Consider a group homomorphism $G_1 \xrightarrow{\varphi} G_2$. There is trouble: there exists a $\varphi : \mathbb{R} \to \mathbb{R}$ which is a homomorphism but which is not differentiable. You can construct nice ones with Galois theory, but a simpler one is given by writing $\mathbb{R}$ as a $\mathbb{Q}$-vector space with an uncountable basis and mapping the basis around in a strange way.

Definition .0.1

If $\varphi : G_1 \to G_2$ between Lie groups is a group homomorphism and C^∞, then we call it a Lie group homomorphism

Remark .0.1

It is good enough to assume φ is measurable (more strongly, continuous). When we say measurable we mean to be with respect to charts. This is very very important, although we won't use it. The thought: it's generally much easier to prove something is measurable than to prove something is differentiable.

This induces a Lie algebra homomorphism

$$D\varphi_1 : \mathrm{Lie}(G_1) = T_1 G_1 \to T_1 G_2 = \mathrm{Lie}(G_2).$$

Call these $\mathfrak{g}_1, \mathfrak{g}_2$. We want this to respect the bracket.

Let $X \in \mathfrak{g}_1$ be some left-invariant vector field on G_1. $D\varphi_1 X(1) \in \mathfrak{g}_2$, and corresponds to some left invariant vector field Y on $\mathfrak{g}_2$.

Claim

X, Y are φ-related.

Proof. Take some $g \in G_1$. We must show that

$$D_g\varphi \cdot X(g) = Y(\varphi(g)).$$

Well, we know that

$$\begin{aligned} D_g\varphi \cdot X(g) &= D_g\varphi \cdot D_1 L_g \cdot X(1) \\ Y(\varphi(g)) &= D_1 L_{\varphi(g)} \cdot Y(1) = D_1 L_{\varphi(g)} \cdot D_1\varphi \cdot X(1). \end{aligned}$$

The result then follows since $L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$ since this is a group homomorphism.

THE FOLLOWING THEOREM IS WRONG, CORRECTED NEXT TIME

Theorem .0.1

There is a bijective correspondence between Lie group homomorphisms $G_1 \to G_2$ and Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$.

Proof. The forward direction we just did (modulo 1-1 business). For the converse we need a trick. Namely if $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism we see that

$$\operatorname{graph}\psi = \{(X, \psi(X)) \mid X \in \mathfrak{g}_1\}$$

is in fact Lie subalgebra of $\mathfrak{g}_1 \times \mathfrak{g}_2$, which is a Lie algebra with bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2]).$$

We then see that

$$[(X_1, \psi(X_1)), (X_2, \psi(X_2))] = ([X_1, X_2], [\psi(X_1), \psi(X_2)]) = ([X_1, X_2], \psi([X_1, X_2])).$$

We now have a Lie subalgebra, so by the main result last time there is a Lie subgroup $H \subseteq G_1 \times G_2$ (note: $\mathrm{Lie}(G_1 \times G_2) = \mathrm{Lie}(G_1) \times \mathrm{Lie}(G_2)$).

Claim

H is the graph of a homomorphism $\Psi : G_1 \to G_2$.

I.e., $\Psi(g_1) = g_2$ if $(g_1, g_2) \in H$. One must check that this is well-defined and a homomorphism

Exercise .0.1

Check well-definedness and homomorphism. We'll come back to it later.

.1. Exponential Map

Let $X \in \mathfrak{g}$, where G is a Lie group with Lie algebra $\mathfrak{g}$. Then $\{tX \mid t \in \mathbb{R}\}$ is a Lie subalgebra since

$$[sX, tX] = st[X, X] = 0.$$

Thus there exists a connected Lie subgroup of G corresponding to $X \in \mathfrak{g}$.

This is extremely abstract. Lets get down to Earth again. Le t $X \in \mathfrak{g}$ be a left invariant vector field. This gives us a local flow φ_t on G.

We can consider $1 \in G$ and define $g_t \coloneqq \varphi_t(1)$. Then

$$g_t \cdot g_s = \varphi_t \cdot \varphi_s(1) = \varphi_{t+s}(1) = g_{t+s}.$$

We also have

Claim

φ_t is a global flow, i.e. defined for all t.

Proof. Appeal to the subgroups argument. Or more simply, we know the local flow of X through g is simply

$$L_g(\varphi_t(1)) = g \cdot g_t.$$

Thus if local flow at 1 is defined on $(-\varepsilon, \varepsilon)$ so is it at g. We can then define it globally, around each point in $(-\varepsilon, \varepsilon)$ the flow is defined in $(-\varepsilon, \varepsilon)$ about it, and then we can continue, defining the flow on $(-2\varepsilon, 2\varepsilon)\ldots$

Since $\varepsilon > 0$ is fixed this gives us a global flow.

Example .1.1

We want to look at this very concretely. Prime Example is $G = \mathrm{GL}_n(\mathbb{R})$. We see that

$$\mathfrak{gl}_n(\mathbb{R}) = T_1 \mathrm{GL}_n(\mathbb{R}) = M_{n,n}.$$

If $X \in M_{n,n}$ then what is φ_t, well

$$e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!},$$

converges sicne

$$\left\| \sum_{n=0}^{\infty} \frac{(tX)^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{|t|^n \|X\|^n}{n!} = e^{|t| \cdot \|X\|},$$

where $\|\cdot\|$ is the operator norm, and it is easy to check that $\|AB\| \leq \|A\| \cdot \|B\|$, whcih gives $\|X^n\| \leq \|X\|^n$.

Finally note that $\frac{\mathrm{d}}{\mathrm{d}t}(e^{tX}) = Xe^{tX}$. We also must show $e^{tX} \in \mathrm{GL}(n, \mathbb{R})$. This will be because if A, B commute then $e^A e^B = e^{A+B}$, so $e^{tX} e^{-tX} = \mathrm{Id}$.

Example .1.2

This also works for any subgroups of $H \subseteq \mathrm{GL}_n(\mathbb{R})$, namely if we have a flow for $X \in T_1 H$ lying in $\mathrm{GL}_n(\mathbb{R})$, then of course the flow lies in H.