Lemma .0.1

Let μ, ν be finite positive measures on $(X, \mathcal{A})$. Then either

(1) $\nu \perp \mu$.

(2) There exists an $\varepsilon > 0$, an $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for the measure $\nu - \varepsilon\mu$.

I.e., for all $G \subseteq F$, $\nu(G) \geq \varepsilon\mu(G)$.

Proof. Let $\kappa_n = \nu - (1/n)\mu$. By gthm:hahn-decomposition we have$X = P_n \cup N_n$ for P_n positive for κ_n, N_n negative for K_n.

Let $P = \bigcup_n P_n, N = \bigcap_n N_n = X \setminus P$. Then $X = P \cup N$.

We see that for any N we have $\kappa_n(N) \leq 0$ because $N \subseteq N_n$. Thus

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N).$$

This implies $\nu(N) = 0$. Because ν is positive for any $N' \subseteq N$ we have $0 \leq \nu(N') \leq \nu(N)$, and thus $\nu(N') = 0$. This shows N is null for N.

If $\mu(P) = 0$, then $\nu \perp \mu$.

If $\mu(P) \neq 0$, then we have $\mu(P_n) > 0$ for some n.

With $F = P_n$, $\varepsilon = 1/n$, then F is a positive set for $\kappa_n = \nu - (1/n)\mu$ as desired.

Theorem .0.2 (Lebesgue-Radon-Nikodym)

Let μ be a σ-finite positive measure, ν a σ-finite signed measure on $(X, \mathcal{A})$.

Then there are unique λ, ρ σ-finite signed measures on $(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$, $\nu = \lambda + \rho$.

Furthermore, there exists a measurable function $f : X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\rho = f\,\mathrm{d}\mu$ (that is for all $E \in \mathcal{A}$, $\rho(E) = \int_E f\,\mathrm{d}\mu$).

And if there is another g such that $\mathrm{d}\rho = g\,\mathrm{d}\mu$, then $f = g$, μ-a.e.

Notationally we may write $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}\mu$, where $\mathrm{d}\lambda$ and $\mathrm{d}\mu$ are singular to each other.

Proof. Lets go!

(a) Assume μ, ν are finite positive measures. Let

$$\begin{aligned}\mathscr{F} &= \left\{g : X \to [0, \infty] \mid \int_E g\,\mathrm{d}\mu \leq \nu(E), \forall E \in \mathcal{A}\right\} \\ &= \{g : X \to [0, \infty] \mid \mathrm{d}\nu - g\,\mathrm{d}\mu \text{ is a positive measure}\}.\end{aligned}$$

This set is nonempty since $g = 0 \in \mathscr{F}$. Let $s = \sup\{\int_X g\,\mathrm{d}\mu \mid g \in \mathscr{F}\}$.

Claim

There is an $f \in \mathscr{F}$ such that $s = \int_X f\,\mathrm{d}\mu$.

If $g, h \in \mathscr{F}$, we can define $u(x) = \max\{g(x), h(x)\}$. Then $u \in \mathscr{F}$. Why? Well let $A = \{x \mid g(x) \geq h(x)\}$. Then

$$\int_E u\,\mathrm{d}\mu = \int_{E \cap A} g\,\mathrm{d}\mu + \int_{E \cap A^c} h\,\mathrm{d}\mu$$

$$\leq \nu(E \cap A) + \nu(E \cap A^c) = \nu(E).$$

There exist measurable functions $g_1, g_2, \ldots \in \mathscr{F}$ such that $\lim_{n\to\infty} \int_X g_n \, \mathrm{d}\mu = s$. We can replace g_2 by $\max(g_1, g_2)$, g_3 by $\max(g_1, g_2, g_3)$, so that we may assume $0 \leq g_1 \leq g_2 \leq \cdots$.

Then we still know that $\lim_{n\to\infty} \int_X g_n \, \mathrm{d}\mu = s$, as all the relevant integrals are bounded above by s. Now let $f(x) = \sup_n g_n(x) = \lim_{n\to\infty} g_n(x)$. By Monotone convergence theorem,

$$\int_E f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_E g_n \, \mathrm{d}\mu \leq \nu(E)$$

Thus $f \in \mathscr{F}$. When $E = X$ we get $\int_X f \, \mathrm{d}\mu = s$ as desired.

Great! Let $\rho(E) \coloneqq \int_E f \, \mathrm{d}\mu$. We of course have $\rho \ll \mu$. And also we know

$$0 \leq \rho(X) = \int_X f \, \mathrm{d}\mu \leq \nu(X) < \infty.$$

Thus ρ is a finite positive measure. We can define $\lambda(E) \coloneqq \nu(E) - \rho(E)$. Then

$$\lambda(E) = \nu(E) - \int_E f \, \mathrm{d}\mu \geq 0$$

because $f \in \mathscr{F}$. Thus λ is also a positive measure, and $\lambda(X) \leq \nu(X) < \infty$. It remains to show the following.

Claim

$\lambda \perp \mu$

Suppose not, by glemma:finite-singular, there exists $\varepsilon > 0$, $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for $\lambda - \varepsilon\mu$.

Then this says that $\mathrm{d}\lambda - \varepsilon 1_F \, \mathrm{d}\mu$ is a positive measure, that is $\mathrm{d}\nu - f \, \mathrm{d}\mu - \varepsilon 1_F \, \mathrm{d}\mu$ is a positive measure. This will break maximality of f.

Explicitly, let $g(x) = f(x) + \varepsilon 1_F(x)$. Then for all $E \in \mathcal{A}$ we have

$$\begin{aligned}
\int_E g \, \mathrm{d}\mu &= \int_E f \, \mathrm{d}\mu + \varepsilon\mu(E \cap F) \\
&= \nu(E) - \lambda(E) + \varepsilon\mu(E \cap F) \\
&\leq \nu(E) - \lambda(E \cap F) + \varepsilon\mu(E \cap F) \leq \nu(E)
\end{aligned}$$

since $\lambda(E \cap F) - \varepsilon\mu(E \cap F) \geq 0$. Thus $g \in \mathscr{F}$. We then see that

$$\begin{aligned}
s \geq \int_X g \, \mathrm{d}\mu &= \int_X f \, \mathrm{d}\mu + \int_X \varepsilon 1_F \, \mathrm{d}\mu \\
&= s + \varepsilon\mu(F) > s.
\end{aligned}$$

This is a contradiction! Perfect!

There are now technical things, such as extending to σ-finite measures and uniqueness. These are relatively easy compared to this part.