On the suggestion of Sarah. Here is a discord link for everyone!

https://discord.gg/SFc3QmXMhm

We now move to proving the Casorati-Weierstrass Theorem from last time

Proof of **??**. The converse is immediate from our characterization of poles/removable singularities.

We prove the forward direction via contrapositive. Suppose $w_0 \in \mathbb{C}$ is not the limit of such a sequence $f(z_n)$ (where $z_n \to z_0$). Then the image of f avoids some neighborhood of w_0 when z is near z_0.

In other words, there exists an $\varepsilon > 0$ such that $|f(z) - w_0| > \varepsilon$ for all z near z_0. We may define

$$h(z) = \frac{1}{f(z) - w_0}.$$

This is bounded near z_0 and analytic on a punctured disk around z_0, and so by Riemann's theorem, $h(z)$ can be extended analytically near z_0. We may then write

$$h(z) = (z - z_0)^N g(z)$$

for some $N \geq 0$, and some analytic $g(z)$ with $g(z_0) \neq 0$.

This immediately implies that

$$f(z) - w_0 = (z - z_0)^{-N} \cdot \frac{1}{g(z)},$$

with $\frac{1}{g(z)}$ analytic on a disk around $z = z_0$. If $N = 0$, then $f(z) = w_0 + \frac{1}{g(z)}$ and z_0 is a removable singularity of f. If $N > 0$, then $f(z)$ has a pole of order N at z_0.

In either case, z_0 is NOT an essential singularity of $f(z)$.

Stay tuned for the Great Picard Theorem, to be proved later!!!

Theorem .0.1 (Great Picard Theorem)

If an analytic function $f(z)$ has an essential singularity at z_0, then on any punctured neighborhood of z_0, $f(z)$ takes on all possible complex values with at most one exception!

Example .0.1

$\exp(1/z)$, with its essential singularity at 0. The only point not hit on a neighborhood of 0 is 0 itself (since the exponential is always nonzero).

Why is it only one point? There is some sort of intuition that Sarah has about $\mathbb{C}, \mathbb{C} \setminus \{a\}$ both being Euclidean, whereas $\mathbb{C} \setminus \{a, b\}$ (or more) is hyperbolic... hmmmmm

.1. Singularities at ∞

We want to define what it means for $f(z)$ to have isolated singularities at ∞. We analyze $f(1/w)$ at $w = 0$, and just look there. Compare with Gamelin discussion of analytic functions at ∞, namely [Gam03, V.5, p149]

.2. Partial Fractions decompositions

We say a function $f(z)$ is meromorphic on $D \subseteq \widehat{\mathbb{C}}$ provided that $f(z)$ is analytic on D except possibly at isolated singularities each of which is a pole.

Möbius transformations!!! We once made a claim that

$$\mathrm{Aut}(\widehat{\mathbb{C}}) = \text{Möb},$$

we proved $\supseteq$, but we have not shown $\subseteq$. Can we do it now?

Theorem .2.1

A meromorphic function on $\widehat{\mathbb{C}}$ must be a rational map.

Proof. See HW 10!

I. Residue Calculus

This section will give you the ability to evaluate real integrals by shifting to the complex plane. A cautionary quote from Ahlfors:

"Even complete mastery does not guarantee success" ☹

I.1. The Residue Theorem

Suppose $f(z)$ has an isolated singularity at z_0 and write $f(z)$ as a Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

which is valid for some annulus $0 < |z - z_0| < \rho$. Then we say that

$$a_n = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta$$

where $0 < r < \rho$. One of these is more special than the others! Namely

$$a_{-1} = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} f(\zeta) \, \mathrm{d}\zeta.$$

In fact, a_{-1} is special because it is an invariant of the one-form $f(\zeta)\,\mathrm{d}\zeta$. It does not change when we change coordinates!

Definition I.1.1

We define the residue of $f(z)$ at z_0 to be the coefficient of a_{-1} of $\frac{1}{z-z_0}$ in the Laurent series. We define notation for the residue as

$$\mathrm{Res}[f(z), z_0] \coloneqq a_{-1}.$$

Example I.1.1

We have

$$\mathrm{Res}\left[\frac{1}{z-57}, 57\right] = 1$$
$$\mathrm{Res}\left[\frac{1}{(z-53i)^2}, 53i\right] = 0$$
$$\mathrm{Res}\left[\frac{z^3+z+1}{z^2+1}, -i\right]?$$

Well, use partial fractions

$$f(z) = \frac{z^3 + z + 1}{z^2 + 1} = z - \frac{1}{2i} \cdot \frac{1}{z + i} + \frac{1}{2i} \cdot \frac{1}{z - i}.$$

Then

$$\operatorname{Res}\left[\frac{z^3 + z + 1}{z^2 + 1}, -i\right] = -\frac{1}{2i} = \frac{i}{2}.$$

Example I.1.2

Let's look at

$$\begin{aligned} f(z) = \frac{\sin(z)}{z^6} &= \frac{1}{z^6}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{7!}z^7 + \cdots\right) \\ &= \frac{1}{z^5} - \frac{1}{6}\frac{1}{z^3} + \frac{1}{120} \cdot \frac{1}{z} - \frac{z^2}{7!} + \cdots \end{aligned}$$

this has a pole at $z = 0$ of order 5, and

$$\operatorname{Res}[f(z), 0] = \frac{1}{120}.$$

Theorem I.1.1 (Residue Theorem)

Let $D \subseteq \mathbb{C}$ be bounded, open, and connected with ∂D being piecewise smooth. Now suppose that $f(z)$ is analytic on $D \cup \partial D$ except for a finite number of isolated singularities $z_1, z_2, \ldots, z_m$ in D. Then

$$\oint_{\partial D} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j].$$

Proof. Punch out tiny ε-disks about each singularity, and call the new region D_ε. By Cauchy's Theorem, we have that

$$\oint_{\partial D_\varepsilon} f(z)\,\mathrm{d}z = 0 <$$

since f is holomorphic here. But then

$$\oint_{\partial D_\varepsilon} f(z)\,\mathrm{d}z = \oint_{\partial D} f(z)\,\mathrm{d}z - \sum_{j=1}^{m} \oint_{|z - z_j| = \varepsilon} f(z)\,\mathrm{d}z.$$

The latter piece is equal to the residues as desired. The minus sign comes from an orientation flip.

Residue Rules/Recipes:

Rule 1: If $f(z)$ has a simple pole at z_0 then $\operatorname{Res}[f(z), z_0]$ then

$$\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} f(z)(z - z_0),$$

which we can derive from the Laurent expansion

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots.$$

Rule 2: If $f(z)$ has a double pole at z_0, then

$$\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{\mathrm{d}}{\mathrm{d}z}((z-z_0)^2 f(z)).$$

This can be seen since $(z-z_0)^2 f(z)$ has the form

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \cdots . \frac{\mathrm{d}}{\mathrm{d}z}((z-z_0)^2 f(z)) \quad = a_{-1} + 2a_0(z-z_0) + \cdots .$$

Rule 3: If $f(z)$ and $g(z)$ are analytic at z_0 and if $g(z)$ has a simple zero at z_0, then

$$\operatorname{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}.$$

Why? Well $f(z)/g(z)$ has at "worst" a simple pole at z_0, and then apply rule #1.

Rule 4: While this is just rule 3, Gamelin says it is so darn useful. If $g(z)$ is analytic and has a simple zero at z_0, then

$$\operatorname{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}.$$

Rule 1: We see that $\frac{1}{z^2+1}$ has a simple pole at i, so

$$\begin{aligned}\operatorname{Res}\left[\frac{1}{z^2+1}, i\right] &= \lim_{z \to i}(z-i)\frac{1}{z^2+1} \\ &= \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}.\end{aligned}$$

Rule 2: We see $\frac{1}{(z^3+1)z^2}$ has a double pole at 0, so

$$\begin{aligned}\operatorname{Res}\left[\frac{1}{(z^3+1)z^2}, 0\right] &= \lim_{z \to 0} \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{z^3+1} \\ &= \lim_{z \to 0}(z^3+1)^{-2}(-3z^2) = 0.\end{aligned}$$

Rule 3: Since $\sin(z)$ has a simple zeto at $z = \pi$ we see

$$\operatorname{Res}\left[\frac{e^z}{\sin z}, \pi\right] = \frac{e^\pi}{\cos \pi} = -e^\pi$$

Exercise I.1.3

Compute the residues of $f(z) = \frac{1}{z^n+1}$ at its poles. Recall that it has poles at the $2n$-th roots of unity which are not also n-th roots of unity since $z^{2n} - 1 = (z^n+1)(z^n-1)$.

I.2. Integrals of Rational Functions

We want to compute something like $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$. Consider $f(z) = \frac{1}{1+z^2}$. Consider a contour ∂D_R consisting of a semi-circle Γ_R from R to $-R$ of radius R about 0 and a line segment $[-R, R]$ (with the counterclockwise orientation). This encloses a region D_R which contains i if $R > 1$, so

$$\begin{aligned}\oint_{\partial D_R} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}[f(z), i] &= 2\pi i \lim_{z \to i} \frac{z-i}{z^2+1} \\ &= 2\pi i \lim_{z \to i} \frac{1}{z+i} = \pi.\end{aligned}$$

We know that

$$\pi = \oint_{\partial D_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^2} + \int_{\Gamma_R} \frac{\mathrm{d}z}{1+z^2}.$$

We claim that as $R \to \infty$ that $\int_{\Gamma_R} \frac{\mathrm{d}z}{1+z^2} \to 0$. This comes from the ML-estimate, if $z \in \Gamma_R$ then for $R > 1$,

$$\left|\frac{1}{1+z^2}\right| \leq \frac{1}{R^2-1}.$$

Then

$$\left|\int_{\Gamma_R} \frac{\mathrm{d}z}{1+z^2}\right| \leq \frac{1}{R^2-1} 2\pi R,$$

which goes to 0 as $R \to \infty$. This tells us that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^2} = \pi.$$