Stuff:

- Problem 3.2 is incorrect as stated. If $S \subseteq M$ is a submanifold and X is a vector field on M which is tangent to S, then if X is tangent to S then for all $p \in S$, the integral curve $\theta^{(p)}(t)$ (the flow) of X is contained in S for *small* values of t.

  The problem stated was for all values of t, obviously false.
- Hint for Problem 2: If $M + v, N$ intersect, then $v = y - x$ for some $y \in N, x \in M$. Consider the map $F : M \times N \to \mathbb{R}^n$ given by $F(x, y) = y - x$ and apply Sard's Theorem.
- Last time: distributions (k-plane fields), $V(p) \subseteq T_pM$ a k-dimensional subspace..

There are two kinds of distributions

- integrable (tractable)
- non-integrable (more fun)

Definition .0.1

We call a k-plane V integrable provided that for all $p \in M$ there exists a coordinate chart (U, φ) such that for all $x \in M$,

$$\{(x_1, \ldots, x_k, 0, \ldots, 0) \mid x_i \in \mathbb{R}\} = D\varphi(V(x)) \subseteq T_{\varphi(x)}\mathbb{R}^n \cong \mathbb{R}^n.$$

Example .0.1

Take $M = \mathbb{R}^n$, and $V(p) = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle$.

This is in particular an example of a "foliation," which we will define now. Namely the foliation is given by the partition of $\mathbb{R}^2$ as

$$\mathbb{R}^2 = \bigcup_{x \in \mathbb{R}} \mathbb{R} \times \{x\}.$$

Non-Example .0.2

The Heisenberg group from last time, namely if

$$\mathrm{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Then if we take

$$V(1) = \left\langle \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

We can of course set $V(g) = DL_g(V(1))$.

Definition .0.2

Let M be a C^∞ manifold. A foliation is a partition $\mathcal{F}$ of M such that for all $x \in M$ we have F_x is an immersed submanifold of M, and

$$F_x \cap F_y \neq \emptyset \iff F_x = F_y.$$

(i.e., F defines an equivalence relation).

Furthermore, we require that for all $p \in M$ there exists a coordinate chart (U, φ) such that for each $x \in U$, if V_x is the connected component of $F_x \cap U$ in U, then $\varphi(V)$ locally looks like $\mathbb{R}^k \times \{0\} + \varphi(x)$ (aka looks locally like the above example).

We take the connected component in case F_x "loops back" into U.

If $\mathcal{F}$ is a foliation, then call F_x the leaf of $\mathcal{F}$ through x. Then we can define a distribution $V(p) = T_p F_p$, which is a k-dimensional distribution, C^∞.

Consider: Let X, Y be vector fields on M such that for all $p \in M$, we have $X(p), Y(p) \in V(p) := T_p F_p$ for some foliation F. By the Homework 7 Problem 3c we know that $[X, Y](p) \in V(p)$.

Fact: If $V(p)$ is the tangent distribution to a foliation $\mathcal{F}$ (i.e., $V(p) = T_p F_p$), then for any two vector fields X, Y with $X(p), Y(p) \in V(p)$ for all p, we have $[X, Y](p) \in V(p)$.

Definition .0.3

Given any smooth k-dimensional distribution V on a C^∞-manifold M, we call V involutive if for any two vector fields X, Y with $X(p), Y(p) \in V(p)$ (tangent to V) for all p we have $[X, Y](p) \in V(p)$ for all p.

Theorem .0.1 (Frobenius Theorem)

A distribution is involutive if and only if it is integrable (defined with charts).

Proof. This is in [Lee10], p490. The $\Longleftarrow$ direction we just did with HW 7 Problem 3

Example .0.3

Give $p + \mathbb{R}^2$ as a foliation on $\mathbb{R}^3$, with V its corresponding distribution. Now quotient out by $\mathbb{R}^3$, so then $\overline{V}(\overline{p}) = D\pi_p \cdot V(p)$ where $\pi : \mathbb{R}^3 \to \mathbb{R}^3/\mathbb{Z}^3 =: \mathbb{T}^3$.

This gives us 2-tori foliating $\mathbb{T}$.

Mess it up a little, rotate $V(p)$ by an angle irratioanl with the embedded $\mathbb{R}^2$. Namely consider a foliation $\mathbb{R} \cdot v_1 + \mathbb{R} \cdot v_2$ where v_1, v_2 are irrational with respect to $\mathbb{Z}^2$.

We may then push this down to $\mathbb{T}^3$ as before (check this is well-defined...). Then $\mathbb{T}^3$ is foliated by "planes" (they cannot close up) densely.

Proof of Frobenius, in special case. Suppose for all p there exists a U neighborhood of p with vector fields $X_1, \dots, X_k$ with $\langle X_1(q), \dots, X_k(q) \rangle = V(q)$ such that for all i, j we have $[X_i, X_j] = 0$.

Then Frobenius holds. By last Friday, local flows φ_i associated with X_i commute. We can then build an immersion

$$(t_1, \dots, t_k) \mapsto \varphi_k(t_k) \circ \cdots \circ \varphi_2(t_2) \circ \varphi_1(t_1) \cdot p.$$