Stuff:

- Office Hours moved to Monday 5pm and Tuesday 4pm.
- Final: Thursday December 15th from 4pm to 6pm

Last Time: We proved for C a singular $(k-1)$-chain and α a $(k-1)$-form that

$$\int_C \mathrm{d}\alpha = \int_{\partial C} \alpha.$$

Corollary .0.1 (Manifold Stokes)

$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha$ for α an $(n-1)$-form, where $n = \dim M$.

But how is $\int_M \omega$ defined for an n-form ω? We want M to be oriented, compact, and we'll let $n \coloneqq \dim M$.

Well take an open cover U_i such that each $U_i \subseteq \operatorname{Image}(C_i)$ for C_i an orientation-preserving singular n-cube. We may also take a partition of unity subordinate to U_i. Write $\omega = \sum f_i \omega$ where $f_i\omega$ is supported on U_i. Then we define

$$\int_M \omega \coloneqq \sum_i \int_{C_i} f_i \omega.$$

Lemma .0.2

This is in fact well-defined, i.e. does not depend on U_i, C_i, f_i.

Proof. Essentially the change of variables formula.

Proof Idea of Manifold Stokes. In the definition of $\int_M \omega$ use C_i such that only one face of C_i lies in ∂M. Write $\alpha = \sum_i f_i \alpha$. Then

$$\mathrm{d}\alpha = \sum_i (\mathrm{d}f_i) \wedge \alpha + \sum_i f_i \,\mathrm{d}\alpha.$$

Then we see by how we wrote the

$$\begin{aligned} \int_{\partial M} \alpha &= \sum_i \int_{\partial C_i} f_i \alpha \\ &= \sum_i \int_{C_i} \end{aligned}$$

Degree and $H^n_{\mathrm{dR}}(M)$.

Theorem .0.3

If M is a compact, oriented, manifold then $H^n_{\mathrm{dR}}(M) = \mathbb{R}$.

Idea of Proof. We see that $\mathbb{R} \subseteq H^n_{\mathrm{dR}}(M)$ by orientation.

We then have a map precisely $H^n_{\mathrm{dR}}(M) \to \mathbb{R}$ given by $\omega \mapsto \int_M \omega$. Since M is oriented, if ν is a volume form then $\int_M \nu > 0$. Thus the map is onto.

Now suppose $\int_M \omega = 0$. The claim is that $\omega = \mathrm{d}\beta$. To prove this claim, you cover M by open sets U_i contained in some singular n-cubes C_i. We do this in such a way that

$$(U_1 \cup \cdots \cup U_k) \cap U_{k+1} \neq \emptyset.$$

Call $M_k = U_1 \cup \cdots \cup U_k$. We know $M = M_k$ for some k since M is compact.

It suffices to prove that if ω on M_k has zero integral then $\omega = \mathrm{d}\eta$ for some η defined on M_k. We prove this by induction on k.

For $k = 1$, we're on a chart so this is just the Poincare Lemma. Suppose the result holds for k. We see that

$$\int_{M_{k+1}} \omega = 0,$$

Let θ be a form supported in $M_k \cap U_{k+1}$ such that $\int_{M_{k+1}} \theta = 1$. Let $\{\varphi, \psi\}$ be a partition of unity subordinate to $\{M_k, U_{k+1}\}$ and let $c \coloneqq \int_{M_{k+1}} \varphi\omega$.

We see that $\varphi\omega - c\theta$ is zero on M_k, thus $\mathrm{d}\alpha = \varphi\omega - c\theta$ for some α. Likewise $\psi\omega + c\theta$ has integral zero on U_{k+1} so is $\mathrm{d}\beta = \psi\omega + c\theta$ (it must have this integral since ω has integral zero on M_{k+1}).

Then we see that $\mathrm{d}(\alpha + \beta) = \omega$!

Suppose M, N are dimension n, oriented and compact. Let ν be a volume form on N.

Definition .0.1

The degree of a map $f : M \to N$ is some map then the degree of f is defined by

$$\int_M f^*\nu = (\deg f) \int_N \nu,$$

since this number is uniquely defined

Example .0.1

The map $z \mapsto z^k$ on $S^1 \to S^1$ has degree k. Furthermore, if $f : S^1 \to S^1$ is an orientation preserving diffeomorphism then it has degree 1.

Proposition .0.4

$\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Theorem .0.5

Brouwer's Fixed Point theorem. Let D^n be a closed ball in $\mathbb{R}^n$. Then if $f : D^n \to D^n$ is continuous, then f has a fixed point.

Proof. Only prove for f smooth.

Claim

It is enough to prove for f smooth.

Approximate f continuous by smooth maps homotopic to it. Then see Lee.

A homotopy between f, g is a map $F : X \times [0,1] \to X$ so that $F(x,0) = f(x), F(x,1) = g(x)$. It turns out the action on de Rham cohomology for a continuous map can be defined by approximating with a smooth map this way, and is independent of the approximation chosen.

Suppose $f : D \to D$ is smooth and has no fixed points. Then define

$$G(x) = \frac{x - f(x)}{\|x - f(x)\|}.$$

This map is then well-defined and continuous. G is a map from $D \to \partial D = S^{n-1}$. We can also let $H(t,x)$ on S^{n-1} be given by

$$H_t(x) = \frac{x - tf(x)}{\|x - tf(x)\|}.$$

We see that if $0 \le t < 1$ then $\|x\| = 1$, $\|tf(x)\| \le t < 1$, so this is well-defined. Likewise we know $x - f(x) \neq 0$ so it's well-defined for $t = 1$ as well.

Then H_0 is the identity map, so $\deg H_0 = 1$. On the other hand, if ν is the volume form on S^{n-1}, then

$$\int_{S^{n-1}} H_0^* \nu = \int_{S^{n-1}} H_1^* \nu$$

But then since $H_1 = g$, $H_1^* \nu$ can be defined over D as $G^* \nu$. But wait! Applying Stokes yields

$$\begin{aligned} \int_{S^{n-1}} g^* \nu = \int_{S^{n-1}} G^* \nu &= \int_D \mathrm{d} G^* \nu \\ &= \int_D G^*(\mathrm{d}\nu) = \int_D 0 = 0. \end{aligned}$$

Thus the degree is zero! Contradiction ☺.