Stuff:

- Last Class: Good Job! ☺
- Homework due tonight
- No Office Hours tomorrow (stop by atrium tonight!)
- Volunteers (10am tomorrow)! Free t-shirts.
- DRP Presentations at Math Club 4-5pm.
  - Chip Firing Game
  - Dynamical Systems
  - Elliptic Curves
- **Final Next Week!**
  - Friday December 16th 1:30-3:30pm in this classroom
  - 1 double-sided Sheet of Notes allowed
  - Evenly distributed through content of whole course
- Office Hours next week! Thursday December 15th 1-4pm.

One final topic to cover!

I. Infinite Products

This is [Gam03] page 353.

Definition I.0.1

An infinite product is an expression of the form

$$\prod_{j=1}^{\infty} p_j$$

where each p_j is a complex number.

We say this converges provided that

(1) p_j converges to 1.

(2) $\sum_{j=1}^{\infty} \operatorname{Log}(p_j)$ converges as a series. We only sum over terms where $p_j \neq 0$.

This is a strange definition (think about why it's strange), but it is convenient.

If the infinite product converges, we define its value to be 0 if one of the p_j is zero. Otherwise we define the limit to be

$$\prod_{j=1}^{\infty} p_j := \exp\left(\sum_{j=1}^{\infty} \operatorname{Log} p_j\right).$$

Observations:

(1) If $\prod p_j$ converges then at most finitely many of p_j are 0.

(2) If $\prod p_j$ converges, then in fact

$$\prod_{j=1}^{\infty} p_j = \lim_{m\to\infty} \prod_{j=1}^{m} p_j$$

(3) We can always factor out a finite # of terms from a convergent infinite product

$$\prod_{j=1}^{\infty} p_j = p_1 \cdots p_m \prod_{j=m+1}^{\infty} p_j.$$

(4) If an infinite product converges, and if none of the factors is zero, then the product is not zero.

Note: This excludes a product like $\prod_{j=1}^{\infty} \frac{1}{2} = 0$, even though this makes sense sort of intuitively.

Example I.0.1

Consider the following product

$$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{k}\right) = (1+1)\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\cdots.$$

Using the power series of the logarithm we can see that this converges. Then there is a subsequence of the partial products which is always 1. If we call the terms p_j, then because

$$\left(1 + \frac{1}{2k-1}\right)\left(1 - \frac{1}{2k}\right) = 1,$$

the product of the first m terms is always 1 if m is even. If m is odd then it is equal to $\frac{1}{1+\frac{1}{m}}$.

General idea: It is helpful to write $p_j = 1 + a_j$ and look at the product as $\prod(1 + a_j)$, as one can often exploit the power series expansion of Log near 1 to compute $\sum \operatorname{Log}(1 + a_j)$.

Gamelin Notes:

- If $0 \le t \le 1$, then $\frac{t}{2} \le \log(1+t) \le t$. Exercise in basic analysis.
- As a consequence, if $t_j \ge 0$ then $\sum t_j$ if and only if $\sum \log(1 + t_j)$ converges.

  Why? Well in either case $t_j \to 0$ so eventually $0 \le t \le 1$, so we're in business to use the comparison test of the above.

Theorem I.0.1

If $t_j \ge 0$, then

$$\prod_{j=1}^{\infty}(1 + t_j) \text{ converges} \iff \sum t_j \text{ converges}$$

Application: Let $\alpha > 0$. Then consider

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{k^{\alpha}}\right).$$

Claim

This converges for $\alpha > 1$ and diverges for $0 \le \alpha \le 1$.

Direct from p-series test from real analysis and the test above.

Definition I.0.2

The infinite product $\prod_{j=1}^{\infty}(1 + a_j)$ is said to converge absolutely provided that $a_j \to 0$ as $j \to \infty$ and $\sum \operatorname{Log}(1 + a_j)$ converges absolutely. Note: if the product converges absolutely then the product itself converges.

Remark I.0.1

It is very unnatural to look at $\prod |1+a_j|$. Why is that unnatural? Explain it as an exercise.

Theorem I.0.2

The infinite product $\prod_{j=1}^{\infty}(1+a_j)$ converges absolutely if and only if $\sum a_j$ converges absolutely, which occurs if and only if $\prod(1+|a_j|)$ converges.

Proof. See Gamelin, routine.

Example I.0.2

We know $\prod\left(1+\frac{(-1)^{k+1}}{k}\right)$ converges, but does not converge absolutely.

Application: 3blue1brown video and the Riemann zeta (ζ) function. For more, see the Prime Number Theorem section in Gamelin [Gam03].

Definition I.0.3

For $\mathrm{Re}(s) > 1$ we define

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In fact this converges absolutely and it converges uniformly for $\mathrm{Re}(s) \geq 1+\varepsilon$ for any $\varepsilon > 0$. It will have *severe* problems at $s = 1$.

This function can be continued across the entire plane except at $s = 1$. Call this extension ξ.

Corollary I.0.3 (Riemann Hypothesis)

If s is a zero of the Riemann zeta then s is a negative even integer or $\mathrm{Re}(s) = 1/2$.

Theorem I.0.4 (Euler)

If $\mathrm{Re}(s) > 1$, then

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right).$$

Proof Idea. Consider $\frac{1}{1-p^{-s}}$. We like this because it is a geometric series

$$\frac{1}{1-p^{-s}} = \sum_{n=0}^{\infty} p^{-ns}.$$

Take your favorite prime numbers $p_1, \ldots, p_m$, and multiply these expressions

$$\prod_{k=1}^{m} \frac{1}{1-p_k^{-s}} = \sum_{\ell_1,\ldots,\ell_m=0}^{\infty} \left(p_1^{\ell_1} \cdots p_m^{\ell_m}\right)^{-s}.$$

This is in fact $\zeta(s)$ as $m \to \infty$ by the Fundamental Theorem of Arithmetic.