Final Stuff:

- Extra Office Hours: Next Monday 2pm, Next Tuesday 4pm, plus by appointment/drop by.
- Material will concentrate on things since the midterms, but of course mathematics is largely cumulative
- Old QR problems are good practice. As are old HW problems.
- Each Question will be worth 20pts.

Last time we began discussing singular cubes, i.e. smooth maps $I^k \coloneqq I^k \to M$.

Definition .0.1

A smooth map $C : I^k \to M$ is called a singular k-cube in M.

New idea, do something crazy: look at formal sums of singular k-cubes. Say

$$\mathcal{C}_k \coloneqq \left\{ \sum_{i=1}^{m} a_i c_i \mid a_i \in \mathbb{R}, m \in \mathbb{N}, c_i k\text{-cubes} \right\}.$$

This is then the free $\mathbb{R}$-module (i.e. vector space) with basis $\{c \mid ck\text{-cube}\}$. This space is infinite-dimensional.

Definition .0.2

An form sum $\sum_{i=1}^m a_i c_i$ as above is called a singular k-chain in M.

Goal: manifolds to algebra. We need some sort of map between these things.

Note: $\mathcal{C}_0$ is formal sums of points in M, as 0-cubes are points in M. Now, we have a map $\partial : \mathcal{C}_k \to \mathcal{C}_{k-1}$. What is it? It's the "boundary" operation (with signs)!

Want: $\partial^2 = 0$ (this whole thing is "dual" to what we do with forms).

We'll think of I^n as both $[0,1]^n$ and $I^n = \mathrm{Id} : [0,1]^n \to [0,1]^n$. We now take

$$I^n_{i,0}(x) \coloneqq I(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$
$$I^n_{i,1}(x) \coloneqq I(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}).$$

If $i = 0$ we kick the 0 to the last coordinate (in some sense this is all modulo n). where $x = (x_1, \ldots, x_{n-1})$. Well

Example .0.1

$I^1_{(1,0)}() = (0)$. Then $I^2_{(1,0)}(x_1) = (0, x_1)$ and $I^2_{(1,1)} = (1, x_1)$. Likewise

$$I^2_{(0,0)}(x_1) = (x_1, 0)$$
$$I^2_{(0,1)}(x_1) = (x_1, 1).$$

Crucial Fact: Each $(n-2)$-dimensional face of I^n is the $(n-2)$-face of two $(n-1)$-faces of I^n. Must figure out a combinatorial way of assigning opposite signs to get $\partial^2 = 0$.

Definition .0.3 (Formal Definition of ∂)

We define

$$\partial I^n \coloneqq \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}.$$

This works generally. For $C : I^n \to M$ define

$$\partial C \coloneqq C \circ \partial I^n \coloneqq \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha}(C \circ I^n_{(i,\alpha)}),$$

(this is essentially defined by pushing forward ∂I^n from I^n to M along C).

Proposition .0.1

$\partial^2 I^n = 0$.

Proof. More formally, suppose $i \leq j$. Let $x = (x_1, \dots, x_{n-2})$. We compute

$$\begin{aligned}(I^n_{i,\alpha})_{j,\beta}(x) &= I^n_{(i,\alpha)}(x_1, \dots, x_{j-1}, \beta, x_j, \dots, x_{n-2}) \\ &= (x_1, \dots, x_{i-1}, \alpha, x_i, \dots, x_{j-1}, \beta, x_j, \dots, x_{n-2}).\end{aligned}$$

We likewise compute that

$$\begin{aligned}(I^n_{j+1,\beta})_{i,\alpha}(x) &= I^n_{j+1,\beta}(x_1, \dots, x_{i-1}, \alpha, x_i, \dots, x_{n-2}) \\ &= (x_1, \dots, x_{i-1}, \alpha, x_i, \dots, x_{j-1}, \beta, x_j, \dots, x_{n-2}).\end{aligned}$$

Thus these maps are equal! But the signs associated to them, i.e. $(-1)^{i+\alpha}(-1)^{j+\beta}$ and $(-1)^{j+\beta+1}(-1)^{i+\alpha}$ are opposite! Thus these will cancel in $\partial^2 I^n$.

Then extend ∂ linearly to $\mathcal{C}_k$ to $\mathcal{C}_{k-1}$.

Lemma .0.2

$\partial^2 = 0$.

Proof. $\partial^2(I^k) = 0$. Then $\partial^2(C) = C \circ \partial^2(I^k) = C \circ 0 = 0$. Then including sums gives zero.

We now want to integrate over singular k-chains. The setup: if $C : I^k \to M$ is a k-cube, $\omega \in \Omega^k M$ a k-form, then

$$\int_C \omega \coloneqq \int_{I^k} C^*(\omega).$$

We know $C^*(\omega) = f \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. We can then just take

$$\int_{I^k} C^*(\omega) = \int_{I^k} f \,\mathrm{d}x_1 \,\mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

(In fact: you can integrate on the interior of I^k, since this is a Lebesgue integral and the boundary has measure zero). For chains $\sum_{i=1}^m \alpha_i c_i$ we take

$$\int_{\sum_{i=1}^m \alpha_i c_i} \omega = \sum_{i=1}^{m} \alpha_i \int_{c_i} \omega$$

Theorem .0.3 (Basic Stokes)

Suppose $\omega \in \Omega^{k-1}(A)$ where $A \subseteq \mathbb{R}^n$ is open, then if C is a k-chain then

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega.$$