Stuff:

- HW 3B due tonight!
- HW 4 – Due Thursday (4A) and Tuesday (4B)
- Lots of Department Stuff!
  - Math S^1
  - Super Saturdays
  - Bagel Sundays
  - 20 mile walk 9/24, 40 mile walk 10/1
  - U(M) Student Seminar EH 3096 4-5pm this Friday: Circle Method and Waring's problem by Xun
  - Math Mental Health Hour EH1866 every 2 weeks Sunday afternoon

We had a problem on the homework which shows that

Claim

If p is a complex polynomial whose zeroes lie in the half plane $\mathrm{Re}(z) > 0$, then the zeros of p' lie in this half-plane as well.

Theorem .0.1 (Gauss-Lucas Theorem)

If p is a complex polynomial, then the zeros of p' lie in the convex hull of the zeros of p.

Recall .0.1

A subset A of $\mathbb{R}^n$ is convex provided that for any two points $a, b \in \mathbb{R}^n$ the line between a, b is a subset of A.

The convex hull of a set of points $\{z_1, \ldots, z_m\}$ is the set of all combinations $\sum t_j z_j$ such that $0 \leq t_j \leq 1$ and $\sum t_j = 1$.

Back to integration!

Lemma .0.2

If $P, Q : D \to \mathbb{C}$ are continuous functions on a connected open set, then $\int P\,\mathrm{d}x + Q\,\mathrm{d}y$ is path independent if and only if the form is exact.

See Gamelin [Gam03, Chapter III].

Lemma .0.3

Exact differentials are closed

Proof. Suppose $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact. Then there exists h such that $\mathrm{d}h = P\,\mathrm{d}x + Q\,\mathrm{d}y$, that is

$$\frac{\partial h}{\partial x} = P \qquad\qquad \frac{\partial h}{\partial y} = Q.$$

Then we have that

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\frac{\partial h}{\partial y} = \frac{\partial Q}{\partial x}$$

Definition .0.1

The connected open set $D \subseteq \mathbb{C}$ is called star shaped provided that there exists some $z_0 \in D$ such that for all $z \in D$, the line between z_0 and z is contained in the set.

It is convenient to call z_0 the "spectator" as a nice piece of terminology. Note that this is not necessarily unique.

This implies simply connected (contract everything to the spectator), but the converse is not true. Consider a horseshoe, which is simply connected but not star-shaped.

Theorem .0.4

Let P, Q be continuous differentiable functions on a connected open subset $D \subseteq \mathbb{C}$. Suppose

(1) D is star-shaped (simply connected suffices, but we won't prove it).

(2) $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is closed on D.

Then $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact.

Proof Sketch. Let A be a spectator for D. Define

$$\begin{aligned} h : D &\to \mathbb{C} \\ B &\mapsto \int_A^B P\,\mathrm{d}x + Q\,\mathrm{d}y. \end{aligned}$$

We take the path of integration $A \to B$ to be the straight line from A to B. We must then check that this h works. See [Gam03]

Example .0.2

Consider the 1-form

$$P\,\mathrm{d}x + Q\,\mathrm{d}y = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}$$

on $\mathbb{C} \setminus \{0\}$. This is closed by a simple computation, but it isn't exact/path-independent! Why? Well integrate about the unit circle C in the counterclockwise direction.

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = 2\pi.$$

However, $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact on $\mathbb{C} \setminus (-\infty, 0]$. $P\,\mathrm{d}x + Q\,\mathrm{d}y = \mathrm{d}\,\mathrm{Arg}(x + iy)\ldots$

We'll now apply this discussion to harmonic conjugates (Faye's question). Take $u \in C^1$ with second order partials with Δu, then does there exist a v harmonic so that $f = u + iv$ is holomorphic.

Lemma .0.5

If $u(x, y)$ is harmonic on an open connected set $D \subseteq \mathbb{C}$, then the differential

$$-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y$$

is closed.

Proof. A simple computation.

Id D is star-shaped, then we know that is exact, so

$$\mathrm{d}v = -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y.$$

But then we have that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad\qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

so that $f = u + iv$ is holomorphic!

Theorem .0.6

Any harmonic function $u(x, y)$ on a star-shaped domain (can promote this to simply connected) $D \subseteq \mathbb{C}$ has a harmonic conjugate $v(x, y)$ on D.

Example .0.3

Consider $u = \log|z|$ on $\mathbb{C} \setminus (-\infty, 0]$. We showed this was harmonic (hw). By the previous discussion there exists a harmonic conjugate $v(x, y)$ on $\mathbb{C} \setminus (-\infty, 0]$. Then

$$\begin{aligned} u(x, y) &= \frac{1}{2}\log(x^2 + y^2) \\ \mathrm{d}u &= \frac{x}{x^2 + y^2}\,\mathrm{d}x + \frac{y}{x^2 + y^2}\,\mathrm{d}y \\ \mathrm{d}v &= \frac{-y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y \\ v(z) &= \int_1^z \mathrm{d}v. \end{aligned}$$

This is choosing the conjugate from the proof above, normalized so that $v(1) = 0$. This tells us that

$$\operatorname{Arg}(z) = \int_1^z \frac{-y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y,$$

as harmonic conjugates are unique up to adding constants.

We're now in [Gam03, Gamelin III.4], the Mean Value Property.

Definition .0.2

Take $D \subseteq \mathbb{C}$ to be a connected open subset containing the disk $\{|z - z_0| < \rho\}$. Let $h : D \to \mathbb{R}$ be continuous.

We define the average value of h on the circle $\{|z - z_0| = r\}$ for $0 < r < \rho$ to be

$$A_h(r, z_0) \coloneqq \frac{1}{2\pi}\int_0^{2\pi} h\left(z_0 + re^{i\theta}\right)\mathrm{d}\theta.$$

Theorem .0.7

If $u(z)$ is a harmonic function on D (as defined above), then

$$A_u(r, z_0) \coloneqq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,\mathrm{d}\theta = u(z_0).$$

Proof. We see because $-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}$ is closed/exact/path-independent (on $|z - z_0| < \rho$) that

$$0 = \int_{|z - z_0| = r} -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y.$$

We parameterize the circle as $x(\theta) = x_0 + r\cos(\theta), y(\theta) = y_0 + r\sin(\theta)$. We see that thsi is exactly

$$0 = \int_0^{2\pi} \left(\frac{\partial u}{\partial y} \cdot (r \sin\theta) + \frac{\partial u}{\partial x} \cdot (r\cos\theta) \right) \mathrm{d}\theta.$$

We then see via the chain rule that this is

$$0 = r \int_0^{2\pi} \frac{\partial u}{\partial r} \, \mathrm{d}\theta$$

Divide both sides by $2\pi r$ and exchanging the integral with the differentation, we see that

$$0 = \frac{\partial}{\partial r} \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} = \frac{\partial}{\partial r} A_u(r, z_0).$$

Thus $A_u(r, z_0)$ is constant in r, and since u is continuous at z_0, as $r \to 0$, the average value of the function tends to $u(z_0)$ as $r \to 0$. Thus $A_u(r, z_0) = u(z_0)$.

Definition .0.3

We say that a continuous function $h(z)$ on a connected open domain $D \subseteq \mathbb{C}$ has the mean value property provided that for all $z_0 \in D$, $h(z_0)$ is the average of its values over any small circle centered at z_0. I.e., for all $z_0 \in D$, there exists an $\varepsilon > 0$ such that for all $0 < r < \varepsilon$, we have

$$h(z_0) = \int_0^{2\pi} h(z_0 + re^{i\theta}) \frac{\mathrm{d}\theta}{2\pi}.$$

Now we move to [Gam03, Gamelin III.5].

Definition .0.4

Let $u(z)$ be a real-valued continuous function with the mean value property (so including harmonic functions!) on the connected open subset $D \subseteq \mathbb{C}$. Suppose there exists $M \in \mathbb{R}$ such that for all $z \in D$, $u(z) \leq M$. If $u(z_0) = M$ for some $z_0 \in D$, then u is constant.

Proof Strategy. Let $S_M \coloneqq \{z \in D \mid u(z) = M\}$ and $S_{<M} \coloneqq \{z \in D \mid u(z) < M\}$. We see that $D = S_M \coprod S_{<M}$. We know that $z_0 \in S_M M$

We know that $S_{<M}$ is open, so it suffices to show that S_M is open, as then $D = S_M$. Suppose that $u(z_1) = M$, i.e., $z_1 \in S_M$.

We use the Mean Value property to write

$$M = u(z_1) = \int_0^{2\pi} u(z_1 + re^{i\theta} \frac{\mathrm{d}\theta}{2\pi}.$$

We may then rearrange to give that

$$0 = \int_0^{2\pi} [M - u(z_1 + re^{i\theta})] \frac{\mathrm{d}\theta}{2\pi}.$$

But wait! This is a non-negative continuous integrand!!! So the integral is zero if and only if $M = u(z_1 + re^{i\theta})$ for all θ. Thus for $0 < r < \rho$ we have

$$u(z + re^{i\theta}) = M,$$

so S_M is open as desired.