Stuff:

- HW 12 due 12/8, 11:59pm.
- 4 Classes Left. Here's the Plan
- Today: Prove Riemann Mapping Theorem
- After Today: Finish some syllabus topics
- If we have time: Prime Number Theorem? Dynamics? Elliptic Functions?

Recall .0.1 (Schwarz Lemma)

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map and suppose $|f(z)| \leq 1$. If $f(z) = 0$, then

$$|f(z)| \leq |z|$$

for all $z \in \mathbb{D}$. Furthermore, if there exists a $z_0 \in \mathbb{D}$ for which $|f(z_0)| = |z_0|$, then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Infinitesimal version: Same hypotheses, $|f'(0)| \leq 1$, with equality if and only if $f(z) = e^{i\theta} z$.

Using these we characterized the automorphisms of $\mathbb{D}$ as

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a} z}$$

for $a \in \mathbb{D}$.

These together gave us Pick's lemma,

Recall .0.2 (Pick's Lemma and Hyperbolic Geometry)

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

for all $z \in \mathbb{D}$, with equality when $f \in \mathrm{Aut}(\mathbb{D})$. This told us that

$$\frac{|\mathrm{d}w|}{1 - |w|^2} = \frac{|\mathrm{d}z|}{1 - |z|^2}$$

for $w = f(z)$ where $f \in \mathrm{Aut}(\mathbb{D})$. This allowed us to define

$$\text{hyplength}_\gamma := 2 \int_\gamma \frac{|\mathrm{d}z|}{1 - |z|^2},$$

and taking the infimum along such paths gives a metric $\rho(z_0, z_1)$.

Theorem .0.1

For any two distinct points z_0, z_1 there exists a unique geodesic from z_0 to z_1 in the hyperbolic metric. Namely, it is the arc of the circle passing through z_0, z_1 which is perpendicular to $\partial\mathbb{D}$.

Proof. Let $w = f(z)$ be a conformal automorphism $\mathbb{D} \to \mathbb{D}$ with $f(z_0) = 0$. Rotate to move $f(z_1)$ to the positive real axis. Call that point $r = |f(z_1)| > 0$. We want the geodesic from 0 to r to be the line segment from 0 to r, as then we'll be done!

Now suppose $\gamma(t) = x(t) + iy(t)$ is a path in $\mathbb{D}$ connecting 0 to r, with $0 \le t \le 1$. Then $x(t)$ defines a path from 0 to r also, along the real axis. Set $\alpha(t) = x(t)$. We see that

$$\int_\alpha \frac{|\mathrm{d}z|}{1-|z|^2} = \int_0^1 \frac{|\mathrm{d}x(t)|}{1-x(t)^2} \le \int_0^1 \frac{|\mathrm{d}x(t)|}{1-|\gamma(t)|^2} \le \int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2}.$$

Thus if $y(t) \neq 0$ then for some $t \in [0,1]$, $|\gamma(t)| > |x(t)|$, and so the path $\alpha = x(t)$ is strictly shorter than $\gamma(t)$.

Moreover, if α backtracks, then α is decreasing on some interval, and we could make $\int_\alpha \frac{|\mathrm{d}z|}{1-|z|^2}$ even smaller, by removing these segments. Thus the integral is minimized uniquely when α is a straight line from 0 to r.

Recall .0.3 (Arzelà-Ascoli Theorem, the late 19th century.)

Let $E \subseteq \mathbb{C}$ be compact and let $\mathcal{F} = \{f_i : E \to \mathbb{C} \mid i \in I\}$ be a family of continuous functions on E which is uniformly bounded. Then the following are equivalent

(1) $\mathcal{F}$ is equicontinuous at all $z_0 \in E$.

(2) Each sequence of functions $f_n \in \mathcal{F}$ has a subsequence which converges uniformly on E. (NOTE: the limit of the subseqence may or may not belong to $\mathcal{F}$).

Gamelin uses this on p307 to prove Montel's theorem. Some mathematical genealogy

$$\text{Borel} \longrightarrow \text{Montel} \longrightarrow \text{Henri Cartan} \longrightarrow \text{Douady} \longrightarrow \text{Hubbard} \longrightarrow \text{Sarah}$$

Theorem .0.2 (Montel's Theorem)

Suppose $\mathcal{F}$ is a family of analytic functions on a connected, nonempty, open $D \subseteq \mathbb{C}$ such that $\mathcal{F}$ is uniformly bounded on all compact subsets of D.

Then every sequence in $\mathcal{F}$ has a subsequence that converges normally on D; that is, it converges uniformly on compact subsets.

Proof. We want to apply Arzelà-Ascoli. We need to get that $\mathcal{F}$ is equicontinuous at all points of D.

Fix $z_0 \in D$. Since D is open there exists an $r > 0$ so that $\{|z - z_0| \le r\} \subseteq D$. By hypothesis, $\mathcal{F}$ is uniformly bounded on this closed disk by some M. The Cauchy estimate tells us that the derivatives of functions in $\mathcal{F}$ are uniformly bounded on a slighly smaller disk $\{|z - z_0| \le r - \varepsilon\} \subseteq D$. This gives us equicontinuity via the mean value theorem.

We now apply Arzelà-Ascoli with a diagonalization argument. We define

$$E_n \coloneqq \left\{ z \in D \mid |z| \le n \text{ and } \operatorname{dist}(z, \partial D) \ge \frac{1}{n} \right\}$$

The E_n are compact and increase to fill up D, that is $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup_n E_n = D$. Even better, any compact subset of D is contained in some E_n.

Let $\{f_m\}$ be a sequence of points (aka functions) in $\mathcal{F}$. By Arzelà-Ascoli there is a subsequence $f_{11}, f_{12}, f_{13}, \ldots$ that converges uniformly on E_1.

By Arzelà-Ascoli there is a subsequence of f_{1m} that converges unifromly on E_2 to get f_{2m}. Lather rinse repeat

Claim

The diagonal sequence f_{mm} is a subsequence of $\{f_n\}$ converging uniformly on each E_n, hence uniformly on each compact subset of D.

The Deepest Result in $\mathbb{C}$-analysis. The Riemann Mapping Theorem.

- Riemann, partial proof 1851. Details kept getting filled in
- Schwarz 1870
- Osgood 1900
- Koeloe 1907
- Carathéodory 1912
- Modern Proof: Riesz + Fejer 1923.

Theorem .0.3 (Riemann Mapping Theorem)

Every nonempty, connected, open, simply connected, proper subset $U \subseteq \mathbb{C}$ is conformally isomorphic to $\mathbb{D}$.

Proof. Idea: Solve an extremal problem. We'll take $U \subsetneq \mathbb{C}$, $p \in U$, and we'll build the conformal map $(U, p) \overset{f}{\mapsto} (\mathbb{D}, 0)$. This map is unique up to rotating the right hand side (since we insisted p maps to 0). f is called a Riemann map

Uniqueness is easy. Suppose $f, g : (U, p) \to (\mathbb{D}, 0)$ are both conformal maps. Then $g \circ f^{-1}$ is a conformal map $(\mathbb{D}, 0) \to (\mathbb{D}, 0)$, so it is a rotation, and so $g = \text{rot} \circ f$.

Existence is hard. We'll do it in 3 steps. The map we seek will be a solution to an extremal problem (Hubbard: "This is fantastically nonconstructive").

Plan: Find holomorphic injective maps $(U, p) \overset{f}{\mapsto} (\mathbb{D}, 0)$, want $|f'(p)|$ to be as large as possible among all such holomorphic injective maps $(U, p) \to (\mathbb{D}, 0)$. Secretly: There's something called the Kobayashi metric...google it.

Consider the family

$$\mathcal{F} := \{f : (U, p) \to (\mathbb{D}, 0) \mid f \text{ is holomorphic and injective}\}.$$

Let's go!

Step 1) If U were $\mathbb{C}$, then $\mathcal{F}$ would be empty by Liouville's theorem. But since $U \neq \mathbb{C}$, there is an $a \in \mathbb{C} \setminus U$, and since U is simply connected the function $z \mapsto z - a$ is nonvanishing and has a holomorphic square root defined on U.

Call this square root function h, with $(h(z))^2 = z - a$ for all $z \in U$. We claim h is injective, which follows since $z \mapsto z - a$ is injective. By the open mapping theorem, $h(U) \subseteq \mathbb{C}$ is open. Since $(h(z))^2$ is injective and $a \notin U$, $h(U), -h(U)$ are disjoint.

Take a ball $\overline{B_r(q)} \subseteq -h(U)$. The Möbius transformation $\varphi(z) = \frac{r}{z-q}$ maps $B_r(q)$ onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. So where does φ send $h(U)$?

It sends $h(U)$ into $\mathbb{D}$. Now take an automorphism $\psi \in \text{Aut}(\mathbb{D})$ taking $\varphi(h(p))$ to 0. Then $\psi \circ \varphi \circ h$ lies in $\mathcal{F}$.

Step 2) We'll show that if $f \in \mathcal{F}$ and if $f(U) \neq \mathbb{D}$ (i.e. f is not surjective), then there is a $g \in \mathcal{F}$ such that $|g'(p)| > |f'(p)|$. Why?

We'll use the "square root trick" due to Carathéodory and Koebe. We'll adopt some convenient notation

$$\varphi_a(z) = \frac{z-a}{1-\overline{a}z} \in \mathrm{Aut}(\mathbb{D})$$

sending a to 0. Moreover, $(\varphi_a)^{-1} = \varphi_{-a}$.

Suppose $f \in \mathcal{F}$ with $f(U) \neq \mathbb{D}$, and take $a \in \mathbb{D} \setminus f(U)$. Then look at $\varphi_a \circ f \mid U \to \mathbb{D}$ omits zero. Thus $\varphi_a \circ f : U \to \mathbb{C}$ admits a homolormphic square root $h : U \to \mathbb{C}$, let s be the squaring function. Then

$$s \circ h = \varphi_a \circ f$$

on U. We know h is injective and maps U into $\mathbb{D}$ (since f does). If $b = h(p)$ then the composition $g = \varphi_b \circ h$ belongs to $\mathcal{F}$, and

$$f = \varphi_{-a} \circ s \circ h = (\varphi_{-a} \circ s \circ \varphi_{-b}) \circ g.$$

But then $\psi \coloneqq \varphi_{-a} \circ s \circ \varphi_{-b}$ sends $(\mathbb{D}, 0) \to (\mathbb{D}, 0)$ and is NOT injective on $\mathbb{D}$ implying it's not a rotation (since it has a squaring part). Thus by the Schwarz Lemma, $|\psi'(0)| < 1$. But wait!

$$|f'(p)| = |\psi'(0)|\,|g'(p)| < |g'(p)|\,.$$

Step 3) Let's finish[1] the proof! We need to cook up an extremal map $f \in \mathcal{F}$ such that $|f'(p)|$ is as big as possible. Let

$$M \coloneqq \sup\{|f'(p)| \mid f \in \mathcal{F}\}$$

We'll note some facts about M

(a) $M > 0$ because $f \in \mathcal{F}$ are injective, so $|f'(p)| > 0$.

(b) $M < +\infty$. If $B_r(p) \subseteq U$, then $|f'(p)| \leq \frac{1}{r}$ by the Cauchy estimates since $|f(z)| < 1$ for all $z \in U$. Thus $M \leq \frac{1}{r}$.

Let $\{f_n\}$ be a sequence of functions in $\mathcal{F}$ such that $|f_n'(p)|$ converges to M. Since $\mathcal{F}$ is uniformly bounded (outputs to $\mathbb{D}$, so by the constant 1), Montel's theorem applies.

Thus there is a subsequence f_{n_k} which converges normally on U. Call this limit function f. We know f is an analytic function since the convergence is normal.

> **Claim**
>
> $f \in \mathcal{F}$.

We need to check a bunch of things

(a) $f(p) = 0$ since $f_{n_k}(p) = 0$ for all k.

(b) We know $f(U) \subseteq \overline{\mathbb{D}}$, by taking limits of the inequalities. If there was a $q \in U$ with $|f(q)| = 1$, then f is not open, so f must be constant. But wait! $|f'(p)| = M > 0$.

Thus $f(U) \subseteq \mathbb{D}$, so $f : (U, p) \to (\mathbb{D}, 0)$ is analytic.

[1] Technically this was done on December 1st, but I didn't want to split it up

(c) We must check f is injective. Well, all f_{n_k} are injective and f is nonconstant, so by some version of Hurwitz's theorem (see [**thm:hurwitz**])

Thus $f \in \mathcal{F}$. By Step 2, since $|f'(p)| = M$, f is surjective. Done! $f : (U, p) \to (\mathbb{D}, 0)$ is a conformal isomorphism.