Announcements

- HW 3 posted.
- Piazza.

.1. Borel measures on $\mathbb{R}$

Definition .1.1

A function $F : \mathbb{R} \to \mathbb{R}$ is an inreasing function provided that for $x \le y$ we have $F(x) \le F(y)$.

A function $F : \mathbb{R} \to \mathbb{R}$ which is increasing and right-continuous (that is $\lim_{x\to a^+} F(x) = F(a)$ for all a) is called a distribution function.

Example .1.1

These functions are distributions

- $F(x) = x$
- $F(x) = e^x$
- $F(X) = 1$ for $x \ge 0$ and $F(x) = 0$ for $x < 0$.
- Let $\mathbb{Q} = \{r_1, r_2, \dots\}$. Then

$$F_n(x) = \begin{cases} 1 & \text{if } x \ge r_n \\ 0 & \text{if } x < r_n \end{cases}$$

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}$$

F is a distribution function **(HW)**

Note: If F is increasing, we have that

$$F(\infty) := \lim_{x\to\infty} F(x) \qquad F(-\infty) = \lim_{x\to-\infty} F(x)$$

exist in $[-\infty, \infty]$.

Definition .1.2

In probability theory, cumulative distribution function is a distribution function with $F(\infty) = 1, F(-\infty) = 0$.

There are distributions [Fol99], but these are different from distribution functions.

Definition .1.3

If X is a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called locally finite if $\mu(K) < \infty$ for all compact sets $K \subseteq X$.

Lemma .1.1

Let μ be a locally finite Borel measure on $\mathbb{R}$. From this we can define

$$F_\mu(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}.$$

Definition .1.4

The h-intervals are $\emptyset, (a,b), (a,\infty), (-\infty,b], (-\infty,\infty)$.

Lemma .1.2

Let $\mathcal{H}$ be the collection of finite disjoint unions of h-intervals. Then $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. DIY

Proposition .1.3 (Distribution function defines a Pre-measure)

Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. For all the h-intervals I define $\ell(I) \coloneqq \ell_F(I)$

$$\begin{aligned}
\ell_F(\emptyset) &= 0 \\
\ell_F((a,b]) &= F(b) - F(a) \\
\ell_F((a,\infty)) &= F(\infty) - F(a) \\
\ell_F((-\infty,b)) &= F(b) - F(-\infty) \\
\ell_F((-\infty,\infty)) &= F(\infty) - F(-\infty).
\end{aligned}$$

We now define $\mu_0 \coloneqq \mu_{0,F} : \mathcal{H} \to [0,\infty]$ by

$$\mu_0(A) = \sum_{k=1}^{N} \ell_F(I_k)$$

if A may be written as a finite disjoint union $\bigcup_{k=1}^N I_k$ of h-intervals.

Then μ_0 is well-defined and a pre-measure on $\mathcal{H}$.

Proof. There are a few conditions to verify

(a) μ_0 is well-defined. This can be shown by taking a common "refinement" of two expressions $I_1, \ldots, I_N$ and $J_1, \ldots, J_M$ which both union to $A \subseteq \mathcal{H}$.

(b) $\mu_0(\emptyset) = 0$ ✓.

(c) μ_0 is finitely additive ✓.

(d) μ_0 is countably additive within $\mathcal{H}$. That is suppose $A \in \mathcal{H}$ and $A = \bigcup_{i=1}^\infty A_i$, disjoint union, $A_i \in \mathcal{H}$. These cases look something like

$$(0,1] = \bigcup_{i=1}^\infty \left(\frac{1}{i+1}, \frac{1}{i} \right].$$

It is enough to consider the case where $A = I$, $A_k = I_k$ all h-intervals. (why?)

Furthermore the statement is easy to extend to the infinite cases, so we focus on $I = (a,b]$ **(HW)**

Suppose that $(a,b] = \bigcup_{n=1}^\infty (a_n, b_n]$, disjoint. We must check that

$$F(b) - F(a) \stackrel{?}{=} \sum_{n=1}^\infty (F(b_n) - F(a_n)).$$

We know that for all N

$$\begin{aligned}
(a, b] &\supseteq \bigcup_{n=1}^{N} (a_n, b_n] \\
F(b) - F(a) &\geq \sum_{n=1}^{N} (F(b_n) - F(a_n)) \\
F(b) - F(a) &\geq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).
\end{aligned}$$

Fix $\varepsilon > 0$. Since F is right-continuous, there exists $a' > a$ such that $F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, ther eis a point $b'_n > b_n$ such that $F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n}$. We then see that

$$\begin{aligned}
[a', b] &\subseteq \bigcup_{n=1}^{\infty} (a_n, b'_n) \\
[a', b] &\subseteq \bigcup_{n=1}^{N} (a_n, b'_n) \\
(a', b] &\subseteq \bigcup_{n=1}^{N} (a_n, b'_n] \\
F(b) - F(a') &\leq \sum_{n=1}^{N} F(b'_n) - F(a_n) \\
F(b) - F(a) &\leq F(b) = F(a') + \varepsilon \\
&\leq \varepsilon + \sum_{n=1}^{\infty} F(b'_n) - F(a_n) \\
&\leq \varepsilon + \sum_{n=1}^{\infty} F(b_n) - F(a_n) + \frac{\varepsilon}{2^n} \\
&= 2\varepsilon + \sum_{n=1}^{\infty} F(b_n) - F(a_n).
\end{aligned}$$

taking $\varepsilon \to 0$ yields the result.