Stuff:

- Math Club Today!
- Math S^1 tonight 6:30pm-8pm!
- Popcorn 4:30pm
- Bagel Sunday at 11:30am
- Free voting t-shirts
- Super Saturdays!
- Extra Halloween Shirts / Free voting shirts
- Student seminar Friday 4pm EH3096 "Combinatorial reciprocity via Möbius functions."
- Undergrad student advisory council 1-2pm atrium.

Last time: Residue theorem! Evaluating real integrals!

The same techniques from last time can evaluate integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x,$$

where P, Q are real polynomials where Q has no real zeros and $\deg Q \geq \deg P + 2$. In this case we'll have

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x = 2\pi i \sum \operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_j\right],$$

where each z_j is a zero of Q within the upper half-plane. This method can be used to evaluate other integrals too! Consider

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos ax\,\mathrm{d}x,$$

where $a > 0$ and p, q are polynomials of the form mentioned above. We would like to complexify. The simplest candidate is

$$\frac{p(z)}{q(z)} \cos az = \frac{p(z)}{q(z)} \cdot \frac{e^{iaz} + e^{-iaz}}{2}.$$

But $\cos az$ is unbounded in the upper half plane… this causes problems for us. Instead we'll use e^{iz} and apply real and imaginary parts at the end of the calculation. In particular, we'll look at $f(z) = \frac{p(z)}{q(z)} e^{iaz}$.

Example .0.1

Show $\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,\mathrm{d}x = \pi e^{-a}$ when $a > 0$. We can look at $f(z) = \frac{e^{iaz}}{1+z^2}$. We'll look at our favorite contour ∂D given by the semi-circle Γ_R and the interval $[-R, R]$.

We see by the residue theorem and our Rule 3 that

$$\begin{aligned}\int_{\partial D} \frac{e^{iaz}}{1+z^2}\,\mathrm{d}z &= 2\pi i \operatorname{Res}[f(z), i] = 2\pi i \frac{e^{iaz}}{2z}\Big|_{z=i} \\ &= \frac{2\pi i e^{-a}}{2i} = \pi e^{-a}.\end{aligned}$$

Now we see via the ML-estimate that since $\left|e^{iaz}\right| \leq 1$ in the upper half plane (since $a > 0$) that

$$\int_{\Gamma_R} \frac{e^{iaz}}{1+z^2}\,\mathrm{d}z \leq \pi R \cdot \frac{1}{R^2 - 1},$$

which goes to 0 as $R \to \infty$. Thus

$$\lim_{R\to\infty} \int_{\partial D} \frac{e^{iaz}}{1+z^2}\,\mathrm{d}z = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iaz}}{1+z^2}\,\mathrm{d}z.$$

Applying what we've already done, this yields

$$\pi e^{-a} = \int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,\mathrm{d}x,$$

which upon taking Real parts of both sides yields the desired result.

.1. Integrals of Trig Functions

Previous plan: Start with real integral + complexify, integrate over a "good" contour, take limit and we're happy.

Now we have things of the form

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\sin\theta}.$$

How can we use complex analysis to integrate this. Well we use the common substitution $z = e^{i\theta}$. Then $\mathrm{d}z = iz\,\mathrm{d}\theta$. Furthermore, we have some nice identities for $|z| = 1$, namely $\overline{z} = 1/z = e^{-i\theta}$ and

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Example .1.1

Let's actually compute $\int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\sin\theta}$. Armed with the identities, we have

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\sin\theta} &= \int_{|z|=1} \frac{1}{5-2i(z-1/z)} \frac{\mathrm{d}z}{iz} \\ &= \int_{|z|=1} \frac{\mathrm{d}z}{2z^2-2+5iz}.\end{aligned}$$

We factor the denominator (or use the quadratic formula) to get

$$2z^2 + 5iz - 2 = (2z+i)(z+2i).$$

Thus there are simple zeros at $-i/2, -2i$. We then see that

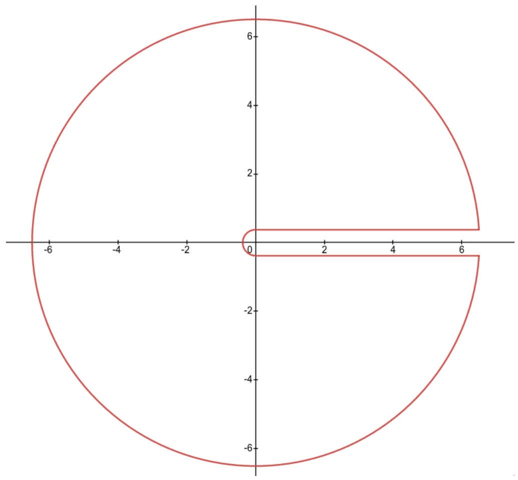
$$\int_{|z|=1} \frac{\mathrm{d}z}{2z^2-2+5iz} = 2\pi i \operatorname{Res}[f(z), -i/2] = \frac{2\pi}{3}.$$

.2. Integrands with Branch Cuts

We would like to evaluate things like $\int_0^{\infty} \frac{x^a}{(1+x)^2}\,\mathrm{d}x$ where $0 < |a| < 1$ is real. We recall that

$$z^a = \exp(a \log z)$$

is not well-defined in the complex plane. Thus instead we have to take branch cuts... Take a branch of the log where the argument ranges from 0 to 2π. The contour we're going to use is something called a keyhole contour of the following shape



The inner semi-circle will be called γ_ε (it has radius ε), and the outer arc of a circle $\Gamma_{R,\varepsilon}$ and will have radius $R_\heartsuit := \sqrt{R^2+\varepsilon^2}$. The lines are from $i\varepsilon$ to $R+i\varepsilon$ and $-i\varepsilon$ to $R-i\varepsilon$. We'll call the top one $L^+(\varepsilon, R)$, and the one on the bottom $L^-(\varepsilon, R)$ (in opposite orientation). We'll also take $\varepsilon < 1/2$ and $R > 2$ so that the singularity at -1 is included. We'll call the region bounded by these $D(\varepsilon, R)$.

We'll show that

$$\int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = \frac{\pi a}{\sin(\pi a)}.$$

We'll use the function $f(z) = \frac{z^a}{(1+z)^2}$. f has a double pole at $z=-1$, and we have

$$\operatorname{Res}[f(z), -1] = \frac{\mathrm{d}}{\mathrm{d}z}(1+z)^2\frac{z^a}{(1+z)^2}\Big|_{z=-1} = -ae^{\pi i a}.$$

The residue theorem then gives

$$\int_{\partial D(\varepsilon,R)} f(z)\,\mathrm{d}z = 2\pi i(-ae^{\pi i a}).$$

This breaks into four pieces

$$\int_{\partial D(\varepsilon,R)} = \int_{L^+(\varepsilon,R)} + \int_{\Gamma_{R,\varepsilon}} + \int_{L^-(\varepsilon,R)} + \int_{\gamma_\varepsilon}.$$

If $|z| = R_\heartsuit$ then

$$|f(z)| = \left|\frac{z^a}{(1+z)^2}\right| \le \frac{R_\heartsuit^a}{(R_\heartsuit - 1)^2}$$

abd if $|z| = \varepsilon$ then

$$|f(z)| = \left|\frac{z^a}{(1+z)^2}\right| \le \frac{\varepsilon^a}{(1-\varepsilon)^2}.$$

We then have that

$$\left|\int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z\right| \leq \frac{R_\heartsuit^a}{(R_\heartsuit - 1)^2} 2\pi R_\heartsuit,$$

which goes to 0 as $R \to \infty$ since $R_\heartsuit \geq R$ and $a \in (-1, 1)$. Also

$$\left|\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z\right| \leq \frac{\varepsilon^a}{(1-\varepsilon)^2},$$

which goes to 0 as $\varepsilon \to 0$.

If $z \in L^+(\varepsilon, R)$ then as $\varepsilon \to 0$ the argument is close to 0, and if $z \in L^-(\varepsilon, R)$ then as $\varepsilon \to 0$ the argument is close to 2π. In the limit, as $\varepsilon \to 0$ and $R \to \infty$ (which we'll denote with L^+),

$$\begin{aligned} \int_{L^+} f(z)\,\mathrm{d}z &= \int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x \\ -\int_{L^-} f(z)\,\mathrm{d}z &= \int_0^\infty \frac{x^a e^{2\pi i a}}{(1+x)^2}\,\mathrm{d}x. \end{aligned}$$

Writing ∂D for the limit as $\varepsilon \to 0, R \to \infty$ we have

$$\begin{aligned} -2\pi i a e^{\pi i a}) = \int_{\partial D} f(z)\,\mathrm{d}z &= \int_{L^+} f(z)\,\mathrm{d}z + \int_{L^-} f(z)\,\mathrm{d}z \\ &= \int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x - \int_0^\infty \frac{x^a e^{2\pi i a}}{(1+x)^2}\,\mathrm{d}x \\ &= (1 - e^{2\pi i a}) \int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x. \end{aligned}$$

Putting this all together gives

$$\int_0^\infty \frac{x^a}{(1+x)^2}\,\mathrm{d}x = -\frac{2\pi i a e^{\pi i a}}{1 - e^{2\pi i a}}.$$

This simplifies down to $\frac{\pi a}{\sin \pi a}$.

.3. Fractional Residues

Idea: What can we do when our path of integration crashes into a singularity?

If $f(z)$ has a simple pole at z_0, then we can do something cool!

Theorem .3.1 (Fractional Residue Theorem)

If z_0 is a simple pole of $f(z)$, and C_ε is an arc of the circle $\{|z - z_0| = \varepsilon\}$ of angle α, then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,\mathrm{d}z = \alpha i \operatorname{Res}[f(z), z_0].$$

Here the integration is taken with the orientation where the singularity is on the left (as usual). This nearly generalizes the full Residue Theorem when z_0 is simple.

Clarification: we're integrating from angle 0 to angle α around the circle. If α is like 4π then we're integrating over the full circle twice.

Proof. Write $f(z) = \frac{A}{z-z_0} + g(z)$ where g is analytic at z_0 and $A = \operatorname{Res}[f(z), z_0]$. Parameterize C_ε as $z = z_0 + \varepsilon e^{i\theta}$ where $\theta_0 < \theta < \theta_0 + \alpha$. Then

$$\int_{C_\varepsilon} \frac{A\,\mathrm{d}z}{z - z_0} = iA \int_{\theta_0}^{\theta_0+\alpha} \mathrm{d}\theta = \alpha i A.$$

Furthermore as $\varepsilon \to 0$ we see $\int_{C_\varepsilon} g(z)\,\mathrm{d}z \to 0$ since $g(z)$ is bounded near z_0 and the length of C_ε is $\alpha\varepsilon$. Combining these two results yields the theorem.