Stuff:

- Exam on Tuesday October 25th in class (full time)
- HW 7 due tomorrow
- Office Hours on Friday 1-2:30pm EH3855
- Math Club!
- Popcorn Thursdays/Bagel Sundays!
- Super Saturdays!
- Voting t-shirts
- Student seminar tomorrow (Topic: Conway's topograph by Xuyan), EH 3096 4pm-5pm.

No content from today on the midterm. HW8 will be based on today and will be received tuesday after midterm.

I. Laurent Series

I.1. Laurent Decomposition

This is Gamelin Ch. VI, 1-4 [Gam03], and [Ahl53] 5.1.

Motivating Question: Let $D \subseteq \mathbb{C}$ be open and connected, and let

$$A(D) := \{f : D \to \mathbb{C} \mid f \text{ is analytic on } D\}.$$

How do we understand this?

Example I.1.1

If $D = B_R(z_0)$ or $D = \mathbb{C}$ then

$$A(D) = \left\{ f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \mid \limsup_{k\to\infty} |a_k|^{1/k} \le \frac{1}{R} \right\}.$$

Example I.1.2

Consider $D = \widehat{\mathbb{C}} \setminus \overline{B_r(0)}$. That is analytic on an open disk about ∞.

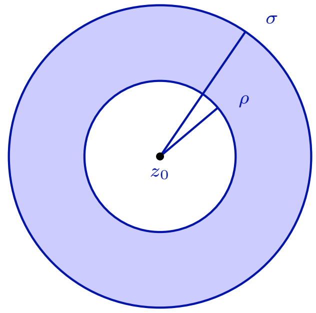
This is $D = \{|z| > r\} \cup \{\infty\}$. Setting $w = 1/z$. This is the same as

$$\begin{aligned} A(D) &= \{f \mid f(z) \text{ analytic on } |z| > r \text{ and } z = \infty\} \\ &\cong \{g \mid g(w) \text{ analytic on } |w| < 1/r\}. \end{aligned}$$

What about the intersection of these? This will be an annulus, and the space $A(D)$ will be larger, as the domain is smaller and so it is *easier* to be analytic here.

Theorem I.1.1 (Laurent Decomposition Theorem)

Suppose $0 \le \rho < \sigma \le \infty$ and suppose $f(z)$ is analytic for $\rho < |z - z_0| < \sigma$. That is, suppose f is analytic on an annulus



Then $f(z)$ can be decomposed as a sum $f(z) = f_0(z) + f_1(z)$ where $f_0(z)$ is analytic on $|z - z_0| < \sigma$ and $f_1(z)$ is analytic on $|z - z_0| > \rho$ and at $z = \infty$.

If we normalize so that $f_1(\infty) = 0$, then this decomposition is unique.

Proof of Uniqueness. Suppose we have $f(z) = f_0(z) + f_1(z) = g_0(z) + g_1(z)$, and normalize so that $f_1(\infty) = g_1(\infty)$. Then we have that

$$g_0(z) - f_0(z) = f_1(z) - g_1(z)$$

for all $\rho < |z - z_0| < \sigma$. We know that $g_0(z) - f_0(z)$ is analytic on $|z - z_0| < \sigma$. Likewise $f_1(z) - g_1(z)$ is analytic in $|z - z_0| > \rho$ and at $z = \infty$. We may define

$$\begin{aligned} h &: \mathbb{C} \to \mathbb{C} \\ z &\mapsto \begin{cases} g_0(z) - f_0(z) & \text{if } |z - z_0| < \sigma \\ f_1(z) - g_1(z) & \text{if } |z - z_0| > \rho \end{cases} . \end{aligned}$$

We see that h is entire, and

$$\lim_{z \to \infty} h(z) = f_1(\infty) - g_1(\infty) = 0.$$

Thus h is bounded on $\mathbb{C}$, and so it is a constant by Liouville's Theorem. Thus $h = 0$, showing that $g_0 = f_0, f_1 = g_1$.

Proof of Existence. We will use the Cauchy integral formula. Choose r and s so that $\rho < r < s < \sigma$. Call the annulus $r < |\zeta| < s$, A. Then keeping track of orientations we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial A} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \left(- \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta + \int_{|\zeta - z_0| = s} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \right). \end{aligned}$$

Define

$$\begin{aligned} f_0(z) &\coloneqq \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \\ f_1(z) &\coloneqq -\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta. \end{aligned}$$

We know f_0 is defined on $|z - z_0| < s$, and f_1 is defined on $|z - z_0| > r$.

HW 4A (up to small changes) shows that they're analytic on these areas (note we're only plugging in ζ on the region where f is continuous). Although this depends on r, s, because of the uniqueness of the decomposiion in fact f_0, f_1 must not depend on our choice of r, s!

Question: Can we get independence of the decomposition with only Cauchy integral formula?

Express $f_1(z)$ as a power series in $\frac{1}{z-z_0}$, then

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k, |z-z_0| > \rho$$

Note $f_1(\infty)$ is the constant term, in this series which is 0. The series for $f_1(z)$ converges absolutely for any $r > \rho$, and it converges uniformly for $|z - z_0| \geq r$.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \lim_{K\to\infty} \left(\sum_{k=0}^{K} a_k (z-z_0)^k + \sum_{k=-K}^{-1} a_k (z-z_0)^k \right).$$

when $\rho < |z - z_0| < \sigma$. This converges absolutely and converges uniformly for $\rho < r \leq |z - z_0| \leq s < \sigma$.

How do we get coefficients? Namely what's a formula for a_k? Well let $\rho < r < s < \sigma$. We know

$$f(z) = \sum_{-\infty}^{\infty} a_k (z-z_0)^k$$

How do we extract a_n for $n \in \mathbb{Z}$, divide both sides by $(z-z_0)^{n+1}$ and integrate. Then

$$\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z = \int_{|z-z_0|=r} \frac{1}{(z-z_0)^{n+1}} \sum_{-\infty}^{\infty} a_k (z-z_0)^k.$$

We have uniform convergence, so we can swap integral and series to get

$$\sum_{-\infty}^{\infty} a_k \int_{|z-z_0|=r} (z-z_0)^{k-n-1} \, \mathrm{d}z = 2\pi i a_n,$$

because only the integral where $k - n - 1 = -1$ does not vanish! Therefore

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}$$

Example I.1.3

Let $f(z) = \frac{1}{z-z^2}$, which is defined for $z \neq 0, 1$.

Let $z_0 = 0$. We want to find the Laurent series decomposition. We do this using partial fractions, but in fact this depends on the annulus we consider!!! If we take $\rho = 0, \sigma = 1$, then we get

$$f(z) = \underbrace{\frac{1}{z}}_{f_1} + \underbrace{\frac{1}{1-z}}_{f_0} = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$$

$$=$$

which is valid for $|z| < 1$.

If we take $\rho = 1, \sigma = \infty$, then we get $f_0 = 0$ and $f_1 = \frac{1}{z-z^2}$. Then

$$f(z) = \frac{-1}{z^2} \cdot \frac{1}{1-\frac{1}{z}} = \sum_{k=0}^{\infty} -z^{-k-2}.$$

which converges for $|z| > 1$.

Theorem I.1.2 (Laurent Series Expansion)

Suppose $0 \leq \rho < \sigma \leq \infty$ and suppose $f(z)$ is analytic on the annulus $\rho < |z - z_0| < \sigma$. Then $f(z)$ has a Laurent series expansion that converges absolutely at each point in the annulus and converges uniformly on each subannulus $r \leq |z - z_0| \leq s$, where $\rho < r < s < \sigma$.

The coefficients are uniquely determined by $f(z)$ and given as

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z.$$

I.2. Isolated Singularities

Definition I.2.1

A point $z_0 \in \mathbb{C}$ is an isolated singularity of $f(z)$ if $f(z)$ is analytic in a punctured disk centered at z_0.

Example I.2.1

$f(z) = 1/z$

Non-Example I.2.2

$\mathrm{Log}(z)$ does not have an isolated singularity at $z = 0$. There is no argument function on any neighborhood of 0.

Isolated singularities come in 3 types.

Definition I.2.2

The isolated singularity of $f(z)$ at z_0 is said to be removable if a Laurent series about z_0 has $a_k = 0$ for all $k < 0$.

In this case, the Laurent series becomes an honest power series, and $f(z)$ can be continued analytically to z_0 with $f(z_0) = a_0$.

Example I.2.3

$f(z) = \frac{\sin z}{z}$ has an isolated singularity at $z = 0$. This is a removable singularity since

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots.$$