Definition .0.1

A manifold M is called closed if it is compact and has no boundary.

For HW11 #4, assume M is oriented and closed. We want to integrate $F_t^*(\mu)$ over $\partial(M \times [0,1])$. But then how do we apply stokes, i.e. how do we differentiate $\mathrm{d}F_t^*(\mu)$. The instinct is to use commutativity to get $F_t^*(\mathrm{d}\mu) = 0$. However d here lives in $M \times [0,1]$, not in M, so this is not quite immediate.

Stokes says that for C a k-chain, α a $(k-1)$-form.,

$$\int_C \mathrm{d}\alpha = \int_{\partial C} \alpha.$$

Corollary is stokes for manifolds themselves (cover with k-chain), regular stokes (curl), divergence (div), and Green's theorem

Proof of Stokes for simplicial k-chains. Good enough to check for singular k-cubes since

$$\int_C \mathrm{d}\alpha = \int_{[0,1]^k} C^*(\mathrm{d}\alpha) = \int_{[0,1]^k} \mathrm{d}(C^*\alpha)$$

$$\int_{\partial C} \alpha = \sum_{i=1}^{k} \sum_{\beta=0,1} (-1)^{\beta+i} \int_{I^{k-1}} C_{1,i}^*(\alpha).$$

Now we see that α is

$$\alpha = \sum_{i=1}^{k} f_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k.$$

Then it is good enough to check on $\alpha = f \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \mathrm{d}x_n$.

Claim

$\int_{I^k} \mathrm{d}\alpha = \int_{\partial I^k} \alpha$

Note that

$$\int_{[0,1]^{k-1}} I_{j,\beta}^*(f \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k) = \begin{cases} 0 & \text{if } j \neq i \\ \int_{[0,1]}^{k-1} f(x_1, \ldots, \beta, x_i, \ldots, x_k) & \text{if } j = i \end{cases}$$

because $\mathrm{d}x_j$ restricted is just 0, and when $j \neq i$ it shows up. The other one is a bit harder but not too bad. and

$$\int_{\partial I^k} \alpha = \sum_{j=1}^{k} \sum_{\beta=0,1} (-1)^{j+\beta} \int_{[0,1]^{k-1}} I_{(j,\beta)}^*(\alpha).$$

Thus the right hand side of the about equation is

$$\sum_{i=1}^{k} (-1)^{i+1} \int_{[0,1]^{k-1}} f(x_1, \ldots, 1, \ldots, x_k) + (-1)^i \int_{[0,1]^{k-1}} f(x_1, \ldots, 0, \ldots, x_k).$$

On the other hand we have that

$$\int_{I^k} \mathrm{d}(f \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k) = \int_{I^k} \mathrm{d}f \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \mathrm{d}x_k.$$

Then $\mathrm{d}f = \sum_{j=1}^{k} \frac{\partial f}{\partial x_j} \,\mathrm{d}x_j$, and so the wedge is nonzero only when $j = i$ so we have, switching things to be in standard order that this is

$$\int_{I^k} (-1)^{i-1} \frac{\partial f}{\partial x_i} \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k.$$

Then we apply Fubini's Theorem to get

$$\begin{aligned} &(-1)^{i-1} \int_0^1 \int_0^1 \cdots \left(\int_0^1 \frac{\partial f}{\partial x_i} \,\mathrm{d}x_i \right) \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k \\ &= \int_0^1 \int_0^1 \cdots (f(x_1, \ldots, 1, \ldots, x_k) - f(x_1, \ldots, 0, \ldots, x_k)) \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_k \end{aligned}$$

via the fundamnetal theorem of calculus.

Via the above reductions we win!