Stuff:

- HW 4B due today
- Purple HW 5
- Office Hours Wednesday/Friday
- Halloween shirts for Michigan Math \$15 (walk around EH for info!)
- Walk 40 miles through rattlesnakes on Saturday
- For HW 4B #7a, prove f is constant on closed unit disk.

Lets go back to the FTC, first we'll do an example

Example .0.1 (Application of **??**)

Consider $z \mapsto \frac{1}{z}$ on a connected open set D that contains the unit circle. Does this have a primitive on D?

No! We know $\int_{S^1} \frac{\mathrm{d}z}{z} = 2\pi i$, which would contradict that $\int_{S^1} \frac{\mathrm{d}z}{z} = F(1) - F(1) = 0$ if there were a primitive F for $\frac{1}{z}$ on D.

Theorem .0.1 (FTC II)

Let $D \subseteq \mathbb{C}$ be an open connected star shaped (can be simply connected) subset of $\mathbb{C}$. Let $f(z)$ be holomorphic on D. Then $f(z)$ has a primitive on D, and the primitive is unique up to adding a constant. The primitive can be given as

$$F(z) = \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta$$

for $z \in D$, where we take any available path.

Proof. Write $f = u + iv$. Consider the differential form $u\,\mathrm{d}x - v\,\mathrm{d}y$. Since f is holomorphic, Cauchy-Riemann implies that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial u}$. Thus $u\,\mathrm{d}x - v\,\mathrm{d}y$ is a closed differential form.

Since D is open, connected, and simply connected, we know $u\,\mathrm{d}x - v\,\mathrm{d}y$ is exact on D. Thus there is a continuously differentiable function U on D so that $\mathrm{d}U = u\,\mathrm{d}x - v\,\mathrm{d}y$. That is $\frac{\partial U}{\partial x} = u$, $\frac{\partial U}{\partial y} = -v$.

Applying Cauchy-Riemann yields

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0.$$

Thus U is harmonic on D. Since D is simply connected, there exists a harmonic conjugate V for U on D such that $G = U + iV$ is holomorphic on D.

We see that

$$G' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = u + iv = f.$$

Perfect! This shows that there is a primitive G for f. To show it is unique up to adding a constant, let H be another primitive, then $G - H$ has derivative zero, so $G - H$ is constant on D.

Finally, if z_0 is any point of D, then by **??** we see that

$$F(z) := \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta = G(z) - G(z_0).$$

Differentiating both sides yields $F'(z) = G'(z) = f$. This completes the proof of all the pieces given above of the theorem.

Corollary .0.2

Integrals of holomorphic functions in star-shaped regions are path-independent.

.1. Cauchy's Theorem

Setting: $f = u + iv$ is holomorphic on a connectedopen set $D \subseteq \mathbb{C}$. Then

$$f(z)\,\mathrm{d}z = (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) = (u + iv)\,\mathrm{d}x + (-v + iu)\,\mathrm{d}y$$

Exercise .1.1

The condition that $f(z)\,\mathrm{d}z$ is closed is exactly the Cauchy-Riemann equations. That is

$$\frac{\partial[u + iv]}{\partial y} = \frac{\partial[-v + iu]}{\partial x}.$$

Theorem .1.1 (Morera)

A continuously differentiable function $f(z)$ on D is holomorphic if and only if the differential $f(z)\,\mathrm{d}z$ is closed on D.

Theorem .1.2 (Cauchy)

Let D be a bounded connected open subset of $\mathbb{C}$ with piecewise smooth boundary. If $f(z)$ is holomorphic on D and if it extends smoothly to ∂D, then

$$\int_{\partial D} f(z)\,\mathrm{d}z = 0.$$

Proof. Apply Green's theorem!

Corollary .1.3

If $f(z)$ is holomorphic on a region that contains an annulus D with inner radius r and outer radius R about z then

$$\int_{|z-w|=r} f(w)\,\mathrm{d}w = \int_{|z-w|=R} f(w)\,\mathrm{d}w.$$

Proof. We know that

$$0 = \int_{\partial D} f(w)\,\mathrm{d}w = \int_{|z-w|=R} f(w)\,\mathrm{d}w - \int_{|z-w|=r} f(w)\,\mathrm{d}w.$$

Theorem .1.4 (Cauchy's Integral Formula for $f(z)$)

Let $D \subseteq \mathbb{C}$ be a bounded, connected, open subset with piecewise smooth boundary. If $f(z)$ is holomorphic on D and $f(z)$ extends continuously to ∂D, then for each $z \in D$ we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w - z}\,\mathrm{d}w.$$

Compare this with $\int_{|w|=1} \frac{\mathrm{d}w}{w} = 2\pi i$. This is the $f(w) = 1, z = 0$ case. In fact we're going to use this to steal the game.

Proof. Fix $z \in D$ and choose $\varepsilon > 0$ so that $\overline{B(z, \varepsilon)} \subseteq D$ (the closed ball).

Define $D_\varepsilon \coloneqq D \setminus \overline{B(z, \varepsilon)}$. We know $w \mapsto \frac{f(w)}{w-z}$ is holomorphic on D_ε and extends smoothly to $\partial D_\varepsilon = \partial D \cup \{|w - z| = \varepsilon\}$, with reversed orientations.

We then have that

$$\int_{D_\varepsilon} \frac{f(w)}{w-z} \, \mathrm{d}w = 0$$

$$\int_{D_\varepsilon} \frac{f(w)}{w-z} = \int_{\partial D} \frac{f(w)}{w-z} \, \mathrm{d}w - \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z} \, \mathrm{d}w$$

$$\int_{\partial D} \frac{f(w)}{w-z} \, \mathrm{d}w = \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z} \, \mathrm{d}w.$$

Thus we can reduce the problem to evaluating the integral over a small circle $|w - z| = \varepsilon$. There are a number of different proofs. The simplest uses the mean value property. Parameterizing $|w - z| = \varepsilon$ as $z + \varepsilon e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ yields

$$\int_{\partial D} \frac{f(w)}{w-z} = \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) i \, \mathrm{d}\theta$$

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} = \int_0^{2\pi} f(z + \varepsilon e^{i\theta} \frac{\mathrm{d}\theta}{2\pi}.$$

Applying the mean value property to u, v where $f = u + iv$ gives us the result.

If we want to approach this using analysis directly, we may do so as below

Note that the value of this integral cannot depend on $\varepsilon > 0$. Fix some $\delta > 0$, we wish to show that

$$\left| 2\pi i f(z) - \int_{\partial D} \frac{f(w)}{w-z} \, \mathrm{d}w \right| < \delta.$$

Well, we may write this as

$$2\pi i f(z) - \int_{|z-w|=\varepsilon} \frac{f(w)}{w-z} \, \mathrm{d}w = \int_{|z-w|=\varepsilon} \frac{f(z) - f(w)}{z-w} \, \mathrm{d}w,$$

because

$$\int_{|z-w|=\varepsilon} \frac{\mathrm{d}w}{z-w} = 2\pi i.$$

Then using the ML-inequality (see [**thm:ml-theorem**]) we have that

$$\left| 2\pi i f(z) - \int_{|z-w|=\varepsilon} \frac{f(w)}{w-z} \, \mathrm{d}w \right| = \left| \int_{|z-w|=\varepsilon} \frac{f(z) - f(w)}{z-w} \, \mathrm{d}w \right| = 2\pi\varepsilon \cdot \sup_{|z-w|=\varepsilon} |f(z) - f(w)| \, .$$

Taking $\varepsilon \to 0$ takes the right hand side to zero, so we win!

Theorem .1.5 (Cauchy's Generalized Integral Formula)

Let $D \subseteq \mathbb{C}$ be a bounded connected open subset with piecewise smooth boundary. Suppose that $f(z)$

is holomorphic on$D$ and $f(z)$ extends smoothly to ∂D, then f has complex derivatives of all orders on D, which are given by

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,\mathrm{d}w.$$

Proof. Proof is via induction on m. Going from $m = 0$ to $m = 1$ is similar enough to going from $m = n$ to $m = n + 1$, so we'll do the first only.

Consider that

$$\frac{1}{w-(z+h)} - \frac{1}{w-z} = \frac{w-z-(w-(z+h)}{(w-(z+h))(w-z)} = \frac{h}{(w-(z+h))(w-z)}.$$

Then we have by Theorem .1.4 that

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{1}{2\pi i h}\int_{\partial D}\left(\frac{f(w)}{w-(z+h)} - \frac{f(w)}{w-z}\right)\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-(z+h))(w-z)}\,\mathrm{d}w.\end{aligned}$$

As we take $h \to 0$, the integrand converges to $\frac{f(w)}{(w-z)^2}$ uniformly on $w \in \partial D$. Thus the integrals converge and we obtain

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^2}\,\mathrm{d}w$$

for $z \in D$.

Corollary .1.6

If $f(z)$ is holomorphic on D, then $f(z)$ is infinitely differentiable and all derivatives are also holomorphic on D.

Example .1.2

We can simply compute

$$\int_{|z|=2}\frac{\sin(2z)}{(z-i)^6}\,\mathrm{d}z.$$

We know $f(z) = \sin(2z)$ is holomorphic on this region, and so Cauchy's integral formula tells us that

$$f^{(5)}(i) = \frac{5!}{2\pi i}\int_{|z|=2}\frac{\sin(2z)}{(z-i)^6}\,\mathrm{d}z.$$

Taking 5 derivatives of $\sin(2z)$ yields $2^5\cos(2z)$. Thus this is

$$2^5\cos(2i) = \frac{5!}{2\pi i}\int_{|z|=2}\frac{\sin(2z)}{(z-i)^6}\,\mathrm{d}z.$$

Noting that $2\cos(\theta) = e^{i\theta} + e^{-i\theta}$, we have that

$$\int_{|z|=2}\frac{\sin(2z)}{(z-i)^6}\,\mathrm{d}z = \frac{2^5\pi i(e^{-2}+e^2)}{5!}$$