I. Series!

Stuff:

- HW 5B due tonight!
- HW 6A due Thursday
- HW 6B due next week
- Office Hours: Wednesday + Friday (email with questions!)
- Popcorn? T-shirts Halloween and voting
- Super Saturdays
- Student Seminar Friday 10/7 4pm-5pm EH 3096: Partition Statistics and the S^1 method by Faye Jackson.

Hint for $n, m < 0$ in HW 5B Q1. Take small circles around 0. Also for HW 5B Q1 note

$$z\overline{z} = r^2 \qquad\qquad |\mathrm{d}z| = \frac{-ir}{z}\,\mathrm{d}z.$$

I.1. Very Quick Review

A sequence of complex numbers is a function $n \mapsto a_n \in \mathbb{C}$ for $n \in \mathbb{N}$. Associated to this is a new sequence that we can build

$$m \mapsto (s_m := \sum_{k=1}^{m} a_k)$$

called the sequence of partial sums. If $\lim_{m\to\infty} s_m$ exists, then we say series $\sum_{1}^{\infty} a_n$ converges and is equal to that limit. There are a number of nice tests from real analysis and many extend to complex analysis

- Divergence Test: If $\sum_{1}^{\infty} a_n$ converges then $a_n \to 0$. Caution: Remember the harmonic series.

Proposition I.1.1

If $|z| < 1$ then the series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

If $|z| \geq 1$ then the series diverges.

Proof. Very lucky! Work out what the partial sums are explicitly.

Definition I.1.1

A complex series $\sum a_k$ converges absolutely provided that $\sum |a_k|$ converges.

Theorem I.1.2

If $\sum a_k$ converges absolutely, then $\sum a_k$ converges.

Proof. Write $\mathrm{Re}(a_k) = \mathrm{Re}(a_k) + |a_k| - |a_k|$. Then since $|\mathrm{Re}(a_k)| \leq |a_k|$, we know that

$$0 \leq \mathrm{Re}(a_k) + |a_k| \leq 2\,|a_k|$$

Similarly

$$0 \le \mathrm{Im}(a_k) + |a_k| \le 2\,|a_k|\,.$$

We then know that the series of non-negative real numbers $\sum \mathrm{Re}(a_k) + |a_k|$ and $\sum \mathrm{Im}(a_k) + |a_k|$ both converge by the monotone convergence theorem, since the partial sums are bounded above by $2\sum |a_k|$.

Then we see that

$$\begin{aligned}\sum \mathrm{Re}(a_k) &= \left(\sum \mathrm{Re}(a_k) + |a_k|\right) - \sum |a_k| \\ \sum \mathrm{Im}(a_k) &= \left(\sum \mathrm{Im}(a_k) + |a_k|\right) - \sum |a_k| \\ \sum a_k &= \sum \mathrm{Re}(a_k) + i \sum \mathrm{Im}(a_k).\end{aligned}$$

Thus $\sum a_k$ converges!

Next: Series of FUNctions!

In Gamelin this is [Gam03, pp. V .2].

Definition I.1.2

The sequence $n \mapsto (f_n : E \to \mathbb{C})$ converges pointwise to $f : E \to \mathbb{C}$ on E provided that for all $x \in E$, the sequence $n \mapsto f_n(x)$ converges to $f(x)$.

Definition I.1.3

The sequence of fuctions $n \mapsto (f_n : E \to \mathbb{C})$ converges uniformly to $f : E \to \mathbb{C}$ on E provided that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in E$ we have $|f_n(x) - f(x)| < \varepsilon$.

Importantly, the choice of N does not depend on x. An equivalent formulation is to define for $f, g : E \to \mathbb{C}$

$$\|f - g\| = \sup_{z \in \mathbb{C}} |f(z) - g(z)|\,.$$

Example I.1.1

If we pick $f_n : [0,1] \to [0,1]$ with $f_n : x \mapsto x^n$ then $f_n(x) \to 0$ for $x \in [0,1)$ and $f_n(1) \to 1$. Thus it converges, but it's clear it doesn't converge uniformly.

Why do we like uniform convergence?

Theorem I.1.3

Let $n \mapsto (f_n : E \to \mathbb{C})$ be a sequence of functions converging uniformly to $f : E \to \mathbb{C}$. If all the f_n are continuous on E, then f is continuous on E.

Theorem I.1.4

Let $\gamma \subseteq \mathbb{C}$ be a piecewise smooth curve in the plan. If $j \mapsto (f_j : E \to \mathbb{C})$ is a sequence of continuous functions on γ converging uniformly to $f : E \to \mathbb{C}$ then

$$\int_\gamma f_j(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z.$$

We will apply this whole discussion to series! We'll look at a series of functions $\sum g_j(x)$ and consider when the partial sums $s_n(x) = \sum_{j=0}^{n} g_j(x)$ converges uniformly to some limit function G.