Let V be a vector space over scalar field $K = \mathbb{R}$ or $K = \mathbb{C}$

Definition .0.1

A seminorm on V is a map $\|\cdot\| : V \to [0, \infty)$ satisfying

(1) $\|v + w\| \le \|v\| + \|w\|$

(2) $\|cv\| = |c| \, \|v\|$

A norm additionally satisfies

(3) $\|v\| = 0 \iff v = 0$

A norm gives a metric $\rho(v, w) = \|v - w\|$, and we have

$$v_n \to v \iff \lim_{n\to\infty} \|v_n - v\| = 0$$

Example .0.1

We have the following examples

(1) $L^1(X, \mathcal{A}, \mu)$ with $\|f\|_1 = \int |f| \, \mathrm{d}\mu$

(2) $C([0,1])$ with $\|f\|_1 = \int_0^1 |f(x)| \, \mathrm{d}x$, $\|f\|_\infty = \max_{0\le x\le 1} |f(x)|$.

(3) For $\mathbb{R}^d$ we have for $0 < p < \infty$

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}$$
$$\|x\|_\infty = \max_{1\le i\le d} |x_i|$$

.1. L^p spaces

Definition .1.1 (L^p spaces)

For $(X, \mathcal{A}, \mu)$ a measure space, f a measurable function. For $0 < p < \infty$ define

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p}$$
$$L^p(X, \mathcal{A}, \mu) = \{ f \mid \|f\|_p < \infty \}.$$

This is a seminorm, and a norm if we identify functions which are equal almost everywhere.

Example .1.1

$(\mathbb{R}, \mathcal{L}, m)$ has $f(x) = x^{-\alpha} 1_{(1,\infty)}(x) \in L^p$ if and only if $\alpha p > 1$.

In contrast, $g(X) = x^{-\beta} 1_{(0,1)}(x) \in L^p$ if and only if $\beta p < 1$.

Definition .1.2 (ℓ^p spaces)

If $(X, \mathcal{P}(X), \nu)$ is the counting measure, then define

$$\ell^p(X) := L^p(X, \mathcal{P}(X), \nu).$$

Of particular interest is $\ell^p(\mathbb{N})$. Here we have

$$\ell^p := \ell^p(\mathbb{N}) = \left\{ a = (a_1, a_2, \ldots) \mid \|a\|_p = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} < \infty \right\}$$

Lemma .1.1

$L^p(X, \mathcal{A}, \mu)$ is a vector space, for all $p \in (0, \infty)$.

Proof. Note that

$$\|cf\|_p = \left(\int |cf|^p \, \mathrm{d}\mu \right)^{1/p} = |c| \, \|f\|_p < \infty \iff \|f\|_p < \infty.$$

Note that for any real numbers α, β we have

$$(\alpha + \beta)^p \leq (2 \cdot \max(|\alpha|, |\beta|))^p = 2^p \cdot \max(|\alpha|^p, |\beta|^p) \leq 2^p(|\alpha|^p + |\beta|^p).$$

Therefore for f, g we have

$$\|f + g\|_p < \infty \iff \|f + g\|_p^p = \int |f + g|^p \, \mathrm{d}\mu \leq 2^p \int (|f|^p + |g|^p) < \infty \impliedby \|f\|_p, \|g\|_p < \infty.$$

But this is not quite satisfactory, as it does not give us the triangle inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

For this we need a new result

Theorem .1.2 (Hölder's inequality)

Let $1 < p < \infty$, and let $q = p/(p-1)$ so that $1/p + 1/q = 1$.

Then we have that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Example .1.2

For $p = q = 2$, $X = \{1, \ldots, d\}$, μ counting measure, then for $x, y \in \mathbb{R}^d$

$$\sum_{i=1}^{d} |x_i y_i| \leq \sqrt{\sum_{i=1}^{d} x_i^2} \sqrt{\sum_{i=1}^{d} y_i^2}.$$

Proof. We do this in steps

(1) Note that

$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p} = \frac{t^p}{p} + \frac{1}{q}$$

for all $t \geq 0$, by taking $F(t) = t - t^p/p$, $t \geq 0$, and using calculus to find the maximum.

(2) Note that

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

This follows by taking $t = \alpha/\beta^{q-1}$. This inequality is known as Young's Inequality.

(3) WLOG, assume $0 < \|f\|_p, \|g\|_q < \infty$. Now consider $F(x) = f(x)/\|f\|_p, G(x) = g(x)/\|g\|_q$. We know that $\|F\|_p = 1 = \|G\|_q$. Then by Young's Inequality

$$\begin{aligned}\int |F(x)G(x)| \, \mathrm{d}\mu &\leq \int \frac{|F(x)|^p}{p} + \frac{|G(x)|^q}{q} \\ \frac{\|fg\|_1}{\|f\|_p\|g\|_q} &\leq \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

Theorem .1.3 (Minkowski's Inequality)

Let $1 \leq p < \infty$. For $f, g \in L^p$, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. For $p = 1$¡ it's easy (just triangle inequality). Now assume $1 < p < \infty$ and WLOG assume $\|f+g\| \neq 0$. Then

$$\begin{aligned}\int |f(x)+g(x)|^p &\leq \int |f(x)+g(x)|^{p-1} \left(|f(x)| + |g(x)|\right) \\ &\leq \left(\int |f+g|^{(p-1)q}\right)^{1/q} \left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right] \\ &\leq \left(\int |f+g|^p\right)^{1/q} \left[\|f\|_p + \|g\|_p\right] \\ \left(|f(x)+g(x)|^p\right)^{1-1/q} &\leq \|f\|_p + \|g\|_p \\ \left(|f(x)+g(x)|^p\right)^{1/p} &\leq \|f\|_p + \|g\|_p \\ \|f+g\|_p &\leq \|f\|_p + \|g\|_p\end{aligned}$$