

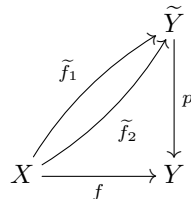
Announcements

- Quiz on Wednesday
 - Know definition and basic properties (simply connected) of the universal cover
 - Know lifting properties (existence / uniqueness) for covers.

Back to Math!

Proposition .0.1

Let $p : \tilde{Y} \rightarrow Y$ be a covering map with X a connected space, then if two lifts \tilde{f}_1, \tilde{f}_2 of the same map f agree at a single point then they agree everywhere.



Proof. Let $S = \{x \in X \mid \tilde{f}_1(x) = \tilde{f}_2(x)\}$. Our goal is to show that S is both open and closed. Since X is connected, the only sets that are open and closed are X and \emptyset , and S is nonempty by hypothesis.

Choose $x \in X$ and let U be a neighborhood of $f(x)$ so that $p^{-1}(U)$ is a disjoint union of open subsets $\{U_\alpha\}$ each mapped homeomorphically to U by p . (Aside: we say that U is evenly covered by p , and that each open subset U_α is a slice of the preimage).

Now since $f = p \circ \tilde{f}_1 = p \circ \tilde{f}_2$ we must have that $\tilde{f}_1(x), \tilde{f}_2(x) \in p^{-1}(f(x))$. Let $\tilde{f}_1(x) \in U_1$ and $\tilde{f}_2(x) \in U_2$.

Exercise .0.1

Since \tilde{f}_1, \tilde{f}_2 are continuous there exists a neighborhood $N \subseteq X$ so that $\tilde{f}_1(x) \in \tilde{f}_1(N) \subseteq U_1$ and $\tilde{f}_2(x) \in \tilde{f}_2(N) \subseteq U_2$.

There are two cases:

- Suppose that $\tilde{f}_1(x) \neq \tilde{f}_2(x)$. Then U_1 and U_2 are disjoint because each U_α contains only one preimage of $f(x)$, so \tilde{f}_1 and \tilde{f}_2 must differ on every point of N . Therefore $X \setminus S$ is open (aka S is closed) because $x \in N \subseteq X \setminus S$.
- Suppose that $\tilde{f}_1(x) = \tilde{f}_2(x)$, that is $x \in S$. Then $U_1 = U_2$, so for all $n \in N$ we have:

$$\tilde{f}_1(n) = p^{-1}(f(n)) = \tilde{f}_2(n)$$

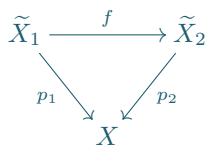
Where we're using p^{-1} here to mean $p^{-1} : U \rightarrow U_1 = U_2$, the inverse of the restriction $p|_{U_1} : U_1 \rightarrow U \rightarrow U$. This shows that S is open since $x \in N \subseteq S$.



1. Deck Transformations

Definition .1.1

Given covering maps $p_1 : \tilde{X}_1 \rightarrow X$ and $p_2 : \tilde{X}_2 \rightarrow X$, then an isomorphism of covers is a homeomorphism $f : \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $p_1 = p_2 \circ f$:



We can actually talk about a category of covering spaces, but we won't delve into that too much.

Exercise .1.1

This defines an equivalence relation on covers.

Definition .1.2

Fix a cover $p : \tilde{X} \rightarrow X$. The isomorphisms of the cover $\tilde{X} \rightarrow \tilde{X}$ are called deck transformations. We'll let $G(\tilde{X})$ be the set of deck transformations. Notice that we've suppressed the data of p in the notation, but this data is essential to what a deck transformation is, when this is unclear we write $G(\tilde{X}, p)$.

Exercise .1.2

Deck transformations $G(\tilde{X})$ are a subgroup of the group of homeomorphisms of \tilde{X} .

Example .1.3

Consider the cover $p : \mathbb{R} \rightarrow S^1$, then $G(\mathbb{R}) \cong \mathbb{Z}$, and $n \in \mathbb{Z}$ acts on \mathbb{R} by translating n units.

Example .1.4

There are covers $p_n : S^1 \rightarrow S^1$ where we “wind n times.” Then $G(S^1, p_n) \cong \mathbb{Z}/n\mathbb{Z}$ which acts by rotation.

Exercise .1.5

Notice that a deck transformation $\tau : \tilde{X} \rightarrow \tilde{X}$ is a lift of $p : \tilde{X} \rightarrow X$:

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow \tau & \downarrow p \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

Then by the unique lifting property (Proposition .0.1), τ is determined by the image of a single point when X is connected.

Corollary .1.1

If a deck transformation has a fixed point, it is the identity transformation.

Exercise .1.6

Let X be connected. Given a deck transformation $\tau : \tilde{X} \rightarrow \tilde{X}$, and $x_0 \in X$, τ defines a permutation of $p^{-1}(x_0)$. If this permutation has a fixed point, then it is the identity.