Announcements

- Move midterm by 1 or 2 weeks.
- Bonus 5 + HW6 Due Friday 11PM

Last time: Take a chart $\varphi : U \to \mathbb{R}^n$ which takes p to 0. We really want to take a smooth f on $\varphi^{-1}(B_\varepsilon(0))$ to a smooth f which is 0 outside $\varphi^{-1}(B_{2\varepsilon}(0))$.

For this, we take a bump function $\rho : \mathbb{R}^n \to \mathbb{R}$ which is 0 outside $B_{2\varepsilon}(0)$ and which is 1 inside $B_\varepsilon(0)$.

Now back to immersions, submersions, and regular values.

Recall .0.1

Let M, N be C^∞ manifolds, $q \in N$ is a regular value of $f : M \to N$ if $Df_p : T_pM \to T_qN$ is surjective for all $p \in f^{-1}(q)$. Note that if $q \notin \text{Image}(f)$ then q is regular

We recall a bit of measure theory. Let $A \subseteq \mathbb{R}^n$ and define $\text{vol}(A) = \int_A 1 \,\mathrm{d}x$. Lebesgue measure is preferable, but we say A mas measure zero if $\text{vol}(A) = 0$.

Definition .0.1

Suppose M is a C^1 manifold. Consider $B \subseteq U$ some chart. We say B has measure zero if $\varphi(B)$ has measure zero.

Note this is well-defined since the transition maps are C^1, which gives for two charts φ, ψ that

$$\begin{aligned}\text{vol}(\psi(B)) &= \int_{\psi(B)} 1 \,\mathrm{d}x \\ &= \int_{\varphi(B)} \det(D(\psi \circ \varphi^{-1})_x \,\mathrm{d}x\end{aligned}$$

Note then that volume changes across charts; but zero volume is well-defined.

Say $B \subseteq M$ has zero measure if for all charts (U, φ) we have $\text{vol}(\varphi(B \cap U)) = 0$.

We say $A \subseteq M$ has full measure if $M \setminus A$ has 0 measure.

Theorem .0.1 (Sard's Theorem)

Let M, N be C^∞ manifolds and $f : M \to N$ be smooth, then the set of regular values has full measure.

Warning: needs C^∞ (at least some C^k for k large enough).

Proof Idea. Approximate f by a linear map $Df = L$. Actually lol Professor Spatzier doesn't know

Example .0.2

Here's an application. To show $\text{SL}_n(\mathbb{R})$ is a manifold it suffices to show 1 is a regular value of det since $\text{SL}_n(\mathbb{R}) = \det^{-1}(1)$.

Note $\det(\lambda A) = \lambda^n \det(A)$ for $A \in \text{GL}_n(\mathbb{R})$.

Claim

If 1 is not a regular value, then neither is λ^n for $\lambda \neq 0$.

If λ is nonzero, then $A \xmapsto{m_\lambda} \lambda \cdot A$ is invertible. Suppose λ^n is a regular value. Then fix A so that $\det(A) = 1$. Then we see that $D_A \det$ is surjective if and only if

$$D_A(\det \circ m_\lambda) = D_{\lambda A} \det \circ D_A(m_\lambda)$$

is surjective, which follows by regularity and invertibility of m_λ.

Now by Sard, since any $\{\lambda^n \mid \lambda \neq 0\}$ doe snot have measure zero, 1 must be a regular value.

The simplest immersion is given for $k \leq n$ as

$$\mathbb{R}^k \hookrightarrow \mathbb{R}^n$$
$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0).$$

Proposition .0.2

Suppose $f : M \to N$ where $k = \dim M, n = \dim N$ is an immersion at p, so that Df_p is injective. Then there exist charts (U, φ) at p and (V, ψ) at $f(p)$ so that $\psi \circ f \circ \varphi^{-1}$ has the form given above.

Proof. Fix arbitrary charts (U, φ) and (V, ψ) as well. We'll work on the charts, and this is good enough. From now on conflate f with its coordinate map.

We know Df_p is injective and $n \times k$ so we can look at

$$Df_p = A = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_k \\ \vdots \end{pmatrix} =: \widehat{A}.$$

We know that the rank of A is k, so there exists k linearly independent rows. We can compose with an inverse to these rows to get

$$\begin{pmatrix} \mathrm{Id} & * \\ * & *. \end{pmatrix}$$

Then $\widehat{A} \circ F : \mathbb{R}^k \to \mathbb{R}^n$ extends to $\mathbb{R}^n \to \mathbb{R}^n$.

This gives us for the derivative

$$\begin{pmatrix} \mathrm{Id} & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

We can then get a local diffeomorphism to define coordinates in $\mathbb{R}^n$. By construction in this chart f has the desired form...