Convention: All manifolds are assumed to be differentiable (in fact C^1). Later we will prove that

Theorem .0.1

If M has a C^1 structure then it has a compatible C^∞ structure.

Proposition .0.2 (Chain Rule)

Let $f : M \to N, g : N \to O$ be differentiable maps, then $g \circ f$ is differentiable and

$$D_p(g \circ f) = D_{f(p)} g \circ D_p f.$$

Writing this diagramatically

$$\begin{tikzcd}
T_pM \arrow[rr, "D_p(g\circ f)"] \arrow[dr, "D_pf"'] & & T_{(g\circ f)(p)}O \\
& T_{f(p)}N \arrow[ur, "D_{f(p)}g"'] &
\end{tikzcd}$$

Proof. Use curves (aka hide the coordinate charts in the equivalence of curves with charts)! To do this, let c be a curve then

$$D_{f(p)} g \circ D_p f([c]) = D_{f(p)} g([f \circ c]) = [g \circ f \circ c] = [(g \circ f) \circ c] = D_p(g \circ f).$$

What is a diffeomorphism?

Definition .0.1

A differentiable map $f : M \to N$ is called a diffeomorphism provided that it is bijective and its inverse $g : N \to M$ is differentiable.

A map $f : M \to N$ is a local diffeomorphism at $p \in M$ if there exists open neighborhoods U of p and V of $f(p)$ usch that $f : U \to V$ is a diffeomorphism.

Remark .0.1

In this case we have

$$\mathrm{Id} = D_p(\mathrm{Id}) = D_p(g \circ f) = D_{f(p)} g \circ D_p f.$$

That is $D_p f$ has an inverse map.

Furthermore, coordinate charts $\varphi : U \to \widetilde{U} \subseteq \mathbb{R}^n$ are always invertible.

Corollary .0.3

If f is a local diffeomorphism at p, then $D_p f : T_pM \to T_{f(p)}N$ has an inverse. This implies that $\dim M = \dim N$ about p.

Theorem .0.4 (Inverse Function Theorem from Real Analysis)

If $f : U \to V$ with $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ has an invertible derivative $D_p f$ at p, then there is a neighborhood $U' \subseteq U$ such that $f : U' \to f(U')$ is a diffeomorphism.

Theorem .0.5 (Inverse Function Theorem for Manifolds)

Suppose $f : M \to N$ is C^1 and suppose $Df_p : T_pM \to T_{f(p)}N$ is invertible (as a linear map), then f is a local diffeomorphism at p.

Proof. Fix charts $(U, \varphi), (V, \psi)$ about $p, f(p)$, with $U \subseteq \mathbb{R}^n$, so that $f(U) \subseteq V \subseteq \mathbb{R}^m$ (this requires minor yoga). Call $T = \psi \circ f \circ \varphi^{-1}$.

By the chain rule, $D_{\varphi(p)}T$ is invertible, and so $n = m$. By the inverse funct

Definition .0.2

Suppose M is a C^1 manifold, we say that $S \subseteq M$ is called a embedded submanifold of M provided that for all $p \in S$, there exists a coordinate chart (U, φ) about M such that $\varphi\big|_S : S \cap U \to \mathbb{R}^k \subseteq \mathbb{R}^n$. That is

$$S = \{q \in U \mid \varphi(q) = (*, \dots, *, 0, 0, \dots, 0).$$

We call such a thing an adapted chart (adapted to S).

Note: S is a C^1-manifold in its own right.

Example .0.1

$\emptyset, M \subseteq M$, $\mathbb{R}^\ell \subseteq \mathbb{R}^n$, $S^\ell \subseteq S^n \subseteq \mathbb{R}^{n+1}$, $\mathbb{R}\mathrm{P}^\ell \subseteq \mathbb{R}\mathrm{P}^n$.

Example .0.2

Consider $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \subseteq S^1$, and let $p : \mathbb{R}^2 \to T^2$. Let $\ell \subseteq \mathbb{R}^2$ be a line forming an angle of α to the origin. If $\alpha = 0$ this is $S^1 \times \{1\}$. If $\alpha \notin \pi\mathbb{Q}$ then $p(\ell)$ is dense in T^2.

This means that $p(\ell)$ will not be a submanifold when $\alpha \notin \pi\mathbb{Q}$. This doesn't cross over itself, but the density prevents you from taking a small open chart making the rest a line.

Definition .0.3

Suppose $f : M \to N$ is a C^1 map. Then f is called an immersion provided that for all $p \in M$ we have that D_pf is injective.

This will imply that $f(M)$ is "locally" a submanifold.