### Announcements

- HW 10 due tomorrow
- HW 11 posted
- Final exam on 4/27 Wednesday 1:30-3:30 here (in the classroom). Cumulative, with emphasis on material covered after the midterm.

From last time we have that if $\rho \ll m$ is regular then

$$\lim_{r \to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for Lebesgue almost every $x$, where $E_r$ shrinks nicely to $x$. Likewise if $\lambda \perp m$ regular (positive measure) then

$$\lim_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} = 0.$$

for Lebesgue almost every $x$, where $E_r$ shrinks nicely to $x$.

**Theorem .0.1** (Lebesgue Differentiation Theorem for Regular measures)

Let $\nu$ be a regular Borel signed measure on $\mathbb{R}^d$. Then $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}m$, $\lambda \perp m$ by **??**.

Then for Lebesgue almost every $x$,

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every $E_r \to x$ nicely.

*Proof.* It must be checked that $\nu$ regular implies $\lambda, f\,\mathrm{d}m$ are regular (check!)

## .1. Monotone Differentiation Theorem

This is from [Fol99] section 3.5.

**Definition .1.1**

For $F : \mathbb{R} \to \mathbb{R}$ that is monotonically increasing, denote

$$F(x^+) = \lim_{y \to x^+} F(y) \qquad F(x^-) = \lim_{y \to x^-} F(y).$$

These exist and are

$$\inf_{y > x} F(y) \qquad \sup_{y < x} F(y).$$

So they always exist (being bounded below/above respectively by $F(x)$).

**Lemma .1.1**

If $F$ is monotonically increasing, then $D = \{x \mid F \text{ is discontinuous at } x\}$ is a countable set.

*Proof.* $x \in D$ if and only if $F(x^+) > F(x^-)$. For each $x \in D$, let $I_x = (F(x^-), F(x^+))$, not empty.

Also if $x, y \in D$, $x \neq y$, then $I_x, I_y$ are disjoint. Say if $x < y$ then

$$F(x^-) < F(x^+) \leq F(x) \leq F(y) \leq F(y^-) < F(y^+).$$

Taking a rational number in each interval gives an injective map $D \to \mathbb{Q}$, so $D$ is countable.

**Theorem .1.2** (Monotone Differentiation Theorem)

Let $F$ be increasing. Then

- $F$ is differentiable Lebesgue almost everywhere.
- $G(x) = F(x^+)$ (which is right-continuous) is differentiable almost everywhere.
- $G' = F'$ almost everywhere.

*Proof.* Start with $G$, which is increasing and right-continuous on $\mathbb{R}$. There is then a Lebesgue-Stieltjes measure $\mu_G$ on $\mathbb{R}$. Thus it is regular on $\mathbb{R}$. We see

$$\frac{G(x+h) - G(x)}{h} = \begin{cases} \frac{\mu_G((x,x+h])}{m((x,x+h])} & \text{if } h > 0 \\ \frac{\mu_G((x+h,x])}{m((x+h,x])} & \text{if } h < 0 \end{cases}$$

These both shrink nicely to $x$. By Theorem .0.1 (since these shrink nicely) we know then that these both converge for Lebesgue almost every $x$ to some common limit $f(x)$. Thus $G'$ exists Lebesgue almost everywhere.

Define $H(x) = G(x) - F(x) \geq 0$. We see that

$$\{x \mid H(x) > 0\} \subseteq \{x \mid F \text{ is discontinuous at } x\}$$

This is then countable by the lemma above, and we can write $\{x \mid H(x) > 0\} = \{x_n\}$. Then let

$$\mu \coloneqq \sum_n H(x_n)\delta_{x_n}.$$

This is a Borel measure, so we check if it is locally finite. That is we check

$$\mu((-N, N)) = \sum_{-N < x_n < N} H(x_n) \leq G(N) - F(-N) < \infty$$

checking the inequality just consists of seeing that the intervals $(F(x_n), G(x_n))$ are disjoint and a subset of $(F(-N), G(N))$ so

$$\sum_{-N<x_n<N} H(x_n) = \mu\left(\bigcup_n (F(x_n), G(x_n))\right) \leq \mu((F(-N), G(N))).$$

Thus $\mu$ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, so it is regular

**Special to $\mathbb{R}$ that locally finite Borel $\implies$ Lebesgue-Stieltjes $\implies$ regular $\implies$ outer regularity**

Then we have that

$$\left|\frac{H(x+h) - H(x)}{h}\right| \leq \frac{H(x+h) + H(x)}{|h|} \leq \frac{\mu((x - 2h, x + 2h))}{|h|}.$$

This goes to 0 for Lebesgue almost every $x$ by Theorem .0.1 and that $\mu \perp m$ (check!)

Thus $H$ is differentiable almost everywhere and $H' = 0$ almost everywhere. Thus $F$ is differentiable almost everywhere and $F' = G'$ almost everywhere.

**Proposition .1.3**

Suppose $F$ is an increasing function. Then $F'$ exists almost everywhere and is measurable. We have that

$$\int_a^b F'(x)\,\mathrm{d}x \le F(b) - F(a).$$

**Example .1.1**

Take $F(x)$ to be 0 on $x \le 0$, 1 on $x > 0$. Then $F'(x) = 0$ almost everywhere. So

$$\int_{-1}^1 F'(x)\,\mathrm{d}x = 0 < 1 = F(1) - F(-1).$$

Even if $F$ is continuous we might not have equality. Take $F(x)$ to be the cantor function. Then $F'(x) = 0$ almost everywhere, but

$$\int_0^1 F'(x)\,\mathrm{d}x = 0 < 1 = F(1) - F(0).$$