Stuff:

- Math Club!
- Popcorn 4:30pm
- Bagel Sunday 11:30am
- Super Saturdays
- HW due next week
- Last student seminar of the semester at 4pm in EH 3096, Drew Keisling on $S$-LID sequences (generalizations of fibonacci stuff)

Exciting Stuff! We were proving Riemann Mapping Theorem[1].

The proof was incredibly nonconstructive. We had something like

$$M \coloneqq \sup\{|f'(p)| \mid f \in \mathcal{F}\}$$

and we needed a sequence $f_n$ with $|f_n'(p)| \to M$. As we all know, supremums are awful to compute.

> **Example .0.1** (Classic 295/296 style example)
>
> $\sup\{\cos(10^n) \mid n \in \mathbb{N}\}$ is not known, but it is almost certainly 1 (related to properties of $\pi$, such as being normal).

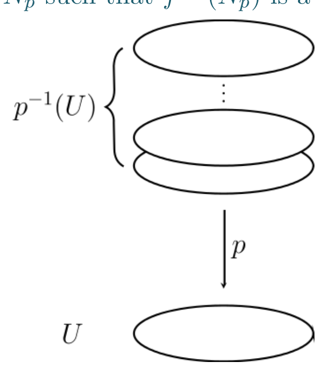
To fully appreciate Hubbard's constructive proof, we need to know some stuff from 592 (algebraic topology).

Question: Great. We know that if $U \subsetneq \mathbb{C}$ is proper, simply connected, then there exists a conformal isomorphism $U \to \mathbb{D}$. What if $U$ is not simply connected? Well since any conformal map is a homeomorphism, this will cause problems (simply connected is a topological invariant).

Cool, we won't prove it! Every annulus has something called an annulus. This is gotten by thinking of an annulus as a projection of a cylinder. The modulus is then the height/circumference... This turns out to be an invariant in complex analysis.

Fisher-Hubbard-Wittner: Given a domain $U \subsetneq \mathbb{C}$, possibly not simply connected, there exists a local biholomorphism $\mathbb{D} \to U$ called a covering map

> **Definition .0.1**
>
> Let $f : X \to Y$ (where $X, Y$ are topological spaces). We call $f$ a covering map provided that for all points $p \in Y$ there is a neighborhood $N_p$ such that $f^{-1}(N_p)$ is a "pile of plates"




[1] For users of the notes, the rest of the proof has been uploaded on the document for November 29th

These are called "sheets." Formally, $f^{-1}(N_p)$ is a disjoint union of open sets $\widetilde{N}_p^i$ such that $f : \widetilde{N}_p^i \to N_p$ is a homeomorphism.

Riemann surfaces are then very closely related to covering maps. In fact, there is a Grandpa to the RMT, which we will state but not prove.

**Theorem .0.1** (Uniformization Theorem)

Let $X$ be a Riemann surfaces ($\mathbb{C}$ manifold with $\dim_{\mathbb{C}} X = 1$. Then there is exactly one cover with the following total space (the top one):

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[d, "\text{spherical}"] & \mathbb{C} \arrow[d, "\text{Euc}"] & \mathbb{D} \arrow[d, "\text{hyp}"] \\ X & X & X. \end{tikzcd}$$

These are called universal covers, since they are simply connected (see more 592!). This determines the geometry, since $\mathbb{D}$ is hyperbolic, $\mathbb{C}$ is Euclidean, $\widehat{\mathbb{C}}$ is spherical.

The first case $X = \widehat{\mathbb{C}}$, second is one of torus, $\mathbb{C} \setminus \{0\}, \mathbb{C}$, and third is a wild zoo.

Now how do we compute Riemann maps when they exist. Let $U \subsetneq \mathbb{C}$ be simply connected. You could use Hubbard's method (a limit, for the computers). In general it's very hard. In fact, going from a square to a disk is super difficult!

Weirdly enough, it is *much* easier to compute a Riemann map from the complement of the Mandelbrot set to the complement of a disk (by inversion ideas, this is the same as an appropriate Riemann map).

Suppose we have a Riemann map $(U, p) \to (\mathbb{D}, 0)$. When does it extend to a continuous map $(\overline{U}, p) \to (\overline{\mathbb{D}}, 0)$.

**Theorem .0.2** (Carathéodory,1913)

A Riemann map $\mathbb{D} \to U$ extends to a continuous map $\overline{\mathbb{D}} \to \overline{U}$ if and only if $\partial U$ is locally connected.

**Recall .0.2**

A topological space $X$ is locally connected at $x \in X$ provided that for all neighborhoods $U$ of $x$, there exists a connected neighborhood $V \subseteq U$ of $x$.

Not Known: is the Mandelbrot set $M$ locally connected (if bad things happen it would be at the boundary...hmmm).

HW: Asked to prove $M$ is connected and full (a set is full provided that $\mathbb{C} \setminus M$ is connected).

Fun fact: A Riemann map $\mathbb{D} \to U$ extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{U}$ if and only if $\partial U$ is a Jordan Curve.

## .1. Mandelbrot Set Things

We'll look at maps $z \mapsto z^2 + c$ where $c \in \mathbb{C}$ is a parameter. Is this too specific? No in fact. Any $az^2+bz+d$ can be conjugated with an element of $\operatorname{Aut}(\mathbb{C})$ to a polynomial $z \mapsto z^2 + c$. Thus from the point of view of dynamics, all complex polynomials look like $z^2 + c$.

The key idea of the Mandelbrot set is to look at the $c$-plane (the parameter space). Generally, in dynamics we like to study orbits. Given a map $f : X \to X$ and a point $z_0$, the orbit is the sequence $z_n = f(z_{n-1})$. In $\mathbb{C}$-analysis, we really like looking at it as polynomials! We can think of these as maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\infty$ as a fixed point (a special point for a polynomial).

**Recall .1.1** (Filled Julia sets, **??**)

Given a polynomial $f : \mathbb{C} \to \mathbb{C}$ we define the filled Julia set

$$K_f \coloneqq \{z_0 \in \mathbb{C} \mid \text{the orbit of } z_0 \text{ is bounded}\}$$

Fun facts:

- $K_f \neq \emptyset$, except when $f$ is a non-identity translation, This is since $f(z) - z$ will be a nonconstant polynonmial, so $f$ will have a fixed point.
- $K_f$ is bounded. Can show if $f(z) = z^2 + c$ then they are in disk of $r = 2$.
- $K_f$ is closed.

$\mathbb{C} \setminus K_f$ is the basin of $\infty$, and consists of all points $z_0$ whose orbit diverges to $\infty$. Can show this is open explicitly.

The big question: How does the shape change when you change the polynomial? What subsets of $\mathbb{C}$ arise as filled Julia sets of $z \mapsto z^2 + c$ as $c$ varies? Good thing to look up, there are also pictures earlier in the notes (see around **??**)!

**Definition .1.1** (Mandelbrot Set)

The Mandelbrot Set is defined as

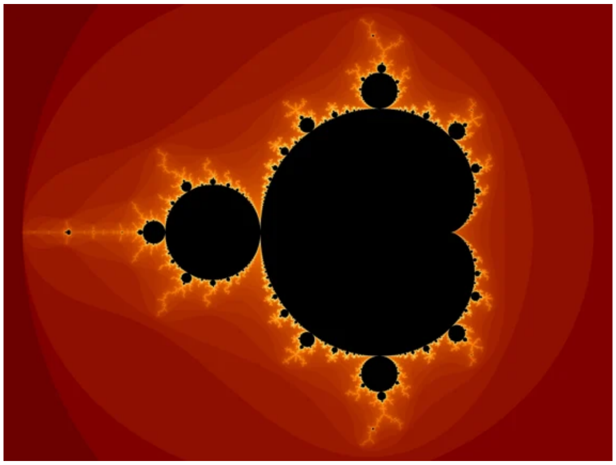
$$\mathcal{M} \coloneqq \{c \in \mathbb{C} \mid \text{Filled Julia set of } z \mapsto z^2 + c \text{ is connected}\}$$

We can ask the computer to draw filled Julia sets (look at the orbit and see if it escapes outside of disk of radius 2, do it for long enough)... How the hell do you draw the Mandelbrot set? We could draw a quilt (tile complex plane with julia sets). This gets us close! But we need to do better. We need a theorem that relates connectivity of the filled Julia set to something easy for a computer to check

**Theorem .1.1** (Proof, basic $\mathbb{C}$-analysis)

The Filled Julia set of $z^2 + c$ is connected if and only if it contains 0, in other words if and only if the orbit of 0 is bounded.

WHY ZERO: well one idea, it's a critical point since $2 \cdot 0 = 0$. But if you're thinking geometrically it's also sort of clear. It's the *center of symmetry* of the Julia set, since $z_0, -z_0$ have the same orbit. Here is the Mandelbrot set!



Facts about the Mandelbrot set:

- $\mathcal{M} \neq \emptyset$ since $0 \in \mathcal{M}$ since the filled Julia set of $z \mapsto z^2$ is $\overline{\mathbb{D}}$.
- Compact, bounded inside closed disk of $r = 2 \ldots$ Going to keep going.