Definition .0.1

We call $S \subseteq M$ an immersed submanifold if S is a manifold, and $\iota : S \hookrightarrow M$ is an immersion.

Definition .0.2

Call a map $f : M \to N$ which is C^1 a submersion provided that every value $y \in N$ is a regular value. That is, for every $x \in M$ such that $f(x) = y$ we have $D_x f : T_x M \to T_y N$ is surjective.

Convention: If $f : M \to N$ and $y \notin f(M)$ then y is a regular value of f.

Note: If $M \to N$ is a submersion, then $\dim M = \dim T_x M \geq \dim T_y N = \dim N$.

Question: Why is $\dim M = \dim T_x M$ for all $x \in M$? Well it is clear that if φ is a chart we have

$$\dim T_x M = \dim T_{\varphi(x)} \mathbb{R}^{\dim M} = \dim M.$$

Note that once we pick a coordinate chart φ, it induces an isomorphism of vector spaces

$$T_x M \cong T_{\varphi(x)} \mathbb{R}^{\dim M} \cong \mathbb{R}^{\dim M}$$

This isomorphism is given by taking representatives of the equivalence classes by which $T_x M$ is defined. The isomorphism is intimately related to φ.

Theorem .0.1

Suppose $f : M \to N$ and $q = f(p)$ is a regular value, then $f^{-1}(q)$ is an embedded submanifold of M. In fact $f^{-1}(q)$ has dimension $\dim M - \dim N$.

Proof Idea. Really, work with a coordinate chart for M at $p \in M$. Select a chart (W_β, ψ_β) about $q = f(p)$ to $\mathbb{R}^n$ with $\dim N = n$. Now take a chart $(U_\alpha, \varphi_\alpha)$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ with $\dim M = m$ and $U_\alpha \subseteq f^{-1}(W_\beta)$. For convenience let $\boldsymbol{p} = \varphi_\alpha(p)$ and $\boldsymbol{q} = \psi_\beta(q)$.

Now consider the map $F_{\alpha\beta} = \psi_\beta \circ f \circ \varphi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$. It now suffices to check the claim for the coordinate map $F_{\alpha\beta}$. We want $F^{-1}(\boldsymbol{q})$ to be a submanifold. Well, we know that $D_{\boldsymbol{p}} F : T_{\boldsymbol{p}} \mathbb{R}^m \to T_{\boldsymbol{q}} \mathbb{R}^n$ is surjective.

This means $m \geq n$ and $\ker D_{\boldsymbol{p}} F \subseteq \mathbb{R}^m$ has dimension $m - n$. Put this kernel into $\mathbb{R}^m$ as the last $m - n$ coordinates, to do this use an invertible linear map B with $B^{-1}(\ker D_{\boldsymbol{p}} F) = \mathbb{R}^{m-n}$.

We may then precompose to get $\overline{F} = F \circ B$.We know $D_{B^{-1}(\boldsymbol{p})} \overline{F} = D_{\boldsymbol{p}} F \circ B$, and so this is surjective with kernel $\mathbb{R}^{m-n}$. We define an extended map

$$G : R^m \to R^n \times R^{m-n}$$

$$G(x_1, \ldots, x_m) = (\overline{F}(x_1, \ldots, x_m), x + m - n + 1, \ldots, x_m),$$

then $D_{B^{-1}(\boldsymbol{p})}(G)$ is an isomorphism. Why? Well it has the form $(D_{B^{-1}(\boldsymbol{p})} \overline{F}, \mathrm{Id}_{\mathbb{R}^{m-n}})$. This is clearly surjective with zero kernel.

Now use inverse function theorem on G. G is a local diffeomorphism, so $G^{-1}(q) \to \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$. We then use G to get an adapted coordinate chart!

Example .0.1

$\mathrm{SL}(n, \mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$ is an embedded submanifold (seen on HW). To show this, we proved $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ has $D_p \det$ is surjective for any $p \in \mathrm{SL}(n, \mathbb{R})$, and so $\mathrm{SL}(n, \mathbb{R}) = \det^{-1}(1)$ is a submanifold.