Stuff:

- Office Hours Wednesday 10:30-12 EH3855.
- HW 2B due tonight 10pm
- HW 3A, 3B is the next round. For 3A, possibly look at Gamelin for inspiration.
- Walk: Carol + snowcones!

Back to Möbius transformations. Recall that

$$\text{Möb} := \{f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid f(z) = \frac{az+b}{cz+d}, ad - bc \neq 0\}.$$

This is a group with respect to function composition, we saw an isomorphism with familiar groups earlier. Furthermore we have

$$\text{Möb} \subseteq \text{Aut}(\widehat{\mathbb{C}}) := \{g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid g \text{ is a biholomorphism}\}.$$

In fact, $\text{Möb} = \text{Aut}(\widehat{\mathbb{C}})$. We will show this eventually.

What about

$$\text{Aut}(\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} \mid f \text{ is biholo}\}.$$

Note that affine maps $z \mapsto \alpha z + \beta$ where $\alpha \in \mathbb{C}^\times$ are biholomorphisms of the plane. These are special examples of Möbius transformations. In fact the affine transformations are exactly those Möbius transformations that send ∞ to ∞. Let Aff be the group of these transformations.

Fact: $\text{Aff} = \text{Aut}(\mathbb{C})$.

We can also consider the open disk

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}.$$

This will become one of our close friends. The disk has a nice cousin

$$\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$$

which is the upper half plane. Who is this! Well consider the map

$$\begin{aligned} \varphi : \mathbb{H} &\to \mathbb{D} \\ z &\mapsto \frac{i-z}{i+z}. \end{aligned}$$

One must check this is holomorphic, bijective, and its inverse is holomorphic. This preserves angles and so we call it conformal. A picture is below.

Definition .0.1

We define the <u>cross ratio</u> of an ordered quadruple of distinct points $p_1, p_2, p_3, p_4 \in \widehat{\mathbb{C}}$ is

$$[p_1, p_2, p_3, p_4] := \frac{(p_3 - p_1)(p_4 - p_2)}{(p_2 - p_1)(p_4 - p_3)}.$$

If $p_i = \infty$, the definition is interpreted as the appropriate limit.

Example .0.1

$$[0, p_2, p_3, p_4] = \frac{p_4 - p_2}{p_4 - p_3}$$

Exercise .0.2

Compute for $p_i = 0, 1, \infty, z$. What are all possible outputs with all possible permutations

$$[0, 1, \infty, z] = \lim_{w \to \infty} \frac{(w - 0)(z - 1)}{(1 - 0)(z - w)} = \lim_{w \to \infty} \frac{z - 1}{\frac{z}{w} - 1} = 1 - z.$$

According to a classmate (Zach), there are six distinct outputs

$$z, 1 - z, \frac{1}{1 - z}, \frac{1}{z}, \frac{z}{1 - z}, \frac{1 - z}{z}$$

Cool properties!

Theorem .0.1

Consider these neat properties

(i) If $f \in$ Möb is the unique element sending $(p_1, p_2, p_4) \mapsto (0, 1, \infty)$ then

$$[p_1, p_2, p_3, p_4] = f(p_3).$$

In particular, $[p_1, p_2, p_3, p_4]$ takes values in $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$.

(ii) Two quadruples (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) can be sent to each other by Möbius transformations if and only if

$$[p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4].$$

(iii) If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism that preserves cross ratios of ALL quadruples, then f is a Möbius transformation.

(iv) Four points p_1, p_2, p_3, p_4 lie on the same circle in $\mathbb{C}$ if and only if $[p_1, p_2, p_3, p_4] \in \mathbb{R}$.

Back to holomorphic discussion!

Recall .0.3

$f : U \to \mathbb{C}$, $z_0 \in U$, we say that f is holomorphic at z_0 means f is holomorphic on a neighborhood of z_0, that is for any z within that neighborhood the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \in \mathbb{C}$$

exists. We showed that if $f = u + iv$ then

- $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0$
- f satisfies Cauchy-Riemann equations on U

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- For $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ we have $\Delta u, \Delta v = 0$ under regularity assumptions on u, v (which will be unnecessary later). This means u, v are harmonic, and in fact they are harmonic conjugates.

Theorem .0.2

Let $u, v : U \to \mathbb{R}$ have contiuous first order partials on U and satisfy the Cauchy-Riemann equations on U. Then $f = u + iv$ is holomorphic on U.

Proof. We will use Taylor's Theorem for real variables. Consider some point $z_0 = (x_0, y_0) \in U$, and consider some small $h = (h_1, h_2)$. We see that

$$u(x_0 + h, y_0 + k) - u(x_0, y_0) = \left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)} \cdot h_1 + \left.\frac{\partial u}{\partial y}\right|_{(x_0, y_0)} \cdot h_2 + \varepsilon_1$$

and

$$v(x_0 + h, y_0 + k) - v(x_0, y_0) = \left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)} \cdot h_1 + \left.\frac{\partial v}{\partial y}\right|_{(x_0, y_0)} \cdot h_2 + \varepsilon_2$$

where $\varepsilon_1, \varepsilon_2$ tend to 0 more rapidly than $h + ik$ in the sense that

$$\frac{\varepsilon_1}{h_1 + ih_2}, \frac{\varepsilon_2}{h_1 + ih_2} \to 0 \iff \frac{|\varepsilon_1|^2}{h_1^2 + h_2^2}, \frac{|\varepsilon_2|^2}{h_1^2 + h_2^2} \to 0$$

as $h = h_1 + ih_2 \to 0$. Using the Cauchy-Riemann equations, we have that

$$f(z_0 + h) - f(z_0) = \left(\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)} \right) \cdot h + \varepsilon_1 + i\varepsilon_2. \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)}.$$

Thus $f'(z_0)$ exists.

Using this converse, one may show the exponential $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic. Recall the definition below

$$\begin{aligned} \exp : \mathbb{C} &\to \mathbb{C} \\ x + iy &\mapsto e^x \cdot e^{iy} \\ &= e^x \cos y + i \sin y. \end{aligned}$$

Exercise .0.4

Show that exp is holomorphic by showing it satisfies the Cauchy-Riemann.

For the logarithm, we need to use the inverse function theorem. This gets an upgrade in the setting of complex analysis!

Theorem .0.3

Suppose f is holomorphic on the open set $\Omega \subseteq \mathbb{C}$ and $f'(z_0) \neq 0$ for $z_0 \in \Omega$. Then there exists a neighborhood U containing z_0 on which

- f is injective.
- The image $V \coloneqq f(U)$ is open in $\mathbb{C}$.
- The inverse $f^{-1} : V \to U$ is holomorphic on V and satisfies

$$\left(f^{-1}\right)'(f(z)) = \frac{1}{f'(z)}.$$

Proof. The first two bullet points come from analysis of real variables by using the identification $\mathbb{R}^2 \cong \mathbb{C}$. Take $g = f^{-1}$, and let $w = f(z), w_1 = f(z_1)$. Then we want to show $g'(w_1)$ exists. Consider

$$\lim_{w \to w_1} \frac{g(w) - g(w_1)}{w - w_1} = \lim_{z \to z_1} \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{f'(z_1)}.$$

We must show $\operatorname{Log}(z) : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}$ is holomorphic, where

$$\operatorname{Log}(z) = \log |z| + i \operatorname{Arg}(z).$$

Well, consider $\exp : \mathbb{R} \times (-\pi, \pi) \to \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and use the inverse function theorem... Note exp is already bijective on this domain.

Consider your homework, that a linear map $\mathbb{R}^2 \xrightarrow{L_M} \mathbb{R}^2$ which has positive determinant descends to a $\mathbb{C}$-linear map $z \mapsto \alpha z$ as

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "L_M"] \arrow[d, leftrightarrow] & \mathbb{R}^2 \arrow[d, leftrightarrow] \\
\mathbb{C} \arrow[r, dashed] & \mathbb{C} \\
z \arrow[r, dashed, mapsto] & \alpha z
\end{tikzcd}$$

if and only if L_M preserves angles between vectors. Thus if $f : U \to \mathbb{C}$ then $f'(z_0)$ exists if and only if the derivative preserves angles or it is zero.

Definition .0.2

Let $U \subseteq \mathbb{C}$ be open. The function $f : U \to \mathbb{C}$ is conformal at $z_0 \in U$ provided that it preserves angles in the since that for any pair of smooth curves $\gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$, the angle between $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ is equal to the angle between $(f \circ \gamma_1)'(t_1)$, $(f \circ \gamma_2)'(t_2)$ at $f(z_0)$.

More precisely,

$$\frac{|\langle \gamma_1'(t_1), \gamma_2'(t_2) \rangle|}{\|\gamma_1'(t_1)\| \cdot \|\gamma_2'(t_2)\|} = \frac{|\langle (f \circ \gamma_1)'(t_1), (f \circ \gamma_2)'(t_2) \rangle|}{\|(f \circ \gamma_1)'(t_1)\| \cdot \|(f \circ \gamma_2)'(t_2)\|}.$$

If we further require that the signs match, that is

$$\frac{\langle \gamma_1'(t_1), \gamma_2'(t_2) \rangle}{\|\gamma_1'(t_1)\| \cdot \|\gamma_2'(t_2)\|} = \frac{\langle (f \circ \gamma_1)'(t_1), (f \circ \gamma_2)'(t_2) \rangle}{\|(f \circ \gamma_1)'(t_1)\| \cdot \|(f \circ \gamma_2)'(t_2)\|}.$$

This makes a statement about the *orientation* being preserved as well, we call these orientation-preserving.

Caution: Gamelin insists that all maps that are conformal at z_0 are orientation-preserving at z_0. We will adopt Gamelin's convention.

Example .0.5

Complex conjugation will preserve the angles between 2 vectors, but not the directed angle.

Theorem .0.4

If $f : U \to \mathbb{C}$ is holomorphic then f is conformal (and orientation-preserving) at all points $z_0 \in U$ such that $f'(z_0) \neq 0$.

Proof. We use the chain rule, set $z_0 = \gamma(a) = \delta(c)$ for smooth curves γ, δ. Then

$$(f \circ \gamma)'(a) = f'(\gamma(a))\gamma'(a) \qquad\qquad (f \circ \delta)'(c) = f'(\delta(c))\delta'(a).$$

But wait! We then have that

$$\langle f'(z_0)\gamma'(a), f'(z_0)\delta'(c)\rangle = |f'(z_0)|^2 \langle \gamma'(a), \delta'(c)\rangle$$

because $h \mapsto f'(z_0) \cdot h$ is a $\mathbb{C}$-linear map, and thus an angle/orientation-preserving linear map $\mathbb{R}^2 \to \mathbb{R}^2$ from homework.