Stuff:

- HW 2B due September 13th by 10PM.
  - For 6, to say L_M is $\mathbb{C}$-linear means there exists $\alpha \in \mathbb{C}$ so that the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "L_M"] \arrow[d, leftrightarrow] & \mathbb{R}^2 \arrow[d, leftrightarrow] \\ \mathbb{C} \arrow[r, dashed] & \mathbb{C} \\ z \arrow[r, mapsto, dashed] & \alpha z \end{tikzcd}$$

  - For 9, the set $U \subseteq \mathbb{C}$ which is the domain of $f : U \to \mathbb{C}$ should be connected.

To say f is holomorphic at a point, we will always mean f is holomorphic on a neighborhood of z_0.

We're headed to Gamelin, II.4-7, Ahlfors 3.2-3.3. Now back to Möbius transformations

If we have a Möbius transformation $f(z) = \frac{az+b}{cz+d}$ we note that

$$f(\infty) = \frac{a}{c}, c \neq 0 \qquad f(\infty) = \infty, c = 0$$

$$f^{-1}(\infty) = \frac{-d}{c}, c \neq 0 \qquad f^{-1}(\infty) = \infty, c = 0.$$

One can compute that

$$f^{-1}(z) = \frac{dz - b}{-cz + a}$$

which is also a Möbius transformation.

Fact: Möbius transformations are holomorphic.

Question: what does it mean for a function $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ to be holomorphic

Answer: Use local charts around each point. The relevant charts are $\widehat{\mathbb{C}} \setminus \{\infty\} \to \mathbb{C}, \widehat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$ given by stereographic projection about each pole. The transition map on the intersection is given by $z \mapsto 1/z$. Consider $\mathrm{inv} : z \mapsto 1/z$

Definition .0.1

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be continuous and $a \in \widehat{\mathbb{C}}$. We say that f is holomorphic at a provided that

(i) When $a = \infty, f(a) = \infty$, the map $\mathrm{inv} \circ f \circ \mathrm{inv}$ is holomorphic at $z = 0$.

(ii) When $a = \infty, f(a) \neq \infty$, the map $f \circ \mathrm{inv}$ is holomorphic at $z = 0$.

(iii) When $a \neq \infty, f(a) = \infty$, the map $\mathrm{inv} \circ f$ is holomorphic at $z = a$.

(iv) When $a \neq \infty, f(a) \neq \infty$, the map $\circ f$ is holomorphic at $z = a$.

Consequence: all rational functions $R(z) = \frac{P(z)}{Q(z)}$ are holomorphic maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Corollary .0.1

Möbius transformations are holomorphic.

Exercise .0.1

Prove that polynomials extend to holomorphic maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. It is clear for constant polynomials, so let $p(z)$ be nonconstant.

Take $p(z) = \sum_{j=1}^{d} a_j z^j$ with $a_j \in \mathbb{C}$ and $a_d \neq 0$, it is clear that p is holomorphic at $z \neq \infty$. We just need to check around $z = \infty$. Here we have $p(\infty) = \infty$ unless p is constant. We thus must look at

$$(\mathrm{inv} \circ p \circ \mathrm{inv})(z) = \frac{1}{p\left(\frac{1}{z}\right)} = \frac{1}{\sum_j a_j z^{-j}}.$$

Cleaning this up gives

$$\frac{z^d}{a_d + \cdots + a_1 z^{d-1} + a_0 z^d}.$$

This is holomorphic at zero since $a_d \neq 0$. Its derivative is indeed

$$p'(\infty) = \begin{cases} 0 & \text{if } d > 1 \\ \frac{1}{a_d} & \text{if } d = 1 \end{cases}.$$

Even better: Möbius transformations are biholomorphisms on the Riemann sphere (aka a holomorphic bijection with holomorphic inverse).

We collect these into a group

$$\text{Möb} := \{\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid \mu \text{ is a Möbius trnasformation}\}$$

Algebraically, Möb is a group with respect to the binary operation of composition. We can think of this as a matrix group via $\mathrm{GL}_2(\mathbb{C})$. Namely via the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f_A(z) = \frac{az+b}{cz+d}.$$

The determinant being nonzero corresponds to $ad - bc \neq 0$, which we require. One can check this is a surjective homomorphism. The kernel is

$$\ker(\mathrm{GL}_2(\mathbb{C}) \to \text{Möb}) = \left\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{C} \setminus \{0\} \right\},$$

which may be easily checked. We often call the quotient of $\mathrm{GL}_2(\mathbb{C})$ by this kernel the "projective general lienar group"

$$\mathrm{PGL}_2(\mathbb{C}) := \mathrm{GL}_2(\mathbb{C}) \Big/ \left\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{C} \setminus \{0\} \right\} \cong \text{Möb}$$

We could normalize our matrices to have determinant one...

$$\mathrm{SL}_2(\mathbb{C}) = \{A \in \mathrm{GL}_2(\mathbb{C}) \mid \det(A) = 1\}$$

There is then a homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \text{Möb}$ with kernel $\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. This gives us

$$\mathrm{PSL}_2(\mathbb{C}) := \mathrm{SL}_2(\mathbb{C}) \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \text{Möb}.$$

This is in some sense 3-dimensional, as we have four variables and one condition, $ad - bc = 1$.

There are three fundamental types of Möbius transformations

(1) Linear, $z \mapsto \alpha z$ where $\alpha \in \mathbb{C} \setminus \{0\}$.
(2) Translation, $z \mapsto z + \beta$ for some $\beta \in \mathbb{C}$.
(3) Inversion, $z \mapsto \frac{1}{z}$.

Theorem .0.2

We have that

(i) The group Möb is generated by translation, linear maps, and inversion.
(ii) The action of Möb on $\widehat{\mathbb{C}}$ is "simply 3-transitive" i.e. for any two triples of distinct points $(p_1, p_2, p_3), (q_1, q_2, q_3)$ on $\widehat{\mathbb{C}}$ there exists a unique Möbius transformation taking p_j to q_j.
(iii) The action of Möb on $\widehat{\mathbb{C}}$ preserves circles.

Proof of (ii). For existence, it suffices to show that any triple (p_1, p_2, p_3) can be sent to $(0, 1, \infty)$, then take

$$f(z) = \frac{(p_2 - p_3)(z - p_1)}{(p_2 - p_1)(z - p_3)}.$$

Caution: Breaking the rules a bit if one of the p_i is ∞... but just adjust and change formula a bit. Namely one of these three formulas

$$\frac{p_2 - p_3}{z - p_3}, p_1 = \infty \qquad \frac{z - p_1}{z - p_3}, p_2 = \infty \qquad z \mapsto \frac{z - p_1}{p_2 - p_1}, p_3 = \infty.$$

To prove uniqueness, suppose $g \in$ Möb that sends $(p_1, p_2, p_3) \mapsto (0, 1, \infty)$. We must examine $f \circ g^{-1}$.

This is a Möbius transformation fixing $0, 1, \infty$. Check that the only such map is the identity.

Something cool: Suppose p_1, p_2, p_3, p_4 are distinct points in $\mathbb{C}$ that lie on a circle $\Gamma \subseteq \mathbb{C}$. The Möbius transformation $f(z) = \frac{1}{z - p_1}$ sends the circle Γ to a line $L = f(\Gamma)$.

Let $q_k = f(p_k)$, $k = 2, 3, 4$. Choose the ordering so on the circle p_3 is between p_2, p_4. Then q_3 is between q_2, q_4. This gives that

$$|q_2 - q_4| = |q_2 - q_3| + |q_3 - q_4|$$

plug in $q_k = \frac{1}{p_k - 1}$ and simplyify to get

$$|p_1 - p_3| \cdot |p_2 - p_4| = |p_1 - p_2| \cdot |p_3 - p_4| + |p_1 - p_4| \cdot |p_2 - p_3|$$

Theorem .0.3 (Ptolemy's Theorem)

A quadrilateral can be inscribed in a circle if and only if the sum of products of lengths of opposite edges is equal to the product of the lengths of the diagonals.