Stuff:

- HW 11A due today
- Math Circle
- Super Saturdays
- Bagel Sundays
- Student Seminar 11/18: **Dedekind and the Axiom of Choice** by Dhruv Kul in EH 3096 4-5pm.

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Definition .0.1

X is a Riemann surface provided that it is a $\mathbb{C}$-manifold with $\dim_{\mathbb{C}} X = 1$.

Example .0.1

$U \subseteq \mathbb{C}$ where U is open. Complex tori (which are elliptic curves). We can do similarly for a genus g surface. Also $\widehat{\mathbb{C}}$.

We are in the process of classifying all Riemann surfaces. There are three types of Riemann surfaces

(1) $\widehat{\mathbb{C}}$
(2) $\mathbb{C}, \mathbb{C}^*$, and complex tori.
(3) all others

In particular $\mathbb{D}, \mathbb{H}$ lie in category three.

Warm up: There are exactly three simply connected Riemann surfaces up to conformal isomorphism.

Last time we showed the conformal maps $\mathbb{D} \to \mathbb{D}$ are

$$z \mapsto e^{i\theta} \frac{z-a}{1-\overline{a}z}$$

for $a \in \mathbb{D}, \theta \in [0, 2\pi]$. In fact $\operatorname{Aut}(\mathbb{D})$ is a "Lie group" (for those of us taking 591), and has dimension 3 over $\mathbb{R}$.

Lets cook up more examples of holomorphic maps $\mathbb{D} \to \mathbb{D}$. There are the power maps $z \mapsto z^n$ for $n > 0$.

Consider

$$f_1(z) = \frac{z-a_1}{1-\overline{a}_1 z} \qquad f_2(z) = \frac{z-a_2}{1-\overline{a}_2 z},$$

where $a_1, a_2 \in \mathbb{D}$. We could build a new map $\mathbb{D} \to \mathbb{D}$ by multiplying, since $|f_1(z)|, |f_2(z)| < 1$.

Definition .0.2

A Blashcke product $B : \mathbb{D} \to \mathbb{D}$ is a rational function of the form

$$f(z) = e^{i\theta} \prod_{j=1}^{d} \frac{z-a_j}{1-\overline{a}_j z}.$$

It turns out that every proper analytic map $\mathbb{D} \to \mathbb{D}$ is a Blaschke product.

Definition .0.3

Let X, Y be topological spaces. $f : X \to Y$ is proper provided that for all $K \subseteq Y$ compact we have $f^{-1}(K) \subseteq X$ is compact.

Non-Example .0.2

$z \mapsto 57$ is not compact since $f^{-1}(\{57\}) = \mathbb{C}$.

Example .0.3

For any homeomorphism $f : X \to X$, this is proper, since the image of a compact set under a continuous function is compact (the Extreme Value Theorem)

Proposition .0.1

Let X, Y be metric spaces. Suppose Y is connected, locally compact. Assume f is continuous, open, proper, then in fact f is surjective.

Proof. Let $V = f(X)$. We will show V is both open and closed. We know V is open by assumption (that f is an open map).

We'll show V is closed. Let y_0 be an accumulation point of V. Let $y_n \in V$ be a sequence of points converging to y_0, with $f(x_n) = y_n$. Take K a compact neighborhood of y_0 in Y. Consider $f^{-1}(K)$ in X, which is compact since f is proper.

Now for $n > N$, where $N \in \mathbb{N}$, we have $y_n \in K$, so $x_n \in f^{-1}(K)$. By compactness there is a subsequence x_{n_k} which is convergent to some x, and so then y_{n_k} converges to $f(x)$ and to y! Perfect!

Theorem .0.2

Every proper analytic map $f : \mathbb{D} \to \mathbb{D}$ is a Blaschke product.

Lemma .0.3

Let $f : \mathbb{D} \to \mathbb{D}$ be

$$f(z) = e^{i\theta} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a}_i z}$$

Then in fact f is proper.

Proof. Note that a closed subset C of $\mathbb{D}$ is compact if and only if C does not intersect $|z| = 1$.

It thus suffices to show that f maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$ (Exercise: think about why). This follows from arguments on tuesday since $z \mapsto \frac{z-a}{1-\overline{a}z}$ maps S^1 to S^1 for any $a \in \mathbb{D}$.

*Proof of [**thm:blaschke**].* A proper map $f : \mathbb{D} \to \mathbb{D}$ is surjective. So f has at least one zero. Say a. Let $M(z) = \frac{z-a}{1-\overline{a}z}$.

Then $g(z) = \frac{f(z)}{M(z)} : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic, with one fewer zero than f. The result follows by induction on the degree of f (which we would need more details for).

Corollary .0.4

A proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ extends to a holomorphic map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (because Blaschke products are rational functions).

We know what $\mathrm{Aut}(\mathbb{D})$ is. How do we get $\mathrm{Aut}(\mathbb{H})$. Well we use the Cayley map to move things around

$$\begin{tikzcd} \mathbb{H} \arrow[r] \arrow[d, "\varphi"'] & \mathbb{H} \arrow[d, "\varphi"] \\ \mathbb{D} \arrow[r] & \mathbb{D} \end{tikzcd}$$

With this you can show that

$$\mathrm{Aut}(\mathbb{H}) \simeq \left\{ z \mapsto \frac{az+b}{cz+d} \mid ad - bc \neq 0, a, b, c, d \in \mathbb{R} \right\} = \mathrm{PSL}_2(\mathbb{R})$$

Likewise $\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathrm{PSL}_2(\mathbb{C})$

To do: We need to classify the geodesics on $\mathbb{D}$, which we claimed were arcs of circles intersecting S^1 at right angles (and diameters of the disk).

I. Riemann Mapping Theorem

I.1. Arzelà-Ascoli and the Proof

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We also want to prove a *huge* theorem

Theorem I.1.1 (Riemann Mapping Theorem)

If $D \subseteq \mathbb{C}$ is open, connected, and simply connected and $D \neq \mathbb{C}$, then D is conformally isomorphic to $\mathbb{D}$.

Corollary I.1.2

A simply connected domain in $\widehat{\mathbb{C}}$ is $\widehat{\mathbb{C}}$, conformally isomorphic to $\mathbb{C}$, or conformally isomorphic to $\mathbb{D}$.

Such an isomorphism is often called "the" Riemann map (this is clearly not unique). It is unique up to postcomposition with $\mathrm{Aut}(\mathbb{D})$.

We need some ingredients to prove the Riemann Mapping Theorem. They're pretty heavy.

Preliminaries: We need to study equicontinuity. This is in Gamelin, [Gam03, p. XI.5] page 306. Let $E \subseteq \mathbb{C}$ be a set and let

$$\mathcal{F} \subseteq \{f : E \to \mathbb{C}\}$$

a family of functions $E \to \mathbb{C}$.

Definition I.1.1

We say $\mathcal{F}$ is <u>equicontinuous</u> at $z_0 \in E$ provided that for any $\varepsilon > 0$ there is a $\delta > 0$ so that if $z \in E$, $|z - z_0| < \delta$ and $f \in \mathcal{F}$ then $|f(z) - f(z_0)| < \varepsilon$.

Essentially, the continuity has a form of uniformity over the functions in $\mathcal{F}$ (not necessarily over the inputs, that would be uniform continuity).

Definition I.1.2

We say $\mathcal{F}$ is <u>uniformly bounded</u> on E provided that there is an $M > 0$ so that $|f(z)| \leq M$ for all $z \in E, f \in \mathcal{F}$.

Question: Suppose $\mathcal{F}$ is a family of differentiable functions $D \to \mathbb{C}$, and further suppose the family of derivatives is uniformly bounded on D. What can we say about $\mathcal{F}$? Well then $\mathcal{F}$ is equicontinuous (or if you'd like, equi-Lipschitz... at least if D is convex).

Proposition I.1.3

Suppose $\mathcal{F}$ is a family of holomorphic functions $D \to \mathbb{C}$ whose derivatives are uniformly bounded. Then $\mathcal{F}$ is equicontinuous at any $z_0 \in D$.

Proof. Let $M > 0$ with $|f'(z)| \leq M$ for all $z \in D$ and $f \in \mathcal{F}$. So take $z \in D$ close to z_0, so that the straight line z to z_0 is contained in D, and then integrate

$$|f(z) - f(z_0)| = \left| \int_{z_0}^{z} f'(\zeta)\, \mathrm{d}\zeta \right| \leq M \cdot |z - z_0|\,.$$

More general–equi-Lipschitz, with the metric on the codomain defined by the infimum of paths contained in D.

Big Deal:

Theorem I.1.4 (Arzelà-Ascoli Theorem, the late 19th century.)

Let $E \subseteq \mathbb{C}$ be compact. Let $\mathcal{F} = \{f_i : E \to \mathbb{C} \mid i \in I\}$ be a family of continuous functions on E that is uniformly bounded. Then the following are equivalent

(1) $\mathcal{F}$ is equicontinuous at all $z_0 \in E$.

(2) Each sequence of functions $f_n \in \mathcal{F}$ has a subsequence which converges uniformly on E.

Recall I.1.1

Sarah's General Advice: If you're trying to prove something exists, change the problem to a question about fixed points, inverse function theorem, or use compactness (build a sequence of approximations, then use compactness).

Next Time: Arzelà-Ascoli Theorem, Hubbard Stuff, donuts, fun!!! Riemann Mapping Theorem after Thanksgiving!