Stuff:

- HW 2 (A due tomorrow, B due next week)
- HW 1B due tonight
- Bagels! WAlks!

Recall .0.1

The principal values of the argument are $\operatorname{Arg}(z) \in (-\pi, \pi]$ for $z \neq 0$.

The principal branch is $U \to \mathbb{C}$ for $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. We may also define the principal branch of the logarithm as $\operatorname{Log}(z) = \log|z| + i \operatorname{Arg}(z)$.

Back to calculus!

Remark .0.1

If f is complex differentiable at $z_0 \in U$, then it is continuous at z_0. Look at

$$f(z_0 + h) - f(z_0) = h\left(\frac{f(z_0+h) - f(z_0)}{h}\right)$$

and take limits as $h \to 0$, to see that $\lim_{h \to 0} f(z_0 + h) - f(z_0) = 0$.

Lets now consider what the Cauchy-Riemann equations imply about $|f'(z)|$ for some f holomorphic at z. Well, if $f = u + iv$ we have that

$$\begin{aligned} |f'(z)| &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}. \end{aligned}$$

This is the determinant of the Jacobian matrix of partials.

Remark .0.2

Gamelin's definition of holomorphic requires not only that $f'(z)$ exists, but also that $f'(z)$ is continuous at z. This is redundant! But it makes some things easier to phrase early on.

Eventually, we will show that the derivative of a holomorphic function at z is also holomorphic at z, which will give us lots of extra stuff.

This will later show that if $f = u + iv$ is holomoprhic, it will have continuous partial derivatives of every order and so the mixed partials will be equal! Taking partial derivatives of the left and right hand sides of the Cuachy-Riemann equations yields

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{-\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{-\partial^2 v}{\partial y \partial x}$$

A consequence if the mixed partials are equal is that

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial x^2}.$$

We see that

$$\Delta u \coloneqq \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\Delta v \coloneqq \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Definition .0.1

A smooth function $u : \mathbb{R}^2 \to \mathbb{R}$ that satisfies Laplace's equation

$$\Delta u \coloneqq \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is said to be harmonic

The real and imaginary parts of a holomorphic function are thus harmonic.

Definition .0.2

If two harmonic functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy Cauchy-Riemann equations, then v is said to be "the" harmonic conjugate of u (unique up to an additive constant).

Example .0.2

Let $u = x^2 - y^2$. This is harmonic. Can we build a harmonic conjugate? Well we know

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y.$$

We're led to consider $v = 2xy + \text{const}$.

Building the function $f = u + iv$ yields $f(z) = (x^2 - y^2) + i(2xy + \text{const}) = z^2 + i \cdot \text{const}$.

Formal + Helpful:

Consider $f(x, y) = u(x, y) + iv(x, y)$, $z = x + iy, \overline{z} = x - iy$. Then $x = \frac{1}{2}(z + \overline{z}), y = -\frac{i}{2}(z - \overline{z})$. We want to change variables from (x, y) to $(z, \overline{z})$. We define new operators

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

To say that f is holomorphic (aka u, v satisfy the Cauchy-Riemann equations) is exactly to say $\frac{\partial f}{\partial \overline{z}} = 0$, and this gives $\frac{\partial f}{\partial z} = f'$.

If f is holomorphic, then is $1/f$ holomorphic? Yes, provided that f is nonzero.

Rational functions! Let $R(z) = P(z)/Q(z)$ where P, Q are polynomials and P, Q have no common roots. The zeros of Q are called poles of R. We extend R to a function $\widehat{R} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by taking $\widehat{R}(z) = \infty$ for z a pole of R. We could also consider

$$\widehat{R}(\infty) = \lim_{z \to 0} R(z).$$

It is nicer to use a related function $R_1(z) = R(1/z)$, and define $\widehat{R}(\infty) = \widehat{R}_1(0)$. Note that $R(1/z)$ is a rational function

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_m z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

$$R_1(z) = z^{m-n}\left(\frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}\right).$$

If $m > n$, $R(z)$ has a zero of order $m-n$ at ∞, define $\widehat{R}(\infty) = 0$. If $m < n$, the point at infinity is a pole of order $n-m$ so $\widehat{R}(\infty) = \infty$. If $m = n$, then $\widehat{R}(\infty) = \frac{a_n}{b_m} \neq 0, \infty$.

Consider $R(z) = \frac{z^2+57i}{z-53}$. The zeros of $\widehat{R}$ are $\pm\sqrt{-57i}$, and the poles are $z = 53, \infty$.

Fact: The total number of zeros of a rational function is equal to $\max(n, m)$ which is also equal to the number of poles wher we count with multiplicity. Find it in your book!

Definition .0.3

The degree of a rational function $R(z) = P(z)/Q(z)$ is $\max(\deg P, \deg Q)$.

This will agree with the topological degree, which you might know about.

Definition .0.4 (Möius tranformations)

A Möbius transformation is a rational function of degree 1.

Möbius transformations are in fact the automorphisms (bijective, holomorphic, with holomorphic inverse) of $\widehat{\mathbb{C}}$. To think about defining whether a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic at ∞, consider testing if $f(1/z) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic at 0.

Example .0.3

When if $f(z) = \frac{az+b}{cz+d}$ a Möbius tranformation? Maybe we should think about if

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0.$$

We say a Möbius transformation g is affine provided that $g(\infty) = \infty$, and we can then express $g(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$. The affine group is then

$$\{z \mapsto \alpha z + \beta \mid \alpha \neq 0, \beta \in \mathbb{C}\} = \mathrm{Aut}(\mathbb{C}).$$