Stuff:

- HW 6A due tonight (Problem 1 erased).
- HW 6B due Tuesday.
- Math Club
- Math S^1.
- Super Saturdays starts this weekend.
- Student Seminar Friday 4pm EH 3096: Circle Method and Applications to Partitions by Faye Jackson.
- HW Hint: (9) Gamelin p119 4. $f : \mathbb{C} \to \mathbb{C}$ is entire and there is an $R > 0$ so that $f(z)/z^n$ is bounded for all $|z| \geq R$. We must show $f(z)$ is a polynomial. What does it mean to show that f is a polynomial? Show there exists $N \in \mathbb{N}$ such that $f^{(m)}(z) = 0$ for all $m \geq N$. Fix $z \in R$ from hypothesis, choose $r > \max(R, |z|)$. . . .

Theorem .0.1 (Weierstrass M-test)

Suppose $M_k \geq 0$ and $\sum M_k$ converges. If $g_k(z)$ are $\mathbb{C}$-valued on some set $E \subseteq \mathbb{C}$ and $|g_k(z)| \leq M_k$ for all $z \in E$, then $\sum g_k(z)$ converges unifornly on E.

Proof. A Real Analysis Course. See p135 of [Gam03] if you would still like a proof.

Example .0.1

Consider $\sum\limits_{0}^{\infty} z^k \frac{1}{1-z}$ for $|z| < 1$. Is the convergence of the partial sums $S_N(z) = \frac{1-z^{N+1}}{1-z}$ uniform for $|z| < 1$?

No! However, if we take $|z| \leq r$ for $0 < r < 1$, then the convergence is uniform! Namely for $M_k = r^k$ we have $\sum M_k$ converges, so $\sum z^k$ converges uniformly by the Weierstrass M-test because $\left|z^k\right| \leq r^k$.

Theorem .0.2

If $\{f_k(z)\}$ is a sequence of holomorphic functions on a connected open $D \subseteq \mathbb{C}$ that converges uniformly to $f : D \to \mathbb{C}$, then f is holomorphic.

Proof. Consider any closed region R in D. Then

$$\int_{\partial R} f(z)\,\mathrm{d}z = \lim_{k\to\infty} \int_{\partial R} f_k(z)\,\mathrm{d}z = 0$$

Furthermore f is continuous since each f_k is continuous. Thus f is holomorphic by **??**.

Theorem .0.3

Suppose $\{f_k(z)\}$ is a sequence of functions holomorphic on $|z - z_0| < R$ and suppose f_k converges uniformly to f on $|z - z_0| < R$. Then for each fixed $0 < r < R$ and each fixed $m \geq 1$, the sequence of m-th derivatives $\{f_k^{(m)}(z)\}_k$ converges uniformly to $f^{(m)}(z)$ on $|z - z_0| \leq r$.

Proof. Suppose we have $\varepsilon_k \to 0$ as $k \to \infty$ is a sequence such that $|f_k(z) - f(z)| < \varepsilon_k$ for all $|z - z_0| < R$. Fix $m \geq 1$.

Fix r, s such that $0 < r < s < R$. We use the Cauchy integral formula to get

$$f_k^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|z-z_0|=s} \frac{f_k(\zeta) - f(\zeta)}{(\zeta - z)^{m+1}}\,\mathrm{d}\zeta$$

for $|z - z_0| \leq r < s$. By the triangle inequality we know $|\zeta - z| \geq s - r$. Thus

$$\left| \frac{f_k(\zeta) - f(\zeta)}{(\zeta - z)^{m+1}} \right| \leq \frac{\varepsilon_k}{(s-r)^{m+1}}.$$

The triangle inequality and the ML-estimate yields

$$\left| f_k^{(m)}(z) - f^{(m)}(z) \right| \leq \frac{m!}{2\pi(s-r)^{m+1}} \cdot \varepsilon_k \cdot 2\pi s$$

for *any* z satisfying $|z - z_0| \leq r$. Sending $k \to \infty$ gives uniform convergence of m-th derivatives on $|z - z_0| \leq r$.

Definition .0.1

We say that for a sequence of holomorphic functions $f_k : D \to \mathbb{C}$ on a connected open set $D \subseteq \mathbb{C}$ converges normally to $f : D \to \mathbb{C}$ on D provided that it converges uniformly to $f(z)$ on every closed disk contained in D.

Some mathematicians instead say "converges locally uniformly."

Example .0.2

$\sum z^k$ converges normally on $|z| < 1$.

Theorem .0.4

Suppose $f_k 9z)$ is a sequence of holomorphic functions on a connected open $D \subseteq \mathbb{C}$. Suppose $f_k(z)$ converges uniformly to a holomorphic $f : D \to \mathbb{C}$. Then for each $m \geq 1$ the sequence of m-th derivatives $f_k^{(m)}(z)$ converges normally to $f^{(m)}(z)$ on$D$.

.1. Power Series

In Gamelin this is [Gam03, p. V.3]

Definition .1.1

A power series (centered at $z_0 \in \mathbb{C}$) is a series of the form $\sum_{k=0}^{\infty} a_k(z - z_0)^k$.

Claim

We can reduce just about all conversations of power series centered at z_0 to power series centered at $z_0 = 0$ via the change of variables $w = z - z_0$.

Theorem .1.1

Let $\sum a_k z^k$ be a power series. Then there exists an R, $0 \leq R \leq +\infty$ so that $\sum a_k z^k$ converges absolutely if $|z| < R$, and $\sum a_k z^k$ does not converge if $|z| > R$.

Furthermore, for each fixed r satisfying $0 \leq r < R$, the series $\sum a_k z^k$ converges uniformly on $|z| \leq r$. That is $\sum a_k z^k$ converges normally on $|z| < R$.

Definition .1.2

This R is called the radius of convergence of the power series.

Proof. Consider the sequence $k \mapsto |a_k| r^k$. If this sequence is bounded for some $r =_0$, then it is bounded for all values of r satisfying $0 \leq r \leq r_0$. Define

$$R \coloneqq \sup\{r \geq 0 \mid \text{the sequence } k \mapsto |a_k| r^k \text{ is bounded}\}.$$

Here we take sup to be ∞ if the right hand set is unbounded. We just need to show this R has the right properties.

By construction, for all $r < R$, $k \mapsto |a_k| r^k$ is bounded, and for all $s > R$, $k \mapsto |a_k| s^k$ is unbounded.

Back to our series $\sum a_k z^k$. If $|z| > R$, then the terms $a_k z^k$ do not go to 0 as $k \to \infty$, so the series does not converge.

Now suppose $|z| \leq r < R$. Choose s such that $r < s < R$. Then the sequence $k \mapsto |a_k| s^k$ is bounded by some $C \in \mathbb{R}$. If $|z| \leq r$, then for all $k \geq 0$ we have

$$\left|a_k z^k\right| \leq |a_k| r^k = |a_k| s^k \left(\frac{r}{s}\right)^k \leq C \left(\frac{r}{s}\right)^k.$$

Set $M_k = C\left(\frac{r}{s}\right)^k$. Does $\sum M_k$ converge? Yes because $s > r > 0$. By the M-test $\sum \left|a_k z^k\right|$, $\sum a_k z^k$ both converge uniformly for $|z| \leq r$. Thus $\sum a_k z^k$ converges absolutely and unifomrly on $|z| \leq r$.

Gamelin: Examples on p139 [Gam03].

Remark .1.1

The partial sums of the power series $\sum a_k z^k$ are all polynomial functions! Thus they are holomorphic functions and everything is awesome!

Theorem .1.2

Suppose $\sum a_k z^k$ is a power series with radius of convergence $R > 0$. Then the function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

for $|z| < R$ is holomorphic on $|z| < R$. Furthermore the derivatives $f^{(m)}(z)$ are obtained as power series by differentiating the power series term by term. By example

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$$

$$f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k z^{k-2}.$$

Moreover, the coefficients $a_k = \frac{f^{(k)}(0)}{k!}$. This is given by evaluating the derivatives above at 0.

Proof. This follows from all of the above.

Example .1.1

$\sum z^k = \frac{1}{1-z}$, $|z| < 1$. We also have

$$\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1} = \sum_{m=0}^{\infty} (m+1) z^m, |z| < 1.$$

Note, we can also integrate term by term because we have uniform convergence on subdisks $|z| \leq r$ for $r < R$.

Example .1.2

We can write that for $|z| < 1$

$$\begin{aligned} -\operatorname{Log}(1-z) &= \int_0^\infty \frac{\mathrm{d}\zeta}{1-\zeta} = \int_0^z \sum_{k=0}^\infty \zeta^k \,\mathrm{d}\zeta \\ &= \sum_{k=0}^\infty \int_0^z \zeta^k \,\mathrm{d}z = \sum_{k=0}^\infty \frac{z^{k+1}}{k+1}. \end{aligned}$$

Therefore if we set $w = 1 - z$, then for $|w-1| < 1$ we have

$$\operatorname{Log} w = \sum_{k=1}^\infty \frac{(-1)^k}{k}(w-1)^k.$$

How do we compute R?

Example .1.3

Sarah's advisor (Hubbard) likes to ask people to compute $\sup\{\cos(10^n) \mid n \in \mathbb{N}\}$. This exists but is so difficult to compute you'd probably win a fields medal if you did…it's almost surely equal to 1…

Try to reinterpret this as a famous hard problem (it's not too difficult to do). Hint: $2k\pi$ being somehow close to 10^n…

How do we compute it? Recall the Ratio and Root tests from calculus!

Theorem .1.3

Let $\sum a_k z^k$ be a power series. If $\left|\frac{a_k}{a_{k+1}}\right|$ has a limit as $k \to \infty$ (the limit is allowed to be $+\infty$), then the limit is equal to R, where R is the radius of convergence of $\sum a_k z^k$.

Proof. Set $L = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$. If $r < L$, then $\left|\frac{a_k}{a_{k+1}}\right| > r$ eventually, say for $k \geq N$. Then we have

$$|a_k| > r\,|a_{k+1}|$$

for all $k \geq N$. Thus

$$|a_N|\, r^N \geq |a_{N+1}|\, r^{N+1} \geq \cdots$$

so the sequence $k \mapsto |a_k|\, r^k$ is bounded. Thus $r \leq R$. From this we have that $L \leq R$ (as if $L > R$ we could pick an $L > r > R$).

Suppose next that $s > L$, then $\left|\frac{a_k}{a_{k+1}}\right| < s$ eventually, say for all $k \geq N$. Then

$$|a_k| < s\,|a_{k+1}|.$$

Therefore

$$|a_N|\, s^N < |a_{N+1}|\, s^{N+1} \leq \cdots$$

and $a_k z^k$ does not go to zero if $|z| \geq s$! Thus $\sum a_k z^k$ does not converge for $|z| \geq s$. It follows then that $L \geq R$. Together we have $L = R$.

Theorem .1.4

If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$ (we allow limit to be $+\infty$). Then

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}},$$

where R is the readius of convergence of $\sum a_k z^k$.

Proof. See Gamelin.

Remark .1.2

In the above two theorems we can replace the limit with the limit superior and they still work.

We have seen that $\sum a_k(z-z_0)^k$ converges to a holomorphic function inside a disk of convergence $\{|z-z_0| < R\}$.

We will now see that ANY function that is holomorphic on a disk can be represented locally by power series. That is for each z within the disk we can find a power series which converges to the function on a small ball around z. This latter property is called being analytic, and so we can now use holomorphic/analytic interchangeably.

Theorem .1.5

Suppose that $f(z)$ is holomorphic for $|z-z_0| < \rho$. Then $f(z)$ is represented by the power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

for $a_k = \frac{f^{(k)}(z_0)}{k!}$ on $|z-z_0| < \rho$, and the radius of convergence R of the power series satisfies $R \geq \rho$.

Furthermore, for any fixed r with $0 < r < \rho$ we have

$$a_k = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \,\mathrm{d}\zeta,$$

and if $|f(z)| \leq M$ on $|z-z_0| < r$ then

$$|a_k| \leq \frac{M}{r^k}.$$

We first need a lemma

Lemma .1.6

Fix z satisfying $|z| < r$. For $|\zeta| = r$ we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\frac{1}{1-\frac{z}{\zeta}} = \sum_{k=0}^{\infty} \frac{z^k}{\zeta^{k+1}}$$

We claim this converges uniformly for $|\zeta| = r$.

Proof. Weierstrass M-test!

Proof. It suffices to take $z_0 = 0$ by translation. For any fixed r, satisfying $0 < r < \rho$, we have

$$a_k = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{k+1}} \,\mathrm{d}\zeta,$$

by the Cauchy integral formula. Furthermore if $|f(z)| \le M$ on $|z| \le r$, then by the Cauchy estimates.

$$|a_k| \le \frac{M}{r^k}$$

for $k \ge 0$. Furthermore for $|z| < r$ we have convergence of $\sum a_k z^k$ then by the Weierstrass M-test! Thus the radius of convergence $R \ge \rho$.

Now fix $|z| < r$, we have by the Cauchy integral formula and uniform convergence that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_{|\zeta|=r} \sum_{k=0}^{\infty} \frac{f(\zeta) z^k}{\zeta^{k+1}} \,\mathrm{d}\zeta \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{k+1}} \,\mathrm{d}\zeta \right) z^k \\
&= \sum_{k=0}^{\infty} a_k z^k,
\end{aligned}$$

and everything must converge.