I. Introduction and Administration

See the syllabus (see **??**)! As a summary

- Prereqs: Analysis at the level of 451, multivariable calculus, and familiarity with complex numbers.
- HW: Generally due Monday (proofy homework, B homework) and Wednesday (calculationy homework, A homework). Two lowest homeworks in both series are dropped (four total).
- Goal: Prepare PhD students for the qualifying exams.
- Book: Gamelin *Complex Analysis* (see [Gam03]) and an older book by Ahlfors (see [Ahl53]).
- Grading Scheme
  - 45% Homework
  - 25% Midterm (in class Oct 25th)
  - 30% Final (Dec 15th 1:30pm-3:30pm)

The first A homework is due tomorrow (8/31) and the first B homework is due next Tuesday due to Labor Day (9/6).

II. The Basics

II.1. Motivation and Recollections

$\mathbb{C}$-analysis is a nexus for lots of fields:

- Algebra (fields and solving equations)
- Algebraic geometry and complex manifolds
- Geometry (platonic solids, flat tori, hyperbolic manifolds in dimensions 2 and 3)
- Lie Groups
- ♡ dynamics ♡
- Number theory (automorphic forms, elliptic functions, zeta functions)
- Riemann surfaces (Teichmuller theory, curves and their Jacobians)
- Several complex variables and complex manifolds
- Real analysis and PDEs (harmonic functions, elliptic equations, distributions)

The complex numbers are formally defined as the field $\mathbb{C} = \mathbb{R}[i]$ where $i^2 = -1$, They are represented in the Euclidean plane by $z = (x, y) = x + iy$. There are 2 square roots of -1 in $\mathbb{C}$, the number i is the one with positive imaginary part.

Definition II.1.1

If $z = x + iy$ we define the real and imaginary parts as

$$\Re(z) \coloneqq x \qquad\qquad \Im(z) \coloneqq y$$

We should recall the definition of addition, multiplication, and division in $\mathbb{C}$. If $z_j = x_j + iy_j$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i.$$

Because $\mathbb{C} = \mathbb{R}[i]$ there is a Galois automorphism for this field extension

$$z = x + iy \mapsto \overline{z} \coloneqq x - iy$$

called complex conjugation which fixes $\mathbb{R}$. We know

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}.$$

We then define the norm squared of z to be the multiplication of all its Galois conjugates (works for any finite Galois extension), that is

$$|z|^2 \coloneqq z \cdot \overline{z} = x^2 + y^2 \in \mathbb{R}_{\geq 0}.$$

Compatibility of $|z| = \sqrt{|z|^2}$ with the Euclidean metric on $\mathbb{R}^2$. This along with the fact that $\mathbb{C} \cong \mathbb{R}^2$ as a vector space justifies the identification of $\mathbb{C}$ with $\mathbb{R}^2$

If $z \neq 0$, $z = x + iy$ then

$$\frac{z}{|z|} = \frac{x}{|z|} + i\frac{y}{|z|}$$

lies on the unit circle S^1. Thus we can write this complex number as

$$\frac{z}{|z|} = \cos\theta + i\sin\theta$$

for some $\theta \in \mathbb{R}$. Please note θ is uniquely defined only up to adding integer multiples of 2π and only when $z \neq 0$. This number θ is called the argument of z, denoted $\arg(z)$.

We let

$$e^{i\theta} \coloneqq \cos\theta + i\sin\theta$$

so that $z = |z|\, e^{i\theta}$ in these "polar" coordinates. The nice thing about polar coordinates is that if $z = re^{i\alpha}, w = \rho e^{i\beta}$ then $zw = r\rho \cdot e^{i(\alpha+\beta)}$. Thus

$$|zw| = r_1 r_2.$$

II.2. Solving Polynomial Equations

A critical feature of $\mathbb{C}$ is that it is algebraically closed. In other words, we have

Theorem II.2.1 (The Fundamental Theorem of Algebra)

Every nonconstant polynomial in $\mathbb{C}[X]$ has a root in $\mathbb{C}$.

This is the historical origin of the complex numbers. In the 16th century, Cardano and others were solving cubic equations over $\mathbb{R}$ which have solutions in $\mathbb{R}$! However, their calculations/algorithms included complex numbers which *cancelled* in the end to give real solutions.

Example II.2.1

Fix $a \in \mathbb{C}$. We must solve $z^2 = a$. If $a = 0$ there is one solution, $z = 0$.

If $a \neq 0$, write $a = re^{i\theta}$, then we may take $\omega = e^{i\theta/2}, -\omega = e^{i(\theta/2+\pi)}$ so that $(\pm\omega)^2 = e^{i\theta}$. Then $\pm\omega\sqrt{r}$ are two roots of a, and neither can be "preferred"

Proposition II.2.2 (see HW)

There is no continuous (topology!) function $f : \mathbb{C} \to \mathbb{C}$ such that $(f(z))^2 = z$ for all $z \in \mathbb{C}$. In other words, there is no continuous choice of a square root.

Proof. HW.

Definition II.2.1

Let $n \in \mathbb{N}$ and consider the equation $z^n = 1$. The solutions of this equation are called the roots of unity of order n.

Proof there are n n-th roots of unity. Note if $z^n = 1$, then $|z|^n = 1$, so $|z| = 1$. Note $z^n - 1$ has polynomial derivative nz^{n-1}, whose only solution is 0, so no roots are repeated.

Explicitly, we have solutions $z_j = e^{i\theta_j}$ where

$$\theta_j = \frac{2\pi j}{n}$$

for $j \in \mathbb{Z}$. Up to integer multiples of 2π, there are n such arguments, and n such z_j.

A similar arguments shows that any nonzero complex number z has n different n-th roots that are exactly spaced around the circle of radius $|z|^{1/n}$.

II.3. Topology Time

$\mathbb{C}$ is a metric space as $d(z, w) = |z - w|$. Refresher on the induced topology is

- If $a \in \mathbb{C}$ and $r > 0$ then $B(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$ is the open ball centered at a of radius r.
- We call a set $U \subseteq \mathbb{C}$ open provided that for every $z \in U$ there exists an $r_z > 0$ such that $B(z, r_z) \subseteq U$.
- A subset $A \subseteq \mathbb{C}$ is closed provided that $\mathbb{C} \setminus A$ is open. This is equivalent to the statement that for every convergent sequence $z_n \to z$ with $z_n \in A$, we have $z \in A$ (that is the set of limit points $A' \subseteq A$).

$$A' := \{\omega \in \mathbb{C} \mid \omega \text{ is a limit point of } A\} := \{\omega \in \mathbb{C} \mid \text{There exists } \{a_n\}_{n\in\mathbb{N}} \subseteq A \text{ converging to } \omega\}.$$

- We say a sequence $a_n \in \mathbb{C}$ converges to $\ell \in \mathbb{C}$ provided that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that for $n \geq N$ we have $|a_n - \ell| < \varepsilon$.

We know that $\mathbb{C}$ is Cauchy complete as it is essentially the same as $\mathbb{R}^2$.

If $f : \mathbb{C} \to \mathbb{C}$ and $a \in \mathbb{C}$ we say that f is continuous at a provided that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|z - a| < \delta$ then $|f(z) - f(a)| < \varepsilon$.

We may also take limits to infinity and negative infinity in real analysis. In complex analysis, we only have **ONE** direction of infinity.

For $f : \mathbb{C} \to \mathbb{C}$ let $\ell \in \mathbb{C}$. We define $\lim\limits_{z\to\infty} f(z) = \ell$ to mean for all $\varepsilon > 0$ there exists an $M \in \mathbb{R}$ such that if $|z| > M$ then $|f(z) - \ell| < \varepsilon$.

II.4. The Riemann Sphere

We add a single point to $\mathbb{C}$, called the point at infinity and denoted ∞ to get $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

Definition II.4.1

$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the Riemann sphere

We can naturally identify $\widehat{\mathbb{C}}$ with the unit 2-sphere $S^2 \coloneqq \{v \in \mathbb{R}^3 \mid \|v\| = 1\} \subseteq \mathbb{R}^3$ via stereographic projection. That is if N is the north pole, we can cook up a map $\phi : S^2 \to \{N\} \to \mathbb{C}$ via taking $\phi(p)$ to be the intersection of the line through N and p with the plane $\{(x, y, 0) \in \mathbb{R}^3\} \cong \mathbb{C}$.

We can extend ϕ to a bijection $\widehat{\phi} : S^2 \to \widehat{\mathbb{C}}$ via $N \mapsto \infty$. The topology on S^2 exactly matches the limit definition we gave above for $\lim_{z\to\infty}$, and also this is a homeomorphism where $\widehat{\mathbb{C}}$ is given the topology of the one-point compactification.

Formally, if $f : \mathbb{C} \to \mathbb{C}$m we can compose to make $g = f \circ \phi : S^2 \setminus \{N\} \to \mathbb{C}$. Then for $\ell \in \mathbb{C}$

$$\lim_{z\to\infty} f(z) = \ell \iff \lim_{p\to N} g(p) = \ell.$$

This second limit is defined using Euclidean distance in $S^2 \subseteq \mathbb{R}^3$. If C is a circle in S^2 (the intersection of S^2 with a plane in $\mathbb{R}^3$) then $\widehat{\phi}(C)$ is a circle in $\mathbb{C}$ if $N \notin C$ and the image in $\mathbb{C}$ is a line if $N \in \mathbb{C}$. Conversely, every circle and every line in $\mathbb{C}$ is obtained this way.