I. Integration

I.1. Measurable functions

Definition I.1.1 (Measurable Functions)

Suppose $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measure spaces. We call $f : X \to Y$ $(\mathcal{A}, \mathcal{B})$-measurable if for all $B \in \mathcal{B}$ we have $f^{-1}(\mathcal{B}) \in \mathcal{A}$.

Lemma I.1.1

Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$ measurable if and only if for all $E \in \mathcal{E}$ we have $f^{-1}(E) \in \mathcal{A}$.

Proof. The forward direction is clear.

For the converse, let $\mathcal{D} = \{E \subseteq Y \mid f^{-1}(E) \in \mathcal{A}\}$. We see that $\mathcal{E} \subseteq \mathcal{D}$ by assumption. It is not difficult to check that $\mathcal{D}$ is a σ-algebra.

Then $\mathcal{B} = \langle \mathcal{E} \rangle \subseteq \mathcal{D}$, proving the claim.

Definition I.1.2

Let $(X, \mathcal{A})$ be a measurable space

- $f : X \to \mathbb{R}$
- $f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$
- $f : X \to \mathbb{C}$

is $\mathcal{A}$-measurable if

- f is $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable
- f is $(\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable
- $\Re f, \Im f : X \to \mathbb{R}$ are $\mathcal{A}$-measurable.

where we have $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subseteq \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$.

Example I.1.1

For $\mathcal{A} = \mathcal{P}(X)$, every function is $\mathcal{A}$-measurable. if $\mathcal{A} = \{\emptyset, X\}$, the only $\mathcal{A}$-measurable functions are the constant functions.

Lemma I.1.2

$f : X \to \mathbb{R}$, then the following are are equivalent

(a) f is $\mathcal{A}$-measurable.
(b) For all $a \in \mathbb{R}$, $f^{-1}((a, \infty)) \in \mathcal{A}$.
(c) For all $a \in \mathbb{R}$, $f^{-1}([a, \infty)) \in \mathcal{A}$.
(d) For all $a \in \mathbb{R}$, $f^{-1}((-\infty, a)) \in \mathcal{A}$
(e) For all $a \in \mathbb{R}$, $f^{-1}((-\infty, a])) \in \mathcal{A}$.

As a consequence, continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ are $\mathcal{B}(\mathbb{R})$-measurable (Borel measurable).

Proof. Lemma I.1.1.

Properties Let $f, g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable

(a) Let $\phi : \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R})$-measurable (i.e., Borel measurable). Then $\phi \circ f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

(b) As a consequence, $-f, 3f, f^2, |f|$ are $\mathcal{A}$-measurable, and $1/f$ is $\mathcal{A}$-measurable if $f(x) \neq 0$ for all $x \in X$.

(c) $f + g$ is $\mathcal{A}$-measurable, as given by the following equality of sets

$$(f+g)^{-1}((a,\infty)) = \bigcup_{r \in \mathbb{Q}} (f^{-1}((r,\infty)) \cap g^{-1}((a-r,\infty))).$$

For a quick proof of this equality.

> If $f(x) + g(x) > a$, then $f(x) > a - g(x)$. And thus there exists some $r \in \mathbb{Q}$ so that $f(x) > r > a - g(x)$.
>
> Likewise if $f(x) > r$ and $g(x) > a - r$, then $f(x) + g(x) > a$.

(d) fg is $\mathcal{A}$-measurable because

$$f(x)g(x) = \frac{1}{2}\left((f(x)+g(x))^2 - f(x)^2 - g(x)^2\right)$$

(e) $(f \vee g)(x) = \max\{f(x), g(x)\}$, $(f \wedge g)(x) = \min\{f(x), g(x)\}$ are both $\mathcal{A}$-measurable because

$$(f \vee g)^{-1}((a,\infty)) = f^{-1}((a,\infty)) \cup g^{-1}((a,\infty))$$
$$(f \vee g)^{-1}((a,\infty)) = f^{-1}((a,\infty)) \cap g^{-1}((a,\infty)).$$

(f) If $f_n : X \to \overline{\mathbb{R}}$ are $\mathcal{A}$-measurable, then $\sup_n f_n, \inf_n f_n, \limsup_{n\to\infty} f_n, \liminf_{n\to\infty} f_n$ are all $\mathcal{A}$-measurable. We give a quick proof

> Call $\sup_n f_n$ as g. We need to check that
>
> $$g^{-1}((a,\infty]) = \bigcup_{n \in \mathbb{N}} f^{-1}((a,\infty]).$$
>
> For $\subseteq$, $\sup_n f_n(x) = g(x) > a$, and so necessarily there is an $n \in \mathbb{N}$ such that $f_n(x) > a$. The other direction is easy. Also $\inf_n f_n$ is exactly the same with the opposite type of interval.
>
> Now we note that $\limsup_n f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n$. Thus this also must be measurable.

(g) Let $f_n : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If $f(x) = \lim_{n\to\infty} f_n(x)$ converges (in $\mathbb{R}$) for every $x \in X$, then f is $\mathcal{A}$-measurable.

Definition I.1.3

For $f : X \to \overline{\mathbb{R}}$ define $\operatorname{supp} f = \{x \in X \mid f(x) \neq 0\}$. This is called the support of f.

Definition I.1.4

For $f : X \to \overline{\mathbb{R}}$, let $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

We call f^+, f^- the positive and negative part of f respectively. Note that $\operatorname{supp}(f^+) \cap \operatorname{supp} f^- = \emptyset$.

Furthermore, $f = f^+ - f^-$, and so f is $\mathcal{A}$-measurable if and only if f^+, f^- are $\mathcal{A}$-measurable.

Definition I.1.5 (Characteristic Function)

For $E \subseteq X$, the characteristic (indicator) function of E is given by

$$\chi_E(x) \coloneqq 1_E(x) \coloneqq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Note that 1_E is $\mathcal{A}$-measurable if and only if $E \in \mathcal{A}$.

Definition I.1.6 (Simple Function)

A simple function $\phi : X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes finitely many values.

If $\phi(X) = \{c_1, \dots, c_N\}$, then $E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A}$, and $\phi = \sum_{i=1}^N c_i 1_{E_i}$.

Note that $c_i \neq \pm\infty$.