Lemma .0.1 (Best approximation)

Let $e_1, \ldots, e_N$ be orthonormal. For $x \in V$, let $\alpha_i = \langle x, e_i \rangle$, then

$$\left\| x - \sum_{i=1}^{N} \alpha_i e_i \right\| \leq \left\| x - \sum_{i=1}^{N} \beta_i e_i \right\|$$

for all $\beta_1, \ldots, \beta_N \in \mathbb{C}$. Aka this is the best approximation to x within the span of $e_1, \ldots, e_N$. We can also think of it as an orthogonal projection

Proof. Let $z = x - \sum_{i=1}^{N} \alpha_i e_i, w = \sum_{i=1}^{N} (\alpha_i - \beta_i) e_i$.

Note that for all $n = 1, \ldots, N$ we have

$$\langle z, e_n \rangle = \langle x, e_n \rangle - \alpha_n = 0.$$

Thus $\langle z, w \rangle = 0$. So by the Pythagorean theorem

$$\|z + w\|^2 = \|z\|^2 + \|w\|^2 \geq \|z\|$$

proving the result!

Lemma .0.2

Let $\{e_i\}_1^\infty$ be an orthonormal set. For $x \in V$, let $\alpha_i = \langle x, e_i \rangle$. Then,

(1) We have that

$$\|x\|^2 = \left\| x - \sum_{i=1}^{N} \alpha_i e_i \right\|^2 + \sum_{i=1}^{N} |\alpha_i|^2$$

for all $N \in \mathbb{N}$.

(2) $\sum_{i=1}^{\infty} |\alpha_i|^2 \leq \|x\|^2$, referred to as Bessel's inequality.

These actually hold even for an uncountable collection.

Proof. (2) follows from (1), for (1), we see that

$$\begin{aligned}
\left\| x - \sum_{i=1}^{N} \alpha_i e_i \right\|^2 &= \|x\|^2 - 2\,\mathrm{Re}\left\langle x, \sum_{i=1^N} \alpha_i e_i \right\rangle + \left\| \sum_{i=1}^{N} \alpha_i e_i \right\|^2 \\
&= \|x\|^2 - 2 \sum_{i=1}^{N} \mathrm{Re}\, \overline{\alpha_i} \langle x, e_i \rangle + \sum_{i=1}^{N} |\alpha_i|^2 \\
&= \|x\|^2 - 2 \sum_{i=1}^{N} |\alpha_i|^2 + \sum_{i=1}^{N} |\alpha_i|^2 \\
&= \|x\|^2 - \sum_{i=1}^{N} |\alpha_i|^2 .
\end{aligned}$$

Great!

Definition .0.1

An orthonormal set $\{e_i\}$ is said to be an orthonormal basis of V provided that $\overline{W} = V$, where

$$W = \left\{ \sum_{i=1}^{N} \beta_i e_i \mid N \in \mathbb{N}, \beta_1, \ldots, \beta_N \in \mathbb{C} \right\}$$

is the subspace of finite linear combinations. In other words, for all $x \in V$ and for every $\varepsilon > 0$, there exists $w \in W$ such that $\|x - w\| < \varepsilon$.

Example .0.1

For $\mathbb{C}^d$, the orthonormal basis is $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $i = 1, \ldots, d$

For ℓ^2 the orthonormal basis is the countably many $e_i = (0, \ldots, 0, 1, 0, \ldots)$ for $i \in \mathbb{N}$.

Definition .0.2 (Hilbert Space)

A Hilbert space is a complete inner product space (a Banach space with an inner product).

Example .0.2

$\mathbb{R}^d, \mathbb{C}^d, L^2(X, \mathcal{A}, \mu), \ell^2$ are Hilbert spaces.

$C([0, 1]) \subseteq L^2(X, \mathcal{A}, \mu)$ is not a Hilbert space (it is not complete). Take a function f_n so that f_n is zero from 0 to $1/2$ and 1 from $1/2 + 1/n$ to 1, connected continuously line.

Then f_n is Cauchy, but its natural limit is discontinuous.

Theorem .0.3

Ket $\mathcal{H}$ be a Hilbert space. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal set. The following are equivalent

(1) $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis.

(2) If $x \in \mathcal{H}$ and $\langle x, e_i \rangle = 0$ for all i, then $x = 0$.

(3) If $x \in \mathcal{H}$, then $s_N := \sum_{i=1}^{N} \alpha_i e_i \to x$ strongly where $\alpha_i = \langle x, e_i \rangle$.

(4) If $x \in \mathcal{H}$, then $\|x\|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2$ (Plancherel identity).

Proof. Let's go!

(3) $\implies$ (4) We have by Lemma .0.2 that

$$\|x\|^2 = \|x - s_N\|^2 + \sum_{i=1}^{N} |\alpha_i|^2 .$$

Taking $N \to \infty$ and noting $s_N \to x$ strongly gives

$$\|x\|^2 = \lim_{N \to \infty} \sum_{i=1}^{N} |\alpha_i|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2 .$$

(4) $\implies$ (1) Using the same equality

$$\|x\|^2 = \|x - s_N\|^2 + \sum_{i=1}^{N} |\alpha_i|^2 .$$

and taking $N \to \infty$ yields $\|x - s_N\|^2 \to 0$ so $\|x - s_N\| \to 0$. Therefore $s_N \to x$ strongly, yielding that x can be approximated by finite linear combinations as desired.

(1) $\implies$ (2) Fix $x \in \mathcal{H}$, and fix $\varepsilon > 0$. Then by (1), there exists a $y = \sum_{i=1}^{k} \beta_i e_i$ such that $\|x - y\| < \varepsilon$.

By the best approximation lemma (see Lemma .0.1), $\|x - s_k\| \le \|x - y\| < \varepsilon$. If $\langle x, e_i\rangle = 0$ for all i, then $s_k = 0$, so $\|x\| < \varepsilon$.

Taking $\varepsilon \to 0$ would yield $\|x\| = 0$, implying $x = 0$.

(2) $\implies$ (3) Bessel's inequality gives $\sum_{i=1}^{\infty} |\alpha_i|^2 \le \|x\|^2 < \infty$. We now see that for $N > M$

$$\|s_N - s_M\|^2 = \left\| \sum_{i=M+1}^{N} \alpha_i e_i \right\|^2 = \sum_{i=M+1}^{N} |\alpha_i|^2 \to 0$$

as $N > M \to \infty$, by convergence of the series. This implies that $\{s_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}$.

Since $\mathcal{H}$ is complete, there is a vector y such that $s_N \to y$ strongly. Question is, is $y = x$?

Fix $i \in \mathbb{N}$, consider $\langle y - x, e_i\rangle$. We see that

$$\langle y - x, e_i\rangle = \langle y - s_N, e_i\rangle + \langle s_N - x, e_i\rangle.$$

We can compute that for $N > i$ that

$$\langle s_N - x, e_i\rangle = \alpha_i - \langle x, e_i\rangle = 0.$$

Therefore $\langle y - x, e_i\rangle = \langle y - s_N, e_i\rangle$. Because strong convergence implies weak convergence, taking $N \to \infty$ yields that $\langle y - x, e_i\rangle = 0$ for all $i \in \mathbb{N}$.

Therefore by the assumption of (2) $y - x = 0$, so $x = y$ and we're done.

Note that for everything except (2) $\implies$ (3) we did not use the Hilbert space property. When $\mathcal{H}$ is replaced by any inner product space V we only have

$$(3) \implies (4) \implies (1) \implies (2).$$

Definition .0.3

A metric space is called separable if there exists a countable dense subset.

Example .0.3

$\mathbb{R}^d \supseteq \mathbb{Q}^d$, ℓ^p, $1 \le p < \infty$, but not $p = \infty$. To do this consider sequences of rational numbers.

$L^p(\mathbb{R}, m)$ is separable for $1 \le p < \infty$. Take step functions with rational heights and rational endpoints to intervals.

Theorem .0.4

Every separable Hilbert space has a countable orthonormal basis.

Proof. Gram-Schmidt.

Note: The cardinality of an orthonormal basis is determined by the space, and we can call this the dimension of the Hilbert space.