Lemma .0.1

Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle$ for all $\alpha \in I$.

(a) $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b) We have

$$\bigotimes_\alpha \mathcal{A}_\alpha = \left\langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle.$$

(c) If I is countable, then

$$\bigotimes_{i=1}^\infty \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^\infty E_i \mid E_i \in \mathcal{E}_i \right\} \right\rangle$$

Proof. DIY (b), (c), as they should not be difficult from (a).

Let's prove part (a). In general, if $f : Y \to Z$ and $\mathcal{B}$ is a σ-algebra on Z. Then $f^{-1}(\mathcal{B})$ is a σ-algebra Why is that?

- $f^{-1}(\emptyset) = \emptyset$.
- $f^{-1}(B)^c = f^{-1}(B^c)$
- $\bigcup_n f^{-1}(B_n) = f^{-1}\left(\bigcup_n B_n\right)$.

Hence $\pi_\alpha^{-1}(\mathcal{A}_\alpha)$ is a σ-algebra on X. Furthermore it is clear that $\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subseteq \pi_\alpha^{-1}(\mathcal{A}_\alpha)$. Therefore $\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subseteq \pi_\alpha^{-1}(\mathcal{A}_\alpha)$.

Now we prove the other direction. Consider

$$\mathcal{M} \coloneqq \{B \subseteq X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha \rangle\}.$$

We must then simply prove that $\mathcal{A}_\alpha \subseteq \mathcal{M}$. Since $\langle \mathcal{E}_\alpha \rangle = \mathcal{A}_\alpha$, it suffices to show that

- $\mathcal{M}$ is a σ-algebra. This is easy, because we're taking preimages with set operations.
- $\mathcal{E}_\alpha \subseteq \mathcal{M}$. This is trivial by definition of $\mathcal{M}$.

Thus $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle \subseteq \mathcal{M}$.

Thus if $E \in \mathcal{A}_\alpha$, then $E \in M$, so $\pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$. Therefore $\pi_\alpha^{-1}(\mathcal{A}_\alpha) \subseteq \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

Theorem .0.2

Suppose $X_1, \ldots, X_d$ are metric spaces. Let $X = \prod_{i=1}^\infty X_i$ with product metric. For concreteness say $\rho(x, y) = \sum_i \rho_i(x_i, y_i)$ where ρ_i is the metric on X_i.

Then,

(a) We have that

$$\bigotimes_{i=1}^d \mathcal{B}(X_i) \subseteq \mathcal{B}(X).$$

(b) If, in addition, each X_i has a countable dense subset, then

$$\bigotimes_{i=1}^d \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof. DIY (see Homework 1 while doing part (b)).

Example .0.1

We have that

$$\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}).$$

Consider $f = u + iv : X \to \mathbb{C}$, with $\mathcal{A}$ a σ-algebra on X. Then $u^{-1}(E), v^{-1}(E) \in \mathcal{A}$ for all $E \in \mathcal{B}(\mathbb{R})$ if and only if $f^{-1}(F) \in A$ for all F in $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

All of this so far was pages 22-23. Now we pick back up at page 65 of [Fol99].

Definition .0.1

Let X, Y are sets. Now

(a) For $E \subseteq X \times Y$, $E_x = \{y \in Y \mid (x, y) \in E$, $E^y = \{x \in X \mid (x, y) \in E\}$.

(b) For $f : X \times Y \to Z$, define $f_x : Y \to Z$ and $f^y : X \to Z$ by $f_x(y) = f(x, y) = f^y(x)$.

Example .0.2

$(1_E)_x = 1_{E_x}$, similarly $(1_E)^y = 1_{E^y}$.

Proposition .0.3

Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces. Then

(a) If $E \in \mathcal{A} \otimes \mathcal{B}$, then $E_x \in \mathcal{B}$, $E^y \in \mathcal{A}$ for all $x \in X, y \in Y$.

(b) If $f : X \times Y \to Z$ is measurable $(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$ for some measurable space $(Z, \mathcal{C})$. Then f_x is $(\mathcal{B}, \mathcal{C})$ measurable, f^y is $(\mathcal{A}, \mathcal{C}$-measurable for all $x \in X, y \in Y$.

Proof. (b) follows from (a). We prove (a). Let

$$\mathcal{F} = \{E \subseteq X \times Y \mid \forall\, x \in X, y \in Y\ E_x \in \mathcal{B}, E^y \in \mathcal{A}\}.$$

Then

- $\mathcal{F}$ is a σ-algebra. This works because $(E^c)_x = E_x^c$, and similar statements hold for unions.
- We recall that

$$\mathcal{R}_0 \coloneqq \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}\mathcal{A} \otimes \mathcal{B} \qquad = \langle \mathcal{R}_0 \rangle.$$

It is not difficult to show $\mathcal{R}_0 \subseteq \mathcal{F}$.

Then $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{F}$, and we're done with part (a).

Midterm may use things up to this point

.1. Product Measures

Definition .1.1

Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces. A (measurable) rectangle is $R = A \times B$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let $\mathcal{R}_0$ be the set of all (measurable) rectangles. Then let

$$\mathcal{R} = \left\{ \bigcup_{i=1}^{N} R_i \mid N \in \mathbb{N}, R_1, \ldots, R_N \text{ are disjoint rectangles} \right\}$$

Lemma .1.1

$\mathcal{R}$ is an algebra, and $\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$.

Proof. DIY, noting that $(A \times B)^c = (A^c \times Y) \sqcup (A \times B^c)$.