Comment on HW4B problem 5: Given a Möbius transformation $f(z) = \frac{az+b}{cz+d}$, we wish to show the number of g such that $g(g(g(g(g(z))))) = f(z)$ is $1, 5$ or ∞. It would be very difficult to work with a general f. Perhaps instead we should work with particular f and show this is enough. One should think about conjugation in the group, namely consider for $f, h \in$ Möb, the conjugate $h \circ f \circ h^{-1}$.

It is enough to solve the problem for a conjugate. Use g^5 to genote 5 copies of g composed, then

$$g^5 = f \iff (hgh^{-1})(hgh^{-1})(hgh^{-1})(hgh^{-1})(hgh^{-1}) = hfh^{-1}.$$

Since $g \mapsto hgh^{-1}$ is a bijection between the set of rational functions with itself, we're good!

We now need to understand the conjugacy classes in Möb. What is somethingthat is invariant under conjugation. The # of fixed points of $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is invariant under conjugation. Consider

$$\frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0.$$

It turns out there are three possibilities

- f has exactly one fixed point (conjugate to $z \mapsto z+1$).
- f has exactly two fixed points (conjugate to $z \mapsto \lambda z$, $\|\lambda\| = 1$).
- f has infinitely many fixed points (conjugate to $z \mapsto z$).

In Sarah's research area, she takes rational functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and considers the behavior of iterations $f \circ \cdots \circ f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. A "simple" example is $f(z) = z^2 + c$ where $c \in \mathbb{C}$ is some parameter.

Definition .0.1 (Filled Julia set)

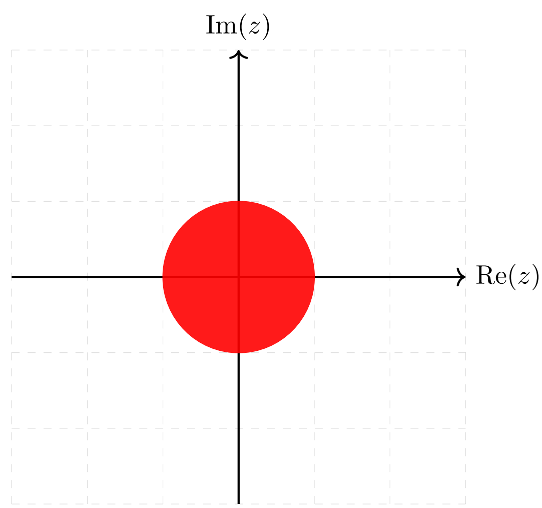
The filled Julia set K_f of f is

$$K_f := \{z_0 \in \mathbb{C} \mid \text{ orbit of } z_0 \text{ under } f \text{ is bounded}\}.$$

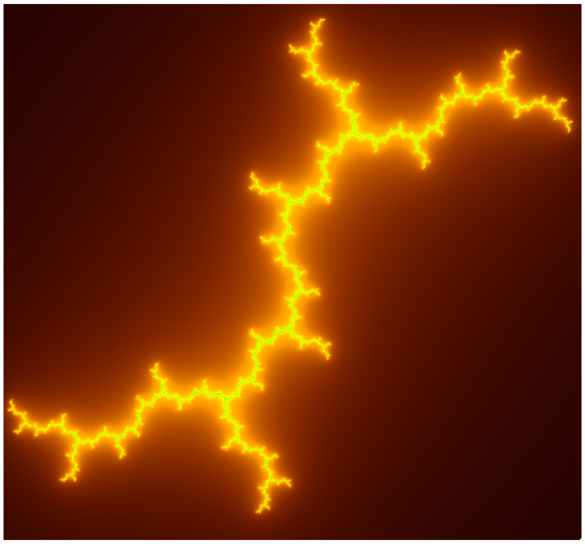
The orbit here is the sequence z_n defined by $z_n = f(z_{n-1})$.

Example .0.1

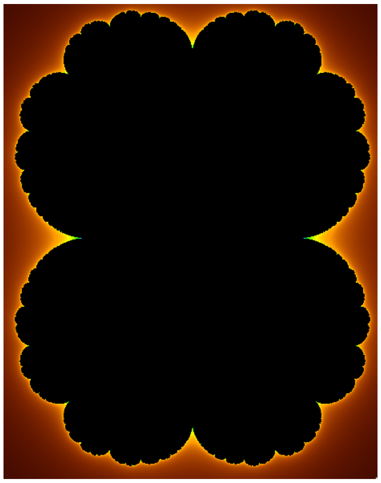
For $z \mapsto z^2$ the filled Julia set is simple



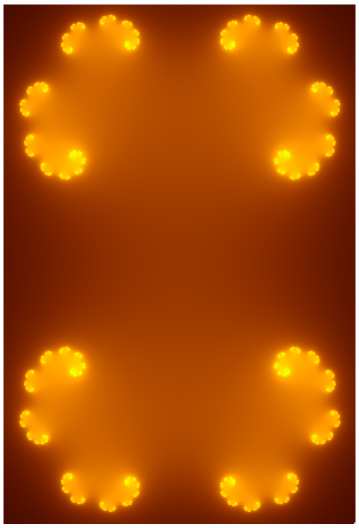
Generally filled Julia sets are crazy. For $z \mapsto z^2 + i$ we have



For $z \mapsto z^2 + 1/4$ we have a cauliflower shape



For $z \mapsto z^2 + 1/2$ we get cantor dust!



All filled julia sets for polynomials are full, that is their complement is connected. Why? Well it's a corollary of our discussion using a rate of escape function which is harmonic...

.1. Defining Complex Integrals

Why do we want to do integration? Well holomorphic maps have amazing properties. It is much easier to prove that they have these properties with integrals. We integrate 1-forms, and we define formally

$$\mathrm{d}z \coloneqq \mathrm{d}x + i\,\mathrm{d}y.$$

If γ is a curve in $\mathbb{C}$, $f = u + iv$, with u, v continuous on γ, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma u\,\mathrm{d}y - v\,\mathrm{d}x.$$

We can also parameterize as

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\frac{\mathrm{d}\gamma}{\mathrm{d}t}\,\mathrm{d}t.$$

As well we have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}x + i\int_\gamma f(z)\,\mathrm{d}y.$$

Remark .1.1

All of the basic theorems concernining linearity from real integrals work! Namely

$$\int_\gamma c(f(z) + g(z))\,\mathrm{d}z = c\int_\gamma f(z)\,\mathrm{d}z + \int_\gamma g(z)\,\mathrm{d}z.$$

Example .1.1

Consider $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$. Then we wish to calculate $\int_\gamma \frac{\mathrm{d}z}{z}$. Then we see that $\mathrm{d}z = ie^{it}\,\mathrm{d}t$, and so

$$\int_\gamma \frac{\mathrm{d}z}{z} = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i.$$

Example .1.2

Let L be the line segment in $\mathbb{C}$ parameterized by $[0, 1] \to \mathbb{C}$. $t \mapsto p + t(q - p)$.

Fix $n \in \mathbb{Z}, n \neq -1$, then

$$\int_L z^n\,\mathrm{d}z = \int_0^1 (p + t(q - p))^n \cdot (q - p)\,\mathrm{d}t = \left(\frac{(p + t(q - p))^{n+1}}{n + 1}\right]_0^1 = \frac{q^{n+1} - p^{n+1}}{n + 1}.$$

Exercise .1.3

Fix $m \in \mathbb{Z}$ and $R > 0$, compute

$$\int_{|z-z_0|=R} (z - z_0)^m\,\mathrm{d}z = \begin{cases} 0 & \text{if } m \neq -1 \\ 2\pi i & \text{if } m = -1 \end{cases}$$

Arc length, we denote as

$$|\mathrm{d}z| := \mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}.$$

if γ is parameterized by $\gamma(t) = x(t) + iy(t)$ then

$$\int_\gamma h(z)\,|\mathrm{d}z| = \int_a^b h(\gamma(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Theorem .1.1

[ML Theorem] Suppose γ is a piecewise smooth curve in the plane. If $h(z)$ is continuous on γ, then

(1) $\left|\int_\gamma h(z)\,\mathrm{d}z\right| \leq \int_\gamma |h(z)|\,|\mathrm{d}z|$.

(2) Furthermore, if γ has length L and $|h(z)| \leq M$ on γ then

$$\left|\int_\gamma h(z)\,\mathrm{d}z\right| \leq \int_\gamma |h(z)|\,|\mathrm{d}z| \leq ML.$$

Proof. Use triangle inequality and Riemann sums.

.2. The Complex Fundamental Theorem of Calculus

Definition .2.1

Let $f(z)$ be a continuous function on a connected open subset $D \subseteq \mathbb{C}$. $F : D \to \mathbb{C}$ is called a complex primitive for $f(z)$ provided that $F(z)$ is holomorphic on D and $F'(z) = f(z)$.

Theorem .2.1 (FTC I)

If $f(z)$ is continuous on a connected open subset D and if $F(z)$ is a primitive for $f(z)$, then

$$\int_\gamma f(z)\,\mathrm{d}z = F(B) - F(A),$$

where γ is any path from A to B.

Proof. Fix γ a path between A and B. We see that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}z}\,\mathrm{d}z = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}z}\,\mathrm{d}x + i\int_\gamma \frac{\mathrm{d}F}{\mathrm{d}z}\,\mathrm{d}y = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}x}\,\mathrm{d}x + i\int_\gamma -i\frac{\mathrm{d}F}{\mathrm{d}y}\,\mathrm{d}y$$

using the Cauchy-Riemann Equations! We then have that

$$\int_\gamma f(z) = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}x}\,\mathrm{d}x + \frac{\mathrm{d}F}{\mathrm{d}y}\,\mathrm{d}y = \int_\gamma \mathrm{d}F = F(B) - F(A).$$