Midterm in class on Wednesday October. Things you should know:

- Charts
- Tangent Stuff: T_pM, TM, T^*M.
- Vector bundles and sections
- Basic Examples/Counterexamples
- Constructions
  - Products
  - Group Actions
  - Level sets via Regular Value Theorem (Remember: Sard's Theorem, easy proof that SL_n is a manifold)

Midterm: 50 minutes, ≈4-5 questions, should be able to answer questions in ≈10-15 minutes.

CONTENT FOR MIDTERM I STOPS HERE
(does not include flows)

Recall that uniqueness of ODEs tells us that if $q = \varphi_{t_0}(p)$ then

$$\varphi_s(q) = \varphi_{t_0+s}(p)$$

by uniqueness of ODEs. Therefore

$$\varphi_{s+t_0}(p) = \varphi_s(\varphi_{t_0}(p))$$
$$\varphi_{s+t_0} = \varphi_s \circ \varphi_{t_0},$$

where φ_{s+t_0} is defined. If X is complete, we get an $\mathbb{R}$-action on X, commonly called a flow on X.

Definition .0.1

Now suppose X is a vector field and $F : M \to N$ is a diffeomorphism. Then we can define the <u>pushforward of X by F</u>.

$$(F_*(X))(p) \coloneqq D_qF(X(q)).$$

where $p = F(q)$. Then $F_*(X)$ is a vector field on M.

If $Y = F_*(X)$ we say that X, Y are F-related (still makes sense for local diffeomorphisms). We also say X and Y are $C^?$-conjugate if F is $C^?$.

Let φ_t, ψ_t be flows for X, Y (vector fields on M, N). If $F : M \to N$ is a diffeomorphism and $Y = F_*(X)$, what can we say about the flows?

Fix $q \in M$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(F(\varphi_t(q)))\Big|_{t=t_0} &= DF_{\varphi_{t_0}(q)} \cdot X(\varphi_{t_0}(q)).\\ &= Y(F(\varphi_{t_0}(q)))\end{aligned}$$

From this and the uniqueness of ODEs we can see that

$$\psi_t = F \circ \varphi_t \circ F^{-1}.$$

Namely, by the chain rule again we have for $p \in N$ that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\varphi_t(F^{-1}(p)))\Big|_{t=t_0} = Y(F(\varphi_{t_0}(F^{-1}(p)))).$$

More generally: one might have a map $\pi : M \to N$ and flows φ_t, ψ_t on M, N where $\psi_t \circ \pi = \pi \circ \varphi_t$. Something like this would be called a "quotient of φ_t or a factor of φ_t."

Suppose X, Y are vector fields on M. Recall that a vector field X can be thought of as a derivation $X : C^\infty(M) \to C^\infty(M)$. We had an unproved lemma from last time we discussed commutators

Lemma .0.1

$[X, Y] := Y \circ X - X \circ Y$ is a vector field, that is the Lie bracket of two vector fields is a vector field.

Proof. We must verify the product rule, since linearity of $[X, Y]$ is clearly. Thus we compute

$$\begin{aligned}
(X \circ Y)(fg) &= X(Y(f)g + fY(g)) \\
&= (X \circ Y)(f)g + Y(f)X(g) + X(f)Y(g) + f(X \circ Y)(g) \\
(Y \circ X)(fg) &= (Y \circ X)(f)g + X(f)Y(g) + Y(f)X(g) + f(Y \circ X)(g) \\
[X, Y](fg) &= [X, Y](f) \cdot g + f \cdot [X, Y](g).
\end{aligned}$$

Perfect!

Message: $[X, Y]$ measures how much X, Y do not commute. What does it mean in terms of vector fields?

Suppose φ_t, ψ_s are local flows of X, Y respectively. Consider the following sequence of moves starting at $p \in M$

$$\begin{aligned}
&\varphi_t(p) \\
&\psi_t(\varphi_t(p)) \\
&\varphi_{-t}(\psi_t(\varphi_t(p))) \\
&\psi_{-t}(\varphi_{-t}(\psi_t(\varphi_t(p)))).
\end{aligned}$$

As $t \to 0$ this goes to p by a continuity argument. But what about the *derivative* at $t = 0$.

Example .0.1

For $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$ this commutator is zero.