Stuff:

- HW 11B due tonight, for #11 assume nonconstant.
- There are 2 drops in both the A and B series (so 2 drops *each*!).
- Happy Early Thanksgiving!

. Fun stuff today: applications of ideas we have seen.

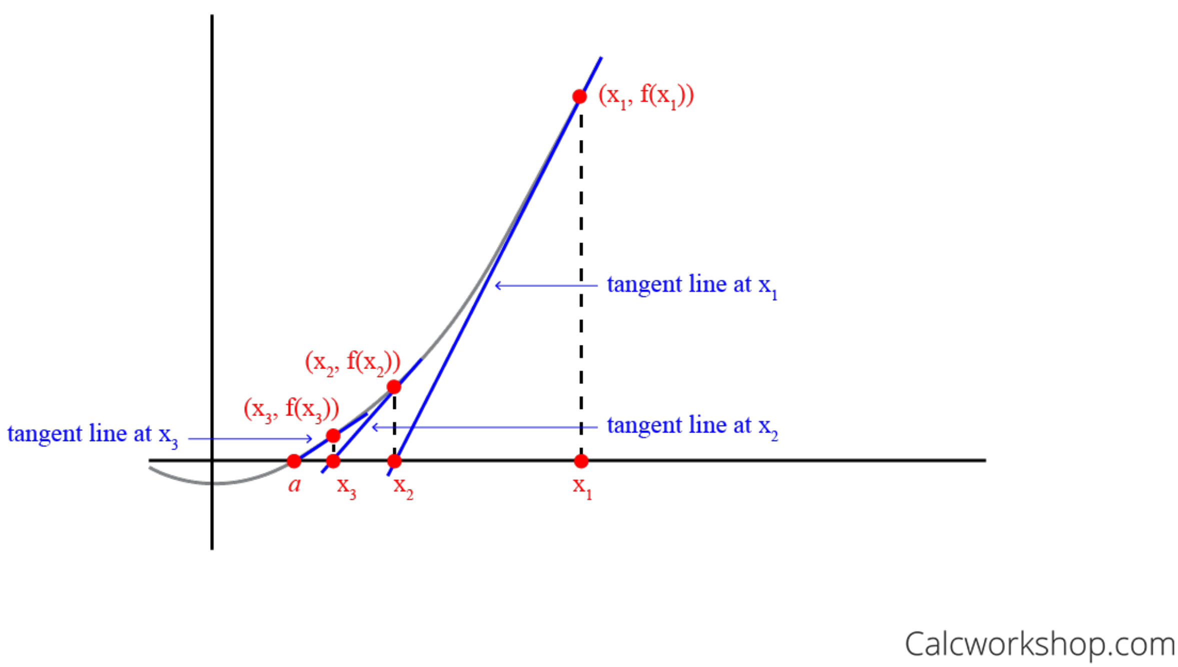
Today, we're going to talk about Newton's Method, which is a way of finding zeros of differentiable functions. Next week we will prove the Riemann Mapping Theorem, which says that if $D \subsetneq \mathbb{C}$ is open and simply connected, then D is conformally isomorphic to $\mathbb{D}$. THere is a constructive proof of this with an iteration scheme.

Thurston Idea: Set up iteration scheme such that the object you need is a fixed point of this scheme, and appeal to fixed point theorems.

The sketch of Newton's method in words:

- Start with an initial guess x_0.
- Draw the tangent line $\ell_1(x)$ to $(x_0, f(x_0))$, and solve for $\ell_1(x) = 0$ to get x_1.
- Lather rinse repeat.

Newton's Method in pictures



In formulas we have a map

$$N_f(x) = x - \frac{f(x)}{f'(x)}.$$

Then we let $x_{n+1} = N_f(x_n)$. Question: when does this converge and what to? Well a weird example, $f(x) = e^x$ is never zero and $N_f(x) = x - 1$. We can also have cycles

$$\begin{tikzcd} x_0 \arrow[r] & x_1 \arrow[r] & \cdots \arrow[r] & x_{157} \arrow[r, bend left] & x_{158} \arrow[l, bend left] \end{tikzcd}$$

, and bad things happen when $f'(x) = 0$.

Steven Smale and Curt McMullen have done incredible work with this. Question: Are there any generally convergent iterative algorithms for finding roots of polynomials? Generally convergent meaning the set of bad guesses is sparse (small measure)¿ For Newton's method there are big open sets of bad guesses. The answer is no, for degree ≥ 4. There are also theorems that give good algorithms for finding *good* guesses.

Cayley: work over $\mathbb{C}$. Look at $N_p(z) = z - \frac{p(z)}{p'(z)}$ again. Cayley wondered how the initial guess affects the convergence. For example, you can look at $p(z) = z^2 - 1$. There are two roots, color z_0 black if it goes to 1, and red if it goes to 0. If it does not converge, color it blue. What do the pictures look like in general?

Cayley figured it out for quadratics. . . without a computer!!! This was in 1879, and he published his results in a 1-page paper. We'll do this now, and assume monic. Let

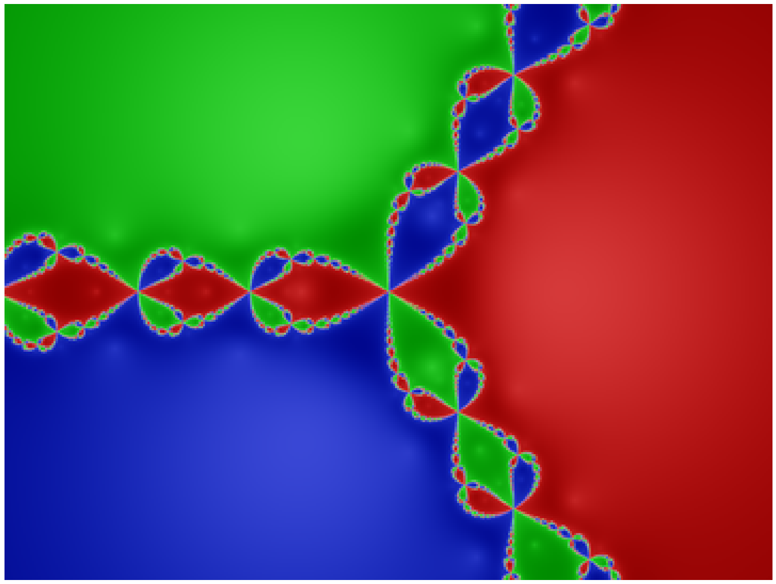
$$\begin{aligned} p(z) &= (z - r_1)(z - r_2) \\ N_p(z) &= z - \frac{(z - r_1)(z - r_2)}{(z - r_1) + (z - r_2)} \\ &= \frac{z^2 - r_1 z + z^2 - r_2 z - (z^2 - r_1 z - r_2 z + r_1 r_2)}{2z - r_1 - r_2} \\ &= \frac{z^2 - r_1 r_2}{2z - r_1 - r_2}. \end{aligned}$$

If we plug in $r_1$we get

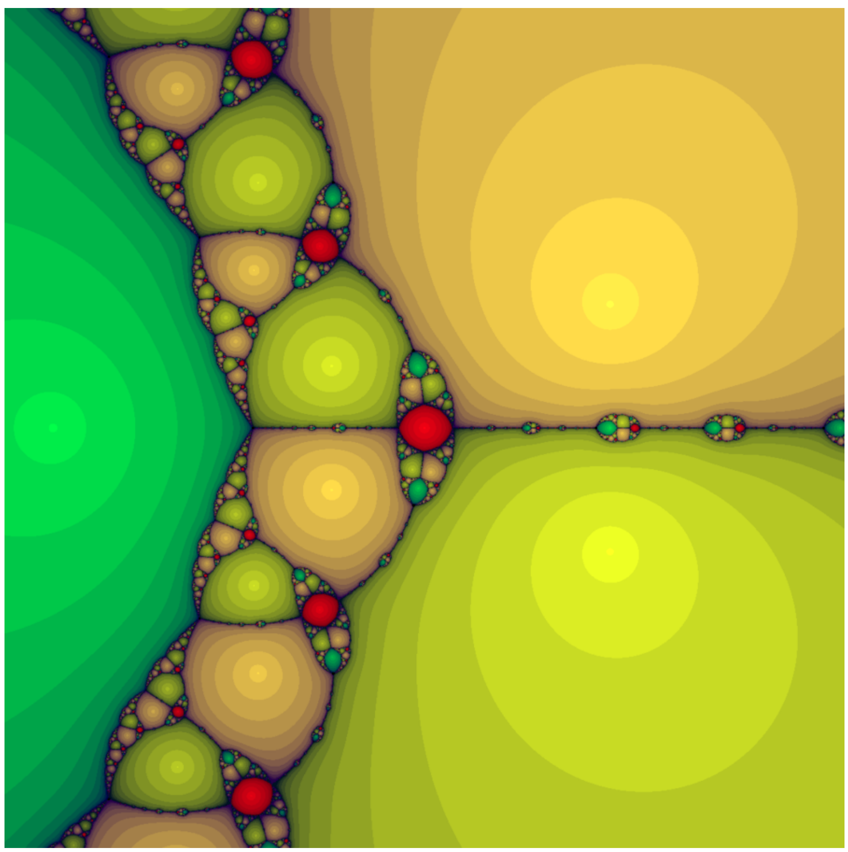
$$N_p(r_1) = \frac{r_1^2 - r_1 r_2}{r_1 - r_2} = r_1.$$

Then $N_p(z)$ has r_1, r_2 as fixed points. In fact they'll be superattracting fixed points. If r_1, r_2 are simple roots, then $N_p'(r_1), N_p'(r_2) = 0$. Locally, the Newton map must look like $z \mapsto z^2$. We see then a tiny neighborhood of r_1 maps to a smaller neighborhood of r_1. The colors correspond to splitting the plane in half based on which one its closest to. Take a line connected them and the perpendicular bisector (proximity based).

The bad initial guesses are along the perpendicular bisector, and stay on the line forever (very thin set). Everyone thinks that for three roots r_1, r_2, r_3 you get a pizza, a similar proximity based thing. It wasn't so. Let's look at a picture for $p(z) = z^3 - 1$, with roots at the roots of unity, then we get



For bad cubics we can get red basins where there is no convergence, say $p(z) = z^3 - 2z + 2$, we get a picture



This is actually how computers solve equations. There's a trick where you can go far out enough and use equidistant spacings to get good guesses (use "channels").

Exercise .0.1

Let $p(z)$ be a quadratic polynomial with Newton map $N(z)$. Prove that there exists a Möbius transformation μ so that $\mu \circ N \circ \mu^{-1}$ is the squaring function. This is just changing coordinates in a nice way.