Stuff:

- HW 7 Due Friday October 21st, 11:59pm
- Problem 3
  - Setup: $j \mapsto g_j$, nowhere vanishing entire, and we have $j \mapsto p_j$ polynomials with $\deg(p_j) \leq 10$.
  - $f_j := g_j p_j$ converges locally uniformly on $\mathbb{C}$ to $f : \mathbb{C} \to \mathbb{C}$. Then $f = g \cdot p$, where g is nowehre vanishing and entire, p is a polynomial. What do we know about $\deg(p)$?
  - This requires Hurwitz's theorem, so we will delay this problem until later!

Last time: Powerful result! Analytic functions.

We saw that if $f(z)$ is not identically zero on a connected open set $D \subseteq \mathbb{C}$, and if $z_0 \in D$ is a zero of f, then z_0 has a finite order as a zero; i.e., $f(z) = (z - z_0)^N \cdot h(z)$ where $h(z_0) \neq 0$ is analytic locally about z_0.

We used this to show that the zeros of f (when f is not identically zero) are isolated from each other.

We were then able to prove the uniqueness principle: if f, g are analytic on a connected open set $D \subseteq \mathbb{C}$ and $f(z) = g(z)$ for z belonging to a set with an accumulation point then $f = g$ on D.

Cool application: Let $g(z)$ be an entire function such that $g(x) = \exp(x)$ for all $x \in \mathbb{R}$, and then $g(z) = \exp(z)$.

.1. The Open Mapping Theorem

Pause: Topology Break!

Definition .1.1

Let X, Y be topological spaces, a map $f : X \to Y$ is called open provided that for every open $U \subseteq X$ we have $f(U) \subseteq Y$ is open.

Example .1.1

Projection $X \times Y \to X$ taking $(x, y) \mapsto x$. The identity map. Conway's base 13 function is an example of an open map which is not continuous. It is given by writing a real number x in base 13, using the additional symbols $\{+, -, .\}$, and saying x maps to a number if some tail of the base 13 expansion is a valid base 10 number (and we take the longest such tail). If no such tail exists then we send x to 0.

The function $\mathbb{R} \to \mathbb{R}$ taking $x \mapsto x^2$ is not open since the image of $\mathbb{R}$ is $[0, \infty)$. Similarly the map $F : \mathbb{C} \to \mathbb{R}$ with $F(z) = |z|$ is not open, since $F(\mathbb{C}) = [0, \infty)$. However these are "close" to being open in some sense.

Another nonexample is the constant function $z \mapsto 57 + 53i$.

An example of an open map are affine maps $f : \mathbb{C} \to \mathbb{C}$ wiht $z \mapsto az + b$ for $a \neq 0$ since they are homeomorphisms.

Remark .1.1

If $f : U \to \mathbb{C}$ is holomorphic with $f'(z) \neq 0$ for all $z \in U$, then f is locally invertible. Thus f is a local homeomorphism, and so f must be open. This is a consequence of the inverse function theorem.

Exercise .1.2

Show a local homoemorphism must be open. More generally show that being open is a "local" property (appropriately define this as well).

Example .1.3

The map $z \mapsto z^k$ on $\mathbb{C}$ for $k \geq 1$ is open. Yes! We only have to worry about points where $f'(z) = 0$. Thus we only need to worry about $z = 0$.

Working with a basis of the topology, take a small open disk of radius $r > 0$ about 0, this maps to a small open disk of radius r^k about 0, with k preimages for each point. Great!

Theorem .1.1 (Open Mapping Theorem)

Let $D \subseteq \mathbb{C}$ be open and connected and let $f : D \to \mathbb{C}$ be a nonconstant holomorphic function. Then f is an open map.

Proof. We only need to worry about $z_0 \in D$ where $f'(z_0) = 0$, since it is a local homeomorphism elsewhere.

Since f is nonconstant, we know that $f' \not\equiv 0$ (not identically zero). Thus there exists a minimal $k \geq 1$ such that $f^{(k)}(z_0) \neq 0$. There is then some disk $|z - z_0| < \rho$ in D so that

$$f(z) = f(z_0) + a_k(z - z_0)^k \cdot h(z)$$

with $a_k \neq 0$, $h(z_0) = 1$, and $h(z)$ analytic. We know the map $z \mapsto z^k$ is locally invertible in a neighborhood of $z = 1$. Let g be a local inverse.

When z is close to z_0, $h(z)$ is close to 1, and so in a neighborhood of z_0 we have $(g(h(z)))^k = h(z)$. Thus we can look at

$$f(z) = f(z_0) + a_k((z - z_0) \cdot g(h(z)))^k$$

This is a composition of open maps near z_0 (translation, powering, and a mystery function) since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(z - z_0)g(h(z))\Big|_{z=z_0} &= \Big[(z - z_0)g'(h(z))h'(z) + g(h(z))\Big]_{z=z_0} \\ &= g(h(z_0)) = 1. \end{aligned}$$

Perfect! This shows that f is open near z_0 as desired!

Back to Gamelin!

.2. Analytic Continuation

This is section V.8 in [Gam03]. There are no homework problems / QR problems on this part especially anything with paths/monodromy.

Definition .2.1

Let $U \subseteq V \subseteq \mathbb{C}$ be open and connected. Now let $f : U \to \mathbb{C}$ be analytic. We call $F : V \to \mathbb{C}$ an analytic continuation provided that $F\big|_U = f$.

Example .2.1

Define $f(z) = \sum_k \left(\frac{z}{2}\right)^k$ for $|z| < 2$. Well when $|z| < 2$ we have

$$f(z) = \frac{2}{2 - z}.$$

We can expand $f(z)$ at $z_0 = -1$ to get a different series

$$\begin{aligned} f(z) &= \frac{2}{2-(z+1-1)} = \frac{2}{3-(z+1)} = \frac{2}{3} \cdot \frac{1}{1-\frac{(z+1)}{3}} \\ &= \frac{2}{3} \cdot \sum_{k=0}^{\infty} \left(\frac{z+1}{3}\right)^k, \end{aligned}$$

which is valid for $|z+1| < 3$.

This gives an analytic continuation!

END OF MIDTERM I MATERIAL

MIDTERM I is in class October 25th

How do we extend analytic functions? Especially important for things like the Riemann ζ function.

Lemma .2.1

Let $D \subseteq \mathbb{C}$ be open and connected and let $f(z)$ be analytic on D. Now let $R(z_1)$ be the radius of convergence of the power series expansion about $z_1 \in D$. Then in fact

$$|R(z_1) - R(z_2)| \leq |z_1 - z_2|.$$

Proof. Gamelin!

We say that $f(z)$ is analytically continuable along $\gamma \subseteq \mathbb{C}$ if for each $t \in [a, b]$ there exists a convergent power series

$$f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z - \gamma(t))^n$$

for $|z - \gamma(t)| < r(t)$ such that $f_a(z)$ is the power series representation for $f(z)$ at $z_0 = \gamma(a)$ and when $s \in [a, b]$ is near $t \in [a, b]$, then $f_s(z) = f_t(z)$ for z in the intersection of the disks in convergence.

By the uniqueness principle, the series $f_t(z)$ determines uniquely each of the series $f_s(z)$ for s near t.

Theorem .2.2

Suppose $f(z)$ can be continued analytically along the path γ for $t \in [a, b]$. Then the analytic continuation is unique.

Example .2.2

Take $f(z)$ to be the principal branch of the square root function, and $\gamma(t) = e^{it}$.

In a neighborhood of $z = 1$ we have

$$f(z) = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \cdots$$

We may then change centers to get

$$f_t(z) = e^{it/2} + \frac{e^{-it/2}}{2}(z - e^{it}) - \frac{e^{-3it/2}}{8}(z - e^{it})^2 + \cdots$$

$$f_{2\pi}(z) = -1 - \frac{1}{2}(z-1) + \frac{1}{8}(z-1)^2 - \cdots .$$

It turns out $f_{2\pi}$ gives us the *other branch* of the square root. The fancy way of saying this is we picked up *monodromy*.