.1. Riemann Integrability

Definition .1.1 (Riemann Integral)

Let f be a bounded function $f : [a, b] \to \mathbb{R}$. Now fix some partition $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$. We define the upper and lower Riemann sums

$$L(f, P) = \sum_{i=1}^{k} (t_i - t_{i-1}) \cdot \inf_{[t_{i-1}, t_i]} f$$

$$U(f, P) = \sum_{i=1}^{k} (t_i - t_{i-1}) \cdot \sup_{[t_{i-1}, t_i]} f.$$

Then note that if P' is a refinement of P then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

And if P, Q are any partitions with common refinement $P \cup Q$ then

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

Thus we can define the lower/upper Riemann integrals as

$$\underline{I}(f) = \sup_P L(f, P) \qquad \overline{I}(f) = \inf_P U(f, P).$$

We say that f is Riemann integrable provided that

$$\underline{I}(f) = \overline{I}(f).$$

and we call this common value $\int_a^b f(x)\,\mathrm{d}x$ the Riemann integral.

Theorem .1.1

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then we see that

(a) If f is Riemann integrable, then f is Lebesgue measurable (and so Lebesgue integrable because f is bounded). Furthermore the two integrals agree

$$\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}m$$

(b) f is Riemann integrable if and only if f is continuous almost everywhere.

Proof. Pick partitions P_n such that $L(f, P_n)$ converges to $\underline{I}$ upwards and $U(f, P_n)$ converges to $\overline{I}$ downwards (taking refinements if needed).

Define functions for $P_n = \{a = t_0 < \cdots < t_k\}$ by

$$\phi_n = \sum_{i=1} \left(\inf_{[t_{i-1}, t_i]} f \right) 1_{(t_{i-1}, t_i]}$$

$$\psi_n = \sum_{i=1} \left(\inf_{[t_{i-1}, t_i]} f \right) 1_{(t_{i-1}, t_i]}$$

$$\phi = \sup_n \phi_n$$

$$\psi = \inf_n \psi_n.$$

THen ϕ, ψ are Lebesgue (Borel) measurable functions. Note there exists $M > 0$ such that $|f| < M1_{[a,b]}$ and so $|\phi_n|, |\psi_n| \leq M1_{[a,b]}$. Then

$$\int \phi_n \,\mathrm{d}m = L(f, P_n) \qquad \int \psi_n \,\mathrm{d}m = U(f, P_n).$$

Now by the dominated convergence theorem

$$\underline{I} = \lim_{n\to\infty} \int \phi_n \,\mathrm{d}m = \int \phi \,\mathrm{d}m$$
$$\overline{I} = \lim_{n\to\infty} \int \psi_n \,\mathrm{d}m = \int \psi \,\mathrm{d}m.$$

Thus f is Riemann integrable if and only if $\int \psi = \int \psi$ which holds if and only if $\int(\psi - \phi) = 0$ which holds if and only if $\psi = \phi$ Lebesgue almost everywhere.

Recall that $\phi \leq f \leq \psi$, so this holds if and only if $f = \phi$ almost everywhere (which implies f is Lebesgue measurable because the Lebesgue measure is complete).

This proves part (a). Part (b) follows by similar arguments.

.2. Mode of Convergence

Definition .2.1

Say that $f_n, f : X \to \mathbb{C}$ and $S \subseteq X$. We can say that

- $f_n \to f$ pointwise on S provided that for all $x \in S$, and for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|f_n(x) - f(x)| < \varepsilon$.
- $f_n \to f$ uniformly on S provided that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in S$ we have $|f_n(x) - f(x)| < \varepsilon$.

Note: We can change for every $\varepsilon > 0$ to for every $k \in \mathbb{N}$ we have $|f_n(x) - f(x)| < 1/k$ by the Archimedean principle.

Lemma .2.1

Let $B_{n,k} = \{x \in X \mid |f_n(x) - f(x)| < 1/k\}$. Then we have that

(a) $f_n \to f$ pointwise on S if and only if

$$S \subseteq \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}$$

(b) $f_n \to f$ uniformly on S if and only if there exist integers $N_1, N_2, \ldots \in \mathbb{N}$ such that

$$S \subseteq \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}$$

Definition .2.2

Let $(X, \mathcal{A}, \mu)$ be a measure space

(a) $f_n \to f$ almost everywhere provided that there is a null set E such that $f_n \to f$ pointwise on E^c.

(b) $f_n \to f$ in L^1 provided that

$$\lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \int |f_n - f| = 0.$$

Example .2.1

Consider $(\mathbb{R}, \mathcal{L}, m)$, we'll have $f = 0$.

(1) $f_n = 1_{(n,n+1)}$.
(2) $f_n = 1/n \cdot 1_{(0,n)}$.
(3) $f_n = n1_{(0,1/n)}$
(4) the typewriter functions. We define f_1 supported on $[0,1]$, f_2 supporterd on $[0,1/2]$, f_3 supported on $[1/2,1]$, f_4 supported on $[0,1/4]$, f_5 supported on $[1/4,1/2]$, f_5 supported on $[1/2,3/4]$...

Then (1)-(3) we have $f_n \to f$ pointwise, $f_n \not\to f$ in L^1.

For (4) we have $f_n \to f$ in L^1, but $f_n \not\to f$ almost everywhere. Note that (4) has a convergent subsequence to f almost everywhere.