.1. Examples of Functions

We take $\arg(z)$ to be multivalued, and $\operatorname{Arg}(z)$ is the principal values of $\arg(z)$ lying in $(-\pi, \pi]$.

We can consider the map $z \mapsto z^2$. If we take square root to be the one in the right half-plane then we have a discontinuity across the negative real axis.

Definition .1.1

The principal value of the square root function is $\omega \mapsto |\omega|^{1/2} e^{i \operatorname{Arg}(\omega)/2}$.

We also have polynomials $p(z) = a_d z^d + \cdots + a_1 z + a_0$, $a_i \in \mathbb{C}$. And furthermore we have the exponential $\exp : \mathbb{C} \to \mathbb{C}$ taking $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$. Modulus of e^z is $e^{\operatorname{Re}(z)}$ and argument is $\operatorname{Im}(z)$

The image of e^z is $\mathbb{C} \setminus \{0\}$ and it is not injective. Defining the logarithm will give multiple vlaues.

We can take $\log w = \log |w| + i \arg(w)$ for $w \neq 0$, which is multivalued. We can also take a principal branch $\operatorname{Log}(w) = \log |w| + i \operatorname{Arg}(w)$, which is an inverse of $\exp(z)$.

exp is periodic of period $2\pi i$

Definition .1.2

$f : U \to \mathbb{C}$ is periodic with period $\lambda \in \mathbb{C}$ if $f(z + \lambda) = f(z)$ for all $z \in U$.

Fact: $\exp(z + w) = \exp(z) \exp(w)$, and $\log(1 + i) = \log \sqrt{2} + (\pi/4 + 2k\pi)i$ for all $k \in \mathbb{Z}$.

We can also consider power functions. Fix $\alpha \in \mathbb{C}$, and define for $z \neq 0$

$$z^\alpha := \exp(\alpha \log(z))$$

this is multivalued unless $\alpha \in \mathbb{Z}$.

$$i^i = e^{-\pi/2} e^{-2\pi k} \in \mathbb{R}$$

for $k \in \mathbb{Z}$

I. Complex Differentiation

Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$, we're going to define holomorphic functions (in Gamelin [Gam03], this is "analytic")

Definition I.0.1

The function f is holomorphic at $z_0 \in U$ provided that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, and in that case we call that limit the derivative $f'(z_0)$.

The function f is holomorphic on U provided that it's holomorphic at all points inside U.

If $C \subseteq U$ is closed, then we say $f : C \to \mathbb{C}$ is holomorphic on C provided that there is an open set containing C on which f is holomorphic.

f is said to be entire provided that f is holomorphic on all of $\mathbb{C}$.

Proposition I.0.1

If $f, g : U \to \mathbb{C}$ are holomorphic at some $z_0 \in U$ then

(1) $f+g$ is holomorphic, $(f+g)' = f' + g'$.
(2) fg is holomorphic, $(fg)' = f'g + fg'$.
(3) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$$
(4) If $f : \Omega \to U$ is holomorphic at z_0, $g : U \to \mathbb{C}$ is holomorphic at $f(z_0)$, $g \circ f$ is holomorphic at z_0 and
$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

Proof. Same as in $\mathbb{R}$! Just manipulating limits with each other.

Example I.0.1

Polynomials are entire! The proof is now easy. Constants and the identity map are both entire (exercise), and polynomials are sums/products of these.

Question: How can we tell if a function is holomorphic at a given point $z_0 \in U$? In the case of complex differenitation, the derivative is a complex number...

Consider $f : \mathbb{C} \to \mathbb{C}$, we can view this as a map $F : \mathbb{R}^2 \to \mathbb{R}^2$, and then the derivative of F is a linear transformation with standard basis on $\mathbb{R}^2$, namely
$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}.$$

Proposition I.0.2 (Cauchy-Riemann Equations)

Writing $f = u + iv$ which is holomorphic at some $z_0 = x_0 + iy_0$, we have
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
in other words
$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Proof. Consider the limit
$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$
Write $h = h_1 + ih_2$, and approach along real/imaginary axes
$$f'(z_0) = \lim_{h_1\to 0} \frac{f(z_0+h_1) - f(z_0)}{h_1} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$
Similarly
$$f'(z_0) = \lim_{h_2\to 0} \frac{f(z_0+ih_2) - f(z_0)}{ih_2} = -i\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$
Equating these gives the Cauchy-Riemann equations above.

Remark I.0.1

If f is complex differentiable at $z_0 \in U$, then it is continuous at z_0. Write

$$f(z_0 + h) - f(z_0) = h\left(\frac{f(z_0 + h) - f(z_0)}{h}\right)$$

and take the limit as $h \to 0$.

Note:

$$\begin{aligned}
|f'(z_0)|^2 &= \left|\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\right|^2 \\
&= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\
&= \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial y}
\end{aligned}$$

which is the determinant of the Jacobian when we view this as a map $\mathbb{R}^2 \to \mathbb{R}^2$.

Gamelin's definition requires that $f'(z_0)$ exists and also that $f'(z)$ is continuous at z_0. Later we will show that the derivative of a holomorphic function (at $z_)$) is also holomorphic at z_0, which will give us lots of extra stuff

If we assume this, then the functions u, v in $f = u + iv$ will have continuous partial derivatives of every order, and so the mixed partials will agree...this is useful to keep in mind.