Stuff:

- Handout: Ahlfor's Guide to Contours
- HW 9B due tonight!
- HW 10A + 10B due Thursday and next Tuesday.

Last time: Evaluating real integrals using the residue theorem, and the fractional residue theorem.

Example .0.1 (Example of Fractional Residue Theorem)

Look at

$$\int_0^\infty \frac{\log x}{x^2-1}\,\mathrm{d}x.$$

Consider $f(z) = \frac{\log z}{z^2-1}$. We'll integrate along a semicircular contour of radius R with indents at -1 and 0. This will split into *six* integrals, and we'll use a branch cut of log with argument from $-\pi/2$ to $3\pi/2$.

We'll call the non-linear parts Γ_R,C_ε (indent at -1),γ_δ (indent at 0). We know

$$\int_{\partial D(R,\varepsilon)} f(z)\,\mathrm{d}z = 0.$$

Furthermore $\int_{\Gamma_R} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$. The C_δ^0 piece will also go to 0. By the fractional residue theorem

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)\,\mathrm{d}z = -\pi i\,\mathrm{Res}[f(z),-1] = -\pi i \frac{i\pi}{2(-1)} = -\frac{\pi^2}{2}.$$

Take real and imaginary parts and let $\varepsilon \to 0$ to get

$$\int_0^\infty \frac{\log x}{x^2-1} + \int_{-\infty}^0 \frac{\log|x|}{x^2-1}\,\mathrm{d}x - \frac{\pi^2}{2} = 0.$$

Therefore we get

$$\int_0^\infty \frac{\log x}{x^2-1}\,\mathrm{d}x = \frac{\pi^2}{4}$$

.1. Principal Values

Definition .1.1

Suppose $f(x)$ is continuous for $a \le x < x_0$ and $x_0 < x \le b$. We define the principal value of

$$\int_a^b f(x)\,\mathrm{d}x$$

to be

$$\mathrm{PV}\int_a^b f(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\left(\int_a^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^b f(x)\,\mathrm{d}x\right),$$

provided that this limit exists.

.2. Jordan's Lemma

Recall .2.1

We know all about

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x,$$

well, we needed zeros of Q not on $\mathbb{R}$ and $\deg Q \geq 2 + \deg P$. Jordan's Lemma will allow us to change this to $\deg Q \geq 1 + \deg P$, by circumventing the ML-estimate portion of this proof.

Lemma .2.1 (Jordan)

If Γ_R is the semicircular contour $z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$, then

$$\int_{\Gamma_R} \left|e^{iz}\right| \cdot |\mathrm{d}z| < \pi.$$

Proof. Rewrite this as $z = Re^{i\theta}$, $\mathrm{d}z = zi\,\mathrm{d}\theta$, so thaen $|\mathrm{d}z| = R\,\mathrm{d}\theta$. The lemma boils down to

$$\int_0^{\pi} e^{-R\sin\theta}\,\mathrm{d}\theta < \frac{\pi}{R},$$

Notice that $y = \frac{2}{\pi}\theta$ and $y = \sin\theta$ both go through the points $(0,0)$ and $(\pi/2, 1)$, but $\sin\theta$ is above this line for all $\theta \in [0, \pi/2]$. Therefore

$$\begin{aligned}\int_0^{\pi} e^{-R\sin\theta}\,\mathrm{d}\theta &= 2\int_0^{\pi/2} e^{-R\sin\theta}\,\mathrm{d}\theta \\ &\leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,\mathrm{d}\theta \\ &= \frac{2(-1)\pi}{2R} e^{-2\pi R\theta/\pi}\Big]_0^{\pi/2} = \frac{-\pi}{R}\cdot\left(\frac{1}{e^R} - 1\right)\end{aligned}$$

In other words, this is $\frac{\pi}{R} -$ positive, which is less than $\frac{\pi}{R}$.

.3. Exterior Domains

Question: Can we defined the residue of f at ∞ on $\widehat{\mathbb{C}}$.

Definition .3.1

Let $D \subseteq \widehat{\mathbb{C}}$ be open, connected, and suppose it contains a neighborhood of ∞. That is there exists an $R > 0$ such that D contains $|z| > R$. Then D is called an exterior domain

Theorem .3.1

Let $D \subseteq \mathbb{C}$ be an exterior domain with piecewise smooth boundary. Suppose $f(z)$ is analytic on $D \cup \partial D$ except for a finite # of isolated singularities $z_1, \ldots, z_m \in D$. Let a_{-1} be the coefficient of $\frac{1}{z}$ in the series of f at ∞. Then

$$\oint_{\partial D} f(z)\,\mathrm{d}z = -2\pi i a_{-1} + 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j]$$

Definition .3.2

We define residue of $f(z)$ at ∞ to be $\operatorname{Res}[f(z), \infty] = -a_{-1}$. Why the negative!!!

Proof. Apply the standard residue theorem to the new region $D_R = D \setminus (\text{disk } |z| > P)$, and follow nose and see Gamelin. Cool HW problem on 10B.

.4. Logarithmic Integral

Gamelin: Let $D \subseteq \mathbb{C}$ be bounded, open, connected subset. Suppose $f(z)$ is meromorphic on D that extends to be analytic on ∂D such that $f(z) \neq 0$ for all $z \in \partial D$.

Then: f has finitely many zeros in D. Question: How many zeros in D, and how many poles in D?

Suppose $f(z)$ has a zero of order $N > 0$, then $f(z) = (z - z_0)^N g(z)$ at z_0. Then

$$f'(z) = (z - z_0)^N g'(z) + g(z)N(z - z_0)^{N-1}$$

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{(z - z_0)^{N-1} N \cdot g(z)}{(z - z_0)^N g(z)} = \frac{g'(z)}{g(z)} + \frac{N}{z - z_0}.$$

We call $\frac{f'(z)}{f(z)}$ the logarithmic derivative of f, since if $\log f(z)$ is defined when we take the derivative we get this.

But wait! We have

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)}, z_0\right] = N,$$

this is amazing!!! In fact we have the same thing for if we started with any $N \in \mathbb{Z}$! This can also detect poles!

Theorem .4.1

Let $f(z)$ be as above. Then we have that

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)} \, dz = (\text{\# of zeros of } f \text{ in } D) - (\text{\# of poles of } f \text{ in } D),$$

where we count with multiplicity.

What is f'/f!!! See Curt McMullen notes rs.pdf and search "good cocycle"

Now we go to the Argument Principle.

Definition .4.1

If $f(z)$ is analytic on $D \subseteq \mathbb{C}$, then for a closed curve $\gamma \subseteq D$ such that $f(z) \neq 0$ for all $z \in \gamma$, we call

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} d(\log f(z)) = \frac{1}{2\pi i} \int_\gamma d \log |f(z)| + \frac{1}{2\pi} \int_\gamma d \arg f(z).$$

Thus the left hand side is 0 because the differential is exact...

But the argument is something different!!!

Theorem .4.2 (Argument Principle)

Suppose $D \subseteq \mathbb{C}$ is bounded, connected, open, and with piecewise boundary and let $f(z)$ be meromorphic on D that extends to be analytic on ∂D. Suppose further that $f(z) \neq 0$ for all $z \in \partial D$. Then the increase in the argument of $f(z)$ around the boundary of D is

$$2\pi \left[(\text{\# zeros of } f \text{ in } D - \text{\# of poles}\right].$$

Example .4.1

Let $f(z) = z^N$ for $N \in \mathbb{N}$¡ we know the number of zeros of in D minus the number of poles in D is N. So as we traverse the boundary of any D containing 0, then the argument increases by $2\pi N$.

Application: Show that $p(z) = z^4 + 2z^2 - z + 1$ has exactly one zero in the first quadrant. Solution: Apply the arugment principle. There are no poles!

Go around the quadrant in three pieces. If $z \in [0, R]$, then one can check that $p(z) > 0$ there, so there are no zeros. For R large enough there are no zeros when $|z| = R$, $0 \leq \arg z \leq \frac{\pi}{2}$, and $p(z) \approx z^4$. Thus as z goes along this quater circle the change in the argument is 2π.