Stuff:

- HW 12 due Thursday!
- Office Hours tomorrow 10:30am-12pm!
- No Office Hours on Friday!
- Decorating for WOLOG party on Thursday night
- Talk about final
- Hint on 8, show $f$ maps $\overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Show that $f$ is proper, then it's a Blaschke product...

Last time: Mandelbrot stuff! This time: More mandelbrot stuff.

**Recall .0.1**

$\mathcal{M}$ is the "connectedness locus" of $z \mapsto z^2 + c$. We said last time that $\mathcal{M} \neq \emptyset$ (since $0 \in \mathcal{M}$), $\mathcal{M}$ is compact (why!), $\mathcal{M}$ is connected (WHY! Relevant for HW), and $\mathcal{M}$ is full (also HW).

See Gamelin for details, but

**Theorem .0.1** (Fatou-Julia, 100 years ago)

Let $P(z)$ be a polynomoial of degree $d \geq 2$. The filled julia set of $P(z)$ is connected if and only if the orbits of each critical point of $P(z)$ is bounded.

**Theorem .0.2** (Fatou-Julia)

The filled Julia set $K_c$ of $z \mapsto z^2 + c$ is disconnected if and only if it is a Cantor set (up to homeomorphism).

For other degrees, there is intermediate behavior, essentially because there are multiple critical points, so one can have bounded orbit while the other does not.

Take $c \in \mathcal{M}$. Then the filled Julia set $K_c$ of $z^2 + c$ is connected and <u>full</u> ($\mathbb{C} \setminus K_c$ is connected, the proof is maximum modulus principle).

**Definition .0.1**

The <u>basin of $\infty$</u> of $z^2 + c$ is $\mathbb{C} \setminus K_c$.

FACT: There is an explicit conformal isomorphism on the basis of $\infty$ for $z \mapsto z^2 + c$ to the basin of $\infty$ of $z \mapsto z^2$. Furthermore, it conjugates $z^2 + c$ to $z^2$. It's important here that $c \in \mathcal{M}$.

Idea: lift the identity iteratively, let $s(z) = z^2$, $P(z) = z^2 + c$,

$$\begin{tikzcd}
(U, \infty) \arrow[r, dashed, "\phi_2"] \arrow[d, "P"'] & (U, \infty) \arrow[d, "s"] \\
(U, \infty) \arrow[r, dashed, "\phi_1"] \arrow[d, "P"'] & (U, \infty) \arrow[d, "s"] \\
(U, \infty) \arrow[r, dashed, "\mathrm{Id}"'] & (U, \infty)
\end{tikzcd}$$

Then $\phi_n \circ P = s \circ \phi_{n+1}$. This will converge normally to some map $\phi$, and we'll have $\phi \circ P = s \circ \phi$. These are called Böttcher coordinates for the polynomial about $\infty$. When $c \in \mathcal{M}$, this can be extended to all of $\widehat{\mathbb{C}} \setminus K_c$.

If $c \notin \mathcal{M}$, then we can extend the Böttcher coordinates until we hit the critical point in the dynamical plane of $z^2 + c$.

We can cook up a map $\Phi$ from $\mathbb{C} \setminus \mathcal{M}$ to $\mathbb{C} \setminus \overline{\mathbb{D}}$ by $c \mapsto \varphi_c(c)$ where $\varphi_c$ is the conformal isomorphism defined on the exterior of the figure 8 (aka where Böttcher coordinates apply).

Prove by hand that this is the Riemann map (aka it is a conformal isomorphism). There are three parts to that

(1) $\Phi$ is holomorphic.
(2) $\Phi$ is proper. It follows that $\Phi$ is surjective.
(3) To get injectivity, look near $\infty$ and show that $\Phi$ has degree 1.

**Corollary .0.3**

As a corollary the Mandelbrot set is connected and full.

For HW: explain a piece of this proof sketch you find interesting.

Trick: Now you can try to label the boundary of the Mandelbrot set by angles... the labeling being unique even for irrationals is exactly the claim that $\mathcal{M}$ is locally connected, a huge conjecture.