Stuff:

- HW 8A due tonight
- HW 8B due Tuesday
- Math Club 4pm-5pm today
- Math Circle 6:30-8pm today
- Bagel Sunday/Popcorn Thursdays
- Super Saturdays 9:30am-12pm Saturday

Recall .0.1

A point $z_0 \in \mathbb{C}$ is an isolated singularity of $f(z)$ provided that $f(z)$ is holomorphic in a punctured disk $\{0 < |z - z_0| < r\}$.

Example .0.2

Let $f(z) = \frac{1}{z-53}$, this has an isolated singularity at $z_0 = 53$.

Non-Example .0.3

The complex logarithm $\mathrm{Log}(z)$. One cannot define the logarithm on any neighborhood of 0.

There are three types of isolated singularities. Set up: Let $f(z)$ have an isolated singularity at z_0. Expand $f(z)$ in a Laurent series about z_0

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

which is valid for $0 < |z - z_0| < r$.

Definition .0.1

The three types of singularities are

(I) $a_k = 0$ for all $k < 0$, in which case we say z_0 is a <u>removable singularity</u>

(II) $a_k \neq 0$ for finitely many $k < 0$ (and at least one), in which case we say z_0 is a <u>pole</u>.

(III) $a_k \neq 0$ for infinitely many $k < 0$, in which case we call z_0 an <u>essential singularity</u>

Note: If z_0 is a removable singularity, then we can define $f(z_0) = a_0$, and this make f analytic in the whole disk $|z - z_0| < r$ since the Laurent series is just a power series.

Example .0.4

If $f(z) = \frac{\sin(z)}{z}$ then the Laurent series is

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots ,$$

and the singularity at $z = 0$ is removable, so f can be extended to an entire function with $f(0) = 1$.

What can we say at $f(z)$ when z is close to a removable singularity z_0? The limit should exist!

- It should have $\lim_{z \to z_0} f(z)$ is some complex number.
- f can be extended continuously

Even better!

Theorem .0.1 (Riemann's Removable Singularity Theorem)

Let z_0 be an isolated singularity of $f(z)$. f is bounded near z_0 if and only if $f(z)$ has a removable singularity at z_0.

Proof. The converse is immediate from the above discussion. For the forward direction, expand $f(z)$ in a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

on $0 < |z - z_0| < \rho$. We know from before that

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{k+1}} \, \mathrm{d}z$$

where $0 < r < \rho$. We want $a_k = 0$ for all $k < 0$. We know $f(z)$ is bounded near z_0 so there exists an M so that $|f(z)| \le M$ for all $0 < |z - z_0| < \rho$ (possibly making ρ smaller).

Using the ML-estimate yields

$$|a_k| = \frac{2\pi r}{2\pi} \frac{M}{r^{k+1}} = \frac{M}{r^k}.$$

If $k < 0$ then this tends to 0 as $r \to 0$.

Definition .0.2

The isolated singularity of $f(z)$ at z_0 is called a pole of order N if there exists an $N > 0$ such that $a_{-N} \neq 0$ but $a_k = 0$ for all $k < -N$.

In this case

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - z_0)^k.$$

In this case, we collect the terms with negative powers of $(z - z_0)$:

$$P(z) \coloneqq P_f(z; z_0) \coloneqq \sum_{k=-N}^{-1} a_k (z - z_0)^k$$

which we call the principal part of f at z_0. This is a piece of the Laurent decomposition from last week.

Example .0.5

$f(z) = 1/z$ has a pole of order 1 at $z_0 = 0$, and

$$g(z) = \frac{1}{(z - 53)^2 (z + 57)}$$

which has a pole of order 2 at $z_0 = 53$ and a pole of order 1 at $z_0 = -57$.

Theorem .0.2

Let z_0 be an isolated singularity of $f(z_0)$. Then z_0 is a pole of order N if and only if we may write

$$f(z) = \frac{g(z)}{(z - z_0)^N}$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

Proof. First for the forward direction. Give the Laurent expansion as

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n = \frac{1}{(z - z_0)^N} \sum_{n=0}^{\infty} a_{n-N} (z - z_0)^n,$$

let $g(z)$ be the right hand power series, then $g(z_0) = a_{-N} \neq 0$, and we win!

For the other direction, just expand $g(z)$ as a power series about z_0 and then divide through to get a Laurent series for f.

Example .0.6

$f(z) = \frac{e^z}{(z-1)^5}$ has a pole at $z_0 = 1$ of order 5.

Theorem .0.3

Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order N if and only if $\frac{1}{f(z)}$ is analytic at z_0 with a zero of order N.

Proof. Use the previous theorem! [Gam03].

Example .0.7

Consider $f(z) = \frac{1}{\sin(z)}$. This has isolated singularities at all $z_0 = n\pi$, for $n \in \mathbb{N}$. What kind of singularities are they?

Well we can use the previous theorem! First check that $\frac{1}{\sin(z)}$ is unbounded near these points, so it's not removable (work over $\mathbb{R}$). The previous theorem tells us to look at $\sin(z)$, which has simple zeros (zeros of order 1) at each $z = n\pi$, so $\frac{1}{\sin(z)}$ has simple poles at $z = n\pi$.

Quick calculation:

$$\sin(z) = \sin(z - n\pi + n\pi) = \cos(n\pi)\sin(z - n\pi) = (-1)^n \left[(z - n\pi) - \frac{(z - n\pi)^3}{3!} \cdots \right].$$

Definition .0.3

We say that a function $f : D \to \mathbb{C}$ is meromorphic on a connected open set $D \subseteq \mathbb{C}$ provided that $f(z)$ is analytic on D except possibly at isolated singularities, each of which is a pole.

Sums/Products/Quotients (as long as denominator is not identically zero).

Theorem .0.4

Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Proof. Suppose z_0 is a pole of $f(z)$ of order N. Write $f(z) = \frac{g(z)}{(z-z_0)^N}$. Then

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \left| \frac{g(z)}{(z - z_0)^N} \right| = \infty$$

since $g(z_0) \neq 0$, and g is continuous.

For the other direction, suppose $|f(z)|$ goes to ∞ as $z \to z_0$. Then since f is not identically zero, we know $f(z)$ is nonzero in a punctured disk around z_0. Here set

$$h(z) := \frac{1}{f(z)},$$

which is analytic on this punctured disk and has $h(z) \to 0$ as $z \to z_0$. Then Riemann's theorem applies and $h(z_0) = 0$. If N is the order of the zero that h has at z_0, then

$$f(z) = \frac{1}{h(z)}$$

has a pole of order N at z_0.

Definition .0.4

The isolated singularity of $f(z)$ at z_0 is said to be essential provided that $a_k \neq 0$ for infinitely many $k < 0$.

Example .0.8

$f(z) = \exp(1/z)$ has an essential singularity at $z = 0$.

We now state some theorems to prove next time.

Theorem .0.5 (Casorati-Weierstrass, 1868)

Suppose z_0 is an isolated singularity of f. Then z_0 is an essential singularity if and only if for every complex number w, there exists a sequence $z_n \to z_0$ so that $f(z_n) \to w$.