Stuff:

- HW 3B due Tuesday
- Due dates for 4A/4B to be decided
- There exists Math Club (Constructing $\mathbb{R}$!)
- There exists Math S^1 (Thursdays 6:30pm-8pm, starts next week 9/22)
- There is a 40 mile walk on Saturday October 1st!
- There is Super Saturdays (starts 10/8, 9:30am-12pm)
- Math Mental Health Hour Sunday (2-3pm), EH 1866
- Bagels on Sunday 10am-11:30am
- U(M) Undergrad Mathematics Seminar EH 3096

Recall .0.1

If $f = u + iv$, we know if $f : U \to \mathbb{C}$ is holomorphic, then u is harmonic on U, that is $\Delta u = 0$ and u has continuous first and second order partials.

Faye's question: Given $u : U \to \mathbb{R}^2$ harmonic, does there exist a harmonic conjugate?

No! Take $u(z) = \operatorname{Log}|z|$, which is harmonic on $\mathbb{C} \setminus \{0\}$. This does not have a harmonic conjugate on $\mathbb{C} \setminus \{0\}$, but does have a harmonic conjugate on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, namely $\operatorname{Arg}(z)$. Then $\operatorname{Log}(z) = \log|z| + i \operatorname{Arg}(z)$ is harmonic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

What's the difference in domains? $\mathbb{C} \setminus \{0\}$ is *not* simply connected, while $\mathbb{C} \setminus (-\infty, 0]$ is simply connected.

Proposition .0.1

If $u : U \to \mathbb{R}$ is harmonic on U and U is simply connected, then a harmonic conjugate exists.

For Gamelin, he constructs a harmonic conjugate p57 on rectangles (and will eventually do star-shaped regions).

Back to conformal maps: We saw that if $f'(z_0) \neq 0$, then f maps orthogonal curves then z_0 to orthogonal curves at $f(z_0)$.

Definition .0.1

The map $f : U \to V$ is conformal on U provided that

(1) f is conformal at all points $z_0 \in U$.

(2) f is bijective.

Example .0.2

$\exp : \mathbb{C} \to \mathbb{C}$ satisfies (1) but not (2). We can also consider $z \mapsto z^2$ as a map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$.

I. Complex Integration!

Chapter 3/III in Gamelin.

I.1. Review of prerequisites

Definition I.1.1

A path in the plane is a continuous function $\gamma : [a, b] \to \mathbb{C}$, and we say it is a path from $\gamma(a)$ to $\gamma(b)$.

A path γ is simple provided that $\gamma\big|_{[a,b)}$ is injective. The path γ is closed provided that $\gamma(a) = \gamma(b)$.

All paths γ have an orientation, $\gamma(a)$ is the initial point and $\gamma(b)$ is the end point.

A path is called smooth if it is smooth as a function.

If we have paths $\gamma : [0,1] \to \mathbb{C}$ from $A \in \mathbb{C}$ to $B \in \mathbb{C}$ and $\delta : [0,1] \to \mathbb{C}$ from B to $C \in \mathbb{C}$, we can construct a path

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le 1/2 \\ \delta(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

from A to C. This is called the concatenation.

A piecewise smooth path is a concatenation of smooth paths. A curve is a smooth path or piecewise smooth.

Let γ be a path in $\mathbb{C}$ from A to B and let $P(x,y), Q(x,y)$ be continuous complex-valued functions on the image of γ. Break up the image of γ into pieces (x_i, y_i) and form the sum

$$\sum P(x_j, y_j)(x_{j+1} - x_j) + \sum Q(x_j, y_j)(y_{j+1} - y_j).$$

where we require $\gamma(t_j) = (x_j, y_j)$ where $a = t_0 < t_1 < \cdots < t_n = b$.

Definition I.1.2

If these sums have a limit as distance between points $(x_j, y_j) \to 0$ then we define the limit to be the line integral of $P\,\mathrm{d}x + Q\,\mathrm{d}y$ along γ, denoted

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

More precisely, let $\gamma(t) = (x(t), y(t))$ with $a \le t \le b$. Suppose $t_j \in [a,b]$ satisfies $\gamma(t_j) = (x_j, y_j)$ with $a \le t_0 < t_1 < \cdots < t_n = b$.

Apply the Mean Value Theorem to find points $t_j^* \in [t_j, t_{j+1}]$ so that $x(t_{j+1}) - x(t_j) = x'(t_j^*)(t_{j+1} - t_j)$. Likewise for y. Plugging into the above sums this gives

$$\begin{aligned}&\sum P(x(t_j), y(t_j))x'(t_j^*)(t_{j+1} - t_j) + \sum Q(x(t_j), y(t_j))y'(t_j^*)(t_{j+1} - t_j) \\ &= \sum (P(x(t_j), y(t_j))x'(t_j^*) + Q(x(t_j), y(t_j))y'(t_j^*))(t_{j+1} - t_j).\end{aligned}$$

As $t_{j+1} - t_j$ go to zero we have this is equal to

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_a^b P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\,\mathrm{d}t.$$

Theorem I.1.1 (Green's Theorem)

Consider some region $\Omega \subseteq \mathbb{C}$ which is a connected bounded open set whose boundary consists of a finite # of disjoint piecewise smooth curves.

Let P, Q be continuously differentiable on $\Omega \cup \partial\Omega$, then

$$\int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

Definition I.1.3

If $h(x,y)$ is a continuously differentiable $\mathbb{C}$-valued function, we define its differential as

$$\mathrm{d}h = \frac{\partial h}{\partial x}\,\mathrm{d}x + \frac{\partial h}{\partial y}\,\mathrm{d}y.$$

The differential $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is called exact if $P\,\mathrm{d}x + Q\,\mathrm{d}y = \mathrm{d}h$ for some h.

Theorem I.1.2 (FTC for Line Integrals)

If γ is a piecewise smooth curve from A to B and if $h(x,y)$ is continuously differentiable on γ then

$$\int_\gamma \mathrm{d}h = h(B) - h(A).$$

Proof. Chain rule! See Gamelin!

Example I.1.1

Exact differentials are nice and very easy to integrate♡

Natural question: which differentials $P\,\mathrm{d}x + Q\,\mathrm{d}y$ are exact? hmmm...

Definition I.1.4

As before, let P, Q be complex-valued and continuously differentiable on $U \subseteq \mathbb{C}$, the integral

$$\int P\,\mathrm{d}x + Q\,\mathrm{d}y$$

is said to be path independent provided that for any paths $A, B \in U$, and for any pair of paths γ, δ from A to B we have

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_\delta P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Note: This is equivalent to the statement that given any simple closed curve in U, call it $\mu \subseteq D$, we have

$$\int_\mu P\,\mathrm{d}x + Q\,\mathrm{d}y = 0.$$

Lemma I.1.3

Let $P, Q : U \to \mathbb{C}$ be continuous. Then $\int P\,\mathrm{d}x + Q\,\mathrm{d}y$ is independent of path if and only if $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact.

Proof. The converse follows from Theorem I.1.2. For the forward direction, fix a basepoint $z_0 \in U$, then define $h(z) = \int_{z_0}^{z} P\,\mathrm{d}x + Q\,\mathrm{d}y$.

It does not take much effort to show $\mathrm{d}h = P\,\mathrm{d}x + Q\,\mathrm{d}y$.

Definition I.1.5

Let P, Q be continuously differentiable on a connected open set U. The differential $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is closed on U provided that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

that is any Green's theorem type integral will be zero.

If $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is a closed form, then

$$\int_{\partial U} P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$$

for any bounded connected open set $U \subseteq \mathbb{C}$. We have

$$\text{independence of path} \iff \text{exact} \iff \text{closed}$$

Theorem I.1.4

If U is simply connected (often we will use star-shaped spaces, which are simply connected) then closed implies exact.