Notes on
MATH 592
(Algebraic Topology)
Syllabus

May 28, 2021

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Contents

I. Foundations

This week: Some foundations.

- Homotopies of maps
- Homotopy Equivalence of spaces. A coarser notion of equivalence of spaces than homeomorphism
- CW Complexes. A class of topological spaces that is "the right setting" to do algebraic topology. They are more general than manifolds but still very well-behaved and also combinatorial.

I.1. Homotopies

I.1.1. Basic Definitions

Definition I.1.1

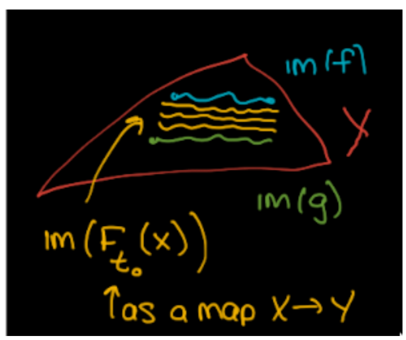
Let X, Y be topological spaces and f, g be continuous maps $X \to Y$. By definiton a homotopy from f to g is a continuous 1-parameter family of maps that we can view as continuously deforming the map f to the map g.

Concretely, a homotopy from f to g it is a map $F : X \times I \to Y$, where $I = [0, 1]$ is a closed interval, subject to the conditions that for all $x \in X$:

$$F(x, 0) = f(x) \qquad\qquad F(x, 1) = g(x)$$

We often write $F_t(x)$ for $F(x, t)$.

We should think of t as a time parameter, and the map F as giving a deformation of the map f into a map g. In other words, this is a family of maps $X \to Y$ interpolating between f and g. In pictures, this looks like:



Definition I.1.2

If a homotopy exists from f to g, we say that f and g are homotopic and write $f \simeq g$.

If f is homotopic to a constant map, then we write $f \simeq *$ and we call f nullhomotopic

Example I.1.1

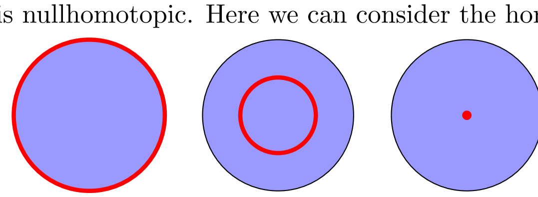
Any two maps $f, g : X \to \mathbb{R}$ are homotopic. We can deform f to g by the "straight line homotopy":

$$F_t(x) = tf(x) + (1 - t)g(x)$$

Example I.1.2

Let S^1 be the unit circle in $\mathbb{R}^2$ and let D^2 be the closed unit ball in $\mathbb{R}^2$.

The inclusion $S^1 \hookrightarrow D^2$ is nullhomotopic. Here we can consider the homotopy $F_t(x) = (1 - t)f(x)$



$t = 0$ $\qquad$ $t = 1/2$ $\qquad$ $t = 1$

Example I.1.3

The maps:

$$\begin{aligned} S^1 &\to S^1 \\ \theta &\mapsto \theta \end{aligned} \qquad\qquad \begin{aligned} S^1 &\to S^1 \\ \theta &\mapsto -\theta \end{aligned}$$

are not homotopic.

Exercise I.1.4

On homework, you will prove that "homotopic" is an equivalence relation on maps $X \to Y$

Breakout Rooms

Exercise I.1.5

A subset of $\mathbb{R}^n$ is called star-shaped if there exists some $x_0 \in S$ so that for all $x \in S$, the line segment from x to x_0 is contained in S. Show that any map from a space to S is nullhomotopic.

Solution. We will show that any map $f : X \to S$ is homotopic to the constant map $x_0 : X \to S$. This is given by the straight line homotopy:

$$F_t(x) = (1 - t)f(x) + tx_0$$

We know that $f(x) \in S$, so this straight line is contained in S because S is star-shaped. Furthermore this is continuous since it is a convex combination of continuous functions. Of course $F_0(x) = f(x)$ and $F_1(x) = x_0$, and so f is nullhomotopic.

Exercise I.1.6

Suppose that we have the following maps:

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

And further that $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then show that $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Solution. Write $F : X \times [0, 1] \to Y$ and $G : Y \times [0, 1] \to Z$ as the homotopies from f_0 to f_1 and g_0 to g_1 respectively. Then consider the map:

$$H_t(x) = G_t(F_t(x))$$

By writing out this map more explicitly we can show that it is continuous:

$$H(x, t) = G(F(x, t), t)$$

Note then this is a composition of the continuous maps given as:

$$(x, t) \mapsto (F(x, t), t) \mapsto G(F(x, t), t)$$

Of course, note that since the map $(x, t) \mapsto (F(x, t), t)$ is continuous in each component it is continuous overall. Therefore H is continuous since it is a composition of continuous functions. Thus, H gives a homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$ since F and G are homotopies and we know:

$$\begin{aligned} H_0(x) = G_0(F_0(x)) = g_0(f_0(x)) = (g_0 \circ f_0)(x) \\ H_1(x) = G_1(F_1(x)) = g_1(f_1(x)) = (g_1 \circ f_1)(x) \end{aligned}$$

Therefore $g_0 \circ f_0 \simeq g_1 \circ f_1$ just as desired.

Exercise I.1.7

How could you prove that two maps are not homotopic

Resume Breakout Rooms

Last time we were working on the following problem:

Exercise I.1.8
How could you prove that two maps are not homotopic

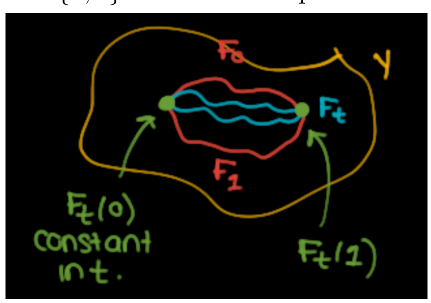
Solution. We had a few ideas for the example maps from last time:

- We can embed S^1 in a larger space like $\mathbb{R}^2$ and show that any homotopy between the two maps will land outside the circle at some point
- We can use the fact that one map is orientation-preserving and one map is orientation-reversing and show that homotopic maps will have the same behavior on orientation.
- Show that any homotopy in S^1 will become a bad homotopy in $\mathbb{R}$ when we lift it back by using the fact that $S^1 = \mathbb{R}/\mathbb{Z}$.

Resume Lecture

Definition I.1.3
Let X, Y be spaces and let $B \subseteq X$ be a subspace. A homotopy $F_t(x) : X \times [0, 1] \to Y$ is called a homotopy relative to B ("rel B") if $F_t(b)$ is independent of t for every $b \in B$.

Example I.1.9
Homotopies of paths $[0, 1] \to Y$ rel $\{0, 1\}$. Here's a nice picture courtesy of Jenny!



I.2. Homotopy Equivalence

Definition I.2.1
A map $f : X \to Y$ is a homotopy equivalence if there exists a $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. You can say these are inverses "up to homotopy"

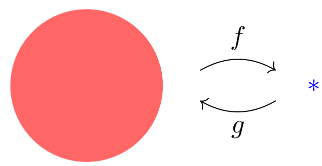
X and Y are called homotopy equivalent and we say they have the same homotopy type provided that there exists a homotopy equivalence between them. We write $X \simeq Y$.

Exercise I.2.1 (Homework)
You will prove that "homotopy equivalent" is an equivalence relation

Example I.2.2
If we look at D^n (the closed n-ball), then we see that $D^n \simeq *$, aka it is homotopy eqivalent to a point.

The map $f : D^n \to *$ is trivial, and we can choose any map $g : * \to D^n$, but for simplicitly we'll choose the map $g : * \mapsto 0$. Then $f \circ g = \mathrm{Id}_*$ so that's easy. On the other side $g \circ f : D^n \to D^n$ is the constant map which maps the entire disk to the origin.

Note that by using the straight line homotopy we can see that $g \circ f \simeq \mathrm{Id}_{D^n}$, and so D^n is homotopic to a point. Here's a picture



Definition I.2.2
A space X is contractible if it is homotopy equivalent to a point.

Example I.2.3
$\mathbb{R}^n \simeq *$, using a proof similar to the above.

Take-aways:
- $\mathbb{R}^n \simeq \mathbb{R}^m \simeq *$ for all n, m
- Homotopy equivalence does not preserve dimension
- It does not preserve compactness since $D^2 \simeq * \simeq \mathbb{R}^2$

Example I.2.4
The inclusion $S^1 \hookrightarrow D^2$ is not a homotopy equivalence. Right now this would be fairly hard to prove.

Definition I.2.3
Given a space X and a subspace $B \subseteq X$. A (strong) deformation retraction F_t is a homotopy rel B to a map with image in B from Id_X.
In plain terms:
- $F_0(x) = x$ for all $x \in X$
- $F_1(x) \in B$ for all $x \in X$
- $F_t(b) = b$ for all $b \in B$ and $t \in [0, 1]$.

Exercise I.2.5
When X deformation retracts to a subspace B, X is homotopy equivalent to the subspact B.

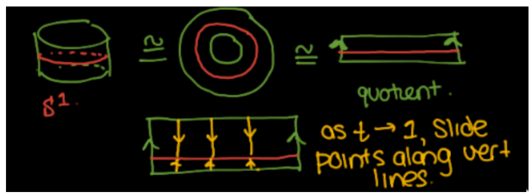
Announcements

- Quiz #1 is on Wednesday. Here are your hints!:
  - Know definition and examples from Lecture I on homotopies
  - You may assume the result from homework: "homotopic" is an equivalence relation.

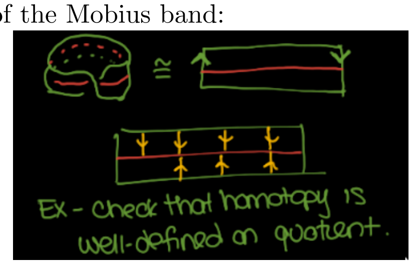
Lecture Time!

Example I.2.6
The point $\{0\}$ is a deformation retract of $\mathbb{R}^n$ via the straight-line homotopy. (That is $\mathbb{R}^n$ deformation retracts onto $\{0\}$, to make things unambiguous)

Example I.2.7
S^1 is the circle, and the circle is a deformation retract of the cylinder.



S^1 is also a deformation retract of the Mobius band:



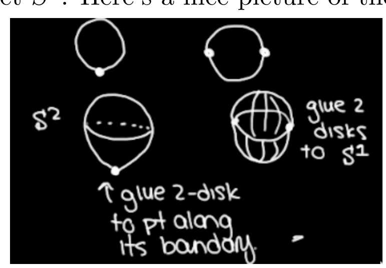
Take-away: "Homotopy equivalence" does not respect orientability, since the cylinder is orientable but the Mobius band is not.

Exercise I.2.8
Prove that any homotopy equivalence induces a bijection on path components, and thus the number of path components is a homotopy invariant. This is in a sense the most basic homotopy invariant, and much of our course is focused on building more of these invariants.

I.3. CW Complexes

I.3.1. Examples of CW complexes

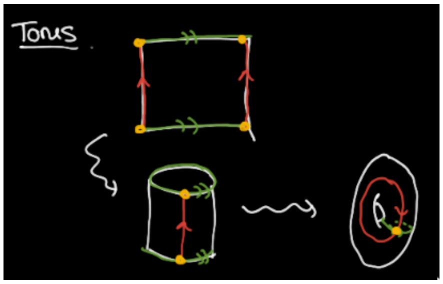
Example I.3.1 (S^1 and S^2)
We can take an interval and glue the two points of its boundary together to get S^1. Similarly we can cosnstruct S^2 by gluing the boundary of 2-disk together.
We could also take two intervals and glue them at their boundaries to make S^1, or two 2-disks and glue them at their boundaries to get S^2. Here's a nice picture of these four constructions:



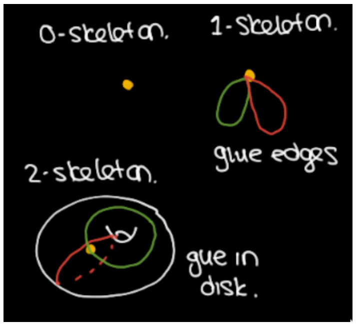
Here are some more explicit instructions for the S^2 construction, since it can be a little bit unclear:

Example I.3.2 (Torus)

Here's the traditional method of building a torus as a quotient space. Notice that the four corners are identified:



We can also build a torus inductively by gluing in edges then gluing in disks:



We can now view the square above as giving us gluing instructions for gluing in the edges to the point in the 1-skeleton, and the disk to the edges in the 2-skeleton.

Breakout Rooms

Exercise I.3.3

Prove that if $X \simeq Y$ then X is path connected if and only if Y is.

Solution. Note that it suffices to prove that when $X \simeq Y$ and X is path connected that Y is path connected because homotopy equivalence is an equivalence relation. Let the homotopy equivalence be given by $f : X \to Y$ and $g : Y \to X$, and let $F : Y \times [0,1] \to Y$ be the homotopy from $f \circ g$ to Id_Y. Note that for any $y \in Y$ this induces a path from y to $f(g(y))$ by holding the first input to the homotopy fixed:

$$\phi_y(t) = F_t(y) \qquad \phi_y(0) = f(g(y)) \qquad \phi_y(1) = 1$$

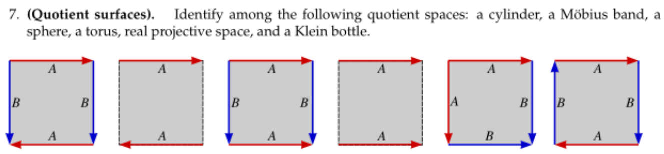
Great!

Now fix two points $y, z \in Y$. We know since X is path connected that there is some path $p : [0,1] \to X$ from $g(y)$ to $g(z)$, and we can compose this with f to get a path from $f(g(y))$ to $f(g(z))$ given by $f \circ p$. We then know that there is a path from y to $f(g(y))$ and a path from $f(g(z))$ to z given by the above, and pasting these paths appropriately we get a path from y to z as desired! Therefore Y is path connected!

Awesome! We now have our first homotopy invariant!

Exercise I.3.4

Identify these quotient spaces!



Let's go in order!

(1) Klein bottle
(2) Mobius band
(3) Torus
(4) Cylinder
(5) Sphere
(6) Real projective space

Great ☺

Definition I.3.1

D^n is the closed n-disk and $S^{n-1} = \partial D^n$

A 0-cell is a point, and a n-cell for $n \geq 1$ is the interior of D^n.

I.3.2. The CW Complex Definitions

Definition I.3.2

A CW complex (cell complex) is a topological space constructed as follows:

- X^0 (0-skeleton) is a set of discrete points
- We build X^n (n-skeleton) from X^{n-1} by attaching n-cells e^n_α (α is an index). The instructions for how to "glue" e^n_α are given by the attaching map
$$\phi_\alpha : S^{n-1} \to X^{n-1}$$
This tells us where to glue the boudary of an n-disk ∂D^n, which will be a boundary of our n-cell e^n_α. Formally, we take:
$$X^n = \left(X^{n-1} \coprod_\alpha D^n_\alpha \right) / \ (x \sim \phi_\alpha(x) \ \ \forall x \in \partial D^n_\alpha)$$
As a set then:
$$X^n = X^{n-1} \coprod_\alpha e^n_\alpha$$
- We define $X = \bigcup_{n \geq 0} X^n$ with the weak topology. That is U is open in X if and only if $U \cap X^n$ is open in X^n for all $n \geq 0$.

Announcements

- First office hours tonight, 8pm-9pm, use the "Zoom Lounge"
- Homework #1 due 8pm Friday on Gradescope.

Definition I.3.3

We need a few definitions for working with CW complexes!

- A CW-complex is called finite if it involves a finite number of cells.
- A subcomplex of a CW complex is a closed subset consisting of a union of cells.

Exercise I.3.5

A subcomplex is itself a CW complex.

I.3.3. Operations on CW Complexes

Definition I.3.4

We can consider the product of two CW complexes can be given a CW complex structure.

Namely, given X and Y CW complexes, we can take two cells e^n_α from X and e^m_β from Y we can form the product space $e^n_\alpha \times e^m_\beta$ which is homeomorphic to an $(n+m)$-cell. We take these products as the cells for $X \times Y$

Warning It is possible (in "pathological" cases) that the product topology on $X \times Y$ does not agree with the weak topology. They do agree if either X or Y is locally compact or if X and Y have at most countably many cells

Exercise I.3.6

The torus is $S^1 \times S^1$. Write down the CW complex structure on the torus that comes from the CW complex structure on S^1 with one point and one edge.

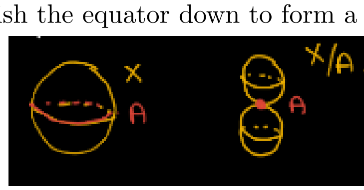
Definition I.3.5

If X is a CW complex and A is a subcomplex, then the quotient X/A (A is identified to a point) inherits a CW complex structure. Namely

- The 0-skeleton is points in $X^0 - A^0$ unioned with one point for A
- Each cell in $X^n - A$ is attached to $(X/A)^n$ by the attaching map defined by composing with the quotient map $S^n \to X^n \to X^n/A^n$.

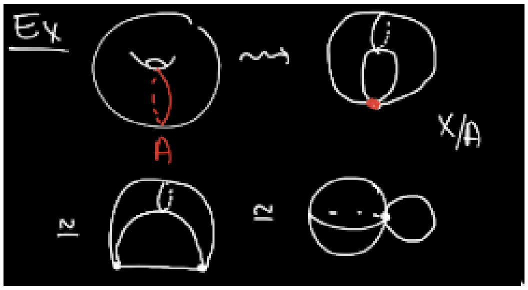
Example I.3.7

We can take the sphere and squish the equator down to form a wedge of two spheres as follows:



Example I.3.8

We can take the torus and squish down a ring around the hole like in this picture:



The above is homotopy equivalent to a 2-sphere wedged with a 1-sphere via the extending the red point into a line, and then sliding the left point of the line along the two-sphere towards the other point, forming a circle.

Definition I.3.6

Take X, Y to be CW complexes, and let $x_0 \in X^0, y_0 \in Y^0$. Then we can consider $X \vee Y$ which is the quotient of $X \cup Y$ by identifying x_0 and y_0 to one point, called the wedge sum.

Announcements

- Quiz 1 now graded - Gradescope, and solutions are posted on the webpage
- Homework #1 due 8pm tonight on Gradescope, be sure to select the pages
- Office hours 2:30-4:30 in "Lounge Zoom"

Back to Lecture!

I.4. Intro to Category Theory

I.4.1. Our Definitions

Definition I.4.1

A category $\mathscr{C}$ is three pieces of data with two conditions. Here's the data first:

- A class of objects, $\mathrm{Ob}(\mathscr{C})$.
- For all $X, Y \in \mathrm{Ob}(\mathscr{C})$ we have a class of morphisms (or arrows) denoted $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- For every $X, y, Z \in \mathrm{Ob}(\mathscr{C})$ we have a composition law, that is a map:

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}}(X,Z)$$
$$(f,g) \mapsto g \circ f$$

All of this satisfies the following two conditions:

- Associativity of composition:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

  And this holds for all f, g, h such that these compositions make sense.
- For every object $X \in \mathrm{Ob}(\mathscr{C})$ there should exist some $\mathrm{Id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that;

$$f \circ \mathrm{Id}_X = f \qquad \mathrm{Id}_X \circ g = g$$

  For all f, g such that these compositions are defined.

Example I.4.1

Let's get some examples down.

$\mathscr{C}$	$\mathrm{Ob}(\mathscr{C})$	$\mathrm{Mor}(\mathscr{C})$
Set	Sets	Functions
Grp	Groups	Group Homomorphisms
Ab	Abelian Groups	Group Homomorphisms
k-Vect	Vector spaces over k	k-linear maps
Rng	Rings	Ring Homomorphisms
Top	Top. Spaces	continuous maps
Haus	Hausdorff Spaces	continuous maps
hTop	Top Spaces	homotopy classes of continuous maps
Top*	Top spaces with a distinguished basepoint	Continuous maps that preserve the basepoint

Example I.4.2

Any "diagram" with a composition law defines a category. Just consider:

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{Id}_A"] \arrow[r, "f"] & B \arrow[loop right, "\mathrm{Id}_B"] \end{tikzcd}$$

Definition I.4.2

A morphism $f : M \to N$ in a category $\mathscr{C}$ is monic if for all $g_1, g_2 : X \to M$ with the same domain:

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_1"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, f is epic if for all $g_1, g_2 : N \to X$ we have:

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_1"'] & \bullet \end{tikzcd}$$

Breakout Rooms

Lemma I.4.1
In **Set**, **Ab**, **Top** a map is monic if and only if it is injective.

Solution. We'll do the problem in abelian groups. We go in both directions!

($\Rightarrow$) Suppose that $f : G \to H$ is a monic map of abelian groups. We wish to show that $f : G \to H$ is injective. Fix any two elements $a, b \in G$, and construct morphisms $\phi_a : \mathbb{Z} \to G$ and $\phi_b : \mathbb{Z} \to G$ as follows:

$$\phi_a(n) = n \cdot a \qquad \phi_b(n) = n \cdot b$$

Since we know that abelian groups are $\mathbb{Z}$-modules. Now suppose that $f(a) = f(b)$. Then consider that:

$$(f \circ \phi_a)(n) = f(n \cdot a) = n \cdot f(a) = n \cdot f(b) = f(n \cdot b) = (f \circ \phi_b)(n)$$

And therefore $f \circ \phi_a = f \circ \phi_b$. This shows since f is monic that $\phi_a = \phi_b$. However then we're in business since:

$$a = \phi_a(1) = \phi_b(1) = b$$

And so $a = b$. This shows that f is injective. Awesome ☺

($\Leftarrow$) Suppose that the map $f : G \to H$ is injective. We will show that f is monic. To do so, fix two maps $g_1, g_2 : A \to G$ where A is an abelian group, and suppose that $f \circ g_1 = f \circ g_2$. Then for any $a \in A$ we know that $f(g_1(a)) = f(g_2(a))$, giving us since f is injective that $g_1(a) = g_2(a)$. Since this holds for arbitrary $a \in A$ we know that $g_1 = g_2$!!! Great! ☺

With this we've finished the problem

I.4.2. **Functors**

Definition I.4.3
For $\mathscr{C}, \mathscr{D}$ categories a (covariant) functor $F : \mathscr{C} \to \mathscr{D}$ is

(a) A map of objects:

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X) \end{aligned}$$

(b) A map of morphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ f &\mapsto F(f) \end{aligned}$$

In another way we can say that if we have $f : X \to Y$ in $\mathscr{C}$ then we get a new map lying in $\mathscr{D}$, $F(f) : F(X) \to F(Y)$.

With the extra conditions that:

(1) $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for all X in C
(2) $F(f \circ g) = F(f) \circ F(g)$ for all maps f, g in $\mathscr{C}$ for which the composition makes sense

For a contravariant functor we replace some conditions:

(b)' A map of morphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))\\ f &\mapsto F(f)\end{aligned}$$

In another way we can say that if we have $f : X \to Y$ in $\mathscr{C}$ then we get a new map lying in $\mathscr{D}$, $F(f) : F(Y) \to F(X)$.

(2)' $F(f \circ g) = F(g) \circ F(f)$ for all maps f, g in $\mathscr{C}$ for which the composition makes sense

Announcements

- Homework #2 posted – course website
- Quiz #2 Wednesday
  - Know definitions of category, functor
  - Review our examples of functors

Examples of Functors

Example I.4.3
We have an identity functor $\mathscr{C} \to \mathscr{C}$ for any category $\mathscr{C}$.

Example I.4.4 (Forgetful functors)
For example:

$$\begin{aligned} \mathscr{F} : \underline{\text{Grp}} &\to \underline{\text{Set}} \\ (G, \star) &\mapsto G \\ [f : (G, \star) \to (H, *)] &\mapsto \underbrace{[f : G \to H]}_{\text{same function}} \end{aligned}$$

There are lots of such examples. Consider:

$$\begin{aligned} \mathscr{F} : \underline{\text{Top}} &\to \underline{\text{Set}} \\ (X, \mathcal{T}) &\mapsto X \\ [f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)] &\mapsto \underbrace{[f : X \to Y]}_{\text{same function}} \end{aligned}$$

Similarly we have:

$$\begin{aligned} \mathscr{F} : \underline{\text{Top}^*} &\to \underline{\text{Top}} \\ (X, x_0) &\mapsto X \\ [f : (X, x_0) \to (Y, y_0)] &\mapsto [f : X \to Y] \end{aligned}$$

Example I.4.5 (Free Functors)
For every ring R we have a free functor:

$$\begin{aligned} F : \underline{\text{Set}} &\to \underline{R\text{-mod}} \\ A &\mapsto F(A) \\ [f : A \to B] &\mapsto \text{map of } R\text{-modules} \\ &\qquad F(A) \to F(B) \text{ that extends the map} \\ &\qquad f : A \to B \text{ on their bases} \end{aligned}$$

We similarly get free group constructions:

$$\begin{aligned} F : \underline{\text{Set}} &\to \underline{\text{Grp}} \\ A &\mapsto \text{free group on } A \end{aligned}$$

Example I.4.6
The dual space construction. Given a field k we have a contravariant functor:

$$\begin{aligned} \underline{k\text{-vect}} &\to \underline{k\text{-vect}} \\ V &\mapsto V^* = \operatorname{Hom}_k(V, k) \\ [A : V \to W] &\mapsto [A^* : \operatorname{Hom}_k(W, k) \to \operatorname{Hom}_k(V, k)] \\ &\qquad [\phi : W \to k] \mapsto [\phi \circ A : V \to K] \end{aligned}$$

I.5. Free Groups

Definition I.5.1

Let S be a set. The free group is a group F_S equipped with a map $S \to F_S$ satisfying the following universal property.

If G is any group, and $f : S \to G$ is any map of sets, then f extends uniquely to a group homomorphism $\overline{f} : F_S \to G$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & G \\ F_S \arrow[ur, dashed, "\exists!\ \overline{f}"'] & \end{tikzcd}$$

Aside: This defines a (natural) bijection:

$$\mathrm{Hom}_{\underline{\mathrm{Set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\mathrm{Grp}}}(F_S, G)$$

Where $\mathscr{U}$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoints; specifically that the free functor is the left adjoint (appears on the left in the Hom's above).

Remark I.5.1

Whenever we state a universal property (UP) for an object (+ map) in a category, an object (+ map) satisfying the UP may or may not eist.

However, if it does exist, the UP determines it "uniquely up to unique isomorphism." Sow we may take the UP to be the definition of the object (+ map)

Claim

The Universal Property determines F_S (+ map $S \to F(S)$) uniquely up to unique isomorphism

Proof. Fix S. Suppose $F_S, \widetilde{F}_S$ with maps $S \to F_S$ and $S \to \widetilde{F}_S$ which both satisfy the universal property.

There must exist unique maps filling in the bottom two diagrams by the universal property for $\widetilde{F}_S$ and F_S:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F_S \\ \widetilde{F}_S \arrow[ur, dashed, "\exists!\ f"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[d] & \widetilde{F}_S \\ F_S \arrow[ur, dashed, "\exists!\ g"'] & \end{tikzcd}$$

The goal is to show that f and g are inverses (and hence isomorphisms). The uniqueness follows from the condition that f and g are the only group homomorphisms making the above diagrams commute.

We now paste the above diagrams together in two different ways:

$$\begin{tikzcd} & \widetilde{F}_S \\ S \arrow[ur] \arrow[r] \arrow[dr] & F_S \arrow[u, dashed, "g"] \\ & \widetilde{F}_S \arrow[u, dashed, "f"] \arrow[uu, dashed, bend right=60, "g\circ f"'] \end{tikzcd} \qquad \begin{tikzcd} & F_S \\ S \arrow[ur] \arrow[r] \arrow[dr] & \widetilde{F}_S \arrow[u, dashed, "f"] \\ & F_S \arrow[u, dashed, "g"] \arrow[uu, dashed, bend right=60, "f\circ g"'] \end{tikzcd}$$

We then observe that the outer triangle in each case is a Universal Property diagram for $\widetilde{F}_S$ and F_S respectively. Since the identity makes these outer triangles commute, we can conclude from the commutativity of the above diagrams and uniqueness that these are isomorphisms, aka:

$$g \circ f = \mathrm{Id}_{\widetilde{F}_S} \qquad f \circ g = \mathrm{Id}_{F_S}$$

Announcements

- Homework #1 feedback is on gradescope
- Homework #2 due 8pm Friday
- Office hours 8pmm-9pm today

I.5.1. Construction of free groups

Proposition I.5.1
The free group defined via the universal property before exists. We will give a construction below.

Definition I.5.2
Fix a set S. A word is a sequence (possibly empty) of formal symbols $\{s, s^{-1} \mid s \in S\}$.

Proof. Fix S, F_S is equivalence classes of words:

$$vss^{-1}w \sim vw$$
$$vs^{-1}sw \sim vw$$

For every words v, w. The group operation is concatenation of words.

Example I.5.1
Given words ab^{-1}, bba their product is:

$$ab^{-1} \cdot bba = ab^{-1}bba = aba$$

Exercise I.5.2
This product is well-defined on equivalence classes.

Exercise I.5.3
Every equivalence class of words has a unique "reduced form."

Exercise I.5.4
F_S satisfies the Universal Property with respect to the map:

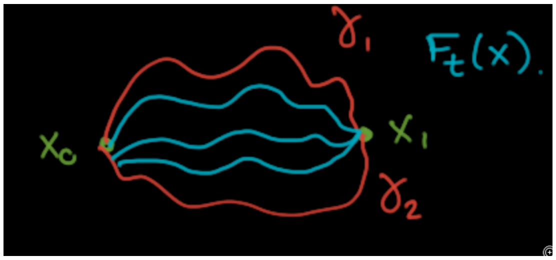
$$S \to F_S$$
$$s \mapsto s$$

II. The Fundamental Group π_1

II.1. Basic Definitions

Definition II.1.1
A path in a space X is a continuous map $\gamma : I \to X$
γ is a loop if $\gamma(0) = \gamma(1)$

Definition II.1.2
A homotopy of paths γ_1, γ_2 is a homotopy from γ_1 to γ_2 rel $\{0, 1\}$.

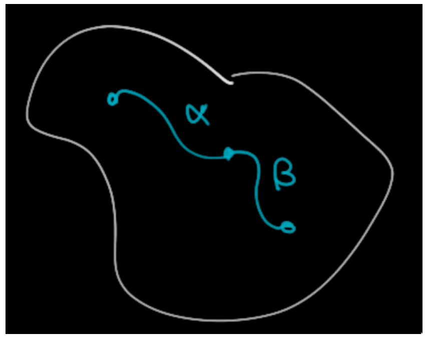


Definition II.1.3

For paths α, β with $\alpha(1) = \beta(0)$, the composition, product, or concatenation of these paths is defined as follows:

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le 1/2 \\ \beta(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

By the pasting lemma this is continuous. Thus $\alpha \cdot \beta$ is a path from $\alpha(0)$ to $\beta(1)$.



Definition II.1.4

A reparameterization of γ is a path:

$$I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

Such that $\varphi(0) = 0$ and $\varphi(1) = 1$, and $\varphi : I \to I$ is continuous

Exercise II.1.1 (Homework)

A path γ is homotopic rel $\{0, 1\}$ to all of its reparameterizations.

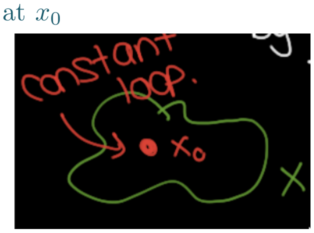
Exercise II.1.2

Fix $x_0, x_1 \in X$. "Homotopy of paths" (relative $\{0, 1\}$) is an equivalence relation on paths from x_0 to x_1.

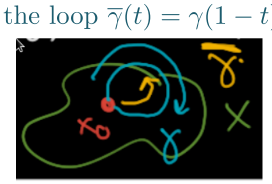
Definition II.1.5

Let X be a space, and $x_0 \in X$. The fundamental group of X based at x_0 (denoted $\pi_1(X, x_0)$) is a group:

- Elements are homotopy classes rel $\{0, 1\}$ of loops γ with $\gamma(0) = \gamma(1) = x_0$ (we say γ is based at x_0)
- The operation is composition of paths
- The identity is the constant loop at x_0



- The inverse $[\gamma]^{-1}$ is represented by the loop $\overline{\gamma}(t) = \gamma(1-t)$:



The proof that this is a group is Homework

Exercise II.1.3
Composition of paths is well-defined on homotopy classes rel $\{0, 1\}$.

Theorem II.1.1 (Homework)
If X is path-connected then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for every $x_0, x_1 \in X$. So we can write $\pi_1(X)$ up to isomorphism.

Exercise II.1.4
If X is a contractible space, then X is path connected and $\pi_1(X)$ is trivial.

II.2. Calculations with $\pi_1(S^n)$

Theorem II.2.1
$\pi_1(S^1) \cong \mathbb{Z}$, and this identification is given by the following paths:

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))]$$

Intuitively this winds around S^1 n times. The key to this proof was to understand S^1 via the covering space $\mathbb{R} \to S^1$. We will talk about covering spaces more in class later.

Proof. See Homework

Theorem II.2.2
There is a natural identification $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. The identfication is exactly:

$$[\gamma : I \to X \times Y] \leftrightarrow ([p_X \circ \gamma], [p_Y \circ \gamma])$$

Where p_X and p_Y are the projections.

Exercise II.2.1
Give a proof of ths result. The key is that a map:

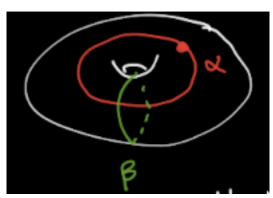
$$Z \xrightarrow{f} X \times Y$$

$$z \xmapsto{f} (f_X(z), f_Y(z))$$

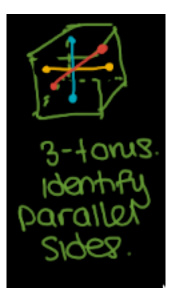
f is continuous if and only if f_X and f_Y are. The proof should go like:

- Apply this principle to paths $I \to X \times Y$
- Apply this principle again to homotopies of paths $I \times I \to X \times Y$.

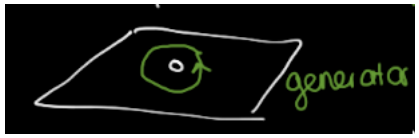
Corollary II.2.3
The torus $T \cong S^1 \times S^1$ has fundamental group $\pi_1(T) \cong \mathbb{Z}^2$. This will in fact be generated by the loops around each of the factors:



Furthermore the n-torus $\underbrace{S^1 \times \cdots \times S^1}_{n}$ has fundamental group $\mathbb{Z}^n$. One way to think of the n-torus is as an n-dimensional cube with opposite $(n-1)$-dimensional faces identified by translation. We include a picture of the 3-torus with the generators:



Corollary II.2.4
$\mathbb{R}^2 - \{0\} \cong S^1 \times \mathbb{R}$ must have fundamental group $0 \times \mathbb{Z} \cong \mathbb{Z}$. Intuitively the generators are just loops around the hole:



Theorem II.2.5

$\pi_1(S^n) \cong 0$ for all $n \geq 2$. The picture for the 2-sphere is fairly simple:

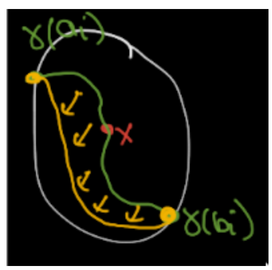
Great!

Proof. Let $\gamma : I \to S^n$ be loop based at x_0. Choose $x \neq x_0$.

Goal: Homotope γ (rel $\{0,1\}$) so that x is not in its image. Then we know $S^n \setminus \{x\} \cong \mathbb{R}^n$, which deformation retracts to x_0, and we can use this deformation retraction to push our loop to a point.

Strategy: Choose a small open ball B about x. Then we know that $\gamma^{-1}(B)$ is open in I, so it is a union of disjoint open intervals (a_i, b_i). The set $\gamma^{-1}(x)$ is compact, since it is a closed subset of I, so it is contained in finitely many (a_i, b_i)

The next observation is that by continuity of γ we have $\gamma([a_i, b_i]) \subseteq \overline{\gamma((a_i, b_i))} \subseteq \overline{B}$, and thus the endpoints a_i, b_i must map to ∂B. So then for all i for which $\gamma((a_i, b_i))$ passes through x choose a new path α_i in $\overline{B}$ from $\gamma(a_i)$ to $\gamma(b_i)$ that misses x:



We homotope $\gamma\big|_{[a_i, b_i]}$ to α_i rel $\{0,1\}$. We can always do this because D^n is contractible and has enough "room" to not pass through the point x for $n \geq 2$. We do this finitely many times by the compactness argument above, and so we're done! We have a path homotopic to γ that misses x.

Breakout Rooms: π_1 is a functor

Theorem II.2.6

$\pi_1 : \underline{\text{Top}*} \to \underline{\text{Grp}}$ is a functor. Recalle the objects in $\underline{\text{Top}*}$ are spaces with a distinguished basepoint (X, x_0), and the maps are base point preserving maps.

The functor is defined on morphisms by considering that a map $f : X \to Y$ taking x_0 to y_0 induces:

$$\begin{aligned} f_* : \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

Proof: In Breakout Groups. We need to check four things:

(1) f_* is well-defined on homotopy classes
(2) f_* is a group homomorphism
(3) $(\text{Id}_{(X,x_0)})_* = \text{Id}_{\pi_1(X,x_0)}$
(4) $(f \circ g)_* = f_* \circ g_*$.

Let's go in order!

(1) Take two homotopic paths γ, γ', then there's a relative homotopy between them $\Gamma : I \times I \to X$. Then we can postcompose this to get a homotopy $f \circ \Gamma$ from $f \circ \gamma$ to $f \circ \gamma'$. This will be a relative homotopy since f preserves basepoints.

(2) This is a pretty simple calculation. Fix two loops α, β at x_0. Then we see that:

$$\begin{aligned}(f \circ (\alpha \cdot \beta))(t) &= \begin{cases} f(\alpha(2t)) & \text{if } 0 \leq t \leq 1/2 \\ f(\beta(2t-1)) & \text{if } 1/2 \leq t \leq 1 \end{cases} \\ &= ((f \circ \alpha) \cdot (f \circ \beta))(t)\end{aligned}$$

Great! Thus this is a group homomorphism since:

$$f_*([\alpha][\beta]) = f_*([\alpha \cdot \beta]) = [f \circ (\alpha \cdot \beta)] = [(f \circ \alpha) \cdot (f \circ \beta] = f_*(\alpha) f_*(\beta)$$

(3) We see that for any loop γ at x_0:

$$(\mathrm{Id}_{(X,x_0)})_*([\gamma]) = [\mathrm{Id}_{(X,x_0)} \circ \gamma] = [\gamma]$$

And so this map is the identity on the group!

(4) The calculation here is similar to the quiz, using associativity of composition. Namely given a loop γ at x_0 we see that for $f : (Y, y_0) \to (Z, z_0)$ and $g : (X, x_0) \to (Y, y_0)$ that :

$$\begin{aligned}(f \circ g)_*([\gamma]) &= [(f \circ g) \circ \gamma] = [f \circ (g \circ \gamma)] \\ &= f_*([g \circ \gamma]) = f_*(g_*([\gamma])) = (f_* \circ g_*)([\gamma])\end{aligned}$$

Great!

With this established we've seen our first really cool example of a functor ☺.

Announcements

- Quiz #3 Wednesday. Hints:
  - Know our calculation of π_1 for spheres and contractible spaces (don't need proof)
  - Know our result on π_1 of a product
  - Know the definition of a retraction
  - Understand solution to Homework #3, Assignment Questions 1a and 1b
- Midterm next week, February 18th 7-8pm

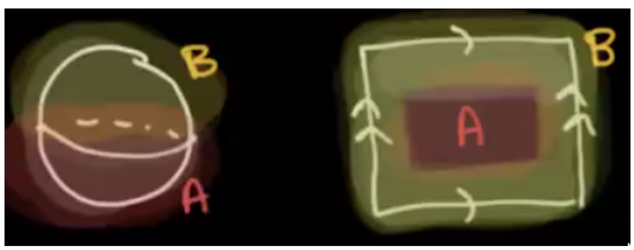
Definition II.2.1 (From Homework)
Let X be a topological space, and $A \subseteq X$ a subspace. A retraction $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$. The subspace A is called a retract of X.

Lecture

Last time, we showed in breakout rooms that π_1 is a functor from Top∗ to Grp. Note that parts (3) and (4) were very similar to the quiz!

II.3. The Van Kampen Theorem

Goal: Compute $\pi_1(X)$ where $X = A \cup B$ using the data of $\pi_1(A)$, $\pi_1(B)$, $\pi_1(A \cap B)$.



Definition II.3.1 (Free product of groups with amalgamation)
Given some collection of groups $\{G_\alpha\}_\alpha$, the free product $\underset{\alpha}{*} G_\alpha$ is a group:

- Elements are words $g_1 g_2 \cdots g_n$ where $g_i \in G_\alpha$ for some α. Modulo the equivalence relation generated by:
  - First we have
  $$w g_i g_j v \sim w(g_i g_j)v$$
  Whenever both $g_i, g_j \in G_\alpha$. And also:
  - We also want to deal with identities 1_α for $1_\alpha \in G_\alpha$ the identity element
  $$w 1_\alpha v = wv$$
  Great!
- Operation is concatenation of words.

If groups G_α and G_β have a common subgroup H (inclusion maps $i_\alpha : H \to G_\alpha$ and $i_\beta : H \to G_\beta$) then the free product with amalgamation ${}_\alpha *_H G_\alpha$ is defined as $\underset{\alpha}{*}$ modulo the subgroup generated by the words:

$$i_\alpha(h) i_\beta(h)^{-1}$$

Aka, $i_\alpha(h)$ and $i_\beta(h)$ will be identified in the quotient.
We can then write out words as such as $g_1 g_2 h g_3$ for $h \in H$, and view h as an element of G_α or G_β. In fact, we can do this construction even when i_α and i_β are not injective, though this means we are not working with a subgroup.

Exercise II.3.1
Check that ${}_\alpha *_H G_\alpha$ is well-defined as a group under concatenation.

Exercise II.3.2

${}_{\alpha}*_H\, G_\alpha$ contains each group G_α as a subgroup in a canonical way.

Exercise II.3.3

$G_1 *_H G_2$ satisfies the universal property–which is called a pushout–meaning that whenever we have $f_1 : G_1 \to Z$ and $f_2 : G_2 \to Z$ such that $f_1 \circ i_1 = f_2 \circ i_2$ then there is a unique $u : G_1 *_H G_2$ making the diagram commute:

$$\begin{tikzcd}
H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d] \arrow[ddr, bend left, "f_1"] & \\
G_2 \arrow[r] \arrow[drr, bend right, "f_2"'] & G_1 *_H G_2 \arrow[dr, dashed, "\exists!\ u"'] & \\
& & Z
\end{tikzcd}$$

Awesome!

Note on Notation: The α in ${}_{\alpha}*_H$ is the indexing set, and the amalgamating group is H, with maps $H \to G_\alpha$, $H \to G_\beta$ left implicit. This notation may only be standard for working with two groups.

Analogue: If we have sets A, B with common subset C (i.e. $A \cap B = C$), then we sometimes write $A \cup_C B = A \cup B$, then again we have this universal property:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \cup_C B \arrow[dr, dashed, "\exists!\ u"'] & \\
& & Z
\end{tikzcd}$$

The universal property is actually a bit more general if we take any maps $C \to A$ and $C \to B$. In this case we have:

$$A \cup_C B = A \sqcup B / [f(c) \sim g(c)]$$

Theorem II.3.1 (Van Kampen)

Here are the preconditions:

- Suppose we have a space X with base point x_0.
- We have $X = \bigcup_\alpha A_\alpha$
- A_α are each open, path-connected, and contain x_0
- $A_\alpha \cap A_\beta$ is path-connected.

Then there exists a surjective homomorphism $*_\alpha\, \pi_1(A_\alpha) \to \pi_1(X)$.

If we additionally assume that if $A_\alpha \cap A_\beta \cap A_\gamma$ are all path connected, then:

$$\pi_1(X) \cong {}_{\alpha}*_{\pi_1(A_\alpha \cap A_\beta)}\, \pi_1(A_\alpha)$$

associated to all maps $\pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha), \pi_1(A_\beta)$ induced by inclusions of spaces. I.e. $\pi_1(X)$ is a quotient of the free product $*_\alpha\, \pi_1(A_\alpha)$ where we have:

$$(i_{\alpha\beta})_* : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$$

which is induced by the inclusion $i_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$. We quotient by the normal subgroup generated by:

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_*(\gamma)^{-1} \mid \gamma \in \pi_1(A_\alpha \cap A_\beta)\}$$

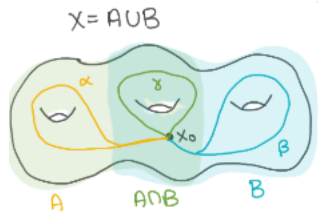
We're often interested in the special case with two sets:

Theorem II.3.2 (Van Kampen for two sets)

For $X = A \cup B$ and A, B open path connected sets containing x_0 with $A \cap B$ path connected, then:

$$\pi_1(X) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

Here's a great visualization of the Van Kampen Theorem:



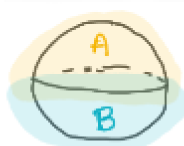
Intuitively we see the fundamental group of X–which is built by gluing A and B along their intersection–as the fundamental group of A and B glued along the fundamental group of their intersection. In essence, $\pi_1(X, x_0)$ is the quotient of $\pi_1(A) * \pi_1(B)$ by relations to impose the condition that loops like γ lying in $A \cap B$ can be viewed as elements of either $\pi_1(A)$ or $\pi_1(B)$.

Announcements

- Midterm – Thursday February 18th
- Shorter homework next week on Van Kampen to review for the exam
- No quiz next week.
- Extra Office Hours next Wednesday 17th February from 7pm-9pm (Midterm review)

Back to Van Kampen

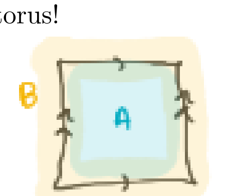
Example II.3.4
Lets compute the fundamental group of S^2 again using Van Kampen.



$\pi_1(S^2)$ must be a quotient of $\pi_1(A) * \pi_1(B)$, but since $A, B \simeq D^2$ we know $\pi_1(A)$ and $\pi_1(B)$ are both zero groups. Thus $\pi_1(A) * \pi_1(B)$ is the zero group, and $\pi_1(S^2)$ is the zero group.

Remark II.3.1
Something to note, the inclusion of $A \cap B \to A$ induces the zero map $\pi_1(A \cap B) \to \pi_1(A)$, which cannot be an injection. In fact we know that $\pi_1(A \cap B) \cong \mathbb{Z}$ since $A \cap B \simeq S^1$.

Example II.3.5
Now let's do the same thing with the torus!



Now note that $A \simeq D^2$ and $B \simeq S^1 \vee S^1$, since it is a thickening of the two loops around the torus in both ways. This suggests the question of how do we find $\pi_1(B)$? We grab a bit of knowledge from Van Kampen before we continue.

Exercise II.3.6
Suppose we have path connected spaces (X_α, x_α) and we take their wedge sum $\bigvee_\alpha X_\alpha$ by identifying the points x_α to a single point x. We also suppose a mild condition for all α, the point x_α is a deformation retract of some neighborhood of x_α.

For example, this doesn't work if we choose the "bad point" on the Hawaiian earring. Then we can use Van Kampen to show that:

$$\pi_1\left(\bigvee_\alpha X_\alpha, x\right) = \mathop{*}_\alpha \pi_1(X_\alpha, x_\alpha)$$

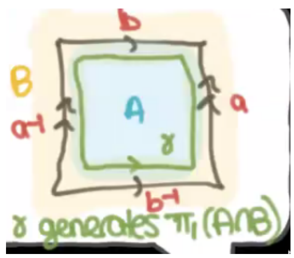
Proof idea. Take $A_\alpha = X_\alpha \bigcup_\beta U_\beta \simeq X_\alpha$ where U_β is a neighborhood of x_β which deformation retracts to x_β. This makes A_α open as desired.

Corollary II.3.3
The wedge sum of circles $\pi_1\left(\bigvee_{\alpha\in A} S^1\right) = *_\alpha \mathbb{Z}$ is the free group on A. In particular when A is finite, the fundamental group of a bouquet of circles is the free group on $|A|$ generators

Returning to Example II.3.5 we have that:

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$.

We know that $\pi_1(A\cap B) \to \pi_1(A)$ is the zero map. We need to understand $\pi_1(A\cap B) \to \pi_1(B)$. To do so we need to understand how we're able to identify $\pi_1(S^1 \vee S^1$ with F_2 and how we identify $\pi_1(S^1)$ with $\mathbb{Z}$. We update our picture to talk about this



From picture we have that:

$$\pi_1(A\cap B) \to \pi_1(B) \cong F_{a,b}$$
$$\gamma \mapsto aba^{-1}b^{-1}$$

By Van Kampen: identify the image of γ in $\pi_1(B)$ $[aba^{-1}b^{-1}]$ with its image in $\pi_1(A)$ (trivial). Therefore:

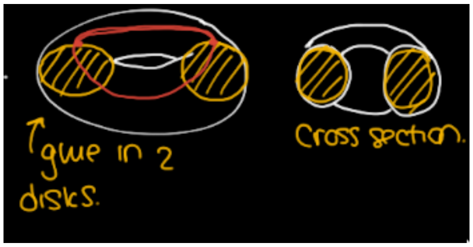
$$\pi_1(T^2) = F_{a,b}/\langle aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2$$

Announcements

- Midterm on next Thursday the 18th
- Fill out "when to meet" for midterm study groups
- Extra Office Hours 7pm-9pm Wednesday.

Back to Van Kampen!

Example II.3.7
Start with a torus, and glue in two disks into the hollow inside:



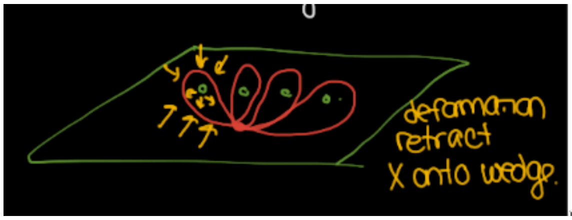
We'll call this space X, and our goal is to find $\pi_1(X)$. We can place a CW complex structure on this space so that each disk is a subcomplex. Then by homework we can quotient eeach disk to a point without changing the homotopy type:

By the same property, we can expand one of these points into an interval, and then contract the red path:

This is exactly $S^2 \vee S^2 \vee S^1$. OUr work with Van Kampen told us that:

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}$$

Exercise II.3.8
Consider $\mathbb{R}^2 \setminus \{x_1, \ldots, x_n\}$, that is the plane punctured at n points. Then $X \simeq \bigvee_n S^1$, so then $\pi_1(X) \cong F_n$.
One way to do this is to convince yourself that you can deformation retract the plane onto this wedge:



II.4. Group Presentations

Definition II.4.1

A presentation $\langle S \mid R \rangle$ of a group G consists of:

- S, which is a generating set, generators
- R, a set of relations (words in generators) such that

Such that:

$$G \cong F_S / N$$

Where F_S is the free group on S, and N is the subgroup normally generated by the elements of R.

A presentation is finite if S, R are finite. G is finitely presented if it admits a finite presentation. One way to think about the relations is that if r is a word in R then $r = 1$, where 1 is the identity of G. People often do this.

Example II.4.1

We have some nice examples!

Group	Presentation
F_2	$\langle a, b \mid \rangle$
$\mathbb{Z}^2$	$\langle a, b \mid aba^{-1}b^{-1} \rangle$
$\mathbb{Z}/3\mathbb{Z}$	$\langle a \mid a^3 \rangle$
$\mathrm{PSL}_2 \mathbb{Z}$	$\langle a, b \mid a^2, b^3 \rangle$
S_3	$\langle s, t \mid s^2, t^2, (st)^3 \rangle$

Theorem II.4.1

Every group has a presentation

Proof. We'll give an outline:

- Choose generators $S \subseteq G$, we could even choose $S = G$
- There exists a surjective map $\varphi : F_S \to G$ which is given by $s \mapsto s$ for $s \in S$
- Choose R to be a generating set for $\ker \varphi$. By the first isomorphism theorem $G \cong F_S / \ker \varphi$.

Great

Advantages

Exercise II.4.2

If $G = \langle S \mid R \rangle$ and we have a map $\varphi : S \to H$, then φ defines a group homomorphism $G \to H$ if and only if $\varphi(r) = 0$ for all $r \in R$. By this we mean something like if we have $G = \langle a, b \mid aba \rangle$, a map $\varphi : \{a, b\} \to H$ gives a group homomorphism if and only if:

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H$$

This essentially uses the universal property of quotients.

Exercise II.4.3

Suppose all relations in R are commutators, so $R \subseteq [G, G]$. Then:

$$G^{ab} = (F_S)^{ab} = \bigoplus_S \mathbb{Z}$$

Disadvantages

Computationally very difficult.

Example II.4.4

Show that $\langle a, b \mid aba^{-1}b^{-2}, a^{-2}b^{-1}ab \rangle$. This is a presentation of the trivial group, but this is entirely unclear.

Announcements

- Midterm Thursday 7pm ET
- Study Groups
  - 7pm tonight ET
  - 6pm Tuesday ET
- Midterm review package posted on webpage under "Exams"
- Midterm
  - You are responsible for material up to / including today
  - You are responsible for material on homework.
  - Two questions, many parts
  - True and Counterexample Questions
  - Spaces and presentations for the fundamental groups.

Back to Lecture

Exercise II.4.5

Consider $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$ where $f_1 : H \to G_1$ and $f_2 : H \to G_2$. Then we have

$$G_1 *_H G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{f_1(h) f_2(h)^{-1} \mid h \in H\} \rangle$$

This is super useful!

II.5. Presentations for π_1 of CW Complexes

Outline: For X a CW complex:

(a) A 1-dimensional CW complex has free π_1 (call its generators $a_1, \ldots, a_n$)

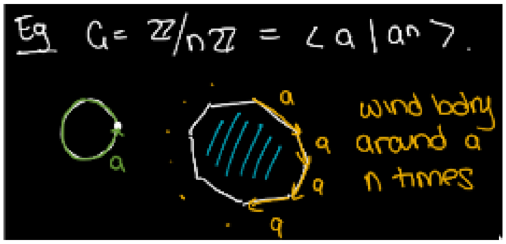
(b) Gluing a 2-disk by its boundary along a word w in the generators "kills" w in π_1
We then get a presentation for $\pi_1(X^2)$ given by:

$$\langle a_1, \ldots, a_n \mid w \text{ for each 2-cell in } X^2 \rangle$$

(c) Gluing in any higher dimensional cells along their boundary will not change π_1. That is in a CW complex we have $\pi_1(X) = \pi_1(X^2)$.

Example II.5.1

$G = \mathbb{Z}/n\mathbb{Z} = \langle a, a^n \rangle$, then we take a loop and then wind a 2-disk around the loop a n times.



Consequence: Given a group G with presentation $\langle S \mid R \rangle$ one can construct a 2-dimensional CW complex with $\pi_1 = G$:

- Set $X^1 = \bigvee_{s \in S} S^1$
- For each relation $r \in R$ glue in a 2-disk along loops specified by the word r.

Every group is then π_1 of some space.

This theorem will give us part c)

Theorem II.5.1 (From Homework)

If X is a CW complex and $\iota_1 : X^1 \hookrightarrow X$ and $\iota_2 : X^2 \hookrightarrow X$, then $(i_1)_*$ surjects onto π_1 and $(i_2)_*$ is an iso on π_1.

Definition II.5.1

We import some topological definitions of graph theoeretic concepts:

- A graph is a 1-dimensional CW complex.
- A subgraph is a subcomplex
- A tree is a contractible graph.
- A tree in a graph X (necessarily a subgraph) is maximal or spanning if it contains all the vertices.

Theorem II.5.2

Every connected graph has a maximal tree. Every tree is contained in a maximal tree.

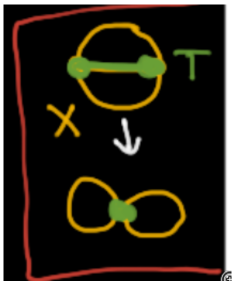
Corollary II.5.3

Suppose X is a connected graph with basepoint x_0. Then $\pi_1(X, x_0)$ is a free group.

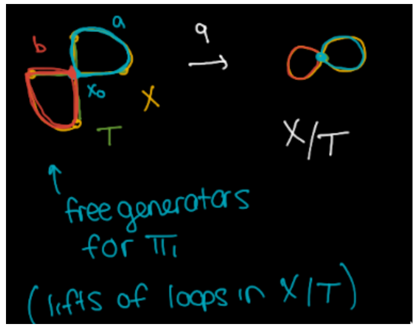
Furthermore, we can give a presentation for $\pi_1(X, x_0)$ by finding a spanning tree T in X. The generators of π_1 will be indexed by cells $e_\alpha \in X - T$.

e_α will correspond to a loop that passes through T, traverses e_α once, then returns to the basepoint x_0 through T.

Idea: X is homotopy equivalent to X/T. via previous work on the homework. T contains all the vertices, so the quotient has a single vertex. Thus it is a wedge of circles, and each e_α projects to a loop in X/T. Here's a picture illustrating the process:



The current program is to calculate the fundamental groups of CW complexes. For now, we need to see that the fundamental group of a 1-skeleton (a graph) can be found by taking a maximal tree, and then quotienting the space by that tree to get a wedge of circles:



Proof (Maximal Trees Exist). Recall X is a quotient of $X^0 \coprod_\alpha I_\alpha$. Each subset U is open if and only if it intersects each edge $\overline{e_\alpha}$ in an open subset. A map $X \to Y$ if and only if its restriction to each edge $\overline{e}_\alpha$ is continuous.

Take X_0 to be a subgraph.

Goal: Construct a subgraph Y with

- $X_0 \subseteq Y \subseteq X$
- Y deformation retracts to X_0
- Y contains all vertices of X.

So if we take X_0 to be a vertex, then Y is our tree and we're done!

Strategy: Build sequence $X_0 \subseteq X_1 \subseteq \cdots$ and corresponding $Y_0 \subseteq Y_1 \subseteq \cdots$. We start with X_0 and inductively define:

$$X_i = X_{i-1} \bigcup \text{all edges } \overline{e_\alpha} \text{ with one or both vertices in } X_{i-1}$$

Exercise II.5.2

Check that $X = \bigcup_i X_i$. In Hatcher we do this by arguing the union on the right is both open and closed.

Now let $Y_0 = X_0$. By induction, we will assume that Y_i is a subgraph of X_i such that:

- Y_i contains all vertices of X_i
- Y_i deformation retracts to Y_{i-1}

We can then construct Y_{i+1} by taking Y_i and adding to it one edge to adjoin every vertex of X_{i+1}:

$$Y_{i+1} = Y_i \bigcup \text{one edge to adjoint every vertex of } X_i$$

This is possible by using the axiom of choice.

Exercise II.5.3

Check that Y_{i+1} deformation retracts to Y_i (just smush down each edge).

Exercise II.5.4

Y deformation retracts to $Y_0 = X_0$ by performing the deformation retraction fromY_i to Y_{i-1} during the time interval $[1/2^i, 1/2^{i-1}]$

Awesome! We win!

Announcements

- Midterm 1 is over!!! Here are the statistics (grades were out of 20)
  - Ranged from 13.5 to 20
  - Median: 15.5
  - Average: 16.11
- The exam was made more difficult in order to prevent searchability, since algebraic topology is very searchable. Grades will be interpreted accordingly
- The grades for homework this week may come back a bit late due to grading of the midterm taking precedence.

II.6. **Proof of the Van Kampen Theorem**

Van Kampen: Proof Outline. Let $X = \bigcup_\alpha A_\alpha$ where the A_α are open, path-connected, and contain the basepoint x_0. We also must guarantee that $A_\alpha \cap A_\beta$ is path-connected.

Step 1) We have a map induced by the inclusions:

$$\Phi : \mathop{*}_{\alpha} \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

We want to show that Φ surjects. Take some $\gamma : I \to X$. You use compactness of the interval I to show that you can partition I into pieces, each of which is mapped completely into one A_α. In particular lets partition I with $s_1 < \cdots < s_n$ so that $\gamma\big|_{s_i, s_{i+1}} =: \gamma_i$ has image in A_{α_i} for some α_i. We'll leave the full point-set argument as an exercise, but as some hints:
- A_α is open for all α
- I is compact

Fora ll i, we choose a path h_i from x_0 to $\gamma(s_i)$ in $A_{\alpha_{i-1}} \cap A_{\alpha_i}$, using path-connectedness of the pairwise intersections. Now take γ and write it as follows:

$$\gamma = (\gamma_1 \cdot \overline{h_1}) \cdot (h_1 \cdot \gamma_2) \cdots (\gamma_{n-1} \cdot \overline{h_{n-1}}) \cdot (h_{n-1} \cdot \gamma_n)$$

Great! Each of these paths is fully contained in A_{α_i}, and so this shows that $\gamma \in \operatorname{im}(\Phi)$. Therefore Φ surjects.

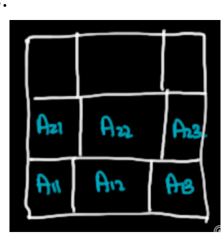
Step 2) For the next step, showing the second part of Van Kampen, we assume that our triple intersections are path connected.

We want to show that $\ker(\Phi)$ is generated by $(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)^{-1}$, where $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ for all loops $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$.

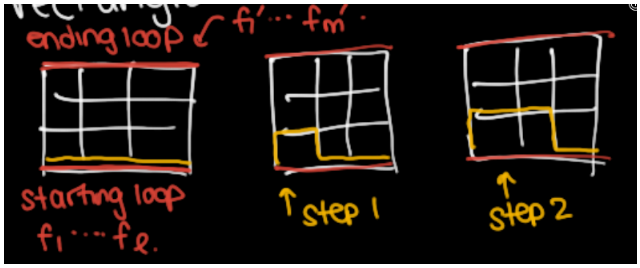
> **Definition II.6.1**
> A factorization of a homotopy class $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1][f_2] \cdots [f_\ell]$ with $[f_i] \in \pi_1(A_{\alpha_i}, x_0)$, such that $f \simeq f_1 \cdot f_2 \cdots f_\ell$.

We showed that every $[f]$ has a factorization in Step 1. Now we want to show that two facotrizations $[f_1] \cdots [f_\ell]$ and $[f_1'] \cdots [f_m']$ of $[f]$ must be related by two moves:
- $[f_i] \cdot [f_{i+1}] = [f_i \cdot f_{i+1}]$ if $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$. Aka, the relation defining the free product of groups.
- $[f_i]$ can be viewed as an element of $\pi_1(A_\alpha, x_0)$ or $\pi_1(A_\beta, x_0)$ whenever $[f_i] \in \pi_1(A_\alpha \cap A_\beta, x_0)$. This is the relation defining the amalgamated free product.

Now let $F_t : I \times I \to X$ be a homotopy from $f_1 \cdots f_\ell$ to $f_1' \cdots f_m'$, since they both represent $[f]$. We subdivide $I \times I$ into rectangles R_{ij} so that $F(R_{ij}) \subseteq A_{\alpha_{ij}} =: A_{ij}$ for some α_{ij}, using compactness. We also argue that we can perturb the corners of the squares so that a corner lies in only three of the A_α's indexed by adjacent rectangles:

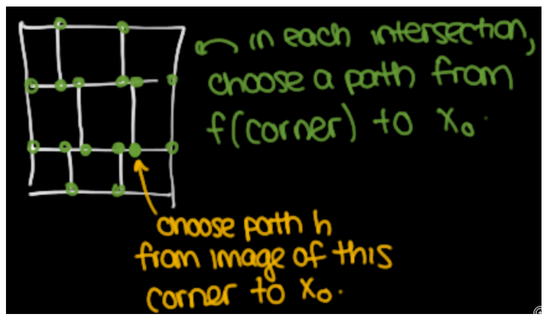


We also argue that we can set up our subdivision so that the partition of the top and bottom intervals must correspond with the two factorizations of $[f]$. We then perform our homotopy one rectangle at a time:



Idea: Argue that homotoping over a single rectangle has the effect of using allowable moves to modify the factorization.

At each triple intersection choose a path from $f(\text{corner})$ to x_0 which lies in the triple intersection, so we use the assumption that the triple intersections are path connected.



Along the top and bottom we make choices compatible with the two factorizations. It's now an exercise to check that these choices result in homotoping across a rectangle gives a new factorization related by an allowable move.

Announcements

- No class Wednesday, Office Hour moved to 8pm on Thursday
- Homework #5 corrected.

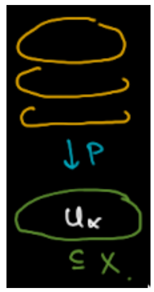
Resume Math!

III. Covering Spaces

III.1. Definitions and Lifting Properties

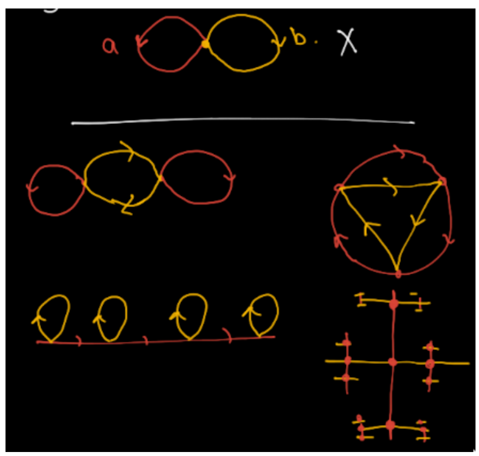
Definition III.1.1

A covering space $\widetilde{X}$ of X is a space $\widetilde{X}$ equipped with a map $p : \widetilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X so that for all α, $p^{-1}(U_\alpha)$ is a disjoint union (possibly empty) of open subsets in $\widetilde{X}$, each of which is mapped homeomorphically by p to U_α. Here's the picture:



Example III.1.1

Covers of $S^1 \vee S^1$, lifted from Hatcher:



Proposition III.1.1

Covering spaces (say $\widetilde{Y}$ over Y) satisfy the homotopy lifting property. That is, we may fill in diagrams in the following way:

$$\begin{tikzcd}
X \times \{0\} \cong X \arrow[r, "\widetilde{F}_0"] \arrow[d] & \widetilde{Y} \arrow[d] \\
X \times I \arrow[r, "F_t"'] \arrow[ur, dashed, "\widetilde{F}_t", "\exists!"'] & Y
\end{tikzcd}$$

That is given a lift $\widetilde{F}_0$ of F_0, there is a unique lift $\widetilde{F}_t$ of F_t extending $\widetilde{F}_0$.

Corollary III.1.2

Covering spaces satisf the path-lifting property:

For each path $I \xrightarrow{\gamma} Y$ and for each preimage $\widetilde{y}_0$ of $\gamma(0) = y_0$ there exists a unique path $I \xrightarrow{\widetilde{\gamma}} \widetilde{Y}$ lifting γ and starting at $\widetilde{y}_0$.

Proof. See Homework.

Proposition III.1.3

Suppose that $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map (i.e. p is a covering map and $p(\widetilde{x}_0) = x_0$). Then we have the following relationships between the fundamental groups:

(i) $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective

(ii) $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$ picks out the subset:

$$\{[\gamma] \mid \text{Lift } \widetilde{\gamma} \text{ of } \gamma \text{ starting at } \widetilde{x}_0 \text{ is a loop}\}$$

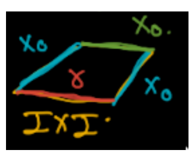
Proof. Suppose that $\widetilde{\gamma} \in \ker p_*$. Then $\gamma = p \circ \widetilde{\gamma}$. Let γ_t be a null-homotopy from γ to the constant loop c_{x_0} rel $\{0,1\}$. Then we can lift γ_t to $\widetilde{\gamma}_t$ where $\widetilde{\gamma}_0 = \widetilde{\gamma}$. We then claim that, using a similar proof as in Homework 2:

- $\widetilde{\gamma}$ is a homotopy rel $\{0,1\}$
- $\widetilde{\gamma}_1$ is the constant loop $c_{\widetilde{x}_0}$.

In diagrams and pictures:

$$\begin{tikzcd} & \widetilde{X} \arrow[d] \\ I \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{X} \arrow[d] \\ I \times I \arrow[ur, "\widetilde{\gamma}_t"] \arrow[r, "\gamma_t"'] & X \end{tikzcd}$$

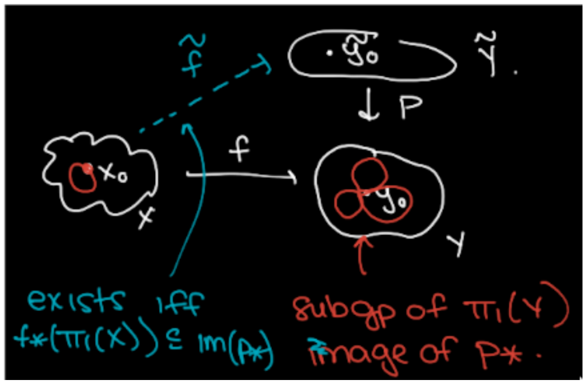
This picture provides a proof of the first claim, we know that the left and right edge of $I \times I$ maps to x_0 under γ_t, and $c_{\widetilde{x}_0}$ lifts this, so by uniqueness $t \mapsto \widetilde{\gamma}_t(0)$ and $t \mapsto \widetilde{\gamma}_t(1)$ must be constant paths at $\widetilde{x}_0$ as desired.



This shows that $\ker p_*$ is trivial. Proving part (i). We leave part (ii) as an exercise. The proof uses similar ideas.

Proposition III.1.4

Suppose we have a covering map $p : (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$, a continuous function $f : (X, x_0) \to (Y, y_0)$, with X path-connected and locally path-connected. Then there exists a lift $\widetilde{f} : (X, x_0) \to (\widetilde{Y}, \widetilde{y}_0)$ if and only if $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(\widetilde{Y}, \widetilde{y}_0))$. In a picture:

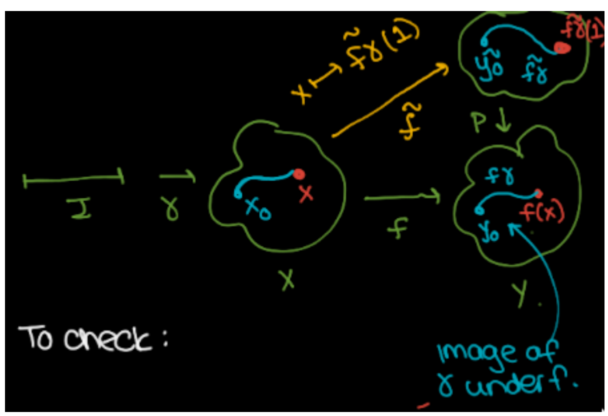


Proof. The "only if" portion is straightforward from the factorization $f_* = p_* \circ \widetilde{f}_*$ due to functoriality.

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "p"] \\ X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(\widetilde{Y}, \widetilde{y}_0) \arrow[d, "p_*"] \\ \pi_1(X, x_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(Y, y_0) \end{tikzcd}$$

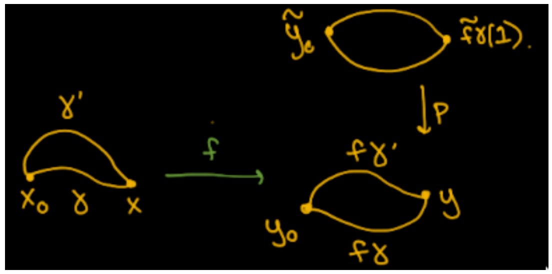
Then we have that $f_*(\pi_1(X, x_0)) = p_*(\widetilde{f}_*(\pi_1(X, x_0))) \subseteq p_*(\pi_1(\widetilde{Y}, \widetilde{y}_0))$.

Now lets do the "if" portion. Let $x \in X$. Choose a path γ from x_0 to x. The path $f \circ \gamma$ has a unique lift starting at $\widetilde{y}_0$. Define $\widetilde{f}(x) = \widetilde{f\gamma}(1)$. Consider the following picture:



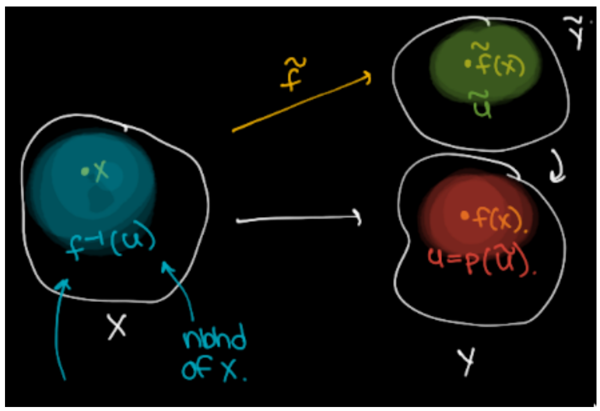
We must show that $\widetilde{f}$ is well-defined and that $\widetilde{f}$ is continuous:

(1) Let γ' be some other path from x_0 to x. We want to show that $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. Since $\gamma \cdot \overline{\gamma'}$ is a loop in X at x_0, we know that $f\gamma \cdot \overline{f\gamma'}$ is a loop in Y in $\operatorname{im}(f_*)$. Thus this loop is in $\operatorname{im}(p_*)$ by assumption, so it must form a loop when lifted there. Here's the picture:



But wait! By uniqueness of lifts, the loop lifting $(f\gamma)\cdot\overline{(f\gamma')}$ to $\widetilde{Y}$ must be equal to the lifts $\widetilde{f\gamma}\cdot\overline{\widetilde{f\gamma'}}$ with a common value at $t=1/2$. And then $\widetilde{f\gamma}(1)=\widetilde{f\gamma'}(1)$ as desired. We'll leave the details of this use of uniqueness as an exercise.

(2) Choose $x\in X$ and choose a neighborhood $\widetilde{U}$ of $\widetilde{f}(x)$ in $\widetilde{Y}$. We may shrink $\widetilde{U}$ so that $p|_{\widetilde{U}}$ is a homeomorphism to $p(\widetilde{U})=U$, via the definition of a covering space. We know that $f^{-1}(U)$ is an open neighborhood of x. It suffices to show that $f^{-1}(U)\subseteq\widetilde{f}^{-1}(\widetilde{U})$, and so we may show $\widetilde{f}(f^{-1}(U))\subseteq\widetilde{U}$, and we actually pass to a smaller neighborhood V and show that $\widetilde{f}(V)\subseteq\widetilde{U}$. Here's the picture:



Replace $f^{-1}(U)$ with a p(ossibly smaller) path-connected open neighborhood $V\subseteq f^{-1}(U)$ using the fact that X is locally path-connected. Now for any $x'\in V$ choose a path α from x to x'. If γ is some path from x_0 to x, then we get a path $\gamma\cdot\alpha$ from x_0 to x'. Now $f\gamma\cdot f\alpha$ in Y has a lift $\widetilde{f\gamma}\cdot\widetilde{f\alpha}$ where $\widetilde{f\alpha}=p^{-1}(f\alpha)$, since $f\alpha$ is contained entirely in U, and so p is invertible here. But then necessarily, $\widetilde{f}(x')=\widetilde{f\alpha}(1)\in\widetilde{U}$. But this is exactly what we wanted! $\widetilde{f}(V)\subseteq\widetilde{U}$.

Exercise III.1.2

On Homework we had that $\pi_1(\mathbb{RP}^2)=\mathbb{Z}/2\mathbb{Z}$. Prove that every map $\mathbb{RP}^2\to S^1$ is nullhomotopic.

Solution. Consider the cover $\mathbb{R}\xrightarrow{p}S^1$ given on previous homework. For convenience, choose a presentation $\pi_1(\mathbb{RP}^2)=\langle a\mid a^2\rangle$. Now for any function $f:\mathbb{RP}^2\to S^1$, we know that $f_*(a)+f_*(a)=f_*(a^2)=0$, but necessarily because we are working in $\mathbb{Z}$ this means $f_*(a)=0$. Therefore $f_*(\pi_1(\mathbb{RP}^2))$ is trivial because f_* sends the generators to 0, and this must be contained in the trivial image of $\pi_1(\mathbb{R})$ in $\pi_1(S^1)$.

Therefore, by the proposition (Proposition III.1.4), we know that f extends to some map $\widetilde{f}:\mathbb{RP}^2\to\mathbb{R}$. Any such map into $\mathbb{R}$ is nullhomotopic to some constant map c because $\mathbb{R}$ is contractible. Thus $f=p\circ\widetilde{f}$ is nullhomotopic, because composition respects homotopies, so $f=p\circ\widetilde{f}\simeq p\circ c$, and $p\circ c$ is a constant map.

Announcements

- Quiz on Wednesday
  - Know definition and basic properties (simply connected) of the universal cover
  - Know lifting properties (existence / uniqueness) for covers.

Back to Math!

Proposition III.1.5

Let $p : \widetilde{Y} \to Y$ be a covering map with X a connected space, then if two lifts $\widetilde{f}_1, \widetilde{f}_2$ of the same map f agree at a single point then they agree everywhere.

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "p"] \\ X \arrow[ur, bend left, "\widetilde{f}_1"] \arrow[ur, bend right, "\widetilde{f}_2"'] \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. Let $S = \{x \in X \mid \widetilde{f}_1(x) = \widetilde{f}_2(x)\}$. Our goal is to show that S is both open and closed. Since X is connected, the only sets that are open and closed are X and $\emptyset$, and S is nonempty by hypothesis.

Choose $x \in X$ and let U be a neighborhood of $f(x)$ so that $p^{-1}(U)$ is a disjoint union of open subsets $\{U_\alpha\}$ each mapped homeomorphically to U by p. (Aside: we say that U is evenly covered by p, and that each open subset U_α is a slice of the preimage).

Now since $f = p \circ \widetilde{f}_1 = p \circ \widetilde{f}_2$ we must have that $\widetilde{f}_1(x), \widetilde{f}_2(x) \in p^{-1}(f(x))$. Let $\widetilde{f}_1(x) \in U_1$ and $\widetilde{f}_2(x) \in U_2$.

Exercise III.1.3

Since $\widetilde{f}_1, \widetilde{f}_2$ are continuous there exists a neighborhood $N \subseteq X$ so that $\widetilde{f}_1(x) \in \widetilde{f}_1(N) \subseteq U_1$ and $\widetilde{f}_2(x) \in \widetilde{f}_2(N) \subseteq U_2$.

There are two cases:

- Suppose that $\widetilde{f}_1(x) \neq \widetilde{f}_2(x)$. Then U_1 and U_2 are disjoint because each U_α contains only one preimage of $f(x)$, so $\widetilde{f}_1$ and $\widetilde{f}_2$ must differ on every point of N. Therefore $X \setminus S$ is open (aka S is closed) because $x \in N \subseteq X \setminus S$.
- Suppose that $\widetilde{f}_1(x) = \widetilde{f}_2(x)$, that is $x \in S$. Then $U_1 = U_2$, so for all $n \in N$ we have:

  $$\widetilde{f}_1(n) = p^{-1}(f(n)) = \widetilde{f}_2(n)$$

  Where we're using p^{-1} here to mean $p^{-1} : U \to U_1 = U_2$, the inverse of the restriction $p\big|_{U_1} : U_1 = U_2 \to U$. This shows that S is open since $x \in N \subseteq S$.

III.2. Deck Transformations

Definition III.2.1

Given covering maps $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$, then an isomorphism of covers is a homeomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_1 = p_2 \circ f$:

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

We can actually talk about a category of covering spaces, but we won't delve into that too much.

Exercise III.2.1

This defines an equivalence relation on covers.

Definition III.2.2
Fix a cover $p : \widetilde{X} \to X$. The isomorphims of the cover $\widetilde{X} \to \widetilde{X}$ are called deck transformations. We'll let $G(\widetilde{X})$ be the set of deck transformations. Notice that we've suppressed the data of p in the notation, but this data is essential to what a deck transformation is, when this is unclear we write $G(\widetilde{X}, p)$.

Exercise III.2.2
Deck transformations $G(\widetilde{X})$ are a subgroup of the group of homeomorphisms of $\widetilde{X}$.

Example III.2.3
Consider the cover $p : \mathbb{R} \to S^1$, then $G(\mathbb{R}) \cong \mathbb{Z}$, and $n \in \mathbb{Z}$ acts on $\mathbb{R}$ by translating n units.

Example III.2.4
There are covers $p_n : S^1 \to S^1$ where we "wind n times." Then $G(S^1, p_n) \cong \mathbb{Z}/n\mathbb{Z}$ which acts by rotation.

Exercise III.2.5
Notice that a deck transformation $\tau : \widetilde{X} \to \widetilde{X}$ is a lift of $p : \widetilde{X} \to X$:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ \widetilde{X} \arrow[ur, dashed, "\tau"] \arrow[r, "p"'] & X \end{tikzcd}$$

Then by the unique lifting property (Proposition III.1.5), τ is determined by the image of a single point when X is connected.

Corollary III.2.1
If a deck transformation has a fixed point, it is the identity transformation.

Exercise III.2.6
Let X be connected. Given a deck transformation $\tau : \widetilde{X} \to \widetilde{X}$, and $x_0 \in X$, τ defines a permutation of $p^{-1}(x_0)$. If this permutation has a fixed point, then it is the identity.

We'll assume that $\widetilde{X}$ is connected for now.

Definition III.2.3

A covering space $p : \widetilde{X} \to X$ is normal or regular if for every $x_0 \in X$ and every pair of lifts $\widetilde{x}_1, \widetilde{x}_2 \in p^{-1}(x_0)$, there exists a Deck transformation mapping $\widetilde{x}_1$ to $\widetilde{x}_2$.

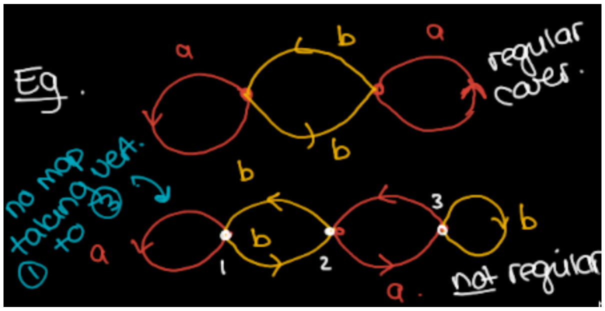
Slogan: A regular cover is "as symmetric as possible"

Exercise III.2.7

"Regular" means that the group $G(\widetilde{X})$ acts transitively on $p^{-1}(x_0)$. Explain why we cannot ask for more than this–$G(\widetilde{X})$ cannot (eg) induce the full symmetric group on $p^{-1}(x_0)$ (key: uniqueness) provided that $\left|p^{-1}(x_0)\right| > 2$.

Example III.2.8

Lets go back to the wedge of two circles!



Exercise III.2.9

A Deck transformation of covers of $S^1 \vee S^1$ is precisely a graph automorphism that preserves the labels / directed edges.

Definition III.2.4

If G is a group and H is a subgroup, then the normalizer of H is:

$$N(H) = \{g \in G \mid gH = Hg\}$$

Exercise III.2.10

Check that:

- $N(H)$ is a subgroup containing H
- H is normal in $N(H)$.
- $N(H)$ is the largest subgroup of G which contains H in which H is normal.

Theorem III.2.2

Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map. We assume that $\widetilde{X}, X$ are path-connected, and locally path-connected. For convenience, let $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$. Then:

(a) p is a normal cover if and only if H is normal in $\pi_1(X, x_0)$.

(b) The group of Deck Transformations $G(\widetilde{X}) \cong N(H)/H$, where the normalizer of H is taken in $\pi_1(X, x_0)$.

Corollary III.2.3

If p is a normal covering, then $G(\widetilde{X}) \cong \pi_1(X, x_0)/H$.

Corollary III.2.4

If $\widetilde{X}$ is the universal cover, then $G(\widetilde{X}) \cong \pi_1(X, x_0)$

Exercise III.2.11

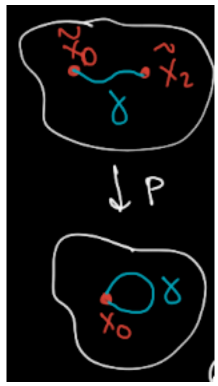
Consider $\mathbb{R}^2 \to T^2$ and $\mathbb{R} \to S^1$.

Proof of Theorem. Notation: Let (X, x_0) be the base space and $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$ where $p : \widetilde{X} \to X$ is a covering map. Further let $H := p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

On Homework: (X, x_0), $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$, if we change the basepoint from $\pi_1(\widetilde{X}, \widetilde{x}_0)$ to $\pi_1(\widetilde{X}, \widetilde{x}_1)$. Then we have the the induced subgroups of the base space's fundamental group are conjugate by some loop $[\gamma] \in \pi_1(X, x_0)$, that is:

$$p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = [\gamma] \cdot p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \cdot [\gamma]^{-1}$$

Where γ lifts to a path from $\widetilde{x}_0$ to $\widetilde{x}_1$.



Therefore $[\gamma] \in N(H)$ if and only if $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, and this holds if and only if there is a deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$ by the classification of based covering spaces in the homework (alternatively use the lifting criterion).

Therefore p is a normal cover if and only if H is normal, proving (a).

We then define a map to help us out:

$$\Phi : N(H) \to G(\widetilde{X})[\gamma] \qquad\qquad \mapsto \tau$$

Where τ lifts to a path $\widetilde{x}_0$ to $\widetilde{x}_1$ and τ is a deck transformation mapping $\widetilde{x}_0$ to $\widetilde{x}_1$, which will be uniquely defined by uniquness of lifts with specified base points. We need to check some things

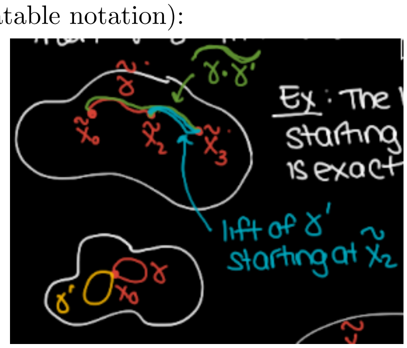
(i) Check that Φ is a group homomorphism
(ii) Φ is surjective
(iii) $\ker(\Phi) = H$

If we can prove these things, then the first isomorphism theorem gives us the desired result.

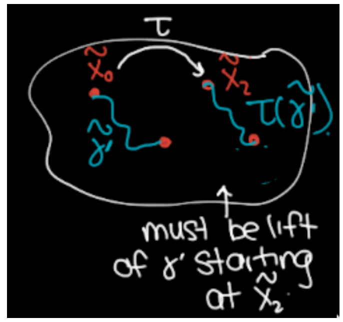
(ii) We've proved that Φ is surjective before in our work above.
(iii) $\Phi([\gamma])$ is the identity if and only if τ sends $\widetilde{x}_0$ to $\widetilde{x}_0$, meaning that $[\gamma]$ lifts to a loop, well then by our characterization of the fundamental group downstairs:

$$\ker(\Phi) = \{[\gamma] \mid \gamma \text{ lifts to a loop}\} = H$$

(i) Suppose we have loops $[\gamma_1] \xmapsto{\Phi} \tau_1$ and $[\gamma_2] \xmapsto{\Phi} \tau_2$. We claim that $\gamma_1 \cdot \gamma_2$ lifts to $\widetilde{\gamma}_1 \cdot \tau(\widetilde{\gamma}_2)$. Here's our motivating picture (with translatable notation):



It's an exercise to check that the lift of γ_2 starting at $\widetilde{x}_1$ is exactly $\tau_1(\widetilde{\gamma}_2)$, where $\widetilde{\gamma}_2$ is a lift starting at $\widetilde{x}_0$. The picture of the claim is below:



The idea is that by uniqueness of lifts we'll have the desired claim. We then just observe that this path $\widetilde{\gamma}_1 \cdot \tau_1(\widetilde{\gamma}_2)$ is a path from $\widetilde{x}_0$ to $\tau_1(\widetilde{\gamma}_2(1)) = \tau_1(\tau_2(\widetilde{x}_0))$, so the image must be a deck transformation sending $\widetilde{x}_0$ to $\tau_1(\tau_2(\widetilde{x}_0))$. But then $\tau_1 \circ \tau_2$ maps $\widetilde{x}_0$ to this same point, and since deck transformations are determined by where they send a single point, we're done ☺.

Announcements

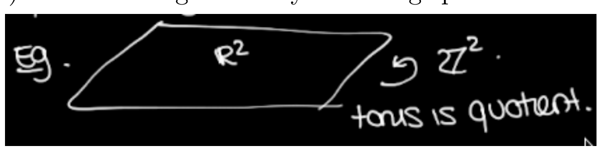
- Corrections to Homework #7
- Quiz on Wednesday on covering spaces
  - Know definition of action of the fundamental group on a fiber
  - Know the definition of a regular cover
  - Know the result on Deck Transformation group of regular cover
  - Know the action of $N(p_*(\pi_1(\widetilde{X}, \widetilde{x})))$ by Deck transformations.
- Midterm II - 1 week from Thursday

III.3. **Covering Space Recap so far**

- Lifting Properties
  - Homotopy lifting property (extending $\widetilde{F}_0$, existence / uniqueness)
  - Path lifting (existence / uniqueness given a preimage of $\gamma(0)$)
  - Lifts of $f : X \to Y$ (given conditions on $f_*(\pi_1(X))$ in $\pi_1(Y)$.
- Classification given by:

$$\{\text{basepoint-preserving isomorphism classes of covers } p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)\} \to \{\text{Subgroups of } \pi_1(X, x_0)\}$$
$$p \mapsto p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$$

  - Constructed the universal cover of X (corresponds to trivial subgroup of $\pi_1(X)$ whenever X is path connected, locally path connected, and simply connected
  - Constructed cover X_H corresponding to a subgroup H
  - Proved uniqueness up to isomorphism
- Deck Transformations
  - Classified regular covers using normal subgroups of $\pi_1(X, x_0)$
  - Showed that $G(\widetilde{X}) \cong N(H)/H$
- Next (current homework): Constructing covers by "covering space actions"

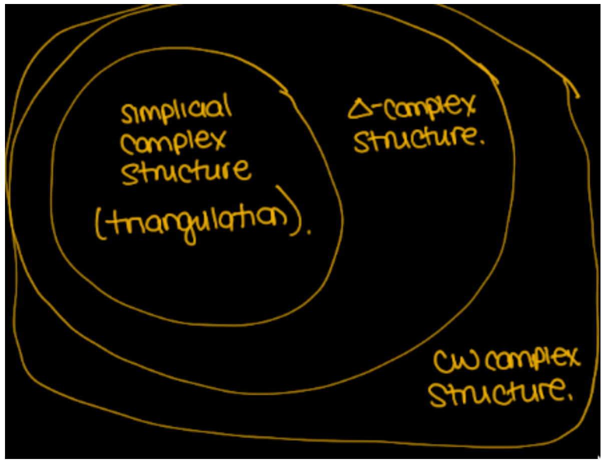


Then we can view covers as quotients by group actions that satisfy the properties of a "covering space action"

IV. **Homology**

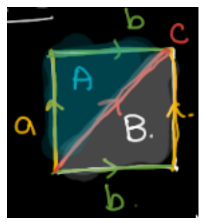
IV.1. **Δ-complexes**

This is a stricter version of a CW complex which allows us to decompose our spaces into cells. In terms of how things fit together, we have this diagram:

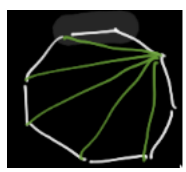


Example IV.1.1

The torus with the following edges a, b, c, and gluing in triangles A and B



For this delta complex notice we've glued down a triangle whose vertices are all identified, this is not allowed in a simplicial complex / triangulation. We can also do it for genus 2 surfaces:



Definition IV.1.1 (Simplices)

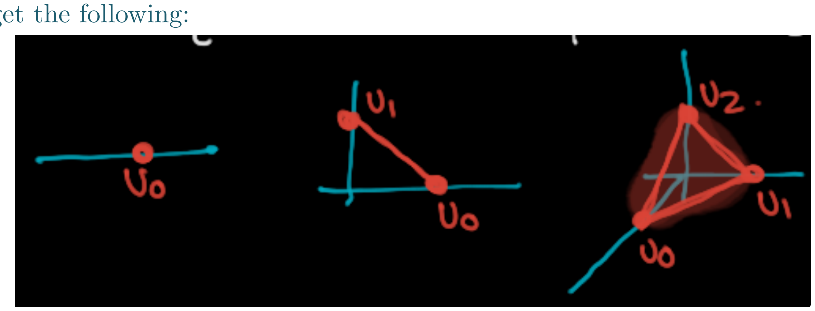
A 0-simplex is a point. A 1-simplex is an interval. A 2-simplex is a triangle. A 3-simplex is a tetrahedron. . . so what's a simplex?

Well, in general, a n-simplex is always the convex hull of $(n + 1)$ points in $\mathbb{R}^n$. We can view simplices as both combinatorial and topological objects.

The standard n-simplex is given by:

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0 \;\forall i\}$$

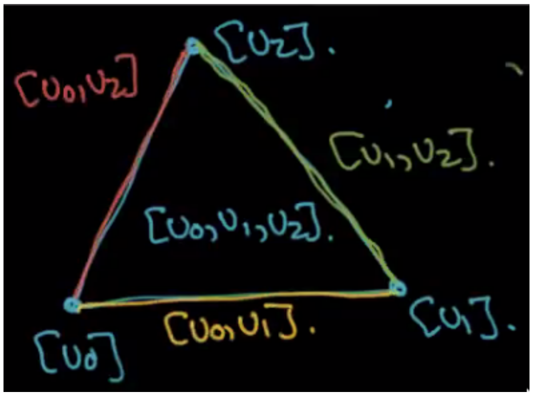
In pictures we get the following:



Our simplices will implcitly come with a choice of ordering of the vertices as $\Delta^n = [v_0, \dots, v_n]$ (the convex hull of these points with this ordering).

Definition IV.1.2

A face of a simplex $\Delta^n = [v_0, \ldots, v_n]$ is a subsimplex spanned by any n of the $n+1$ vertices with the induced order.



Exercise IV.1.2

In general, any subset of k vertices in Δ^n spans a subsimplex of dimension $k-1$.

The order on the vertices of Δ^n also induces an order on the vertices of every subsimplex.

Definition IV.1.3

A subsimplex of Δ^n is:

- Combinatorially, a subset of the vertices
- Topologically, we can identify with a smaller dimensional simplex

A face is a subsimplex of 1 dimension lower than Δ^n ("codimension 1").

Announcements

- Midterm II - 1 week from tomorrow!
  - Practice package available this weekend
  - Let me know about conflicts ASAP
  - Covers material up to / including this Friday + Homework 8 (focus on material since Midterm I)
  - Study groups will be set up with When2Meet
- Extra OH next week (7-9pm on Mar 17)
- Corrections to Homework #7 Problem #6 posted

Back to Math!

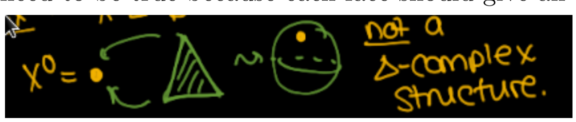
Definition IV.1.4
The boundary $\partial\Delta^n$ of Δ^n is the union of its faces. The open simplex $\mathring{\Delta}^n$ is $\Delta^n \setminus \partial\Delta^n$.

Definition IV.1.5 (Δ-complex)
A Δ-complex structure on X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ (n depends on α) such that:

(i) $\sigma_\alpha\big|_{\mathring{\Delta}^n}$ is injective, and each point in X is in the image of exactly one such restriction.
(ii) Each restriction of σ_α to a face of Δ^n must coincide with a map $\sigma_\beta : \Delta^{n-1} \to X$.
(iii) A set A in X is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α. (i.e., X is the quotient space $\coprod_{n,\alpha} \Delta^n \to X$)

Exercise IV.1.3
The Δ-complex structure is a CW-complex structure. But with condition that attaching maps must be injective on the interior of each face individually (must glue faces onto existing simplices).

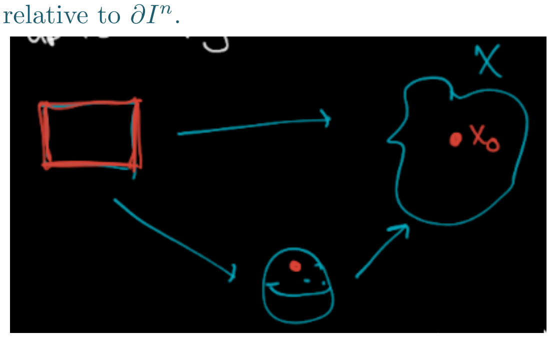
Non-Example IV.1.4
Take $X = S^2$. A CW Complex structure can be a 0-skeleton of a point, and then glue on a 2-cell by mapping the entire boundary to a single point. This is not injective on each of the faces of the triangle given below (which would need to be true because each face should give an attaching map). Nice!



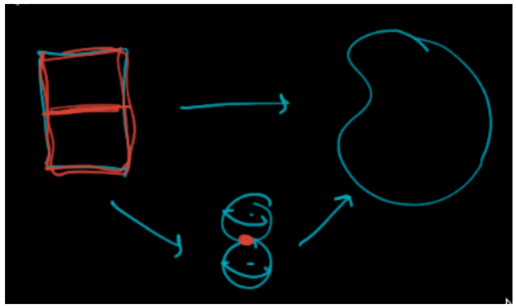
There is a Δ-complex structure on S^2, but this particular structure doesn't work.

IV.2. Motivation for Homology

Definition IV.2.1 (Informal Higher Homotopy Groups)
Define π_n to be homotopy classes of based maps from $I^n \to (X, x_0)$ which maps the boundary to the basepoint, up to homotopy relative to ∂I^n.



We get a group structure (and even nicer for $n \geq 2$ it's abelian!!!)



Problem: Although these are the natural, and very useful, they are **really** hard to compute. So hard that the higher homotopy groups of the k-sphere $\pi_n(S^k)$ is an open question for $n \geq k$.

Instead, we will study homology groups. They are much easier to compute–however their definition is a bit less intuitive.

Instead of the higher homotopy groups π_n, we will study "higher-dimensional holes" in our space using homology groups.

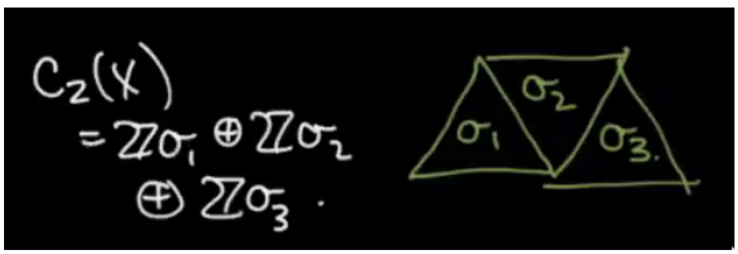
Homology Pros	Homology Cons
Homotopy invariants (like π_n)	Definition (at first) seems less natural
Functorial (like π_n)	
Abelian (like π_n, $n > 1$)	
No basepoints	
Lots of computational tools	
Can compute from cell structure on X	
Good properties like $H_n = 0$ if $n > \dim X$	

Idea for the Homology Definition

Fix a space X, which is a Δ-complex. We define $C_n(X)$ to be the free abelian group on the n-simplices of X. That is:

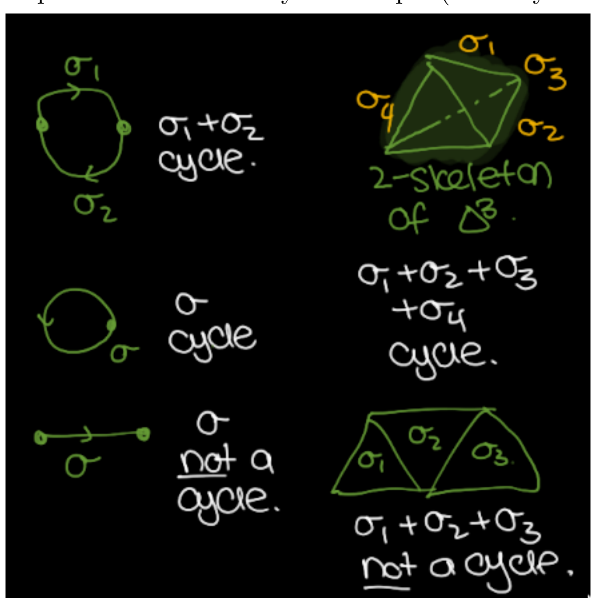
$$C_n(X) = \{\text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha : \Delta^n \to X\}$$

In a picture:

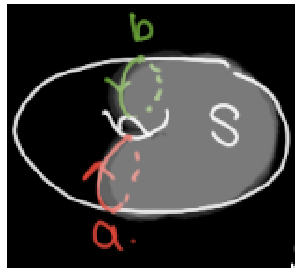


The n-th homology group will be a subquotient of $C_n(X)$. The Heuristic / imprecise idea is:

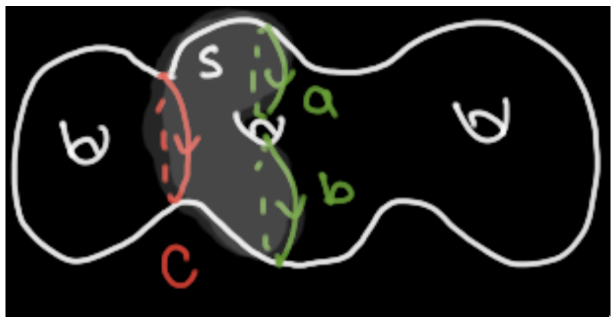
- Take subgroup of C_n of "cycles." These are sums of simplices satisfying a combinatorial condition on the boundary gluing maps to ensure that they "close up." (i.e. they have no boundary)



- To take the quotient, we consider two cycles to be equivalent if their difference is a boundary. For example, in this picture of the torus, a is homologous to b since $a - b$ is the boundary of the shaded subsurface S.



In fact, a and b are homotopic (which will imply they're homologous), but two loops do not need to be homotopic to be homologous. For example:



$a + b$ is homologous to c, since $a + b - c$ is the boundary of S ($a + b$ [which isn't even a loop] and c are not homotopic).

Formal Definition

For the duration, take X with a Δ-complex structure.

Definition IV.2.2

We define the chain group $C_n(X)$ of order n to be the free abelian group on the n-simplices of X. Formally:

$$C_n(X) = \{\text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha : \Delta^n \to X\}$$

Definition IV.2.3

We now define the boundary homomorphism, which will be a map $\partial_n : C_n(X) \to C_{n-1}(X)$. We'll first give this in lower dimensions to motivate the general definition:

$$\begin{aligned}
\partial_1 &: C_1(X) \to C_0(X) \\
[\sigma_\alpha : [v_0, v_1] \to X] &\mapsto \sigma_\alpha\big|_{[v_1]} - \sigma_\alpha\big|_{[v_0]} \\
\partial_2 &: C_2(X) \to C_1(X) \\
[\sigma_\alpha : [v_0, v_1, v_2] \to X] &\mapsto \sigma_\alpha\big|_{[v_1,v_2]} - \sigma_\alpha\big|_{[v_0,v_2]} + \sigma_\alpha\big|_{[v_0,v_1]}
\end{aligned}$$

So in general what we have is:

$$\begin{aligned}
\partial_n &: C_n(X) \to C_{n-1}(X) \\
[\sigma_\alpha] &\mapsto \sum_{i=1}^{n} (-1)^i \sigma_\alpha\big|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_n]}
\end{aligned}$$

And this defines the map on the basis, and we extend linearly ☺.

Lemma IV.2.1

For any $n \geq 2$ we have that:

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n"] \arrow[rr, bend right, "\partial_{n-1}\circ\partial_n=0"'] & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] & C_{n-2}(X) \end{tikzcd}$$

Definition IV.2.4

A chain complex (C_*, d_*) is a collection of maps:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of abelian groups and group homomorphism such that $d_{n-1} \circ d_n = 0$. We call C_n the n-th chain group and d_n the n-th differential.

This means that $\ker(d_n)$ contains $\operatorname{im}(d_{n+1})$, since $d_n \circ d_{n+1} = 0$.

The sequence is exact at C_n provided that $\ker(d_n) = \operatorname{im}(d_{n+1})$. A chain complex is exact if it is exact at each point. The previous lemma guarantees that our simplicial chain groups form a chain complex.

Definition IV.2.5

The n-th homology group of a chain complex (C_*, d_*) is written H_n or $H_n(C_*)$. It is the quotient:

$$H_n = \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})}$$

It measures how far the chain complex is from being exact at C_n.

Definition IV.2.6

This means that we may now define the homology groups of spaces X with a Δ-complex structure. Namely $\ker(\partial_n)$ is the subgroup of cycles in $C_n(X)$, and $\operatorname{im}(\partial_{n+1})$ is the subgroup of boundaries in $C_n(X)$. We then set:

$$H_n(X) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})} = \frac{\text{cycles}}{\text{boundaries}}$$

I.e., it is the homology of our chain complex:

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

Where we take it to be 0 in all negative indices.

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

Elements of $H_n(X)$ are called homology classes

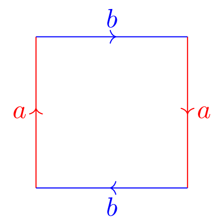
Announcements

- Midterm II on Thursday.
- Extra Office Hours Wendesday: 7-9pm ET
- Student study group: Tuesday 4pm-6pm ET
- Review package posted
- HW #8 warm-up + Problem 1 is exam-relevant

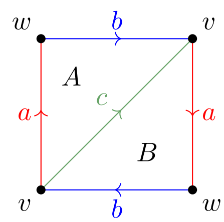
IV.3. Computing Simplicial Homology

Example IV.3.1
$X = \mathbb{RP}^2$. Goal: Compute simiplical homology groups.

Consider the fundamental polygon given below for $\mathbb{RP}^2$:



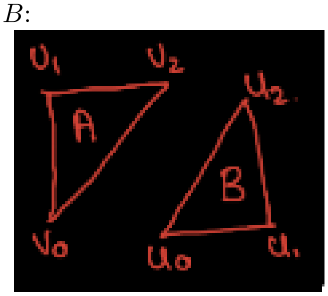
Now make this into a Δ-complex structure as below:



So now we have three nonzero chain groups, where we use $\langle f_1, \ldots, f_n \rangle$ to denote the free abelian group on n generators:

$$C_0(\mathbb{RP}^2) = \langle v, w \rangle$$
$$C_1(\mathbb{RP}^2) = \langle a, b, c \rangle$$
$$C_2(\mathbb{RP}^2) = \langle A, B \rangle$$

We must choose orientations on A and B:



Let $A = [v_0, v_1, v_2]$ and take $B = [v_0, v_1, v_2]$ and so then:

$$\partial_2 A = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = b - c + a$$
$$\partial_2 B = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = -a - c - b$$

Now for the boundaries on our edges:

$$\partial_1 a = w - v \qquad \partial_1 b = v - w \qquad \partial_1 c = v - v = 0$$

Now that we've computed this we want to look at our chain complex:

$$0 \xrightarrow{\partial_3} C_2(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_2} C_1(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_1} C_0(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_0} 0$$

We know the following images and kernels:

$$\begin{aligned}\operatorname{im}\partial_3 &= 0\\ \operatorname{im}\partial_2 &= \langle b-c+a, -a-b-c\rangle\\ \operatorname{im}\partial_1 &= \langle v-w\rangle\\ \ker\partial_0 &= C_0 = \langle v, w\rangle\end{aligned}$$

Now to compute $\ker\partial_2$ we see that:

$$\begin{aligned}\partial_2(mA+nB) &= 0\\ m(b-c+a)+n(-a-b-c) &= 0\\ a(m-n)+b(m-n)-c(m+n) &= 0\end{aligned}$$

And so $m-n=0$ and $m+n=0$. This means that we need to have $m=n=0$, and so:

$$\ker\partial_2 = 0$$

We can also check what the kernel of ∂_1 is as below:

$$\begin{aligned}\partial_1(\alpha a+\beta b+\gamma c) &= 0\\ \alpha(w-v)+\beta(v-w) &= 0\\ (\beta-\alpha)v+(\alpha-\beta)w &= 0\end{aligned}$$

And so to have this we need to have $\alpha-\beta=\beta-\alpha=0$, this happens when $\alpha=\beta$ and we have no conditions on γ, and therefore:

$$\ker\partial_1 = \langle c, a+b\rangle$$

Now all we need to do is take the quotients to get the homology groups.

$$\begin{aligned}H_2(\mathbb{R}\mathrm{P}^2) &= \frac{\ker\partial_2}{\operatorname{im}\partial_3} = \frac{0}{0} = 0\\ H_1(\mathbb{R}\mathrm{P}^2) &= \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle c, a+b\rangle}{\langle a+b-c, -a-b-c\rangle}\\ H_0(\mathbb{R}\mathrm{P}^2) &= \frac{\ker\partial_0}{\operatorname{im}\partial_1} = \frac{\langle v, w\rangle}{\langle v-w\rangle} \cong \mathbb{Z}\end{aligned}$$

Lets think about how to compute the quotient in $H_1(\mathbb{R}\mathrm{P}^2)$. We can use row operations from linear algebra (ways to change from one basis to another) to get that:

$$H_1(\mathbb{R}\mathrm{P}^2) = \frac{\langle c, a+b\rangle}{\langle a+b-c, -a-b-c\rangle} = \frac{\langle c, a+b-c\rangle}{\langle a+b-c, -2c\rangle} \cong \mathbb{Z}/2\mathbb{Z}$$

Key: Given a basis for a free abelian group $\langle b_1, \ldots, b_n\rangle$ we can replace b_i with

$$b_i \pm m_1 b_1 \pm \cdots \pm \widehat{m_i b_i} \pm \cdots \pm m_n b_n$$

Exercise IV.3.2

If b_1, b_2 is a basis for $A \subseteq \mathbb{Z}^n$, then $b_1 - b_2, b_1 + b_2$ is not a basis, it is an index-2 subgroup. The key to this is that $\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ has determinant 2 (not unit in $\mathbb{Z}$).

Principle

We can transform a basis for a free group into a different basis by applying a matrix of determinant ± 1. If we apply a matrix of determinant D we will obtain generators for a subgroup of index $|D|$..

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \pm m_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots 0 & \pm m_2 & 0 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \pm m_{i-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \pm m_{i+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \pm m_n & 0 & \cdots & 1 \end{bmatrix}$$

Summary of Procedure

(1) Choose Δ-complex structure on X. (Later: We will prove $H_*(X)$ is independent of the choice of Δ-complex structure)
(2) Choose orientations on each simplex (Any choice is okay but you must commit to a choice or you will make a sign error!)
(3) For each n-simplex σ compute $\partial_n(\sigma)$ (careful with signs!)
(4) $\operatorname{im} \partial_n = \langle \partial_n(\sigma) \mid \sigma \text{ an } n\text{-simplex} \rangle$. Use linear algebra to compute $\ker(\partial_n)$
(5) For each n compute $H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$. Be careful that any change-of-variables map you apply is invertible over $\mathbb{Z}$.

Announcements

- Midterm tomorrow!
- Extended Office Hours tonight 7pm-9pm

IV.4. **Singular Homology**

Definition IV.4.1
A singular n-simplex in a space X is a continuous map $\sigma : \Delta^n \to X$.

Definition IV.4.2
Let $C_n(X)$ be the free group on singular n-simplices in X. The singular n-chains with boundary maps:

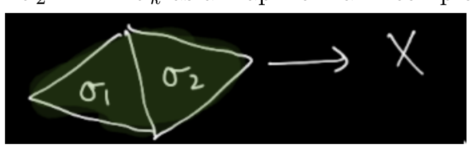
$$\partial_n : C_n(X) \to C_{n-1}(X)$$
$$\sigma \xmapsto{\sum} {}_{i=1}^{n} (-1)^i \sigma\big|_{[v_0,\dots,\widehat{v}_i,\dots,v_n]}$$

This gives us a singular chain complex

Definition IV.4.3
The singular homology groups are the homology groups of this singular chain complex given as $H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$.

Since the generating sets for $C_n(X)$ are almost always hugely uncountable, it's almost impossible to compute with these. However it does give us a definition that does not depend on any other structure than the topology of X, making it useful for developing theory ☺.

Heuristic: Interpret a chain $\sigma_1 \pm \sigma_2 \pm \cdots \pm \sigma_k$ as a map from a Δ-complex to X. For example with $\sigma_1 + \sigma_2$.



Where we've glued $[v_1, v_2]$ of σ_1 to $[v_0, v_2]$ of σ_2 if $\sigma_1\big|_{[v_1,v_2]}$ and $\sigma_{[v_0,v_2]}$ are the same singular n-chain with opposite signs.

Goals:

- Singular homology is a homotopy invariant
- Singular and simplicial homology groups are isomorphic.

Exercise IV.4.1
Check that if X has path components $\{X_\alpha\}$ then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$.

Exercise IV.4.2
If $X = *$ then

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

Exercise IV.4.3
If X is path-connected, then $H_0(X) \cong \mathbb{Z}$

IV.5. **Functoriality and Homotopy Invariance**

Definition IV.5.1
For a given continuous map $f : X \to Y$ we can consider the following map:

$$f_\# : C_n(X) \to C_n(Y)$$
$$[\sigma : \Delta^n \to X] \mapsto [f \circ \sigma : \Delta^n \to Y]$$

Definition IV.5.2

Given two chain complexes (C_*, ∂_*) and (D_*, δ_*), a chain map between them is a collection of group homomorphisms $g_n : C_n \to D_n$ such that the below diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "g_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r, "\delta_{n+2}"] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots
\end{tikzcd}$$

i.e. we have that $\delta_n \circ f_n = f_{n-1} \circ \partial_n$.

Exercise IV.5.1

We have that $f_\# \partial = \partial f_\#$. In other words, $f_\#$ is a chain map. Thus by the homework $f_\#$ induces a group homomorphism on the homology groups. We write this as $f_* : H_n(X) \to H_n(Y)$ for all n.

Exercise IV.5.2

We have functoriality, i.e. $(f \circ g)_* = f_* \circ g_*$. Also we have that $(\mathrm{Id}_X)_* = \mathrm{Id}_{H_n(X)}$.

Theorem IV.5.1

The n-th homology group $H_n : X \mapsto H_n(X)$ gives a functor from Top to Ab. This follows from the two exercises above.

Theorem IV.5.2

If $f, g : X \to Y$ are homotopic, then they will induce the same map on homology $f_* = g_* : H_n(X) \to H_n(Y)$.

Exercise IV.5.3

These two theorems imply that H_n is a homotopy invariant.

To prove the second theorem, we introduce some homological algebra.

Definition IV.5.3

Given chain complexes (A_*, d_*^A) and (B_*, d_*^B) and chain maps $f_*, g_* : A_* \to B_*$. A chain homotopy from f to g is a sequence of group homomorphisms $\psi_n : A_n \to B_{n+1}$ such that:

$$f_n - g_n = d_{n+1}^B \circ \psi_n + \psi_{n-1} d_n^A$$

In a diagram, letting $h_n = f_n - g_n$:

$$\begin{tikzcd}
\cdots \arrow[r, "d^A_{n+2}"] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "h_{n+1}"] & A_n \arrow[r, "d^A_n"] \arrow[dl, "\psi_n"'] \arrow[d, "h_n"] & A_{n-1} \arrow[r, "d^A_{n-1}"] \arrow[dl, "\psi_{n-1}"'] \arrow[d, "h_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^B_{n+2}"] & B_{n+1} \arrow[r, "d^B_{n+1}"] & B_n \arrow[r, "d^B_n"] & B_{n-1} \arrow[r, "d^B_{n-1}"] & \cdots
\end{tikzcd}$$

This diagram does not commute, but it shows everything that is going on. However the **red** map is the sum of the **green** maps composed up.

Theorem IV.5.3
If there is a chain homotopy ψ from f_* to g_*, then the induced maps on homology are equal.

Proof. Let $\sigma \in A_n$ be an n-cycle, i.e. $\partial_n^A \sigma = 0$. Then we compute that:

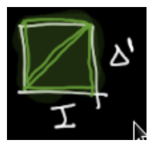
$$(f_n - g_n)(\sigma) = d_{n+1}^B(\psi_n(\sigma)) + \psi_{n-1}(d_n^A(\sigma)) = d_{n+1}^B(\psi_n(\sigma)) \in \operatorname{im} d_{n+1}^B$$

This tells us that $(f_n - g_n)(\sigma)$ is a boundary, and so $(f_n - g_n)(\sigma) = 0$ when considered as an element of the homology group. Thus $f_n(\sigma) = g_n(\sigma)$ in the homology group, and so f, g induce the same map as desired.

We now sketch the proof of Theorem IV.5.2 given in Hatcher. From this point in the course many of the theorems require much more algebraic work than we are interested in. We instead want to learn how to use the computational tools.

Proof idea. Suppose we have some homotopy $F : I \times X \to Y$ from f to g. The most difficulty in this proof is the combinatorial difficulty involved in the fact that the product of a simplex in X and I is not a simplex.

Key: Subdivide $\Delta^n \times I$ into $(n+1)$ dimensional subsimplices.



We define the prism operator:

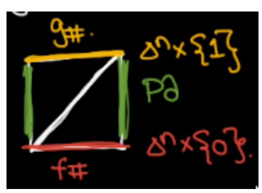
$$P_n : C_n(X) \to C_{n+1}(Y)$$

$$[\sigma : \Delta^n \to X] \mapsto \begin{bmatrix} \text{alternating sums of restrictions} \\ \Delta^n \times I \xrightarrow{\sigma \times \mathrm{Id}} X \times I \xrightarrow{F} Y] \end{bmatrix}$$

We now need to check that

$$\partial_{n+1}^Y P_n = \boxed{g_\#} - \boxed{f_\#} - \boxed{P_{n-1}\partial_n^X}$$

We have the following diagram.



Thus P is a chain homotopy and we're done.

IV.6. **Relative Homology**

Definition IV.6.1 (Studied on Homework)
The reduced homology groups $\widetilde{H}_n(X) = H_n(X)$ when $n > 0$. When $n = 0$ we have that:

$$\widetilde{H}_0(X) \oplus \mathbb{Z} = H_0(X)$$

The usefulness of this is that for path-connected space X we have $\widetilde{H}_0(X) = 0$, and for contractible spaces X we have $\widetilde{H}_n(X) = 0$.

Definition IV.6.2
Let X be a space, and $A \subseteq X$. Then (X, A) is a good pair if A is closed and nonempty, and also it is a deformation retract of a neighborhood in X.

Example IV.6.1

If X is a CW complex and A is a subcomplex, then (X, A) is a good pair.

The proof is given in the Appendix of Hatcher and requires some point-set topology.

Non-Example IV.6.2

(Hawaiian earring, bad point) is a bad pair.

Theorem IV.6.1

If (X, A) is a good pair, then there exists a long exact sequence (exact at every n) on reduced homology groups given by:

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A)$$

$$\xrightarrow{\delta} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \xrightarrow{j_*} \widetilde{H}_{n-1}(X/A)$$

$$\xrightarrow{\delta} \cdots \xrightarrow{j_*} \widetilde{H}_0(X/A) \longrightarrow 0$$

Where $i : A \hookrightarrow X$ is the inclusion and $j : X \to X/A$ is the quotient map. We will define each δ in the proof. The fact that this sequence is exact often means that if we know the homology groups of two of the spaces we can compute the homology of the remaining space.

Announcements

- Midterm Exams graded
- Correction to Homework Problem # 1
- Quiz Wednesday
  - Know definition of singular homology

Application of the quotient Long Exact Sequence.

Proposition IV.6.2
We have that:

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Proof. Some facts we need:

- $(D^n, \partial D^n)$ is a good pair (since it is a CW complex and a subcomplex)
- $D^n/\partial D^n \cong S^n$ (previous homework)
- $\widetilde{H}_n(D^n) = 0$ for all n since D^n is contractible
- $\partial D^n \cong S^{n-1}$

We then proceed by induction on n.

Exercise IV.6.3
Verify the theorem in the case $n = 0$, so $S^0 = 2$ points.

Now using the long exact sequence, we have:

$$\cdots \longrightarrow \widetilde{H}_n(\partial D^n) \xrightarrow{i_*} \widetilde{H}_n(D^n) \xrightarrow{j_*} \widetilde{H}_n(S^n)$$

$$\xrightarrow{\delta} \widetilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{j_*} \widetilde{H}_{n-1}(S^n)$$

$$\xrightarrow{\delta} \cdots \xrightarrow{j_*} \widetilde{H}_0(S^n) \longrightarrow 0$$

By induction we can fill in some of these groups as follows:

$$\cdots \longrightarrow 0 \xrightarrow{i_*} 0 \xrightarrow{j_*} \widetilde{H}_n(S^n)$$

$$\xrightarrow{\delta} \mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{j_*} \widetilde{H}_{n-1}(S^n)$$

$$\xrightarrow{\delta} \cdots \xrightarrow{j_*} \widetilde{H}_0(S^n) \longrightarrow 0$$

Summary: We have an exact sequence:

$$0 \longrightarrow \widetilde{H}_n(S^n) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0$$

By exactness, δ is an isomorphism, thus $\widetilde{H}_n(S^n) \cong \mathbb{Z}$. Now we must verify $\widetilde{H}_i(S^n) = 0$ when $i \neq n$. In that case the exact sequence looks like:

$$\longrightarrow \widetilde{H}_i(D^n) \longrightarrow \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_{i-1}(\partial D^n)$$

$$\longrightarrow 0 \longrightarrow \widetilde{H}_i(S^n) \longrightarrow 0$$

Exactness then tells us that $\widetilde{H}_i(S^n) = 0$.

Theorem IV.6.3 (Brouwer's Fixed Point Theorem)
∂D^n is not a retract of D^n. Hence every continuous map $f : D^n \to D^n$ has a fixed point.

Proof. If $r : D^n \to \partial D^n$ were a retraction, then by definition this would give us that:

$$\begin{tikzcd} \partial D^n \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{Id}_{\partial D^n}"'] & D^n \arrow[r, "r"] & \partial D^n \end{tikzcd}$$

Functoriality of homology tells us that:

$$\begin{tikzcd} \widetilde{H}_{n-1}(\partial D^n) \arrow[r, "i_*"] \arrow[rr, bend right, "\mathrm{Id}"'] & \widetilde{H}_{n-1}(D^n) \arrow[r, "r_*"] & \widetilde{H}_{n-1}(\partial D^n) \end{tikzcd}$$

So then:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "\mathrm{Id}"'] & 0 \arrow[r, "r_*"] & 0 \end{tikzcd}$$

This is impossible.

Exercise IV.6.4
As with D^2, if $f : D^n \to D^n$ had no fixed point, we could build a retraction.

Tool for proving Theorem: diagram chase

Lemma IV.6.4 (The Short Five Lemma)
Suppose we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\psi'"'] & B' \arrow[r, "\varphi'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

So that the rows are exact. Then:

(1) If α, γ are injective then β is injective.
(2) If α, γ are surjective then β is surjective.
(3) If α, γ are isomorphisms then β is an isomorphism

Proof. (1) and (2) imply (3). We leave (2) as an exercise. We fix $b \in B$ such that $\beta(b) = 0$. We want to show that $\beta = 0$. Well, we draw a diagram chase:

$$\begin{tikzcd} 0 \arrow[r, mapsto] & \bullet \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\beta"] & \varphi(b) \arrow[r, mapsto] \arrow[d, mapsto, "\gamma"] & 0 \\ 0 \arrow[r, mapsto] & \bullet \arrow[r, mapsto, "\psi'"'] & 0 \arrow[r, mapsto, "\varphi'"'] & 0 \arrow[r, mapsto] & 0 \end{tikzcd}$$

And thus by injectivity of γ we know $\varphi(b) = 0$. By exactness, $b \in \operatorname{im} \psi$. We then may write for some $a \in A$:

$$\begin{tikzcd} 0 \arrow[r, mapsto] & a \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\beta"] & 0 \arrow[r, mapsto] \arrow[d, mapsto, "\gamma"] & 0 \\ 0 \arrow[r, mapsto] & \alpha(a) \arrow[r, mapsto, "\psi'"'] & 0 \arrow[r, mapsto, "\varphi'"'] & 0 \arrow[r, mapsto] & 0 \end{tikzcd}$$

Therefore $\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) = 0$ by commutativity. By exactness of the bottom row we know that ψ' is an injection. Thus $\alpha(a) = 0$, so since α is injective, $a = 0$. With this $b = \psi(a) = \psi(0) = 0$. Great! With this $\ker(\beta) = 0$, and β injects. This ends the proof! ☺

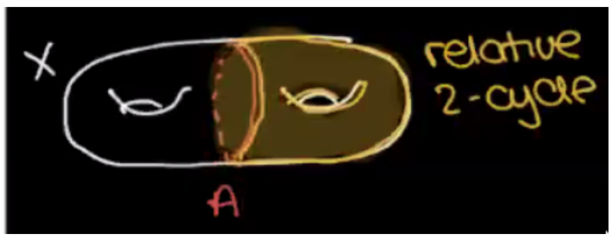
Definition IV.6.3
Let X be a space and let $A \subseteq X$ be a subspace. Then we define the relative chain complex

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}$$

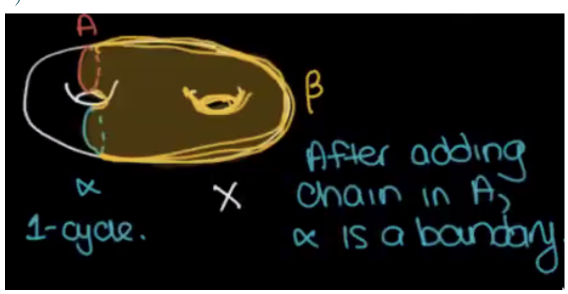
Exercise IV.6.5
The boundary map $\partial : C_n(X) \to C_{n-1}(X)$ induces a well-defined map $\partial : C_n(X, A) \to C_{n-1}(X, A)$. Since $\partial^2 = 0$ we can conclude that these groups will in fact form a chain complex $(C_*(X, A), \partial)$.

Definition IV.6.4
The homology groups of $(C_*(X, A), \partial)$ are denoted by $H_n(X, A)$, and they are called relative homology groups

Elements in $\ker \partial_n$ are called relative n-cycles. These are elements $\alpha \in C_n(X)$ such that $\partial_n \alpha \in C_{n-1}(A)$.



Likewise elements in $\operatorname{im} \partial_{n+1}$ are called relative n-boundaries. This means that $\alpha = \partial\beta + \gamma$ where $\beta \in C_n(X)$ and $\gamma \in C_{n-1}(A)$.



Theorem IV.6.5 (LES of a pair)
Let $A \subseteq X$ be spaces, then there exists a long exact sequence

$$\longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow H_n(X, A)$$

$$\longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow H_0(X, A) \longrightarrow 0$$

Later: We will prove that when (X, A) is a good pair, then $H_n(X, A) = \widetilde{H}_n(X/A)$. Then Theorem IV.6.1 is a special case of Theorem IV.6.5. The key to the proof of Theorem IV.6.5 above is the following slogan.

Remark IV.6.1
Slogan A short exact sequence of chain complexes gives rise to a long exact sequence of homology groups. This will be proved on homework. Then Theorem IV.6.5 will follow from a short exact sequence:

$$0 \longrightarrow \widetilde{C}_*(A) \longrightarrow \widetilde{C}_*(X) \longrightarrow C_*(X, A) \longrightarrow 0$$

where $\widetilde{C}_*$ denotes the augmented chain complex (the one with $\mathbb{Z}$ after it).

Exercise IV.6.6
If A is a single point in X, then $H_n(X, A) = \widetilde{H}_n(X/A) = \widetilde{H}_n(X)$.

Theorem IV.6.6 (Excision)
Suppose we have subspace $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq \text{Int}(A)$. Then the inclusion:

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

induces isomorphisms:

$$H_n(X - Z, A - Z) \xrightarrow{\cong} H_n(X, A)$$

Exercise IV.6.7
Equivalently for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion:

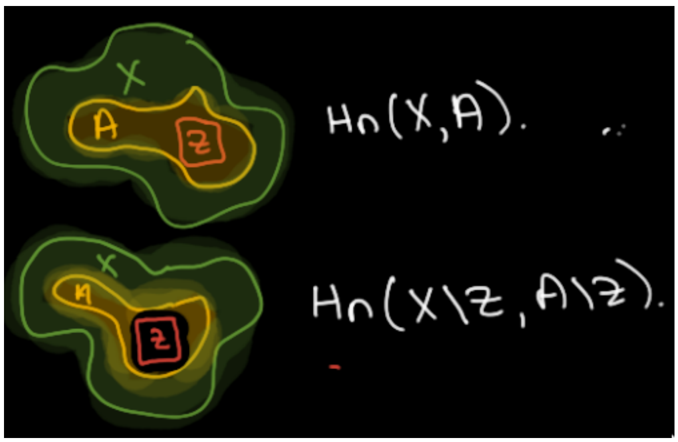
$$(B, A \cap B) \hookrightarrow (X, A)$$

induces an isomorphism:

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

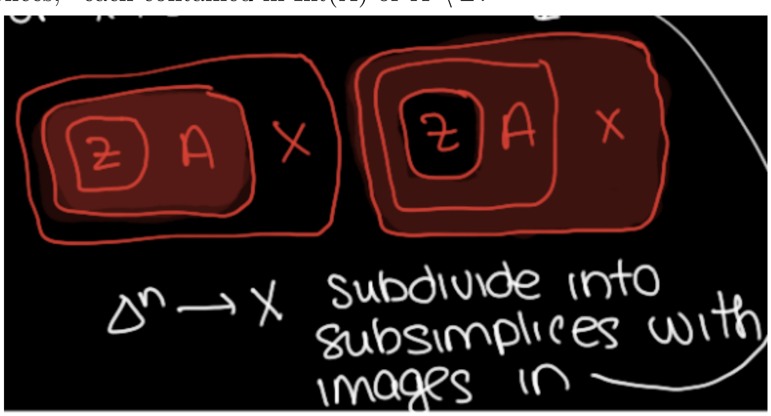
Hint: $B = Z \setminus Z$, $Z = X \setminus B$.

Picture!



Proof Sketch. We sketch the proof here, which is notorious for being hairy.

- Given a relative cycle x in (X, A), subdivide the simplices to make x a linear combination of chains on "smaller simplices," each contained in $\text{Int}(A)$ or $X \setminus Z$.



  This means x is homologous to sum of subsimplices with images in $\text{Int}(A)$ or $X \setminus Z$. One of the things we use is that simplices are compact, so this process takes finite time.

  Key: "Subdivision operator" is chain homotopic to the identity.
- Since we are working relative to A, the chains with image in A are zero. Thus we have a relative cycle homologous to x with all simplices contained in $X \setminus Z$.

Exercise IV.6.8
$H_*(Y, y_0) \cong \widetilde{H}(Y)$.

Theorem IV.6.7
Let (X, A) be a good pair. Then the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism:

$$H_n(X, A) \xrightarrow{\cong} H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$$

where the last equality is from the exercise.

Proof Outline. Let $A \subseteq V \subseteq X$ where V is a neighborhood of A that deformation retracts onto A. Using excision, we obtain a commutative diagram:

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\cong"] \arrow[d, "q_*"'] & H_n(X, V) & H_n(X - A, V - A) \arrow[l, "\cong"'] \arrow[d, "\cong"', "q_*"] \\
H_n(X/A, A/A) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "\cong"']
\end{tikzcd}$$

Done if we can prove all the colored isos.

- $\cong$ is an isomrphism by excision
- $\cong$ is an isomorphism by direct calculation (since q is a homeomorphism on the complement of A)
- $\cong$ on Homework, since V deformation retracts to A.

Announcements

- Quiz #7 Wednesday
  - Know the long exact sequence of a good pair, applications like Homework 9 #6
  - Know homology/reduced homology of basic spaces. (contractible space, spheres S^n, discrete set of points, $\mathbb{R}\mathrm{P}^2$, torus, $\ldots$, spaces homotopy equivalent to any of the above).
  - The following fact below

Fact: If M is a smooth manifold and N is an embedded smooth closed submanifold, then (M, N) is a good pair. Why? Well this follows from the tubular neighborhood theorem, which should be proven in a course like 591. We will only use the result in obvious cases, and simply assert that certain pairs are good pairs.

Upshot: With pairs like $(\mathbb{R}^{n+1}, S^n)$, you can just assert that this is a good pair (and do not need to prove that S^n is a smooth submanifold of $\mathbb{R}^{n+1}$. Another good example is manifolds and their boundary always form a good pair.

Theorem IV.6.8

Let X be a Δ-complex. We use $\Delta_n(X)$ to represent the simplicial chain groups on X, and $C_n(X)$ to denote the singular chain groups. Likewise $\Delta_n(X, A) = \Delta_n(X)/\Delta_n(A)$ and $C_n(X, A) = C_n(X)/C_n(A)$.

With this notation, we claim that the inclusion $\Delta_n(X, A) \hookrightarrow C_n(X, A)$ given by:

$$[\sigma : \Delta^n \to X] \mapsto [\sigma : \Delta^n \to X]$$

induces isomorphisms on homology.

$$H_n^\Delta(X, A) \cong H_n(X, A)$$

If we consider the case that $A = \emptyset$, we recover the case of absolute homology.

$$H_n^\Delta(X) \cong H_n(X)$$

Comment: This says, given a singular homology class x, we can assume x is represented by a simplicial n-cycle.

The Proof uses the Five Lemma:

Lemma IV.6.9 (The Five Lemma)

If I have a commutative diagram with exact rows:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "\ell"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \arrow[r, "k'"'] & D' \arrow[r, "\ell'"'] & E'
\end{tikzcd}$$

If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

Proof. Diagram chase!

Proof Sketch of the Theorem. Here's the idea

- We can use the long exact sequence of a pair and the five lemma to reduce to proving the result for absolute homology groups (and we will recover the general result).
- Because the image $\Delta^n \to X$ is compact, it is contained in some finite skeleton X^k. Use this ot reduce the proof to the finite skeleta X^k of X

From the LES of a pair we get:

$$\begin{tikzcd}
H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "\alpha"] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d, "\beta"] & H_n^\Delta(X^k) \arrow[r] \arrow[d, "\gamma"] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "\delta"] & H_{n-1}^\Delta(X^{k-1}) \arrow[d, "\varepsilon"] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

The Goal is to prove γ is an isomorphism using the 5-lemma.

We assume that β, ε are isomorphisms by induction, checking the case manually for X^0 (which will be a discrete set of points). It remains to show that α, δ are isomorphisms.

We know then that:

$$\begin{aligned}\Delta_n(X^k, X^{k-1}) &= \begin{cases} \mathbb{Z}[k - \text{simplices}] & \text{if } k = n \\ 0 & \text{otherwise} \end{cases} \\ &= H_n^{\Delta}(X^k, X^{k-1})\end{aligned}$$

Claim: $H_n(X^k, X^{k-1})$ are also free abelian on the singular k-simplices defined by the characteristic maps $\Delta^k \to X^k$ when $n = k$, and 0 otherwise. Consider the map:

$$\Phi : \coprod_{\alpha} (\Delta^k_\alpha, \partial\Delta^k_\alpha) \to (X^k, X^{k-1})$$

Defined by the characteristic map. This induces an isomorphism on homology since:

$$\frac{\coprod_\alpha \Delta^k_\alpha}{\coprod \partial\Delta^k_\alpha} \xrightarrow{\cong} \frac{X^k}{X^{k-1}}$$

This reduces to checking that:

$$H_n(\Delta^k, \partial\Delta^k) = \begin{cases} 0 & \text{if } n \neq k \\ \mathbb{Z} & \text{if } n = k \end{cases}$$

generated by the identity map $\Delta^k \to \Delta^k$.

IV.7. Degree

Definition IV.7.1

Let $f : S^n \to S^n$. Then $f_* : \mathbb{Z} \cong H_n(S^n) \to H_n(S^n) \cong \mathbb{Z}$. From group theory, this map must be multiplication by some integer $d \in \mathbb{Z}$, which we call the degree $\deg(f)$ of f

Last time we defined degree (Definition IV.7.1). Now we list some of its properties
Properties of Degree:

(a) $\deg(\mathrm{Id}_{S^n}) = 1$ sinc $(\mathrm{Id}_{S_n})_* = \mathrm{Id}_{\mathbb{Z}}$.

(b) If $f : S^n \to S^n$ is not surjective, then $\deg(f) = 0$. To see this, we know that f_* factors as:

$$H_n(S^n) \longrightarrow H_n(S^n - \{*\}) = 0 \longrightarrow H_n(S^n)$$

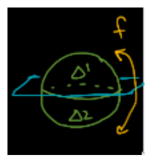
And since the middle group is zero, $f_* = 0$.

(c) If $f \simeq g$, then $f_* = g_*$, so $\deg(f) = \deg(g)$.
Later: The converse is true!

(d) $(f \circ g)_* = f_* \circ g_*$, and so $\deg(f \circ g) = \deg(f)\deg(g)$.
Consequently: If f is a homotopy equivalence then $\deg f = \pm 1$.

Exercise IV.7.1
It is possible to put a Δ-complex structure with 2 n-cells, Δ_1 and Δ_2 glued together along their boundary ($\cong S^{n-1}$), and $H_n(S^n) = \langle \Delta_1, \Delta_2 \rangle$.

(e) Consequences: If f is a reflection fixing the equator, and swapping the 2-cells, then $\deg f = -1$.



(f) We now have the following linear algebra exercise.

Exercise IV.7.2
The map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $x \mapsto -x$ is the composite of $(n+1)$ reflections.

So the antipodal map $S^n \to S^n$ given by $x \mapsto -x$ has degree which is the product of $n+1$ copies of (-1), and so it has degree $(-1)^{n+1}$.

(g) We again start with an exercise

Exercise IV.7.3
If f has no fixed points, then we can homotope f to the antipodal map via:

$$f_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$$

Therefore $\deg f = (-1)^{n+1}$.

Theorem IV.7.1 (Hairy Ball Theorem)
See the homework. This essentially says that there is no nonvanishing continuous tangent vector field on even-dimensional spheres.

Theorem IV.7.2 (Groups acting on S^{2n})
If G acts on S^{2n} freely, then $G = \mathbb{Z}/2\mathbb{Z}$ or $G = 1$

Corollary IV.7.3
S^{2n} is only the trivial cover $S^{2n} \to S^{2n}$ or degree 2 cover (for example, $S^{2n} \to \mathbb{RP}^{2n}$). This follows since any covering space action acts freely.

Proof. There exists a homomorphism given by:

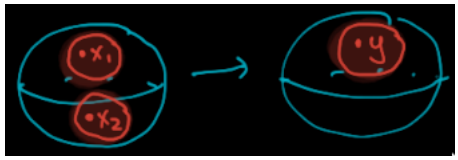
$$\begin{aligned} G &\to \{\pm 1\} \\ g &\mapsto \deg(\tau_g) \end{aligned}$$

Where τ_g is the action of $g \in G$ on S^{2n} as a map $S^{2n} \to S^{2n}$. We know this map is well-defined since τ_g is invertible (simply take $\tau_{g^{-1}}$) for each $g \in G$. Our note on composites shows this is a homomorphism.

We want to show that the kernel is trivial, since then by the first isomorphism theorem $G \cong \text{im}$, and the image is either trivial or $\mathbb{Z}/2\mathbb{Z}$. Suppose that g is a nontrivial element of g, then since G acts freely we know that τ_g has no fixed points. With this in mind we have $\deg \tau_g = (-1)^{2n+1} = -1$. Thus $g \notin \ker$. Therefore the kernel is trivial as desired.

Definition IV.7.2

Let $f : S^n \to S^n$ $(n > 0)$. Suppose there exists $y \in S^n$ such that $f^{-1}(y)$ is finite, say, $\{x_1, \ldots, x_m\}$. Then let $U_1, \ldots, U_m$ be disjoint neighborhoods of $x_1, \ldots, x_m$ that are mapped by f to some neighborhood V of y. In a picture



The local degree of f at x_i (denoted $\deg f\big|_{x_i}$) is the degree of the map

$$f_* : \mathbb{Z} \cong H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\}) \cong \mathbb{Z}$$

Theorem IV.7.4

Let $f : S^n \to S^n$ with $f^{-1}(y) = \{x_1, \ldots, x_m\}$ as above, then:

$$\deg f = \sum_{i=1}^{m} \deg f\big|_{x_i}$$

Thus we can compute the degree of f by computing these degrees.

Let's grab some intuition. What really is local homology?

Well, by excision, there is an isomorphism $H_n(S^n, S^n \setminus \{x_i\}) \cong H_n(U, U \setminus \{x_i\})$ for any open neighborhood U of x_i.

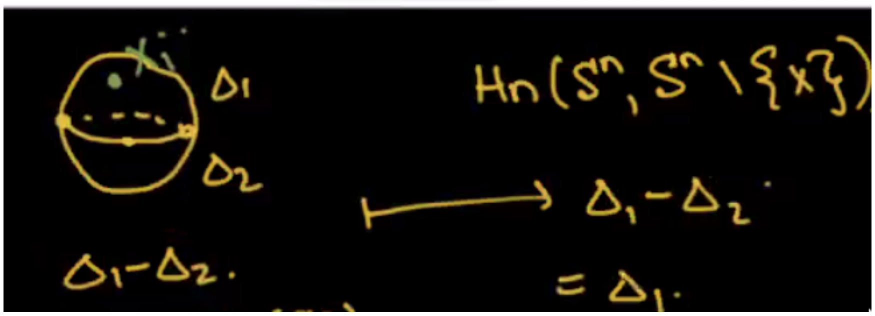
The long exact sequence of a pair also gives us:

$$\cdots \longrightarrow H_k(S^n \setminus \{x_i\}) \longrightarrow H_k(S^n) \xrightarrow{j_*} H_k(S^n, S^n \setminus \{x_i\}) \longrightarrow H_{k-1}(S^n \setminus \{x_i\}) \longrightarrow \cdots$$

Since $S^n \setminus \{x_i\}$ is homeomorphic to an open n-ball, we see that $H_k(S^n \setminus \{x_i\}) = H_{k-1}(S^n \setminus \{x_i\}) = 0$. With this in mind, j_* is an isomorphism.

We want to think about what j_* does when $k = n$, aka when this is an isomorphism $\mathbb{Z} \cong H_n(S^n) \to H_n(S^n, S^n \setminus \{x_i\}) \cong \mathbb{Z}$.

We see that $\Delta_1 - \Delta_2$ generate $H_n(S^n)$, where Δ_1, Δ_2 are the top and bottom hemisphere indicated here:



We then understand that $j_*(\Delta_1 - \Delta_2) = \Delta_1 - \Delta_2 = \Delta_1$¡ since $\Delta_2 = 0$ in $C_n(S^n)/C_n(S^n \setminus \{x_i\})$.

Upshot: $H_n(S^n, S^n \setminus \{x\})$ is generated by an n-simplex with x in its interior.

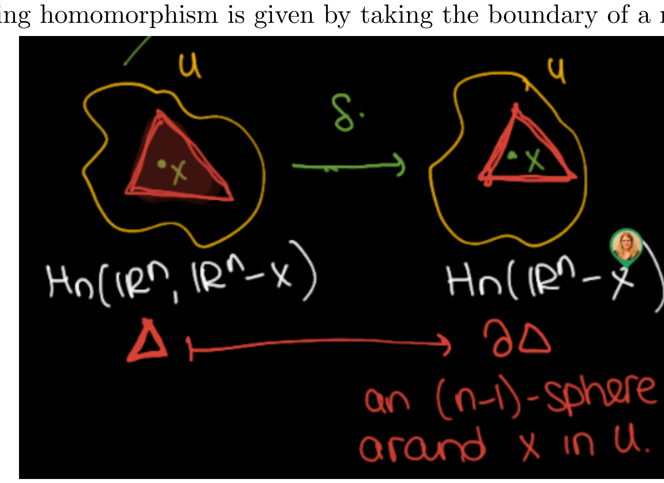
Suppose M is an n-manifold. Then $H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$, where U is a small ball around x. Because U is a ball homeomrphic to $\mathbb{R}^n$, we see that:

$$H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$

By the long exact sequence of a pair:

$$0 = H_n(\mathbb{R}^n) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \xrightarrow{\cong} H_{n-1}(\mathbb{R}^n \setminus \{x\}) \longrightarrow H_{n-1}(\mathbb{R}^n) = 0$$

And since $\mathbb{R}^n \setminus \{x\}$ is homotopy equivalent to an $n-1$ sphere, this means that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$. By homework, this connecting homomorphism is given by taking the boundary of a relative cycle as below:



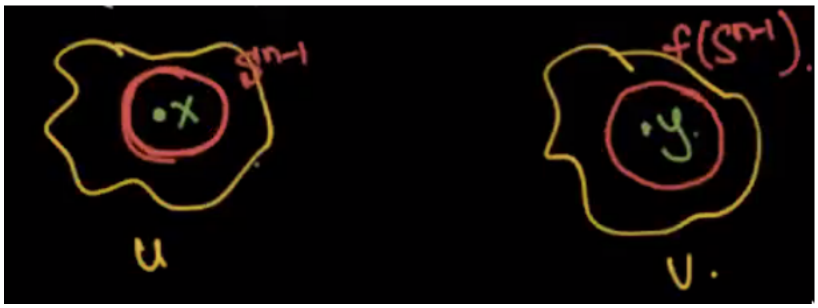
We intuitively want to use this idea to compute degree using this idea. We use naturality of the long exact sequence, namely the fact that where $f : (U_i, U_i \setminus \{x_i\}) \to (V, y)$ is a map of pairs, then the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & H_n(U_i, U_i \setminus \{x_i\}) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(U_i, U_i \setminus \{x_i\}) \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & H_n(V, V \setminus \{y\}) \arrow[r] & H_{n-1}(V, V \setminus \{y\}) \arrow[r] & \cdots \end{tikzcd}$$

By naturality of the LES and the isomorphism discussed above, we can compute the local degree of a map $S^n \to S^n$ at a point x by computing the degree of the map:

$$H_{n-1}(U \setminus \{x\}) \longrightarrow H_{n-1}(V - \{y\})$$

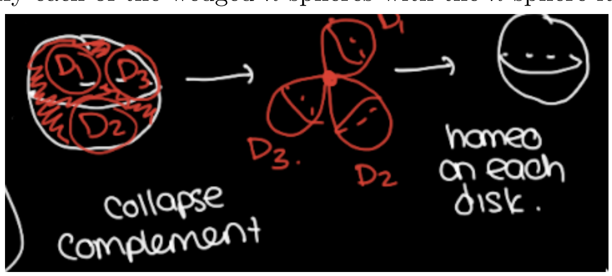
In fact the local degree will be the degree restricted to a small S^{n-1} in the neighborhood U.



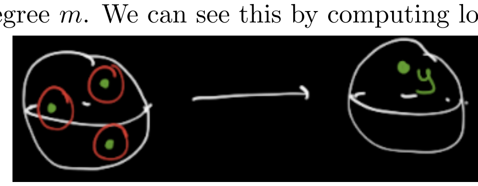
Let's work with some examples for our edification

Example IV.7.4

Consider S^n and choose m disks in S^n. Namely we first collapse the complement of the m disks to a point, and then we identify each of the wedged n-spheres with the n-sphere itself



The result will be a map of degree m. We can see this by computing local degree:



By choosing a good point in the codomain, we get one point for each disk in the preimage, and the map is a local homeomorphism around these points which is orientation preserving. Perfect! We could likewise compose the maps to S^n from the wedge with a reflection to construct a map of degree $-m$.

Example IV.7.5

Consider the composition of the quotient maps below $S^n \to \mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n$. We want to compute the degree of this map.

Note that this restricts to a homeomorphism on each component of $S^n \setminus$ equator as a map to $\mathbb{RP}^n \setminus \mathbb{RP}^{n-1}$. Suppose we've oriented our copies of S^n in such a way that the homeomorphism on the top hemisphere is orientation-preserving. The homeomorphism on the bottom hemisphere is given by taking the antipodal map and composing with the homeomorphism of the top hemisphere

$$\deg = \deg(\mathrm{Id}) = \deg(\text{antipodal}) = 1 + (-1)^{n+1} = \begin{cases} 0 & \text{if } n \text{ even} \\ 2 & \text{if } n \text{ odd} \end{cases}$$

IV.8. **Cellular Homology**

Suppose that X is a CW complex. Then (X^n, X^{n-1}) is a good pair for all $n > 1$, and X^n/X^{n-1} is a wedge of n-spheres, one for each n-cell e^n_α. Hence:

$$H_k(X^n, X^{n-1}) \cong \begin{cases} 0 & \text{if } k \neq n \\ \langle e^n_\alpha \mid e^n_\alpha \text{ is an } n\text{-cell} \rangle & \text{if } k = n \end{cases}$$

Definition IV.8.1

The cellular chain complex of X has chain groups $H_n(X^n, X^{n-1})$ with $X^{-1} = \emptyset$.

The boundary maps are given as:

$$\begin{aligned} d_1 : H_1(X^1, X^0) &\to H_0(X^0) \\ \langle \text{1-cells} \rangle &\to \langle \text{0-cells} \rangle \end{aligned}$$

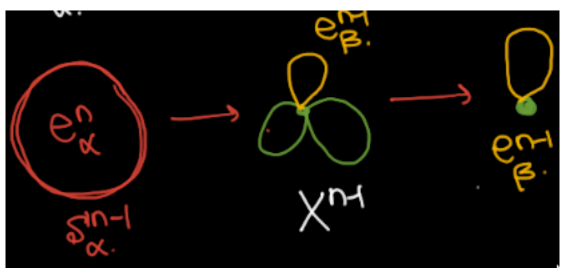
is the usual simplicial boundary map. FOr $n > 1$, the boundayr map:

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

where $d_{\alpha\beta}$ is the degree of the map:

$$\partial e^n_\alpha = S^{n-1}_\alpha \xrightarrow{\text{attaching map}} X^{n-1} \xrightarrow{\text{quotient by } X^{n-1} \setminus e^{n-1}_\beta} S^{n-1}_\beta$$

In pictures, this is given as:



Theorem IV.8.1

The homology groups of the cellular chain complex (cellular homology groups) coincide with the singular homology groups.

Announcements

Quiz 8 Wednesday

- Compute cellular homology of a CW complex
- Similar to examples in today's lecture

Corollary IV.8.2 (of Theorem IV.8.1)

We get a good bit of mileage out of this theorem:

- $H_n(X) = 0$ if X has a CW-complex structure with no n-cells.
- If X has a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements.
- If $H_n(X)$ is a group with a minimum of k generators, then any CW complex structure on X must have at least k n-cells.
- If X has a CW complex with no cells in conscecutive dimensions, then its homology is free abelian on its n-cells. For example $S^n, n \geq 2$ or $\mathbb{C}\mathrm{P}^n$.

Example IV.8.1

S^n with $n \geq 2$, using the CW coplex structure of e^n attached to a single point x_0. The cellular chain complex is given as:

$$0 \longrightarrow 0 \longrightarrow \langle e^n \rangle \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \langle x_0 \rangle$$

So then all the boundary maps are zero and we see that:

$$H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Exercise IV.8.2

Redo this calculation with other CW complex structure on S^n, e.g. glue 2 n-cells onto S^{n-1} and proceed inductively.

Example IV.8.3

Let's do this with the torus

The chain complex looks like:

$$0 \longrightarrow \langle D \rangle \xrightarrow{\partial_2} \langle a, b \rangle \xrightarrow{\partial_1} \langle x \rangle \longrightarrow 0$$

Note that $a \mapsto x - x = 0$ and $b \mapsto x - x = 0$ and so $\partial_1 = 0$. Now D is glued along $aba^{-1}b^{-1}$, so we look at the composed up map

We wind forwards then backwards around a, so the degree is zero. The same thing happens for b so:

$$\partial_2 D = 0 \cdot a + 0 \cdot b = 0$$

This gives a nice principle: If a 2-cell D is glued down via some word w (this only makes sense for 2-cells), then the coefficient to a letter b in $\partial_2 D$ is the sum of the exponents of b in w.

Great! Now we just have that the homology groups are equal to the chain groups because the boundary maps are all zero.

Example IV.8.4
A genus g surface Σ_g has the CW complex strucutre:

- 1 0-cell x
- 2g 1-cells $a_1, b_1, a_2, b_2, \ldots$
- 1 2-cell D glued along $[a_1, b_2][a_2, b_2] \cdots [a_g, b_g]$ (a product of commutators)

We obtain the result that:

$$\partial_1(a_i) = \partial_1(b_i) = x - x = 0$$

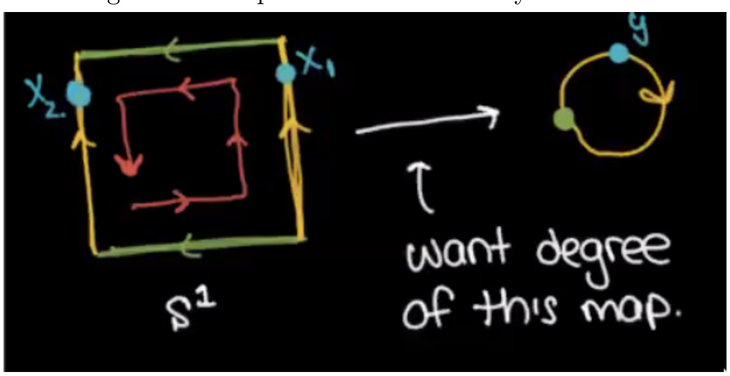
Furthermore by the principle discussed above, we know that every 1-cell appears once in the word, and its inverse appears once, so all the coefficients of 1-cells in $\partial_2(D)$ are zero, so $\partial_2(D) = 0$. This means we have a chain complex:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

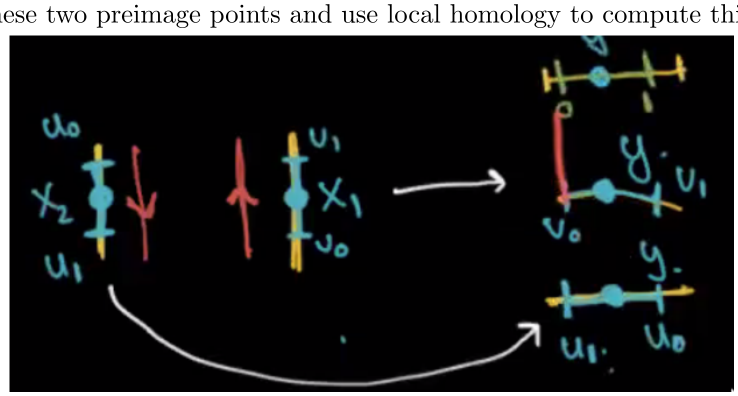
And so then we have that:

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example IV.8.5 (Torus example: ∂_2 in more detail)
We're going to work through this example a bit more carefully.



Let's zoom in on these two preimage points and use local homology to compute this:



Announcements

- Math 695
  - Cohomology
  - Poincare duality
  - Spectral Sequences
  - "Modern Perspectives"
  - Homotopy groups.
- Evaluations: Please give your feedback!
- Today: Some proof outlines

Proof outline for local degree computations. If $f : S^n \to S^n$ and we have some $y \in S^n$ with $f^{-1}(\{y\}) = \{x_1, \ldots, x_m\}$ then:

$$\deg f = \sum_i \deg f\big|_{x_i}$$

We have a nice commutative diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[rr] \arrow[d] & & H_n(S^n) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x_1, \ldots, x_m\}) & & H_n(S^n, S^n \setminus \{y\}) \\
H_n\left(\bigsqcup_i U_i, \bigsqcup U_i \setminus \{x_i\}\right) \arrow[u, "\cong"'] & & \\
\bigoplus_i H_n(U_i, U_i \setminus \{x_i\}) \arrow[u, "\cong"'] \arrow[rr] & & H_n(V, V \setminus \{y\}) \arrow[uu, "\cong"]
\end{tikzcd}$$

Where we have the isomorphisms $\cong$ by excision and maps / isomorphisms blue by the LES of a pair. And we also have $\cong$ from the homology of a disjoint union.

But then tracing around the outside of the diagram we get:

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[d, mapsto] & \deg f \arrow[d, mapsto] \\
(1, \ldots, 1) \arrow[r, mapsto] & \deg f = \sum_i \deg f\big|_{x_i}
\end{tikzcd}$$

And this proves the result. Perfect! ☺

We're now going to work towards proving that cellular homology agrees with singular homology. First we need some nontrivial preliminaries:

(1) $H_k(X^n, X^{n-1}) = \begin{cases} 0 & \text{if } k \neq n \\ \langle n - \text{cells} \rangle & \text{if } k = n \end{cases}$

(2) $H_k(X^n) = 0$ for all $k > n$. If X is finite dimensional, then $H_k(X^n) = 0$ for all $k > \dim X$.

(3) The inclusion $X^n \hookrightarrow X$ induces $H_k(X^n) \to H_k(X)$. Then this map is
- An isomorphism for $k < n$.
- Surjective for $k = n$
- Zero for $k > n$.

Exercise IV.8.6

Check (2) and (3) directly in the case that the CW-complex structure is a Δ-complex structure using simplicial chains

Proof of (2) + (3). We consider the Long Exact Sequence of a pair for fixed n:

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \longrightarrow \widetilde{H}_k(X^{n-1}) \longrightarrow \widetilde{H}_k(X^n) \longrightarrow H_k(X^n, X^{n-1}) \longrightarrow \cdots$$

When $k+1 < n$ or $k > n$ then $H_{k+1}(X^n, X^{n-1}) = 0$ and $H_k(X^n, X^{n-1}) = 0$, so the above map $\widetilde{H}_k(X^{n-1}) \to \widetilde{H}_k(X^n)$ is an isomorphism. We also get sequences telling us the injective and surjective maps when $k = n$ or $k = n-1$:

$$0 = H_{n+1}(X^n, X^{n-1}) \longrightarrow \widetilde{H}_n(X^{n-1}) \longrightarrow \widetilde{H}_n(X^n) \longrightarrow H_n(X^n, X^{n-1})$$

$$\xrightarrow{\quad\cong\quad} \widetilde{H}_{n-1}(X^{n-1}) \longrightarrow \widetilde{H}_{n-1}(X^n) \longrightarrow H_{n-1}(X^n, X^{n-1}) = 0$$

So the maps $\widetilde{H}_n(X^{n-1}) \to \widetilde{H}_n(X^n)$ is injective, and the map $\widetilde{H}_{n-1}(X^{n-1}) \to \widetilde{H}_{n-1}(X^n)$ is surjective.

Fix k, then we a pile of maps induced by the inclusions $X^n \hookrightarrow X^{n+1}$:

$$\widetilde{H}_k(X^0) \xrightarrow{\cong} \widetilde{H}_k(X^1) \xrightarrow{\cong} \widetilde{H}_k(X^2) \xrightarrow{\cong} \cdots$$

$$\cdots \longrightarrow \widetilde{H}_k(X^{k-1}) \xrightarrow{\text{inj.}} \widetilde{H}_k(X^k) \xrightarrow{\text{surj.}} \widetilde{H}_k(X^{k+1})$$

$$\xrightarrow[\cong]{} \widetilde{H}_k(X^{k+2}) \xrightarrow[\cong]{} \widetilde{H}_k(X^{k+3}) \xrightarrow[\cong]{} \cdots$$

Note: This sequence is not exact. Descriptions of maps (in red) follow from our analysis of the LES of a pair above.

To prove (2):

- $k = 0$, we do by hand
- $k \geq 1$, then $\widetilde{H}_k(X^0) = 0$, so we have that $\widetilde{H}_k(X^0), \ldots, \widetilde{H}_k(X^{k-1})$ are all zero from the isomorphisms above. That is the k-th homology $\widetilde{H}_k(X^n) = H_k(X^n)$ is zero for every n-skeleton where $n < k$, just as desired.

We also have the following collection of maps for fixed k:

$$H_k(X^k) \xrightarrow{surj.} H_k(X^{k+1}) \xrightarrow{\cong} H_k(X^{k+2}) \xrightarrow{\cong} \cdots$$

This implies (3) when X is finite dimensional. For general X, we use the fact that every simplex has image contained in some finite skeleton (since image is compact).

Proof that Cellular Homology $\cong$ Singular Homology. We get some exact sequences from our preliminaries last time:

$$0 = H_{n+1}(X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1})$$

$$H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) = 0$$

These come from the long exact sequences of a pair combined with the things we've deduced in the preliminaries. We can paste these together into a diagram:

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Hatcher tells us this diagram commutes, and what we've done here tells us that the two red diagonal pieces crossing at $H_n(X^n)$ are exact. We also have exactness of the bottom right diagonal by just going down a degree.

Then this has to at least be a chain complex. Why? Well the diagram commutes because of Hatcher. We then know that:

$$d_{n+1} \circ d_n = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0$$

By exactness, we know that if $\iota_* : H_n(X^n) \to H_n(X^{n+1})$ then using the first isomorphism theorem:

$$H_n(X) \cong H_n(X^{n+1}) = \operatorname{im} \iota_* \cong \frac{H_n(X^n)}{\ker \iota_*} = \frac{H_n(X^n)}{\operatorname{im} \partial_{n+1}}$$

Since j_n injects by exactness,

$$j_n : H_n(X^n) \xrightarrow{\cong} j_n(H_n(X^n))$$

$$\operatorname{im} \partial_{n+1} \xrightarrow{\cong} \operatorname{im}(j_n \circ \partial_{n+1}) = \operatorname{im} d_{n+1}$$

j_{n-1} must also inject by exactness, and so applying exactness:

$$\ker d_n = \ker \partial_n = \operatorname{im} j_n$$

Then we just do some group theory, the n-th cellular homology group is:

$$\frac{\ker d_n}{\operatorname{im} d_{n+1}} \cong \frac{\operatorname{im} j_n}{\operatorname{im}(j_n \circ \partial_{n+1})} \cong \frac{H_n(X^n)}{\operatorname{im} \partial_{n+1}} \cong H_n(X)$$

There is one thing left to show, namely commutativity of this map. That is

Claim

The differentials $d_n = j_n \circ \partial_{n+1}$ satisfy the formula (in terms of degree) that we stated. This is done by direct analysis of definitions of maps; details in Hatcher.

IV.9. **The Formal Viewpoint: Eilenberg-Steenrod axioms**

Definition IV.9.1

Given two functors $F, G : C \to D$, a natural transformation $\eta : F \to G$ is a collection of maps $\eta_X : F(X) \to G(X)$ lying in D for every $X \in C$ so that for any map $f : X \to Y$ we have a commutative diagram:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

Definition IV.9.2

A homology theory is a sequence of functors:

$$H_n : \text{pairs } (X, A) \text{ of spaces} \to \text{abelian groups}$$

Equipped with natural transformations $\partial : H_n(X, A) \to H_{n-1}(A)$, where $H_{n-1}(A) := H_{n-1}(A, \emptyset)$ called the boundary map. Naturality here means that for any map $f : (X, A) \to (Y, B)$ we have a commutative diagram:

$$\begin{tikzcd} H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[d, "f_*"] \\ H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \end{tikzcd}$$

These must satisfy these axioms:

1. (Homotopy) If $f, g : (X, A) \to (Y, B)$ and $f \simeq g$, then $f_* = g_*$
2. (Excision) If $U \subseteq A \subseteq X$ so that $\overline{U} \subseteq \mathrm{Int}(A)$ then $\iota : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces isomorphisms on homology
3. (Dimension) $H_n(*) = 0$ for all $n \neq 0$, where $*$ denotes some arbitrary point
4. (Additivity) $H_n\left(\bigsqcup_\alpha X_\alpha\right) = \bigoplus_\alpha H_n(X_\alpha)$.
5. (Exactness) If we have an inclusion $\iota : A \hookrightarrow$ and $j : X \to (X, A)$ induces a long exact sequence on homology:

$$\cdots \longrightarrow H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

If H_* satisfies all axioms but dimension, it is called an extraordinary homology theory

Example IV.9.1

Topological K-theory and cobordism.

Theorem IV.9.1

If H_n : CW pairs $\to$ Ab is a homology theory and $H_0(*) = \mathbb{Z}$, then H_n are exactly the singular homology functors up to a natural isomorphism of functors

More generally, without the assumption that $H_0(*) = \mathbb{Z}$, then H_n are exactly the singular homology functors with coefficients in the abelian group $H_0(*)$.

Proof. Reconstruct the cellular homology groups using the axioms. The exact same argument we did today follows. We then check that the cellular homology groups we just constructed satisfies the degree formula as in our last step. This is a bit more difficult, but we won't get into it.

Announcements

- Bonus problem added to Homework 12 on orientability and homology
- Quiz on Wednesday
  - Calculation using the Mayer-Vietoris Long Exact Sequence

V. Lefschetz Fixed Point Theorem

V.1. Statement

Definition V.1.1

Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ be a group homomorphism, we may represent this with a matrix $A = \{a_{ij}\}$. The trace is the sum $a_{11} + \cdots + a_{nn}$.

For a group homomorphism $\varphi : M \to M$ where M is a finitely generated abelian group, we define the trace of φ to be the trace of the induced map $\overline{\varphi} : M/M_T \to M/M_T$, where M_T is the torsion subgroup of M.

Exercise V.1.1

We have

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

Thus, matrices related by a change of basis have the same trace.

Definition V.1.2

Let X be a space with the assumption that $\oplus_k H_k(X)$ is finitely generated. That is, each homology group is finitely generated, and there are finitely many nonzero homology groups. For example X could be a finite CW complex.

The Lefschetz number $\tau(f)$ of a map $f : X \to X$ is:

$$\tau(f) := \sum_k (-1)^k \operatorname{tr}(f_* : H_k(X) \to H_k(X))$$

Example V.1.2

When $f \simeq \mathrm{Id}_X$. Then $f_* = \mathrm{Id}_{H_k(X)}$ for all k. Then $\operatorname{tr}(f_* : H_k(X) \to H_k(X)) = \operatorname{rank}(H_k(X))$. Therefore:

$$\tau(f) = \sum_k \operatorname{rank}(H_k(X)) = \chi(X)$$

Where $\chi(X)$ is the Euler characteristic (see homework).

Theorem V.1.1 (Lefschetz Fixed Point Theorem)

Suppose X admits a finite triangulation (i.e. a finite simplicial complex structure). Or more generally, X is a retract of a finite simplicial complex.

Then if $f : X \to X$ is a map with $\tau(f) \neq 0$, then f has a fixed point. Note that the converse does not hold.

Theorem V.1.2 (Hatcher's Appendix A.7)

If X is a compact, locally contractible space that can embed in $\mathbb{R}^n$ for some n, then X is a retract of a finite simplicial complex.

This includes:

- Compact Manifolds
- Finite CW complexes

Definition V.1.3

Let $\mathbb{F}$ be a field, and let $H_k(X;\mathbb{F})$ be the k-th homology of X with coefficients in $\mathbb{F}$ (see homework).

THen $H_k(X;\mathbb{F})$ is always a vector space over $\mathbb{F}$. Define $\tau^{\mathbb{F}}(X)$ be:

$$\sum_k (-1)^k \operatorname{tr}(f_* : H_k(X;\mathbb{F}) \to H_k(X;\mathbb{F}))$$

The Lefschetz fixed point theorem still holds if we replace "$\tau(x) \neq 0$" with "$\tau^{\mathbb{F}} \neq 0$"

Example V.1.3
Let $f : S^n \to S^n$ be a degree d map. Then $\tau(f)$ is:

$$(-1)^0 \operatorname{tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \operatorname{tr}(f_* : H_n(S^n) \to H_n(S^n))$$

Then $f_* : H_0(S^n) \to H_0(S^n)$ is the identity, and $f_* : H_n(S^n) \to H_n(S^n)$ is given by the 1×1 matrix with entry d. And then we have:

$$\tau(f) = 1 + (-1)^n d$$

Corollary V.1.3
f has a fixed point whenever $1 + (-1)^n \neq 0$. Aka whenever $d \neq (-1)^{n+1}$. THat is f has a fixed point if its degree is not equal to the degree of the antipodal map.

Exercise V.1.4
If $f : X \to X$, then $\operatorname{tr}(f_* : H_0(X) \to H_0(X))$ is equal to the # of path-components of X mapped to themselves

Exercise V.1.5
If X is contractible, then its homology is concentrated in degree zero, so $\tau(f) = 1$.

If X is a compact manifold or finite CW-complex, every f has a fixed point (in particular, this recovers Brouwer's Fixed Point Theorem)

Example V.1.6
If we consider the map $f : \mathbb{R} \to \mathbb{R}$ given by translation by $x \neq 0$, then $\tau(f) = 1$, but f does not have a fixed point. The key here is that $\mathbb{R}$ is not compact.

Example V.1.7 (Qual, May 2016)

Let X be a finite, connected CW complex. $\widetilde{X}$ is its universal cover, and $\widetilde{X}$ is compact. Show that $\widetilde{X}$ cannot be contractible unless X is contractible.

Solution. By homework, we then know that, since $\widetilde{X}$ is contractible and $\widetilde{X}$ has finitely many sheets d over X:

$$1 = \chi(\widetilde{X}) = d \cdot \chi(X)$$

Therefore, $\chi(X) = d = 1$, and so $p : \widetilde{X} \to X$ is a 1-sheeted cover, so it is a homeomorphism. Therefore X is contractible. Perfect!

Solution. Since $\widetilde{X}$ is contractible, $\tau(f) = 1$ for all $f : \widetilde{X} \to \widetilde{X}$. Furthermore, because $\widetilde{X}$ is compact and covers a finite CW complex, it is a finite CW complex. Therefore the Lefschetz Fixed Point Theorem applies, so any such map has a fixed point. If f is a deck map, then that means that $f = \mathrm{Id}_{\widetilde{X}}$ from our covering space theory. Great!

We have proved then that $X \cong \widetilde{X}/G(\widetilde{X})$ because $p : \widetilde{X} \to X$ is normal, but then the deck group $G(\widetilde{X})$ is trivial, so $X \cong \widetilde{X}$, and we are done.

Exercise V.1.8

A 1-sheeted cover is always injective and surjective. Furthermore, it's a local homeomorphism. This suffices to show that a 1-sheeted cover is a homeomorphism.

Theorem V.1.4

If X is a finite CW complex, with cellular chain groups $H_n(X^n, X^{n-1})$. If we have a cellular map $f : X \to X$, so f induces maps $f_* : H_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1})$. Then:

$$\tau(f) = \sum_n (-1)^n \operatorname{tr}(f_* : H_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1}))$$

Proof. Do some algebra! This is a purely algebraic fact

Exercise V.1.9

Given a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

Then $\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$.

Using the exercise, the theorem follows by an argument analogous to the argument for Euler Characteristic on Homework 12.

Announcements

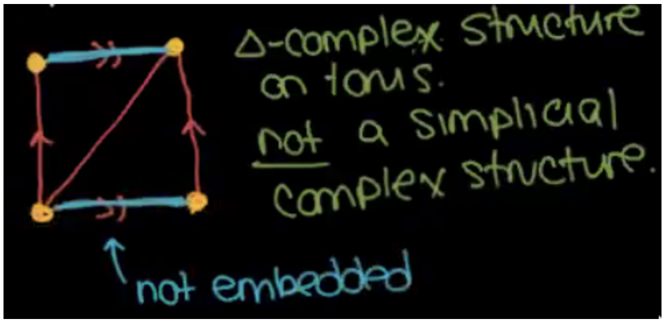
- Fill our the Office Hour / Review Session scheduling survey
- No quiz next week
- No more homeworks
- Final Exam–April 28 1:30pm ET

Definition V.1.4 (Simplicial Complexes)

A simplicial complex is a Δ-complex with the conditions that:

- Each simplex is embedded
- Intersection of simplices $\sigma_1 \cap \sigma_2$ must be $\emptyset$ or a single subsimplex o both σ_1 and σ_2

Let's look at our Δ-complex structure on the torus:



A simplicial complex structure on the torus requires at least fourteen triangles.

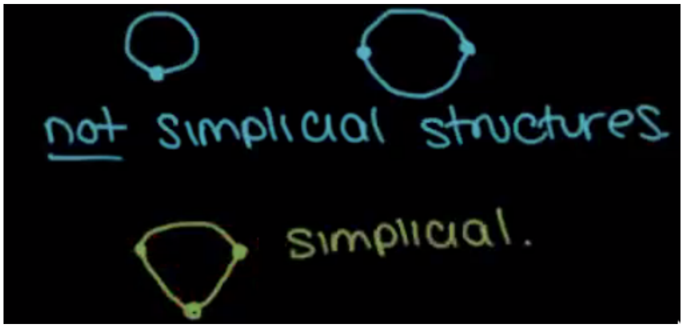
To specify a simplicial complex, we can do the following

- Start with X^0 (discrete set)
- Indicate which subsets of X^0 span a simplex.

This completely specifies the data of a simplicial complex.

Example V.1.10

For S^1 we have the ollowing picture:



Definition V.1.5

A *simplicial map* $f : K \to L$ is a continuous map that sends each simplex of K to a (possibly smaller dimensional) simplex of L by a linear maps as follows

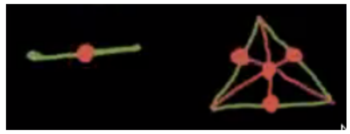
$$\sum t_i v_i \mapsto \sum t_i f(v_i)$$

A simplicial map is completely determined by its restriction to the vertex set.

Theorem V.1.5 (Simplicial Approximation)

Given any continuous map $f : K \to L$ where K is a finite simplicial complex and L is any simplicial complex. Then f is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K

Here is barycentric subdivision in pictures:



That is, we add a new vertex to the center of every subsimplex, filling things in like the above. For an n-simplex we end up with $(n+1)!$ simplices which replace it.

Proof Outline of Lefschetz Fixed Point Theorem. We now prove Theorem V.1.1. Fix a space X which is a finitee simplicial complex (or a retract of a fintie simplicial complex) and a map $f : X \to X$.

(Step 1) Reduce to the case of a finite simplicial complex X. Suppose K is a finite simplicial complex, with $r : K \to X$ a retraction. First notice that the following composite of maps

$$K \xrightarrow{r} X \xrightarrow{f} X \xrightarrow{\iota} K$$

has the same fixed points as f.

Exercise V.1.11

$r_* : H_n(K) \to H_n(X)$ is split surjective (see ι_*), and so it has to be a projection onto a direct summand

Exercise V.1.12

It follows that $\mathrm{tr}(\iota_* \circ f * \circ r_*) = \mathrm{tr}(f_*)$ on degree n homology.

This implies that $\tau(f) = \tau(\iota \circ f \circ r)$. Therefore if we can prove the result for a simplicial complex then we are done.

(Step 2) Let X be a finite simplicial complex. We show that if $f : X \to X$ has no fixed points then $\tau(f) = 0$.

Goal: Find subdivisions K, L of X and $g : K \to L$ so that:

- g is simplicial
- $g \simeq f$, $\tau(f) = \tau(g)$
- $g(\sigma) \cap \sigma = \emptyset$ for all simplices σ.

So this becomes a few steps, none of which we'll justify too formally:

- Choose a metric d on X
- Since X is compact, and f has no fixed point, then $d(x, f(x))$ has some minimum value $\varepsilon > 0$.
- Subdivide all simplices of X until simplices have diameter smaller than $\frac{\varepsilon}{53}$. Call this subdivision L.
- Use the simplicial approximation theorem to obtain a map $g : K \to L$, where K is a subdivision of L, $g \simeq f$
- By proof of simplicial approximation theorem, we can construct g so that for all simplices σ, $g(\sigma)$ is not too far from $f(\sigma)$. We can then conclude that $g(\sigma) \cap \sigma = \emptyset$.
- So then g is a cellular map $K \to K$ that moves every cell. We can then check that:

$$\tau(f) = \tau(g) = \sum (-1)^n \,\mathrm{tr}(g_* : \text{cellular } n\text{-chains} \to \text{cellular } n\text{-chains}) = 0$$

Because each g_* has vanishing diagonal entries.

Then we're done! Great ☺