Recall .0.1

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. be a differentiable map. Recall from multivariable calculus that if $p \in U$ then we define $\mathrm{d}f_p$ to be the best linear approximation of f at p, that is we require

$$\lim_{\varepsilon \to 0} \frac{f(p+\varepsilon) - f(p) - \mathrm{d}f_p(\varepsilon)}{\|\varepsilon\|} = 0.$$

We can compute that in the standard coordinates

$$\mathrm{d}f_p = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Lets make the definition of tangent vectors with curves more explicit

Definition .0.1

Call two differentiable curves c, d in M with $c(0), d(0) = p$ equivalent if for one (and hence every chart in the atlast) U_α, φ_α we have

$$(\varphi_\alpha \circ c)'(0) = (\varphi_\alpha \circ d)'(0).$$

Then

$$T_pM = \{[c] \mid c : (-\varepsilon, \varepsilon) \to M \text{ differentiable at } 0, c(0) = p\}.$$

Claim

T_pM has a vector space vector of dimension $\dim M$.

Proof. Use the coordinate chart definition. Take $A, B \in T_pM$. Then in coordinates A, B correspond to $v_\alpha, w_\alpha \in T_{\varphi_\alpha(p)}\mathbb{R}^n$ for some chart $(U_\alpha, \varphi_\alpha)$.

Then $v_\alpha + w_\alpha \in T_{\varphi_\alpha(p)}\mathbb{R}^n$, take $A + B = [v_\alpha + w_\alpha]$. We should check that it doesn't matter where we do the addition. Well let v_β, w_β represent A, B in $T_{\varphi_\beta(p)}\mathbb{R}^n$.

We check that

$$(\mathrm{d}T_{\beta\alpha})_{\varphi_\beta(p)}(v_\beta + w_\beta) = (\mathrm{d}T_{\beta\alpha})_{\varphi_\beta(p)}(v_\beta) + (\mathrm{d}T_{\beta\alpha})_{\varphi_\beta(p)}(w_\beta) = v_\alpha + w_\alpha.$$

Scalar multiplication is quite similar.

Recall .0.2

Recall **??** of the derivative of a map $f : M \to N$ which is differentiable at p.

Note on notation first: We can do all of

$$(f_*)_p = D_pf = Df_p = (\mathrm{d}f_p = \mathrm{d}_pf)$$

But we **really** shouldn't be using $\mathrm{d}f_p = \mathrm{d}_pf$, as later it will confuse us with differential forms.

Now we give the definition in terms of charts. Take a chart $(U_\alpha, \varphi_\alpha)$ about p and take $A \in T_pM$, then A is represented by some $v_\alpha \in T_{\varphi_\alpha(p)}\mathbb{R}^n$.

Take some other chart (V_γ, ψ_γ) about $f(p)$ in N. Then we take $Df_p(A)$ to be represented by

$$D(\psi_\gamma \circ f \circ \varphi_\alpha^{-1})_{\varphi_\alpha(p)}(v_\alpha).$$

Theorem .0.1

If $f : U \to V$ is differentiable at $p \in U$ where $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^\ell$ are open, and Df_p is onto, then $f^{-1}(f(p))$ (aka a level set) is a manifold.

Might be nice on homework...