Recall the claim from last time: TM is a vector bundle over M. To do this we need to show that TM carries a manifold structure. It turns out we lose regularity (summed up below)

Remark .0.1

If M has a C^r-manifold structure, then TM has a C^{r-1}-manifold structure. Thus M must have at least C^2-structure to get TM with C^1-structure (that is a manifold).

Note: In the end this is not a problem, as we will later show that every C^1 manifold has a C^∞ structure

Proposition .0.1

If M is a C^r-manifold then $TM := \coprod_p T_pM$ is a C^{r-1}-manifold which is a vector bundle over M.

Proof the Tangent Bundle is a Vector Bundle. Take M to be an abstract differentiable manifold, and $\pi : TM \to M$ the obvious map. Call $m := \dim M$, then we'll take $F := \mathbb{R}^m$. We must do two things

(a) Give TM a manifold structure.

(b) Show it can be endowed with a vector bundle structure.

We take a covering of M by charts $(U_\alpha, \varphi_\alpha)$, $\varphi_\alpha : U_\alpha \to W_\alpha \subseteq \mathbb{R}^m$. Note φ_α is in fact a diffeomorphism, by how we've set up the definition of differentiability on manifolds.

Then we'll do each of the above

(a) We know that $TW_\alpha \subseteq \mathbb{R}^{2m}$ is an open subset, as $TW_\alpha = W_\alpha \times \mathbb{R}^m$.

Then

$$TU_\alpha = \coprod_{p \in U_\alpha} T_pU_\alpha \xrightarrow{D\varphi_\alpha} \coprod_{p \in U_\alpha} T_{\varphi_\alpha(p)}W_\alpha = TW_\alpha.$$

We take this as a coordinate chart on TM. Namely, take TM to have a topological strcture with basis the open sets $\{TU_\alpha\}$.

Then we claim $(TU_\alpha, D\varphi_\alpha)$ is an atlas. We must look at the transition map, that is

$$\begin{tikzcd} & TU_\alpha \cap TU_\beta \arrow[dl, "D\varphi_\alpha"'] \arrow[dr, "D\varphi_\beta"] & \\ D\varphi_\alpha(TU_\alpha \cap TU_\beta) \arrow[rr, "D\varphi_\beta \circ D\varphi_\alpha^{-1}"'] & & D\varphi_\beta(TU_\alpha \cap TU_\beta) \end{tikzcd}$$

By the chain rule this is $D(\varphi_\beta \circ \varphi_\alpha^{-1})$, which is C^{r-1} by assumption. Thus this is a C^{r-1}-atlas.

(b) We now will show this is a vector bundle. Note that

$$\pi^{-1}(U_\alpha) = \coprod_{p \in U_\alpha} T_pM.$$

We then have $U_\alpha \xrightarrow{\varphi_\alpha} W_\alpha \subseteq \mathbb{R}^m$. Note then

$$\pi^{-1}(U_\alpha) \xrightarrow{D\varphi_\alpha} TW_\alpha = W_\alpha \times \mathbb{R}^m.$$

This is nearly our trivialization. Follow up with $(\varphi_\alpha^{-1}, \mathrm{Id})$ to get

$$\pi^{-1}(U_\alpha) \xrightarrow{(\varphi_\alpha^{-1}, \mathrm{Id}) \circ D\varphi_\alpha} U_\alpha \times \mathbb{R}^m.$$

Call this ψ_α. Clearly $\mathrm{proj} \circ \psi_\alpha = \pi$, so this is a fiber bundle.

Last thing to check is that ψ_α is linear on fibers. This comes from the fact that T_pU_α inherits its linear structure from $T_{\varphi_\alpha(p)}W_\alpha$.

That is

$$\psi_\alpha : T_pU_\alpha = T_pM \xrightarrow{(\varphi_\alpha^{-1},\mathrm{Id})\circ D_p\varphi_\alpha} \{p\} \times \mathbb{R}^m$$

is linear because $D_p\varphi_\alpha$ is linear by construction of the linear structure on T_pM.

We'll now do constructions with vector bundles! Take $\mathcal{V}, \mathcal{W}$ to be vector bundles over M with maps π_1, π_2 to M.

- HW5: Define $\mathcal{V} \oplus \mathcal{W} \to M$, and if $V_p = \pi_1^{-1}(p)$ and W_p are similar then the fiber over p should be $V_p \oplus W_p$.
- We can take $\mathcal{V}^*$ with fibers V_p^*, where $*$ denotes the dual space. For this one looking fiber by fiber

$$\phi_\alpha : \pi^{-1}(p) \to \{p\} \times V$$

is linear, and we have

$$\phi_\alpha^* : \{p\} \times V^* \to (\pi^{-1}(p))^*.$$

This goes in the opposite direction as desired, but ϕ_α is invertible! Thus we can look at

$$(\phi_\alpha^*)^{-1} : (\pi^{-1}(p))^* \to \{p\} \times V^*.$$

- If $\mathcal{V}, \mathcal{W}$ are two vector bundles you can look at $\mathcal{V} \otimes \mathcal{W}$.
- Important one down the road: Given one specific vector space V. We cal look at the k-fold tensor product $\underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}}$. Hiding inside of this is something important, the k-alternating linear forms $\bigwedge^k V$.

  We can of course do this with $\mathcal{V}$ as a vector bundle $\bigwedge^k \mathcal{V}$. Later then $\bigwedge^k TM$ will be differential k-forms, which will lead to de-Rham cohomology at the end.

Recall .0.1

A multilinear form on $V_1, \ldots, V_k$ is a map

$$V_1 \oplus V_2 \oplus \cdots \oplus V_k \to \mathbb{R}$$

if $\lambda(v_1, \ldots, v_k)$ is linear in each coordinate. If $V = V_1 = \cdots = V_k$ then λ is called alternating when

$$\lambda(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\lambda(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Prime example: Determinant, $\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$.