Stuff:

- Math club today 4pm
- Math S^1 this evening 6:30-8pm
- Halloween shirts! Order by 10/8
- Bagel Sunday at 11:30
- MMHH Sunday afternoon
- 40 Mile Walk this saturday.

Exercise .0.1 (Warmup)

Compute the following for $n \in \mathbb{N}$

$$\int_{|w|=1} \frac{e^w}{w^n} \, \mathrm{d}w.$$

Answer: $\frac{2\pi i}{(n-1)!}$, using the Cauchy integral formula.

.1. Liouville's Theorem

This is given in [Gam03, Gamelin IV.5].

Setting: $f(z)$ is holomorphic on a closed disk $\{|z - z_0| \leq \rho\}$. By our convention, f is holomorphic on a neighborhood of that closed disk.

The Cauchy integral formula is

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(w)}{(w-z)^{m+1}} \, \mathrm{d}w.$$

We may then parameterize the circle as $w = z_0 + \rho e^{i\theta}$, and since $\mathrm{d}w = \rho i e^{i\theta}$ we get that

$$f^{(m)}(z_0) = \frac{i \cdot m!}{\rho^m} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) e^{-im\theta} \frac{\mathrm{d}\theta}{2\pi}.$$

Then applying the triangle inequality yields that

$$\left| f^{(m)}(z_0) \right| \leq \frac{m!}{\rho^m} \int_0^{2\pi} \left| f(z_0) + \rho e^{i\theta} \right| \frac{\mathrm{d}\theta}{2\pi}.$$

Theorem .1.1 (Cauchy estimates)

Suppose $f(z)$ is holomorphic for $|z - z_0| \leq \rho$. Then if $|f(z)| \leq M$ for $|z - z_0| = \rho$, then

$$\left| f^{(m)}(z_0) \right| \leq \frac{m!}{\rho^m} M,$$

for $m \geq 0$.

Proof. ML estimate.

Theorem .1.2 (Liouville's Theorem)

Let $f(z)$ be an entire function. If $f(z)$ is bounded, then it is constant!

Proof. We show the derivative is zero at any $z_0 \in \mathbb{C}$, say f is bounded by M on $\mathbb{C}$. Take $m = 1$ and send $\rho \to \infty$ in the Cauchy estimate, then $|f'(z_0)| \leq \frac{M}{\rho}$, the right hand side goes to zero, so $f'(z_0) = 0$.

We'll now see an application of this theorem to the theory of Riemann surfaces.

Recall: Sarah said earlier that there are exactly 3 "different" types of simply connected Riemann surfaces (complex manifolds of dimension 1).

Examples of Riemann surfaces: $\mathbb{C}, \widehat{\mathbb{C}}, \mathbb{H}, \mathbb{D}$, and any open $U \subseteq \mathbb{C}$.

We call 2 Riemann surfaces X, Y equivalent provided there is a biholomorphism (that is a bijective holomorphic map with holomorphic inverse) $\varphi : X \to Y$.

Lets look at some examples

$$
\begin{tikzcd}
\text{Spherical} & \text{Hyperbolic} & \text{Euclidean} \\
\widehat{\mathbb{C}} & \mathbb{H} \arrow[d, leftrightarrow, "\text{Cayley map}"'] & \mathbb{C} \\
& \mathbb{D} &
\end{tikzcd}
$$

Because $\widehat{\mathbb{C}}$ is compact we know it is not equivalent to $\mathbb{H}$ or $\mathbb{C}$. Liouville's theorem tells us that since any map $\mathbb{C} \xrightarrow{\varphi} \mathbb{D}$ would be entire and bounded. . . it would be constant! Thus $\mathbb{C}$ is not equivalent to $\mathbb{D}$ either.

How does one find a conformal map $\varphi : U \to \mathbb{D}$ when $U \subsetneq$ is open and simply connected?

2-dimensional manifolds it's much much much harder. Bill Thurston's geometrization program was all about this, and led to the proof of the Poincare conjecture by Perelmann.

Exercise .1.1

Simpler simpler case, take a square and find a conformal isomorphism to the disk. This is hard.

Take a fractal (say a julia set!) and find a conformal isomorphism from $\widehat{\mathbb{C}} \setminus$fractal $\to \widehat{\mathbb{C}} \setminus$closed unit disk. This is in fact easier than the square problem. . .

If you've taken 592: Fix a Riemann surface X. The universal cover of X is either $\widehat{\mathbb{C}}, \mathbb{C}$, or $\mathbb{D}$.

We'll see some proofs of statements like this later. Another application!

Theorem .1.3 (Fundamental Theorem of Algebra)

Every nonconstant polynomial $p(z)$ has a root in $\mathbb{C}$.

Proof. Let $p(z)$ be a nonconstant polynomial with no root in $\mathbb{C}$. Then $z \mapsto \frac{1}{p(z)}$ is entire on $\mathbb{C}$. Is it bounded on $\mathbb{C}$? p is a polynomial, so on a large disk, p "looks likes" z^n. Namely with some Pain in the Ass Estimates

$$
\begin{aligned}
\lim_{z \to \infty} p(z) &= \infty \\
\lim_{z \to \infty} \frac{1}{p(z)} &= 0.
\end{aligned}
$$

Thus on outside a large disk $\frac{1}{p(z)}$ is smaller than 53, and inside of the disk it is bounded by the Extreme Value Theorem.

.2. Morera's Theorem

Theorem .2.1 (Morera)

Let $f(z)$ be a <u>continuous</u> function on a connected open subset $D \subseteq \mathbb{C}$. If $\int_{\partial R} f(z)\, \mathrm{d}z = 0$ for every closed rectangle $R \subseteq D$ with sides parallel to the real/imaginary axes, then f is holomorphic on D with continuous derivative.

Proof. Suppose D is a disk with center z_0 (this is sufficient by openness, since everything is local).

Define $F : D \to \mathbb{C}$ as

$$F : z \mapsto \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta.$$

The path of integration is taxicabs, and is well-defined by the assumption above. We now compute the derivative. Fix $z \in D$, and take $|h|$ to be small enough so that $z + h \in D$.

Then we have that

$$\begin{aligned} F(z+h) - F(z) &= \int_{z}^{z+h} f(\zeta)\,\mathrm{d}\zeta \\ &= \int_{z}^{z+h} f(\zeta)\,\mathrm{d}\zeta + \int_{z}^{z+h} (f(z) - f(z))\,\mathrm{d}\zeta \\ &= hf(z) + \int_{z}^{z+h} f(\zeta) - f(z)\,\mathrm{d}\zeta \\ \frac{F(z+h) - F(z)}{h} &= f(z) + \frac{1}{h}\int_{z}^{z+h} f(\zeta) - f(z)\,\mathrm{d}\zeta. \end{aligned}$$

Now using the ML-inequality we know that

$$\begin{aligned} \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \frac{1}{|h|}\left|\int_{z}^{z+h} f(\zeta) - f(z)\,\mathrm{d}\zeta\right| \\ &\le 2M_h, \end{aligned}$$

where M_h is the maximum value of $|f(\zeta) - f(z)|$ for ζ satisfying $|\zeta - z| \le |h|$. Notice: the 2 comes from the taxicab metric.

Since $f(\zeta)$ is continuous at z, $M_h \to 0$ as $h \to 0$.

Note: We assume f is continuous, so since $F' = f$, we know F' is continuous. This means $F(z)$ is holomorphic and it has continuous derivative. Apply Cauchy integral formula to get f' exists and is continuous.

.3. Goursat's Theorem

We're going to get rid of Gamelin assumption in definition of holomorphic. Recall: Gamelin assumes $f'(z_0)$ exists and f' is continuous in a neighborhood of z_0.

Theorem .3.1 (Goursat)

If $f(z)$ is a complex-valued function on a connected open set D such that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists at each $z_0 \in D$, then f' is continuous on$D$. Thus f is holomorphic in the sense of Gamelin.

Proof. Idea: Use Morera.

Let R_0 be a closed rectangle in D with sides parallel to the coordinate axes. Divide R_0 into 4 equally sized subrectangles $S_{11}, \ldots, S_{14}$. Let R_1 be the subrectangle for which

$$\left|\int_{\partial(\text{subrect})} f(z)\,\mathrm{d}z\right|$$

is maximal. Note

$$\begin{aligned}\left|\int_{\partial R} f(z)\,\mathrm{d}z\right| &= \left|\int_{\partial S_{11}} f(z)\,\mathrm{d}z + \int_{\partial S_{12}} f(z)\,\mathrm{d}z + \int_{\partial S_{13}} f(z)\,\mathrm{d}z + \int_{\partial S_{14}} f(z)\,\mathrm{d}z\right| \\ &\le 4\left|\int_{\partial R_1} f(z)\,\mathrm{d}z\right|.\end{aligned}$$

Induct! Divide R_1 into 4-subrectangles and call R_2 the subrectangle (of R_1) for which $\left|\int_{\partial(\text{subrect})} f(z)\,\mathrm{d}z\right|$ is maximal.

In this way, we get a sequence of nested rectangles $R =: R_0 \supseteq R_1 \supseteq R_2 \supseteq R_3$ such that

$$\begin{aligned}\left|\int_{\partial R_j} f(z)\,\mathrm{d}z\right| &\le 4\left|\int_{\partial R_{j+1}} f(z)\,\mathrm{d}z\right| \\ \left|\int_{\partial R} f(z)\,\mathrm{d}z\right| &\le 4^n\left|\int_{\partial R_n} f(z)\,\mathrm{d}z\right|.\end{aligned}$$

As $n \to \infty$, the rectangles shrink down to a single point (since their diameters shrink to 0), which we call z_0.

Furthermore, if L is the perimeter of R, then $\frac{L}{2^n}$ is the length of ∂R_n. Now since $f(z)$ is complex differentiable at z_0, we know that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that for all $n > N$ and $z \in R_n$ we have

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \le \varepsilon.$$

Thus if we let

$$\varepsilon_n := \sup_{z \in R_n}\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right|,$$

we know $\varepsilon_n \to 0$ as $n \to \infty$.

Now we write

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon_n\,|z - z_0|$$

Consider $z \mapsto f(z_0) + f'(z_0)(z - z_0)$. This is an affine function of z, so it is holomorphic in z, and it has a primitive $G(z)$ on R_n, so we have

$$\int_{\partial R_n} f(z_0) + f'(z_0)(z - z_0)\,\mathrm{d}z = 0$$

$$\begin{aligned}\left|\int_{\partial R_n} f(z)\,\mathrm{d}z\right| &= \left|\int_{\partial R_n} f(z) - f(z_0) - f'(z_0)(z - z_0)\,\mathrm{d}z\right| \\ &\le \int_{\partial R_n} \varepsilon_n\,|z - z_0|\,\mathrm{d}z.\end{aligned}$$

Since $|z - z_0|$ is at most $P_n/2$¡ where $P_n = P/2^n$ is the perimeter of R_n and P is the perimeter of R, we have that

$$\left|\int_{\partial R_n} f(z)\,\mathrm{d}z\right| \le \varepsilon_n \frac{P}{2^n} \cdot \frac{P}{2 \cdot 2^n}.$$

Then we know that

$$\left|\int_{\partial R} f(z)\,\mathrm{d}z\right| \le 4^n\left|\int_{\partial R_n} f(z)\,\mathrm{d}z\right| \le \frac{\varepsilon_n P^2}{2}.$$

Taking $n \to \infty$ takes the right hand quantity to 0 and the left hand quantity does not depend on n! Perfect! Thus the integral is zero and we win by applying Morera!