Stuff:

- Math Club Today 4-5pm, random graphs social networks and the internet.
- Math S^1 6:30-8pm
- Super Saturdays!
- Popcorn Thursday!
- Career Fair tomorrow!
- Mass undergrad peer advising 7-9:30pm atrium Monday!

Last time, we had a lemma

Lemma .0.1

Suppose $D \subseteq \mathbb{C}$ is an open connected set, and $f : D \to f(D) \subseteq \mathbb{C}$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in D$.

Thus the inverse is holomorphic on $f(D)$!

Proof. Compare this with the proof that nonconstant holomorphic maps are open. Suppose $f'(z_0) = 0$ for some $z_0 \in D$. Then near z_0 we have

$$f(z) - f(z_0) = (z - z_0)^n g(z)$$

where $n \geq 2$, $g(z)$ is holomorphic at z_0, and $g(z_0) \neq 0$. Then we can find some analytic $h(z)$ near z_0 such that $g(z) = (h(z))^n$. Then

$$f(z) - f(z_0) = ((z - z_0)h(z))^n.$$

We know $f(z) - f(z_0)$ will map a small open set around z_0 to a small open set about 0 injectively.

We know $((z - z_0) \cdot h(z))^n$ is not injective because $(z - z_0)h(z)$ maps to a small open set about 0 and $n \geq 2$. Thus these can't be equal!

Last time! Logarithmic Integrals! We were interested in the Argument Principle, **??**. We restate it fully here for convenience

Theorem .0.2

Suppose $D \subseteq \mathbb{C}$ is bounded, connected, open, and with piecewise smooth boundary and let $f(z)$ be meromorphic on D that extends to be analytic on ∂D.

Suppose further that $f(z) \neq 0$ for all $z \in \partial D$. Let N_0 be the number of zeros of f in D, N_∞ be the number of poles in D counted with multiplicity. Then

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'}{f} \, \mathrm{d}z = N_0 - N_\infty.$$

We got the Argument Principle, which is that the increase in the argument of $f(z)$ around the boundary of D is

$$\int_{\partial D} \mathrm{d} \arg(f(z)) = 2\pi(N_0 - N_\infty).$$

.1. Rouché's Theorem

Theorem .1.1

Let $D \subseteq \mathbb{C}$ be connected, open, and bounded with piecewise smooth boundary. Let $f(z)$ and $h(z)$ be anlytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for all $z \in \partial D$ then $f(z)$ and $f(z) + h(z)$ have the same number of zeros in D counting with multiplicity.

Example .1.1

An example from Kurt McMullin. Question: Where in $\mathbb{C}$ are the zeros of $p(z) = z^5 + 14z + 1$? Let $f(z) = z^5, h(z) = 14z + 1$. We need to find a region D where for $z \in \partial D$ we have

$$|h(z)| < |f(z)|,$$

Lets try D as the ball of radius 2. Then when $|z| = 2$ we have

$$|f(z)| = |2|^5 = 32 > 29 \geq |14z + 1| = |h(z)|.$$

Now Rouché tells us that $f(z)$ and $p(z) = f(z) + h(z)$ have the same number of zeros in D. Since $f(z) = z^5$ has five zeros in D, this shows $p(z)$ attains all of its zeros in D.

Can we make the answer more precise? Now consider $|z| = 3/2$, and break up $p(z)$ as $h(z) = z^5 + 1, f(z) = 14z$. Then we have

$$|h(z)| \leq \left(\frac{3}{2}\right)^5 + 1 < 9 < |14z|$$

when $|z| = 3/2$. Then since $f(z)$ has one zero inside the disk of radius $3/2$, so does $p(z)$.

Thus $p(z)$ has one zero in $|z| < 3/2$ and 4 zeros in the annulus $3/2 < |z| < 2$.

Proof. We know since $|h(z)| < |f(z)|$ for all $z \in \partial D$, then this implies $f(z) \neq 0$ and $f(z) + h(z) \neq 0$ (reverse triangle inequality) for all $z \in \partial D$.

This sets us up to consider their arguments! We can rewrite

$$f(z) + h(z) = f(z)\left[1 + \frac{h(z)}{f(z)}\right].$$

We then know that

$$\arg(f(z) + h(z)) = \arg(f(z)) + \arg\left(1 + \frac{h(z)}{f(z)}\right).$$

Since $\frac{|h(z)|}{|f(z)|} < 1$ on ∂D. Then the values $w = 1 + \frac{h(z)}{f(z)}$ lie in a disk of radius 1 about 1, so $\mathrm{Re}(w) > 0$.

We can then use the argument principle. How does the argument of $w = 1 + \frac{h(z)}{f(z)}$ change as z moves around in closed loops? It can't!!! The outputs lie in the right half-plane so $\mathrm{d}\arg w$ is exact! In other words we can't wrap around 0 to pick up a change in argument.

Thus we have

$$\oint_{\partial D} \mathrm{d}\arg(f(z) + h(z)) = \oint_{\partial D} \mathrm{d}\arg f(z) + \mathrm{d}\arg(w) = \oint_{\partial D} \mathrm{d}\arg f(z).$$

The result then follows from the argument principle.

It is clear from the proof that we can extend to f, h meromorphic on D, analytic on $D \cup \partial D$, and then the number of poles/zeros in D.

Corollary .1.2

The Fundamental Theorem of Algebra. Find a large enough disk so that the leading term dominates, just as in the example.

.2. Hurwitz's Theorem

Recall HW7 #3, which will show up on HW 11. We talked about $n \mapsto (f_n : D \to \mathbb{C})$ a sequence of functions converging to $f : D \to \mathbb{C}$. What can we say about how the zeros of f_n compare to zeros of f?

Theorem .2.1 (Hurwitz's Theorem)

Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a connected open set D. Suppose $\{f_k(z)\}$ converges normally (on compact subsets/locally uniformly) to $f : D \to \mathbb{C}$. Further f has a zero of order N at z_0.

Then there exists a small $\rho > 0$ such that for k large, $f_k(z)$ has exactly N zeros on $\{|z - z_0| < \rho\}$, counting with multiplicity. And these zeros converge to z_0 as $k \to \infty$.

Proof. The hypothesis implies that f is not identically zero. So take $\rho > 0$ so that $\{|z - z_0| \leq \rho\} \subseteq D$ and $f(z) \neq 0$ for all z on the punctured disk $\{0 < |z - z_0| \leq \rho\}$.

Now choose $\delta > 0$ so that $|f(z)| \geq \delta$ for all z on the boundary circle $|z - z_0| = \rho$. Since $\{f_k\}$ converges uniformly to f on our closed sets, we know there exists an M so that for all $k \geq M$ we have $|f_k(z)| > \frac{\delta}{2}$ for all z on $|z - z_0| = \rho$.

Furthermore, the sequence of functions $\frac{(f_k)'(z)}{f_k(z)}$ converges uniformly to $\frac{f'(z)}{f(z)}$ on the boundary circle $|z - z_0| = \rho$... so... apply the logarithmic integrals!

$$\lim_{k \to \infty} \frac{1}{2\pi i} \oint_{|z - z_0| = \rho} \frac{(f_k)'(z)}{f_k(z)} \, dz = \frac{1}{2\pi i} \oint_{|z - z_0| = \rho} \frac{f'(z)}{f(z)} \, dz.$$

The left hand side counts the number of zeros of f_k inside $|z - z_0| < \rho$, which we'll call N_k. The right hand side is equal to N, since f is nonzero on $0 < |z - z_0| < \rho$, and has a zero of order N at z_0.

Since these are integers, they are discrete, so for large enough k, we have $N_k = N$! This is exactly eh first part of the result. What about the second? Play the same game with a smaller ρ, shrinking ρ to zero and running the argument again.

Definition .2.1

We say f is univalent on a domain $D \subseteq \mathbb{C}$ provided that it is analytic and injective on D.

Theorem .2.2 (Another version of Hurwitz)

Suppose $\{f_k(z)\}$ is a sequence of univalent functions on a connected open $D \subseteq \mathbb{C}$ that converge normally to $f : D \to \mathbb{C}$. Then $f(z)$ is either univalent OR $f(z)$ is constant.

Example .2.1

Consider $f_k(z) = \frac{z}{k}$ converging to the zero function.

Proof. See Gamelin.

.3. Winding Numbers

Definition .3.1 (Winding Number)

Let γ be a piecewise smooth path in $\mathbb{C}$. For $z_0 \notin \gamma$ define the winding number as

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}. = \frac{1}{2\pi} \int_\gamma \mathrm{d} \arg(z - z_0).$$

Note: $W(\gamma, z_0)$ depends analytically on z_0. For $\mathbb{C} \setminus \{\gamma\} \to \mathbb{Z}$ given by $z_0 \mapsto W(\gamma, z_0)$. Thus $W(\gamma, z_0)$ is constant on connected components of $\mathbb{C} \setminus \{\gamma\}$.

Proposition .3.1

Gamelin p254, then let $D \subseteq \mathbb{C}$ be open, connected. Then the following are equivalent

(1) D is simply connected.

(2) Every closed differential form is exact on D.

(3) For each $z_0 \in \mathbb{C} \setminus D$, there exists an anlytic branch of $\operatorname{Log}(z - z_0)$ defined on D.

(4) Each closed curve $\gamma \in D$ has winding number $W(\gamma, z_0)$ for $z_0 \in \mathbb{C} \setminus D$.

(5) The complement of D in $\widehat{\mathbb{C}}$ is connected.

(5) is easiest to check in practice. Proof is in the book.

I. Schwarz Lemma and Hyperbolic Geometry

The Schwarz Lemma is central to the theory of analytic maps between Riemann surfaces. We'll state it as a theorem because it is so important.

Theorem I.0.1 (Schwarz Lemma)

Let $f(z)$ be analytic on $|z| < 1$. Suppose $|f(z)| \leq 1$ for all $|z| < 1$. Suppose further that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $|z| < 1$.

Furthermore, if $|f(z_0)| = |z_0|$ for some point $z_0 \neq 0$, then $f(z) = \lambda z$ for some $\lambda = e^{i\theta}$.