Announcements

- Get to know you Video
- HW1 Due Thursday 9pm
- Office Hours (not today)
  - M 12:30-1:30, T 1:30-2:30 in-person EH5838
  - Thursday 1-2, online

Recall: **??**

Note: For $A, B \in \mathcal{A}$, $A \subseteq B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B).$$

And thus $\mu(A) \leq \mu(B)$ and $\mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$. We must always be careful with ∞ when we subtract, because $\infty - \infty$ is not well-defined.

Theorem .0.1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then we have the following properties

(1) Monotonicity: $A \subseteq B \in \mathcal{A} \implies \mu(A) \leq \mu(B)$.

(2) Countable subadditivity: If $A_1, A_2, \ldots \in \mathcal{A}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

(3) Continuity from below / Montone Convergence Theorem (MCT) for sets: Given $A_1, A_2, \ldots \in \mathcal{A}$ satisfying $A_1 \subseteq A_2 \subseteq \cdots$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$$

(4) Continuity from above: Given $A_1, A_2, \ldots \in \mathcal{A}$satisfying $A_1 \supseteq A_2 \supseteq \cdots$ and $\mu(A_1) < \infty$ then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$$

Proof. (1) and (2) DIY.

For part (3), let $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$. Then we know that

$$\bigcup_{i=1}^{\infty} A_i = \bigsqcup_{i=1}^{\infty} B_i$$

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) \\ &= \lim_{n\to\infty} \mu\left(\bigsqcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

For part (4), let $E_i = A_1 \setminus A_i$. Then $E_1 \subseteq E_2 \subseteq \cdots$. Then

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_1 \setminus A_i = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right)$$

Now note that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty.$$

Therefore we have that

$$\begin{aligned}
\bigcap_{i=1}^{\infty} A_i &= A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) \\
\mu\left(\bigcap_{i=1}^{\infty} A_i\right) &= \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \\
&= \mu(A_1) - \lim_{n\to\infty} \mu(E_n) \\
&= \mu(A_1) - \lim_{n\to\infty} \mu(A_1) - \mu(A_n) \\
&= \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

Example .0.1

TAke $\mathbb{N}, \mathcal{P}(\mathbb{N})$ with the counting measure. Then let $A_n = \{n, n+1, n+2, \ldots\}$. Then note that $A_1 \supseteq A_2 \supseteq \cdots$ and

$$\bigcap_{i=1}^{\infty} A_i = \emptyset \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$$

But $\mu(A_n) = \infty$ for each n. This shows that finiteness is necessary for part (4).

Definition .0.1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

- $A \subseteq X$ is a <u>μ-null</u> set if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subseteq X$ is a <u>μ-subnull</u> set if there exists a μ-null set B with $A \subseteq B$. Note: A is not necessarily $\mathcal{A}$-measurable.
- $(X, \mathcal{A}, \mu)$ is a <u>complete</u> measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Definition .0.2 (Almost everywhere)

A statement $P(x)$ quantified over $x \in X$, holds <u>μ-a.e. (almost everywhere)</u> if the set $\{x \in X \mid P(x) \text{ does not hold}\}$ is μ-null.

Definition .0.3

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

- μ is a <u>finite measure</u> if $\mu(X) < \infty$.
- μ is a <u>σ-finite measure</u> if $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in \mathcal{A}$ and $\mu(X_n) < \infty$.

HW: Every measure space can be "completed" by expanding the relevant σ-algebra and expanding the definition of the measure.

.1. Building Measures

Definition .1.1 (Outer measure)

An outer measure on X is $\mu^* : P(X) \to [0, \infty]$ such that

- $\mu^*(\emptyset) = 0$
- Monotonicity: If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$.
- Countable subadditivity: That is

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

For every $A_1, A_2, \ldots \subseteq X$.

Example .1.1

For $A \subseteq \mathbb{R}$,

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) \mid \bigcup_{i=1}^{\infty}(a_i, b_i) \supseteq A\right\}$$

is an outer measure due to Proposition .1.1 by taking $\mathcal{E} = \{(a, b) \mid -\infty \leq a \leq b \leq \infty\}$ and $\rho((a, b)) = b - a$.

This is called the Lebesgue outer measure on $\mathbb{R}$.

Proposition .1.1

Let $\mathcal{E} \subseteq P(X)$ such that $\emptyset, X \in \mathcal{E}$. Then let $\rho : \mathcal{E} \to [0, \infty]$ such that $\rho(\emptyset) = 0$.

Then

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \mid E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supseteq A\right\}$$

is an outer measure on X. Note! It may fail that μ^* may not be ρ when restricted to $\mathcal{E}$. We need more conditions to guarantee that!

The proof of this proposition will introduce two very important tricks that we will use over and over.

Proof of Proposition .1.1: The easy parts. We will not have time to do the proof today, but we will sketch out the easy steps

(1) μ^* is well-defined: This is easy, since inf is taken over a non-empty set bounded below by zero.

(2) $\mu^*(\emptyset) = 0$. Just take all the $E_i = \emptyset$ to get a minimum

(3) $A \subseteq B$ implies $\mu^*(A) \leq \mu^*(B)$ because every cover of B by elements of $\mathcal{E}$ also covers A.

Next class: we will prove countable subadditivity.