Announcements

- HW 2 due tomorrow
- HW 1 solutions (by you) – Canvas/HW 1 page
- Piazza made for the class

Definition .0.1

We call $(\mathcal{A}, \mu)$ the Hahn-Kolmogorov (HK) extension of $(\mathcal{A}_0, \mu_0)$ where $\mathcal{A}_0$ is an algebra and μ_0 is a premeasure.

Namely, we define

$$\begin{aligned}
\mu^* &: \mathcal{P}(X) \to [0, \infty] \\
\mu^*(E) &:= \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \bigcup_{i=1}^{\infty} B_i \supseteq E \right\} \\
\mathcal{A} &:= \{ A \subseteq X \mid \forall E \subseteq X, \ \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \} \\
\mu &:= \mu^* \big|_{\mathcal{A}}.
\end{aligned}$$

We have by **??** that $\mathcal{A}_0 \subseteq \mathcal{A}$ and that $\mu\big|_{\mathcal{A}_0} = \mu_0$.

Theorem .0.1 (Uniqueness of HK extension)

Let $\mathcal{A}_0$ be an algebra on X, μ_0 a pre-measure on $\mathcal{A}_0$.

Let $(\mathcal{A}, \mu)$ be the HK extension of $(\mathcal{A}_0, \mu_0)$.

Let $(\mathcal{A}', \mu')$ be some other extension of $(\mathcal{A}_0, \mu_0)$.

If μ_0 is σ-finite (recall **??**), then $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Corollary .0.2

Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite.

Then there exists a unique measure μ on $\langle \mathcal{A}_0 \rangle$ that extends μ_0.

Furthermore,

(i) the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$ **(HW)**

(ii) We have a formula for all $A \in \overline{\langle \mathcal{A} \rangle}$

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \bigcup_{i=1}^{\infty} B_i \supseteq A \right\}$$

Proof of Theorem .0.1. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show that $\mu^*(A) = \mu(A) = \mu'(A)$. Again we prove two inequalities

(a) Show $\mu^*(A) \geq \mu'(A)$ **(HW)**

(b) We will show $\mu^*(A) \leq \mu'(A)$. First

(i) Assume $\mu^*(A) < \infty$. Then fix $\varepsilon > 0$, then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup_{i=1}^{\infty} B_i \supseteq A$ so that

$$\begin{aligned}
\mu(A) + \varepsilon \geq \sum_{i=1} \mu_0(B_i) &= \sum_{i=1}^{\infty} \mu(B_i) \\
&\geq \mu(B)
\end{aligned}$$

Then since $A \subseteq B, \mu(A) < \infty$, we know that

$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon.$$

On the other hand using continuity from below

$$\begin{aligned}\mu(B) &= \lim_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) \\ &= \lim_{N\to\infty} \mu'\left(\bigcup_{i=1}^{N} B_i\right) = \mu'(B)\end{aligned}$$

Then we have by part (a) that

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \varepsilon$$

Perfect! We win by taking $\varepsilon \to 0$.

(ii) Assume $\mu(A) = \infty$. Because μ_0 is σ-finite we know $X = \bigcup_{i=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ satisfying $\mu_0(X_n) < \infty$.

Replacing X_n by $X_1 \cup \cdots \cup X_n \in \mathcal{A}_0$, we may assume $X_1 \subseteq X_2 \subseteq \cdots$.

Then note $\mu(A \cap X_n) < \infty$ so by part (i) we have

$$\mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

Now by continuity of the measure

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) \leq \lim_{n\to\infty} \mu'(A \cap X_n) = \mu'(A).$$

This finishes the proof!