Stuff:

- HW 6B due this evening!
- HW 7 is due the 20th, some Gamelin and Qual Exam problems.
- Midterm Tuesday October 25th.

Midterm Information

- Admin:
  - In class
  - Bring a sheet of notes
  - Solo Exam
- Topics:
  - Gamelin, everything we have covered through + including stuff in Section V [Gam03]
  - Likely problems from Gamelin will be stolen...

Definition .0.1

A function $f(z)$ is analytic on $|z - z_0| < r$ provided that there exists a sequence $k \mapsto a_k$ such that $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$. for all z satisfying $|z - z_0| < r$.

A function $f(z)$ is called analytic at z_0 provided there is a neighborhood of f on which f is analytic (i.e, agrees with a power series).

A function f is holomorphic on $|z - z_0| < r$ if and only if f is analytic on $|z - z_0| < r$.

Example .0.1

Let $z \mapsto \exp(z)$. This is entire, and so it has a power series about 0 which converges everywhere. In fact

$$\exp(z) = \frac{z^k}{k!}$$

Corollary .0.1 (of **??**)

Syppose $f(z)$ and $g(z)$ are analytic for $|z - z_0| < r$. If $g^{(k)}(z_0) = f^{(k)}(z_0)$ for all $k \geq 0$, then $g = f$ on $|z - z_0| < r$.

The above is known as a rigidity theorem. The tagline is that

"Analytic functions are extremely rigid"

Corollary .0.2 (p146, [Gam03])

Suppose $f(z)$ is analytic at z_0 with power series expansion $f(z) = \sum a_k(z - z_0)^k$. Then the radius of convergence of the power series is the largest R such that $f(z)$ extends to be analytic on the disk $\{|z - z_0| < R\}$.

That is there is an analytic g on $|z - z_0| < R$ given by the power series which agrees with f on the restriction to any disk about z_0 where both f and g are defined.

Tagline: "Radius of convergence is the distance from z_0 to the nearest singularity."

Warning: f may be defined on a larger domain and *disagree* with the extension g at some points.

Example .0.2

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto \frac{1}{1+x^2}$. Is this $\mathbb{R}$-analytic at $x_0 = 0$. Yes! The series is given by $\sum(-x^2)^k$ whenever $\left|-x^2\right| < 1$, that is $|x| < 1$. There are *different* power series expansions about say 2.

Why do you hit an obstruction at ± 1??? **YOU DON'T.** You are actually crashing into $\pm i$ for the complex function $F : \mathbb{C} \setminus \{\pm i\} \to \mathbb{C}$ given by $z \mapsto \frac{1}{1+z^2}$.

The complex numbers are showing us things that the real numbers cannot show us!!!

There is a definition of power series at ∞ given in Gamelin. Recall that $f(z)$ is holomorphic at $z = \infty$ means that $g(w) = f\left(\frac{1}{w}\right)$ is holomoprhic at $w = 0$.

Section V.6 of [Gam03] is about algebraic manipulation of series and details of this.

.1. Zeros of Analytic Functions

Definition .1.1

We say that $f(z)$ has a zero of order N at z_0 provided that

$$f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$$

and $f^{(N)}(z_0) \neq 0$. This happens if and only if we can write $f(z) = (z - z_0)^N h(z)$ for some holomorphic function $h(z)$ and $h(z_0) = \frac{f^{(N)}(z_0)}{N!} \neq 0$. Note h is only defined about some disk about z_0.

This all happens if and only if the power series expansion has the form

$$f(z) = a_N(z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + \cdots$$

where $a_N \neq 0$.

Definition .1.2

Let $E \subseteq \mathbb{C}$. We say that a point $x \in \mathbb{C}$ is an accumulation point of $\mathbb{C}$ provided that for all open $U \subseteq \mathbb{C}$ which contain x, we have $E \cap (U \setminus \{x\}) \neq \emptyset$.

To say E is isolated is to say that it has no accumulation points.

Restricting our attention to points in E, we have

$$E = \text{accumulation points of } E \text{ in } E \sqcup \text{ isolated points in } E.$$

For a closed set S, to say S is isolated is equivalent to saying that for any $s \in S$ there is an open ball $B(s, \varepsilon)$ so that $B(s, \varepsilon)$ contains no points of S except for s.

I.e. it is equivalent to saying S consists only of isolated points.

Theorem .1.1

Let $D \subseteq \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be analytic. Now suppose that f is not identically zero. Then the zeros of f are isolated, that is they have no accumulation points.

That is given any zero z_0 of f, we can find a neighborhood of z_0 that contains no other zero of f.

Proof. Note that any accumulation point would necessarily be a zero of f by continuity of f. We start by showing that

Claim

If z_0 is a zero of f, then it has finite order (as a zero).

Let $U = \{z \in D \mid f^{(m)}(z) = 0 \text{ for all } m\}$. If $z_0 \in U$, then the power series $\sum a_k(z - z_0)^k$ has zero coeffficients, and is equal to f on a disk centered at z_0. Thus U is open. To see that U is closed, note that

$$U = \bigcap_m (f^{(m)})^{-1}(\{0\}),$$

and all the $f^{(m)}$ are continuous, so U is closed. Thus either $U = \emptyset$ or $U = D$ by the connectedness of D. if $U = D$ then f is identically zero, so $U = \emptyset$.

Now suppose $z_0 \in D$ is a zero of f, necessarily of finite order N. We may write $f(z) = (z - z_0)^N h(z)$ for $h(z)$ analytic at z_0 and $h(z_0) \neq 0$.

For ρ sufficiently small we have $h(z) \neq 0$ for all $|z - z_0| < \rho$. Therefore $f(z)$ cannot be zero for those z satisfying $0 < |z - z_0| < \rho$! Thus the zeros at z_0 are separated from other zeros of f.

Theorem .1.2 (Uniqueness Principle)

If $f(z)$ and $g(z)$ are analytic on a connected open set $D \subseteq \mathbb{C}$, and if $f(z) = g(z)$ on a set that has an accumulation point lying in D, then $f(z) = g(z)$ everywhere.

Proof. Apply the above!