.1. Integration of complex functions

Definition .1.1

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to \overline{R}, g : X \to \mathbb{C}$ be measurable (for g, this means that both $\Re g, \Im g$ are measurable).

Then f, g are called <u>integrable</u> if $\int |f| < \infty$. Then we define

$$\int f = \int f^+ - \int f^- \qquad \int g = \int \Re g + i \int \Im g.$$

For $f : X \to \overline{\mathbb{R}}$ we can define

$$\int f = \begin{cases} \infty & \text{if } \int f^+ = \infty, \int f^- < \infty \\ -\infty & \text{if } \int f^+ < \infty, \int f^- = \infty \end{cases}$$

Lemma .1.1

Let $f, g : X \to \overline{R}$ or $\mathbb{C}$ integrable. Assume that $f(x) + g(x)$ is well-defined for all $x \in X$. That is we never see $\infty + (-\infty)$ or $(-\infty) + \infty$.

Then we have that

(a) $f + g$, cf for all $c \in \mathbb{C}$ are integrable.

(b) $\int f + g = \int f + \int g$. This is non-trivial because $(f + g)^+ \neq f^+ + g^+$.

(c) $\left|\int f\right| \leq \int |f|$.

Proof. Check [Fol99] pg 53.

Lemma .1.2

Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be an integrable function on X. Then

(a) f is finite almost everywhere (i.e. $\{x \in X \mid |f(x)| = \infty\}$ is a null set).

(b) The set $\{x \in X \mid f(x) \neq 0\}$ is σ-finite

Proof. HW5 Q8 (think Markov).

Proposition .1.3

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

(a) If h is integrable on X, then

$$\int_E h = 0, \quad \forall E \in \mathcal{A} \iff \int |h| = 0 \iff h = 0 \text{ almost everywhere}$$

the second $\iff$ was done last class.

(b) If f, g are integrable on X, then

$$\int_E f = \int_E g, \quad \forall E \in \mathcal{A} \iff f = g \text{ almost everywhere}$$

Proof. Let's go!

(a) We have that

$$\int |h| = 0 \implies \left|\int_E h\right| \leq \int_E |h| \leq \int |h| = 0.$$

This handles one implication. Now assume $\int_E h = 0$ for all $E \in \mathcal{A}$. Then write

$$h = u + iv = u^+ - u^- + i(v^+ - v^-).$$

Then let $B = \{x \mid u^+(x) > 0\}$. By assumption

$$0 = \int_B h = \Re \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+.$$

Therefore $u^+ = 0$ almost everywhere. Similarly, u^-, v^+, v^- are zero almost everywhere. This gives us that h is zero almost everywhere as desired.

Theorem .1.4 (Dominated Convergence Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Here is the setup!

(1) Let f_n be integrable on X.

(2) $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere.

(3) There is a $g : X \to [0, \infty]$ such that g is integrable and

$$|f_n(x)| \leq g(x) \text{ almost everywhere for all } n \in \mathbb{N}.$$

Then we have that

$$\lim_{n\to\infty} \int f_n = \int f = \int \lim_{n\to\infty} f_n$$

Proof. Let F be the countable union of null sets on which (1)-(3) may fail. Modifying the definition of f_n, f, g on F, we may assume (1)-(3) hold everywhere because modifying on a null set does not change the integral.

We consider the $\overline{\mathbb{R}}$-valued case only ($\mathbb{C}$-valued case, check yourself). Note that (2),(3) imply that f is integrable, because $|f| \leq g(x)$.

Then $g + f_n \geq 0$ and $g - f_n \geq 0$ because $-g \leq f_n \leq g$. Then Fatou's Lemma tells us that

$$\int g + f \leq \liminf_{n\to\infty} \int g + f_n$$
$$\int g - f \leq \liminf_{n\to\infty} \int g - f_n.$$

Using linearity and cancellation (because $\int g < \infty$) this shows that

$$\int f \leq \liminf_{n\to\infty} \int f_n \qquad\qquad -\int f \leq \liminf \int -f_n = -\limsup \int f_n.$$

Therefore

$$\int f \leq \liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f.$$

This shows the limit exists as well as the desired result!

Corollary .1.5 (Fubini's Theorem for series and integrals)

Suppose f_n are integrable functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty.$$

Then we have that

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Take $G(x)$ to be $\sum_{n=1}^{\infty} |f_n(x)|$. Then $G(x) \geq |F_N(x)|$ where $F_N(x) = \sum_{n=1}^{N} f_n(x)$.

By **??** we have that

$$\int G(x) = \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Then the Dominated Convergence Theorem hands us the result