Notes on
MATH 596
($\mathbb{C}$omplex Analysis)

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I. Introduction and Administration

See the syllabus (see **??**)! As a summary

- Prereqs: Analysis at the level of 451, multivariable calculus, and familiarity with complex numbers.
- HW: Generally due Monday (proofy homework, B homework) and Wednesday (calculationy homework, A homework). Two lowest homeworks in both series are dropped (four total).
- Goal: Prepare PhD students for the qualifying exams.
- Book: Gamelin *Complex Analysis* (see [Gam03]) and an older book by Ahlfors (see [Ahl53]).
- Grading Scheme
  - 45% Homework
  - 25% Midterm (in class Oct 25th)
  - 30% Final (Dec 15th 1:30pm-3:30pm)

The first A homework is due tomorrow (8/31) and the first B homework is due next Tuesday due to Labor Day (9/6).

II. The Basics

II.1. Motivation and Recollections

$\mathbb{C}$-analysis is a nexus for lots of fields:

- Algebra (fields and solving equations)
- Algebraic geometry and complex manifolds
- Geometry (platonic solids, flat tori, hyperbolic manifolds in dimensions 2 and 3)
- Lie Groups
- ♡ dynamics ♡
- Number theory (automorphic forms, elliptic functions, zeta functions)
- Riemann surfaces (Teichmuller theory, curves and their Jacobians)
- Several complex variables and complex manifolds
- Real analysis and PDEs (harmonic functions, elliptic equations, distributions)

The complex numbers are formally defined as the field $\mathbb{C} = \mathbb{R}[i]$ where $i^2 = -1$, They are represented in the Euclidean plane by $z = (x, y) = x + iy$. There are 2 square roots of -1 in $\mathbb{C}$, the number i is the one with positive imaginary part.

Definition II.1.1

If $z = x + iy$ we define the real and imaginary parts as

$$\Re(z) \coloneqq x \qquad\qquad \Im(z) \coloneqq y$$

We should recall the definition of addition, multiplication, and division in $\mathbb{C}$. If $z_j = x_j + iy_j$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i.$$

Because $\mathbb{C} = \mathbb{R}[i]$ there is a Galois automorphism for this field extension

$$z = x + iy \mapsto \overline{z} := x - iy$$

called complex conjugation which fixes $\mathbb{R}$. We know

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}.$$

We then define the norm squared of z to be the multiplication of all its Galois conjugates (works for any finite Galois extension), that is

$$|z|^2 := z \cdot \overline{z} = x^2 + y^2 \in \mathbb{R}_{\geq 0}.$$

Compatibility of $|z| = \sqrt{|z|^2}$ with the Euclidean metric on $\mathbb{R}^2$. This along with the fact that $\mathbb{C} \cong \mathbb{R}^2$ as a vector space justifies the identification of $\mathbb{C}$ with $\mathbb{R}^2$

If $z \neq 0$, $z = x + iy$ then

$$\frac{z}{|z|} = \frac{x}{|z|} + i\frac{y}{|z|}$$

lies on the unit circle S^1. Thus we can write this complex number as

$$\frac{z}{|z|} = \cos\theta + i\sin\theta$$

for some $\theta \in \mathbb{R}$. Please note θ is uniquely defined only up to adding integer multiples of 2π and only when $z \neq 0$. This number θ is called the argument of z, denoted $\arg(z)$.

We let

$$e^{i\theta} := \cos\theta + i\sin\theta$$

so that $z = |z|\, e^{i\theta}$ in these "polar" coordinates. The nice thing about polar coordinates is that if $z = re^{i\alpha}, w = \rho e^{i\beta}$ then $zw = r\rho \cdot e^{i(\alpha+\beta)}$. Thus

$$|zw| = r_1 r_2.$$

II.2. **Solving Polynomial Equations**

A critical feature of $\mathbb{C}$ is that it is algebraically closed. In other words, we have

Theorem II.2.1 (The Fundamental Theorem of Algebra)

Every nonconstant polynomial in $\mathbb{C}[X]$ has a root in $\mathbb{C}$.

This is the historical origin of the complex numbers. In the 16th century, Cardano and others were solving cubic equations over $\mathbb{R}$ which have solutions in $\mathbb{R}$! However, their calculations/algorithms included complex numbers which *cancelled* in the end to give real solutions.

Example II.2.1

Fix $a \in \mathbb{C}$. We must solve $z^2 = a$. If $a = 0$ there is one solution, $z = 0$.

If $a \neq 0$, write $a = re^{i\theta}$, then we may take $\omega = e^{i\theta/2}, -\omega = e^{i(\theta/2+\pi)}$ so that $(\pm\omega)^2 = e^{i\theta}$. Then $\pm\omega\sqrt{r}$ are two roots of a, and neither can be "preferred"

Proposition II.2.2 (see HW)

There is no continuous (topology!) function $f : \mathbb{C} \to \mathbb{C}$ such that $(f(z))^2 = z$ for all $z \in \mathbb{C}$. In other words, there is no continuous choice of a square root.

Proof. HW.

Definition II.2.1

Let $n \in \mathbb{N}$ and consider the equation $z^n = 1$. The solutions of this equation are called the roots of unity of order n.

Proof there are n n-th roots of unity. Note if $z^n = 1$, then $|z|^n = 1$, so $|z| = 1$. Note $z^n - 1$ has polynomial derivative nz^{n-1}, whose only solution is 0, so no roots are repeated.

Explicitly, we have solutions $z_j = e^{i\theta_j}$ where

$$\theta_j = \frac{2\pi j}{n}$$

for $j \in \mathbb{Z}$. Up to integer multiples of 2π, there are n such arguments, and n such z_j.

A similar arguments shows that any nonzero complex number z has n different n-th roots that are exactly spaced around the circle of radius $|z|^{1/n}$.

II.3. Topology Time

$\mathbb{C}$ is a metric space as $d(z, w) = |z - w|$. Refresher on the induced topology is

- If $a \in \mathbb{C}$ and $r > 0$ then $B(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$ is the open ball centered at a of radius r.
- We call a set $U \subseteq \mathbb{C}$ open provided that for every $z \in U$ there exists an $r_z > 0$ such that $B(z, r_z) \subseteq U$.
- A subset $A \subseteq \mathbb{C}$ is closed provided that $\mathbb{C} \setminus A$ is open. This is equivalent to the statement that for every convergent sequence $z_n \to z$ with $z_n \in A$, we have $z \in A$ (that is the set of limit points $A' \subseteq A$).

$$A' \coloneqq \{\omega \in \mathbb{C} \mid \omega \text{ is a limit point of } A\} \coloneqq \{\omega \in \mathbb{C} \mid \text{There exists } \{a_n\}_{n \in \mathbb{N}} \subseteq A \text{ converging to } \omega\}.$$

- We say a sequence $a_n \in \mathbb{C}$ converges to $\ell \in \mathbb{C}$ provided that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that for $n \geq N$ we have $|a_n - \ell| < \varepsilon$.

We know that $\mathbb{C}$ is Cauchy complete as it is essentially the same as $\mathbb{R}^2$.

If $f : \mathbb{C} \to \mathbb{C}$ and $a \in \mathbb{C}$ we say that f is continuous at a provided that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|z - a| < \delta$ then $|f(z) - f(a)| < \varepsilon$.

We may also take limits to infinity and negative infinity in real analysis. In complex analysis, we only have **ONE** direction of infinity.

For $f : \mathbb{C} \to \mathbb{C}$ let $\ell \in \mathbb{C}$. We define $\lim_{z \to \infty} f(z) = \ell$ to mean for all $\varepsilon > 0$ there exists an $M \in \mathbb{R}$ such that if $|z| > M$ then $|f(z) - \ell| < \varepsilon$.

II.4. The Riemann Sphere

We add a single point to $\mathbb{C}$, called the point at infinity and denoted ∞ to get $\widehat{\mathbb{C}} \coloneqq \mathbb{C} \cup \{\infty\}$.

Definition II.4.1

$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the Riemann sphere

We can naturally identify $\widehat{\mathbb{C}}$ with the unit 2-sphere $S^2 \coloneqq \{v \in \mathbb{R}^3 \mid \|v\| = 1\} \subseteq \mathbb{R}^3$ via stereographic projection. That is if N is the north pole, we can cook up a map $\phi : S^2 \to \{N\} \to \mathbb{C}$ via taking $\phi(p)$ to be the intersection of the line through N and p with the plane $\{(x, y, 0) \in \mathbb{R}^3\} \cong \mathbb{C}$.

We can extend ϕ to a bijection $\widehat{\phi} : S^2 \to \widehat{\mathbb{C}}$ via $N \mapsto \infty$. The topology on S^2 exactly matches the limit definition we gave above for $\lim_{z\to\infty}$, and also this is a homeomorphism where $\widehat{\mathbb{C}}$ is given the topology of the one-point compactification.

Formally, if $f : \mathbb{C} \to \mathbb{C}$m we can compose to make $g = f \circ \phi : S^2 \setminus \{N\} \to \mathbb{C}$. Then for $\ell \in \mathbb{C}$

$$\lim_{z\to\infty} f(z) = \ell \iff \lim_{p\to N} g(p) = \ell.$$

This second limit is defined using Euclidean distance in $S^2 \subseteq \mathbb{R}^3$. If C is a circle in S^2 (the intersection of S^2 with a plane in $\mathbb{R}^3$) then $\widehat{\phi}(C)$ is a circle in $\mathbb{C}$ if $N \notin C$ and the image in $\mathbb{C}$ is a line if $N \in \mathbb{C}$. Conversely, every circle and every line in $\mathbb{C}$ is obtained this way.

II.5. **Examples of Functions**

We take $\arg(z)$ to be multivalued, and $\operatorname{Arg}(z)$ is the principal values of $\arg(z)$ lying in $(-\pi, \pi]$.

We can consider the map $z \mapsto z^2$. If we take square root to be the one in the right half-plane then we have a discontinuity across the negative real axis.

Definition II.5.1

The principal value of the square root function is $\omega \mapsto |\omega|^{1/2} e^{i \operatorname{Arg}(\omega)/2}$.

We also have polynomials $p(z) = a_d z^d + \cdots + a_1 z + a_0$, $a_i \in \mathbb{C}$. And furthermore we have the exponential $\exp : \mathbb{C} \to \mathbb{C}$ taking $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$. Modulus of e^z is $e^{\operatorname{Re}(z)}$ and argument is $\operatorname{Im}(z)$

The image of e^z is $\mathbb{C} \setminus \{0\}$ and it is not injective. Defining the logarithm will give multiple vlaues.

We can take $\log w = \log |w| + i \arg(w)$ for $w \neq 0$, which is multivalued. We can also take a principal branch $\operatorname{Log}(w) = \log |w| + i \operatorname{Arg}(w)$, which is an inverse of $\exp(z)$.

exp is periodic of period $2\pi i$

Definition II.5.2

$f : U \to \mathbb{C}$ is periodic with period $\lambda \in \mathbb{C}$ if $f(z + \lambda) = f(z)$ for all $z \in U$.

Fact: $\exp(z + w) = \exp(z) \exp(w)$, and $\log(1 + i) = \log \sqrt{2} + (\pi/4 + 2k\pi)i$ for all $k \in \mathbb{Z}$.

We can also consider power functions. Fix $\alpha \in \mathbb{C}$, and define for $z \neq 0$

$$z^\alpha \coloneqq \exp(\alpha \log(z))$$

this is multivalued unless $\alpha \in \mathbb{Z}$.

$$i^i = e^{-\pi/2} e^{-2\pi k} \in \mathbb{R}$$

for $k \in \mathbb{Z}$

III. Complex Differentiation

Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$, we're going to define holomorphic functions (in Gamelin [Gam03], this is "analytic")

Definition III.0.1

The function f is holomorphic at $z_0 \in U$ provided that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, and in that case we call that limit the derivative $f'(z_0)$.

The function f is holomorphic on U provided that it's holomorphic at all points inside U.

If $C \subseteq U$ is closed, then we say $f : C \to \mathbb{C}$ is holomorphic on C provided that there is an open set containing C on which f is holomorphic.

f is said to be entire provided that f is holomorphic on all of $\mathbb{C}$.

Proposition III.0.1

If $f, g : U \to \mathbb{C}$ are holomorphic at some $z_0 \in U$ then

(1) $f + g$ is holomorphic, $(f + g)' = f' + g'$.
(2) fg is holomorphic, $(fg)' = f'g + fg'$.
(3) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$$

(4) If $f : \Omega \to U$ is holomorphic at z_0, $g : U \to \mathbb{C}$ is holomorphic at $f(z_0)$, $g \circ f$ is holomorphic at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

Proof. Same as in $\mathbb{R}$! Just manipulating limits with each other.

Example III.0.1

Polynomials are entire! The proof is now easy. Constants and the identity map are both entire (exercise), and polynomials are sums/products of these.

Question: How can we tell if a function is holomorphic at a given point $z_0 \in U$? In the case of complex differenitation, the derivative is a complex number...

Consider $f : \mathbb{C} \to \mathbb{C}$, we can view this as a map $F : \mathbb{R}^2 \to \mathbb{R}^2$, and then the derivative of F is a linear transformation with standard basis on $\mathbb{R}^2$, namely

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}.$$

Proposition III.0.2 (Cauchy-Riemann Equations)

Writing $f = u + iv$ which is holomorphic at some $z_0 = x_0 + iy_0$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

in other words

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Proof. Consider the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Write $h = h_1 + ih_2$, and approach along real/imaginary axes

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(z_0 + h_1) - f(z_0)}{h_1} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Similarly

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(z_0 + ih_2) - f(z_0)}{ih_2} = -i\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Equating these gives the Cauchy-Riemann equations above.

Remark III.0.1

If f is complex differentiable at $z_0 \in U$, then it is continuous at z_0. Write

$$f(z_0 + h) - f(z_0) = h\left(\frac{f(z_0 + h) - f(z_0)}{h}\right)$$

and take the limit as $h \to 0$.

Note:

$$\begin{aligned} |f'(z_0)|^2 &= \left|\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\right|^2 \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial y} \end{aligned}$$

which is the determinant of the Jacobian when we view this as a map $\mathbb{R}^2 \to \mathbb{R}^2$.

Gamelin's definition requires that $f'(z_0)$ exists and also that $f'(z)$ is continuous at z_0. Later we will show that the derivative of a holomorphic function (at $z_)$) is also holomorphic at z_0, which will give us lots of extra stuff

If we assume this, then the functions u, v in $f = u + iv$ will have continuous partial derivatives of every order, and so the mixed partials will agree... this is useful to keep in mind.

Stuff:

- HW 2 (A due tomorrow, B due next week)
- HW 1B due tonight

- Bagels! WAlks!

Recall III.0.2

The principal values of the argument are $\operatorname{Arg}(z) \in (-\pi, \pi]$ for $z \neq 0$.

The principal branch is $U \to \mathbb{C}$ for $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. We may also define the principal branch of the logarithm as $\operatorname{Log}(z) = \log|z| + i\operatorname{Arg}(z)$.

Back to calculus!

Remark III.0.2

If f is complex differentiable at $z_0 \in U$, then it is continuous at z_0. Look at

$$f(z_0 + h) - f(z_0) = h\left(\frac{f(z_0 + h) - f(z_0)}{h}\right)$$

and take limits as $h \to 0$, to see that $\lim_{h\to 0} f(z_0 + h) - f(z_0) = 0$.

Lets now consider what the Cauchy-Riemann equations imply about $|f'(z)|$ for some f holomorphic at z. Well, if $f = u + iv$ we have that

$$\begin{aligned} |f'(z)| &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}. \end{aligned}$$

This is the determinant of the Jacobian matrix of partials.

Remark III.0.3

Gamelin's definition of holomorphic requires not only that $f'(z)$ exists, but also that $f'(z)$ is continuous at z. This is redundant! But it makes some things easier to phrase early on.

Eventually, we will show that the derivative of a holomorphic function at z is also holomorphic at z, which will give us lots of extra stuff.

This will later show that if $f = u + iv$ is holomoprhic, it will have continuous partial derivatives of every order and so the mixed partials will be equal! Taking partial derivatives of the left and right hand sides of the Cuachy-Riemann equations yields

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{-\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{-\partial^2 v}{\partial y \partial x}$$

A consequence if the mixed partials are equal is that

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial x^2}.$$

We see that

$$\Delta u \coloneqq \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta v \coloneqq \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Definition III.0.2

A smooth function $u : \mathbb{R}^2 \to \mathbb{R}$ that satisfies Laplace's equation

$$\Delta u \coloneqq \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is said to be harmonic

The real and imaginary parts of a holomorphic function are thus harmonic.

Definition III.0.3

If two harmonic functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy Cauchy-Riemann equations, then v is said to be "the" harmonic conjugate of u (unique up to an additive constant).

Example III.0.3

Let $u = x^2 - y^2$. This is harmonic. Can we build a harmonic conjugate? Well we know

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y.$$

We're led to consider $v = 2xy + \text{const}$.

Building the function $f = u + iv$ yields $f(z) = (x^2 - y^2) + i(2xy + \text{const}) = z^2 + i \cdot \text{const}$.

Formal + Helpful:

Consider $f(x, y) = u(x, y) + iv(x, y)$, $z = x + iy, \overline{z} = x - iy$. Then $x = \frac{1}{2}(z + \overline{z}), y = -\frac{i}{2}(z - \overline{z})$. We want to change variables from (x, y) to $(z, \overline{z})$. We define new operators

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

To say that f is holomorphic (aka u, v satisfy the Cauchy-Riemann equations) is exactly to say $\frac{\partial f}{\partial \overline{z}} = 0$, and this gives $\frac{\partial f}{\partial z} = f'$.

If f is holomorphic, then is $1/f$ holomorphic? Yes, provided that f is nonzero.

Rational functions! Let $R(z) = P(z)/Q(z)$ where P, Q are polynomials and P, Q have no common roots. The zeros of Q are called poles of R. We extend R to a function $\widehat{R} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by taking $\widehat{R}(z) = \infty$ for z a pole of R. We could also consider

$$\widehat{R}(\infty) = \lim_{z \to 0} R(z).$$

It is nicer to use a related function $R_1(z) = R(1/z)$, and define $\widehat{R}(\infty) = \widehat{R}_1(0)$. Note that $R(1/z)$ is a rational function

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_m z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$
$$R_1(z) = z^{m-n}\left(\frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}\right).$$

If $m > n$, $R(z)$ has a zero of order $m-n$ at ∞, define $\widehat{R}(\infty) = 0$. If $m < n$, the point at infinity is a pole of order $n-m$ so $\widehat{R}(\infty) = \infty$. If $m = n$, then $\widehat{R}(\infty) = \frac{a_n}{b_m} \neq 0, \infty$.

Consider $R(z) = \frac{z^2+57i}{z-53}$. The zeros of $\widehat{R}$ are $\pm\sqrt{-57i}$, and the poles are $z = 53, \infty$.

Fact: The total number of zeros of a rational function is equal to $\max(n, m)$ which is also equal to the number of poles wher we count with multiplicity. Find it in your book!

Definition III.0.4

The degree of a rational function $R(z) = P(z)/Q(z)$ is $\max(\deg P, \deg Q)$.

This will agree with the topological degree, which you might know about.

Definition III.0.5 (Möius tranformations)

A Möbius transformation is a rational function of degree 1.

Möbius transformations are in fact the automorphisms (bijective, holomorphic, with holomorphic inverse) of $\widehat{\mathbb{C}}$. To think about defining whether a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic at ∞, consider testing if $f(1/z) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic at 0.

Example III.0.4

When if $f(z) = \frac{az+b}{cz+d}$ a Möbius tranformation? Maybe we should think about if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0.$$

We say a Möbius transformation g is affine provided that $g(\infty) = \infty$, and we can then express $g(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$. The affine group is then

$$\{z \mapsto \alpha z + \beta \mid \alpha \neq 0, \beta \in \mathbb{C}\} = \operatorname{Aut}(\mathbb{C}).$$

Stuff:

- HW 2B due September 13th by 10PM.
  - For 6, to say L_M is $\mathbb{C}$-linear means there exists $\alpha \in \mathbb{C}$ so that the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "L_M"] \arrow[d, leftrightarrow] & \mathbb{R}^2 \arrow[d, leftrightarrow] \\ \mathbb{C} \arrow[r, dashed] & \mathbb{C} \end{tikzcd}$$

$$z \overset{- - - -}{\longmapsto} \alpha z$$

  - For 9, the set $U \subseteq \mathbb{C}$ which is the domain of $f : U \to \mathbb{C}$ should be connected.

To say f is holomorphic at a point, we will always mean f is holomorphic on a neighborhood of z_0.

We're headed to Gamelin, II.4-7, Ahlfors 3.2-3.3. Now back to Möbius transformations

If we have a Möbius transformation $f(z) = \frac{az+b}{cz+d}$ we note that

$$f(\infty) = \frac{a}{c}, c \neq 0 \qquad f(\infty) = \infty, c = 0$$

$$f^{-1}(\infty) = \frac{-d}{c}, c \neq 0 \qquad f^{-1}(\infty) = \infty, c = 0.$$

One can compute that

$$f^{-1}(z) = \frac{dz - b}{-cz + a}$$

which is also a Möbius transformation.

Fact: Möbius transformations are holomorphic.

Question: what does it mean for a function $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ to be holomorphic

Answer: Use local charts around each point. The relevant charts are $\widehat{\mathbb{C}} \setminus \{\infty\} \to \mathbb{C}, \widehat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$ given by stereographic projection about each pole. The transition map on the intersection is given by $z \mapsto 1/z$. Consider $\operatorname{inv} : z \mapsto 1/z$

Definition III.0.6

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be continuous and $a \in \widehat{\mathbb{C}}$. We say that f is holomorphic at a provided that

(i) When $a = \infty, f(a) = \infty$, the map $\operatorname{inv} \circ f \circ \operatorname{inv}$ is holomorphic at $z = 0$.

(ii) When $a = \infty, f(a) \neq \infty$, the map $f \circ \operatorname{inv}$ is holomorphic at $z = 0$.

(iii) When $a \neq \infty, f(a) = \infty$, the map $\operatorname{inv} \circ f$ is holomorphic at $z = a$.

(iv) When $a \neq \infty, f(a) \neq \infty$, the map $\circ f$ is holomorphic at $z = a$.

Consequence: all rational functions $R(z) = \frac{P(z)}{Q(z)}$ are holomorphic maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Corollary III.0.3

Möbius transformations are holomorphic.

Exercise III.0.5

Prove that polynomials extend to holomorphic maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. It is clear for constant polynomials, so let $p(z)$ be nonconstant.

Take $p(z) = \sum_{j=1}^{d} a_j z^j$ with $a_j \in \mathbb{C}$ and $a_d \neq 0$, it is clear that p is holomorphic at $z \neq \infty$. We just need to check around $z = \infty$. Here we have $p(\infty) = \infty$ unless p is constant. We thus must look at

$$(\operatorname{inv} \circ p \circ \operatorname{inv})(z) = \frac{1}{p\left(\frac{1}{z}\right)} = \frac{1}{\sum_j a_j z^{-j}}.$$

Cleaning this up gives

$$\frac{z^d}{a_d + \cdots + a_1 z^{d-1} + a_0 z^d}.$$

This is holomorphic at zero since $a_d \neq 0$. Its derivative is indeed

$$p'(\infty) = \begin{cases} 0 & \text{if } d > 1 \\ \frac{1}{a_d} & \text{if } d = 1 \end{cases}.$$

Even better: Möbius transformations are biholomorphisms on the Riemann sphere (aka a holomorphic bijection with holomorphic inverse).

We collect these into a group

$$\text{Möb} := \{\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid \mu \text{ is a Möbius trnasformation}\}$$

Algebraically, Möb is a group with respect to the binary operation of composition. We can think of this as a matrix group via $\mathrm{GL}_2(\mathbb{C})$. Namely via the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f_A(z) = \frac{az+b}{cz+d}.$$

The determinant being nonzero corresponds to $ad - bc \neq 0$, which we require. One can check this is a surjective homomorphism. The kernel is

$$\ker(\mathrm{GL}_2(\mathbb{C}) \to \text{Möb}) = \left\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{C} \setminus \{0\} \right\},$$

which may be easily checked. We often call the quotient of $\mathrm{GL}_2(\mathbb{C})$ by this kernel the "projective general lienar group"

$$\mathrm{PGL}_2(\mathbb{C}) \coloneqq \mathrm{GL}_2(\mathbb{C}) \Big/ \left\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{C} \setminus \{0\} \right\} \cong \text{Möb}$$

We could normalize our matrices to have determinant one...

$$\mathrm{SL}_2(\mathbb{C}) = \{ A \in \mathrm{GL}_2(\mathbb{C}) \mid \det(A) = 1 \}$$

There is then a homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \text{Möb}$ with kernel $\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. This gives us

$$\mathrm{PSL}_2(\mathbb{C}) \coloneqq \mathrm{SL}_2(\mathbb{C}) \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \text{Möb}.$$

This is in some sense 3-dimensional, as we have four variables and one condition, $ad - bc = 1$.

There are three fundamental types of Möbius transformations

(1) Linear, $z \mapsto \alpha z$ where $\alpha \in \mathbb{C} \setminus \{0\}$.
(2) Translation, $z \mapsto z + \beta$ for some $\beta \in \mathbb{C}$.
(3) Inversion, $z \mapsto \frac{1}{z}$.

Theorem III.0.4

We have that

(i) The group Möb is generated by translation, linear maps, and inversion.
(ii) The action of Möb on $\widehat{\mathbb{C}}$ is "simply 3-transitive" i.e. for any two triples of distinct points $(p_1, p_2, p_3), (q_1, q_2, q_3)$ on $\widehat{\mathbb{C}}$ there exists a unique Möbius transformation taking p_j to q_j.
(iii) The action of Möb on $\widehat{\mathbb{C}}$ preserves circles.

Proof of (ii). For existence, it suffices to show that any triple (p_1, p_2, p_3) can be sent to $(0, 1, \infty)$, then take

$$f(z) = \frac{(p_2 - p_3)(z - p_1)}{(p_2 - p_1)(z - p_3)}.$$

Caution: Breaking the rules a bit if one of the p_i is ∞... but just adjust and change formula a bit. Namely one of these three formulas

$$\frac{p_2 - p_3}{z - p_3}, p_1 = \infty \qquad \frac{z - p_1}{z - p_3}, p_2 = \infty \qquad z \mapsto \frac{z - p_1}{p_2 - p_1}, p_3 = \infty.$$

To prove uniqueness, suppose $g \in$ Möb that sends $(p_1, p_2, p_3) \mapsto (0, 1, \infty)$. We must examine $f \circ g^{-1}$.

This is a Möbius transformation fixing $0, 1, \infty$. Check that the only such map is the identity.

Something cool: Suppose p_1, p_2, p_3, p_4 are distinct points in $\mathbb{C}$ that lie on a circle $\Gamma \subseteq \mathbb{C}$. The Möbius transformation $f(z) = \frac{1}{z - p_1}$ sends the circle Γ to a line $L = f(\Gamma)$.

Let $q_k = f(p_k)$, $k = 2, 3, 4$. Choose the ordering so on the circle p_3 is between p_2, p_4. Then q_3 is between q_2, q_4. This gives that

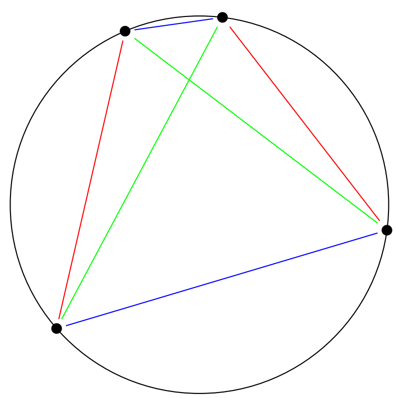
$$|q_2 - q_4| = |q_2 - q_3| + |q_3 - q_4|$$

plug in $q_k = \frac{1}{p_k - 1}$ and simplyify to get

$$|p_1 - p_3| \cdot |p_2 - p_4| = |p_1 - p_2| \cdot |p_3 - p_4| + |p_1 - p_4| \cdot |p_2 - p_3|$$

Theorem III.0.5 (Ptolemy's Theorem)

A quadrilateral can be inscribed in a circle if and only if the sum of products of lengths of opposite edges is equal to the product of the lengths of the diagonals.



Stuff:

- Office Hours Wednesday 10:30-12 EH3855.
- HW 2B due tonight 10pm
- HW 3A, 3B is the next round. For 3A, possibly look at Gamelin for inspiration.
- Walk: Carol + snowcones!

Back to Möbius transformations. Recall that

$$\text{Möb} := \{f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid f(z) = \frac{az + b}{cz + d}, ad - bc \neq 0\}.$$

This is a group with respect to function composition, we saw an isomorphism with familiar groups earlier. Furthermore we have

$$\text{Möb} \subseteq \text{Aut}(\widehat{\mathbb{C}}) := \{g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid g \text{ is a biholomorphism}\}.$$

In fact, $\text{Möb} = \text{Aut}(\widehat{\mathbb{C}})$. We will show this eventually.

What about

$$\text{Aut}(\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} \mid f \text{ is biholo}\}.$$

Note that affine maps $z \mapsto \alpha z + \beta$ where $\alpha \in \mathbb{C}^\times$ are biholomorphisms of the plane. These are special examples of Möbius transformations. In fact the affine transformations are exactly those Möbius transformations that send ∞ to ∞. Let Aff be the group of these transformations.

Fact: $\text{Aff} = \text{Aut}(\mathbb{C})$.

We can also consider the open disk

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}.$$

This will become one of our close friends. The disk has a nice cousin

$$\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$$

which is the upper half plane. Who is this! Well consider the map

$$\begin{aligned} \varphi : \mathbb{H} &\to \mathbb{D} \\ z &\mapsto \frac{i - z}{i + z}. \end{aligned}$$

One must check this is holomorphic, bijective, and its inverse is holomorphic. This preserves angles and so we call it conformal. A picture is below.

Definition III.0.7

We define the cross ratio of an ordered quadruple of distinct points $p_1, p_2, p_3, p_4 \in \widehat{\mathbb{C}}$ is

$$[p_1, p_2, p_3, p_4] := \frac{(p_3 - p_1)(p_4 - p_2)}{(p_2 - p_1)(p_4 - p_3)}.$$

If $p_i = \infty$, the definition is interpreted as the appropriate limit.

Example III.0.6

$$[0, p_2, p_3, p_4] = \frac{p_4 - p_2}{p_4 - p_3}$$

Exercise III.0.7

Compute for $p_i = 0, 1, \infty, z$. What are all possible outputs with all possible permutations

$$[0, 1, \infty, z] = \lim_{w \to \infty} \frac{(w - 0)(z - 1)}{(1 - 0)(z - w)} = \lim_{w \to \infty} \frac{z - 1}{\frac{z}{w} - 1} = 1 - z.$$

According to a classmate (Zach), there are six distinct outputs
$$z, 1-z, \frac{1}{1-z}, \frac{1}{z}, \frac{z}{1-z}, \frac{1-z}{z}$$

Cool properties!

Theorem III.0.6

Consider these neat properties

(i) If $f \in$ Möb is the unique element sending $(p_1, p_2, p_4) \mapsto (0, 1, \infty)$ then
$$[p_1, p_2, p_3, p_4] = f(p_3).$$
In particular, $[p_1, p_2, p_3, p_4]$ takes values in $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$.

(ii) Two quadruples (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) can be sent to each other by Möbius transformations if and only if
$$[p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4].$$

(iii) If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism that preserves cross ratios of ALL quadruples, then f is a Möbius transformation.

(iv) Four points p_1, p_2, p_3, p_4 lie on the same circle in $\mathbb{C}$ if and only if $[p_1, p_2, p_3, p_4] \in \mathbb{R}$.

Back to holomorphic discussion!

Recall III.0.8

$f : U \to \mathbb{C}$, $z_0 \in U$, we say that f is holomorphic at z_0 means f is holomorphic on a neighborhood of z_0, that is for any z within that neighborhood the limit
$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \in \mathbb{C}$$
exists. We showed that if $f = u + iv$ then

- $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0$
- f satisfies Cauchy-Riemann equations on U
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$
- For $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ we have $\Delta u, \Delta v = 0$ under regularity assumptions on u, v (which will be unnecessary later). This means u, v are harmonic, and in fact they are harmonic conjugates.

Theorem III.0.7

Let $u, v : U \to \mathbb{R}$ have contiuous first order partials on U and satisfy the Cauchy-Riemann equations on U. Then $f = u + iv$ is holomorphic on U.

Proof. We will use Taylor's Theorem for real variables. Consider some point $z_0 = (x_0, y_0) \in U$, and consider some small $h = (h_1, h_2)$. We see that
$$u(x_0 + h, y_0 + k) - u(x_0, y_0) = \left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)} \cdot h_1 + \left.\frac{\partial u}{\partial y}\right|_{(x_0, y_0)} \cdot h_2 + \varepsilon_1$$

and

$$v(x_0+h, y_0+k) - v(x_0, y_0) = \frac{\partial v}{\partial x}\bigg|_{(x_0,y_0)} \cdot h_1 + \frac{\partial v}{\partial y}\bigg|_{(x_0,y_0)} \cdot h_2 + \varepsilon_2$$

where $\varepsilon_1, \varepsilon_2$ tend to 0 more rapidly than $h + ik$ in the sense that

$$\frac{\varepsilon_1}{h_1 + ih_2}, \frac{\varepsilon_2}{h_1 + ih_2} \to 0 \iff \frac{|\varepsilon_1|^2}{h_1^2 + h_2^2}, \frac{|\varepsilon_2|^2}{h_1^2 + h_2^2} \to 0$$

as $h = h_1 + ih_2 \to 0$. Using the Cauchy-Riemann equations, we have that

$$f(z_0+h) - f(z_0) = \left(\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}\bigg|_{(x_0,y_0)}\right) \cdot h + \varepsilon_1 + i\varepsilon_2. \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}\bigg|_{(x_0,y_0)}.$$

Thus $f'(z_0)$ exists.

Using this converse, one may show the exponential $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic. Recall the definition below

$$\begin{aligned} \exp &: \mathbb{C} \to \mathbb{C} \\ x + iy &\mapsto e^x \cdot e^{iy} \\ &= e^x \cos y + i \sin y. \end{aligned}$$

Exercise III.0.9

Show that exp is holomorphic by showing it satisfies the Cauchy-Riemann.

For the logarithm, we need to use the inverse function theorem. This gets an upgrade in the setting of complex analysis!

Theorem III.0.8

Suppose f is holomorphic on the open set $\Omega \subseteq \mathbb{C}$ and $f'(z_0) \neq 0$ for $z_0 \in \Omega$. Then there exists a neighborhood U containing z_0 on which

- f is injective.
- The image $V := f(U)$ is open in $\mathbb{C}$.
- The inverse $f^{-1} : V \to U$ is holomorphic on V and satisfies

$$\left(f^{-1}\right)'(f(z)) = \frac{1}{f'(z)}.$$

Proof. The first two bullet points come from analysis of real variables by using the identification $\mathbb{R}^2 \cong \mathbb{C}$. Take $g = f^{-1}$, and let $w = f(z), w_1 = f(z_1)$. Then we want to show $g'(w_1)$ exists. Consider

$$\lim_{w\to w_1} \frac{g(w) - g(w_1)}{w - w_1} = \lim_{z \to z_1} \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{f'(z_1)}.$$

We must show $\mathrm{Log}(z) : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}$ is holomorphic, where

$$\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z).$$

Well, consider $\exp : \mathbb{R} \times (-\pi, \pi) \to \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and use the inverse function theorem... Note exp is already bijective on this domain.

Consider your homework, that a linear map $\mathbb{R}^2 \xrightarrow{L_M} \mathbb{R}^2$ which has positive determinant descends to a $\mathbb{C}$-linear map $z \mapsto \alpha z$ as

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "L_M"] \arrow[d, leftrightarrow] & \mathbb{R}^2 \arrow[d, leftrightarrow] \\ \mathbb{C} \arrow[r, dashed] & \mathbb{C} \\ z \arrow[r, mapsto, dashed] & \alpha z \end{tikzcd}$$

if and only if L_M preserves angles between vectors. Thus if $f : U \to \mathbb{C}$ then $f'(z_0)$ exists if and only if the derivative preserves angles or it is zero.

Definition III.0.8

Let $U \subseteq \mathbb{C}$ be open. The function $f : U \to \mathbb{C}$ is conformal at $z_0 \in U$ provided that it preserves angles in the since that for any pair of smooth curves $\gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$, the angle between $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ is equal to the angle between $(f \circ \gamma_1)'(t_1)$, $(f \circ \gamma_2)'(t_2)$ at $f(z_0)$.

More precisely,

$$\frac{|\langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle|}{\|\gamma_1'(t_1)\| \cdot \|\gamma_2'(t_2)\|} = \frac{|\langle (f \circ \gamma_1)'(t_1), (f \circ \gamma_2)'(t_2)\rangle|}{\|(f \circ \gamma_1)'(t_1)\| \cdot \|(f \circ \gamma_2)'(t_2)\|}.$$

If we further require that the signs match, that is

$$\frac{\langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle}{\|\gamma_1'(t_1)\| \cdot \|\gamma_2'(t_2)\|} = \frac{\langle (f \circ \gamma_1)'(t_1), (f \circ \gamma_2)'(t_2)\rangle}{\|(f \circ \gamma_1)'(t_1)\| \cdot \|(f \circ \gamma_2)'(t_2)\|}.$$

This makes a statement about the *orientation* being preserved as well, we call these orientation-preserving.

Caution: Gamelin insists that all maps that are conformal at z_0 are orientation-preserving at z_0. We will adopt Gamelin's convention.

Example III.0.10

Complex conjugation will preserve the angles between 2 vectors, but not the directed angle.

Theorem III.0.9

If $f : U \to \mathbb{C}$ is holomorphic then f is conformal (and orientation-preserving) at all points $z_0 \in U$ such that $f'(z_0) \neq 0$.

Proof. We use the chain rule, set $z_0 = \gamma(a) = \delta(c)$ for smooth curves γ, δ. Then

$$(f \circ \gamma)'(a) = f'(\gamma(a))\gamma'(a) \qquad (f \circ \delta)'(c) = f'(\delta(c))\delta'(a).$$

But wait! We then have that

$$\langle f'(z_0)\gamma'(a), f'(z_0)\delta'(c)\rangle = |f'(z_0)|^2 \langle \gamma'(a), \delta'(c)\rangle$$

because $h \mapsto f'(z_0) \cdot h$ is a $\mathbb{C}$-linear map, and thus an angle/orientation-preserving linear map $\mathbb{R}^2 \to \mathbb{R}^2$ from homework.

Stuff:

- HW 3B due Tuesday
- Due dates for 4A/4B to be decided
- There exists Math Club (Constructing $\mathbb{R}$!)
- There exists Math S^1 (Thursdays 6:30pm-8pm, starts next week 9/22)
- There is a 40 mile walk on Saturday October 1st!
- There is Super Saturdays (starts 10/8, 9:30am-12pm)
- Math Mental Health Hour Sunday (2-3pm), EH 1866
- Bagels on Sunday 10am-11:30am
- U(M) Undergrad Mathematics Seminar EH 3096

Recall III.0.11

If $f = u + iv$, we know if $f : U \to \mathbb{C}$ is holomorphic, then u is harmonic on U, that is $\Delta u = 0$ and u has continuous first and second order partials.

Faye's question: Given $u : U \to \mathbb{R}^2$ harmonic, does there exist a harmonic conjugate?

No! Take $u(z) = \operatorname{Log}|z|$, which is harmonic on $\mathbb{C} \setminus \{0\}$. This does not have a harmonic conjugate on $\mathbb{C} \setminus \{0\}$, but does have a harmonic conjugate on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, namely $\operatorname{Arg}(z)$. Then $\operatorname{Log}(z) = \log|z| + i \operatorname{Arg}(z)$ is harmonic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

What's the difference in domains? $\mathbb{C} \setminus \{0\}$ is *not* simply connected, while $\mathbb{C} \setminus (-\infty, 0]$ is simply connected.

Proposition III.0.10

If $u : U \to \mathbb{R}$ is harmonic on U and U is simply connected, then a harmonic conjugate exists.

For Gamelin, he constructs a harmonic conjugate p57 on rectangles (and will eventually do star-shaped regions).

Back to conformal maps: We saw that if $f'(z_0) \neq 0$, then f maps orthogonal curves then z_0 to orthogonal curves at $f(z_0)$.

Definition III.0.9

The map $f : U \to V$ is conformal on U provided that

(1) f is conformal at all points $z_0 \in U$.

(2) f is bijective.

Example III.0.12

$\exp : \mathbb{C} \to \mathbb{C}$ satisfies (1) but not (2). We can also consider $z \mapsto z^2$ as a map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$.

IV. Complex Integration!

Chapter 3/III in Gamelin.

IV.1. Review of prerequisites

Definition IV.1.1

A path in the plane is a continuous function $\gamma : [a, b] \to \mathbb{C}$, and we say it is a path from $\gamma(a)$ to $\gamma(b)$.

A path γ is simple provided that $\gamma\big|_{[a,b)}$ is injective. The path γ is closed provided that $\gamma(a) = \gamma(b)$.

All paths γ have an orientation, $\gamma(a)$ is the initial point and $\gamma(b)$ is the end point.

A path is called smooth if it is smooth as a function.

If we have paths $\gamma : [0,1] \to \mathbb{C}$ from $A \in \mathbb{C}$ to $B \in \mathbb{C}$ and $\delta : [0,1] \to \mathbb{C}$ from B to $C \in \mathbb{C}$, we can construct a path

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq 1/2 \\ \delta(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

from A to C. This is called the concatenation.

A piecewise smooth path is a concatenation of smooth paths. A curve is a smooth path or piecewise smooth.

Let γ be a path in $\mathbb{C}$ from A to B and let $P(x,y), Q(x,y)$ be continuous complex-valued functions on the image of γ. Break up the image of γ into pieces (x_i, y_i) and form the sum

$$\sum P(x_j, y_j)(x_{j+1} - x_j) + \sum Q(x_j, y_j)(y_{j+1} - y_j).$$

where we require $\gamma(t_j) = (x_j, y_j)$ where $a = t_0 < t_1 < \cdots < t_n = b$.

Definition IV.1.2

If these sums have a limit as distance between points $(x_j, y_j) \to 0$ then we define the limit to be the line integral of $P\,\mathrm{d}x + Q\,\mathrm{d}y$ along γ, denoted

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

More precisely, let $\gamma(t) = (x(t), y(t))$ with $a \leq t \leq b$. Suppose $t_j \in [a,b]$ satisfies $\gamma(t_j) = (x_j, y_j)$ with $a \leq t_0 < t_1 < \cdots < t_n = b$.

Apply the Mean Value Theorem to find points $t_j^* \in [t_j, t_{j+1}]$ so that $x(t_{j+1}) - x(t_j) = x'(t_j^*)(t_{j+1} - t_j)$. Likewise for y. Plugging into the above sums this gives

$$\begin{aligned} &\sum P(x(t_j), y(t_j))x'(t_j^*)(t_{j+1} - t_j) + \sum Q(x(t_j), y(t_j))y'(t_j^*)(t_{j+1} - t_j) \\ &= \sum (P(x(t_j), y(t_j))x'(t_j^*) + Q(x(t_j), y(t_j))y'(t_j^*))(t_{j+1} - t_j). \end{aligned}$$

As $t_{j+1} - t_j$ go to zero we have this is equal to

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_a^b P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\,\mathrm{d}t.$$

Theorem IV.1.1 (Green's Theorem)

Consider some region $\Omega \subseteq \mathbb{C}$ which is a connected bounded open set whose boundary consists of a finite # of disjoint piecewise smooth curves.

Let P, Q be continuously differentiable on $\Omega \cup \partial\Omega$, then

$$\int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

Definition IV.1.3

If $h(x, y)$ is a continuously differentiable $\mathbb{C}$-valued function, we define its differential as

$$\mathrm{d}h = \frac{\partial h}{\partial x}\,\mathrm{d}x + \frac{\partial h}{\partial y}\,\mathrm{d}y.$$

The differential $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is called exact if $P\,\mathrm{d}x + Q\,\mathrm{d}y = \mathrm{d}h$ for some h.

Theorem IV.1.2 (FTC for Line Integrals)

If γ is a piecewise smooth curve from A to B and if $h(x, y)$ is continuously differentiable on γ then

$$\int_\gamma \mathrm{d}h = h(B) - h(A).$$

Proof. Chain rule! See Gamelin!

Example IV.1.1

Exact differentials are nice and very easy to integrate♡

Natural question: which differentials $P\,\mathrm{d}x + Q\,\mathrm{d}y$ are exact? hmmm...

Definition IV.1.4

As before, let P, Q be complex-valued and continuously differentiable on $U \subseteq \mathbb{C}$, the integral

$$\int P\,\mathrm{d}x + Q\,\mathrm{d}y$$

is said to be path independent provided that for any paths $A, B \in U$, and for any pair of paths γ, δ from A to B we have

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_\delta P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Note: This is equivalent to the statement that given any simple closed curve in U, call it $\mu \subseteq D$, we have

$$\int_\mu P\,\mathrm{d}x + Q\,\mathrm{d}y = 0.$$

Lemma IV.1.3

Let $P, Q : U \to \mathbb{C}$ be continuous. Then $\int P\,\mathrm{d}x + Q\,\mathrm{d}y$ is independent of path if and only if $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact.

Proof. The converse follows from Theorem IV.1.2. For the forward direction, fix a basepoint $z_0 \in U$, then define $h(z) = \int_{z_0}^{z} P\,\mathrm{d}x + Q\,\mathrm{d}y$.

It does not take much effort to show $\mathrm{d}h = P\,\mathrm{d}x + Q\,\mathrm{d}y$.

Definition IV.1.5

Let P, Q be continuously differentiable on a connected open set U. The differential $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is closed on U provided that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

that is any Green's theorem type integral will be zero.

If $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is a closed form, then

$$\int_{\partial U} P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$$

for any bounded connected open set $U \subseteq \mathbb{C}$. We have

$$\text{independence of path} \iff \text{exact} \iff \text{closed}$$

Theorem IV.1.4

If U is simply connected (often we will use star-shaped spaces, which are simply connected) then closed implies exact.

Stuff:

- HW 3B due tonight!
- HW 4 – Due Thursday (4A) and Tuesday (4B)
- Lots of Department Stuff!
  - Math S^1
  - Super Saturdays
  - Bagel Sundays
  - 20 mile walk 9/24, 40 mile walk 10/1
  - U(M) Student Seminar EH 3096 4-5pm this Friday: Circle Method and Waring's problem by Xun
  - Math Mental Health Hour EH1866 every 2 weeks Sunday afternoon

We had a problem on the homework which shows that

Claim

If p is a complex polynomial whose zeroes lie in the half plane $\mathrm{Re}(z) > 0$, then the zeros of p' lie in this half-plane as well.

Theorem IV.1.5 (Gauss-Lucas Theorem)

If p is a complex polynomial, then the zeros of p' lie in the convex hull of the zeros of p.

Recall IV.1.2

A subset A of $\mathbb{R}^n$ is convex provided that for any two points $a, b \in \mathbb{R}^n$ the line between a, b is a subset of A.

The convex hull of a set of points $\{z_1, \ldots, z_m\}$ is the set of all combinations $\sum t_j z_j$ such that $0 \leq t_j \leq 1$ and $\sum t_j = 1$.

Back to integration!

Lemma IV.1.6

If $P, Q : D \to \mathbb{C}$ are continuous functions on a connected open set, then $\int P\,\mathrm{d}x + Q\,\mathrm{d}y$ is path independent if and only if the form is exact.

See Gamelin [Gam03, Chapter III].

Lemma IV.1.7

Exact differentials are closed

Proof. Suppose $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact. Then there exists h such that $\mathrm{d}h = P\,\mathrm{d}x + Q\,\mathrm{d}y$, that is

$$\frac{\partial h}{\partial x} = P \qquad\qquad \frac{\partial h}{\partial y} = Q.$$

Then we have that

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\frac{\partial h}{\partial y} = \frac{\partial Q}{\partial x}$$

Definition IV.1.6

The connected open set $D \subseteq \mathbb{C}$ is called star shaped provided that there exists some $z_0 \in D$ such that for all $z \in D$, the line between z_0 and z is contained in the set.

It is convenient to call z_0 the "spectator" as a nice piece of terminology. Note that this is not necessarily unique.

This implies simply connected (contract everything to the spectator), but the converse is not true. Consider a horseshoe, which is simply connected but not star-shaped.

Theorem IV.1.8

Let P, Q be continuous differentiable functions on a connected open subset $D \subseteq \mathbb{C}$. Suppose

(1) D is star-shaped (simply connected suffices, but we won't prove it).

(2) $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is closed on D.

Then $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact.

Proof Sketch. Let A be a spectator for D. Define

$$h : D \to \mathbb{C}$$
$$B \mapsto \int_A^B P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

We take the path of integration $A \to B$ to be the straight line from A to B. We must then check that this h works. See [Gam03]

Example IV.1.3

Consider the 1-form

$$P\,\mathrm{d}x + Q\,\mathrm{d}y = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}$$

on $\mathbb{C} \setminus \{0\}$. This is closed by a simple computation, but it isn't exact/path-independent! Why? Well integrate about the unit circle C in the counterclockwise direction.

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = 2\pi.$$

However, $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact on $\mathbb{C} \setminus (-\infty, 0]$. $P\,\mathrm{d}x + Q\,\mathrm{d}y = \mathrm{d}\,\mathrm{Arg}(x + iy)\ldots$

We'll now apply this discussion to harmonic conjugates (Faye's question). Take $u \in C^1$ with second order partials with Δu, then does there exist a v harmonic so that $f = u + iv$ is holomorphic.

Lemma IV.1.9

If $u(x, y)$ is harmonic on an open connected set $D \subseteq \mathbb{C}$, then the differential

$$-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y$$

is closed.

Proof. A simple computation.

Id D is star-shaped, then we know that is exact, so

$$\mathrm{d}v = -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y.$$

But then we have that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad\qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

so that $f = u + iv$ is holomorphic!

Theorem IV.1.10

Any harmonic function $u(x, y)$ on a star-shaped domain (can promote this to simply connected) $D \subseteq \mathbb{C}$ has a harmonic conjugate $v(x, y)$ on D.

Example IV.1.4

Consider $u = \log|z|$ on $\mathbb{C} \setminus (-\infty, 0]$. We showed this was harmonic (hw). By the previous discussion there exists a harmonic conjugate $v(x, y)$ on $\mathbb{C} \setminus (-\infty, 0]$. Then

$$\begin{aligned} u(x,y) &= \frac{1}{2}\log(x^2+y^2) \\ \mathrm{d}u &= \frac{x}{x^2+y^2}\,\mathrm{d}x + \frac{y}{x^2+y^2}\,\mathrm{d}y \\ \mathrm{d}v &= \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \\ v(z) &= \int_1^z \mathrm{d}v. \end{aligned}$$

This is choosing the conjugate from the proof above, normalized so that $v(1) = 0$. This tells us that

$$\operatorname{Arg}(z) = \int_1^z \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y,$$

as harmonic conjugates are unique up to adding constants.

We're now in [Gam03, Gamelin III.4], the Mean Value Property.

Definition IV.1.7

Take $D \subseteq \mathbb{C}$ to be a connected open subset containing the disk $\{|z - z_0| < \rho\}$. Let $h : D \to \mathbb{R}$ be continuous.

We define the average value of h on the circle $\{|z - z_0| = r\}$ for $0 < r < \rho$ to be

$$A_h(r, z_0) := \frac{1}{2\pi}\int_0^{2\pi} h\left(z_0 + re^{i\theta}\right)\mathrm{d}\theta.$$

Theorem IV.1.11

If $u(z)$ is a harmonic function on D (as defined above), then

$$A_u(r, z_0) \coloneqq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,\mathrm{d}\theta = u(z_0).$$

Proof. We see because $-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}$ is closed/exact/path-independent (on $|z - z_0| < \rho$) that

$$0 = \int_{|z-z_0|=r} -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y.$$

We parameterize the circle as $x(\theta) = x_0 + r\cos(\theta), y(\theta) = y_0 + r\sin(\theta)$. We see that thsi is exactly

$$0 = \int_0^{2\pi}\left(\frac{\partial u}{\partial y}\cdot(r\sin\theta) + \frac{\partial u}{\partial x}\cdot(r\cos\theta)\right)\mathrm{d}\theta.$$

We then see via the chain rule that this is

$$0 = r\int_0^{2\pi}\frac{\partial u}{\partial r}\,\mathrm{d}\theta$$

Divide both sides by $2\pi r$ and exchanging the integral with the differentation, we see that

$$0 = \frac{\partial}{\partial r}\int_0^{2\pi} u(z_0 + re^{i\theta})\frac{\mathrm{d}\theta}{2\pi} = \frac{\partial}{\partial r}A_u(r, z_0).$$

Thus $A_u(r, z_0)$ is constant in r, and since u is continuous at z_0, as $r \to 0$, the average value of the function tends to $u(z_0)$ as $r \to 0$. Thus $A_u(r, z_0) = u(z_0)$.

Definition IV.1.8

We say that a continuous function $h(z)$ on a connected open domain $D \subseteq \mathbb{C}$ has the mean value property provided that for all $z_0 \in D$, $h(z_0)$ is the average of its values over any small circle centered at z_0. I.e., for all $z_0 \in D$, there exists an $\varepsilon > 0$ such that for all $0 < r < \varepsilon$, we have

$$h(z_0) = \int_0^{2\pi} h(z_0 + re^{i\theta})\frac{\mathrm{d}\theta}{2\pi}.$$

Now we move to [Gam03, Gamelin III.5].

Definition IV.1.9

Let $u(z)$ be a real-valued continuous function with the mean value property (so including harmonic functions!) on the connected open subset $D \subseteq \mathbb{C}$. Suppose there exists $M \in \mathbb{R}$ such that for all $z \in D$, $u(z) \leq M$. If $u(z_0) = M$ for some $z_0 \in D$, then u is constant.

Proof Strategy. Let $S_M \coloneqq \{z \in D \mid u(z) = M\}$ and $S_{<M} \coloneqq \{z \in D \mid u(z) < M\}$. We see that $D = S_M \coprod S_{<M}$. We know that $z_0 \in S_M M$

We know that $S_{<M}$ is open, so it suffices to show that S_M is open, as then $D = S_M$. Suppose that $u(z_1) = M$, i.e., $z_1 \in S_M$.

We use the Mean Value property to write

$$M = u(z_1) = \int_0^{2\pi} u(z_1 + re^{i\theta}\frac{\mathrm{d}\theta}{2\pi}.$$

We may then rearrange to give that

$$0 = \int_0^{2\pi} [M - u(z_1 + re^{i\theta})] \frac{\mathrm{d}\theta}{2\pi}.$$

But wait! This is a non-negative continuous integrand!!! So the integral is zero if and only if $M = u(z_1 + re^{i\theta})$ for all θ. Thus for $0 < r < \rho$ we have

$$u(z + re^{i\theta}) = M,$$

so S_M is open as desired.

Comment on HW4B problem 5: Given a Möbius transformation $f(z) = \frac{az+b}{cz+d}$, we wish to show the number of g such that $g(g(g(g(g(z))))) = f(z)$ is $1, 5$ or ∞. It would be very difficult to work with a general f. Perhaps instead we should work with particular f and show this is enough. One should think about conjugation in the group, namely consider for $f, h \in$ Möb, the conjugate $h \circ f \circ h^{-1}$.

It is enough to solve the problem for a conjugate. Use g^5 to genote 5 copies of g composed, then

$$g^5 = f \iff (hgh^{-1})(hgh^{-1})(hgh^{-1})(hgh^{-1})(hgh^{-1}) = hfh^{-1}.$$

Since $g \mapsto hgh^{-1}$ is a bijection between the set of rational functions with itself, we're good!

We now need to understand the conjugacy classes in Möb. What is somethingthat is invariant under conjugation. The # of fixed points of $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is invariant under conjugation. Consider

$$\frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0.$$

It turns out there are three possibilities

- f has exactly one fixed point (conjugate to $z \mapsto z + 1$).
- f has exactly two fixed points (conjugate to $z \mapsto \lambda z$, $\|\lambda\| = 1$).
- f has infinitely many fixed points (conjugate to $z \mapsto z$).

In Sarah's research area, she takes rational functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and considers the behavior of iterations $f \circ \cdots \circ f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. A "simple" example is $f(z) = z^2 + c$ where $c \in \mathbb{C}$ is some parameter.

Definition IV.1.10 (Filled Julia set)

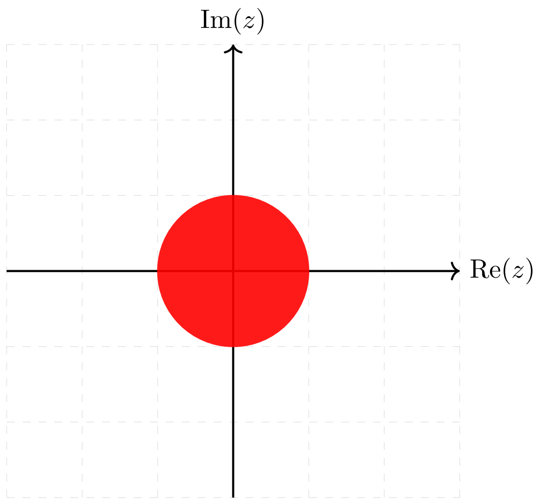
The filled Julia set K_f of f is

$$K_f \coloneqq \{z_0 \in \mathbb{C} \mid \text{ orbit of } z_0 \text{ under } f \text{ is bounded}\}.$$

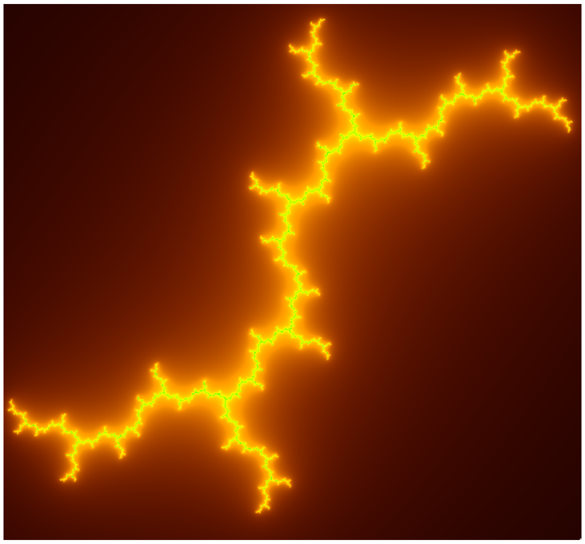
The orbit here is the sequence z_n defined by $z_n = f(z_{n-1})$.

Example IV.1.5

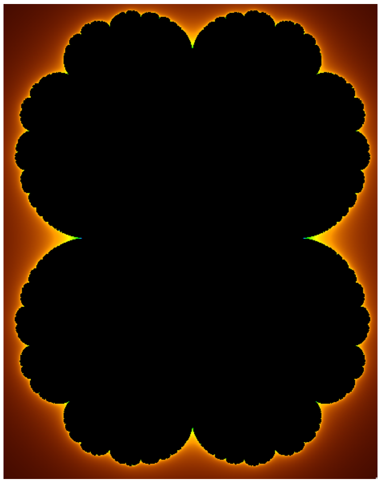
For $z \mapsto z^2$ the filled Julia set is simple



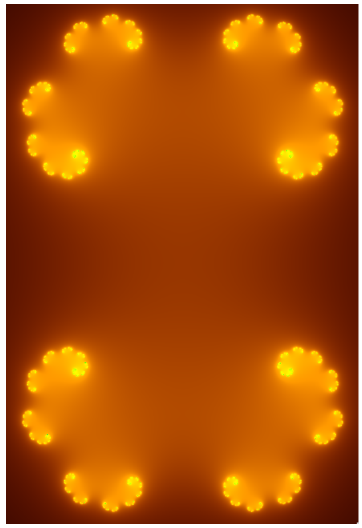
Generally filled Julia sets are crazy. For $z \mapsto z^2 + i$ we have



For $z \mapsto z^2 + 1/4$ we have a cauliflower shape



For $z \mapsto z^2 + 1/2$ we get cantor dust!



All filled julia sets for polynomials are full, that is their complement is connected. Why? Well it's a corollary of our discussion using a rate of escape function which is harmonic...

IV.2. **Defining Complex Integrals**

Why do we want to do integration? Well holomorphic maps have amazing properties. It is much easier to prove that they have these properties with integrals. We integrate 1-forms, and we define formally

$$\mathrm{d}z \coloneqq \mathrm{d}x + i\,\mathrm{d}y.$$

If γ is a curve in $\mathbb{C}$, $f = u + iv$, with u, v continuous on γ, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma u\,\mathrm{d}y - v\,\mathrm{d}x.$$

We can also parameterize as

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\frac{\mathrm{d}\gamma}{\mathrm{d}t}\,\mathrm{d}t.$$

As well we have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}x + i\int_\gamma f(z)\,\mathrm{d}y.$$

Remark IV.2.1

All of the basic theorems concernining linearity from real integrals work! Namely

$$\int_\gamma c(f(z) + g(z))\,\mathrm{d}z = c\int_\gamma f(z)\,\mathrm{d}z + \int_\gamma g(z)\,\mathrm{d}z.$$

Example IV.2.1

Consider $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$. Then we wish to calculate $\int_\gamma \frac{\mathrm{d}z}{z}$. Then we see that

$\mathrm{d}z = ie^{it}\,\mathrm{d}t$, and so

$$\int_\gamma \frac{\mathrm{d}z}{z} = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i.$$

Example IV.2.2

Let L be the line segment in $\mathbb{C}$ parameterized by $[0,1] \to \mathbb{C}$. $t \mapsto p + t(q-p)$.

Fix $n \in \mathbb{Z}, n \neq -1$, then

$$\int_L z^n\,\mathrm{d}z = \int_0^1 (p + t(q-p))^n \cdot (q-p)\,\mathrm{d}t = \left(\frac{(p+t(q-p))^{n+1}}{n+1}\right]_0^1 = \frac{q^{n+1} - p^{n+1}}{n+1}.$$

Exercise IV.2.3

Fix $m \in \mathbb{Z}$ and $R > 0$, compute

$$\int_{|z-z_0|=R} (z-z_0)^m\,\mathrm{d}z = \begin{cases} 0 & \text{if } m \neq -1 \\ 2\pi i & \text{if } m = -1 \end{cases}$$

Arc length, we denote as

$$|\mathrm{d}z| \coloneqq \mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}.$$

if γ is parameterized by $\gamma(t) = x(t) + iy(t)$ then

$$\int_\gamma h(z)\,|\mathrm{d}z| = \int_a^b h(\gamma(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Theorem IV.2.1

[ML Theorem] Suppose γ is a piecewise smooth curve in the plane. If $h(z)$ is continuous on γ, then

(1) $\left|\int_\gamma h(z)\,\mathrm{d}z\right| \le \int_\gamma |h(z)|\,|\mathrm{d}z|$.

(2) Furthermore, if γ has length L and $|h(z)| \le M$ on γ then

$$\left|\int_\gamma h(z)\,\mathrm{d}z\right| \le \int_\gamma |h(z)|\,|\mathrm{d}z| \le ML.$$

Proof. Use triangle inequality and Riemann sums.

IV.3. The Complex Fundamental Theorem of Calculus

Definition IV.3.1

Let $f(z)$ be a continuous function on a connected open subset $D \subseteq \mathbb{C}$. $F : D \to \mathbb{C}$ is called a complex primitive for $f(z)$ provided that $F(z)$ is holomorphic on D and $F'(z) = f(z)$.

Theorem IV.3.1 (FTC I)

If $f(z)$ is continuous on a connected open subset D and if $F(z)$ is a primitive for $f(z)$, then

$$\int_\gamma f(z)\,\mathrm{d}z = F(B) - F(A),$$

where γ is any path from A to B.

Proof. Fix γ a path between A and B. We see that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}z}\,\mathrm{d}z = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}z}\,\mathrm{d}x + i\int_\gamma \frac{\mathrm{d}F}{\mathrm{d}z}\,\mathrm{d}y = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}x}\,\mathrm{d}x + i\int_\gamma -i\frac{\mathrm{d}F}{\mathrm{d}y}\,\mathrm{d}y$$

using the Cauchy-Riemann Equations! We then have that

$$\int_\gamma f(z) = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}x}\,\mathrm{d}x + \frac{\mathrm{d}F}{\mathrm{d}y}\,\mathrm{d}y = \int_\gamma \mathrm{d}F = F(B) - F(A).$$

Stuff:

- HW 4B due today
- Purple HW 5
- Office Hours Wednesday/Friday
- Halloween shirts for Michigan Math \$15 (walk around EH for info!)
- Walk 40 miles through rattlesnakes on Saturday
- For HW 4B #7a, prove f is constant on closed unit disk.

Lets go back to the FTC, first we'll do an example

Example IV.3.1 (Application of Theorem IV.3.1)

Consider $z \mapsto \frac{1}{z}$ on a connected open set D that contains the unit circle. Does this have a primitive on D?

No! We know $\int_{S^1} \frac{\mathrm{d}z}{z} = 2\pi i$, which would contradict that $\int_{S^1} \frac{\mathrm{d}z}{z} = F(1) - F(1) = 0$ if there were a primitive F for $\frac{1}{z}$ on D.

Theorem IV.3.2 (FTC II)

Let $D \subseteq \mathbb{C}$ be an open connected star shaped (can be simply connected) subset of $\mathbb{C}$. Let $f(z)$ be holomorphic on D. Then $f(z)$ has a primitive on D, and the primitive is unique up to adding a constant. The primitive can be given as

$$F(z) = \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta$$

for $z \in D$, where we take any available path.

Proof. Write $f = u + iv$. Consider the differential form $u\,\mathrm{d}x - v\,\mathrm{d}y$. Since f is holomorphic, Cauchy-Riemann implies that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial u}$. Thus $u\,\mathrm{d}x - v\,\mathrm{d}y$ is a closed differential form.

Since D is open, connected, and simply connected, we know $u\,\mathrm{d}x - v\,\mathrm{d}y$ is exact on D. Thus there is a continuously differentiable function U on D so that $\mathrm{d}U = u\,\mathrm{d}x - v\,\mathrm{d}y$. That is $\frac{\partial U}{\partial x} = u$, $\frac{\partial U}{\partial y} = -v$.

Applying Cauchy-Riemann yields

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0.$$

Thus U is harmonic on D. Since D is simply connected, there exists a harmonic conjugate V for U on D such that $G = U + iV$ is holomorphic on D.

We see that

$$G' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = u + iv = f.$$

Perfect! This shows that there is a primitive G for f. To show it is unique up to adding a constant, let H be another primitive, then $G - H$ has derivative zero, so $G - H$ is constant on D.

Finally, if z_0 is any point of D, then by Theorem IV.3.1 we see that

$$F(z) := \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta = G(z) - G(z_0).$$

Differentiating both sides yields $F'(z) = G'(z) = f$. This completes the proof of all the pieces given above of the theorem.

Corollary IV.3.3

Integrals of holomorphic functions in star-shaped regions are path-independent.

IV.4. Cauchy's Theorem

Setting: $f = u + iv$ is holomorphic on a connectedopen set $D \subseteq \mathbb{C}$. Then

$$f(z)\,\mathrm{d}z = (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) = (u + iv)\,\mathrm{d}x + (-v + iu)\,\mathrm{d}y$$

Exercise IV.4.1

The condition that $f(z)\,\mathrm{d}z$ is closed is exactly the Cauchy-Riemann equations. That is

$$\frac{\partial[u + iv]}{\partial y} = \frac{\partial[-v + iu]}{\partial x}.$$

Theorem IV.4.1 (Morera)

A continuously differentiable function $f(z)$ on D is holomorphic if and only if the differential $f(z)\,\mathrm{d}z$ is closed on D.

Theorem IV.4.2 (Cauchy)

Let D be a bounded connected open subset of $\mathbb{C}$ with piecewise smooth boundary. If $f(z)$ is holomorphic on D and if it extends smoothly to ∂D, then

$$\int_{\partial D} f(z)\,\mathrm{d}z = 0.$$

Proof. Apply Green's theorem!

Corollary IV.4.3

If $f(z)$ is holomorphic on a region that contains an annulus D with inner radius r and outer radius R about z then

$$\int_{|z-w|=r} f(w)\,\mathrm{d}w = \int_{|z-w|=R} f(w)\,\mathrm{d}w.$$

Proof. We know that

$$0 = \int_{\partial D} f(w)\,\mathrm{d}w = \int_{|z-w|=R} f(w)\,\mathrm{d}w - \int_{|z-w|=r} f(w)\,\mathrm{d}w.$$

Theorem IV.4.4 (Cauchy's Integral Formula for $f(z)$)

Let $D \subseteq \mathbb{C}$ be a bounded, connected, open subset with piecewise smooth boundary. If $f(z)$ is holomorphic on D and $f(z)$ extends continuously to ∂D, then for each $z \in D$ we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Compare this with $\int_{|w|=1} \frac{\mathrm{d}w}{w} = 2\pi i$. This is the $f(w) = 1, z = 0$ case. In fact we're going to use this to steal the game.

Proof. Fix $z \in D$ and choose $\varepsilon > 0$ so that $\overline{B(z,\varepsilon)} \subseteq D$ (the closed ball).

Define $D_\varepsilon \coloneqq D \setminus \overline{B(z,\varepsilon)}$. We know $w \mapsto \frac{f(w)}{w-z}$ is holomorphic on D_ε and extends smoothly to $\partial D_\varepsilon = \partial D \cup \{|w-z| = \varepsilon\}$, with reversed orientations.

We then have that

$$\begin{aligned} \int_{D_\varepsilon} \frac{f(w)}{w-z}\,\mathrm{d}w = 0 \\ \int_{D_\varepsilon} \frac{f(w)}{w-z} = \int_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}w - \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,\mathrm{d}w \\ \int_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}w = \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,\mathrm{d}w. \end{aligned}$$

Thus we can reduce the problem to evaluating the integral over a small circle $|w-z| = \varepsilon$. There are a number of different proofs. The simplest uses the mean value property. Parameterizing $|w-z| = \varepsilon$ as $z + \varepsilon e^{i\theta}$ for $0 \le \theta \le 2\pi$ yields

$$\begin{aligned} \int_{\partial D} \frac{f(w)}{w-z} = \int_0^{2\pi} f(z+\varepsilon e^{i\theta})i\,\mathrm{d}\theta \\ \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z} = \int_0^{2\pi} f(z+\varepsilon e^{i\theta}\frac{\mathrm{d}\theta}{2\pi}. \end{aligned}$$

Applying the mean value property to u, v where $f = u + iv$ gives us the result.

If we want to approach this using analysis directly, we may do so as below

Note that the value of this integral cannot depend on $\varepsilon > 0$. Fix some $\delta > 0$, we wish to show that

$$\left|2\pi i f(z) - \int_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}w\right| < \delta.$$

Well, we may write this as

$$2\pi i f(z) - \int_{|z-w|=\varepsilon} \frac{f(w)}{w-z}\,\mathrm{d}w = \int_{|z-w|=\varepsilon} \frac{f(z)-f(w)}{z-w}\,\mathrm{d}w,$$

because

$$\int_{|z-w|=\varepsilon} \frac{\mathrm{d}w}{z-w} = 2\pi i.$$

Then using the ML-inequality (see [**thm:ml-theorem**]) we have that

$$\left|2\pi i f(z) - \int_{|z-w|=\varepsilon} \frac{f(w)}{w-z}\,\mathrm{d}w\right| = \left|\int_{|z-w|=\varepsilon} \frac{f(z)-f(w)}{z-w}\,\mathrm{d}w\right| = 2\pi\varepsilon \cdot \sup_{|z-w|=\varepsilon} |f(z)-f(w)|\,.$$

Taking $\varepsilon \to 0$ takes the right hand side to zero, so we win!

Theorem IV.4.5 (Cauchy's Generalized Integral Formula)

Let $D \subseteq \mathbb{C}$ be a bounded connected open subset with piecewise smooth boundary. Suppose that $f(z)$ is holomorphic on$D$ and $f(z)$ extends smoothly to ∂D, then f has complex derivatives of all orders on D, which are given by

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,\mathrm{d}w.$$

Proof. Proof is via induction on m. Going from $m = 0$ to $m = 1$ is similar enough to going from $m = n$ to $m = n + 1$, so we'll do the first only.

Consider that

$$\frac{1}{w-(z+h)} - \frac{1}{w-z} = \frac{w-z-(w-(z+h)}{(w-(z+h))(w-z)} = \frac{h}{(w-(z+h))(w-z)}.$$

Then we have by Theorem IV.4.4 that

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{1}{2\pi i h} \int_{\partial D} \left(\frac{f(w)}{w-(z+h)} - \frac{f(w)}{w-z}\right)\mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-(z+h))(w-z)}\,\mathrm{d}w.\end{aligned}$$

As we take $h \to 0$, the integrand converges to $\frac{f(w)}{(w-z)^2}$ uniformly on $w \in \partial D$. Thus the integrals converge and we obtain

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^2}\,\mathrm{d}w$$

for $z \in D$.

Corollary IV.4.6

If $f(z)$ is holomorphic on D, then $f(z)$ is infinitely differentiable and all derivatives are also holomorphic on D.

Example IV.4.2

We can simply compute

$$\int_{|z|=2} \frac{\sin(2z)}{(z-i)^6}\,\mathrm{d}z.$$

We know $f(z) = \sin(2z)$ is holomorphic on this region, and so Cauchy's integral formula tells us that

$$f^{(5)}(i) = \frac{5!}{2\pi i} \int_{|z|=2} \frac{\sin(2z)}{(z-i)^6} \, \mathrm{d}z.$$

Taking 5 derivatives of $\sin(2z)$ yields $2^5 \cos(2z)$. Thus this is

$$2^5 \cos(2i) = \frac{5!}{2\pi i} \int_{|z|=2} \frac{\sin(2z)}{(z-i)^6} \, \mathrm{d}z.$$

Noting that $2\cos(\theta) = e^{i\theta} + e^{-i\theta}$, we have that

$$\int_{|z|=2} \frac{\sin(2z)}{(z-i)^6} \, \mathrm{d}z = \frac{2^5 \pi i (e^{-2} + e^2)}{5!}$$

Stuff:

- Math club today 4pm
- Math S^1 this evening 6:30-8pm
- Halloween shirts! Order by 10/8
- Bagel Sunday at 11:30
- MMHH Sunday afternoon
- 40 Mile Walk this saturday.

Exercise IV.4.3 (Warmup)

Compute the following for $n \in \mathbb{N}$

$$\int_{|w|=1} \frac{e^w}{w^n} \, \mathrm{d}w.$$

Answer: $\frac{2\pi i}{(n-1)!}$, using the Cauchy integral formula.

IV.5. Liouville's Theorem

This is given in [Gam03, Gamelin IV.5].

Setting: $f(z)$ is holomorphic on a closed disk $\{|z - z_0| \le \rho\}$. By our convention, f is holomorphic on a neighborhood of that closed disk.

The Cauchy integral formula is

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(w)}{(w-z)^{m+1}} \, \mathrm{d}w.$$

We may then parameterize the circle as $w = z_0 + \rho e^{i\theta}$, and since $\mathrm{d}w = \rho i e^{i\theta}$ we get that

$$f^{(m)}(z_0) = \frac{i \cdot m!}{\rho^m} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) e^{-im\theta} \frac{\mathrm{d}\theta}{2\pi}.$$

Then applying the triangle inequality yields that

$$\left| f^{(m)}(z_0) \right| \le \frac{m!}{\rho^m} \int_0^{2\pi} \left| f(z_0) + \rho e^{i\theta} \right| \frac{\mathrm{d}\theta}{2\pi}.$$

Theorem IV.5.1 (Cauchy estimates)

Suppose $f(z)$ is holomorphic for $|z - z_0| \leq \rho$. Then if $|f(z)| \leq M$ for $|z - z_0| = \rho$, then

$$\left| f^{(m)}(z_0) \right| \leq \frac{m!}{\rho^m} M,$$

for $m \geq 0$.

Proof. ML estimate.

Theorem IV.5.2 (Liouville's Theorem)

Let $f(z)$ be an entire function. If $f(z)$ is bounded, then it is constant!

Proof. We show the derivative is zero at any $z_0 \in \mathbb{C}$, say f is bounded by M on $\mathbb{C}$. Take $m = 1$ and send $\rho \to \infty$ in the Cauchy estimate, then $|f'(z_0)| \leq \frac{M}{\rho}$, the right hand side goes to zero, so $f'(z_0) = 0$.

We'll now see an application of this theorem to the theory of Riemann surfaces.

Recall: Sarah said earlier that there are exactly 3 "different" types of simply connected Riemann surfaces (complex manifolds of dimension 1).

Examples of Riemann surfaces: $\mathbb{C}, \widehat{\mathbb{C}}, \mathbb{H}, \mathbb{D}$, and any open $U \subseteq \mathbb{C}$.

We call 2 Riemann surfaces X, Y equivalent provided there is a biholomorphism (that is a bijective holomorphic map with holomorphic inverse) $\varphi : X \to Y$.

Lets look at some examples

$$\begin{tikzcd}
\text{Spherical} & \text{Hyperbolic} & \text{Euclidean} \\
\widehat{\mathbb{C}} & \mathbb{H} \arrow[d, leftrightarrow, "\text{Cayley map}"'] & \mathbb{C} \\
 & \mathbb{D} &
\end{tikzcd}$$

Because $\widehat{\mathbb{C}}$ is compact we know it is not equivalent to $\mathbb{H}$ or $\mathbb{C}$. Liouville's theorem tells us that since any map $\mathbb{C} \xrightarrow{\varphi} \mathbb{D}$ would be entire and bounded... it would be constant! Thus $\mathbb{C}$ is not equivalent to $\mathbb{D}$ either.

How does one find a conformal map $\varphi : U \to \mathbb{D}$ when $U \subsetneq$ is open and simply connected?

2-dimensional manifolds it's much much much harder. Bill Thurston's geometrization program was all about this, and led to the proof of the Poincare conjecture by Perelmann.

Exercise IV.5.1

Simpler simpler case, take a square and find a conformal isomorphism to the disk. This is hard.

Take a fractal (say a julia set!) and find a conformal isomorphism from $\widehat{\mathbb{C}} \backslash$fractal $\to \widehat{\mathbb{C}} \backslash$closed unit disk. This is in fact easier than the square problem...

If you've taken 592: Fix a Riemann surface X. The universal cover of X is either $\widehat{\mathbb{C}}, \mathbb{C}$, or $\mathbb{D}$.

We'll see some proofs of statements like this later. Another application!

Theorem IV.5.3 (Fundamental Theorem of Algebra)

Every nonconstant polynomial $p(z)$ has a root in $\mathbb{C}$.

Proof. Let $p(z)$ be a nonconstant polynomial with no root in $\mathbb{C}$. Then $z \mapsto \frac{1}{p(z)}$ is entire on $\mathbb{C}$. Is it bounded on $\mathbb{C}$? p is a polynomial, so on a large disk, p "looks likes" z^n. Namely with some Pain in the Ass Estimates

$$\lim_{z\to\infty} p(z) = \infty$$
$$\lim_{z\to\infty} \frac{1}{p(z)} = 0.$$

Thus on outside a large disk $\frac{1}{p(z)}$ is smaller than 53, and inside of the disk it is bounded by the Extreme Value Theorem.

IV.6. Morera's Theorem

Theorem IV.6.1 (Morera)

Let $f(z)$ be a continuous function on a connected open subset $D \subseteq \mathbb{C}$. If $\int_{\partial R} f(z)\,\mathrm{d}z = 0$ for every closed rectangle $R \subseteq D$ with sides parallel to the real/imaginary axes, then f is holomorphic on D with continuous derivative.

Proof. Suppose D is a disk with center z_0 (this is sufficient by openness, since everything is local).

Define $F : D \to \mathbb{C}$ as

$$F : z \mapsto \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta.$$

The path of integration is taxicabs, and is well-defined by the assumption above. We now compute the derivative. Fix $z \in D$, and take $|h|$ to be small enough so that $z + h \in D$.

Then we have that

$$\begin{aligned}
F(z+h) - F(z) &= \int_{z}^{z+h} f(\zeta)\,\mathrm{d}\zeta \\
&= \int_{z}^{z+h} f(\zeta)\,\mathrm{d}\zeta + \int_{z}^{z+h} (f(z) - f(z))\,\mathrm{d}\zeta \\
&= hf(z) + \int_{z}^{z+h} f(\zeta) - f(z)\,\mathrm{d}\zeta \\
\frac{F(z+h) - F(z)}{h} &= f(z) + \frac{1}{h}\int_{z}^{z+h} f(\zeta) - f(z)\,\mathrm{d}\zeta.
\end{aligned}$$

Now using the ML-inequality we know that

$$\begin{aligned}
\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \frac{1}{|h|}\left|\int_{z}^{z+h} f(\zeta) - f(z)\,\mathrm{d}\zeta\right| \\
&\le 2M_h,
\end{aligned}$$

where M_h is the maximum value of $|f(\zeta) - f(z)|$ for ζ satisfying $|\zeta - z| \le |h|$. Notice: the 2 comes from the taxicab metric.

Since $f(\zeta)$ is continuous at z, $M_h \to 0$ as $h \to 0$.

Note: We assume f is continuous, so since $F' = f$, we know F' is continuous. This means $F(z)$ is holomorphic and it has continuous derivative. Apply Cauchy integral formula to get f' exists and is continuous.

IV.7. **Goursat's Theorem**

We're going to get rid of Gamelin assumption in definition of holomorphic. Recall: Gamelin assumes $f'(z_0)$ exists and f' is continuous in a neighborhood of z_0.

Theorem IV.7.1 (Goursat)

If $f(z)$ is a complex-valued function on a connected open set D such that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists at each $z_0 \in D$, then f' is continuous on$D$. Thus f is holomorphic in the sense of Gamelin.

Proof. Idea: Use Morera.

Let R_0 be a closed rectangle in D with sides parallel to the coordinate axes. Divide R_0 into 4 equally sized subrectangles $S_{11}, \dots, S_{14}$. Let R_1 be the subrectangle for which

$$\left| \int_{\partial(\text{subrect})} f(z) \, \mathrm{d}z \right|$$

is maximal. Note

$$\begin{aligned} \left| \int_{\partial R} f(z) \, \mathrm{d}z \right| &= \left| \int_{\partial S_{11}} f(z) \, \mathrm{d}z + \int_{\partial S_{12}} f(z) \, \mathrm{d}z + \int_{\partial S_{13}} f(z) \, \mathrm{d}z + \int_{\partial S_{14}} f(z) \, \mathrm{d}z \right| \\ &\le 4 \left| \int_{\partial R_1} f(z) \, \mathrm{d}z \right|. \end{aligned}$$

Induct! Divide R_1 into 4-subrectangles and call R_2 the subrectangle (of R_1) for which $\left| \int_{\partial(\text{subrect})} f(z) \, \mathrm{d}z \right|$ is maximal.

In this way, we get a sequence of nested rectangles $R =: R_0 \supseteq R_1 \supseteq R_2 \supseteq R_3$ such that

$$\left| \int_{\partial R_j} f(z) \, \mathrm{d}z \right| \le 4 \left| \int_{\partial R_{j+1}} f(z) \, \mathrm{d}z \right|$$
$$\left| \int_{\partial R} f(z) \, \mathrm{d}z \right| \le 4^n \left| \int_{\partial R_n} f(z) \, \mathrm{d}z \right|.$$

As $n \to \infty$, the rectangles shrink down to a single point (since their diameters shrink to 0), which we call z_0.

Furthermore, if L is the perimeter of R, then $\frac{L}{2^n}$ is the length of ∂R_n. Now since $f(z)$ is complex differentiable at z_0, we know that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that for all $n > N$ and $z \in R_n$ we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \le \varepsilon.$$

Thus if we let

$$\varepsilon_n := \sup_{z \in R_n} \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right|,$$

we know $\varepsilon_n \to 0$ as $n \to \infty$.

Now we write

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon_n |z - z_0|$$

Consider $z \mapsto f(z_0) + f'(z_0)(z - z_0)$. This is an affine function of z, so it is holomorphic in z, and it has a primitive $G(z)$ on R_n, so we have

$$\int_{\partial R_n} f(z_0) + f'(z_0)(z - z_0)\,\mathrm{d}z = 0$$

$$\begin{aligned} \left|\int_{\partial R_n} f(z)\,\mathrm{d}z\right| &= \left|\int_{\partial R_n} f(z) - f(z_0) - f'(z_0)(z - z_0)\,\mathrm{d}z\right| \\ &\leq \int_{\partial R_n} \varepsilon_n\,|z - z_0|\,\mathrm{d}z. \end{aligned}$$

Since $|z - z_0|$ is at most $P_n/2$¡ where $P_n = P/2^n$ is the perimeter of R_n and P is the perimeter of R, we have that

$$\left|\int_{\partial R_n} f(z)\,\mathrm{d}z\right| \leq \varepsilon_n \frac{P}{2^n} \cdot \frac{P}{2 \cdot 2^n}.$$

Then we know that

$$\left|\int_{\partial R} f(z)\,\mathrm{d}z\right| \leq 4^n \left|\int_{\partial R_n} f(z)\,\mathrm{d}z\right| \leq \frac{\varepsilon_n P^2}{2}.$$

Taking $n \to \infty$ takes the right hand quantity to 0 and the left hand quantity does not depend on n! Perfect! Thus the integral is zero and we win by applying Morera!

V. **Series!**

Stuff:

- HW 5B due tonight!
- HW 6A due Thursday
- HW 6B due next week
- Office Hours: Wednesday + Friday (email with questions!)
- Popcorn? T-shirts Halloween and voting
- Super Saturdays
- Student Seminar Friday 10/7 4pm-5pm EH 3096: Partition Statistics and the S^1 method by Faye Jackson.

Hint for $n, m < 0$ in HW 5B Q1. Take small circles around 0. Also for HW 5B Q1 note

$$z\overline{z} = r^2 \qquad\qquad |\mathrm{d}z| = \frac{-ir}{z}\,\mathrm{d}z.$$

V.1. **Very Quick Review**

A sequence of complex numbers is a function $n \mapsto a_n \in \mathbb{C}$ for $n \in \mathbb{N}$. Associated to this is a new sequence that we can build

$$m \mapsto (s_m \coloneqq \sum_{k=1}^{m} a_k)$$

called the sequence of partial sums. If $\lim_{m\to\infty} s_m$ exists, then we say series $\sum_1^\infty a_n$ converges and is equal to that limit. There are a number of nice tests from real analysis and many extend to complex analysis

- Divergence Test: If $\sum_1^\infty a_n$ converges then $a_n \to 0$. Caution: Remember the harmonic series.

Proposition V.1.1

If $|z| < 1$ then the series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

If $|z| \geq 1$ then the series diverges.

Proof. Very lucky! Work out what the partial sums are explicitly.

Definition V.1.1

A complex series $\sum a_k$ converges absolutely provided that $\sum |a_k|$ converges.

Theorem V.1.2

If $\sum a_k$ converges absolutely, then $\sum a_k$ converges.

Proof. Write $\mathrm{Re}(a_k) = \mathrm{Re}(a_k) + |a_k| - |a_k|$. Then since $|\mathrm{Re}(a_k)| \leq |a_k|$, we know that

$$0 \leq \mathrm{Re}(a_k) + |a_k| \leq 2\,|a_k|$$

Similarly

$$0 \leq \mathrm{Im}(a_k) + |a_k| \leq 2\,|a_k|\,.$$

We then know that the series of non-negative real numbers $\sum \mathrm{Re}(a_k) + |a_k|$ and $\sum \mathrm{Im}(a_k) + |a_k|$ both converge by the monotone convergence theorem, since the partial sums are bounded above by $2 \sum |a_k|$.

Then we see that

$$\begin{aligned}
\sum \mathrm{Re}(a_k) &= \left(\sum \mathrm{Re}(a_k) + |a_k|\right) - \sum |a_k| \\
\sum \mathrm{Im}(a_k) &= \left(\sum \mathrm{Im}(a_k) + |a_k|\right) - \sum |a_k| \\
\sum a_k &= \sum \mathrm{Re}(a_k) + i \sum \mathrm{Im}(a_k).
\end{aligned}$$

Thus $\sum a_k$ converges!

Next: Series of FUNctions!

In Gamelin this is [Gam03, pp. V .2].

Definition V.1.2

The sequence $n \mapsto (f_n : E \to \mathbb{C})$ converges pointwise to $f : E \to \mathbb{C}$ on E provided that for all $x \in E$, the sequence $n \mapsto f_n(x)$ converges to $f(x)$.

Definition V.1.3

The sequence of fuctions $n \mapsto (f_n : E \to \mathbb{C})$ converges uniformly to $f : E \to \mathbb{C}$ on E provided that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in E$ we have $|f_n(x) - f(x)| < \varepsilon$.

Importantly, the choice of N does not depend on x. An equivalent formulation is to define for $f, g : E \to \mathbb{C}$

$$\|f - g\| = \sup_{z\in\mathbb{C}} |f(z) - g(z)|.$$

Example V.1.1

If we pick $f_n : [0,1] \to [0,1]$ with $f_n : x \mapsto x^n$ then $f_n(x) \to 0$ for $x \in [0,1)$ and $f_n(1) \to 1$. Thus it converges, but it's clear it doesn't converge uniformly.

Why do we like uniform convergence?

Theorem V.1.3

Let $n \mapsto (f_n : E \to \mathbb{C})$ be a sequence of functions converging uniformly to $f : E \to \mathbb{C}$. If all the f_n are continuous on E, then f is continuous on E.

Theorem V.1.4

Let $\gamma \subseteq \mathbb{C}$ be a piecewise smooth curve in the plan. If $j \mapsto (f_j : E \to \mathbb{C})$ is a sequence of continuous functions on γ converging uniformly to $f : E \to \mathbb{C}$ then

$$\int_\gamma f_j(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z.$$

We will apply this whole discussion to series! We'll look at a series of functions $\sum g_j(x)$ and consider when the partial sums $s_n(x) = \sum_{j=0}^{n} g_j(x)$ converges uniformly to some limit function G.

Stuff:

- HW 6A due tonight (Problem 1 erased).
- HW 6B due Tuesday.
- Math Club
- Math S^1.
- Super Saturdays starts this weekend.
- Student Seminar Friday 4pm EH 3096: Circle Method and Applications to Partitions by Faye Jackson.
- HW Hint: (9) Gamelin p119 4. $f : \mathbb{C} \to \mathbb{C}$ is entire and there is an $R > 0$ so that $f(z)/z^n$ is bounded for all $|z| \geq R$. We must show $f(z)$ is a polynomial. What does it mean to show that f is a polynomial? Show there exists $N \in \mathbb{N}$ such that $f^{(m)}(z) = 0$ for all $m \geq N$. Fix $z \in R$ from hypothesis, choose $r > \max(R, |z|)$. . . .

Theorem V.1.5 (Weierstrass M-test)

Suppose $M_k \geq 0$ and $\sum M_k$ converges. If $g_k(z)$ are $\mathbb{C}$-valued on some set $E \subseteq \mathbb{C}$ and $|g_k(z)| \leq M_k$ for all $z \in E$, then $\sum g_k(z)$ converges unifornly on E.

Proof. A Real Analysis Course. See p135 of [Gam03] if you would still like a proof.

Example V.1.2

Consider $\sum_0^\infty z^k \frac{1}{1-z}$ for $|z| < 1$. Is the convergence of the partial sums $S_N(z) = \frac{1-z^{N+1}}{1-z}$ uniform for $|z| < 1$?

No! However, if we take $|z| \le r$ for $0 < r < 1$, then the convergence is uniform! Namely for $M_k = r^k$ we have $\sum M_k$ converges, so $\sum z^k$ converges uniformly by the Weierstrass M-test because $\left|z^k\right| \le r^k$.

Theorem V.1.6

If $\{f_k(z)\}$ is a sequence of holomorphic functions on a connected open $D \subseteq \mathbb{C}$ that converges uniformly to $f : D \to \mathbb{C}$, then f is holomorphic.

Proof. Consider any closed region R in D. Then

$$\int_{\partial R} f(z)\,\mathrm{d}z = \lim_{k\to\infty} \int_{\partial R} f_k(z)\,\mathrm{d}z = 0$$

Furthermore f is continuous since each f_k is continuous. Thus f is holomorphic by Theorem IV.4.1.

Theorem V.1.7

Suppose $\{f_k(z)\}$ is a sequence of functions holomorphic on $|z - z_0| < R$ and suppose f_k converges uniformly to f on $|z - z_0| < R$. Then for each fixed $0 < r < R$ and each fixed $m \ge 1$, the sequence of m-th derivatives $\{f_k^{(m)}(z)\}_k$ converges uniformly to $f^{(m)}(z)$ on $|z - z_0| \le r$.

Proof. Suppose we have $\varepsilon_k \to 0$ as $k \to \infty$ is a sequence such that $|f_k(z) - f(z)| < \varepsilon_k$ for all $|z - z_0| < R$. Fix $m \ge 1$.

Fix r, s such that $0 < r < s < R$. We use the Cauchy integral formula to get

$$f_k^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|z-z_0|=s} \frac{f_k(\zeta) - f(\zeta)}{(\zeta - z)^{m+1}}\,\mathrm{d}\zeta$$

for $|z - z_0| \le r < s$. By the triangle inequality we know $|\zeta - z| \ge s - r$. Thus

$$\left|\frac{f_k(\zeta) - f(\zeta)}{(\zeta - z)^{m+1}}\right| \le \frac{\varepsilon_k}{(s-r)^{m+1}}.$$

The triangle inequality and the ML-estimate yields

$$\left|f_k^{(m)}(z) - f^{(m)}(z)\right| \le \frac{m!}{2\pi(s-r)^{m+1}} \cdot \varepsilon_k \cdot 2\pi s$$

for *any* z satisfying $|z - z_0| \le r$. Sending $k \to \infty$ gives uniform convergence of m-th derivatives on $|z - z_0| \le r$.

Definition V.1.4

We say that for a sequence of holomorphic functions $f_k : D \to \mathbb{C}$ on a connected open set $D \subseteq \mathbb{C}$ converges normally to $f : D \to \mathbb{C}$ on D provided that it converges uniformly to $f(z)$ on every closed disk contained in D.

Some mathematicians instead say "converges locally uniformly."

Example V.1.3

$\sum z^k$ converges normally on $|z| < 1$.

Theorem V.1.8

Suppose $f_k 9z)$ is a sequence of holomorphic functions on a connected open $D \subseteq \mathbb{C}$. Suppose $f_k(z)$

converges uniformly to a holomorphic $f : D \to \mathbb{C}$. Then for each $m \geq 1$ the sequence of m-th derivatives $f_k^{(m)}(z)$ converges normally to $f^{(m)}(z)$ on$D$.

V.2. Power Series

In Gamelin this is [Gam03, p. V.3]

Definition V.2.1

A power series (centered at $z_0 \in \mathbb{C}$) is a series of the form $\sum\limits_{k=0}^{\infty} a_k (z - z_0)^k$.

Claim

We can reduce just about all conversations of power series centered at z_0 to power series centered at $z_0 = 0$ via the change of variables $w = z - z_0$.

Theorem V.2.1

Let $\sum a_k z^k$ be a power series. Then there exists an R, $0 \leq R \leq +\infty$ so that $\sum a_k z^k$ converges absolutely if $|z| < R$, and $\sum a_k z^k$ does not converge if $|z| > R$.

Furthermore, for each fixed r satisfying $0 \leq r < R$, the series $\sum a_k z^k$ converges uniformly on $|z| \leq r$. That is $\sum a_k z^k$ converges normally on $|z| < R$.

Definition V.2.2

This R is called the radius of convergence of the power series.

Proof. Consider the sequence $k \mapsto |a_k| r^k$. If this sequence is bounded for some $r =_0$, then it is bounded for all values of r satisfying $0 \leq r \leq r_0$. Define

$$R \coloneqq \sup\{r \geq 0 \mid \text{the sequence } k \mapsto |a_k| r^k \text{ is bounded}\}.$$

Here we take sup to be ∞ if the right hand set is unbounded. We just need to show this R has the right properties.

By construction, for all $r < R$, $k \mapsto |a_k| r^k$ is bounded, and for all $s > R$, $k \mapsto |a_k| s^k$ is unbounded.

Back to our series $\sum a_k z^k$. If $|z| > R$, then the terms $a_k z^k$ do not go to 0 as $k \to \infty$, so the series does not converge.

Now suppose $|z| \leq r < R$. Choose s such that $r < s < R$. Then the sequence $k \mapsto |a_k| s^k$ is bounded by some $C \in \mathbb{R}$. If $|z| \leq r$, then for all $k \geq 0$ we have

$$\left|a_k z^k\right| \leq |a_k| r^k = |a_k| s^k \left(\frac{r}{s}\right)^k \leq C \left(\frac{r}{s}\right)^k.$$

Set $M_k = C\left(\frac{r}{s}\right)^k$. Does $\sum M_k$ converge? Yes because $s > r > 0$. By the M-test $\sum \left|a_k z^k\right|$, $\sum a_k z^k$ both converge uniformly for $|z| \leq r$. Thus $\sum a_k z^k$ converges absolutely and unifomrly on $|z| \leq r$.

Gamelin: Examples on p139 [Gam03].

Remark V.2.1

The partial sums of the power series $\sum a_k z^k$ are all polynomial functions! Thus they are holomorphic functions and everything is awesome!

Theorem V.2.2

Suppose $\sum a_k z^k$ is a power series with radius of convergence $R > 0$. Then the function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

for $|z| < R$ is holomorphic on $|z| < R$. Furthermore the derivatives $f^{(m)}(z)$ are obtained as power series by differentiating the power series term by term. By example

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$$
$$f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k z^{k-2}.$$

Moreover, the coefficients $a_k = \frac{f^{(k)}(0)}{k!}$. This is given by evaluating the derivatives above at 0.

Proof. This follows from all of the above.

Example V.2.1

$\sum z^k = \frac{1}{1-z}$, $|z| < 1$. We also have

$$\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1} = \sum_{m=0}^{\infty} (m+1) z^m, |z| < 1.$$

Note, we can also integrate term by term because we have uniform convergence on subdisks $|z| \le r$ for $r < R$.

Example V.2.2

We can write that for $|z| < 1$

$$-\operatorname{Log}(1-z) = \int_0^{\infty} \frac{\mathrm{d}\zeta}{1-\zeta} = \int_0^z \sum_{k=0}^{\infty} \zeta^k \,\mathrm{d}\zeta$$
$$= \sum_{k=0}^{\infty} \int_0^z \zeta^k \,\mathrm{d}z = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1}.$$

Therefore if we set $w = 1 - z$, then for $|w - 1| < 1$ we have

$$\operatorname{Log} w = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (w-1)^k.$$

How do we compute R?

Example V.2.3

Sarah's advisor (Hubbard) likes to ask people to compute $\sup\{\cos(10^n) \mid n \in \mathbb{N}\}$. This exists but is so difficult to compute you'd probably win a fields medal if you did. . . it's almost surely equal to 1. . .

Try to reinterpret this as a famous hard problem (it's not too difficult to do). Hint: $2k\pi$ being somehow close to 10^n. . .

How do we compute it? Recall the Ratio and Root tests from calculus!

Theorem V.2.3

Let $\sum a_k z^k$ be a power series. If $\left|\frac{a_k}{a_{k+1}}\right|$ has a limit as $k \to \infty$ (the limit is allowed to be $+\infty$), then the limit is equal to R, where R is the radius of convergence of $\sum a_k z^k$.

Proof. Set $L = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$. If $r < L$, then $\left|\frac{a_k}{a_{k+1}}\right| > r$ eventually, say for $k \geq N$. Then we have

$$|a_k| > r\,|a_{k+1}|$$

for all $k \geq N$. Thus

$$|a_N|\, r^N \geq |a_{N+1}|\, r^{N+1} \geq \cdots$$

so the sequence $k \mapsto |a_k|\, r^k$ is bounded. Thus $r \leq R$. From this we have that $L \leq R$ (as if $L > R$ we could pick an $L > r > R$).

Suppose next that $s > L$, then $\left|\frac{a_k}{a_{k+1}}\right| < s$ eventually, say for all $k \geq N$. Then

$$|a_k| < s\,|a_{k+1}|\,.$$

Therefore

$$|a_N|\, s^N < |a_{N+1}|\, s^{N+1} \leq \cdots$$

and $a_k z^k$ does not go to zero if $|z| \geq s$! Thus $\sum a_k z^k$ does not converge for $|z| \geq s$. It follows then that $L \geq R$. Together we have $L = R$.

Theorem V.2.4

If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$ (we allow limit to be $+\infty$). Then

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}},$$

where R is the readius of convergence of $\sum a_k z^k$.

Proof. See Gamelin.

Remark V.2.2

In the above two theorems we can replace the limit with the limit superior and they still work.

We have seen that $\sum a_k (z - z_0)^k$ converges to a holomorphic function inside a disk of convergence $\{|z - z_0| < R\}$.

We will now see that ANY function that is holomorphic on a disk can be represented locally by power series. That is for each z within the disk we can find a power series which converges to the function on a small ball around z. This latter property is called being analytic, and so we can now use holomorphic/analytic interchangeably.

Theorem V.2.5

Suppose that $f(z)$ is holomorphic for $|z - z_0| < \rho$. Then $f(z)$ is represented by the power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for $a_k = \frac{f^{(k)}(z_0)}{k!}$ on $|z - z_0| < \rho$, and the radius of convergence R of the power series satisfies $R \geq \rho$.

Furthermore, for any fixed r with $0 < r < \rho$ we have

$$a_k = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, \mathrm{d}\zeta,$$

and if $|f(z)| \leq M$ on $|z - z_0| < r$ then

$$|a_k| \leq \frac{M}{r^k}.$$

We first need a lemma

Lemma V.2.6

Fix z satisfying $|z| < r$. For $|\zeta| = r$ we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \frac{1}{1 - \frac{z}{\zeta}} = \sum_{k=0}^{\infty} \frac{z^k}{\zeta^{k+1}}$$

We claim this converges uniformly for $|\zeta| = r$.

Proof. Weierstrass M-test!

Proof. It suffices to take $z_0 = 0$ by translation. For any fixed r, satisfying $0 < r < \rho$, we have

$$a_k = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)}{\zeta^{k+1}} \, \mathrm{d}\zeta,$$

by the Cauchy integral formula. Furthermore if $|f(z)| \leq M$ on $|z| \leq r$, then by the Cauchy estimates.

$$|a_k| \leq \frac{M}{r^k}$$

for $k \geq 0$. Furthermore for $|z| < r$ we have convergence of $\sum a_k z^k$ then by the Weierstrass M-test! Thus the radius of convergence $R \geq \rho$.

Now fix $|z| < r$, we have by the Cauchy integral formula and uniform convergence that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_{|\zeta| = r} \sum_{k=0}^{\infty} \frac{f(\zeta) z^k}{\zeta^{k+1}} \, \mathrm{d}\zeta \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)}{\zeta^{k+1}} \, \mathrm{d}\zeta \right) z^k \\
&= \sum_{k=0}^{\infty} a_k z^k,
\end{aligned}$$

and everything must converge.

Stuff:

- HW 6B due this evening!
- HW 7 is due the 20th, some Gamelin and Qual Exam problems.
- Midterm Tuesday October 25th.

Midterm Information

- Admin:
  - In class
  - Bring a sheet of notes
  - Solo Exam
- Topics:
  - Gamelin, everything we have covered through + including stuff in Section V [Gam03]
  - Likely problems from Gamelin will be stolen...

Definition V.2.3

A function $f(z)$ is analytic on $|z - z_0| < r$ provided that there exists a sequence $k \mapsto a_k$ such that $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$. for all z satisfying $|z - z_0| < r$.

A function $f(z)$ is called analytic at z_0 provided there is a neighborhood of f on which f is analytic (i.e, agrees with a power series).

A function f is holomorphic on $|z - z_0| < r$ if and only if f is analytic on $|z - z_0| < r$.

Example V.2.4

Let $z \mapsto \exp(z)$. This is entire, and so it has a power series about 0 which converges everywhere. In fact

$$\exp(z) = \frac{z^k}{k!}$$

Corollary V.2.7 (of Theorem V.2.5)

Syppose $f(z)$ and $g(z)$ are analytic for $|z - z_0| < r$. If $g^{(k)}(z_0) = f^{(k)}(z_0)$ for all $k \geq 0$, then $g = f$ on $|z - z_0| < r$.

The above is known as a rigidity theorem. The tagline is that

"Analytic functions are extremely rigid"

Corollary V.2.8 (p146, [Gam03])

Suppose $f(z)$ is analytic at z_0 with power series expansion $f(z) = \sum a_k(z - z_0)^k$. Then the radius of convergence of the power series is the largest R such that $f(z)$ extends to be analytic on the disk $\{|z - z_0| < R\}$.

That is there is an analytic g on $|z - z_0| < R$ given by the power series which agrees with f on the restriction to any disk about z_0 where both f and g are defined.

Tagline: "Radius of convergence is the distance from z_0 to the nearest singularity."

Warning: f may be defined on a larger domain and *disagree* with the extension g at some points.

Example V.2.5

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto \frac{1}{1+x^2}$. Is this $\mathbb{R}$-analytic at $x_0 = 0$. Yes! The series is given by $\sum(-x^2)^k$ whenever $\left|-x^2\right| < 1$, that is $|x| < 1$. There are *different* power series expansions about say 2.

Why do you hit an obstruction at ± 1??? **YOU DON'T.** You are actually crashing into $\pm i$ for the complex function $F : \mathbb{C} \setminus \{\pm i\} \to \mathbb{C}$ given by $z \mapsto \frac{1}{1+z^2}$.

The complex numbers are showing us things that the real numbers cannot show us!!!

There is a definition of power series at ∞ given in Gamelin. Recall that $f(z)$ is holomorphic at $z = \infty$ means that $g(w) = f\left(\frac{1}{w}\right)$ is holomoprhic at $w = 0$.

Section V.6 of [Gam03] is about algebraic manipulation of series and details of this.

V.3. Zeros of Analytic Functions

Definition V.3.1

We say that $f(z)$ has a zero of order N at z_0 provided that

$$f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$$

and $f^{(N)}(z_0) \neq 0$. This happens if and only if we can write $f(z) = (z - z_0)^N h(z)$ for some holomorphic function $h(z)$ and $h(z_0) = \frac{f^{(N)}(z_0)}{N!} \neq 0$. Note h is only defined about some disk about z_0.

This all happens if and only if the power series expansion has the form

$$f(z) = a_N (z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + \cdots$$

where $a_N \neq 0$.

Definition V.3.2

Let $E \subseteq \mathbb{C}$. We say that a point $x \in \mathbb{C}$ is an accumulation point of $\mathbb{C}$ provided that for all open $U \subseteq \mathbb{C}$ which contain x, we have $E \cap (U \setminus \{x\}) \neq \emptyset$.

To say E is isolated is to say that it has no accumulation points.

Restricting our attention to points in E, we have

$$E = \text{accumulation points of } E \text{ in } E \sqcup \text{ isolated points in } E.$$

For a closed set S, to say S is isolated is equivalent to saying that for any $s \in S$ there is an open ball $B(s, \varepsilon)$ so that $B(s, \varepsilon)$ contains no points of S except for s.

I.e. it is equivalent to saying S consists only of isolated points.

Theorem V.3.1

Let $D \subseteq \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be analytic. Now suppose that f is not identically zero. Then the zeros of f are isolated, that is they have no accumulation points.

That is given any zero z_0 of f, we can find a neighborhood of z_0 that contains no other zero of f.

Proof. Note that any accumulation point would necessarily be a zero of f by continuity of f. We start by showing that

Claim

If z_0 is a zero of f, then it has finite order (as a zero).

Let $U = \{z \in D \mid f^{(m)}(z) = 0 \text{ for all } m\}$. If $z_0 \in U$, then the power series $\sum a_k(z-z_0)^k$ has zero coeffficients, and is equal to f on a disk centered at z_0. Thus U is open. To see that U is closed, note that

$$U = \bigcap_m (f^{(m)})^{-1}(\{0\}),$$

and all the $f^{(m)}$ are continuous, so U is closed. Thus either $U = \emptyset$ or $U = D$ by the connectedness of D. if $U = D$ then f is identically zero, so $U = \emptyset$.

Now suppose $z_0 \in D$ is a zero of f, necessarily of finite order N. We may write $f(z) = (z-z_0)^N h(z)$ for $h(z)$ analytic at z_0 and $h(z_0) \neq 0$.

For ρ sufficiently small we have $h(z) \neq 0$ for all $|z - z_0| < \rho$. Therefore $f(z)$ cannot be zero for those z satisfying $0 < |z-z_0| < \rho$! Thus the zeros at z_0 are separated from other zeros of f.

Theorem V.3.2 (Uniqueness Principle)

If $f(z)$ and $g(z)$ are analytic on a connected open set $D \subseteq \mathbb{C}$, and if $f(z) = g(z)$ on a set that has an accumulation point lying in D, then $f(z) = g(z)$ everywhere.

Proof. Apply the above!

Stuff:

- HW 7 Due Friday October 21st, 11:59pm
- Problem 3
  - Setup: $j \mapsto g_j$, nowhere vanishing entire, and we have $j \mapsto p_j$ polynomials with $\deg(p_j) \leq 10$.
  - $f_j := g_j p_j$ converges locally uniformly on $\mathbb{C}$ to $f : \mathbb{C} \to \mathbb{C}$. Then $f = g \cdot p$, where g is nowehre vanishing and entire, p is a polynomial. What do we know about $\deg(p)$?
  - This requires Hurwitz's theorem, so we will delay this problem until later!

Last time: Powerful result! Analytic functions.

We saw that if $f(z)$ is not identically zero on a connected open set $D \subseteq \mathbb{C}$, and if $z_0 \in D$ is a zero of f, then z_0 has a finite order as a zero; i.e., $f(z) = (z-z_0)^N \cdot h(z)$ where $h(z_0) \neq 0$ is analytic locally about z_0.

We used this to show that the zeros of f (when f is not identically zero) are isolated from each other.

We were then able to prove the uniqueness principle: if f, g are analytic on a connected open set $D \subseteq \mathbb{C}$ and $f(z) = g(z)$ for z belonging to a set with an accumulation point then $f = g$ on D.

Cool application: Let $g(z)$ be an entire function such that $g(x) = \exp(x)$ for all $x \in \mathbb{R}$, and then $g(z) = \exp(z)$.

V.4. **The Open Mapping Theorem**

Pause: Topology Break!

Definition V.4.1

Let X, Y be topological spaces, a map $f : X \to Y$ is called open provided that for every open $U \subseteq X$ we have $f(U) \subseteq Y$ is open.

Example V.4.1

Projection $X \times Y \to X$ taking $(x, y) \mapsto x$. The identity map. Conway's base 13 function is an example of an open map which is not continuous. It is given by writing a real number x in base 13, using the additional symbols $\{+, -, .\}$, and saying x maps to a number if some tail of the base 13 expansion is a valid base 10 number (and we take the longest such tail). If no such tail exists then we send x to 0.

The function $\mathbb{R} \to \mathbb{R}$ taking $x \mapsto x^2$ is not open since the image of $\mathbb{R}$ is $[0, \infty)$. Similarly the map $F : \mathbb{C} \to \mathbb{R}$ with $F(z) = |z|$ is not open, since $F(\mathbb{C}) = [0, \infty)$. However these are "close" to being open in some sense.

Another nonexample is the constant function $z \mapsto 57 + 53i$.

An example of an open map are affine maps $f : \mathbb{C} \to \mathbb{C}$ wiht $z \mapsto az + b$ for $a \neq 0$ since they are homeomorphisms.

Remark V.4.1

If $f : U \to \mathbb{C}$ is holomorphic with $f'(z) \neq 0$ for all $z \in U$, then f is locally invertible. Thus f is a local homeomorphism, and so f must be open. This is a consequence of the inverse function theorem.

Exercise V.4.2

Show a local homoemorphism must be open. More generally show that being open is a "local" property (appropriately define this as well).

Example V.4.3

The map $z \mapsto z^k$ on $\mathbb{C}$ for $k \geq 1$ is open. Yes! We only have to worry about points where $f'(z) = 0$. Thus we only need to worry about $z = 0$.

Working with a basis of the topology, take a small open disk of radius $r > 0$ about 0, this maps to a small open disk of radius r^k about 0, with k preimages for each point. Great!

Theorem V.4.1 (Open Mapping Theorem)

Let $D \subseteq \mathbb{C}$ be open and connected and let $f : D \to \mathbb{C}$ be a nonconstant holomorphic function. Then f is an open map.

Proof. We only need to worry about $z_0 \in D$ where $f'(z_0) = 0$, since it is a local homeomorphism elsewhere.

Since f is nonconstant, we know that $f' \not\equiv 0$ (not identically zero). Thus there exists a minimal $k \geq 1$ such that $f^{(k)}(z_0) \neq 0$. There is then some disk $|z - z_0| < \rho$ in D so that

$$f(z) = f(z_0) + a_k(z - z_0)^k \cdot h(z)$$

with $a_k \neq 0$, $h(z_0) = 1$, and $h(z)$ analytic. We know the map $z \mapsto z^k$ is locally invertible in a neighborhood of $z = 1$. Let g be a local inverse.

When z is close to z_0, $h(z)$ is close to 1, and so in a neighborhood of z_0 we have $(g(h(z)))^k = h(z)$. Thus we can look at

$$f(z) = f(z_0) + a_k((z - z_0) \cdot g(h(z)))^k$$

This is a composition of open maps near z_0 (translation, powering, and a mystery function) since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}(z - z_0)g(h(z))\Big|_{z=z_0} &= \Big[(z - z_0)g'(h(z))h'(z) + g(h(z))\Big]_{z=z_0} \\ &= g(h(z_0)) = 1.\end{aligned}$$

Perfect! This shows that f is open near z_0 as desired!

Back to Gamelin!

V.5. **Analytic Continuation**

This is section V.8 in [Gam03]. There are no homework problems / QR problems on this part especially anything with paths/monodromy.

Definition V.5.1

Let $U \subseteq V \subseteq \mathbb{C}$ be open and connected. Now let $f : U \to \mathbb{C}$ be analytic. We call $F : V \to \mathbb{C}$ an analytic continuation provided that $F\big|_U = f$.

Example V.5.1

Define $f(z) = \sum_k \left(\frac{z}{2}\right)^k$ for $|z| < 2$. Well when $|z| < 2$ we have

$$f(z) = \frac{2}{2 - z}.$$

We can expand $f(z)$ at $z_0 = -1$ to get a different series

$$\begin{aligned}f(z) &= \frac{2}{2 - (z + 1 - 1)} = \frac{2}{3 - (z + 1)} = \frac{2}{3} \cdot \frac{1}{1 - \frac{(z+1)}{3}} \\ &= \frac{2}{3} \cdot \sum_{k=0}^{\infty} \left(\frac{z + 1}{3}\right)^k,\end{aligned}$$

which is valid for $|z + 1| < 3$.

This gives an analytic continuation!

END OF MIDTERM I MATERIAL
MIDTERM I is in class October 25th

How do we extend analytic functions? Especially important for things like the Riemann ζ function.

Lemma V.5.1

Let $D \subseteq \mathbb{C}$ be open and connected and let $f(z)$ be analytic on D. Now let $R(z_1)$ be the radius of convergence of the power series expansion about $z_1 \in D$. Then in fact

$$|R(z_1) - R(z_2)| \leq |z_1 - z_2|.$$

Proof. Gamelin!

We say that $f(z)$ is analytically continuable along $\gamma \subseteq \mathbb{C}$ if for each $t \in [a,b]$ there exists a convergent power series

$$f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z-\gamma(t))^n$$

for $|z-\gamma(t)| < r(t)$ such that $f_a(z)$ is the power series representation for $f(z)$ at $z_0 = \gamma(a)$ and when $s \in [a,b]$ is near $t \in [a,b]$, then $f_s(z) = f_t(z)$ for z in the intersection of the disks in convergence.

By the uniqueness principle, the series $f_t(z)$ determines uniquely each of the series $f_s(z)$ for s near t.

Theorem V.5.2

Suppose $f(z)$ can be continued analytically along the path γ for $t \in [a,b]$. Then the analytic continuation is unique.

Example V.5.2

Take $f(z)$ to be the principal branch of the square root function, and $\gamma(t) = e^{it}$.

In a neighborhood of $z = 1$ we have

$$f(z) = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \cdots$$

We may then change centers to get

$$f_t(z) = e^{it/2} + \frac{e^{-it/2}}{2}(z-e^{it}) - \frac{e^{-3it/2}}{8}(z-e^{it})^2 + \cdots$$
$$f_{2\pi}(z) = -1 - \frac{1}{2}(z-1) + \frac{1}{8}(z-1)^2 - \cdots.$$

It turns out $f_{2\pi}$ gives us the *other branch* of the square root. The fancy way of saying this is we picked up *monodromy*.

Stuff:

- Exam on Tuesday October 25th in class (full time)
- HW 7 due tomorrow
- Office Hours on Friday 1-2:30pm EH3855
- Math Club!
- Popcorn Thursdays/Bagel Sundays!
- Super Saturdays!
- Voting t-shirts
- Student seminar tomorrow (Topic: Conway's topograph by Xuyan), EH 3096 4pm-5pm.

No content from today on the midterm. HW8 will be based on today and will be received tuesday after midterm.

VI. Laurent Series

VI.1. Laurent Decomposition

This is Gamelin Ch. VI, 1-4 [Gam03], and [Ahl53] 5.1.

Motivating Question: Let $D \subseteq \mathbb{C}$ be open and connected, and let

$$A(D) := \{f : D \to \mathbb{C} \mid f \text{ is analytic on } D\}.$$

How do we understand this?

Example VI.1.1

If $D = B_R(z_0)$ or $D = \mathbb{C}$ then

$$A(D) = \left\{ f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \mid \limsup_{k\to\infty} |a_k|^{1/k} \leq \frac{1}{R} \right\}.$$

Example VI.1.2

Consider $D = \widehat{\mathbb{C}} \setminus \overline{B_r(0)}$. That is analytic on an open disk about ∞.

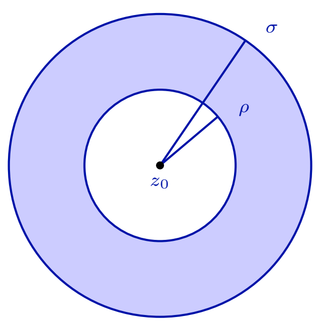
This is $D = \{|z| > r\} \cup \{\infty\}$. Setting $w = 1/z$. This is the same as

$$\begin{aligned} A(D) &= \{f \mid f(z) \text{ analytic on } |z| > r \text{ and } z = \infty\} \\ &\cong \{g \mid g(w) \text{ analytic on } |w| < 1/r\}. \end{aligned}$$

What about the intersection of these? This will be an annulus, and the space $A(D)$ will be larger, as the domain is smaller and so it is *easier* to be analytic here.

Theorem VI.1.1 (Laurent Decomposition Theorem)

Suppose $0 \leq \rho < \sigma \leq \infty$ and suppose $f(z)$ is analytic for $\rho < |z - z_0| < \sigma$. That is, suppose f is analytic on an annulus



Then $f(z)$ can be decomposed as a sum $f(z) = f_0(z) + f_1(z)$ where $f_0(z)$ is analytic on $|z - z_0| < \sigma$ and $f_1(z)$ is analytic on $|z - z_0| > \rho$ and at $z = \infty$.

If we normalize so that $f_1(\infty) = 0$, then this decomposition is unique.

Proof of Uniqueness. Suppose we have $f(z) = f_0(z) + f_1(z) = g_0(z) + g_1(z)$, and normalize so that $f_1(\infty) = g_1(\infty)$. Then we have that

$$g_0(z) - f_0(z) = f_1(z) - g_1(z)$$

for all $\rho < |z - z_0| < \sigma$. We know that $g_0(z) - f_0(z)$ is analytic on $|z - z_0| < \sigma$. Likewise $f_1(z) - g_1(z)$ is analytic in $|z - z_0| > \rho$ and at $z = \infty$. We may define

$$h : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto \begin{cases} g_0(z) - f_0(z) & \text{if } |z - z_0| < \sigma \\ f_1(z) - g_1(z) & \text{if } |z - z_0| > \rho \end{cases}.$$

We see that h is entire, and

$$\lim_{z \to \infty} h(z) = f_1(\infty) - g_1(\infty) = 0.$$

Thus h is bounded on $\mathbb{C}$, and so it is a constant by Liouville's Theorem. Thus $h = 0$, showing that $g_0 = f_0, f_1 = g_1$.

Proof of Existence. We will use the Cauchy integral formula. Choose r and s so that $\rho < r < s < \sigma$. Call the annulus $r < |\zeta| < s$, A. Then keeping track of orientations we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial A} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \left(- \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta + \int_{|\zeta - z_0| = s} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta \right). \end{aligned}$$

Define

$$\begin{aligned} f_0(z) &\coloneqq \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta \\ f_1(z) &\coloneqq -\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta. \end{aligned}$$

We know f_0 is defined on $|z - z_0| < s$, and f_1 is defined on $|z - z_0| > r$.

HW 4A (up to small changes) shows that they're analytic on these areas (note we're only plugging in ζ on the region where f is continuous). Although this depends on r, s, because of the uniqueness of the decomposiion in fact f_0, f_1 must not depend on our choice of r, s!

Question: Can we get independence of the decomposition with only Cauchy integral formula?

Express $f_1(z)$ as a power series in $\frac{1}{z - z_0}$, then

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k, |z - z_0| > \rho$$

Note $f_1(\infty)$ is the constant term, in this series which is 0. The series for $f_1(z)$ converges absolutely for any $r > \rho$, and it converges uniformly for $|z - z_0| \geq r$.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \lim_{K \to \infty} \left(\sum_{k=0}^{K} a_k (z - z_0)^k + \sum_{k=-K}^{-1} a_k (z - z_0)^k \right).$$

when $\rho < |z - z_0| < \sigma$. This converges absolutely and converges uniformly for $\rho < r \leq |z - z_0| \leq s < \sigma$.

How do we get coefficients? Namely what's a formula for a_k? Well let $\rho < r < s < \sigma$. We know

$$f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k$$

How do we extract a_n for $n \in \mathbb{Z}$, divide both sides by $(z-z_0)^{n+1}$ and integrate. Then

$$\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \int_{|z-z_0|=r} \frac{1}{(z-z_0)^{n+1}} \sum_{-\infty}^{\infty} a_k (z-z_0)^k.$$

We have uniform convergence, so we can swap integral and series to get

$$\sum_{-\infty}^{\infty} a_k \int_{|z-z_0|=r} (z-z_0)^{k-n-1}\,\mathrm{d}z = 2\pi i a_n,$$

because only the integral where $k-n-1=-1$ does not vanish! Therefore

$$a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}$$

Example VI.1.3

Let $f(z) = \frac{1}{z-z^2}$, which is defined for $z \neq 0, 1$.

Let $z_0 = 0$. We want to find the Laurent series decomposition. We do this using partial fractions, but in fact this depends on the annulus we consider!!! If we take $\rho = 0, \sigma = 1$, then we get

$$f(z) = \underbrace{\frac{1}{z}}_{f_1} + \underbrace{\frac{1}{1-z}}_{f_0} = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$$

$$=$$

which is valid for $|z| < 1$.

If we take $\rho = 1, \sigma = \infty$, then we get $f_0 = 0$ and $f_1 = \frac{1}{z-z^2}$. Then

$$f(z) = \frac{-1}{z^2} \cdot \frac{1}{1-\frac{1}{z}} = \sum_{k=0}^{\infty} -z^{-k-2}.$$

which converges for $|z| > 1$.

Theorem VI.1.2 (Laurent Series Expansion)

Suppose $0 \leq \rho < \sigma \leq \infty$ and suppose $f(z)$ is analytic on the annulus $\rho < |z-z_0| < \sigma$. Then $f(z)$ has a Laurent series expansion that converges absolutely at each point in the annulus and converges uniformly on each subannulus $r \leq |z-z_0| \leq s$, where $\rho < r < s < \sigma$.

The coefficients are uniquely determined by $f(z)$ and given as

$$a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

VI.2. Isolated Singularities

Definition VI.2.1

A point $z_0 \in \mathbb{C}$ is an isolated singularity of $f(z)$ if $f(z)$ is analytic in a punctured disk centered at z_0.

Example VI.2.1

$f(z) = 1/z$

Non-Example VI.2.2

$\text{Log}(z)$ does not have an isolated singularity at $z = 0$. There is no argument function on any neighborhood of 0.

Isolated singularities come in 3 types.

Definition VI.2.2

The isolated singularity of $f(z)$ at z_0 is said to be removable if a Laurent series about z_0 has $a_k = 0$ for all $k < 0$.

In this case, the Laurent series becomes an honest power series, and $f(z)$ can be continued analytically to z_0 with $f(z_0) = a_0$.

Example VI.2.3

$f(z) = \frac{\sin z}{z}$ has an isolated singularity at $z = 0$. This is a removable singularity since

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots .$$

Stuff:

- HW 8A due tonight
- HW 8B due Tuesday
- Math Club 4pm-5pm today
- Math Circle 6:30-8pm today
- Bagel Sunday/Popcorn Thursdays
- Super Saturdays 9:30am-12pm Saturday

Recall VI.2.4

A point $z_0 \in \mathbb{C}$ is an isolated singularity of $f(z)$ provided that $f(z)$ is holomorphic in a punctured disk $\{0 < |z - z_0| < r\}$.

Example VI.2.5

Let $f(z) = \frac{1}{z-53}$, this has an isolated singularity at $z_0 = 53$.

Non-Example VI.2.6

The complex logarithm $\text{Log}(z)$. One cannot define the logarithm on any neighborhood of 0.

There are three types of isolated singularities. Set up: Let $f(z)$ have an isolated singularity at z_0. Expand $f(z)$ in a Laurent series about z_0

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

which is valid for $0 < |z - z_0| < r$.

Definition VI.2.3

The three types of singularities are

(I) $a_k = 0$ for all $k < 0$, in which case we say z_0 is a removable singularity

(II) $a_k \neq 0$ for finitely many $k < 0$ (and at least one), in which case we say z_0 is a pole.

(III) $a_k \neq 0$ for infinitely many $k < 0$, in which case we call z_0 an essential singularity

Note: If z_0 is a removable singularity, then we can define $f(z_0) = a_0$, and this make f analytic in the whole disk $|z - z_0| < r$ since the Laurent series is just a power series.

Example VI.2.7

If $f(z) = \frac{\sin(z)}{z}$ then the Laurent series is

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots,$$

and the singularity at $z = 0$ is removable, so f can be extended to an entire function with $f(0) = 1$.

What can we say at $f(z)$ when z is close to a removable singularity z_0? The limit should exist!

- It should have $\lim_{z \to z_0} f(z)$ is some complex number.
- f can be extended continuously

Even better!

Theorem VI.2.1 (Riemann's Removable Singularity Theorem)

Let z_0 be an isolated singularity of $f(z)$. f is bounded near z_0 if and only if $f(z)$ has a removable singularity at z_0.

Proof. The converse is immediate from the above discussion. For the forward direction, expand $f(z)$ in a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

on $0 < |z - z_0| < \rho$. We know from before that

$$a_k = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{(z - z_0)^{k+1}} \, \mathrm{d}z$$

where $0 < r < \rho$. We want $a_k = 0$ for all $k < 0$. We know $f(z)$ is bounded near z_0 so there exists an M so that $|f(z)| \leq M$ for all $0 < |z - z_0| < \rho$ (possibly making ρ smaller).

Using the ML-estimate yields

$$|a_k| = \frac{2\pi r}{2\pi} \frac{M}{r^{k+1}} = \frac{M}{r^k}.$$

If $k < 0$ then this tends to 0 as $r \to 0$.

Definition VI.2.4

The isolated singularity of $f(z)$ at z_0 is called a pole of order N if there exists an $N > 0$ such that $a_{-N} \neq 0$ but $a_k = 0$ for all $k < -N$.

In this case

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - z_0)^k.$$

In this case, we collect the terms with negative powers of $(z - z_0)$:

$$P(z) \coloneqq P_f(z; z_0) \coloneqq \sum_{k=-N}^{-1} a_k(z - z_0)^k$$

which we call the principal part of f at z_0. This is a piece of the Laurent decomposition from last week.

Example VI.2.8

$f(z) = 1/z$ has a pole of order 1 at $z_0 = 0$, and

$$g(z) = \frac{1}{(z - 53)^2(z + 57)}$$

which has a pole of order 2 at $z_0 = 53$ and a pole of order 1 at $z_0 = -57$.

Theorem VI.2.2

Let z_0 be an isolated singularity of $f(z_0)$. Then z_0 is a pole of order N if and only if we may write

$$f(z) = \frac{g(z)}{(z - z_0)^N}$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

Proof. First for the forward direction. Give the Laurent expansion as

$$f(z) = \sum_{n=-N}^{\infty} a_n(z - z_0)^n = \frac{1}{(z - z_0)^N} \sum_{n=0}^{\infty} a_{n-N}(z - z_0)^n,$$

let $g(z)$ be the right hand power series, then $g(z_0) = a_{-N} \neq 0$, and we win!

For the other direction, just expand $g(z)$ as a power series about z_0 and then divide through to get a Laurent series for f.

Example VI.2.9

$f(z) = \frac{e^z}{(z-1)^5}$ has a pole at $z_0 = 1$ of order 5.

Theorem VI.2.3

Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order N if and only if $\frac{1}{f(z)}$ is analytic at z_0 with a zero of order N.

Proof. Use the previous theorem! [Gam03].

Example VI.2.10

Consider $f(z) = \frac{1}{\sin(z)}$. This has isolated singularities at all $z_0 = n\pi$, for $n \in \mathbb{N}$. What kind of singularities are they?

Well we can use the previous theorem! First check that $\frac{1}{\sin(z)}$ is unbounded near these points, so it's not removable (work over $\mathbb{R}$). The previous theorem tells us to look at $\sin(z)$, which has simple zeros (zeros of order 1) at each $z = n\pi$, so $\frac{1}{\sin(z)}$ has simple poles at $z = n\pi$.

Quick calculation:

$$\sin(z) = \sin(z - n\pi + n\pi) = \cos(n\pi)\sin(z - n\pi) = (-1)^n \left[(z - n\pi) - \frac{(z - n\pi)^3}{3!} \cdots \right].$$

Definition VI.2.5

We say that a function $f : D \to \mathbb{C}$ is meromorphic on a connected open set $D \subseteq \mathbb{C}$ provided that $f(z)$ is analytic on D except possibly at isolated singularities, each of which is a pole.

Sums/Products/Quotients (as long as denominator is not identically zero).

Theorem VI.2.4

Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Proof. Suppose z_0 is a pole of $f(z)$ of order N. Write $f(z) = \frac{g(z)}{(z-z_0)^N}$. Then

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \left| \frac{g(z)}{(z - z_0)^N} \right| = \infty$$

since $g(z_0) \neq 0$, and g is continuous.

For the other direction, suppose $|f(z)|$ goes to ∞ as $z \to z_0$. Then since f is not identically zero, we know $f(z)$ is nonzero in a punctured disk around z_0. Here set

$$h(z) := \frac{1}{f(z)},$$

which is analytic on this punctured disk and has $h(z) \to 0$ as $z \to z_0$. Then Riemann's theorem applies and $h(z_0) = 0$. If N is the order of the zero that h has at z_0, then

$$f(z) = \frac{1}{h(z)}$$

has a pole of order N at z_0.

Definition VI.2.6

The isolated singularity of $f(z)$ at z_0 is said to be essential provided that $a_k \neq 0$ for infinitely many $k < 0$.

Example VI.2.11

$f(z) = \exp(1/z)$ has an essential singularity at $z = 0$.

We now state some theorems to prove next time.

Theorem VI.2.5 (Casorati-Weierstrass, 1868)

Suppose z_0 is an isolated singularity of f. Then z_0 is an essential singularity if and only if for every complex number w, there exists a sequence $z_n \to z_0$ so that $f(z_n) \to w$.

On the suggestion of Sarah. Here is a discord link for everyone!

https://discord.gg/SFc3QmXMhm

We now move to proving the Casorati-Weierstrass Theorem from last time

Proof of Theorem VI.2.5. The converse is immediate from our characterization of poles/removable singularities.

We prove the forward direction via contrapositive. Suppose $w_0 \in \mathbb{C}$ is not the limit of such a sequence $f(z_n)$ (where $z_n \to z_0$). Then the image of f avoids some neighborhood of w_0 when z is near z_0.

In other words, there exists an $\varepsilon > 0$ such that $|f(z) - w_0| > \varepsilon$ for all z near z_0. We may define

$$h(z) = \frac{1}{f(z) - w_0}.$$

This is bounded near z_0 and analytic on a punctured disk around z_0, and so by Riemann's theorem, $h(z)$ can be extended analytically near z_0. We may then write

$$h(z) = (z - z_0)^N g(z)$$

for some $N \geq 0$, and some analytic $g(z)$ with $g(z_0) \neq 0$.

This immediately implies that

$$f(z) - w_0 = (z - z_0)^{-N} \cdot \frac{1}{g(z)},$$

with $\frac{1}{g(z)}$ analytic on a disk around $z = z_0$. If $N = 0$, then $f(z) = w_0 + \frac{1}{g(z)}$ and z_0 is a removable singularity of f. If $N > 0$, then $f(z)$ has a pole of order N at z_0.

In either case, z_0 is NOT an essential singularity of $f(z)$.

Stay tuned for the Great Picard Theorem, to be proved later!!!

Theorem VI.2.6 (Great Picard Theorem)

If an analytic function $f(z)$ has an essential singularity at z_0, then on any punctured neighborhood of z_0, $f(z)$ takes on all possible complex values with at most one exception!

Example VI.2.12

$\exp(1/z)$, with its essential singularity at 0. The only point not hit on a neighborhood of 0 is 0 itself (since the exponential is always nonzero).

Why is it only one point? There is some sort of intuition that Sarah has about $\mathbb{C}, \mathbb{C} \setminus \{a\}$ both being Euclidean, whereas $\mathbb{C} \setminus \{a, b\}$ (or more) is hyperbolic... hmmmmm

VI.3. Singularities at ∞

We want to define what it means for $f(z)$ to have isolated singularities at ∞. We analyze $f(1/w)$ at $w = 0$, and just look there. Compare with Gamelin discussion of analytic functions at ∞, namely [Gam03, V.5, p149]

VI.4. Partial Fractions decompositions

We say a function $f(z)$ is meromorphic on $D \subseteq \widehat{\mathbb{C}}$ provided that $f(z)$ is analytic on D except possibly at isolated singularities each of which is a pole.

Möbius transformations!!! We once made a claim that

$$\mathrm{Aut}(\widehat{\mathbb{C}}) = \text{Möb},$$

we proved $\supseteq$, but we have not shown $\subseteq$. Can we do it now?

Theorem VI.4.1

A meromorphic function on $\widehat{\mathbb{C}}$ must be a rational map.

Proof. See HW 10!

VII. Residue Calculus

This section will give you the ability to evaluate real integrals by shifting to the complex plane. A cautionary quote from Ahlfors:

"Even complete mastery does not guarantee success" ☹

VII.1. The Residue Theorem

Suppose $f(z)$ has an isolated singularity at z_0 and write $f(z)$ as a Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

which is valid for some annulus $0 < |z - z_0| < \rho$. Then we say that

$$a_n = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta$$

where $0 < r < \rho$. One of these is more special than the others! Namely

$$a_{-1} = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} f(\zeta) \, \mathrm{d}\zeta.$$

In fact, a_{-1} is special because it is an invariant of the one-form $f(\zeta)\,\mathrm{d}\zeta$. It does not change when we change coordinates!

Definition VII.1.1

We define the residue of $f(z)$ at z_0 to be the coefficient of a_{-1} of $\frac{1}{z - z_0}$ in the Laurent series. We define notation for the residue as

$$\operatorname{Res}[f(z), z_0] \coloneqq a_{-1}.$$

Example VII.1.1

We have

$$\operatorname{Res}\left[\frac{1}{z - 57}, 57\right] = 1$$
$$\operatorname{Res}\left[\frac{1}{(z - 53i)^2}, 53i\right] = 0$$
$$\operatorname{Res}\left[\frac{z^3 + z + 1}{z^2 + 1}, -i\right]?$$

Well, use partial fractions

$$f(z) = \frac{z^3 + z + 1}{z^2 + 1} = z - \frac{1}{2i} \cdot \frac{1}{z + i} + \frac{1}{2i} \cdot \frac{1}{z - i}.$$

Then

$$\operatorname{Res}\left[\frac{z^3+z+1}{z^2+1}, -i\right] = -\frac{1}{2i} = \frac{i}{2}.$$

Example VII.1.2

Let's look at

$$\begin{aligned} f(z) = \frac{\sin(z)}{z^6} &= \frac{1}{z^6}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{7!}z^7 + \cdots\right) \\ &= \frac{1}{z^5} - \frac{1}{6}\frac{1}{z^3} + \frac{1}{120}\cdot\frac{1}{z} - \frac{z^2}{7!} + \cdots \end{aligned}$$

this has a pole at $z = 0$ of order 5, and

$$\operatorname{Res}[f(z), 0] = \frac{1}{120}.$$

Theorem VII.1.1 (Residue Theorem)

Let $D \subseteq \mathbb{C}$ be bounded, open, and connected with ∂D being piecewise smooth. Now suppose that $f(z)$ is analytic on $D \cup \partial D$ except for a finite number of isolated singularities $z_1, z_2, \ldots, z_m$ in D. Then

$$\oint_{\partial D} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j].$$

Proof. Punch out tiny ε-disks about each singularity, and call the new region D_ε. By Cauchy's Theorem, we have that

$$\oint_{\partial D_\varepsilon} f(z)\,\mathrm{d}z = 0 <$$

since f is holomorphic here. But then

$$\oint_{\partial D_\varepsilon} f(z)\,\mathrm{d}z = \oint_{\partial D} f(z)\,\mathrm{d}z - \sum_{j=1}^{m} \oint_{|z-z_j|=\varepsilon} f(z)\,\mathrm{d}z.$$

The latter piece is equal to the residues as desired. The minus sign comes from an orientation flip.

Residue Rules/Recipes:

Rule 1: If $f(z)$ has a simple pole at z_0 then $\operatorname{Res}[f(z), z_0]$ then

$$\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} f(z)(z - z_0),$$

which we can derive from the Laurent expansion

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots.$$

Rule 2: If $f(z)$ has a double pole at z_0, then

$$\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{\mathrm{d}}{\mathrm{d}z}((z - z_0)^2 f(z)).$$

This can be seen since $(z-z_0)^2 f(z)$ has the form

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \cdots . \frac{\mathrm{d}}{\mathrm{d}z}((z-z_0)^2 f(z)) \quad = a_{-1} + 2a_0(z-z_0) + \cdots .$$

Rule 3: If $f(z)$ and $g(z)$ are analytic at z_0 and if $g(z)$ has a simple zero at z_0, then

$$\operatorname{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}.$$

Why? Well $f(z)/g(z)$ has at "worst" a simple pole at z_0, and then apply rule #1.

Rule 4: While this is just rule 3, Gamelin says it is so darn useful. If $g(z)$ is analytic and has a simple zero at z_0, then

$$\operatorname{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}.$$

Rule 1: We see that $\frac{1}{z^2+1}$ has a simple pole at i, so

$$\begin{aligned}\operatorname{Res}\left[\frac{1}{z^2+1}, i\right] &= \lim_{z\to i}(z-i)\frac{1}{z^2+1} \\ &= \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}.\end{aligned}$$

Rule 2: We see $\frac{1}{(z^3+1)z^2}$ has a double pole at 0, so

$$\begin{aligned}\operatorname{Res}\left[\frac{1}{(z^3+1)z^2}, 0\right] &= \lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{z^3+1} \\ &= \lim_{z\to 0}(z^3+1)^{-2}(-3z^2) = 0.\end{aligned}$$

Rule 3: Since $\sin(z)$ has a simple zeto at $z=\pi$ we see

$$\operatorname{Res}\left[\frac{e^z}{\sin z}, \pi\right] = \frac{e^\pi}{\cos\pi} = -e^\pi$$

Exercise VII.1.3

Compute the residues of $f(z) = \frac{1}{z^n+1}$ at its poles. Recall that it has poles at the $2n$-th roots of unity which are not also n-th roots of unity since $z^{2n}-1 = (z^n+1)(z^n-1)$.

VII.2. Integrals of Rational Functions

We want to compute something like $\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+x^2}$. Consider $f(z) = \frac{1}{1+z^2}$. Consider a contour ∂D_R consisting of a semi-circle Γ_R from R to $-R$ of radius R about 0 and a line segment $[-R, R]$ (with the counterclockwise orientation). This encloses a region D_R which contains i if $R > 1$, so

$$\begin{aligned}\oint_{\partial D_R} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}[f(z), i] &= 2\pi i \lim_{z\to i}\frac{z-i}{z^2+1} \\ &= 2\pi i \lim_{z\to i}\frac{1}{z+i} = \pi.\end{aligned}$$

We know that

$$\pi = \oint_{\partial D_R} f(z)\,\mathrm{d}z = \int_{-R}^{R}\frac{\mathrm{d}x}{1+x^2} + \int_{\Gamma_R}\frac{\mathrm{d}z}{1+z^2}.$$

We claim that as $R \to \infty$ that $\int_{\Gamma_R} \frac{\mathrm{d}z}{1+z^2} \to 0$. This comes from the ML-estimate, if $z \in \Gamma_R$ then for $R > 1$,

$$\left| \frac{1}{1+z^2} \right| \leq \frac{1}{R^2 - 1}.$$

Then

$$\left| \int_{\Gamma_R} \frac{\mathrm{d}z}{1+z^2} \right| \leq \frac{1}{R^2-1} 2\pi R,$$

which goes to 0 as $R \to \infty$. This tells us that

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^2} = \pi.$$

Stuff:

- Math Club Today!
- Math S^1 tonight 6:30pm-8pm!
- Popcorn 4:30pm
- Bagel Sunday at 11:30am
- Free voting t-shirts
- Super Saturdays!
- Extra Halloween Shirts / Free voting shirts
- Student seminar Friday 4pm EH3096 "Combinatorial reciprocity via Möbius functions."
- Undergrad student advisory council 1-2pm atrium.

Last time: Residue theorem! Evaluating real integrals!

The same techniques from last time can evaluate integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, \mathrm{d}x,$$

where P, Q are real polynomials where Q has no real zeros and $\deg Q \geq \deg P + 2$. In this case we'll have

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, \mathrm{d}x = 2\pi i \sum \operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_j \right],$$

where each z_j is a zero of Q within the upper half-plane. This method can be used to evaluate other integrals too! Consider

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos ax \, \mathrm{d}x,$$

where $a > 0$ and p, q are polynomials of the form mentioned above. We would like to complexify. The simplest candidate is

$$\frac{p(z)}{q(z)} \cos az = \frac{p(z)}{q(z)} \cdot \frac{e^{iaz} + e^{-iaz}}{2}.$$

But $\cos az$ is unbounded in the upper half plane... this causes problems for us. Instead we'll use e^{iz} and apply real and imaginary parts at the end of the calculation. In particular, we'll look at $f(z) = \frac{p(z)}{q(z)} e^{iaz}$.

Example VII.2.1

Show $\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,\mathrm{d}x = \pi e^{-a}$ when $a > 0$. We can look at $f(z) = \frac{e^{iaz}}{1+z^2}$. We'll look at our favorite contour ∂D given by the semi-circle Γ_R and the interval $[-R, R]$.

We see by the residue theorem and our Rule 3 that

$$\begin{aligned}\int_{\partial D} \frac{e^{iaz}}{1+z^2}\,\mathrm{d}z &= 2\pi i \operatorname{Res}[f(z), i] = 2\pi i \frac{e^{iaz}}{2z}\Big|_{z=i}\\ &= \frac{2\pi i e^{-a}}{2i} = \pi e^{-a}.\end{aligned}$$

Now we see via the ML-estimate that since $\left|e^{iaz}\right| \leq 1$ in the upper half plane (since $a > 0$) that

$$\int_{\Gamma_R} \frac{e^{iaz}}{1+z^2}\,\mathrm{d}z \leq \pi R \cdot \frac{1}{R^2-1},$$

which goes to 0 as $R \to \infty$. Thus

$$\lim_{R\to\infty} \int_{\partial D} \frac{e^{iaz}}{1+z^2}\,\mathrm{d}z = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iaz}}{1+z^2}\,\mathrm{d}z.$$

Applying what we've already done, this yields

$$\pi e^{-a} = \int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,\mathrm{d}x,$$

which upon taking Real parts of both sides yields the desired result.

VII.3. **Integrals of Trig Functions**

Previous plan: Start with real integral + complexify, integrate over a "good" contour, take limit and we're happy.

Now we have things of the form

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 4\sin\theta}.$$

How can we use complex analysis to integrate this. Well we use the common substitution $z = e^{i\theta}$. Then $\mathrm{d}z = iz\,\mathrm{d}\theta$. Furthermore, we have some nice identities for $|z| = 1$, namely $\overline{z} = 1/z = e^{-i\theta}$ and

$$\begin{aligned}\cos\theta &= \frac{1}{2}\left(z + \frac{1}{z}\right)\\ \sin\theta &= \frac{1}{2i}\left(z - \frac{1}{z}\right).\end{aligned}$$

Example VII.3.1

Let's actually compute $\int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\sin\theta}$. Armed with the identities, we have

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\sin\theta} &= \int_{|z|=1} \frac{1}{5 - 2i(z - 1/z)} \frac{\mathrm{d}z}{iz}\\ &= \int_{|z|=1} \frac{\mathrm{d}z}{2z^2 - 2 + 5iz}.\end{aligned}$$

We factor the denominator (or use the quadratic formula) to get

$$2z^2 + 5iz - 2 = (2z + i)(z + 2i).$$

Thus there are simple zeros at $-i/2, -2i$. We then see that

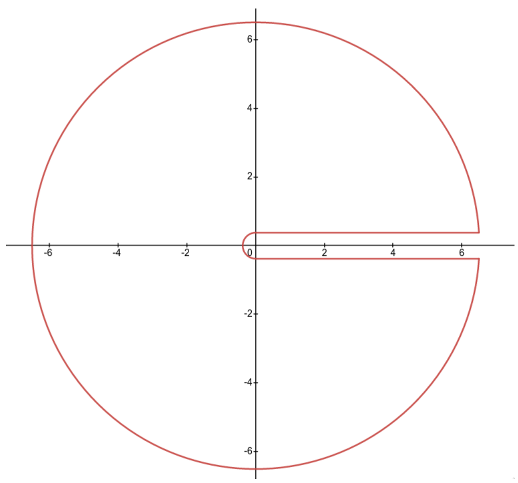
$$\int_{|z|=1} \frac{\mathrm{d}z}{2z^2 - 2 + 5iz} = 2\pi i \operatorname{Res}[f(z), -i/2] = \frac{2\pi}{3}.$$

VII.4. **Integrands with Branch Cuts**

We would like to evaluate things like $\int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x$ where $0 < |a| < 1$ is real. We recall that

$$z^a = \exp(a \log z)$$

is not well-defined in the complex plane. Thus instead we have to take branch cuts... Take a branch of the log where the argument ranges from 0 to 2π. The contour we're going to use is something called a keyhole contour of the following shape



The inner semi-circle will be called γ_ε (it has radius ε), and the outer arc of a circle $\Gamma_{R,\varepsilon}$ and will have radius $R_\heartsuit := \sqrt{R^2 + \varepsilon^2}$. The lines are from $i\varepsilon$ to $R + i\varepsilon$ and $-i\varepsilon$ to $R - i\varepsilon$. We'll call the top one $L^+(\varepsilon, R)$, and the one on the bottom $L^-(\varepsilon, R)$ (in opposite orientation). We'll also take $\varepsilon < 1/2$ and $R > 2$ so that the singularity at -1 is included. We'll call the region bounded by these $D(\varepsilon, R)$.

We'll show that

$$\int_0^\infty \frac{x^a}{1 + x^2}\,\mathrm{d}x = \frac{\pi a}{\sin(\pi a)}.$$

We'll use the function $f(z) = \frac{z^a}{(1+z)^2}$. f has a double pole at $z = -1$, and we have

$$\operatorname{Res}[f(z), -1] = \frac{\mathrm{d}}{\mathrm{d}z}(1 + z)^2 \frac{z^a}{(1 + z)^2}\Big|_{z=-1} = -ae^{\pi i a}.$$

The residue theorem then gives

$$\int_{\partial D(\varepsilon,R)} f(z)\,\mathrm{d}z = 2\pi i(-ae^{\pi i a}).$$

This breaks into four pieces

$$\int_{\partial D(\varepsilon,R)} = \int_{L^+(\varepsilon,R)} + \int_{\Gamma_{R,\varepsilon}} + \int_{L^-(\varepsilon,R)} + \int_{\gamma_\varepsilon}.$$

If $|z| = R_\heartsuit$ then

$$|f(z)| = \left|\frac{z^a}{(1+z)^2}\right| \leq \frac{R_\heartsuit^a}{(R_\heartsuit - 1)^2}$$

abd if $|z| = \varepsilon$ then

$$|f(z)| = \left|\frac{z^a}{(1+z)^2}\right| \leq \frac{\varepsilon^a}{(1-\varepsilon)^2}.$$

We then have that

$$\left|\int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z\right| \leq \frac{R_\heartsuit^a}{(R_\heartsuit - 1)^2} 2\pi R_\heartsuit,$$

which goes to 0 as $R \to \infty$ since $R_\heartsuit \geq R$ and $a \in (-1, 1)$. Also

$$\left|\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z\right| \leq \frac{\varepsilon^a}{(1-\varepsilon)^2},$$

which goes to 0 as $\varepsilon \to 0$.

If $z \in L^+(\varepsilon, R)$ then as $\varepsilon \to 0$ the argument is close to 0, and if $z \in L^-(\varepsilon, R)$ then as $\varepsilon \to 0$ the argument is close to 2π. In the limit, as $\varepsilon \to 0$ and $R \to \infty$ (which we'll denote with L^+),

$$\begin{aligned}\int_{L^+} f(z)\,\mathrm{d}z &= \int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x \\ -\int_{L^-} f(z)\,\mathrm{d}z &= \int_0^\infty \frac{x^a e^{2\pi i a}}{(1+x)^2}\,\mathrm{d}x.\end{aligned}$$

Writing ∂D for the limit as $\varepsilon \to 0, R \to \infty$ we have

$$\begin{aligned}-2\pi i a e^{\pi i a}) = \int_{\partial D} f(z)\,\mathrm{d}z &= \int_{L^+} f(z)\,\mathrm{d}z + \int_{L^-} f(z)\,\mathrm{d}z \\ &= \int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x - \int_0^\infty \frac{x^a e^{2\pi i a}}{(1+x)^2}\,\mathrm{d}x \\ &= (1 - e^{2\pi i a}) \int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x.\end{aligned}$$

Putting this all together gives

$$\int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x = -\frac{2\pi i a e^{\pi i a}}{1 - e^{2\pi i a}}.$$

This simplifies down to $\frac{\pi a}{\sin \pi a}$.

VII.5. **Fractional Residues**

Idea: What can we do when our path of integration crashes into a singularity?

If $f(z)$ has a simple pole at z_0, then we can do something cool!

Theorem VII.5.1 (Fractional Residue Theorem)

If z_0 is a simple pole of $f(z)$, and C_ε is an arc of the circle $\{|z - z_0| = \varepsilon\}$ of angle α, then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,\mathrm{d}z = \alpha i \operatorname{Res}[f(z), z_0].$$

Here the integration is taken with the orientation where the singularity is on the left (as usual). This nearly generalizes the full Residue Theorem when z_0 is simple.

Clarification: we're integrating from angle 0 to angle α around the circle. If α is like 4π then we're integrating over the full circle twice.

Proof. Write $f(z) = \frac{A}{z-z_0} + g(z)$ where g is analytic at z_0 and $A = \operatorname{Res}[f(z), z_0]$. Parameterize C_ε as $z = z_0 + \varepsilon e^{i\theta}$ where $\theta_0 < \theta < \theta_0 + \alpha$. Then

$$\int_{C_\varepsilon} \frac{A\,\mathrm{d}z}{z - z_0} = iA \int_{\theta_0}^{\theta_0+\alpha} \mathrm{d}\theta = \alpha i A.$$

Furthermore as $\varepsilon \to 0$ we see $\int_{C_\varepsilon} g(z)\,\mathrm{d}z \to 0$ since $g(z)$ is bounded near z_0 and the length of C_ε is $\alpha\varepsilon$. Combining these two results yields the theorem.

Stuff:

- Handout: Ahlfor's Guide to Contours
- HW 9B due tonight!
- HW 10A + 10B due Thursday and next Tuesday.

Last time: Evaluating real integrals using the residue theorem, and the fractional residue theorem.

Example VII.5.1 (Example of Fractional Residue Theorem)

Look at

$$\int_0^\infty \frac{\log x}{x^2 - 1}\,\mathrm{d}x.$$

Consider $f(z) = \frac{\log z}{z^2-1}$. We'll integrate along a semicircular contour of radius R with indents at -1 and 0. This will split into *six* integrals, and we'll use a branch cut of log with argument from $-\pi/2$ to $3\pi/2$.

We'll call the non-linear parts Γ_R,C_ε (indent at -1),γ_δ (indent at 0). We know

$$\int_{\partial D(R,\varepsilon)} f(z)\,\mathrm{d}z = 0.$$

Furthermore $\int_{\Gamma_R} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$. The C_δ^0 piece will also go to 0. By the fractional residue theorem

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,\mathrm{d}z = -\pi i \operatorname{Res}[f(z), -1] = -\pi i \frac{i\pi}{2(-1)} = -\frac{\pi^2}{2}.$$

Take real and imaginary parts and let $\varepsilon \to 0$ to get

$$\int_0^\infty \frac{\log x}{x^2 - 1} + \int_{-\infty}^0 \frac{\log |x|}{x^2 - 1}\,\mathrm{d}x - \frac{\pi^2}{2} = 0.$$

Therefore we get

$$\int_0^\infty \frac{\log x}{x^2-1}\,\mathrm{d}x = \frac{\pi^2}{4}$$

VII.6. Principal Values

Definition VII.6.1

Suppose $f(x)$ is continuous for $a \le x < x_0$ and $x_0 < x \le b$. We define the principal value of

$$\int_a^b f(x)\,\mathrm{d}x$$

to be

$$\mathrm{PV}\int_a^b f(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\left(\int_a^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^b f(x)\,\mathrm{d}x\right),$$

provided that this limit exists.

VII.7. Jordan's Lemma

Recall VII.7.1

We know all about

$$\int_{-\infty}^\infty \frac{P(x)}{Q(x)}\,\mathrm{d}x,$$

well, we needed zeros of Q not on $\mathbb{R}$ and $\deg Q \ge 2 + \deg P$. Jordan's Lemma will allow us to change this to $\deg Q \ge 1 + \deg P$, by circumventing the ML-estimate portion of this proof.

Lemma VII.7.1 (Jordan)

If Γ_R is the semicircular contour $z = Re^{i\theta}$ for $0 \le \theta \le \pi$, then

$$\int_{\Gamma_R} \left|e^{iz}\right| \cdot |\mathrm{d}z| < \pi.$$

Proof. Rewrite this as $z = Re^{i\theta}$, $\mathrm{d}z = zi\,\mathrm{d}\theta$, so thaen $|\mathrm{d}z| = R\,\mathrm{d}\theta$. The lemma boils down to

$$\int_0^\pi e^{-R\sin\theta}\,\mathrm{d}\theta < \frac{\pi}{R},$$

Notice that $y = \frac{2}{\pi}\theta$ and $y = \sin\theta$ both go through the points $(0,0)$ and $(\pi/2, 1)$, but $\sin\theta$ is above this line for all $\theta \in [0, \pi/2]$. Therefore

$$\begin{aligned}\int_0^\pi e^{-R\sin\theta}\,\mathrm{d}\theta &= 2\int_0^{\pi/2} e^{-R\sin\theta}\,\mathrm{d}\theta \\ &\le 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,\mathrm{d}\theta \\ &= \frac{2(-1)\pi}{2R} e^{-2\pi R\theta/\pi}\Big]_0^{\pi/2} = \frac{-\pi}{R}\cdot\left(\frac{1}{e^R} - 1\right)\end{aligned}$$

In other words, this is $\frac{\pi}{R}$ − positive, which is less than $\frac{\pi}{R}$.

VII.8. **Exterior Domains**

Question: Can we defined the residue of f at ∞ on $\widehat{\mathbb{C}}$.

Definition VII.8.1

Let $D \subseteq \widehat{\mathbb{C}}$ be open, connected, and suppose it contains a neighborhood of ∞. That is there exists an $R > 0$ such that D contains $|z| > R$. Then D is called an exterior domain

Theorem VII.8.1

Let $D \subseteq \mathbb{C}$ be an exterior domain with piecewise smooth boundary. Suppose $f(z)$ is analytic on $D \cup \partial D$ except for a finite # of isolated singularities $z_1, \dots, z_m \in D$. Let a_{-1} be the coefficient of $\frac{1}{z}$ in the series of f at ∞. Then

$$\oint_{\partial D} f(z)\,\mathrm{d}z = -2\pi i a_{-1} + 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j]$$

Definition VII.8.2

We define residue of $f(z)$ at ∞ to be $\operatorname{Res}[f(z), \infty] = -a_{-1}$. Why the negative!!!

Proof. Apply the standard residue theorem to the new region $D_R = D \setminus (\text{disk } |z| > P)$, and follow nose and see Gamelin. Cool HW problem on 10B.

VII.9. **Logarithmic Integral**

Gamelin: Let $D \subseteq \mathbb{C}$ be bounded, open, connected subset. Suppose $f(z)$ is meromorphic on D that extends to be analytic on ∂D such that $f(z) \neq 0$ for all $z \in \partial D$.

Then: f has finitely many zeros in D. Question: How many zeros in D, and how many poles in D?

Suppose $f(z)$ has a zero of order $N > 0$, then $f(z) = (z - z_0)^N g(z)$ at z_0. Then

$$f'(z) = (z - z_0)^N g'(z) + g(z) N (z - z_0)^{N-1}$$

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{(z - z_0)^{N-1} N \cdot g(z)}{(z - z_0)^N g(z)} = \frac{g'(z)}{g(z)} + \frac{N}{z - z_0}.$$

We call $\frac{f'(z)}{f(z)}$ the logarithmic derivative of f, since if $\log f(z)$ is defined when we take the derivative we get this.

But wait! We have

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)}, z_0\right] = N,$$

this is amazing!!! In fact we have the same thing for if we started with any $N \in \mathbb{Z}$! This can also detect poles!

Theorem VII.9.1

Let $f(z)$ be as above. Then we have that

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)}\,\mathrm{d}z = (\#\text{ of zeros of } f \text{ in } D) - (\#\text{ of poles of } f \text{ in } D),$$

where we count with multiplicity.

What is f'/f!!! See Curt McMullen notes rs.pdf and search "good cocycle"

Now we go to the Argument Principle.

Definition VII.9.1

If $f(z)$ is analytic on $D \subseteq \mathbb{C}$, then for a closed curve $\gamma \subseteq D$ such that $f(z) \neq 0$ for all $z \in \gamma$, we call

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\,\mathrm{d}(\log f(z)) = \frac{1}{2\pi i}\int_\gamma d\log|f(z)| + \frac{1}{2\pi}\int_\gamma \mathrm{d}\arg f(z).$$

Thus the left hand side is 0 because the differential is exact...

But the argument is something different!!!

Theorem VII.9.2 (Argument Principle)

Suppose $D \subseteq \mathbb{C}$ is bounded, connected, open, and with piecewise boundary and let $f(z)$ be meromorphic on D that extends to be analytic on ∂D. Suppose further that $f(z) \neq 0$ for all $z \in \partial D$. Then the increase in the argument of $f(z)$ around the boundary of D is

$$2\pi\left[(\#\text{ zeros of } f \text{ in } D - \#\text{ of poles}\right].$$

Example VII.9.1

Let $f(z) = z^N$ for $N \in \mathbb{N}$¡ we know the number of zeros of in D minus the number of poles in D is N. So as we traverse the boundary of any D containing 0, then the argument increases by $2\pi N$.

Application: Show that $p(z) = z^4 + 2z^2 - z + 1$ has exactly one zero in the first quadrant. Solution: Apply the arugment principle. There are no poles!

Go around the quadrant in three pieces. If $z \in [0, R]$, then one can check that $p(z) > 0$ there, so there are no zeros. For R large enough there are no zeros when $|z| = R$, $0 \leq \arg z \leq \frac{\pi}{2}$, and $p(z) \approx z^4$. Thus as z goes along this quater circle the change in the argument is 2π.

Stuff:

- Math Club Today 4-5pm, random graphs social networks and the internet.
- Math S^1 6:30-8pm
- Super Saturdays!
- Popcorn Thursday!
- Career Fair tomorrow!
- Mass undergrad peer advising 7-9:30pm atrium Monday!

Last time, we had a lemma

Lemma VII.9.3

Suppose $D \subseteq \mathbb{C}$ is an open connected set, and $f : D \to f(D) \subseteq \mathbb{C}$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in D$.

Thus the inverse is holomorphic on $f(D)$!

Proof. Compare this with the proof that nonconstant holomorphic maps are open. Suppose $f'(z_0) = 0$ for some $z_0 \in D$. Then near z_0 we have

$$f(z) - f(z_0) = (z - z_0)^n g(z)$$

where $n \geq 2$, $g(z)$ is holomorphic at z_0, and $g(z_0) \neq 0$. Then we can find some analytic $h(z)$ near z_0 such that $g(z) = (h(z))^n$. Then

$$f(z) - f(z_0) = ((z - z_0)h(z))^n.$$

We know $f(z) - f(z_0)$ will map a small open set around z_0 to a small open set about 0 injectively.

We know $((z - z_0) \cdot h(z))^n$ is not injective because $(z - z_0)h(z)$ maps to a small open set about 0 and $n \geq 2$. Thus these can't be equal!

Last time! Logarithmic Integrals! We were interested in the Argument Principle, Theorem VII.9.2. We restate it fully here for convenience

Theorem VII.9.4

Suppose $D \subseteq \mathbb{C}$ is bounded, connected, open, and with piecewise smooth boundary and let $f(z)$ be meromorphic on D that extends to be analytic on ∂D.

Suppose further that $f(z) \neq 0$ for all $z \in \partial D$. Let N_0 be the number of zeros of f in D, N_∞ be the number of poles in D counted with multiplicity. Then

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'}{f} \, \mathrm{d}z = N_0 - N_\infty.$$

We got the Argument Principle, which is that the increase in the argument of $f(z)$ around the boundary of D is

$$\int_{\partial D} \mathrm{d} \arg(f(z)) = 2\pi(N_0 - N_\infty).$$

VII.10. Rouché's Theorem

Theorem VII.10.1

Let $D \subseteq \mathbb{C}$ be connected, open, and bounded with piecewise smooth boundary. Let $f(z)$ and $h(z)$ be anlytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for all $z \in \partial D$ then $f(z)$ and $f(z) + h(z)$ have the same number of zeros in D counting with multiplicity.

Example VII.10.1

An example from Kurt McMullin. Question: Where in $\mathbb{C}$ are the zeros of $p(z) = z^5 + 14z + 1$? Let $f(z) = z^5, h(z) = 14z + 1$. We need to find a region D where for $z \in \partial D$ we have

$$|h(z)| < |f(z)|,$$

Lets try D as the ball of radius 2. Then when $|z| = 2$ we have

$$|f(z)| = |2|^5 = 32 > 29 \geq |14z + 1| = |h(z)|.$$

Now Rouché tells us that $f(z)$ and $p(z) = f(z) + h(z)$ have the same number of zeros in D. Since $f(z) = z^5$ has five zeros in D, this shows $p(z)$ attains all of its zeros in D.

Can we make the answer more precise? Now consider $|z| = 3/2$, and break up $p(z)$ as $h(z) = z^5 + 1, f(z) = 14z$. Then we have

$$|h(z)| \leq \left(\frac{3}{2}\right)^5 + 1 < 9 < |14z|$$

when $|z| = 3/2$. Then since $f(z)$ has one zero inside the disk of radius $3/2$, so does $p(z)$.

Thus $p(z)$ has one zero in $|z| < 3/2$ and 4 zeros in the annulus $3/2 < |z| < 2$.

Proof. We know since $|h(z)| < |f(z)|$ for all $z \in \partial D$, then this implies $f(z) \neq 0$ and $f(z) + h(z) \neq 0$ (reverse triangle inequality) for all $z \in \partial D$.

This sets us up to consider their arguments! We can rewrite

$$f(z) + h(z) = f(z)\left[1 + \frac{h(z)}{f(z)}\right].$$

We then know that

$$\arg(f(z) + h(z)) = \arg(f(z)) + \arg\left(1 + \frac{h(z)}{f(z)}\right).$$

Since $\frac{|h(z)|}{|f(z)|} < 1$ on ∂D. Then the values $w = 1 + \frac{h(z)}{f(z)}$ lie in a disk of radius 1 about 1, so $\mathrm{Re}(w) > 0$.

We can then use the argument principle. How does the argument of $w = 1 + \frac{h(z)}{f(z)}$ change as z moves around in closed loops? It can't!!! The outputs lie in the right half-plane so $\mathrm{d}\arg w$ is exact! In other words we can't wrap around 0 to pick up a change in argument.

Thus we have

$$\oint_{\partial D} \mathrm{d}\arg(f(z) + h(z)) = \oint_{\partial D} \mathrm{d}\arg f(z) + \mathrm{d}\arg(w) = \oint_{\partial D} \mathrm{d}\arg f(z).$$

The result then follows from the argument principle. ∎

It is clear from the proof that we can extend to f, h meromorphic on D, analytic on $D \cup \partial D$, and then the number of poles/zeros in D.

Corollary VII.10.2

The Fundamental Theorem of Algebra. Find a large enough disk so that the leading term dominates, just as in the example.

VII.11. Hurwitz's Theorem

Recall HW7 #3, which will show up on HW 11. We talked about $n \mapsto (f_n : D \to \mathbb{C})$ a sequence of functions converging to $f : D \to \mathbb{C}$. What can we say about how the zeros of f_n compare to zeros of f?

Theorem VII.11.1 (Hurwitz's Theorem)

Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a connected open set D. Suppose $\{f_k(z)\}$ converges normally (on compact subsets/locally uniformly) to $f : D \to \mathbb{C}$. Further f has a zero of order N at z_0.

Then there exists a small $\rho > 0$ such that for k large, $f_k(z)$ has exactly N zeros on $\{|z - z_0| < \rho\}$, counting with multiplicity. And these zeros converge to z_0 as $k \to \infty$.

Proof. The hypothesis implies that f is not identically zero. So take $\rho > 0$ so that $\{|z - z_0| \leq \rho\} \subseteq D$ and $f(z) \neq 0$ for all z on the punctured disk $\{0 < |z - z_0| \leq \rho\}$.

Now choose $\delta > 0$ so that $|f(z)| \geq \delta$ for all z on the boundary circle $|z - z_0| = \rho$. Since $\{f_k\}$ converges uniformly to f on our closed sets, we know there exists an M so that for all $k \geq M$ we have $|f_k(z)| > \frac{\delta}{2}$ for all z on $|z - z_0| = \rho$.

Furthermore, the sequence of functions $\frac{(f_k)'(z)}{f_k(z)}$ converges uniformly to $\frac{f'(z)}{f(z)}$ on the boundary circle $|z - z_0| = \rho$... so... apply the logarithmic integrals!

$$\lim_{k \to \infty} \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \frac{(f_k)'(z)}{f_k(z)} \, \mathrm{d}z = \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \frac{f'(z)}{f(z)} \, \mathrm{d}z.$$

The left hand side counts the number of zeros of f_k inside $|z - z_0| < \rho$, which we'll call N_k. The right hand side is equal to N, since f is nonzero on $0 < |z - z_0| < \rho$, and has a zero of order N at z_0.

Since these are integers, they are discrete, so for large enough k, we have $N_k = N$! This is exactly eh first part of the result. What about the second? Play the same game with a smaller ρ, shrinking ρ to zero and running the argument again.

Definition VII.11.1

We say f is univalent on a domain $D \subseteq \mathbb{C}$ provided that it is analytic and injective on D.

Theorem VII.11.2 (Another version of Hurwitz)

Suppose $\{f_k(z)\}$ is a sequence of univalent functions on a connected open $D \subseteq \mathbb{C}$ that converge normally to $f : D \to \mathbb{C}$. Then $f(z)$ is either univalent OR $f(z)$ is constant.

Example VII.11.1

Consider $f_k(z) = \frac{z}{k}$ converging to the zero function.

Proof. See Gamelin.

VII.12. **Winding Numbers**

Definition VII.12.1 (Winding Number)

Let γ be a piecewise smooth path in $\mathbb{C}$. For $z_0 \notin \gamma$ define the winding number as

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}. = \frac{1}{2\pi} \int_\gamma \mathrm{d} \arg(z - z_0).$$

Note: $W(\gamma, z_0)$ depends analytically on z_0. For $\mathbb{C} \setminus \{\gamma\} \to \mathbb{Z}$ given by $z_0 \mapsto W(\gamma, z_0)$. Thus $W(\gamma, z_0)$ is constant on connected components of $\mathbb{C} \setminus \{\gamma\}$.

Proposition VII.12.1

Gamelin p254, then let $D \subseteq \mathbb{C}$ be open, connected. Then the following are equivalent

(1) D is simply connected.
(2) Every closed differential form is exact on D.
(3) For each $z_0 \in \mathbb{C} \setminus D$, there exists an anlytic branch of $\mathrm{Log}(z - z_0)$ defined on D.
(4) Each closed curve $\gamma \in D$ has winding number $W(\gamma, z_0)$ for $z_0 \in \mathbb{C} \setminus D$.
(5) The complement of D in $\widehat{\mathbb{C}}$ is connected.

(5) is easiest to check in practice. Proof is in the book.

VIII. Schwarz Lemma and Hyperbolic Geometry

The Schwarz Lemma is central to the theory of analytic maps between Riemann surfaces. We'll state it as a theorem because it is so important.

Theorem VIII.0.1 (Schwarz Lemma)

Let $f(z)$ be analytic on $|z| < 1$. Suppose $|f(z)| \leq 1$ for all $|z| < 1$. Suppose further that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $|z| < 1$.

Furthermore, if $|f(z_0)| = |z_0|$ for some point $z_0 \neq 0$, then $f(z) = \lambda z$ for some $\lambda = e^{i\theta}$.

Stuff:

- John W. Milnor's mathematical writing is excellent. Good book: "Introduction to dynamics in one Complex variable.

Proof of Schwarz's Lemma. Write $f(z) = z \cdot g(z)$ $g(z)$ analytic on $|z| < 1$. Let $r < 1$. Then if $|z| = r$ we have

$$|g(z)| = \frac{|f(z)|}{r},$$

By the maximum principle, $|g(z)| \leq \frac{1}{r}$ when $|z| \leq r$. Letting $r \to 1$ yields $|g(z)| \leq 1$ for all $z \in \mathbb{C}$.

This gives $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Now suppose $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. Then necessarily

$$|f(z_0)| = |z_0| \cdot |g(z_0)| \implies |g(z_0)| = 1,$$

since $z_0 \neq 1$. Thus by the strict maximum principle, we have that g is constant. Say $g(z) = \lambda$ for $\lambda \in \mathbb{C}$. We then know that $|g(z_0)| = |\lambda| = 1$. Thus $f(z) = \lambda \cdot z$.

VIII.1. Conformal self-maps of $\mathbb{D}$

For example. Consider any rotation!

Lemma VIII.1.1

If $g(z)$ is an automorphism of $\mathbb{D}$ with $g(0) = 0$, then $g(z)$ is a rotation.

Proof. Apply the Schwarz lemma twice, once to the function and once to its inverse. $|g(z)| \leq |z|$. Further $\left|g^{-1}(w)\right| \leq |w|$. Plugging in $w = g(z_0)$ for any $z_0 \neq 0$ we have

$$|z_0| \leq \left|g^{-1}(g(z_0))\right| \leq |g(z_0)| \leq |z_0|.$$

Thus we have equality, which tells us that $g(z)$ is a rotation via the Schwarz lemma.

Application:

Theorem VIII.1.2

$\mathrm{Aut}(\mathbb{D})$ is precisely the Möbius transformations of the form

$$f(z) = e^{i\theta}\left(\frac{z-a}{1-\bar{a}z}\right)$$

where $a \in \mathbb{D}$ and $\theta \in [0, 2\pi)$.

Question: Do these Möbius transformations belong to $\mathrm{Aut}(\mathbb{D})$? Consider

$$g(z) = \frac{z-a}{1-\overline{a}z}.$$

Well g maps circles to circles. So g maps unit circle to a unit circle. Fix some $e^{i\alpha} \in S^1$. Then

$$\begin{aligned} \left|e^{i\alpha} - a\right| &= \left|e^{-i\alpha} - \overline{a}\right| = \left|1 - e^{i\alpha}\overline{a}\right| \\ \left|g(e^{i\alpha})\right| &= 1. \end{aligned}$$

Thus g maps the unit circle to itself! Since $g(a) = 0$, g maps $\mathbb{D} \to \mathbb{D}$.

Proof of Theorem. Show if $h(z) \in \mathrm{Aut}(\mathbb{D})$ then $h(z)$ has the desired form.Set $a = h^{-1}(0)$. Then consider the map

$$g(z) = \frac{z-a}{1-\overline{a}z}.$$

We can look at $h \circ g^{-1} \in \mathrm{Aut}(\mathbb{D})$. Then we have

$$h(g^{-1}(0)) = h(a) = 0.$$

Thus $(h \circ g^{-1})(z) = e^{i\theta} z$ by the previous lemma and $h = h \circ g^{-1} \circ g$ so

$$h(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z}.$$

Theorem VIII.1.3 (Pick's Lemma)
If $f(z)$ is analytic and satisfies $|f(z)| < 1$ for $|z| < 1$. Then in fact

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

for $|z| < 1$. If $f(z)$ is conformal, then equality holds for all $z \in \mathbb{D}$. Otherwise, this inequality is strict for all $z \in \mathbb{D}$.

Proof. Want to use Schwarz Lemma, but we don't have $0 \mapsto 0$. The idea is to use clever composition. Let $f(z_0) = w_0$. Take $g, h \in \mathrm{Aut}(\mathbb{D})$ so that $g(0) = z_0, h(w_0) = 0$ (the previous work has shown $\mathrm{Aut}(\mathbb{D})$'s action on $\mathbb{D}$ is transitive).

Then $h \circ f \circ g : \mathbb{D} \to \mathbb{D}$ which maps 0 to 0. Applying Schwarz lemma yields

$$|(h \circ f \circ g)'(0)| = |h'(w_0) f'(z_0) g'(0)| \leq 1,$$

using the definition of the derivative as $\frac{h(f(g(z)))}{z}$ as $z \to 0$. Then

$$|f'(z_0)| \leq \frac{1}{|g'(0)| \cdot |h'(w_0)|}.$$

Using the formula for g, h yields $g'(0) = 1 - |z_0|^2$ and $h'(w_0) = \frac{1}{1-|w_0|^2}$.

Suppose now f is conformal. Then $h \circ f \circ g$ is conformal and fixes 0. Thus by the above $h \circ f \circ g(z)$ is a rotation, so $|h'(w_0) f'(z_0) g'(0)| = 1$. Thus again with the algebra this yields equality.

We will show if equality holds at some $z_0 \in \mathbb{D}$ then f is conformal and equality holds fro all $z_0 \in \mathbb{D}$. Suppose equality holds at some z_0. As above we see $|(h \circ f \circ g)'(0)| = 1$.

By Schwarz lemma (see Gamelin for the derivative version), $h \circ f \circ g$ is a rotation, so f is a conformal self map $\mathbb{D} \to \mathbb{D}$.

VIII.2. **Hyperbolic Geometry**

Suppose $f : \mathbb{D} \to \mathbb{D}$ is a conformal map. Write $w = f(z)$. Then $\mathrm{d}w = f'(z)\,\mathrm{d}z$. Therefore

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = |f'(z)| = \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Rearranged with $w = f(z)$, we have

$$\frac{|\mathrm{d}w|}{1 - |w|^2} = \frac{|\mathrm{d}w|}{1 - |z|^2}.$$

Then if γ is a smooth curve in $\mathbb{D}$ and $w = f(z)$ then

$$\int_{f\circ\gamma} \frac{|\mathrm{d}w|}{1 - |w|^2} = \int_\gamma \frac{|\mathrm{d}z|}{1 - |z|^2}.$$

This tells us that if we want to measure distance in the unit disk, then we should use $\frac{|\mathrm{d}z|}{1-|z|^2}$. This metric will have the miraculous property of being preserved by conformal self-maps of the disk.

Definition VIII.2.1

The length of γ in the hyperbolic metric on $\mathbb{D}$ is

$$\text{hyperbolic lenght of } \gamma := 2\int_\gamma \frac{|\mathrm{d}z|}{1 - |z|^2},$$

where the 2 is innocent, so that the curvature is -1.

NICE: by design, hyperbolic length is invariant under conformal maps $\mathbb{D} \to \mathbb{D}$.

Definition VIII.2.2

Let $z_0, z_1 \in \mathbb{D}$. We define the hyperbolic distance $\rho(z_0, z_1)$ to be

$$\rho(z_0, z_1) := \inf_{\substack{\text{piecewise smooth}\gamma \\ \gamma: z_0 \to z_1}} 2\int_\gamma \frac{|\mathrm{d}z|}{1 - |z|^2}.$$

Since conformal maps $\mathbb{D} \to \mathbb{D}$ preserve hyperbolic lengths of curves, conformal maps preserve the hyperbolic metric too.

Theorem VIII.2.1

For any two distinct points $Z_0, z_1 \in \mathbb{D}$, there exists a unique geodesic in $\mathbb{D}$ from z_0 to z_1 in the hyperbolic metric. This curve is the arc of circle passing through z_0 to z_1 that is orthogonal to the unit circle.

Via an appropriate choice of θ, a we have $z \mapsto e^{i\theta}\frac{z-a}{1-\overline{a}z}$ maps z_0 to 0 (via choosing a) and z_1 to the real axis (via rotation). This makes the problem easier

Stuff:

- HW 11A due today
- Math Circle

- Super Saturdays
- Bagel Sundays
- Student Seminar 11/18: **Dedekind and the Axiom of Choice** by Dhruv Kul in EH 3096 4-5pm.

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Definition VIII.2.3

X is a Riemann surface provided that it is a $\mathbb{C}$-manifold with $\dim_{\mathbb{C}} X = 1$.

Example VIII.2.1

$U \subseteq \mathbb{C}$ where U is open. Complex tori (which are elliptic curves). We can do similarly for a genus g surface. Also $\widehat{\mathbb{C}}$.

We are in the process of classifying all Riemann surfaces. There are three types of Riemann surfaces

(1) $\widehat{\mathbb{C}}$
(2) $\mathbb{C}, \mathbb{C}^*$, and complex tori.
(3) all others

In particular $\mathbb{D}, \mathbb{H}$ lie in category three.

Warm up: There are exactly three simply connected Riemann surfaces up to conformal isomorphism.

Last time we showed the conformal maps $\mathbb{D} \to \mathbb{D}$ are

$$z \mapsto e^{i\theta} \frac{z-a}{1-\overline{a}z}$$

for $a \in \mathbb{D}, \theta \in [0, 2\pi]$. In fact $\operatorname{Aut}(\mathbb{D})$ is a "Lie group" (for those of us taking 591), and has dimension 3 over $\mathbb{R}$.

Lets cook up more examples of holomorphic maps $\mathbb{D} \to \mathbb{D}$. There are the power maps $z \mapsto z^n$ for $n > 0$.

Consider

$$f_1(z) = \frac{z-a_1}{1-\overline{a}_1 z} \qquad f_2(z) = \frac{z-a_2}{1-\overline{a}_2 z},$$

where $a_1, a_2 \in \mathbb{D}$. We could build a new map $\mathbb{D} \to \mathbb{D}$ by multiplying, since $|f_1(z)|, |f_2(z)| < 1$.

Definition VIII.2.4

A Blashcke product $B : \mathbb{D} \to \mathbb{D}$ is a rational function of the form

$$f(z) = e^{i\theta} \prod_{j=1}^{d} \frac{z-a_j}{1-\overline{a}_j z}.$$

It turns out that every proper analytic map $\mathbb{D} \to \mathbb{D}$ is a Blaschke product.

Definition VIII.2.5

Let X, Y be topological spaces. $f : X \to Y$ is proper provided that for all $K \subseteq Y$ compact we have $f^{-1}(K) \subseteq X$ is compact.

Non-Example VIII.2.2

$z \mapsto 57$ is not compact since $f^{-1}(\{57\}) = \mathbb{C}$.

Example VIII.2.3

For any homeomorphism $f : X \to X$, this is proper, since the image of a compact set under a continuous function is compact (the Extreme Value Theorem)

Proposition VIII.2.2

Let X, Y be metric spaces. Suppose Y is connected, locally compact. Assume f is continuous, open, proper, then in fact f is surjective.

Proof. Let $V = f(X)$. We will show V is both open and closed. We know V is open by assumption (that f is an open map).

We'll show V is closed. Let y_0 be an accumulation point of V. Let $y_n \in V$ be a sequence of points converging to y_0, with $f(x_n) = y_n$. Take K a compact neighborhood of y_0 in Y. Consider $f^{-1}(K)$ in X, which is compact since f is proper.

Now for $n > N$, where $N \in \mathbb{N}$, we have $y_n \in K$, so $x_n \in f^{-1}(K)$. By compactness there is a subsequence x_{n_k} which is convergent to some x, and so then y_{n_k} converges to $f(x)$ and to y! Perfect!

Theorem VIII.2.3

Every proper analytic map $f : \mathbb{D} \to \mathbb{D}$ is a Blaschke product.

Lemma VIII.2.4

Let $f : \mathbb{D} \to \mathbb{D}$ be

$$f(z) = e^{i\theta} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a}_i z}$$

Then in fact f is proper.

Proof. Note that a closed subset C of $\mathbb{D}$ is compact if and only if C does not intersect $|z| = 1$.

It thus suffices to show that f maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$ (Exercise: think about why). This follows from arguments on tuesday since $z \mapsto \frac{z-a}{1-\overline{a}z}$ maps S^1 to S^1 for any $a \in \mathbb{D}$.

*Proof of [***thm:blaschke***].* A proper map $f : \mathbb{D} \to \mathbb{D}$ is surjective. So f has at least one zero. Say a. Let $M(z) = \frac{z-a}{1-\overline{a}z}$.

Then $g(z) = \frac{f(z)}{M(z)} : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic, with one fewer zero than f. The result follows by induction on the degree of f (which we would need more details for).

Corollary VIII.2.5

A proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ extends to a holomorphic map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (because Blaschke products are rational functions).

We know what $\mathrm{Aut}(\mathbb{D})$ is. How do we get $\mathrm{Aut}(\mathbb{H})$. Well we use the Cayley map to move things around

$$\begin{tikzcd}
\mathbb{H} \arrow[r] \arrow[d, "\varphi"'] & \mathbb{H} \arrow[d, "\varphi"] \\
\mathbb{D} \arrow[r] & \mathbb{D}
\end{tikzcd}$$

With this you can show that

$$\mathrm{Aut}(\mathbb{H}) \simeq \left\{ z \mapsto \frac{az+b}{cz+d} \mid ad - bc \neq 0, a, b, c, d \in \mathbb{R} \right\} = \mathrm{PSL}_2(\mathbb{R})$$

Likewise $\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathrm{PSL}_2(\mathbb{C})$

To do: We need to classify the geodesics on $\mathbb{D}$, which we claimed were arcs of circles intersecting S^1 at right angles (and diameters of the disk).

IX. Riemann Mapping Theorem

IX.1. Arzelà-Ascoli and the Proof

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We also want to prove a *huge* theorem

Theorem IX.1.1 (Riemann Mapping Theorem)

If $D \subseteq \mathbb{C}$ is open, connected, and simply connected and $D \neq \mathbb{C}$, then D is conformally isomorphic to $\mathbb{D}$.

Corollary IX.1.2

A simply connected domain in $\widehat{\mathbb{C}}$ is $\widehat{\mathbb{C}}$, conformally isomorphic to $\mathbb{C}$, or conformally isomorphic to $\mathbb{D}$.

Such an isomorphism is often called "the" Riemann map (this is clearly not unique). It is unique up to postcomposition with $\mathrm{Aut}(\mathbb{D})$.

We need some ingredients to prove the Riemann Mapping Theorem. They're pretty heavy.

Preliminaries: We need to study equicontinuity. This is in Gamelin, [Gam03, p. XI.5] page 306. Let $E \subseteq \mathbb{C}$ be a set and let

$$\mathcal{F} \subseteq \{f : E \to \mathbb{C}\}$$

a family of functions $E \to \mathbb{C}$.

Definition IX.1.1

We say $\mathcal{F}$ is equicontinuous at $z_0 \in E$ provided that for any $\varepsilon > 0$ there is a $\delta > 0$ so that if $z \in E$, $|z - z_0| < \delta$ and $f \in \mathcal{F}$ then $|f(z) - f(z_0)| < \varepsilon$.

Essentially, the continuity has a form of uniformity over the functions in $\mathcal{F}$ (not necessarily over the inputs, that would be uniform continuity).

Definition IX.1.2

We say $\mathcal{F}$ is uniformly bounded on E provided that there is an $M > 0$ so that $|f(z)| \leq M$ for all $z \in E, f \in \mathcal{F}$.

Question: Suppose $\mathcal{F}$ is a family of differentiable functions $D \to \mathbb{C}$, and further suppose the family of derivatives is uniformly bounded on D. What can we say about $\mathcal{F}$? Well then $\mathcal{F}$ is equicontinuous (or if you'd like, equi-Lipschitz... at least if D is convex).

Proposition IX.1.3

Suppose $\mathcal{F}$ is a family of holomorphic functions $D \to \mathbb{C}$ whose derivatives are uniformly bounded. Then $\mathcal{F}$ is equicontinuous at any $z_0 \in D$.

Proof. Let $M > 0$ with $|f'(z)| \leq M$ for all $z \in D$ and $f \in \mathcal{F}$. So take $z \in D$ close to z_0, so that the straight line z to z_0 is contained in D, and then integrate

$$|f(z) - f(z_0)| = \left| \int_{z_0}^{z} f'(\zeta)\, \mathrm{d}\zeta \right| \leq M \cdot |z - z_0| .$$

More general–equi-Lipschitz, with the metric on the codomain defined by the infimum of paths contained in D.

Big Deal:

Theorem IX.1.4 (Arzelà-Ascoli Theorem, the late 19th century.)

Let $E \subseteq \mathbb{C}$ be compact. Let $\mathcal{F} = \{f_i : E \to \mathbb{C} \mid i \in I\}$ be a family of continuous functions on E that is uniformly bounded. Then the following are equivalent

(1) $\mathcal{F}$ is equicontinuous at all $z_0 \in E$.

(2) Each sequence of functions $f_n \in \mathcal{F}$ has a subsequence which converges uniformly on E.

Recall IX.1.1

Sarah's General Advice: If you're trying to prove something exists, change the problem to a question about fixed points, inverse function theorem, or use compactness (build a sequence of approximations, then use compactness).

Next Time: Arzelà-Ascoli Theorem, Hubbard Stuff, donuts, fun!!! Riemann Mapping Theorem after Thanksgiving!

Stuff:

- HW 11B due tonight, for #11 assume nonconstant.
- There are 2 drops in both the A and B series (so 2 drops *each*!).
- Happy Early Thanksgiving!

. Fun stuff today: applications of ideas we have seen.

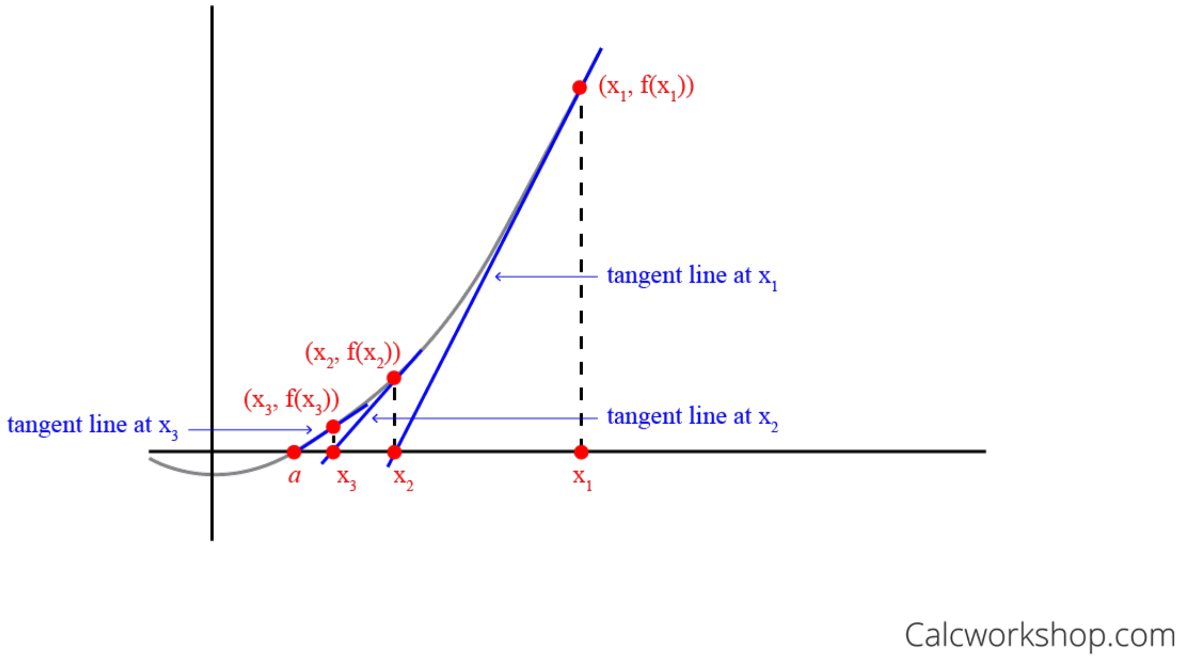
Today, we're going to talk about Newton's Method, which is a way of finding zeros of differentiable functions. Next week we will prove the Riemann Mapping Theorem, which says that if $D \subsetneq \mathbb{C}$ is open and simply connected, then D is conformally isomorphic to $\mathbb{D}$. THere is a constructive proof of this with an iteration scheme.

Thurston Idea: Set up iteration scheme such that the object you need is a fixed point of this scheme, and appeal to fixed point theorems.

The sketch of Newton's method in words:

- Start with an initial guess x_0.
- Draw the tangent line $\ell_1(x)$ to $(x_0, f(x_0))$, and solve for $\ell_1(x) = 0$ to get x_1.
- Lather rinse repeat.

Newton's Method in pictures



In formulas we have a map

$$N_f(x) = x - \frac{f(x)}{f'(x)}.$$

Then we let $x_{n+1} = N_f(x_n)$. Question: when does this converge and what to? Well a weird example, $f(x) = e^x$ is never zero and $N_f(x) = x - 1$. We can also have cycles

$$\begin{tikzcd} x_0 \arrow[r] & x_1 \arrow[r] & \cdots \arrow[r] & x_{157} \arrow[r, bend left] & x_{158} \arrow[l, bend left] \end{tikzcd}$$

, and bad things happen when $f'(x) = 0$.

Steven Smale and Curt McMullen have done incredible work with this. Question: Are there any generally convergent iterative algorithms for finding roots of polynomials? Generally convergent meaning the set of bad guesses is sparse (small measure)¿ For Newton's method there are big open sets of bad guesses. The answer is no, for degree ≥ 4. There are also theorems that give good algorithms for finding *good* guesses.

Cayley: work over $\mathbb{C}$. Look at $N_p(z) = z - \frac{p(z)}{p'(z)}$ again. Cayley wondered how the initial guess affects the convergence. For example, you can look at $p(z) = z^2 - 1$. There are two roots, color z_0 black if it goes to 1, and red if it goes to 0. If it does not converge, color it blue. What do the pictures look like in general?

Cayley figured it out for quadratics... without a computer!!! This was in 1879, and he published his results in a 1-page paper. We'll do this now, and assume monic. Let

$$\begin{aligned} p(z) &= (z - r_1)(z - r_2) \\ N_p(z) &= z - \frac{(z - r_1)(z - r_2)}{(z - r_1) + (z - r_2)} \end{aligned}$$

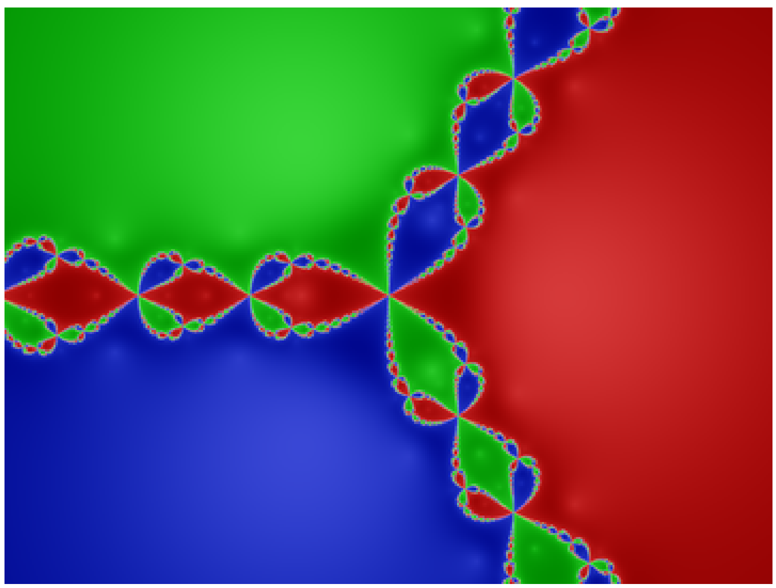
$$= \frac{z^2 - r_1 z + z^2 - r_2 z - (z^2 - r_1 z - r_2 z + r_1 r_2)}{2z - r_1 - r_2}$$
$$= \frac{z^2 - r_1 r_2}{2z - r_1 - r_2}.$$

If we plug in $r_1$we get

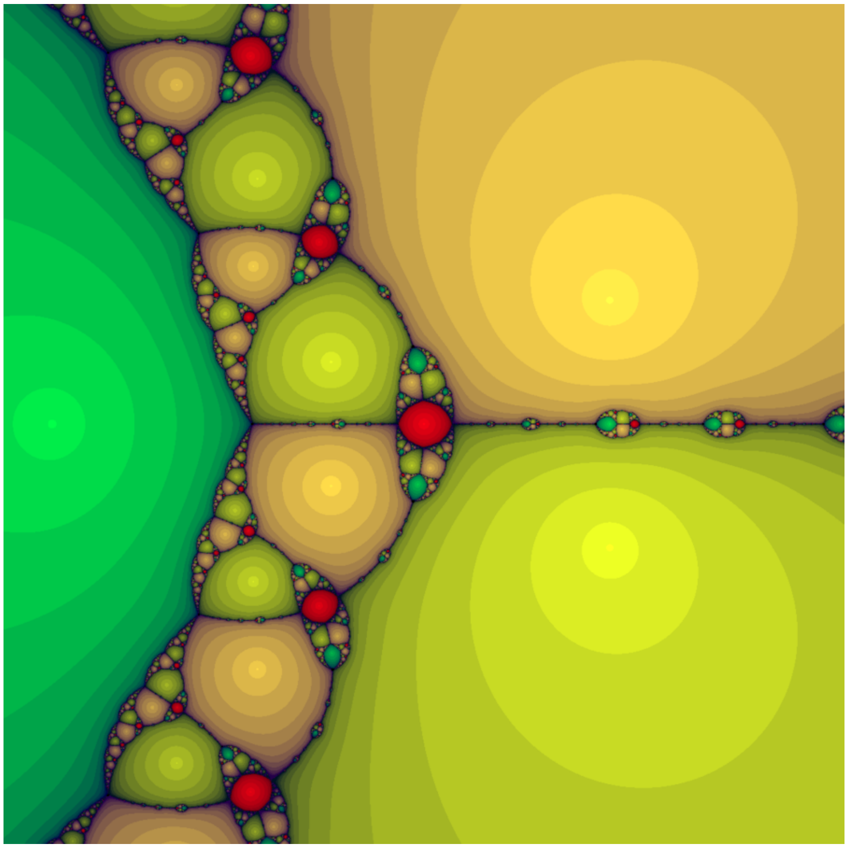
$$N_p(r_1) = \frac{r_1^2 - r_1 r_2}{r_1 - r_2} = r_1.$$

Then $N_p(z)$ has r_1, r_2 as fixed points. In fact they'll be superattracting fixed points. If r_1, r_2 are simple roots, then $N_p'(r_1), N_p'(r_2) = 0$. Locally, the Newton map must look like $z \mapsto z^2$. We see then a tiny neighborhood of r_1 maps to a smaller neighborhood of r_1. The colors correspond to splitting the plane in half based on which one its closest to. Take a line connected them and the perpendicular bisector (proximity based).

The bad initial guesses are along the perpendicular bisector, and stay on the line forever (very thin set). Everyone thinks that for three roots r_1, r_2, r_3 you get a pizza, a similar proximity based thing. It wasn't so. Let's look at a picture for $p(z) = z^3 - 1$, with roots at the roots of unity, then we get



For bad cubics we can get red basins where there is no convergence, say $p(z) = z^3 - 2z + 2$, we get a picture



This is actually how computers solve equations. There's a trick where you can go far out enough and use equidistant spacings to get good guesses (use "channels").

Exercise IX.1.2

Let $p(z)$ be a quadratic polynomial with Newton map $N(z)$. Prove that there exists a Möbius transformation μ so that $\mu \circ N \circ \mu^{-1}$ is the squaring function. This is just changing coordinates in a nice way.

Stuff:

- HW 12 due 12/8, 11:59pm.
- 4 Classes Left. Here's the Plan
- Today: Prove Riemann Mapping Theorem
- After Today: Finish some syllabus topics
- If we have time: Prime Number Theorem? Dynamics? Elliptic Functions?

Recall IX.1.3 (Schwarz Lemma)

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map and suppose $|f(z)| \leq 1$. If $f(z) = 0$, then

$$|f(z)| \leq |z|$$

for all $z \in \mathbb{D}$. Furthermore, if there exists a $z_0 \in \mathbb{D}$ for which $|f(z_0)| = |z_0|$, then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Infinitesimal version: Same hypotheses, $|f'(0)| \le 1$, with equality if and only if $f(z) = e^{i\theta} z$.

Using these we characterized the automorphisms of $\mathbb{D}$ as

$$f(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z}$$

for $a \in \mathbb{D}$.

These together gave us Pick's lemma,

Recall IX.1.4 (Pick's Lemma and Hyperbolic Geometry)

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}$$

for all $z \in \mathbb{D}$, with equality when $f \in \mathrm{Aut}(\mathbb{D})$. This told us that

$$\frac{|\mathrm{d}w|}{1-|w|^2} = \frac{|\mathrm{d}z|}{1-|z|^2}$$

for $w = f(z)$ where $f \in \mathrm{Aut}(\mathbb{D})$. This allowed us to define

$$\mathrm{hyplength}_\gamma := 2\int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2},$$

and taking the infimum along such paths gives a metric $\rho(z_0, z_1)$.

Theorem IX.1.5

For any two distinct points z_0, z_1 there exists a unique geodesic from z_0 to z_1 in the hyperbolic metric. Namely, it is the arc of the circle passing through z_0, z_1 which is perpendicular to $\partial\mathbb{D}$.

Proof. Let $w = f(z)$ be a conformal automorphism $\mathbb{D} \to \mathbb{D}$ with $f(z_0) = 0$. Rotate to move $f(z_1)$ to the positive real axis. Call that point $r = |f(z_1)| > 0$. We want the geodesic from 0 to r to be the line segment from 0 to r, as then we'll be done!

Now suppose $\gamma(t) = x(t) + iy(t)$ is a path in $\mathbb{D}$ connecting 0 to r, with $0 \le t \le 1$. Then $x(t)$ defines a path from 0 to r also, along the real axis. Set $\alpha(t) = x(t)$. We see that

$$\int_\alpha \frac{|\mathrm{d}z|}{1-|z|^2} = \int_0^1 \frac{|\mathrm{d}x(t)|}{1-x(t)^2} \le \int_0^1 \frac{|\mathrm{d}x(t)|}{1-|\gamma(t)|^2} \le \int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2}.$$

Thus if $y(t) \neq 0$ then for some $t \in [0,1]$, $|\gamma(t)| > |x(t)|$, and so the path $\alpha = x(t)$ is strictly shorter than $\gamma(t)$.

Moreover, if α backtracks, then α is decreasing on some interval, and we could make $\int_\alpha \frac{|\mathrm{d}z|}{1-|z|^2}$ even smaller, by removing these segments. Thus the integral is minimized uniquely when α is a straight line from 0 to r.

Recall IX.1.5 (Arzelà-Ascoli Theorem, the late 19th century.)

Let $E \subseteq \mathbb{C}$ be compact and let $\mathcal{F} = \{f_i : E \to \mathbb{C} \mid i \in I\}$ be a family of continuous functions on E which is uniformly bounded. Then the following are equivalent

(1) $\mathcal{F}$ is equicontinuous at all $z_0 \in E$.

(2) Each sequence of functions $f_n \in \mathcal{F}$ has a subsequence which converges uniformly on E. (NOTE: the limit of the subseqence may or may not belong to $\mathcal{F}$).

Gamelin uses this on p307 to prove Montel's theorem. Some mathematical genealogy

$$\text{Borel} \longrightarrow \text{Montel} \longrightarrow \text{Henri Cartan} \longrightarrow \text{Douady} \longrightarrow \text{Hubbard} \longrightarrow \text{Sarah}$$

Theorem IX.1.6 (Montel's Theorem)

Suppose $\mathcal{F}$ is a family of analytic functions on a connected, nonempty, open $D \subseteq \mathbb{C}$ such that $\mathcal{F}$ is uniformly bounded on all compact subsets of D.

Then every sequence in $\mathcal{F}$ has a subsequence that converges normally on D; that is, it converges uniformly on compact subsets.

Proof. We want to apply Arzelà-Ascoli. We need to get that $\mathcal{F}$ is equicontinuous at all points of D.

Fix $z_0 \in D$. Since D is open there exists an $r > 0$ so that $\{|z - z_0| \le r\} \subseteq D$. By hypothesis, $\mathcal{F}$ is uniformly bounded on this closed disk by some M. The Cauchy estimate tells us that the derivatives of functions in $\mathcal{F}$ are uniformly bounded on a slighly smaller disk $\{|z - z_0| \le r - \varepsilon\} \subseteq D$. This gives us equicontinuity via the mean value theorem.

We now apply Arzelà-Ascoli with a diagonalization argument. We define

$$E_n \coloneqq \left\{ z \in D \mid |z| \le n \text{ and } \operatorname{dist}(z, \partial D) \ge \frac{1}{n} \right\}$$

The E_n are compact and increase to fill up D, that is $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup_n E_n = D$. Even better, any compact subset of D is contained in some E_n.

Let $\{f_m\}$ be a sequence of points (aka functions) in $\mathcal{F}$. By Arzelà-Ascoli there is a subsequence $f_{11}, f_{12}, f_{13}, \ldots$ that converges uniformly on E_1.

By Arzelà-Ascoli there is a subsequence of f_{1m} that converges unifromly on E_2 to get f_{2m}. Lather rinse repeat

Claim

The diagonal sequence f_{mm} is a subsequence of $\{f_n\}$ converging uniformly on each E_n, hence uniformly on each compact subset of D.

The Deepest Result in $\mathbb{C}$-analysis. The Riemann Mapping Theorem.

- Riemann, partial proof 1851. Details kept getting filled in
- Schwarz 1870
- Osgood 1900
- Koeloe 1907
- Carathéodory 1912
- Modern Proof: Riesz + Fejer 1923.

Theorem IX.1.7 (Riemann Mapping Theorem)

Every nonempty, connected, open, simply connected, proper subset $U \subseteq \mathbb{C}$ is conformally isomorphic to $\mathbb{D}$.

Proof. Idea: Solve an extremal problem. We'll take $U \subsetneq \mathbb{C}$, $p \in U$, and we'll build the conformal map $(U,p) \overset{f}{\mapsto} (\mathbb{D}, 0)$. This map is unique up to rotating the right hand side (since we insisted p maps to 0). f is called a Riemann map

Uniqueness is easy. Suppose $f, g : (U,p) \to (\mathbb{D}, 0)$ are both conformal maps. Then $g \circ f^{-1}$ is a conformal map $(\mathbb{D}, 0) \to (\mathbb{D}, 0)$, so it is a rotation, and so $g = \mathrm{rot} \circ f$.

Existence is hard. We'll do it in 3 steps. The map we seek will be a solution to an extremal problem (Hubbard: "This is fantastically nonconstructive").

Plan: Find holomorphic injective maps $(U,p) \overset{f}{\mapsto} (\mathbb{D}, 0)$, want $|f'(p)|$ to be as large as possible among all such holomorphic injective maps $(U,p) \to (\mathbb{D}, 0)$. Secretly: There's something called the Kobayashi metric. . . google it.

Consider the family

$$\mathcal{F} \coloneqq \{f : (U,p) \to (\mathbb{D}, 0) \mid f \text{ is holomorphic and injective}\}.$$

Let's go!

Step 1) If U were $\mathbb{C}$, then $\mathcal{F}$ would be empty by Liouville's theorem. But since $U \neq \mathbb{C}$, there is an $a \in \mathbb{C} \setminus U$, and since U is simply connected the function $z \mapsto z - a$ is nonvanishing and has a holomorphic square root defined on U.

Call this square root function h, with $(h(z))^2 = z - a$ for all $z \in U$. We claim h is injective, which follows since $z \mapsto z - a$ is injective. By the open mapping theorem, $h(U) \subseteq \mathbb{C}$ is open. Since $(h(z))^2$ is injective and $a \notin U$, $h(U), -h(U)$ are disjoint.

Take a ball $\overline{B_r(q)} \subseteq -h(U)$. The Möbius transformation $\varphi(z) = \frac{r}{z-q}$ maps $B_r(q)$ onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. So where does φ send $h(U)$?

It sends $h(U)$ into $\mathbb{D}$. Now take an automorphism $\psi \in \mathrm{Aut}(\mathbb{D})$ taking $\varphi(h(p))$ to 0. Then $\psi \circ \varphi \circ h$ lies in $\mathcal{F}$.

Step 2) We'll show that if $f \in \mathcal{F}$ and if $f(U) \neq \mathbb{D}$ (i.e. f is not surjective), then there is a $g \in \mathcal{F}$ such that $|g'(p)| > |f'(p)|$. Why?

We'll use the "square root trick" due to Carathéodory and Koebe. We'll adopt some convenient notation

$$\varphi_a(z) = \frac{z-a}{1-\overline{a}z} \in \mathrm{Aut}(\mathbb{D})$$

sending a to 0. Moreover, $(\varphi_a)^{-1} = \varphi_{-a}$.

Suppose $f \in \mathcal{F}$ with $f(U) \neq \mathbb{D}$, and take $a \in \mathbb{D} \setminus f(U)$. Then look at $\varphi_a \circ f \mid U \to \mathbb{D}$ omits zero. Thus $\varphi_a \circ f : U \to \mathbb{C}$ admits a homolormphic square root $h : U \to \mathbb{C}$, let s be the squaring function. Then

$$s \circ h = \varphi_a \circ f$$

on U. We know h is injective and maps U into $\mathbb{D}$ (since f does). If $b = h(p)$ then the composition $g = \varphi_b \circ h$ belongs to $\mathcal{F}$, and

$$f = \varphi_{-a} \circ s \circ h = (\varphi_{-a} \circ s \circ \varphi_{-b}) \circ g.$$

But then $\psi \coloneqq \varphi_{-a} \circ s \circ \varphi_{-b}$ sends $(\mathbb{D}, 0) \to (\mathbb{D}, 0)$ and is NOT injective on $\mathbb{D}$ implying it's not a rotation (since it has a squaring part). Thus by the Schwarz Lemma, $|\psi'(0)| < 1$. But wait!

$$|f'(p)| = |\psi'(0)|\,|g'(p)| < |g'(p)|\,.$$

Step 3) Let's finish[1] the proof! We need to cook up an extremal map $f \in \mathcal{F}$ such that $|f'(p)|$ is as big as possible. Let

$$M \coloneqq \sup\{|f'(p)| \mid f \in \mathcal{F}\}$$

We'll note some facts about M

(a) $M > 0$ because $f \in \mathcal{F}$ are injective, so $|f'(p)| > 0$.

(b) $M < +\infty$. If $B_r(p) \subseteq U$, then $|f'(p)| \leq \frac{1}{r}$ by the Cauchy estimates since $|f(z)| < 1$ for all $z \in U$. Thus $M \leq \frac{1}{r}$.

Let $\{f_n\}$ be a sequence of functions in $\mathcal{F}$ such that $|f_n'(p)|$ converges to M. Since $\mathcal{F}$ is uniformly bounded (outputs to $\mathbb{D}$, so by the constant 1), Montel's theorem applies.

Thus there is a subsequence f_{n_k} which converges normally on U. Call this limit function f. We know f is an analytic function since the convergence is normal.

> **Claim**
>
> $f \in \mathcal{F}$.

We need to check a bunch of things

(a) $f(p) = 0$ since $f_{n_k}(p) = 0$ for all k.

(b) We know $f(U) \subseteq \overline{\mathbb{D}}$, by taking limits of the inequalities. If there was a $q \in U$ with $|f(q)| = 1$, then f is not open, so f must be constant. But wait! $|f'(p)| = M > 0$.
Thus $f(U) \subseteq \mathbb{D}$, so $f : (U, p) \to (\mathbb{D}, 0)$ is analytic.

(c) We must check f is injective. Well, all f_{n_k} are injective and f is nonconstant, so by some version of Hurwitz's theorem (see [**thm:hurwitz**])

Thus $f \in \mathcal{F}$. By Step 2, since $|f'(p)| = M$, f is surjective. Done! $f : (U, p) \to (\mathbb{D}, 0)$ is a conformal isomorphism.

Stuff:

- Math Club!
- Popcorn 4:30pm
- Bagel Sunday 11:30am
- Super Saturdays
- HW due next week
- Last student seminar of the semester at 4pm in EH 3096, Drew Keisling on S-LID sequences (generalizations of fibonacci stuff)

Exciting Stuff! We were proving Riemann Mapping Theorem[2].

[1] Technically this was done on December 1st, but I didn't want to split it up
[2] For users of the notes, the rest of the proof has been uploaded on the document for November 29th

The proof was incredibly nonconstructive. We had something like

$$M \coloneqq \sup\{|f'(p)| \mid f \in \mathcal{F}\}$$

and we needed a sequence f_n with $|f_n'(p)| \to M$. As we all know, supremums are awful to compute.

Example IX.1.6 (Classic 295/296 style example)

$\sup\{\cos(10^n) \mid n \in \mathbb{N}\}$ is not known, but it is almost certainly 1 (related to properties of π, such as being normal).

To fully appreciate Hubbard's constructive proof, we need to know some stuff from 592 (algebraic topology).

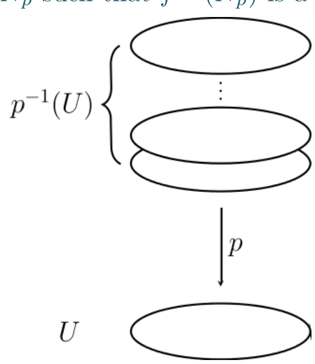
Question: Great. We know that if $U \subsetneq \mathbb{C}$ is proper, simply connected, then there exists a conformal isomorphism $U \to \mathbb{D}$. What if U is not simply connected? Well since any conformal map is a homeomorphism, this will cause problems (simply connected is a topological invariant).

Cool, we won't prove it! Every annulus has something called an annulus. This is gotten by thinking of an annulus as a projection of a cylinder. The modulus is then the height/circumference... This turns out to be an invariant in complex analysis.

Fisher-Hubbard-Wittner: Given a domain $U \subsetneq \mathbb{C}$, possibly not simply connected, there exists a local biholomorphism $\mathbb{D} \to U$ called a covering map

Definition IX.1.3

Let $f : X \to Y$ (where X, Y are topological spaces). We call f a covering map provided that for all points $p \in Y$ there is a neighborhood N_p such that $f^{-1}(N_p)$ is a "pile of plates"



These are called "sheets." Formally, $f^{-1}(N_p)$ is a disjoint union of open sets $\widetilde{N}_p^i$ such that $f : \widetilde{N}_p^i \to N_p$ is a homeomorphism.

Riemann surfaces are then very closely related to covering maps. In fact, there is a Grandpa to the RMT, which we will state but not prove.

Theorem IX.1.8 (Uniformization Theorem)

Let X be a Riemann surfaces ($\mathbb{C}$ manifold with $\dim_{\mathbb{C}} X = 1$. Then there is exactly one cover with the following total space (the top one):

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \mathbb{C} & \mathbb{D} \\ \downarrow{\scriptstyle \text{spherical}} & \downarrow{\scriptstyle \text{Euc}} & \downarrow{\scriptstyle \text{hyp}} \\ X & X & X. \end{array}$$

These are called universal covers, since they are simply connected (see more 592!). This determines the geometry, since $\mathbb{D}$ is hyperbolic, $\mathbb{C}$ is Euclidean, $\widehat{\mathbb{C}}$ is spherical.

The first case $X = \widehat{\mathbb{C}}$, second is one of torus, $\mathbb{C} \setminus \{0\}, \mathbb{C}$, and third is a wild zoo.

Now how do we compute Riemann maps when they exist. Let $U \subsetneq \mathbb{C}$ be simply connected. You could use Hubbard's method (a limit, for the computers). In general it's very hard. In fact, going from a square to a disk is super difficult!

Weirdly enough, it is *much* easier to compute a Riemann map from the complement of the Mandelbrot set to the complement of a disk (by inversion ideas, this is the same as an appropriate Riemann map).

Suppose we have a Riemann map $(U, p) \to (\mathbb{D}, 0)$. When does it extend to a continuous map $(\overline{U}, p) \to (\overline{\mathbb{D}}, 0)$.

Theorem IX.1.9 (Carathéodory,1913)

A Riemann map $\mathbb{D} \to U$ extends to a continuous map $\overline{\mathbb{D}} \to \overline{U}$ if and only if ∂U is locally connected.

Recall IX.1.7

A topological space X is locally connected at $x \in X$ provided that for all neighborhoods U of x, there exists a connected neighborhood $V \subseteq U$ of x.

Not Known: is the Mandelbrot set M locally connected (if bad things happen it would be at the boundary...hmmm).

HW: Asked to prove M is connected and full (a set is full provided that $\mathbb{C} \setminus M$ is connected).

Fun fact: A Riemann map $\mathbb{D} \to U$ extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{U}$ if and only if ∂U is a Jordan Curve.

IX.2. Mandelbrot Set Things

We'll look at maps $z \mapsto z^2 + c$ where $c \in \mathbb{C}$ is a parameter. Is this too specific? No in fact. Any $az^2 + bz + d$ can be conjugated with an element of $\mathrm{Aut}(\mathbb{C})$ to a polynomial $z \mapsto z^2 + c$. Thus from the point of view of dynamics, all complex polynomials look like $z^2 + c$.

The key idea of the Mandelbrot set is to look at the c-plane (the parameter space). Generally, in dynamics we like to study orbits. Given a map $f : X \to X$ and a point z_0, the orbit is the sequence $z_n = f(z_{n-1})$. In $\mathbb{C}$-analysis, we really like looking at it as polynomials! We can think of these as maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with ∞ as a fixed point (a special point for a polynomial).

Recall IX.2.1 (Filled Julia sets, Definition IV.1.10)

Given a polynomial $f : \mathbb{C} \to \mathbb{C}$ we define the filled Julia set

$$K_f := \{z_0 \in \mathbb{C} \mid \text{the orbit of } z_0 \text{ is bounded}\}$$

Fun facts:

- $K_f \neq \emptyset$, except when f is a non-identity translation, This is since $f(z) - z$ will be a nonconstant polynonmial, so f will have a fixed point.
- K_f is bounded. Can show if $f(z) = z^2 + c$ then they are in disk of $r = 2$.
- K_f is closed.

$\mathbb{C} \setminus K_f$ is the basin of ∞, and consists of all points z_0 whose orbit diverges to ∞. Can show this is open explicitly.

The big question: How does the shape change when you change the polynomial? What subsets of $\mathbb{C}$ arise as filled Julia sets of $z \mapsto z^2 + c$ as c varies? Good thing to look up, there are also pictures earlier in the notes (see around Definition IV.1.10)!

Definition IX.2.1 (Mandelbrot Set)

The Mandelbrot Set is defined as

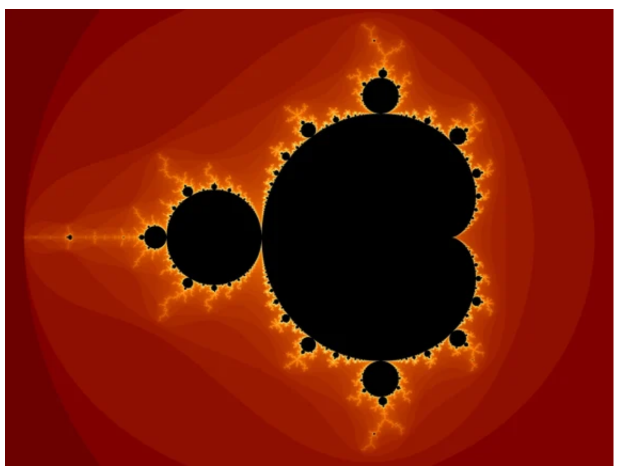
$$\mathcal{M} := \{c \in \mathbb{C} \mid \text{Filled Julia set of } z \mapsto z^2 + c \text{ is connected}\}$$

We can ask the computer to draw filled Julia sets (look at the orbit and see if it escapes outside of disk of radius 2, do it for long enough)... How the hell do you draw the Mandelbrot set? We could draw a quilt (tile complex plane with julia sets). This gets us close! But we need to do better. We need a theorem that relates connectivity of the filled Julia set to something easy for a computer to check

Theorem IX.2.1 (Proof, basic $\mathbb{C}$-analysis)

The Filled Julia set of $z^2 + c$ is connected if and only if it contains 0, in other words if and only if the orbit of 0 is bounded.

WHY ZERO: well one idea, it's a critical point since $2 \cdot 0 = 0$. But if you're thinking geometrically it's also sort of clear. It's the *center of symmetry* of the Julia set, since $z_0, -z_0$ have the same orbit. Here is the Mandelbrot set!



Facts about the Mandelbrot set:

- $\mathcal{M} \neq \emptyset$ since $0 \in \mathcal{M}$ since the filled Julia set of $z \mapsto z^2$ is $\overline{\mathbb{D}}$.
- Compact, bounded inside closed disk of $r = 2$... Going to keep going.

Stuff:

- HW 12 due Thursday!
- Office Hours tomorrow 10:30am-12pm!
- No Office Hours on Friday!
- Decorating for WOLOG party on Thursday night
- Talk about final
- Hint on 8, show f maps $\overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Show that f is proper, then it's a Blaschke product...

Last time: Mandelbrot stuff! This time: More mandelbrot stuff.

Recall IX.2.2

$\mathcal{M}$ is the "connectedness locus" of $z \mapsto z^2 + c$. We said last time that $\mathcal{M} \neq \emptyset$ (since $0 \in \mathcal{M}$), $\mathcal{M}$ is compact (why!), $\mathcal{M}$ is connected (WHY! Relevant for HW), and $\mathcal{M}$ is full (also HW).

See Gamelin for details, but

Theorem IX.2.2 (Fatou-Julia, 100 years ago)

Let $P(z)$ be a polynomoial of degree $d \geq 2$. The filled julia set of $P(z)$ is connected if and only if the orbits of each critical point of $P(z)$ is bounded.

Theorem IX.2.3 (Fatou-Julia)

The filled Julia set K_c of $z \mapsto z^2 + c$ is disconnected if and only if it is a Cantor set (up to homeomorphism).

For other degrees, there is intermediate behavior, essentially because there are multiple critical points, so one can have bounded orbit while the other does not.

Take $c \in \mathcal{M}$. Then the filled Julia set K_c of $z^2 + c$ is connected and full ($\mathbb{C} \setminus K_c$ is connected, the proof is maximum modulus principle).

Definition IX.2.2

The basin of ∞ of $z^2 + c$ is $\mathbb{C} \setminus K_c$.

FACT: There is an explicit conformal isomorphism on the basis of ∞ for $z \mapsto z^2 + c$ to the basin of ∞ of $z \mapsto z^2$. Furthermore, it conjugates $z^2 + c$ to z^2. It's important here that $c \in \mathcal{M}$.

Idea: lift the identity iteratively, let $s(z) = z^2$, $P(z) = z^2 + c$,

$$\begin{tikzcd}
(U, \infty) \arrow[r, dashed, "\phi_2"] \arrow[d, "P"'] & (U, \infty) \arrow[d, "s"] \\
(U, \infty) \arrow[r, dashed, "\phi_1"] \arrow[d, "P"'] & (U, \infty) \arrow[d, "s"] \\
(U, \infty) \arrow[r, dashed, "\mathrm{Id}"'] & (U, \infty)
\end{tikzcd}$$

Then $\phi_n \circ P = s \circ \phi_{n+1}$. This will converge normally to some map ϕ, and we'll have $\phi \circ P = s \circ \phi$. These are called Böttcher coordinates for the polynomial about ∞. When $c \in \mathcal{M}$, this can be extended to all of $\widehat{\mathbb{C}} \setminus K_c$.

If $c \notin \mathcal{M}$, then we can extend the Böttcher coordinates until we hit the critical point in the dynamical plane of $z^2 + c$.

We can cook up a map Φ from $\mathbb{C} \setminus \mathcal{M}$ to $\mathbb{C} \setminus \overline{\mathbb{D}}$ by $c \mapsto \varphi_c(c)$ where φ_c is the conformal isomorphism defined on the exterior of the figure 8 (aka where Böttcher coordinates apply).

Prove by hand that this is the Riemann map (aka it is a conformal isomorphism). There are three parts to that

(1) Φ is holomorphic.
(2) Φ is proper. It follows that Φ is surjective.
(3) To get injectivity, look near ∞ and show that Φ has degree 1.

Corollary IX.2.4

As a corollary the Mandelbrot set is connected and full.

For HW: explain a piece of this proof sketch you find interesting.

Trick: Now you can try to label the boundary of the Mandelbrot set by angles... the labeling being unique even for irrationals is exactly the claim that $\mathcal{M}$ is locally connected, a huge conjecture.

Stuff:

- Last Class: Good Job! ☺
- Homework due tonight
- No Office Hours tomorrow (stop by atrium tonight!)
- Volunteers (10am tomorrow)! Free t-shirts.
- DRP Presentations at Math Club 4-5pm.
  - Chip Firing Game
  - Dynamical Systems
  - Elliptic Curves
- **Final Next Week!**
  - Friday December 16th 1:30-3:30pm in this classroom
  - 1 double-sided Sheet of Notes allowed
  - Evenly distributed through content of whole course
- Office Hours next week! Thursday December 15th 1-4pm.

One final topic to cover!

X. Infinite Products

This is [Gam03] page 353.

Definition X.0.1

An infinite product is an expression of the form

$$\prod_{j=1}^{\infty} p_j$$

where each p_j is a complex number.

We say this converges provided that

(1) p_j converges to 1.

(2) $\sum_{j=1}^{\infty} \mathrm{Log}(p_j)$ converges as a series. We only sum over terms where $p_j \neq 0$.

This is a strange definition (think about why it's strange), but it is convenient.

If the infinite product converges, we define its value to be 0 if one of the p_j is zero. Otherwise we define the limit to be

$$\prod_{j=1}^{\infty} p_j := \exp\left(\sum_{j=1}^{\infty} \mathrm{Log}\, p_j\right).$$

Observations:

(1) If $\prod p_j$ converges then at most finitely many of p_j are 0.

(2) If $\prod p_j$ converges, then in fact

$$\prod_{j=1}^{\infty} p_j = \lim_{m\to\infty} \prod_{j=1}^{m} p_j$$

(3) We can always factor out a finite # of terms from a convergent infinite product

$$\prod_{j=1}^{\infty} p_j = p_1 \cdots p_m \prod_{j=m+1}^{\infty} p_j.$$

(4) If an infinite product converges, and if none of the factors is zero, then the product is not zero.

Note: This excludes a product like $\prod_{j=1}^{\infty} \frac{1}{2} = 0$, even though this makes sense sort of intuitively.

Example X.0.1

Consider the following product

$$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{k}\right) = (1+1)\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\cdots.$$

Using the power series of the logarithm we can see that this converges. Then there is a subsequence of the partial products which is always 1. If we call the terms p_j, then because

$$\left(1 + \frac{1}{2k-1}\right)\left(1 - \frac{1}{2k}\right) = 1,$$

the product of the first m terms is always 1 if m is even. If m is odd then it is equal to $\frac{1}{1+\frac{1}{m}}$.

General idea: It is helpful to write $p_j = 1 + a_j$ and look at the product as $\prod(1 + a_j)$, as one can often exploit the power series expansion of Log near 1 to compute $\sum \operatorname{Log}(1 + a_j)$.

Gamelin Notes:

- If $0 \le t \le 1$, then $\frac{t}{2} \le \log(1+t) \le t$. Exercise in basic analysis.
- As a consequence, if $t_j \ge 0$ then $\sum t_j$ if and only if $\sum \log(1+t_j)$ converges.

  Why? Well in either case $t_j \to 0$ so eventually $0 \le t \le 1$, so we're in business to use the comparison test of the above.

Theorem X.0.1

If $t_j \ge 0$, then

$$\prod_{j=1}^{\infty}(1+t_j) \text{ converges} \iff \sum t_j \text{ converges}$$

Application: Let $\alpha > 0$. Then consider

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{k^{\alpha}}\right).$$

Claim

This converges for $\alpha > 1$ and diverges for $0 \le \alpha \le 1$.

Direct from p-series test from real analysis and the test above.

Definition X.0.2

The infinite product $\prod_{j=1}^{\infty}(1+a_j)$ is said to converge absolutely provided that $a_j \to 0$ as $j \to \infty$ and $\sum \operatorname{Log}(1+a_j)$ converges absolutely. Note: if the product converges absolutely then the product itself converges.

Remark X.0.1

It is very unnatural to look at $\prod |1+a_j|$. Why is that unnatural? Explain it as an exercise.

Theorem X.0.2

The infinite product $\prod_{j=1}^{\infty}(1+a_j)$ converges absolutely if and only if $\sum a_j$ converges absolutely, which occurs if and only if $\prod(1+|a_j|)$ converges.

Proof. See Gamelin, routine.

Example X.0.2

We know $\prod\left(1+\frac{(-1)^{k+1}}{k}\right)$ converges, but does not converge absolutely.

Application: 3blue1brown video and the Riemann zeta (ζ) function. For more, see the Prime Number Theorem section in Gamelin [Gam03].

Definition X.0.3

For $\operatorname{Re}(s) > 1$ we define

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In fact this converges absolutely and it converges uniformly for $\operatorname{Re}(s) \geq 1+\varepsilon$ for any $\varepsilon > 0$. It will have *severe* problems at $s = 1$.

This function can be continued across the entire plane except at $s = 1$. Call this extension ξ.

Corollary X.0.3 (Riemann Hypothesis)

If s is a zero of the Riemann zeta then s is a negative even integer or $\operatorname{Re}(s) = 1/2$.

Theorem X.0.4 (Euler)

If $\operatorname{Re}(s) > 1$, then

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right).$$

Proof Idea. Consider $\frac{1}{1-p^{-s}}$. We like this because it is a geometric series

$$\frac{1}{1-p^{-s}} = \sum_{n=0}^{\infty} p^{-ns}.$$

Take your favorite prime numbers $p_1, \ldots, p_m$, and multiply these expressions

$$\prod_{k=1}^{m} \frac{1}{1-p_k^{-s}} = \sum_{\ell_1,\ldots,\ell_m=0}^{\infty} \left(p_1^{\ell_1} \cdots p_m^{\ell_m}\right)^{-s}.$$

This is in fact $\zeta(s)$ as $m \to \infty$ by the Fundamental Theorem of Arithmetic.

References

[Ahl53] Lars V Ahlfors. "Complex analysis: an introduction to the theory of analytic functions of one complex variable". In: *New York, London* 177 (1953).

[Gam03] Theodore Gamelin. *Complex analysis*. Springer Science & Business Media, 2003.