I. Introduction/Administration

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- Office Hours:
  - Monday 11-12
  - Tuesday 5-6, EH 4088
  - Friday 11-12
  - By appointment.
- HW: Due Wednesdays

II. Definitions and Building Blocks

Definition II.0.1

M is called locally euclidean provided that for all $p \in M$ there exists a neighborhood U of p and a homeomorphism $U \to \mathbb{R}^n$ for some n.

Lemma II.0.1

It is good enough that for all $p \in M$ there exists a neighborhood V of p such that V is homeomorphic to an open subset in $\mathbb{R}^n$.

Proof. Take $V \xrightarrow{\phi} V^* \subseteq \mathbb{R}^n$ with V^* open. Then there is an open ball $U^* \subseteq V^*$ containing p, and so we can take $U = \phi^{-1}(U^*) \cong U^*$. It is clear from real analysis that $U^* \cong \mathbb{R}^n$.

II.1. Paracompactness

Definition II.1.1

Consider a collection of subsets χ of M. χ is called locally finite provided that each point $p \in M$ has a neighborhood U intersecting only finitely many $C \in \chi$.

Definition II.1.2

A topological space M is called paracompact if every open cover χ of M admits a locally finite open subcover.

Recall II.1.1

A cover of M is a collection of χ such that $\bigcup\limits_{C \in \chi} C = M$. A subcover of a cover χ of M is a collection $\chi^* \subseteq \chi$ such that every $C^* \in \chi^*$ is contained in some $C \in \chi$.

χ^* is also called a refinement. A cover χ is open if every element of χ is an open set.

Definition II.1.3

M is called locally compact if every point $p \in M$ and neighborhood U of p there exists a neighborhood $V \subseteq U$ such that $\overline{V} \subseteq U$ (the closure) is compact.

Lemma II.1.1

Topological manifolds are locally compact.

Proof. They are locally euclidean and $\mathbb{R}^n$ is locally compact.

Theorem II.1.2

Topological manifolds are paracompact.

Proposition II.1.3

A 2nd countable locally compact Hausdorff space admits an exhaustion by compact sets.

Definition II.1.4

An exhaustion is a sequence of sets $K_n \subseteq K_{n+1}$ with $\bigcup K_n = M$.

Proof of Proposition II.1.3. In the appendix of Lee [Lee10]. We repeat it here. There is a basis of precompact open sets since M is locally compact. We should extract countably many precompact open sets $\{U_i\}_{i\in\mathbb{N}}$ such that $\bigcup U_i = M$.

By second countability, let $\{W_j\}_{j\in\mathbb{N}}$. Then taking $p \in M$, we know $p \in W_j$. There then exists a precompact neighborhood $\overline{V} \subseteq W_j$ which is compact. Take sets $\{W_{V_i}\}$ whose union contains $\overline{V}$. Then take finitely many such precompact open sets $W_{V_1}, \ldots, W_{V_N}$. It is possible to make the previous argument happen in some neighborhood $\mathcal{O}$ which is precompact. Thus we have $W_{V_i} \subseteq \mathcal{O}$, so the $\overline{W_{V_i}} \subseteq \overline{\mathcal{O}}$ are compact (by Hausdorffness).

Let's define the exhaustion by compact sets. First let $\{U_i\}_{i\in\mathbb{N}}$ be precompact open sets covering M. Let $K_m = \bigcup_{i=1}^{m} \overline{U_i}$.

Exercise II.1.2

Think of how to define a differentiable manifold.

The idea is as follows.

Definition II.1.5

If $p \in M$ and $p \in U \xrightarrow{\varphi} U^* \subseteq \mathbb{R}^n$ where U^* is open is a homeomorphism, then (U, φ) is called a coordinate chart at U.

Definition II.1.6

Call charts (U, φ) and (V, ψ) compatible if $\varphi \circ \psi^{-1}\big|_{U^* \cap V^*}$ is differentiable