X$. Then the function

$$g(x) \coloneqq \int_Y f(x,y)\,\mathrm{d}\nu(y).$$

is defined μ-almost everywhere, and we claim $g(x) \in L^1(X,\mu)$.

(b) $f^y \in L^1(X,\mu)$ for ν-almost every $y \in Y$. Then the function

$$h(y) \coloneqq \int_X f(x,y)\,\mathrm{d}\mu(x).$$

is defined ν-almost everywhere, and we claim $h(x) \in L^1(Y,\nu)$.

(c) The iterated integral formula holds

$$\int_{X\times Y} f\,\mathrm{d}(\mu\times\nu) = \int_X \int_Y f(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

Proof. Read the textbook. Most of the work is done in Theorem IV.4.1, the rest is approximation.

Announcements

- Exam on Wednesday at Chem 1400
  - 6:00-7:50 (110 min)
  - 7:50-8:00 scan + upload to Gradescope (bring computer / phone)
- Office hour this week
  - Today 12:30-1:30, 3:05-3:50
  - Tomorrow 1:30-2:30
- No class Wednesday → office hour 11-12 instead.

IV.5. Lebesgue measure on $\mathbb{R}^d$

$\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}^d)$

Example IV.5.1

$(\mathbb{R}^2, \mathcal{L}\otimes\mathcal{L}, m\times m)$ is not complete.

Let $A \in \mathcal{L}$, $A \neq \emptyset$, $m(A) = 0$. Let $B \subseteq [0,1]$, $B \notin \mathcal{L}$ (Vitali set).

Let $E = A \times B$, $F = A \times [0,1]$. Then $E \subseteq F \in \mathcal{L}\otimes\mathcal{L}$ and $(m\times m)(F) = m(A)m([0,1]) = 0$.

If E were measurable, then every section of E would be measurable. One section is B, so E is not $\mathcal{L}\otimes\mathcal{L}$-measurable.

Definition IV.5.1 (Lebesgue Measure on $\mathbb{R}^d$)

Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the completion of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \cdots \times m)$, which is the same as the completion of $(\mathbb{R}^\lceil, \mathcal{L} \otimes \cdots \otimes \mathcal{L}, m \times \cdots \times m)$.

Recall IV.5.2

Recall from the Caratheodory extension theorem that calling a rectangle in $\mathbb{R}^d$ the product $R = \prod_{i=1}^d E_i$, $E_i \in \mathcal{B}(\mathbb{R})$, we have

$$m^d(E) = \inf\left\{\sum_{k=1}^\infty m^d(R_k) : E \subseteq \bigcup_{k=1}^\infty R_k, R_k \text{ rectangle}\right\}$$

for all $E \in \mathcal{L}^d$.

Theorem IV.5.1 (Regularity of $\mathcal{L}^d$)

Let $E \in \mathcal{L}^d$, then we have the following:

(a) We have a formula from outer/inner regularity

$$m^d(E) = \inf\{m^d(O) \mid \text{ open } O \supseteq E\} = \sup\{m^d(K) \mid \text{ compact } K \subseteq E\} \tag{(a)}$$

(b) We also have for some A_1 a F_δ set, A_2 a G_δ set, N_1, N_2 null sets, that

$$E = A_1 \cup N_1 = A_2 \setminus N_2.$$

(c) If $m^d(E) < \infty$, for all $\varepsilon > 0$ there exists $R_1, \ldots, R_m$ rectangles whose sides are intervals such that

$$m^d\left(E \triangle \left(\bigcup_{i=1}^m R_i\right)\right) < \varepsilon.$$

Theorem IV.5.2

Integrable "step functions" and $C_c(\mathbb{R}^d)$ (compactly supported continuous functions) are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$.

Theorem IV.5.3

Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.

Theorem IV.5.4 (Effect of linear transformation on Lebesgue measure)

If $T \in \mathrm{GL}(\mathbb{R}^d)$, $E \in \mathcal{L}^d$, then $T(E)$ is measurable and $m(T(E)) = |\det T| \cdot m(E)$.

Proof. See pages 71-81 of [Fol99], can skip every part except Theorem IV.5.4.

V. Differentiation on Euclidean Space

If we have $f : [a, b] \to \mathbb{R}$, there are two versions of the fundamental theorem of calculus

- We have
$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$
when f is sufficiently differentiable (later chapter).

- When f is continuous we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

The second statement is the same (by the definition of the derivative) as

$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t = f(x) = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} f(t)\,\mathrm{d}t.$$

We then see that

$$f(x) = \frac{1}{r}\int_x^{x+r} f(x)\,\mathrm{d}t.$$

Thus the above is equivalent to saying that

$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} (f(t)-f(x))\,\mathrm{d}t = 0 = \lim_{r\to 0^+}\int_{x-r}^{x} (f(t)-f(x))\,\mathrm{d}t.$$

Now if $f : \mathbb{R}^d \to \mathbb{R}$, we can instead consider

$$\lim_{r\to 0^+}\frac{1}{\mathrm{vol}(B(x,r))}\int_{B(x,r)} (f(t)-f(x))\,\mathrm{d}t \stackrel{?}{=} 0$$

where $B(x,r)$ is a ball of radius r about x.

This question led to new and interesting techniques, our reference will be [Fol99] section 3.4.

V.1. Hardy-Littlewood maximal function

For an open ball in $\mathbb{R}^d$, $B = B(a,r)$, denote $cB = B(a,cr)$ for $c > 0$.

Lemma V.1.1 (Vitali-type Covering Lemma)

Let $B_1,\ldots,B_k$ be a finite collection of open balls in $\mathbb{R}^d$.

Then there exists a subcollection $B_1',\ldots,B_m'$ of disjoint open balls such that

$$\bigcup_{j=1}^{m}(3B_j') \supseteq \bigcup_{i=1}^{k} B_i$$

Proof. Greedy algorithm, take largest ball in the collection, and then next largest ball not intersecting the first one, etc.

Definition V.1.1 (Hardy-Littlewood Maximal Function)

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ define the Hardy-Littlewood maximal function for f

$$Hf(x) := \sup\{A_r(x) \mid r > 0\}$$
$$A_r(x) := \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)|\,\mathrm{d}y.$$

Lemma V.1.2

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then

(a) $A_r(x)$ is jointly continuous for $(x,r) \in \mathbb{R}^d \times (0,\infty)$.

(b) $Hf(x)$ is Borel measurable.

Proof. WTS that if $(x_n, r_n) \to (x, r)$ then $A_{r_n}(x_n) \to A_r(x)$. Well

$$A_{r_n}(x_n) = \int |f(y)| \, 1_{B(x_n, r_n)} \to \int |f(y)| \, 1_{B(x,r)} = A_r(x)$$

because $1_{B(x_n,r_n)} \to 1_{B(x,r)}$ and we can use the dominated convergence theorem (bound r_n by R, x_n within R' of x, then $B(x_n, r_n) \subseteq B(x, R + R')$).

Now note that $(Hf)^{-1}((a, \infty)) = \bigcup_{r>0} A_r^{-1}((a, \infty))$ for any $a \in \mathbb{R}$. The right hand side is open, and so the preimage of (a, ∞) under Hf is always open, so Hf is Borel.

Recall V.1.1

Markov Lemma III.2.6,

$$m(\{x \in \mathbb{R}^d \mid |f(x)| \geq \alpha\}) \leq \frac{1}{\alpha} \int |f(x)| \, dx.$$

Theorem V.1.3 (Hardy-Littlewood maximal inequality)

There exists $C_d > 0$ depending only on d such that for all $f \in L^1(\mathbb{R}^d)$, for all $\alpha > 0$

$$m(\{x \in \mathbb{R}^d \mid Hf(x) > \alpha\}) \leq \frac{C_d}{\alpha} \int |f(x)| \, dx.$$

We show $C_d = 3^d$ suffices.

Proof. Let $f \in L^1(\mathbb{R}^d)$ and let $\alpha > 0$. let $E = \{x \mid (Hf)(x) > \alpha\}$, which is Borel measurable by the lemma.

Then if $x \in E$, then there exists an $r_x > 0$ so that $A_{r_x}(x) > \alpha$. That is

$$m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x,r_x)} |f(y)| \, dy.$$

Inner regularity gives that $m(E) = \sup\{m(K) \mid K \subseteq E, K \text{ compact}\}$. Let $K \subseteq E$ be compact. Then

$$K \subseteq \bigcup_{x \in K} B_{r_x}(x)$$

Thus $K \subseteq \bigcup_{i=1}^N B_i$. By Vitali (Lemma V.1.1) we may take $K \subseteq \bigcup_{j=1}^m (3B_j')$ where the B_j' are disjoint. Then

$$\begin{aligned} m(K) &\leq \sum_{j=1}^m m(3B_j') = 3^d \sum_{j=1}^m m(B_j') \\ &\leq \frac{3^d}{\alpha} \sum_{j=1}^m \int_{B_j'} |f(y)| \, dy \\ &\leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \, dy \end{aligned}$$

where the last line uses the disjointness. Now taking $\sup_K$ preserves the bound!

Announcements

- HW7 due Thursday
- Avg/Median of Exam will move to an A-

V.2. **Lebesgue Differentiation Theorem**

We should compare the Hardy-Littlewood inequality (Theorem V.1.3) to Markov's inequality (Lemma III.2.6). Namely there exists $C_d > 0$ (can take 3^d) such that for all $f \in L^1(\mathbb{R}^d)$, $\alpha > 0$ we have

$$m(\{x \mid (Hf)(x) > \alpha\}) \leq \frac{C_d}{\alpha} \int |f|$$
$$m(\{x \mid |f(x)| > \alpha\}) \leq \frac{1}{\alpha} \int |f|$$

Theorem V.2.1

Let $f \in L^1$. Then

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0$$

for almost every x.

Proof. The result holds for $f \in C_c(\mathbb{R}^d)$, continuous with compact support (check). Why? Well then for any $\varepsilon > 0$ if r is small $|f(y) - f(x)| < \varepsilon$, so then the quantity

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y < \varepsilon.$$

Great!

Now let $f \in L^1(\mathbb{R}^d)$. Fix $\varepsilon > 0$. By density there exists $g \in C_c(\mathbb{R}^d)$ with $\|f - g\|_1 < \varepsilon$. We have

$$\int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y \leq \int_{B(x,r)} |f(y) - g(y)| \, \mathrm{d}y + \int_{B(x,r)} |g(y) - g(x)| \, \mathrm{d}y + \int_{B(x,r)} |g(x) - f(x)| \, \mathrm{d}y$$

Dividing all of these by $m(B(x,r))$, and taking $\limsup$ as $r \to 0$, we need to understand the error terms

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - g(x)| \, \mathrm{d}y = |g(x) - f(x)|$$
$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g(y)| \, \mathrm{d}y \leq (H(f-g))(x).$$

Define

$$Q(x) = \limsup_{r \to 0} \frac{1}{m(B(x,r))} \int B(x,r) \, |f(y) - f(x)| \, \mathrm{d}y.$$

We want to show $m(\{x \mid Q(x) > 0\}) = 0$. Let $E_\alpha = \{x \mid Q(x) > \alpha\}$. It is enough to show $m(E_\alpha) = 0$ for all $\alpha > 0$, because $\{x \mid Q(x) > 0\} = \bigcup_n E_{1/n}$. We know by the above that

$$Q(x) \leq (H(f-g))(x) + 0 + |g(x) - f(x)|.$$

Therefore

$$E_\alpha \subseteq \{x \mid (H(f-g))(x) > \alpha/2\} \cup \{x \mid |g(x) - f(x)| > \alpha/2\}.$$

By the Hardy-Littlewood maximal inequality and Markov

$$m(\{x \mid (H(f-g))(x) > \alpha/2\}) \leq \frac{2C_d}{\alpha} \int |f - g|$$
$$m(\{x \mid |g(x) - f(x)| > \alpha/2\}) \leq \frac{2}{\alpha} \int |f - g|$$

Thus

$$0 \leq m(E_\alpha) \leq \frac{2C_d}{\alpha}\|f-g\|_1 + \frac{2}{\alpha}\|f-g\|_1 \leq \frac{2(C_d+1)}{\alpha}\varepsilon$$

Taking $\varepsilon \to 0$, $m(E_\alpha)$ does not depend on ε, g so $m(E_\alpha) = 0$.

Corollary V.2.2

This also holds for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$

Proof. DIY, partition $\mathbb{R}^d$ into countably many compact sets K_i then apply the theorem to $f1_{K_i}$ for each i.

Corollary V.2.3

For $f \in L^1_{\text{loc}}$ for almost every x, we have

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,\mathrm{d}y = f(x)$$

Proof. DIY, use that $f(x) = \frac{1}{m(B(x,r))}\int_{B(x,r)} f(x)\,\mathrm{d}y$ and the triangle inequality.

Definition V.2.1

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The point $x \in \mathbb{R}^d$ is called a Lebesgue point of f if

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\,\mathrm{d}y = 0$$

Corollary V.2.2 tells us that almost all points in $\mathbb{R}^d$ are Lebesgue points for f.

Definition V.2.2

We say measurable sets $\{E_r\}_{r>0}$ shrink nicely to x as $r \to 0$ if and only if $E_r \subseteq B(x,r)$ and there exists $c > 0$ such that $c \cdot m(B(x,r) \leq m(E_r)$.

Corollary V.2.4 (Lebesgue Differentiation Theorem)

Suppose E_r shrink nicely to 0, $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, x a Lebesgue point of f. Then

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r+x} |f(y) - f(x)|\,\mathrm{d}y = 0$$
$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r+x} f(y)\,\mathrm{d}y = f(x)$$

Corollary V.2.5

If $f \in L^1_{\text{loc}}(\mathbb{R})$ then $F(x) = \int_0^x f(y)\,\mathrm{d}y$ is differentiable and $F'(x) = f(x)$ almost everywhere.

The rest of Chapter 3 of [Fol99] we will cover later (in 2-3 weeks).

VI. Normed Vector Spaces

Folland sections 5.1,6.1,6.2 [Fol99].

VI.1. **Metric Spaces and Normed Spaces**

Definition VI.1.1

Let Y be a set, a function $\rho : Y \times Y \to [0, \infty)$ is a metric on Y provided that

(1) $\rho(x, y) = \rho(y, x)$

(2) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

(3) $\rho(x, y) = 0$ if and only if $x = y$.

The following make sense in a metric space

- Open/closed balls.
- Open/closed sets.
- Convergence sequences ($x_n \to x$ with respect to ρ if and only if $\lim_{x\to\infty} \rho(x_n, x) = 0$).
- Continuous functions.

Example VI.1.1

We have the following metric spaces

(1) $\mathbb{Q}$, $\rho(x, y) = |x - y|$.

(2) $\mathbb{R}$, $\rho(x, y) = |x - y|$.

(3) $\mathbb{R}_+, \rho(x, y), |\ln(y/x)|$.

(4) $\mathbb{R}^d$, with

$$d_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p}$$
$$d_\infty(x, y) = \max_{1\leq i\leq d} |x_i - y_i|\,.$$

These all give the same open sets (topologically equivalent)

(5) $C([0, 1])$, with

$$d_p(f, g) = \left(\int_0^1 |f - g|^p\right)^{1/p}$$
$$d_\infty(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$$

(6) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Let Y be the set of measurable functions on X. Then

$$\rho(f, g) = \int \min(|f(x) - g(x)|\,, 1)\,\mathrm{d}\mu(x)$$

is a metric and $f_n \to f$ in ρ if and only if $f_n \to f$ in measure.

Let V be a vector space over scalar field $K = \mathbb{R}$ or $K = \mathbb{C}$

Definition VI.1.2

A seminorm on V is a map $\|\cdot\| : V \to [0, \infty)$ satisfying

(1) $\|v + w\| \leq \|v\| + \|w\|$

(2) $\|cv\| = |c|\,\|v\|$

A norm additionally satisfies

(3) $\|v\| = 0 \iff v = 0$

A norm gives a metric $\rho(v, w) = \|v - w\|$, and we have

$$v_n \to v \iff \lim_{n\to\infty} \|v_n - v\| = 0$$

Example VI.1.2

We have the following examples

(1) $L^1(X, \mathcal{A}, \mu)$ with $\|f\|_1 = \int |f| \, d\mu$

(2) $C([0, 1])$ with $\|f\|_1 = \int_0^1 |f(x)| \, dx$, $\|f\|_\infty = \max_{0\le x\le 1} |f(x)|$.

(3) For $\mathbb{R}^d$ we have for $0 < p < \infty$

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$$
$$\|x\|_\infty = \max_{1\le i\le d} |x_i|$$

VI.2. L^p spaces

Definition VI.2.1 (L^p spaces)

For $(X, \mathcal{A}, \mu)$ a measure space, f a measurable function. For $0 < p < \infty$ define

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$$
$$L^p(X, \mathcal{A}, \mu) = \{f \mid \|f\|_p < \infty\}.$$

This is a seminorm, and a norm if we identify functions which are equal almost everywhere.

Example VI.2.1

$(\mathbb{R}, \mathcal{L}, m)$ has $f(x) = x^{-\alpha} 1_{(1,\infty)}(x) \in L^p$ if and only if $\alpha p > 1$.

In contrast, $g(X) = x^{-\beta} 1_{(0,1)}(x) \in L^p$ if and only if $\beta p < 1$.

Definition VI.2.2 (ℓ^p spaces)

If $(X, \mathcal{P}(X), \nu)$ is the counting measure, then define

$$\ell^p(X) := L^p(X, \mathcal{P}(X), \nu).$$

Of particular interest is $\ell^p(\mathbb{N})$. Here we have

$$\ell^p := \ell^p(\mathbb{N}) = \left\{ a = (a_1, a_2, \dots) \mid \|a\|_p = \left(\sum_{i=1}^\infty |a_i|^p\right)^{1/p} < \infty \right\}$$

Lemma VI.2.1

$L^p(X, \mathcal{A}, \mu)$ is a vector space, for all $p \in (0, \infty)$.

Proof. Note that

$$\|cf\|_p = \left(\int |cf|^p \, d\mu\right)^{1/p} = |c| \, \|f\|_p < \infty \iff \|f\|_p < \infty.$$

Note that for any real numbers α, β we have

$$(\alpha+\beta)^p \leq (2 \cdot \max(|\alpha|, |\beta|))^p = 2^p \cdot \max(|\alpha|^p, |\beta|^p) \leq 2^p(|\alpha|^p + |\beta|^p).$$

Therefore for f, g we have

$$\|f+g\|_p < \infty \iff \|f+g\|_p^p = \int |f+g|^p \, \mathrm{d}\mu \leq 2^p \int (|f|^p + |g|^p) < \infty \impliedby \|f\|_p, \|g\|_p < \infty.$$

But this is not quite satisfactory, as it does not give us the triangle inequality

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

For this we need a new result

Theorem VI.2.2 (Hölder's inequality)

Let $1 < p < \infty$, and let $q = p/(p-1)$ so that $1/p + 1/q = 1$.

Then we have that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Example VI.2.2

For $p = q = 2$, $X = \{1, \ldots, d\}$, μ counting measure, then for $x, y \in \mathbb{R}^d$

$$\sum_{i=1}^d |x_i y_i| \leq \sqrt{\sum_{i=1}^d x_i^2} \sqrt{\sum_{i=1}^d y_i^2}.$$

Proof. We do this in steps

(1) Note that

$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p} = \frac{t^p}{p} + \frac{1}{q}$$

for all $t \geq 0$, by taking $F(t) = t - t^p/p$, $t \geq 0$, and using calculus to find the maximum.

(2) Note that

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

This follows by taking $t = \alpha/\beta^{q-1}$. This inequality is known as Young's Inequality.

(3) WLOG, assume $0 < \|f\|_p, \|g\|_q < \infty$. Now consider $F(x) = f(x)/\|f\|_p, G(x) = g(x)/\|g\|_q$. We know that $\|F\|_p = 1 = \|G\|_q$. Then by Young's Inequality

$$\int |F(x)G(x)| \, \mathrm{d}\mu \leq \int \frac{|F(x)|^p}{p} + \frac{|G(x)|^q}{q}$$
$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

Theorem VI.2.3 (Minkowski's Inequality)

Let $1 \le p < \infty$. For $f, g \in L^p$, $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Proof. For $p = 1$¡ it's easy (just triangle inequality). Now assume $1 < p < \infty$ and WLOG assume $\|f+g\| \ne 0$. Then

$$\begin{aligned}
\int |f(x)+g(x)|^p &\le \int |f(x)+g(x)|^{p-1}(|f(x)|+|g(x)|) \\
&\le \left(\int |f+g|^{(p-1)q}\right)^{1/q}\left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right] \\
&\le \left(\int |f+g|^p\right)^{1/q}[\|f\|_p + \|g\|_p] \\
(|f(x)+g(x)|^p)^{1-1/q} &\le \|f\|_p + \|g\|_p \\
(|f(x)+g(x)|^p)^{1/p} &\le \|f\|_p + \|g\|_p \\
\|f+g\|_p &\le \|f\|_p + \|g\|_p
\end{aligned}$$

To summarize last class

- $\|f\|_p = \left(\int |f|^p\right)^{1/p}$ is a norm if $1 \le p < \infty$.
- Hölder's Inequality (Theorem VI.2.2) says that $\|fg\|_1 \le \|f\|_p\|g\|_q$ for $1/p + 1/q = 1$. That is
$$\int |fg| \le \left(\int |f|^p\right)^{1/p}\left(\int |g|^q\right)^{1/q}$$
- Minkowski's Inequality (Theorem VI.2.3) says that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ for $1 \le p < \infty$.

Definition VI.2.3

For a measurable function f on $(X, \mathcal{A}, \mu)$ we define

$$S = \{\alpha \ge 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\} = \{\alpha \ge 0 \mid |f(x)| \le \alpha \text{ almost everywhere}\}$$

Define the essential supremum of f to be $\|f\|_\infty = \inf S$ if $S \ne 0$ and $\|f\|_\infty = \infty$ if $S = \emptyset$.

Let $L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}$, and $\ell^\infty = L^\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ where ν is the counting measure.

Example VI.2.3

Consider $(\mathbb{R}, \mathcal{L}, m)$. Then

$$f(x) = \frac{1}{x}1_{(0,\infty)}(x) \notin L^\infty$$
$$g(x) = x1_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty.$$

If f is continuous on $(\mathbb{R}, \mathcal{L}, m)$ then $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$.

For $a \in \ell^\infty$ we have $\|a\|_\infty = \sup_{i\in\mathbb{N}} |a_i|$, and sequences in ℓ^∞ are exactly the bounded sequences.

Lemma VI.2.4

We have the following

(1) Suppose $f \in L^\infty(X, \mathcal{A}, \mu)$. For $\alpha \geq \|f\|_\infty$, we have $\mu(\{x \mid |f(x)| > \alpha\}) = 0$.
For $\alpha < \|f\|_\infty$ we have $\mu(\{x \mid |f(x)| > \alpha\}) > 0$.
(2) In particular, $|f(x)| \leq \|f\|_\infty$ almost everywhere.
(3) $f \in L^\infty$ if and only if there exists a bounded measurable function g such that $f = g$ almost everywhere.

Proof. DIY.

Theorem VI.2.5

We have that

(1) $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$ (motivation: $1/1 + 1/\infty = 1$).
(2) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
(3) $f_n \to f$ in L^∞ if and only if $f_n \to f$ uniformly almost everywhere (i.e., there is a null set A such that $f_n \to f$ uniformly on A^c).

Proof. DIY. We'll do (3) $\implies$

Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$. Let $A = \bigcup_n A_n$, we see that $\mu(A) = 0$.

For $x \in A^c$ and for every n, we have $|f_n(x) - f(x)| \leq \|f_n - f\|_\infty$. Given $\varepsilon > 0$, there is an N so that $\|f_n - f\|_\infty < \varepsilon$ for all $n \geq N$. But then for all $x \in A^c$, $|f_n(x) - f(x)| < \varepsilon$ as well.

Great! This proves the claim.

Proposition VI.2.6

We have that

(1) For $1 \leq p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.
(2) For $1 \leq p < \infty$, the collection of step functions with finite measure support is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$. So is $C_c(\mathbb{R})$.
(3) For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.
Note: $C_c(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

Proof. DIY.

VI.3. Embedding Properties of L^pspaces

Definition VI.3.1

Two norms $\|\cdot\|, \|\cdot\|'$ on V are equivalent if there exists $c_1, c_2 > 0$ such that

$$c_1 \|v\| \leq \|v\|' \leq c_2 \|v\|$$

for all $v \in V$. Note that these norms give the same topological properties (open sets, closed sets, convergence, etc.)

Note that this is an equivalence relation on norms.

Example VI.3.1

For $\mathbb{R}^d$, we have the norms $\|\cdot\|_p$ for $1 \le p \le \infty$. All of these are equivalent. We see that for $1 \le p < \infty$

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} \le (d\|x\|_\infty^p)^{1/p} = d^{1/p}\|x\|_\infty.$$

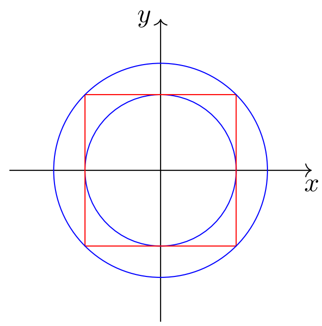
And also

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} \ge (\|x\|_\infty^p)^{1/p} = \|x\|_\infty.$$

Thus $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\infty$ for every $1 \le p < \infty$, transitivity gives that they are all equivalent.

Another way of thinking of this, by assuming $v \neq 0$ and scaling by some t, we may assume v lies on the unit circle in one of the norms. Then we are squeezing a unit circle in $\|\cdot\|'$ between two circles of radius c_1, c_2 in $\|\cdot\|$.

In a picture we have to show that $\|\cdot\|_2, \|\cdot\|_\infty$ are equivalent, we have



Since $\|\cdot\|_\infty$ circles are squares.

Example VI.3.2

For $1 \le p, q \le \infty$, We have $L^p(\mathbb{R}, m)$-norm and $L^q(\mathbb{R}, m)$-norm are not equivalent, even worse, we have that

$$L^p(\mathbb{R}, m) \nsubseteq L^1(\mathbb{R}, m)$$
$$L^p(\mathbb{R}, m) \nsupseteq L^1(\mathbb{R}, m)$$

Proposition VI.3.1

Suppose $\mu(X) < \infty$, then for every $0 < p < q \le \infty$, $L^q \subseteq L^p$.

Proof. You should check the $q = \infty$ case.

Suppose $q < \infty$. We see that

$$\int |f|^p = \int |f|^p \cdot 1 \le \left(\int (|f|^p)^{q/p}\right)^{p/q} \left(\int 1^{q/(q-p)}\right)^{1-p/q}$$
$$= \left(\int |f|^q\right)^{p/q} \mu(X)^{1-p/q} < \infty.$$

Using Hölder's inequality with $q/p > 1$. Thus

$$\|f\|_p \leq \|f\|_q \mu(X)^{1/p-1/q} < \infty.$$

Proposition VI.3.2

If $0 < p < q \leq \infty$ then $\ell^p \subseteq \ell^q$.

Proof. When $q = \infty$ we have

$$\|a\|_\infty^p = \left(\sup_i |a_i|\right)^p = \sup_i |a_i|^p \leq \sum_{i=1}^\infty |a_i|^p .$$

Thus $\|a\|_\infty \leq \|a\|_p$.

When $q < \infty$, we see that

$$\begin{aligned}\sum_{i=1}^\infty |a_i|^q &= \sum_i |a_i|^p \cdot |a_i|^{q-p} \\ &\leq \|a\|_\infty^{q-p} \sum_i |a_i|^p \\ j &\leq \|a\|_p^{q-p} \|a\|_p^p = \|a\|_p^q.\end{aligned}$$

Therefore

$$\|a\|_q \leq \|a\|_p.$$

Proposition VI.3.3

For all $0 < p < q < r \leq \infty$ we have $L^p \cap L^r \subseteq L^q$.

Proof. DIY.

VI.4. **Banach Spaces**

Definition VI.4.1

Let (Y, ρ) be a metric space. We call x_n a Cauchy sequence provided that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that for $n, m \geq N$ we have $\rho(x_n, x_m) < \varepsilon$.

Easy check: convergent sequences are Cauchy.

Definition VI.4.2

A metric space (Y, ρ) is called complete if every Cauchy sequence in Y converges.

Example VI.4.1

$\mathbb{Q}$ with $|x - y|$ is not complete, but $\mathbb{R}$ with the same metric is complete.

$C([0, 1])$, with $\rho(f, g) = \|f - g\|_\infty$ is complete, but with $\rho(f, g) = \int |f - g|$ it is not complete.

Definition VI.4.3 (Banach Space)

A Banach Space is a complete normed vector space (i.e, a vector space equipped with a norm whose metric induced by the norm is complete).

Theorem VI.4.1

Let $(V, \|\cdot\|)$ be a normed space. Then,

$$V \text{ is complete} \iff \text{every absolutely convergent series is convergent}$$

i.e., if $\sum_{i=1}^{\infty} \|v_i\| < \infty$ then $\{\sum_{i=1}^{N} v_i\}_{N\in\mathbb{N}}$ converges to some $s \in V$.

Theorem VI.4.2 (Riesz-Fisher)

For every $1 \le p \le \infty, L^p(X, \mathcal{A}, \mu)$ is complete (hence a Banach space).

Proof. Lets go in pieces

(1) We handle the case where $1 \le p < \infty$ first. Suppose $f_n \in L^p$ and $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

We need to show that there is an $F \in L^p$ such that $\|\sum_{n=1}^{N} f_n - F\|_p \to 0$ as $N \to \infty$. We must show that

(i) $\sum_{n=1}^{\infty} f_n(x)$ is convergent almost everywhere. In fact we can show $\int \sum_{n=1}^{\infty} |f_n(x)| < \infty$.

(ii) $F \in L^p$, where $F(x) := \sum_{n=1}^{\infty} f_n(x)$ almost everywhere and say is zero elsewhere.

(iii) $\|\sum_{n=1}^{N} f_n - F\|_p \to 0$ as $N \to \infty$.

Lets go!

(i) Let $G(x) = \sum_{n=1}^{\infty} |f_n(x)| = \sup_N \sum_{n=1}^{N} |f_n(x)|$, $G : X \to [0, \infty]$.

Let $G_N(x) = \sum_{n=1}^{N} |f_n(x)|$. Then $0 \le G_1 \le G_2 \le \cdots \le G$, $G_N \to G$. Furthermore $0 \le G_1^p \le G_2^p \le \cdots \le G^p$, $G_N^p \to G^p$.

Therefore by the Monotone Convergence Theorem (Theorem III.2.2)

$$\int G^p = \lim_{N\to\infty} \int G_N^p.$$

Now, by Minkowski

$$\|G_N\|_p \le \sum_{n=1}^{N} \|f_n\|_p \le \sum_{n=1}^{\infty} \|f_n\|_p := B < \infty.$$

Thus

$$\int G(x)^p = \lim_{N\to\infty} \int G_N^p = \lim_{N\to\infty} \|G_N\|_p^p \le B^p < \infty.$$

Therefore G is finite almost everywhere as desired. This implies that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ almost everywhere so $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere.

Let $F(x) := \sum_{n=1}^{\infty} f_n(x)$ if it converges, and otherwise $F(x) = 0$.

(ii) Now we see that

$$|F(x)| \le G(x)$$
$$\int |F|^p \le \int G^p < \infty.$$

So $F \in L^p$.

(iii) Now we see that

$$\left|\sum_{n=1}^{N} f_n(x) - F(x)\right|^p \leq \left(\sum_{n=1}^{\infty} |f_n(x)| + |F(x)|\right)^p \leq (2G(x))^p.$$

Well $2G \in L^p$, so $2G^p \in L^1$. Thus by the Dominated Convergence Theorem

$$\lim_{N\to\infty} \int \left|\sum n = 1^N f_n(x) - F(x)\right|^p \mathrm{d}x = 0.$$

And thus $\|\sum_{n=1}^{N} f_n - F\|_p \to 0$ as $N \to \infty$.

We now prove Theorem VI.4.1, completing the proof of Theorem VI.4.2 (which relies on this result).

Proof. Lets go!

($\Longrightarrow$) Suppose V is complete, and fix an absolutely convergent series $\sum_n v_n$. Define $s_N = \sum_{n=1}^{N} v_n$. It suffices to show the partial sums are a Cauchy Sequence.

Fix $\varepsilon > 0$, then because $\sum_{n=1}^{\infty} \|v_n\| < \infty$, there is an $K \in \mathbb{N}$ so that

$$\sum_{n=K}^{\infty} \|v_n\| < \varepsilon.$$

Now let $M > N > K$, we see that

$$\begin{aligned}\|s_M - s_N\| = \left\|\sum_{n=N+1}^{M} v_n\right\| &\leq \sum_{n=N+1}^{M} \|v_n\| \\ &\leq \sum_{n=N}^{\infty} \|v_n\| < \varepsilon.\end{aligned}$$

So this is Cauchy.

($\Longleftarrow$) Now suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. For all $j \in \mathbb{N}$, there exists an $N_j \in \mathbb{N}$ such that

$$\|v_n - v_m\| < \frac{1}{2^j}$$

for all $n, m \geq N_j$. WLOG, may assume $N_1 < N_2 < \cdots$.

Let $w_1 = v_{N_1}$, $w_j = v_{N_j} - v_{N_j - 1}$ for $j \geq 2$. Therefore

$$\sum_{j=1}^{\infty} \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} < \infty.$$

Thus $\sum_{j=1}^{k} w_j \to s \in V$ as $k \to \infty$. But by telescoping

$$v_{N_k} = \sum_{j=1}^{k} w_j \to s.$$

Now we claim that since v_n is Cauchy that $v_n \to s$.

Explicitly, take $\varepsilon > 0$, and let k be large enough so that $\|v_{N_k} - s\| < \varepsilon$ and $1/2^k < \varepsilon$. Then if $n > N_k$ then

$$\|v_n - s\| \leq \|v_n - v_{N_k}\| + \|v_{N_k} - s\| < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $v_n \to s$.

VI.5. Bounded Linear Transformations (BLTs)

Definition VI.5.1

Let $(V, \|\cdot\|), (W, \|\cdot\|')$ be normed spaces. A linear map $T : V \to W$ is called a bounded map if there exists $c \geq 0$ such that

$$\|Tv\|' \leq c\|v\|$$

for all $v \in V$.

Proposition VI.5.1

Suppose $T : (V, \|\cdot\|) \to (W, \|\cdot\|')$ is a linear map. Then the following are equivalent

(a) T is continuous

(b) T is continuous at 0

(c) T is a bounded map

Proof. (a) $\implies$ (b) is clear. For (b) $\implies$ (c) take $\varepsilon = 1$, then there exists a $\delta > 0$ such that $\|Tu\|' < 1$ if $\|u\| < \delta$.

Now take an arbitrary $v \in V$, $v \neq 0$. Let $u = \frac{\delta}{2\|v\|}v$. Then $\|u\| < \delta$. Therefore

$$\begin{aligned} \|Tu\|' &< 1 \\ \frac{\delta}{2\|v\|}\|Tv\|' &< 1 \\ \|Tv\|' &< \frac{2}{\delta}\|v\|. \end{aligned}$$

Then $2/\delta$ is our constant.

For (c) $\implies$ (a). Fix $v_0 \in V$. Then for some constant c

$$\|Tv - Tv_0\|' = \|T(v - v_0)\|' \leq c\|v - v_0\|.$$

Thus T is continuous, as when $v \to v_0$ the right hand side goes to zero, and so $Tv \to Tv_0$.

Example VI.5.1

Example time!

- We can look at

$$\begin{aligned} T &: \ell^1 \to \ell^1 \\ (a_1, a_2, \ldots) &\mapsto (a_2, a_3, \ldots). \end{aligned}$$

Then clearly $\|Ta\|_1 \leq \|a\|_1$, so T is a BLT.

- We can also look at $S : (C([-1,1]), \|\cdot\|_1) \to \mathbb{C}$, where $Sf = f(0)$. S is not a BLT, because we can make

$$\|Tf\| = |f(0)| = n$$
$$\|f\|_1 = 1$$

for every $n \in \mathbb{N}$ (take f's graph to be a skinny triangle shooting up to n at 0).

- But $U : (C([-1,1]), \|\cdot\|_\infty) \to \mathbb{C}$ defined by $Uf = f(0)$ is a BLT, because $|f(0)| \leq \|f\|_\infty$.
- Let A be an $n \times m$ matrix. Then $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $v \mapsto Av$ is a BLT.

  Explicitly this is

$$(Tv)_i = (Av)_i = \sum_{j=1}^{m} A_{ij} v_j$$

- Let $K(x,y)$ be a continuous function on $[0,1] \times [0,1]$. We'll define

$$T : (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$$

by

$$(Tf)(x) = \int_0^1 K(x,y) f(y) \, \mathrm{d}y.$$

This is an analogue of matrix multiplication (K is like a continuous matrix). This is a BLT.

- Lets look at $T : L^1(\mathbb{R}) \to (C(\mathbb{R}), \|\cdot\|_\infty)$ defined by

$$(Tf)(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, \mathrm{d}x$$

that is the Fourier transform of f.

- $T : (C^\infty[0,1], \|\cdot\|_\infty) \to (C^\infty[0,1], \|\cdot\|_\infty)$. Define

$$(Tf)(x) = f'(x).$$

This is not a BLT. In contrast S, defined on the same spaces

$$(Sf)(x) = \int_0^x f(t) \, \mathrm{d}t$$

is bounded.

Definition VI.5.2

Let $L(V,W) = \{T : V \to W \mid T \text{ is a BLT}\}$, which is a vector space. For $T \in L(V,W)$, the operator norm of T is

$$\begin{aligned}
\|T\| &:= \inf\{c \geq 0 \mid \|Tv\| \leq c\|v\| \text{ for all } v \in V\} \\
&= \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \neq 0, v \in V\right\} \\
&= \sup\{\|Tv\| \mid \|v\| = 1, v \in V\}
\end{aligned}$$

check the equalities above.

Lemma VI.5.2

We have that

(a) The three definitions of $\|T\|$ above are all equal.

(b) $(L(V,W), \|\cdot\|)$ is indeed a normed space.

Proof. DIY.

Note that $\|Tv\|'' \le \|T\|\|v\|'$ for $T : (V, \|\cdot\|') \to (W, \|\cdot\|'')$

Theorem VI.5.3

If W is complete, then $L(V,W)$ is complete.

Proof. Suppose T_n is a Cauchy sequence in $L(V,W)$. Fix $v \in V$. Then, let $w_n = T_n v \in W$. Also

$$\|w_n - w_m\| = \|T_n v - T_m v\| = \|(T_n - T_m)v\| \le \|T_n - T_m\|\|v\|.$$

Thus w_n is Cauchy, so it converges since W is complete. We call its unique limit Tv. This makes $T : V \to W$ into a function. We must show it is a BLT and that $\|T_n - T\| \to 0$.

Finish the proof! See book or DIY

VI.6. Dual of L^p spaces

Example VI.6.1

Let $w \in \mathbb{R}^d$. Then we can consider

$$\max\{v \cdot w \mid \|v\|_2 = 1\} = \|w\|_2.$$

If $w \in \mathbb{C}^d$, this is similar we just do

$$\max\{|v \cdot w| \mid \|v\|_2 = 1\} = \|w\|_2.$$

These maxes are achieved by $v = \frac{\overline{w}}{\|w\|_w}$ if $w \ne 0$.

Proposition VI.6.1

Let $1/p + 1/q = 1$ with $1 \le q < \infty$. For every $g \in L^q$,

$$\|g\|_q = \sup\left\{\left|\int fg\right| \mid \|f\|_p = 1\right\}.$$

Suppose μ is σ-finite. Then the result also holds for $q = \infty$, $p = 1$.

Recall VI.6.2

For $\alpha \in \mathbb{C}$, $\operatorname{sgn} \alpha \coloneqq e^{i\theta}$ where $\alpha = |\alpha|\, e^{i\theta}$.

Proof. By Hölder's inequality we know that

$$\left|\int fg\right| \le \int |fg| = \|fg\|_1 \le \|f\|_p \|g\|_q = \|g\|_q.$$

Thus the supremum is $\le \|g\|_q$.

(1) Let

$$f(x) = \frac{\left|g(x)\overline{\mathrm{sgn}(g(x))}\right|^{q-1}}{\|g\|_q^{q-1}}$$

Then $\int |f|^p = 1$, and $\int fg = \|g\|_q$. Check this!

(2) DIY for handling the case when μ is σ-finite and $q = \infty, p = 1$.

Remark VI.6.1

One could use the above to prove Minkowski's inequality (as it only uses Hölder)

Definition VI.6.1

For a normed spave $(V, \|\cdot\|)$ its dual space is $V^* = L(V, \mathbb{R})$ or $V^* = L(V, \mathbb{C})$ (aka BLTs with codomain the scalar field).

$\ell \in V^*$ is called a linear functional on V. This means exactly that

- $\ell : V \to \mathbb{R}$ (or $\mathbb{C}$)
- ℓ linear
- There exists a $c \geq 0$ such that $|\ell(v)| = c\|v\|$.

Note: V^* is always a Banach space (even if V is not complete).

Corollary VI.6.2

We have the following:

(1) Let $1/p + 1/q = 1, 1 \leq q < \infty$. For $g \in L^q$ define $\ell_g \in L^p \to \mathbb{C}$ by

$$\ell_g(f) = \int fg.$$

Then $\ell_g \in (L^p)^*$. Furthermore, $\|\ell_g\| = \|g\|_q$.

(2) If μ is σ-finite then this also holds for $q = \infty, p = 1$.

Proof. ℓ_g is clearly linear in f because the integral is linear. Then Proposition VI.6.1 gives in both (1) and (2) that

$$\|g\|_q = \sup\{|\ell_g(f)| \mid \|f\|_p = 1\} = \|\ell_g\|$$

and so ℓ_g is a BLT with the desired properties.

Theorem VI.6.3

We have the following

(1) Let $1/p + 1/q = 1$, $1 \leq q < \infty$. The map $T : L^q \to (L^p)^*$ given by $Tg = \ell_g$ is an isometric linear isomorphism. (isometric meaning Tg has the same norm as g).

This means that

- T is a BLT.
- T is bijective.
- T is norm-preserving.

(2) If μ is σ-finite then this also holds for $q = \infty, p = 1$.

Even if μ is σ-finite we might not have $L^1 \cong (L^\infty)^*$.

Also note that $L^2 \cong (L^2)^*$. Also for all $1 < p < \infty$ we have $(L^p)^{**} \cong L^p$.

Proof. We have already proved this is isometric in Corollary VI.6.2, it is clearly linear, and isometry implies injectivity.

We will prove that it is surjective later. See

Prove surjectivity

VII. Signed and Complex Measures

See [Fol99] Chapter 3.

Recall VII.0.1

Suppose $f : X \to [0, \infty]$ is a measurable function on $(X, \mathcal{A}, \mu)$.

We can define $\nu(E) = \int_E f \, d\mu$ for $E \in \mathcal{A}$, and ν is a measure on $(X, \mathcal{A})$.

This gives a map from the set of non-negative measurable functions on X to measures on X. This is injective if we identify functions which are equal almost everywhere. But it is not necessarily surjective. We can then think of measures as a generalization of functions.

For an example, think of a dirac delta measure on $\mathbb{R}$. This is not the Lebesgue integral of any non-negative measurable function.

What if instead we took $f : X \to \mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$. We could take the same construction to get $\nu(E) = \int_E f \, d\mu$, but this is no longer a measure as it can take $\mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$ values.

VII.1. Signed Measures

Definition VII.1.1

Let $(X, \mathcal{A})$ be a measurable space. A signed measure is $\nu : \mathcal{A} \to [-\infty, \infty)$ or $\nu : \mathcal{A} \to (-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$.
- If $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint then

$$\nu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \nu(A_i)$$

  where the series on the RHS converges absolutely if $\nu\left(\bigcup_{i=1}^\infty\right) \in (-\infty, \infty)$. This means the series does not depend on re-arrangment.

Example VII.1.1

Consider

(a) ν is a positive measure, then ν is a signed measure.

(b) If we have positive measures μ_1, μ_2 such that either $\mu_1(X) < \infty$ or $\mu_2(X) < \infty$, then $\nu = \mu_1 - \mu_2$ is a signed measure.

(c) If $f : X \to \overline{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ such that $\int_X f^+ \,\mathrm{d}\mu < \infty$ or $\int_X f^- \,\mathrm{d}\mu < \infty$, we can define

$$\nu(E) = \int_E f \,\mathrm{d}\mu$$

and this will be a signed measure.

Note: The following weird things happen with signed measures

(1) $A \subseteq B$ does not imply $\nu(A) \le \nu(B)$, as $\nu(B) = \nu(A) + \nu(B \setminus A)$, and $\nu(B \setminus A)$ may be negative.

(2) If $A \subseteq B$ and $\nu(A) = \infty$, then $\nu(B) = \infty$, because $\nu(B \setminus A) \in (-\infty, \infty]$.

(3) Similarly if $A \subseteq B$ and $\nu(A) = -\infty$ then $\nu(B) = -\infty$.

Lemma VII.1.1

If ν is a signed measure on $(X, \mathcal{A})$, then:

(1) If $E_n \in \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \cdots$ then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{N\to\infty} \nu(E_N).$$

We call this continuity from below.

(2) If $E_n \in \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \cdots$, and $-\infty < \nu(E_1) < \infty$ then

$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{N\to\infty} \nu(E_N).$$

We call this continuity from above.

Proof. DIY, or read [Fol99].

Definition VII.1.2

Let ν be a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$. We say that

(1) E is positive for ν if for all $F \subseteq E$, $\nu(F) \ge 0$.

(2) E is negative for ν if for all $F \subseteq E$, $\nu(F) \le 0$.

(3) E is null for ν if for all $F \subseteq E$, $\nu(F) = 0$.

Note:

(1) If E is a positive set, $F \subseteq E$, then $\nu(F) \le \nu(E)$.

(2) If E is a negative set, $F \subseteq E$, then $\nu(F) \ge \nu(E)$.

Lemma VII.1.2

Let ν be a signed measure on $(X, \mathcal{A})$. Then

(1) If E is positive, $G \subseteq E$ is measurable, then G is positive.

(2) If E is negative, $G \subseteq E$ is measurable, then G is negative.

(3) If E is null, $G \subseteq E$ is measurable, then G is positive.

(4) $E_1, E_2, \ldots$ positive sets then $\bigcup_{i=1}^{\infty} E_n$ positive

Proof. DIY.

Lemma VII.1.3

Suppose that ν is a signed measure with $\nu : \mathcal{A} \to [-\infty, \infty)$. Suppose $E \in \mathcal{A}$ and $0 < \nu(E) < \infty$. Then there exists a measurable $A \subseteq E$ such A is a positive set and $\nu(A) > 0$.

Assuming this lemma we prove

Theorem VII.1.4 (Hahn Decomposition)

If ν is a signed measure on $(X, \mathcal{A})$, then there exist $P, N \in \mathcal{A}$ such that

$$P \cap N = \emptyset$$
$$P \cup N = X.$$

P is positive for ν, N is negative for ν.

If P', N' are another such pair, then $P \triangle P' = N \triangle N'$ is null for ν.

Proof of Uniqueness. We see that $P \setminus P' \subseteq P, P \setminus P' \subseteq N'$. Thus $P \setminus P' \subseteq P \cap N'$ is both positive and negative, hence $P \setminus P'$ is null.

Similarly for $P' \setminus P$, and then their union $P \triangle P'$ is null as well.

Proof of Existence. Without loss of generality suppose $\nu : \mathcal{A} \to [-\infty, \infty)$. If not, consider $-\nu$.

Let

$$s := \sup\{\nu(E) \mid E \in \mathcal{A} \text{ is a positive set}\}$$

which is a nonempty supremum because $\emptyset$ is positive. Then there exist $P_1, P_2, \ldots$ positive sets such that $\lim_{n\to\infty} \nu(P_n) = S$.

Then we have that $P = \bigcup_n P_n$ is positive by Lemma VII.1.2. Then $\nu(P) \le S$, and $\nu(P) = \nu(P_n) + \nu(P \setminus P_n) \ge \nu(P_n)$. Thus

$$\nu(P) \ge \lim_{n\to\infty} \nu(P_n) = s.$$

Hence $\nu(P) = s$ and the supremum is in fact a max. We then know that $s = \nu(P) < \infty$ because ν does not attain the value infinity.

Now let $N = X \setminus P$. We claim that N is negative. If not then there exists a measurable $E \subseteq N$ with $\nu(E) > 0$. By assumption, $\nu(E) < \infty$. Then $0 < \nu(E) < \infty$, so by Lemma VII.1.3 there exists a measurable $A \subseteq E$ such that A is positive and $\nu(A) > 0$.

But wait! We then know that

$$\nu(P \cup A) = \nu(P) + \nu(A) > \nu(P)$$

which is a contradiction since $P \cup A$ is a positive set, and $\nu(P)$ is maximal.

Therefore N is negative, and the theorem holds.

Proof of Lemma VII.1.3. If E is positive, we're done. Otherwise, there exist measurable subsets with negative measure. Let $n_1 \in \mathbb{N}$ be the least such n_1 such that there exists $E_1 \subseteq E$ with $\nu(E_1) < -1/n_1$.

If $E \setminus E_1$ is positive, we're done. Else we can inductively define $n_2, n_3, \ldots$ as well as $E_2, E_3, \ldots$.

Explicitly, if $E \setminus \bigcup_{i=1}^{k-1} E_i$ is not positive, let n_k be the least such that there exists $E_k \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ with $\nu(E_k) < -1/n_k$.

Note then that if $n_k \geq 2$, for all $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ we have that $\nu(B) \geq -\frac{1}{n_k - 1}$.

Now let $A = E \setminus \bigcup_{i=1}^{\infty} E_i$. Since $E = A \cup \bigcup_i E_i$ we have by countable additivity that

$$0 < \nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k) < \nu(A).$$

Furthermore, $\nu(E), \nu(A)$ are both in $(0, \infty)$, and we see that

$$0 < \nu(E) \leq \nu(A) - \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Therefore the sum on the RHS must converge, meaning that $1/n_k \to 0$ as $k \to \infty$. That is $\lim_{k\to\infty} n_k = \infty$.

Now if $B \subseteq A$, then $B \subseteq E \setminus \bigcup_{i=1}^{\infty} E_i$. Therefore $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$. By the note above, for large enough k such that $n_k \geq 2$ we have

$$\nu(B) \geq \frac{-1}{n_k - 1}$$

taking $k \to \infty$ we have $\nu(B) \geq 0$, and so A is a positive set as desired.

Definition VII.1.3

If μ, ν are signed measures on $(X, \mathcal{A})$, then we say $\mu \perp \nu$ (singular to each other) means there exists $E, F \in \mathcal{A}$ such that $E \cap F = \emptyset, E \cup F = X$, F is null for μ, E is null for ν.

Example VII.1.2

Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

(1) The Lebesgue measure m

(2) The Cantor measure μ_C defined by the Cantor function.

(3) The discrete measure $\mu_D = \delta_1 + 2\delta_{-1}$.

We can take $E = \mathbb{R} \setminus \{-1, 1\}, F = \{1, -1\}$ to see that $m \perp \mu_D$.

We can take $E = \mathbb{R} \setminus K$ and $F = K$ where K is the cantor set to see that $m \perp \mu_C$.

We can also see that $\mu_C \perp \mu_D$.

Theorem VII.1.5 (Jordan Decomposition Theorem)

Let ν be a signed measure on $(X, \mathcal{A})$. Then there exists unique positive measures ν^+, ν^- on $(X, \mathcal{A})$ such that for all $E \in \mathcal{A}$ we have

$$\nu(E) = \nu^+(E) - \nu^-(E) \qquad\qquad \nu^+ \perp \nu^-.$$

Proof. For existence take $\nu^+(E) := \nu(E \cap P), \nu^-(E) := -\nu(E \cap N)$. Uniqueness DIY.

Example VII.1.3

For an example of Jordan decomposition, let $(X, \mathcal{A}, \mu)$ be a measure space, $f : X \to \overline{\mathbb{R}}$, and $\nu(E) = \int_E f \, d\mu$. Then

$$\nu^+(E) = \int_E f^+ \, d\mu \qquad\qquad \nu^-(E) = \int_E f^- \, d\mu.$$

Definition VII.1.4

Let ν be a signed measure on $(X, \mathcal{A})$. The total variation measure of ν is $|\nu| := \nu^+ + \nu^-$. This is a positive measure on X.

Example VII.1.4

In the above example, $|\nu|(E) = \int_E |f| \, \mathrm{d}\mu$.

Lemma VII.1.6

We have the following

(1) $|\nu(E)| \leq |\nu|(E)$.

(2) E is ν-null if and only if E is $|\nu|$-null

(3) If κ is another signed measure then

$$\kappa \perp \nu \text{ if and only if } \kappa \perp |\nu| \text{ if and only if } (\kappa \perp \nu^+ \text{ and } \kappa \perp \nu^-).$$

Proof. DIY.

Definition VII.1.5

ν is finite if $|\nu|$ is a finite measure, and similarly for σ-finite.

This holds if and only if ν^+, ν^- are both finite (resp. σ-finite) measures.

VII.2. Absolutely Continuous Measures

Definition VII.2.1

Let μ be a positive measure, ν be a signed measure, both on $(X, \mathcal{A})$. We say that ν if absolutely continuous with respect to μ (written $\nu \ll \mu$) provided that for all $E \in \mathcal{A}, \mu(E) = 0$ implies $\nu(E) = 0$.

This is equivalent to every μ-null set being ν-null.

Example VII.2.1

If $(X, \mathcal{A}, \mu)$, $f : X \to \overline{\mathbb{R}}$, $\nu(E) = \int_E f \, \mathrm{d}\mu$, then $\nu \ll \mu$.

Notation: $\mathrm{d}\nu = f \, \mathrm{d}\mu$ means ν is a signed measure defined by $\nu(E) = \int_E f \, \mathrm{d}\mu$.

Lemma VII.2.1

If μ is a positive measure, ν is a signed measure on $(X, \mathcal{A})$, then

(1) $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $(\nu^+ \ll \mu, \nu^- \ll< \mu)$.

(2) $(\nu \ll \mu$ and $\nu \perp \mu)$ implies $\nu = 0$ (zeno measure)

Proof. DIY (1). For (2), write $X = A \cup B, A \cap B = \emptyset, A$ μ-null, B ν-null. Then

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A).$$

Then $E \cap A \subseteq A$, so $\nu(E \cap A) = 0$. By absolute continuity, $\nu(E \cap A) = 0$. Thus $\nu(E) = 0$.

Theorem VII.2.2 (Radon-Nikodym)

Suppose μ is a σ-finite positive measure, ν is a σ-finite signed measure, and suppose $\nu \ll \mu$. Then there exists $f : X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\nu = f \, \mathrm{d}\mu$, in other words $\nu(E) = \int_E f \, \mathrm{d}\mu$.

> If g is another such function with $\mathrm{d}\nu = g\,\mathrm{d}\mu$ then $f = g$ μ-a.e.

Proof. Next class, we'll prove a more general Lebesgue-Radon-Nikodym theorem (Theorem VII.2.4).

Definition VII.2.2

Suppose $\nu \ll \mu$. The Radon-Nikodym derivative of ν with respect to μ is a function $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} : X \to \overline{\mathbb{R}}$ such that $\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu$ for all $E \in \mathcal{A}$.

i.e. we have $\mathrm{d}\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu$.

Note: By Theorem VII.2.2, such a function exists and is unique up to equivalence μ-a.e. in the σ-finite case.

Example VII.2.2

Say $F(X) = e^{2x} : \mathbb{R} \to \mathbb{R}$. This is continuous and strictly increasing, so we may define a Lebesgue-Stieltjes measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

This is defined to be the unique locally finit emeasure satisfying $\mu_F([a,b]) = F(b) - F(a) = e^{2b} - e^{2a}$. Then one can check that

$$\mu_F(E) = \int_E 2e^{2x}\,\mathrm{d}x$$

by uniqueness and the classical FTC, since the RHS is a locally finite borel measure, and $\kappa([a,b]) = e^{2b} - e^{2a}$. Thus $\mu_F = \kappa$.

Therefore $\mu_F \ll m$ and $\frac{\mathrm{d}\mu_F}{\mathrm{d}m} = 2e^{2x} = \frac{\mathrm{d}F}{\mathrm{d}x}$.

Example VII.2.3

Let $C(X) : \mathbb{R} \to \mathbb{R}$ be the cantor function. Then $C'(x) = 0$ outside the cantor set. But we don't always have

$$\mu_C(E) \neq \int_E 0\,\mathrm{d}x$$

So the candidate derivative is 0, but this fails. In particular

$$C(b) - C(a) \neq \int_a^b C'(x)\,\mathrm{d}x.$$

In fact, $\mu_C \not\ll m$ because $\mu_C \perp m$ and $\mu_C \neq 0$.

Thus the existence of a derivative almost everywhere and continuity is not enough to guarantee a version of the FTC holds.

Lemma VII.2.3

Let μ, ν be finite positive measures on $(X, \mathcal{A})$. Then either

(1) $\nu \perp \mu$.

(2) There exists an $\varepsilon > 0$, an $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for the measure $\nu - \varepsilon\mu$.

I.e., for all $G \subseteq F$, $\nu(G) \geq \varepsilon\mu(G)$.

Proof. Let $\kappa_n = \nu - (1/n)\mu$. By gthm:hahn-decomposition we have$X = P_n \cup N_n$ for P_n positive for κ_n, N_n negative for K_n.

Let $P = \bigcup_n P_n, N = \bigcap_n N_n = X \setminus P$. Then $X = P \cup N$.

We see that for any N we have $\kappa_n(N) \leq 0$ because $N \subseteq N_n$. Thus

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N).$$

This implies $\nu(N) = 0$. Because ν is positive for any $N' \subseteq N$ we have $0 \leq \nu(N') \leq \nu(N)$, and thus $\nu(N') = 0$. This shows N is null for N.

If $\mu(P) = 0$, then $\nu \perp \mu$.

If $\mu(P) \neq 0$, then we have $\mu(P_n) > 0$ for some n.

With $F = P_n$, $\varepsilon = 1/n$, then F is a positive set for $\kappa_n = \nu - (1/n)\mu$ as desired.

Theorem VII.2.4 (Lebesgue-Radon-Nikodym)

Let μ be a σ-finite positive measure, ν a σ-finite signed measure on $(X, \mathcal{A})$.

Then there are unique λ, ρ σ-finite signed measures on $(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$, $\nu = \lambda + \rho$.

Furthermore, there exists a measurable function $f : X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\rho = f\,\mathrm{d}\mu$ (that is for all $E \in \mathcal{A}$, $\rho(E) = \int_E f\,\mathrm{d}\mu$).

And if there is another g such that $\mathrm{d}\rho = g\,\mathrm{d}\mu$, then $f = g$, μ-a.e.

Notationally we may write $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}\mu$, where $\mathrm{d}\lambda$ and $\mathrm{d}\mu$ are singular to each other.

Proof. Lets go!

(a) Assume μ, ν are finite positive measures. Let

$$\begin{aligned}\mathscr{F} &= \left\{g : X \to [0,\infty] \mid \int_E g\,\mathrm{d}\mu \leq \nu(E), \forall E \in \mathcal{A}\right\} \\ &= \{g : X \to [0,\infty] \mid \mathrm{d}\nu - g\,\mathrm{d}\mu \text{ is a positive measure}\}.\end{aligned}$$

This set is nonempty since $g = 0 \in \mathscr{F}$. Let $s = \sup\{\int_X g\,\mathrm{d}\mu \mid g \in \mathscr{F}\}$.

Claim

There is an $f \in \mathscr{F}$ such that $s = \int_X f\,\mathrm{d}\mu$.

If $g, h \in \mathscr{F}$, we can define $u(x) = \max\{g(x), h(x)\}$. Then $u \in \mathscr{F}$. Why? Well let $A = \{x \mid g(x) \geq h(x)\}$. Then

$$\begin{aligned}\int_E u\,\mathrm{d}\mu &= \int_{E \cap A} g\,\mathrm{d}\mu + \int_{E \cap A^c} h\,\mathrm{d}\mu \\ &\leq \nu(E \cap A) + \nu(E \cap A^c) = \nu(E).\end{aligned}$$

There exist measurable functions $g_1, g_2, \ldots \in \mathscr{F}$ such that $\lim_{n\to\infty} \int_X g_n\,\mathrm{d}\mu = s$. We can replace g_2 by $\max(g_1, g_2)$, g_3 by $\max(g_1, g_2, g_3)$, so that we may assume $0 \leq g_1 \leq g_2 \leq \cdots$.

Then we still know that $\lim_{n\to\infty} \int_X g_n\,\mathrm{d}\mu = s$, as all the relevant integrals are bounded above by s. Now let $f(x) = \sup_n g_n(x) = \lim_{n\to\infty} g_n(x)$. By Monotone convergence theorem,

$$\int_E f\,\mathrm{d}\mu = \lim_{n\to\infty} \int_E g_n\,\mathrm{d}\mu \leq \nu(E)$$

Thus $f \in \mathscr{F}$. When $E = X$ we get $\int_X f\,\mathrm{d}\mu = s$ as desired.

Great! Let $\rho(E) \coloneqq \int_E f \, \mathrm{d}\mu$. We of course have $\rho \ll \mu$. And also we know

$$0 \leq \rho(X) = \int_X f \, \mathrm{d}\mu \leq \nu(X) < \infty.$$

Thus ρ is a finite positive measure. We can define $\lambda(E) \coloneqq \nu(E) - \rho(E)$. Then

$$\lambda(E) = \nu(E) - \int_E f \, \mathrm{d}\mu \geq 0$$

because $f \in \mathscr{F}$. Thus λ is also a positive measure, and $\lambda(X) \leq \nu(X) < \infty$. It remains to show the following.

Claim

$\lambda \perp \mu$

Suppose not, by glemma:finite-singular, there exists $\varepsilon > 0$, $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for $\lambda - \varepsilon\mu$.

Then this says that $\mathrm{d}\lambda - \varepsilon 1_F \, \mathrm{d}\mu$ is a positive measure, that is $\mathrm{d}\nu - f \, \mathrm{d}\mu - \varepsilon 1_F \, \mathrm{d}\mu$ is a positive measure. This will break maximality of f.

Explicitly, let $g(x) = f(x) + \varepsilon 1_F(x)$. Then for all $E \in \mathcal{A}$ we have

$$\begin{aligned}
\int_E g \, \mathrm{d}\mu &= \int_E f \, \mathrm{d}\mu + \varepsilon\mu(E \cap F) \\
&= \nu(E) - \lambda(E) + \varepsilon\mu(E \cap F) \\
&\leq \nu(E) - \lambda(E \cap F) + \varepsilon\mu(E \cap F) \leq \nu(E)
\end{aligned}$$

since $\lambda(E \cap F) - \varepsilon\mu(E \cap F) \geq 0$. Thus $g \in \mathscr{F}$. We then see that

$$\begin{aligned}
s \geq \int_X g \, \mathrm{d}\mu &= \int_X f \, \mathrm{d}\mu + \int_X \varepsilon 1_F \, \mathrm{d}\mu \\
&= s + \varepsilon\mu(F) > s.
\end{aligned}$$

This is a contradiction! Perfect!

There are now technical things, such as extending to σ-finite measures and uniqueness. These are relatively easy compared to this part.

Example VII.2.4 (Lebesgue-Radon-Nikodym)

Let $\mu = m$, $\nu = \mu_F$ (Lebesgue-Stieltjes measure for F). We'll define $F(x)$ by

$$F(x) = \begin{cases} e^{3x} & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x < 1 \\ 5 & \text{if } x \geq 1 \end{cases}.$$

Then we will have that

$$\mu_F(E) = \int_{E \cap \mathbb{R}_{<0}} 3e^{3x} \, \mathrm{d}x + 4\delta_1(E).$$

It is enough to check on $(-\infty, x]$ because these are locally finite Borel measures on $\mathbb{R}$.

Then we have $\mu_F = \mathrm{d}\rho + \mathrm{d}\lambda = f\,\mathrm{d}m + \mathrm{d}\lambda$ where $f = 1_{\mathbb{R}_{<0}} 3e^{3x}$ and $\lambda = 4\delta_1$, $\lambda \perp m$.

Read: Theorem 3.5, Proposition 3.9, Corollary 3.10 of section 3.2 of [Fol99]

Skip: Complex measures (section 3.3).

Recall VII.2.5

If $\nu = \nu^+ - \nu^-$, we defined the total variation $|\nu| = \nu^+ + \nu^-$, see Definition VII.1.4.

Then we have $|\nu(E)| \leq |\nu|(E)$.

VII.3. Lebesgue Differentiation Theorem for regular Borel measures on $\mathbb{R}^d$

See page 99 of [Fol99].

Definition VII.3.1

A Borel signed measure ν on $\mathbb{R}^d$ is called regular if

(1) $|\nu|(K) < \infty$ for all compact K.

(2) We have outer regularity

$$|\nu|(E) = \inf\{|\nu|(U) \mid \text{ open } U \supseteq E\}$$

for every Borel set E.

Example VII.3.1

Any Lebesgue-Stieltjes measure on $\mathbb{R}$ has this property (see Section II.7 and theorem II.7.2).

In fact, so is the difference of two of them (at least if one of them is finite).

The Lebesgue measure on $\mathbb{R}^d$ is regular.

Note: From Item (1), if ν is regular then ν is σ-finite. Also if $\mathrm{d}\nu = f\,\mathrm{d}m$ is regular, then

$$|\nu|(K) = \int_K |f|\,\mathrm{d}m < \infty$$

for all compact K. Thus $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Lemma VII.3.1

$f \in L^1_{\text{loc}}(\mathbb{R}^d)$ if and only if $\mathrm{d}\nu = f\,\mathrm{d}m$ is regular

Proof. Skip–read the book.

Recall VII.3.2

Remember the Lebesgue Differentiation theorem (Section V.2 and ??).

Here we had that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ implies that for Lebesgue almost every x,

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\,\mathrm{d}y = f(x)$$

for any E_r shrinking nicely to x (Definition V.2.2, think of $B_r(x)$).

Corollary VII.3.2

Let ρ be a regular signed Borel measure on $\mathbb{R}^d$. Suppose $\rho \ll m$. Then $\mathrm{d}\rho = f\,\mathrm{d}m$ for some

$f \in L^1_{\text{loc}}(\mathbb{R}^d)$, So then for Lebesgue almost every x we have

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, \mathrm{d}y = f(x).$$

Writing this in a nice way, using established notation, this is

$$\lim_{r \to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for every E_r shrinking nicely to x.

Proposition VII.3.3

Let λ be a regular positive Borel measure on $\mathbb{R}^d$. Suppose $\lambda \perp m$.

For Lebesgue almost every x, we have

$$\lim_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for every E_r shrinking to x nicely (equivalently shrinking to 0 nicely).

Proof. It is enough to consider $E_r = B(x, r)$. We wish to prove that

$$\begin{aligned} G \coloneqq G &\coloneqq \left\{ x \mid \limsup_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} \neq 0 \right\} \\ &= \bigcup_{n=1}^{\infty} G_n \\ G_n &\coloneqq \left\{ x \mid \limsup_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} > \frac{1}{n} \right\}. \end{aligned}$$

It is enough to show $m(G_n) = 0$ for all n.

$\lambda \perp m$, so we know $\mathbb{R}^d = A \cup B$ disjoint, $\lambda(A) = 0$, $m(B) = 0$. Thus it suffices to show $m(G_n \cap A) = 0$.

Fix $\varepsilon > 0$, since λ is regular, there exists an open set $U \supseteq A$ such that $\lambda(U) \leq \lambda(A) + \varepsilon = \varepsilon$.

For every $x \in G_n \cap A$, there is an $r_x > 0$ such that $\lambda(B(x, r_x))/m(B(x, r_x)) > 1/n$ and $B(x, r_x) \subseteq U$.

Let $K \subseteq G_n \cap A$, compact. Then $K \subseteq \bigcup_{x \in K} B(x, r_x)$. By compactness, we can take a finite subcover, and then use Vitali (Lemma V.1.1) to find $B_1, B_2, \ldots, B_N$ disjoint each of type $B(x, r_x)$ such that $K \subseteq \bigcup_i 3B_i$.

Therefore

$$\begin{aligned} m(K) &\leq 3^d \sum_{i=1}^{N} m(B_i) \leq 3^d n \sum_{i=1}^{N} \lambda(B_i) \\ &= 3^d n \lambda\left(\bigcup_i B_i \right) \leq 3^d n \lambda(U) = 3^d n \varepsilon. \end{aligned}$$

Thus by inner regularity, $m(G_n \cap A) \leq 3^d n \varepsilon$ for any $\varepsilon > 0$. Taking $\varepsilon \to 0$ yields $m(G_n \cap A) = 0$, so then $m(G_n) = 0$ as desired.

Announcements

- HW 10 due tomorrow
- HW 11 posted

- Final exam on 4/27 Wednesday 1:30-3:30 here (in the classroom). Cumulative, with emphasis on material covered after the midterm.

From last time we have that if $\rho \ll m$ is regular then

$$\lim_{r\to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for Lebesgue almost every x, where E_r shrinks nicely to x. Likewise if $\lambda \perp m$ regular (positive measure) then

$$\lim_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} = 0.$$

for Lebesgue almost every x, where E_r shrinks nicely to x.

Theorem VII.3.4 (Lebesgue Differentiation Theorem for Regular measures)

Let ν be a regular Borel signed measure on $\mathbb{R}^d$. Then $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}m$, $\lambda \perp m$ by Theorem VII.2.4. Then for Lebesgue almost every x,

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every $E_r \to x$ nicely.

Proof. It must be checked that ν regular implies $\lambda, f\,\mathrm{d}m$ are regular (check!)

VII.4. Monotone Differentiation Theorem

This is from [Fol99] section 3.5.

Definition VII.4.1

For $F : \mathbb{R} \to \mathbb{R}$ that is monotonically increasing, denote

$$F(x^+) = \lim_{y\to x^+} F(y) \qquad\qquad F(x^-) = \lim_{y\to x^-} F(y).$$

These exist and are

$$\inf_{y>x} F(y) \qquad\qquad \sup_{y<x} F(y).$$

So they always exist (being bounded below/above respectively by $F(x)$).

Lemma VII.4.1

If F is monotonically increasing, then $D = \{x \mid F \text{ is discontinuous at } x\}$ is a countable set.

Proof. $x \in D$ if and only if $F(x^+) > F(x^-)$. For each $x \in D$, let $I_x = (F(x^-), F(x^+))$, not empty.

Also if $x, y \in D$, $x \neq y$, then I_x, I_y are disjoint. Say if $x < y$ then

$$F(x^-) < F(x^+) \leq F(x) \leq F(y) \leq F(y^-) < F(y^+).$$

Taking a rational number in each interval gives an injective map $D \to \mathbb{Q}$, so D is countable.

Theorem VII.4.2 (Monotone Differentiation Theorem)

Let F be increasing. Then

- F is differentiable Lebesgue almost everywhere.
- $G(x) = F(x^+)$ (which is right-continuous) is differentiable almost everywhere.
- $G' = F'$ almost everywhere.

Proof. Start with G, which is increasing and right-continuous on $\mathbb{R}$. There is then a Lebesgue-Stieltjes measure μ_G on $\mathbb{R}$. Thus it is regular on $\mathbb{R}$. We see

$$\frac{G(x+h) - G(x)}{h} = \begin{cases} \frac{\mu_G((x,x+h])}{m((x,x+h])} & \text{if } h > 0 \\ \frac{\mu_G((x+h,x])}{m((x+h,x])} & \text{if } h < 0 \end{cases}$$

These both shrink nicely to x. By Theorem VII.3.4 (since these shrink nicely) we know then that these both converge for Lebesgue almost every x to some common limit $f(x)$. Thus G' exists Lebesgue almost everywhere.

Define $H(x) = G(x) - F(x) \geq 0$. We see that

$$\{x \mid H(x) > 0\} \subseteq \{x \mid F \text{ is discontinuous at } x\}$$

This is then countable by the lemma above, and we can write $\{x \mid H(x) > 0\} = \{x_n\}$. Then let

$$\mu \coloneqq \sum_n H(x_n)\delta_{x_n}.$$

This is a Borel measure, so we check if it is locally finite. That is we check

$$\mu((-N,N)) = \sum_{-N<x_n<N} H(x_n) \leq G(N) - F(-N) < \infty$$

checking the inequality just consists of seeing that the intervals $(F(x_n), G(x_n))$ are disjoint and a subset of $(F(-N), G(N))$ so

$$\sum_{-N<x_n<N} H(x_n) = \mu\left(\bigcup_n (F(x_n), G(x_n))\right) \leq \mu((F(-N), G(N))).$$

Thus μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, so it is regular

Special to $\mathbb{R}$ that locally finite Borel $\implies$ Lebesgue-Stieltjes $\implies$ regular $\implies$ outer regularity

Then we have that

$$\left|\frac{H(x+h) - H(x)}{h}\right| \leq \frac{H(x+h) + H(x)}{|h|} \leq \frac{\mu((x-2h, x+2h))}{|h|}.$$

This goes to 0 for Lebesgue almost every x by Theorem VII.3.4 and that $\mu \perp m$ (check!)

Thus H is differentiable almost everywhere and $H' = 0$ almost everywhere. Thus F is differentiable almost everywhere and $F' = G'$ almost everywhere.

Proposition VII.4.3

Suppose F is an increasing function. Then F' exists almost everywhere and is measurable. We have that

$$\int_a^b F'(x)\,\mathrm{d}x \le F(b) - F(a).$$

Example VII.4.1

Take $F(x)$ to be 0 on $x \le 0$, 1 on $x > 0$. Then $F'(x) = 0$ almost everywhere. So

$$\int_{-1}^1 F'(x)\,\mathrm{d}x = 0 < 1 = F(1) - F(-1).$$

Even if F is continuous we might not have equality. Take $F(x)$ to be the cantor function. Then $F'(x) = 0$ almost everywhere, but

$$\int_0^1 F'(x)\,\mathrm{d}x = 0 < 1 = F(1) - F(0).$$

Proof of Proposition VII.4.3. Let

$$G(x) \coloneqq \begin{cases} F(a) & \text{if } x < a \\ F(x) & \text{if } a \le x \le b \\ F(b) & \text{if } x > b \end{cases}$$

Then G is increasing. Define

$$g_n(x) = \frac{G(x + 1/n) - G(x)}{1/n} \to F'(x)$$

for almost every $x \in [a, b]$. Also $g_n(x) \ge 0$.

Fatou's Lemma tells us that

$$\int_a^b F'(x)\,\mathrm{d}x = \int_a^b \liminf_{n\to\infty} g_n(x)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_a^b g_n(x)\,\mathrm{d}x.$$

We then evaluate

$$\begin{aligned} \int_a^b g_n(x) &= n\left(\int_{a+1/n}^{b+1/n} G(x)\,\mathrm{d}x - \int_a^b G(x)\,\mathrm{d}x\right) \\ &= n\left(\int_b^{b+1/n} G(x)\,\mathrm{d}x - \int_a^{a+1/n} G(x)\,\mathrm{d}x\right) \\ &\le n\left(G\left(b + \frac{1}{n}\right)\cdot\frac{1}{n} - G(a)\cdot\frac{1}{n}\right) \\ &= F(b) - F(a). \end{aligned}$$

Therefore

$$\int_a^b F'(x)\,\mathrm{d}x \le F(b) - F(a).$$

VII.5. **Functions of bounded variation**

Definition VII.5.1

For $F : \mathbb{R} \to \mathbb{R}$, the total variation function of F is $T_F : \mathbb{R} \to [0, \infty]$ defined by

$$T_F(x) = \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, -\infty < x_0 < x_1 < \cdots < x_n = x \right\}$$

Lemma VII.5.1

We have that

$$T_F(b) = T_F(a) + \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Proof. DIY

Note: T_F is increasing

Definition VII.5.2

We say that $F \in BV$ (F is of bounded variation) provided that

$$T_F(\infty) = \lim_{x \to \infty} T_F(x) < \infty.$$

Similarly $F \in BV([a, b])$ means that

$$\sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b \right\} < \infty.$$

Example VII.5.1

Note that if F is of bounded variation, then F is bounded.

Note that $F(x) = \sin x$ is not of bounded variation. But it is of bounded variation over any $[a, b]$.

Also

$$F(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is not of bounded variation of $[a, b]$ if $a < 0 < b$ because the harmonic series does not converge.

(1) If F, G are of bounded variation, $\alpha F + \beta G$ are of bounded variation.
(2) If F is increasing and bounded, then F is a function of bounded variation.
(3) If F is Lipschitz (see Definition VII.5.3) on $[a, b]$, then $F \in BV([a, b])$.
(4) If F is differentiable, and F' is bounded on $[a, b]$, then F is Lipschitz (mean value theorem), so it is in $BV([a, b])$.
(5) If $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ for $f \in L^1(\mathbb{R})$. Then $F \in BV$.

Namely

$$\begin{aligned} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)\,\mathrm{d}t \right| \\ &\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,\mathrm{d}t \end{aligned}$$

$$= \int_{x_0}^{x_n} |f(t)| \,\mathrm{d}t$$
$$\leq \int_{-\infty}^{\infty} |f(t)| \,\mathrm{d}t < \infty.$$

Definition VII.5.3

A function $F : [a, b] \to \mathbb{C}$ is called Lipschitz provided that there exists an $M \geq 0$ such that $|F(x) - F(y)| \leq M |x - y|$.

Lemma VII.5.2

If $F \in BV$, then T_F is bounded, increasing, $T_F(-\infty) = 0$.

Lemma VII.5.3

$F \in BV$, then $T_F + F$ is increasing/bounded and $T_F - F$ is increasing/bounded. Thus any $F \in BV$ can be written as

$$F = \frac{T_F + F}{2} - \frac{T_F - F}{2}$$

which is a difference of increasing/bounded functions.

Proof. Let $x < y$. Fix $\varepsilon > 0$, then there are points $x_0 < x_1 < \cdots < x_n = x$ such that

$$T_F(x) \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \varepsilon.$$

Furthermore

$$T_F(y) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)|.$$

Then

$$\pm(F(y) - F(x)) \leq |F(y) - F(x)|$$
$$T_F(y) \pm (F(y) - F(x)) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq T_F(x) - \varepsilon.$$
$$T_F(Y) \pm F(y) \geq T_F(x) \pm F(x) - \varepsilon.$$

Taking $\varepsilon \to 0$ yields the result.

Theorem VII.5.4

F is of bounded variation if and only if $F = F_1 - F_2$ for F_1, F_2 increasing and bounded.

Proof. The forward implication is given by the previous lemma. The other direction follows from the examples we gave (check!)

Corollary VII.5.5 (Bounded Variation Differentiation)

$F \in BV$ implies that F is differentiable almost everywhere. Furthermore,

(1) $F(x^+), F(x^-)$ exist for all x as do $F(-\infty), F(\infty)$.

(2) The set of discontinuities of F is countable.
(3) $G(x) = F(x^+)$ is differentiable and $G' = F'$ almost everywhere.
(4) $F' \in L^1(\mathbb{R}, m)$.

Proof. DIY.

Definition VII.5.4

A function $G \in BV$ is said to have normalized bounded variation ($G \in NBV$) provided that G is right continuous and $G(-\infty) = 0$.

Example VII.5.2

If F is increasing and bounded, F right continuous, $F(-\infty) = 0$.
$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$, $f \in L^1(\mathbb{R})$. Midterm gave F is uniformly continuous.

Lemma VII.5.6

If $F \in BV$, right continuous, then $T_F \in NBV$.

Proof. T_F is bounded, increasing, and satisfies $T_F(-\infty) = 0$ by Lemma VII.5.2. Thus $T_F \in BV$.

Thus we just need to check that T_F is right continuous. Suppose not, then there is a point $a \in \mathbb{R}$ such that $c := T_F(a^+) - T_F(a) > 0$.

Fix $\varepsilon > 0$. Since $F(x)$ and $g(x) := T_F(x^+)$ are right-continuous, there exists a $\delta > 0$ such that for $y \in (a, a+\delta]$ we have

$$|F(y) - F(a)| < \varepsilon$$
$$|g(y) - g(a)| < \varepsilon.$$

We then have that

$$T_F(y) - T_F(a^+) \leq T_F(y^+) - T_F(a^+) < \varepsilon.$$

There exist $a = x_0 < x_1 < \cdots < x_n = a + \delta$ such that

$$\begin{aligned}\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &\geq T_F(a+\delta) - T_F(a) - \frac{c}{4} \\ &\geq T_F(a^+) - T_F(a) - \frac{c}{4} = \frac{3c}{4}.\end{aligned}$$

Then $|F(x_1) - F(a)| < \varepsilon$ so we have

$$\sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4} - \varepsilon.$$

There exist $a = t_0 < \cdots < t_k = x_1$ such that

$$\sum_{i=1}^{k} |F(t_i) - F(t_{i-1})| \geq T_F(x_1) - T_F(a) - \frac{c}{4} \geq \frac{3}{4}c.$$

Then as $[a, a+\delta] = [a, x_1] \cup [x_1, a+\delta]$ we see that

$$T_F(a+\delta) - T_F(a) \geq \sum_{j=1}^{k} |F(t_j) - F(t_{j-1})| + \sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}c - \varepsilon + \frac{3}{4}c = \frac{3}{2}c - \varepsilon.$$

Thus

$$\begin{aligned} \varepsilon + c &\geq T_F(a+\delta) - T_F(a^+) + T_F(a^+) - T_F(a) \\ &= T_F(a+\delta) - T_F(a) \geq \frac{3}{2}c - \varepsilon \\ c &\leq 4\varepsilon. \end{aligned}$$

Thus taking $\varepsilon \to 0$ yields $c = 0$, which is a contradiction.

Corollary VII.5.7

$F \in NBV$ if and only if $F = F_1 - F_2$, $F_1, F_2 \in NBV$ and increasing

Proof. $F = (T_F + F)/2 - (T_F - F)/2$.

Theorem VII.5.8

We have that

1. Suppose that μ is a finite signed Borel measure on $\mathbb{R}$, then $F(x) = \mu((-\infty, x]) \in NBV$.
2. $F \in NBV$ implies there exists a unique finite signed Borel measure on $\mathbb{R}$ satisfying $\mu_F((-\infty, x]) = F(x)$.

Proof. We have

1. Let $\mu = \mu^+ - \mu^-$, thenn $F = F^+ - F^-$, where $F^{\pm}(x) = \mu^{\pm}((-\infty, x])$, which are bounded, right continuous, $F^{\pm}(-\infty) = 0$, so $F^{\pm} \in NBV$.
2. Let $F \in NBV$, then $F = F_1 - F_2$, $F_1, F_2 \in NBV$ and increasing. Then define μ_{F_1}, μ_{F_2} by Lebesgue-Stieltjes measure, and set $\mu_F \coloneqq \mu_{F_1} - \mu_{F_2}$.

   Uniqueness? See homework.

Proposition VII.5.9

We have

1. If $F \in NBV$, then F is differentiable almost everywhere, $F' \in L^1(\mathbb{R}, m)$.
2. $d\mu_F + d\lambda + F' \, dm$ for some measure λ satisfying $\lambda \perp m$.
3. $\mu_F \perp m$ if and only if $F' = 0$ Lebesgue almost everywhere.
4. $\mu_F \ll m$ if and only if $\int_{-\infty}^{x} F'(t) \, dt = F(x) - F(-\infty) = F(x)$.

Proof. (1),(2),(3) check.

For part (4), we have

$$\begin{aligned}\mu_F \ll m &\iff \lambda = 0 \iff \mathrm{d}\mu_F = F'\,\mathrm{d}m \\ &\iff \mu_F(E) = \int_E F'\,\mathrm{d}m \ \ \forall \text{ Borel } E \\ &\iff F(x) = \mu_F((-\infty, x]) = \int_{-\infty}^{x} F'(t)\,\mathrm{d}t \ \ \forall x \in \mathbb{R}\end{aligned}$$

The last converse comes from the uniqueness of the theorem above.

Announcements

- HW 12 posted, due next Saturday (4/16), last HW to collect.
- Exam will be April 27th, 1:30-3:30pm.

VII.6. Absolutely Continuous Functions

Definition VII.6.1

We say that $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous ($F \in AC$) means for all $\varepsilon > 0$, there exists a $\delta > 0$, such that if $(a_1, b_1), \dots, (a_N, b_N)$ are finitely many disjoint open intervals satisfying $\sum_{n=1}^{N}(b_n - a_n) < \delta$, then $\sum_{n=1}^{N} |F(b_n) - F(a_n)| < \varepsilon$.

Lemma VII.6.1

We have that

1. If F is absolutely continuous, then it is uniformly continuous (take $N = 1$)
2. If F is Lipschitz then F is absolutely continuous (easy).
3. $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$, $f \in L^1$, is absolutely continuous.

Proof of (3). We write this out as

$$\begin{aligned}\sum_{n=1}^{N} |F(b_n) - F(a_n)| &= \sum_{n=1}^{N} \left| \int_{a_n}^{b_n} f(t)\,\mathrm{d}t \right| \\ &\le \sum_{n=1}^{N} \int_{a_n}^{b_n} |f(t)|\,\mathrm{d}t \\ &= \int_E |f(t)|\,\mathrm{d}t\end{aligned}$$

where $E = \bigcup_{n=1}^{N}(a_n, b_n)$, so $m(E) = \sum_{n=1}^{N}(b_n - a_n)$. By Midterm Q1, if $f \in L^1(X, \mu)$, for all $\varepsilon > 0$, there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |f| < \varepsilon$.

This directly implies that this function is absolutely continuous.

Example VII.6.1

The cantor function F is uniformly continuous. However, we will see that it is not absolutely continuous.

Proposition VII.6.2

Suppose $F \in NBV$, then F is absolutely continuous if and only if $\mu_F \ll m$.

Corollary VII.6.3

$F \in NBV \cap AC$ if and only if $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ for some $f \in L^1(\mathbb{R}, m)$. If this holds, we have $f = F'$ Lebesgue almost everywhere.

Lemma VII.6.4

If $F \in AC([a,b])$, then $F \in NBV([a,b])$

Proof. DIY

Theorem VII.6.5 (Fundamental Theorem of Calculus)

For $F \in [a,b] \to \mathbb{R}$, the following are equivalent

(1) $F \in AC([a,b])$.
(2) $F(x) - F(a) = \int_a^x f(t)\,\mathrm{d}t$ for some $f \in L^1([a,b], m)$.
(3) F is differentiable almost everywhere on $[a,b]$ and $F(x) - F(a) = \int_a^b F'(t)\,\mathrm{d}t$.

This follows directly from the above.

Proof of Proposition VII.6.2. Suppose $\mu_F \ll m$. Then $F(x) = \int_{-\infty}^{x} F'(t)\,\mathrm{d}t$, and $F' \in L^1(\mathbb{R}, m)$, by Proposition VII.5.9. Therefore $F \in AC$.

Now suppose $F \in AC$. Note that since F is continuous,

$$\mu_F((a,b)) = \lim_{n\to\infty} \mu_F((a, b - 1/n]) = \lim_{n\to\infty} F(b - 1/n) - F(a) = F(b) - F(a).$$

Now let E be a Borel set with $m(E) = 0$. Fix $\varepsilon > 0$, we will show $|\mu_F(E)| \le \varepsilon$. Let $\delta > 0$ be the constant from $F \in AC$.

Now there exist open $U_1 \supseteq U_2 \supseteq \cdots \supseteq E$ such that $\lim_{n\to\infty} m(U_n) = m(E) = 0$, and open $V_1 \supseteq V_2 \supseteq \cdots \supseteq E$ such that $\lim_{n\to\infty} \mu_F(V_n) = \mu_F(E)$ by regularity.

Let $O_n = U_n \cap V_n$, then $O_1 \supseteq O_2 \supseteq \cdots \supseteq E$, and by monotonicity (for μ_F decomposing into pos/neg first)

$$\lim_{n\to\infty} m(O_n) = m(E) = 0 \qquad\qquad \lim_{n\to\infty} \mu_F(O_n) = \mu_F(E).$$

Thus without loss of generality, we may assume $m(O_1) < \delta$. Each O_n is a countable union of disjoint intervals

$$O_n = \bigcup_{k=1}^{\infty} (a_k^n, b_k^n).$$

For any N we also have

$$\sum_{k=1}^{N} (b_k^n - a_k^n) \le m(O_n) \le m(O_1) < \delta.$$

Therefore

$$\begin{aligned}\left|\mu_F\left(\bigcup_{k=1}^{N} (a_k^n, b_k^n)\right)\right| &= \left|\sum_{k=1}^{N} \mu_F((a_k^n, b_k^n))\right| \\ &\le \sum_{k=1}^{N} |\mu_F((a_k^n, b_k^n))|\end{aligned}$$

$$\leq \sum_{k=1}^{N} |F(b_k^n) - F(a_k^n)| < \varepsilon.$$

Therefore

$$|\mu_F(O_n)| = \lim_{N\to\infty} \left|\mu_F\left(\bigcup_{k=1}^{N}(a_k^n, b_k^n)\right)\right| \leq \varepsilon$$
$$|\mu_F(E)| = \lim_{n\to\infty} |\mu_F(O_n)| \leq \varepsilon.$$

Therefore, taking $\varepsilon \to 0$, yields $\mu_F(E) = 0$. Therefore $\mu_F \ll m$.

Definition VII.6.2

Let μ be a finite signed Borel measure on $\mathbb{R}$.

- μ is called a discrete measure if there is a countable set $\{x_n\}$ and $c_n \neq 0$ such that $\sum_{n=1}^{\infty} |c_n| < \infty$ and $\mu = \sum_n c_n \delta_{x_n}$ (where δ_{x_n} is the Dirac delta at x_n).
- μ is called continuous if $\mu(\{a\}) = 0$ for all $a \in \mathbb{R}$.

Lemma VII.6.6

Given a finite signed Borel measure μ

(1) Any $\mu = \mu_d + \mu_c$, where μ_d is discrete, μ_c is continuous, uniquely.
(2) μ discrete implies $\mu \perp m$.
(3) $\mu \ll m$ implies μ is continuous.

Corollary VII.6.7

For μ a finite signed Borel measure on $\mathbb{R}$, we have that

$$\mu = \mu_d + \mu_{ac} + \mu_{sc}$$

where μ_d is discrete, μ_{ac} is absolutely continuous, and μ_{sc} is singularly continuous (to m).

VIII. Hilbert Spaces

This is in [Fol99] section 5.5.

VIII.1. Inner Product Spaces

Definition VIII.1.1

Let V be a (complex) vector space. An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying

(1) We have linearity in the first argument

$$\langle \alpha x + \beta y, z \rangle$$

for all $x, y, z \in V$, and $\alpha, \beta \in \mathbb{C}$.
(2) We have that $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for every $x, y \in V$.
(3) $\langle x, x \rangle \in [0, \infty)$.
(4) $\langle x, x \rangle = 0$ if and only if $x = 0$.

Note, we have conjugate linearity in the second argument

$$\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle$$

for any $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.

Example VIII.1.1

We have the following examples

- $\mathbb{R}^d$ with $\langle x, y\rangle = x \cdot y = \sum_{i=1}^d x_i y_i$.
- $\mathbb{C}^d$ with $\langle x, y\rangle = \sum_{i=1}^d x_i \overline{y_i}$.
- $L^2(X, \mu)$ with $\langle f, g\rangle = \int_X f\overline{g}\,\mathrm{d}\mu$. Note by Hölder that

$$\left|\int_X f\overline{g}\right| \leq \|f\overline{g}\|_1 \leq \|f\|_2\|g\|_2 < \infty.$$

  because $1/2 + 1/2 = 1$.
- A special case is ℓ^2, where we have

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

Definition VIII.1.2

Given an inner product space V, let $\|x\| = \sqrt{\langle x, x\rangle}$. We claim this is a norm, called the norm induced from the inner product.

We prove this is a norm below, after proving Theorem VIII.1.1.

Note that

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2\end{aligned}$$

Theorem VIII.1.1 (Cauchy-Schwarz Inequality)

We have that $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Proof. This is clear if $\langle x, y\rangle = 0$. Assume $\langle x, y\rangle \neq 0$. For every $\alpha \in \mathbb{C}$, we know that

$$0 \leq \|\alpha x - y\|^2 = |\alpha|^2 \|x\|^2 - 2\operatorname{Re}\alpha\langle x, y\rangle + \|y\|^2.$$

Write $\langle x, y\rangle = |\langle x, y\rangle| e^{i\theta}$, and take $\alpha = e^{-i\theta}t$ for arbitrary $t \in \mathbb{R}$. Then, the RHS gives

$$0 \leq \|x\|^2 t^2 - 2|\langle x, y\rangle| t + \|y\|^2.$$

Note this is a real quadratic function of t, with at most one real root. Thus the discriminant is ≤ 0. The discriminant is in fact

$$\begin{aligned}4|\langle x, y\rangle|^2 - 4\|x\|^2\|y\|^2 &\leq 0 \\ |\langle x, y\rangle|^2 &\leq \|x\|^2\|y\|^2 \\ |\langle x, y\rangle| &\leq \|x\|\|y\|.\end{aligned}$$

Proof that Definition VIII.1.2 is a norm. We have that $\|x\| = 0 \iff x = 0$ from the definition of an inner product. We also have that

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\overline{\alpha}\langle x, x\rangle} = |\alpha|\,\|x\|.$$

The triangle inequality is less obvious, and comes from Theorem VIII.1.1. Namely

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2\,|\langle x, y\rangle| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \\ \|x+y\| &\le \|x\| + \|y\|.\end{aligned}$$

Perfect!

Theorem VIII.1.2 (Parallelogram law)

Let V be a normed space. Then, $\|\cdot\|$ is induced by an inner product if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

for all $x, y \in V$.

Proof. The forward direction follows from

$$\begin{aligned}\|x \pm y\|^2 &= \|x\|^2 \pm 2\operatorname{Re}\langle x, y\rangle + \|y\|^2. \\ \|x \pm iy\|^2 &= \|x\|^2 \pm 2\operatorname{Im}\langle x, y\rangle + \|y\|^2.\end{aligned}$$

For the backwards direction, define

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

as motivated by the above relationship.

Check this is an inner product inducing the desired norm.

Example VIII.1.2

Consider $L^p(\mathbb{R}, m)$, $f = 1_{(0,1)}$, $g = 1_{(1,2)}$. We see the parallelogram law is satisfied only when $p = 2$. Thus $L^p(\mathbb{R}, m)$ is only an inner product space when $p = 2$.

Definition VIII.1.3 (Weak convergnece)

We say that $x_n \in V$ converges to $x \in V$ <u>weakly</u> provided that for any fixed $y \in V$, $\langle x_n - x, y\rangle \to 0$.

Lemma VIII.1.3 (Strong convergence $\implies$ Weak convergence)

Suppose V is an inner product space. If $x_n \to x$ strongly (i.e. $\|x_n - x\| \to 0$), then $x_n \to x$ <u>weakly</u> in the sense that for any fixed $y \in V$, we have $\langle x_n - x, y\rangle \to 0$.

Proof. Using the Cauchy-Schwarz inequality

$$0 \le |\langle x_n - x, y\rangle| \le \|x_n - x\| \cdot \|y\|.$$

Since $\|x_n - x\| \to 0$ and $\|y\|$ is constant in n, we have by the squeeze theorem that $\langle x_n - x, y\rangle \to 0$.

Example VIII.1.3

Consider ℓ^2, $x_n = (0, \dots, 0, 1, 0, \dots)$ and $x = 0$. Then x_n does not converge strongly to any vector. But, if we fix $y \in \ell^2$, then

$$\langle x_n - x, y\rangle = \overline{y_n}$$

which goes to 0 as $n \to \infty$ because $\sum_n |y_n|^2 < \infty$. Therefore $x_n \to 0$ weakly, but we see that

$$\|x_n - 0\| = \|x_n\| = 1.$$

Thus $x_n \not\to 0$ strongly.

VIII.2. **Orthonormal Bases**

Definition VIII.2.1

We say x, y are orthogonal if $\langle x, y\rangle = 0$, denoted $x \perp y$.

Lemma VIII.2.1 (Pythagorean Theorem)

If $x_1, \dots, x_n \in V$, $\langle x_i, x_j\rangle = 0$ for all $I \neq j$, then

$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2 \tag{1}$$

Proof. Use that $\|x+y\| = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$ and induct.

Definition VIII.2.2

We call $\{e_i\}_{i\in I}$ an orthonormal set if

$$\langle e_i, e_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Lemma VIII.2.2 (Best approximation)

Let $e_1, \dots, e_N$ be orthonormal. For $x \in V$, let $\alpha_i = \langle x, e_i\rangle$, then

$$\left\| x - \sum_{i=1}^N \alpha_i e_i \right\| \le \left\| x - \sum_{i=1}^N \beta_i e_i \right\|$$

for all $\beta_1, \dots, \beta_N \in \mathbb{C}$. Aka this is the best approximation to x within the span of $e_1, \dots, e_N$. We can also think of it as an orthogonal projection

Proof. Let $z = x - \sum_{i=1}^N \alpha_i e_i$, $w = \sum_{i=1}^N (\alpha_i - \beta_i)e_i$.

Note that for all $n = 1, \dots, N$ we have

$$\langle z, e_n\rangle = \langle x, e_n\rangle - \alpha_n = 0.$$

Thus $\langle z, w\rangle = 0$. So by the Pythagorean theorem

$$\|z+w\|^2 = \|z\|^2 + \|w\|^2 \geq \|z\|$$

proving the result!

Lemma VIII.2.3

Let $\{e_i\}_1^\infty$ be an orthonormal set. For $x \in V$, let $\alpha_i = \langle x, e_i\rangle$. Then,

(1) We have that

$$\|x\|^2 = \left\| x - \sum_{i=1}^N \alpha_i e_i \right\|^2 + \sum_{i=1}^N |\alpha_i|^2$$

for all $N \in \mathbb{N}$.

(2) $\sum_{i=1}^\infty |\alpha_i|^2 \leq \|x\|^2$, referred to as Bessel's inequality.

These actually hold even for an uncountable collection.

Proof. (2) follows from (1), for (1), we see that

$$\begin{aligned}
\left\| x - \sum_{i=1}^N \alpha_i e_i \right\|^2 &= \|x\|^2 - 2\operatorname{Re}\left\langle x, \sum_{i=1^N} \alpha_i e_i \right\rangle + \left\| \sum_{i=1}^N \alpha_i e_i \right\|^2 \\
&= \|x\|^2 - 2\sum_{i=1}^N \operatorname{Re} \overline{\alpha_i}\langle x, e_i\rangle + \sum_{i=1}^N |\alpha_i|^2 \\
&= \|x\|^2 - 2\sum_{i=1}^N |\alpha_i|^2 + \sum_{i=1}^N |\alpha_i|^2 \\
&= \|x\|^2 - \sum_{i=1}^N |\alpha_i|^2 .
\end{aligned}$$

Great!

Definition VIII.2.3

An orthonormal set $\{e_i\}$ is said to be an orthonormal basis of V provided that $\overline{W} = V$, where

$$W = \left\{ \sum_{i=1}^N \beta_i e_i \mid N \in \mathbb{N}, \beta_1, \dots, \beta_N \in \mathbb{C} \right\}$$

is the subspace of finite linear combinations. In other words, for all $x \in V$ and for every $\varepsilon > 0$, there exists $w \in W$ such that $\|x - w\| < \varepsilon$.

Example VIII.2.1

For $\mathbb{C}^d$, the orthonormal basis is $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ for $i = 1, \dots, d$

For ℓ^2 the orthonormal basis is the countably many $e_i = (0, \dots, 0, 1, 0, \dots)$ for $i \in \mathbb{N}$.

Definition VIII.2.4 (Hilbert Space)

A Hilbert space is a complete inner product space (a Banach space with an inner product).

Example VIII.2.2

$\mathbb{R}^d, \mathbb{C}^d, L^2(X, \mathcal{A}, \mu), \ell^2$ are Hilbert spaces.

$C([0,1]) \subseteq L^2(X, \mathcal{A}, \mu)$ is not a Hilbert space (it is not complete). Take a function f_n so that f_n is zero from 0 to $1/2$ and 1 from $1/2 + 1/n$ to 1, connected continuously line.

Then f_n is Cauchy, but its natural limit is discontinuous.

Theorem VIII.2.4

Ket $\mathcal{H}$ be a Hilbert space. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal set. The following are equivalent

(1) $\{e_i\}_{i=1}^\infty$ is an orthonormal basis.

(2) If $x \in \mathcal{H}$ and $\langle x, e_i \rangle = 0$ for all i, then $x = 0$.

(3) If $x \in \mathcal{H}$, then $s_N := \sum_{i=1}^N \alpha_i e_i \to x$ strongly where $\alpha_i = \langle x, e_i \rangle$.

(4) If $x \in \mathcal{H}$, then $\|x\|^2 = \sum_{i=1}^\infty |\alpha_i|^2$ (Plancherel identity).

Proof. Let's go!

(3) $\implies$ (4) We have by Lemma VIII.2.3 that

$$\|x\|^2 = \|x - s_N\|^2 + \sum_{i=1}^N |\alpha_i|^2 .$$

Taking $N \to \infty$ and noting $s_N \to x$ strongly gives

$$\|x\|^2 = \lim_{N\to\infty} \sum_{i=1}^N |\alpha_i|^2 = \sum_{i=1}^\infty |\alpha_i|^2 .$$

(4) $\implies$ (1) Using the same equality

$$\|x\|^2 = \|x - s_N\|^2 + \sum_{i=1}^N |\alpha_i|^2 .$$

and taking $N \to \infty$ yields $\|x - s_N\|^2 \to 0$ so $\|x - s_N\| \to 0$. Therefore $s_N \to x$ strongly, yielding that x can be approximated by finite linear combinations as desired.

(1) $\implies$ (2) Fix $x \in \mathcal{H}$, and fix $\varepsilon > 0$. Then by (1), there exists a $y = \sum_{i=1}^k \beta_i e_i$ such that $\|x - y\| < \varepsilon$.

By the best approximation lemma (see Lemma VIII.2.2), $\|x - s_k\| \le \|x - y\| < \varepsilon$. If $\langle x, e_i \rangle = 0$ for all i, then $s_k = 0$, so $\|x\| < \varepsilon$.

Taking $\varepsilon \to 0$ would yield $\|x\| = 0$, implying $x = 0$.

(2) $\implies$ (3) Bessel's inequality gives $\sum_{i=1}^\infty |\alpha_i|^2 \le \|x\|^2 < \infty$. We now see that for $N > M$

$$\|s_N - s_M\|^2 = \left\| \sum_{i=M+1}^N \alpha_i e_i \right\|^2 = \sum_{i=M+1}^N |\alpha_i|^2 \to 0$$

as $N > M \to \infty$, by convergence of the series. This implies that $\{s_N\}_{N=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$.

Since $\mathcal{H}$ is complete, there is a vector y such that $s_N \to y$ strongly. Question is, is $y = x$?

Fix $i \in \mathbb{N}$, consider $\langle y - x, e_i \rangle$. We see that

$$\langle y - x, e_i \rangle = \langle y - s_N, e_i \rangle + \langle s_N - x, e_i \rangle.$$

We can compute that for $N > i$ that

$$\langle s_N - x, e_i \rangle = \alpha_i - \langle x, e_i \rangle = 0.$$

Therefore $\langle y - x, e_i \rangle = \langle y - s_N, e_i \rangle$. Because strong convergence implies weak convergence, taking $N \to \infty$ yields that $\langle y - x, e_i \rangle = 0$ for all $i \in \mathbb{N}$.

Therefore by the assumption of (2) $y - x = 0$, so $x = y$ and we're done.

Note that for everything except (2) $\implies$ (3) we did not use the Hilbert space property. When $\mathcal{H}$ is replaced by any inner product space V we only have

$$(3) \implies (4) \implies (1) \implies (2).$$

Definition VIII.2.5

A metric space is called separable if there exists a countable dense subset.

Example VIII.2.3

$\mathbb{R}^d \supseteq \mathbb{Q}^d$, ℓ^p, $1 \le p < \infty$, but not $p = \infty$. To do this consider sequences of rational numbers.

$L^p(\mathbb{R}, m)$ is separable for $1 \le p < \infty$. Take step functions with rational heights and rational endpoints to intervals.

Theorem VIII.2.5

Every separable Hilbert space has a countable orthonormal basis.

Proof. Gram-Schmidt.

Note: The cardinality of an orthonormal basis is determined by the space, and we can call this the dimension of the Hilbert space.

Announcements

- Final on 4/27 Wednesday 1:30-3:30
- Bring your phone/computer to scan/upload to Gradescope
- Content: Up to Lecture 36, i.e. all but Hilbert spaces & Fourier Analysis

Parseval's Identity says that if e_i is an orthonormal basis then

$$\langle x, y \rangle = \sum_{i=1}^{\infty} \alpha_n \overline{\beta_n}$$

where $\alpha_n = \langle x, e_n \rangle, \beta_n = \langle y, e_n \rangle$.

IX. Intro to Fourier Analysis

IX.1. Fourier Series

We will be considering the Hilbert space $L^2([-\pi, \pi])$ (which by scaling is equivalent to any finite interval).

Lemma IX.1.1

The set

$$\left\{e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx} = \frac{1}{\sqrt{2\pi}}(\cos(nx) + i\sin(nx))\right\}$$

is an orthonormal set in $L^2([-\pi, \pi])$.

Proof. We must evaluate

$$\langle e_n, e_m\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,\mathrm{d}x = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Question: Is $\{e_n\}$ an orthonormal basis?

By Hölder's Inequality on $L^2([-\pi, \pi])$

$$\|f\|_1 \leq \|1\|_2\|f\|_2 = \sqrt{2\pi}\cdot\|f\|_2 < \infty.$$

Likewise

$$\|f\|_2 = \sqrt{\int_{-\pi}^{\pi} |f(t)|^2\,\mathrm{d}t} \leq \sqrt{\|f\|_\infty^2 2\pi}.$$

Therefore

$$\|f\|_1 \leq \sqrt{2\pi}\|f\|_2 \leq 2\pi\|f\|_\infty.$$

Definition IX.1.1

For $f \in L^1([-\pi, \pi])$, its Fourier coefficients are

$$\hat{f}_n := \langle f, e_n\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(y)e^{-iny}\,\mathrm{d}y.$$

To show these form an orthonormal basis we must show that

$$\sum_{n=-M}^{N} \hat{f}_n e_n(x)$$

converges strongly to $f(x)$ as $M, N \to \infty$. Explicitly this is

$$\begin{aligned}\sum_{n=-m}^{N} \hat{f}_n e_n(x) &= \frac{1}{\sqrt{2\pi}}\sum_{n=-M}^{N}\left[\int_{-\pi}^{\pi} f(y)e^{-iny}\,\mathrm{d}y\right]e^{inx} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(y)\left(\sum_{n=-M}^{N} e^{in(x-y)}\right)\mathrm{d}y.\end{aligned}$$

Definition IX.1.2

For $0 \le r < 1$, the Poisson kernel is

$$P_r(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{int} r^{|n|}$$

Explicitly, doing the sum as a geometric series

$$P_r(t) = \frac{1 - r^2}{2\pi(1 - 2r\cos t + r^2)}$$

Lemma IX.1.2

For $f \in L^1([-\pi,\pi])$ and $0 \le r < 1$, $\sum_{n=-\infty}^{\infty} \hat{f}_n e_n(x) r^{|n|}$. This converges absolutely and uniformly for $x \in [-\pi,\pi]$ to $\int_{-\pi} P_r(x-y) f(y)\,\mathrm{d}y$.

Proof. We have that

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}_n e_n(x) r^{|n|}\right| \le \frac{1}{2\pi} \sum_{n=-\infty}^{m=\infty} \left(\int_{-\pi}^{\pi} \left|f(y)e^{-iny}\right| \mathrm{d}y\right) |e_n(x)|\, r^{|n|} = \frac{\|f\|_1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} < \infty.$$

Therefore Fubini's Theorem applies, and

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-\pi}^{\pi} f(y) e^{-iny}\,\mathrm{d}y\,\mathrm{d}y\right) e^{inx} r^{|n|} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(y) \sum_{n=-\infty} e^{in(x-y)} r^{|n|}\right) = \int_{-\pi}^{\pi} P_r(x-y) f(y)\,\mathrm{d}y.$$

Uniform convergence takes a small bit more work.

Note that $P_r(0) = \frac{1-r^2}{2\pi(1-r)^2} = \frac{1+r}{2\pi(1-r)} \to \infty$ as $r \to 1$. For any $t \ne 0$, we have

$$\frac{1 - r^2}{2\pi(1 - 2r\cos t + r^2)} \to 0$$

as the bottom is always nonzero and finite.

Lemma IX.1.3

$P_r(t)$ form a "family of good kernels", i.e.

(1) $P_r(t) \ge 0$

(2) $\int_{-\pi}^{\pi} P_r(t)\,\mathrm{d}t = 1$.

(3) For every $\delta > 0$,

$$\int_{[-\pi,\pi]\setminus[-\delta,\delta]} P_r(t)\,\mathrm{d}t$$

Proof. For (2) use the first formula with Fubini. For (1),(3) use the second formula. Namely we have

$$\int_{[-\pi,\pi]\setminus[-\delta,\delta]} P_r(t)\,\mathrm{d}t \le \frac{1 - r^2}{2\pi(1 - 2r\cos\delta + r^2)} 2\pi \to 0$$

as $r \to 1$

Lemma IX.1.4

For $f \in C([-\pi,\pi])$ satisfying $f(-\pi) = f(\pi)$, then

$$\lim_{r\to 1} \int_{-\pi}^{\pi} P_r(x-y) f(y)\,\mathrm{d}y = f(x)$$

uniformly for $x \in [-\pi,\pi]$.

Proof. Extend f to $f : \mathbb{R} \to \mathbb{C}$ setting $f(x+2\pi) = f(x)$, then f is uniformly continuous and bounded.

$$\begin{aligned}\int_{-\pi}^{\pi} P_r(x-y) f(y)\,\mathrm{d}y - f(x) &= \int_{-\pi}^{\pi} P_r(y) f(x-y)\,\mathrm{d}y - f(x)\\ &= \int_{-\pi}^{\pi} P_r(y) f(x-y)\,\mathrm{d}y - f(x)\int_{-\pi}^{\pi} P_r(y)\,\mathrm{d}y\\ &= \int_{-\delta}^{\delta} P_r(y)(f(x-y)-f(x))\,\mathrm{d}y + \int_{[-\pi,\pi]\setminus\,[-\delta,\delta]} P_r(y)(f(x-y)-f(x))\,\mathrm{d}y.\end{aligned}$$

Fix $\varepsilon > 0$, then f is uniformly continuous, so we can choose a $\delta > 0$ so that $|f(x-y) - f(x)| < \varepsilon$ for any choice of x. Then the left hand term is bounded by ε. For the right hand side, note that f is bounded, so for some M

$$\left|\int_{[-\pi,\pi]\setminus\,[-\delta,\delta]} P_r(y)(f(x-y)-f(x))\right|\mathrm{d}y \le M\int_{[-\pi,\pi]\setminus\,[-\delta,\delta]} P_r(y)\,\mathrm{d}y$$

Therefore, sending $r \to 1$ will send

$$\int_{-\pi}^{\pi} P_r(x-y) f(y)\,\mathrm{d}y - f(x) \to 0.$$

Recall IX.1.1

We have from last time that for $0 \le r < 1$

$$\begin{aligned} e_n(x) &:= \frac{1}{\sqrt{2\pi}} e^{inx}\\ P_r(t) &:= \sum_{n=-\infty}^{\infty} e_n(t) r^{|n|}\quad f \in L^1([-\pi,\pi]) \qquad \Longrightarrow \sum_{n=-\infty}^{\infty} \hat{f}_n e_n(x) r^{|n|} = \int_{-\pi}^{\pi} P_r(x-y) f(y)\,\mathrm{d}y\end{aligned}$$

uniformly in $x \in [-\pi,\pi]$. Furthermore, if $f \in C([-\pi,\pi])$ and $f(\pi) = f(-\pi)$ then

$$\lim_{r\to 1} \int_{-\pi}^{\pi} P_r(x-y) f(y)\,\mathrm{d}y = f(x)$$

uniformly in $x \in [-\pi,\pi]$.

Finally we also have

$$\|f\|_1 \le \sqrt{2\pi}\|f\|_2 \le 2\pi\|f\|_\infty$$

Theorem IX.1.5 (Fourier Series)

The set of $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ for $n \in \mathbb{Z}$ is an orthonormal basis of $L^2([-\pi,\pi])$.

We have from previous work that they are an orthonormal set. Thus we just need to show that for all $f \in L^2([-\pi,\pi])$ that there exists $h = \sum_{n=-M}^{N} \beta_n e_n(x)$ such that $\|f-h\|_2 < \varepsilon$. That is we need to show that the span of the e_n is dense.

Proof. Let $f \in L^2([-\pi,\pi])$. Fix $\varepsilon > 0$. Then there is a function $g \in C([-\pi,\pi])$ with $g(\pi) = g(-\pi)$ such that $\|f-g\|_2 < \frac{\varepsilon}{3}$. Why? Simple density argument.

Let $g_r(x) = \int_{-\pi}^{\pi} P_r(x-y) g(y)\,\mathrm{d}y$. By the above, there exists an $r \in [0,1)$ such that $\|g_r - g\|_\infty < \frac{\varepsilon}{3\sqrt{2\pi}}$.

Therefore $\|g_r - g\|_2 < \frac{\varepsilon}{3}$. Consider $g_{r,N}(x) = \sum_{n=-N}^{N} \hat{g}_n e_n(x) r^{|n|}$. By the above there exists an $N \in \mathbb{N}$ such that $\|g_{r,N} - g_r\|_\infty < \frac{\varepsilon}{3\sqrt{2\pi}}$, thus $\|g_{r,N} - g_r\|_2 < \frac{\varepsilon}{3}$. Therefore

$$\|f - g_{r,N}\| < \varepsilon$$

$g_{r,N}$ is a finite linear combination of e_n's, so these form an orthonormal basis as desired.

Example IX.1.2

Plancherel identity $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left|\hat{f}_n\right|^2$.

For $f(x) = x$, we have

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-inx}\,\mathrm{d}x = \begin{cases} 0 & \text{if } n = 0 \\ \frac{(-1)^n i \sqrt{2\pi}}{n} & \text{if } n \neq 0 \end{cases}.$$

Together these imply that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example IX.1.3

Isoperimetric inequality. Consider a parametric curve $(x(t), y(t)) \in \mathbb{R}^2$ with $t \in [-\pi,\pi]$. Assume that

(1) This is a closed curve, so $(x(-\pi), y(-\pi)) = (x(\pi), y(\pi))$.
(2) We assume these are smooth, but in fact we just need x, y are C^1 functions.
(3) The curve is simple.

Suppose that

$$L = \int_{-\pi}^{\pi} \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t = 2\pi.$$

What is the largest area A enclosed?

By Green's Theorem

$$\begin{aligned} A &= \frac{1}{2} \oint_C (x\,\mathrm{d}y - y\,\mathrm{d}x) \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (x(t)y'(t) - x'(t)y(t))\,\mathrm{d}t. \end{aligned}$$

Arc length parametrization, so that $x'(t)^2 + y'(t)^2 = 1$ for all t. Then the condition $L = 2\pi$ is automatically satisified and can be written as

$$\int_{-\pi}^{\pi} (x'(t)^2 + y'(t)^2)\,\mathrm{d}t = 2\pi.$$

For ease of computation, rewrite using complex numbers, i.e., $z(t) = x(t) + iy(t)$. This is then subject to

$$\int_{-\pi}^{\pi} |z'(t)|^2 \, \mathrm{d}t = 2\pi i.$$

Rewriting our above formula for area, we need to find the maximum of

$$A = \frac{1}{4i} \int_{-\pi}^{\pi} (\overline{z(t)} z'(t) - z(t)\overline{z'(t)}) \, \mathrm{d}t.$$

Note z is C^1 and $z(\pi) = z(-\pi)$. Denote $\hat{z}_n = \alpha_n$. Now

$$\begin{aligned} \widehat{z'}_n &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} z'(t) e^{-int} \, \mathrm{d}t \\ &= \left(z(t)e^{-int}\right]_{\pi}^{\pi} + in \int_{-\pi}^{\pi} z(t) e^{-int} \, \mathrm{d}t \\ &= in\alpha_n. \end{aligned}$$

By Plancherel's identity, we have that

$$\int_{-\pi}^{\pi} |z'(t)|^2 \, \mathrm{d}t = \sum_{n=-\infty}^{\infty} |in\alpha_n|^2 = \sum_{n=-\infty}^{\infty} n^2 |\alpha_n|^2 = 2\pi.$$

We know Parseval's identity says $\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \hat{f}_n \overline{\hat{g}_n}$.

Therefore

$$A = \frac{1}{4i} \sum_{n=-\infty} \infty \overline{\alpha_n}(in\alpha_n) - \alpha_n \overline{(in\alpha_n)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} n |\alpha_n|^2.$$

Great! Then the question becomes what is the maximum of $\frac{1}{2}\sum_{n=-\infty}^{\infty} n |\alpha_n|^2$ subject to $\sum_{n=-\infty}^{\infty} n^2 |\alpha_n|^2 = 2\pi$.

We will show that

$$2\pi - \sum_{-\infty}^{\infty} n |\alpha_n|^2 \geq 0$$

as we guess the maximum should be π, by virtue of the circle being theoretically optimal. Then we have

$$2\pi - \sum_{-\infty}^{\infty} n |\alpha_n|^2 = \sum_{n=-\infty}^{\infty} (n^2 - n) |\alpha_n|^2 \geq 0$$

because term by term every term is non-negative.

Thus the area cannot be more than π. We have $A = \pi$ if and only if equality holds above, that is $\alpha_n = 0$ for $n \neq 0, 1$. This means $z(t) = \alpha_0 + \alpha_1 e_1(x)$, that is

$$z(t) = \alpha_0 + Ce^{it}.$$

This means that $z(t)$ is a circle about α_0.

Professor Baik likes Dym and McKean's Fourier Series & Integrals [DM85].

References

[Axl20] Sheldon Axler. *Measure, integration & real analysis*. Springer Nature, 2020.

[DM85] H. Dym and H. P. McKean. *Fourier Series and Integrals*. Probability and Mathematical Statistics. New York: Academic Press, 1985. ISBN: 0122264509 9780122264504.

[Fol99] Gerald B Folland. *Real analysis: modern techniques and their applications*. Vol. 40. John Wiley & Sons, 1999.

[Tao11] Terence Tao. *An introduction to measure theory*. Vol. 126. American Mathematical Society Providence, RI, 2011.

Todo list