Stuff:

- John W. Milnor's mathematical writing is excellent. Good book: "Introduction to dynamics in one $\mathbb{C}$omplex variable.

Proof of Schwarz's Lemma. Write $f(z) = z \cdot g(z)$ $g(z)$ analytic on $|z| < 1$. Let $r < 1$. Then if $|z| = r$ we have

$$|g(z)| = \frac{|f(z)|}{r},$$

By the maximum principle, $|g(z)| \leq \frac{1}{r}$ when $|z| \leq r$. Letting $r \to 1$ yields $|g(z)| \leq 1$ for all $z \in \mathbb{C}$.

This gives $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Now suppose $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. Then necessarily

$$|f(z_0)| = |z_0| \cdot |g(z_0)| \implies |g(z_0)| = 1,$$

since $z_0 \neq 1$. Thus by the strict maximum principle, we have that g is constant. Say $g(z) = \lambda$ for $\lambda \in \mathbb{C}$. We then know that $|g(z_0)| = |\lambda| = 1$. Thus $f(z) = \lambda \cdot z$.

.1. Conformal self-maps of $\mathbb{D}$

For example. Consider any rotation!

Lemma .1.1

If $g(z)$ is an automorphism of $\mathbb{D}$ with $g(0) = 0$, then $g(z)$ is a rotation.

Proof. Apply the Schwarz lemma twice, once to the function and once to its inverse. $|g(z)| \leq |z|$. Further $\left|g^{-1}(w)\right| \leq |w|$. Plugging in $w = g(z_0)$ for any $z_0 \neq 0$ we have

$$|z_0| \leq \left|g^{-1}(g(z_0))\right| \leq |g(z_0)| \leq |z_0|.$$

Thus we have equality, which tells us that $g(z)$ is a rotation via the Schwarz lemma.

Application:

Theorem .1.2

$\operatorname{Aut}(\mathbb{D})$ is precisely the Möbius transformations of the form

$$f(z) = e^{i\theta}\left(\frac{z-a}{1-\overline{a}z}\right)$$

where $a \in \mathbb{D}$ and $\theta \in [0, 2\pi)$.

Question: Do these Möbius transformations belong to $\operatorname{Aut}(\mathbb{D})$? Consider

$$g(z) = \frac{z-a}{1-\overline{a}z}.$$

Well g maps circles to circles. So g maps unit circle to a unit circle. Fix some $e^{i\alpha} \in S^1$. Then

$$\begin{aligned} \left|e^{i\alpha} - a\right| &= \left|e^{-i\alpha} - \overline{a}\right| = \left|1 - e^{i\alpha}\overline{a}\right| \\ \left|g(e^{i\alpha})\right| &= 1. \end{aligned}$$

Thus g maps the unit circle to itself! Since $g(a) = 0$, g maps $\mathbb{D} \to \mathbb{D}$.

Proof of Theorem. Show if $h(z) \in \mathrm{Aut}(\mathbb{D})$ then $h(z)$ has the desired form.Set $a = h^{-1}(0)$. Then consider the map

$$g(z) = \frac{z-a}{1-\bar{a}z}.$$

We can look at $h \circ g^{-1} \in \mathrm{Aut}(\mathbb{D})$. Then we have

$$h(g^{-1}(0)) = h(a) = 0.$$

Thus $(h \circ g^{-1})(z) = e^{i\theta} z$ by the previous lemma and $h = h \circ g^{-1} \circ g$ so

$$h(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}.$$

Theorem .1.3 (Pick's Lemma)

If $f(z)$ is analytic and satisfies $|f(z)| < 1$ for $|z| < 1$. Then in fact

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}$$

for $|z| < 1$. If $f(z)$ is conformal, then equality holds for all $z \in \mathbb{D}$. Otherwise, this inequality is strict for all $z \in \mathbb{D}$.

Proof. Want to use Schwarz Lemma, but we don't have $0 \mapsto 0$. The idea is to use clever composition. Let $f(z_0) = w_0$. Take $g, h \in \mathrm{Aut}(\mathbb{D})$ so that $g(0) = z_0, h(w_0) = 0$ (the previous work has shown $\mathrm{Aut}(\mathbb{D})$'s action on $\mathbb{D}$ is transitive).

Then $h \circ f \circ g : \mathbb{D} \to \mathbb{D}$ which maps 0 to 0. Applying Schwarz lemma yields

$$|(h \circ f \circ g)'(0)| = |h'(w_0) f'(z_0) g'(0)| \leq 1,$$

using the definition of the derivative as $\frac{h(f(g(z)))}{z}$ as $z \to 0$. Then

$$|f'(z_0)| \leq \frac{1}{|g'(0)| \cdot |h'(w_0)|}.$$

Using the formula for g, h yields $g'(0) = 1 - |z_0|^2$ and $h'(w_0) = \frac{1}{1-|w_0|^2}$.

Suppose now f is conformal. Then $h \circ f \circ g$ is conformal and fixes 0. Thus by the above $h \circ f \circ g(z)$ is a rotation, so $|h'(w_0) f'(z_0) g'(0)| = 1$. Thus again with the algebra this yields equality.

We will show if equality holds at some $z_0 \in \mathbb{D}$ then f is conformal and equality holds fro all $z_0 \in \mathbb{D}$. Suppose equality holds at some z_0. As above we see $|(h \circ f \circ g)'(0)| = 1$.

By Schwarz lemma (see Gamelin for the derivative version), $h \circ f \circ g$ is a rotation, so f is a conformal self map $\mathbb{D} \to \mathbb{D}$.

.2. Hyperbolic Geometry

Suppose $f : \mathbb{D} \to \mathbb{D}$ is a conformal map. Write $w = f(z)$. Then $\mathrm{d}w = f'(z)\,\mathrm{d}z$. Therefore

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = |f'(z)| = \frac{1-|f(z)|^2}{1-|z|^2}.$$

Rearranged with $w = f(z)$, we have

$$\frac{|\mathrm{d}w|}{1-|w|^2} = \frac{|\mathrm{d}w|}{1-|z|^2}.$$

Then if γ is a smooth curve in $\mathbb{D}$ and $w = f(z)$ then

$$\int_{f\circ\gamma} \frac{|\mathrm{d}w|}{1-|w|^2} = \int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2}.$$

This tells us that if we want to measure distance in the unit disk, then we should use $\frac{|\mathrm{d}z|}{1-|z|^2}$. This metric will have the miraculous property of being preserved by conformal self-maps of the disk.

Definition .2.1

The length of γ in the hyperbolic metric on $\mathbb{D}$ is

$$\text{hyperbolic lenght of } \gamma \coloneqq 2\int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2},$$

where the 2 is innocent, so that the curvature is -1.

NICE: by design, hyperbolic length is invariant under conformal maps $\mathbb{D} \to \mathbb{D}$.

Definition .2.2

Let $z_0, z_1 \in \mathbb{D}$. We define the hyperbolic distance $\rho(z_0, z_1)$ to be

$$\rho(z_0, z_1) \coloneqq \inf_{\substack{\text{piecewise smooth}\gamma \\ \gamma : z_0 \to z_1}} 2\int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2}.$$

Since conformal maps $\mathbb{D} \to \mathbb{D}$ preserve hyperbolic lengths of curves, conformal maps preserve the hyperbolic metric too.

Theorem .2.1

For any two distinct points $Z_0, z_1 \in \mathbb{D}$, there exists a unique geodesic in $\mathbb{D}$ from z_0 to z_1 in the hyperbolic metric. This curve is the arc of circle passing through z_0 to z_1 that is orthogonal to the unit circle.

Via an appropriate choice of θ, a we have $z \mapsto e^{i\theta}\frac{z-a}{1-\overline{a}z}$ maps z_0 to 0 (via choosing a) and z_1 to the real axis (via rotation). This makes the problem easier