Definition .0.1

A function $G \in BV$ is said to have normalized bounded variation ($G \in NBV$) provided that G is right continuous and $G(-\infty) = 0$.

Example .0.1

If F is increasing and bounded, F right continuous, $F(-\infty) = 0$.

$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$, $f \in L^1(\mathbb{R})$. Midterm gave F is uniformly continuous.

Lemma .0.1

If $F \in BV$, right continuous, then $T_F \in NBV$.

Proof. T_F is bounded, increasing, and satisfies $T_F(-\infty) = 0$ by **??**. Thus $T_F \in BV$.

Thus we just need to check that T_F is right continuous. Suppose not, then there is a point $a \in \mathbb{R}$ such that $c \coloneqq T_F(a^+) - T_F(a) > 0$.

Fix $\varepsilon > 0$. Since $F(x)$ and $g(x) \coloneqq T_F(x^+)$ are right-continuous, there exists a $\delta > 0$ such that for $y \in (a, a+\delta]$ we have

$$|F(y) - F(a)| < \varepsilon$$
$$|g(y) - g(a)| < \varepsilon.$$

We then have that

$$T_F(y) - T_F(a^+) \leq T_F(y^+) - T_F(a^+) < \varepsilon.$$

There exist $a = x_0 < x_1 < \cdots < x_n = a + \delta$ such that

$$\begin{aligned}\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &\geq T_F(a+\delta) - T_F(a) - \frac{c}{4} \\ &\geq T_F(a^+) - T_F(a) - \frac{c}{4} = \frac{3c}{4}.\end{aligned}$$

Then $|F(x_1) - F(a)| < \varepsilon$ so we have

$$\sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4} - \varepsilon.$$

There exist $a = t_0 < \cdots < t_k = x_1$ such that

$$\sum_{i=1}^{k} |F(t_i) - F(t_{i-1})| \geq T_F(x_1) - T_F(a) - \frac{c}{4} \geq \frac{3}{4}c.$$

Then as $[a, a+\delta] = [a, x_1] \cup [x_1, a+\delta]$ we see that

$$T_F(a+\delta) - T_F(a) \geq \sum_{j=1}^{k} |F(t_j) - F(t_{j-1})| + \sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}c - \varepsilon + \frac{3}{4}c = \frac{3}{2}c - \varepsilon.$$

Thus

$$\varepsilon + c \geq T_F(a+\delta) - T_F(a^+) + T_F(a^+) - T_F(a)$$

$$= T_F(a+\delta) - T_F(a) \geq \frac{3}{2}c - \varepsilon$$
$$c \leq 4\varepsilon.$$

Thus taking $\varepsilon \to 0$ yields $c = 0$, which is a contradiction.

Corollary .0.2

$F \in NBV$ if and only if $F = F_1 - F_2$, $F_1, F_2 \in NBV$ and increasing

Proof. $F = (T_F + F)/2 - (T_F - F)/2$.

Theorem .0.3

We have that

(1) Suppose that μ is a finite signed Borel measure on $\mathbb{R}$, then $F(x) = \mu((-\infty, x]) \in NBV$.
(2) $F \in NBV$ implies there exists a unique finite signed Borel measure on $\mathbb{R}$ satisfying $\mu_F((-\infty, x]) = F(x)$.

Proof. We have

(1) Let $\mu = \mu^+ - \mu^-$, thenn $F = F^+ - F^-$, where $F^\pm(x) = \mu^\pm((-\infty, x])$, which are bounded, right continuous, $F^\pm(-\infty) = 0$, so $F^\pm \in NBV$.
(2) Let $F \in NBV$, then $F = F_1 - F_2$, $F_1, F_2 \in NBV$ and increasing. Then define μ_{F_1}, μ_{F_2} by Lebesgue-Stieltjes measure, and set $\mu_F \coloneqq \mu_{F_1} - \mu_{F_2}$.

   Uniqueness? See homework.

Proposition .0.4

We have

(1) If $F \in NBV$, then F is differentiable almost everywhere, $F' \in L^1(\mathbb{R}, m)$.
(2) $\mathrm{d}\mu_F + \mathrm{d}\lambda + F'\,\mathrm{d}m$ for some measure λ satisfying $\lambda \perp m$.
(3) $\mu_F \perp m$ if and only if $F' = 0$ Lebesgue almost everywhere.
(4) $\mu_F \ll m$ if and only if $\int_{-\infty}^x F'(t)\,\mathrm{d}t = F(x) - F(-\infty) = F(x)$.

Proof. (1),(2),(3) check.

For part (4), we have

$$\begin{aligned}
\mu_F \ll m \iff \lambda = 0 &\iff \mathrm{d}\mu_F = F'\,\mathrm{d}m \\
&\iff \mu_F(E) = \int_E F'\,\mathrm{d}m \ \ \forall \text{ Borel } E \\
&\iff F(x) = \mu_F((-\infty, x]) = \int_{-\infty}^x F'(t)\,\mathrm{d}t \ \ \forall x \in \mathbb{R}
\end{aligned}$$

The last converse comes from the uniqueness of the theorem above.