Proof of **??**. Let

$$G(x) \coloneqq \begin{cases} F(a) & \text{if } x < a \\ F(x) & \text{if } a \le x \le b \\ F(b) & \text{if } x > b \end{cases}$$

Then G is increasing. Define

$$g_n(x) = \frac{G(x+1/n) - G(x)}{1/n} \to F'(x)$$

for almost every $x \in [a,b]$. Also $g_n(x) \ge 0$.

Fatou's Lemma tells us that

$$\int_a^b F'(x)\,\mathrm{d}x = \int_a^b \liminf_{n\to\infty} g_n(x)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_a^b g_n(x)\,\mathrm{d}x.$$

We then evaluate

$$\begin{aligned} \int_a^b g_n(x) &= n\left(\int_{a+1/n}^{b+1/n} G(x)\,\mathrm{d}x - \int_a^b G(x)\,\mathrm{d}x\right) \\ &= n\left(\int_b^{b+1/n} G(x)\,\mathrm{d}x - \int_a^{a+1/n} G(x)\,\mathrm{d}x\right) \\ &\le n\left(G\left(b + \frac{1}{n}\right)\cdot\frac{1}{n} - G(a)\cdot\frac{1}{n}\right) \\ &= F(b) - F(a). \end{aligned}$$

Therefore

$$\int_a^b F'(x)\,\mathrm{d}x \le F(b) - F(a).$$

.1. Functions of bounded variation

Definition .1.1

For $F : \mathbb{R} \to \mathbb{R}$, the total variation function of F is $T_F : \mathbb{R} \to [0,\infty]$ defined by

$$T_F(x) = \sup\left\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, -\infty < x_0 < x_1 < \cdots < x_n = x\right\}$$

Lemma .1.1

We have that

$$T_F(b) = T_F(a) + \sup\left\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\right\}$$

Proof. DIY

Note: T_F is increasing

Definition .1.2

We say that $F \in BV$ (F is of bounded variation) provided that

$$T_F(\infty) = \lim_{x \to \infty} T_F(x) < \infty.$$

Similarly $F \in BV([a, b])$ means that

$$\sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b \right\} < \infty.$$

Example .1.1

Note that if F is of bounded variation, then F is bounded.

Note that $F(x) = \sin x$ is not of bounded variation. But it is of bounded variation over any $[a, b]$.

Also

$$F(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is not of bounded variation of $[a, b]$ if $a < 0 < b$ because the harmonic series does not converge.

(1) If F, G are of bounded variation, $\alpha F + \beta G$ are of bounded variation.

(2) If F is increasing and bounded, then F is a function of bounded variation.

(3) If F is Lipschitz (see Definition .1.3) on $[a, b]$, then $F \in BV([a, b])$.

(4) If F is differentiable, and F' is bounded on $[a, b]$, then F is Lipschitz (mean value theorem), so it is in $BV([a, b])$.

(5) If $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ for $f \in L^1(\mathbb{R})$. Then $F \in BV$.

Namely

$$\begin{aligned} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)\,\mathrm{d}t \right| \\ &\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,\mathrm{d}t \\ &= \int_{x_0}^{x_n} |f(t)|\,\mathrm{d}t \\ &\leq \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t < \infty. \end{aligned}$$

Definition .1.3

A function $F : [a, b] \to \mathbb{C}$ is called Lipschitz provided that there exists an $M \geq 0$ such that $|F(x) - F(y)| \leq M |x - y|$.

Lemma .1.2

If $F \in BV$, then T_F is bounded, increasing, $T_F(-\infty) = 0$.

Lemma .1.3

$F \in BV$, then $T_F + F$ is increasing/bounded and $T_F - F$ is increasing/bounded. Thus any $F \in BV$

can be written as

$$F = \frac{T_F + F}{2} - \frac{T_F - F}{2}$$

which is a difference of increasing/bounded functions.

Proof. Let $x < y$. Fix $\varepsilon > 0$, then there are points $x_0 < x_1 < \cdots < x_n = x$ such that

$$T_F(x) \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \varepsilon.$$

Furthermore

$$T_F(y) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)|.$$

Then

$$\begin{aligned} \pm(F(y) - F(x)) &\leq |F(y) - F(x)| \\ T_F(y) \pm (F(y) - F(x)) &\geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq T_F(x) - \varepsilon. \\ T_F(Y) \pm F(y) &\geq T_F(x) \pm F(x) - \varepsilon. \end{aligned}$$

Taking $\varepsilon \to 0$ yields the result.

Theorem .1.4

F is of bounded variation if and only if $F = F_1 - F_2$ for F_1, F_2 increasing and bounded.

Proof. The forward implication is given by the previous lemma. The other direction follows from the examples we gave (check!)

Corollary .1.5 (Bounded Variation Differentiation)

$F \in BV$ implies that F is differentiable almost everywhere. Furthermore,

(1) $F(x^+), F(x^-)$ exist for all x as do $F(-\infty), F(\infty)$.

(2) The set of discontinuities of F is countable.

(3) $G(x) = F(x^+)$ is differentiable and $G' = F'$ almost everywhere.

(4) $F' \in L^1(\mathbb{R}, m)$.

Proof. DIY.