Stuff:

- HW due Thursday 11pm, November 17th
- The book uses the notation $\Lambda^k V^*$ to refer to the alternating k-multilinear maps on V. We've been using $\Lambda^k V$ to refer to the same thing. Ditto for $\Lambda^k M$ (our notation) versus $\Lambda^n T^* M$ (the book's notation. Make sure to keep this in mind.

  We will try to use the book's notation from now on, but remember that we will always be talking about alternating k-multilinear maps (k-forms).

Recall .0.1

A k-form α is smooth if either

- $X_1, \ldots, X_k$ are smooth vector fields, and then the function
$$p \mapsto \alpha_p(X_1(p), \ldots, X_k(p))$$
is smooth.
- if locally we can write in terms of smooth coordinates $\alpha = \sum \alpha_i (\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k})$ with $\alpha_i : M \to \mathbb{R}$

Example .0.2

We'll look at

$$\begin{aligned}
&(\mathrm{d}x_1 \wedge \mathrm{d}x_2)(\partial x_i, \partial x_j) \\
&= \mathrm{d}x_1(\partial x_i) \cdot \mathrm{d}x_2(\partial x_i) - \mathrm{d}x_1(\partial x_j) \cdot \mathrm{d}x_2(\partial x_i) \\
&= \begin{cases} 0 & \text{if } \{i, j\} \neq \{1, 2\} \\ 1 & \text{if } i = 1, j = 2 \\ -1 & \text{if } i = 2, j = 1 \end{cases} .
\end{aligned}$$

More generally, we can look at

$$\begin{aligned}
&(\mathrm{d}x_1 \wedge \mathrm{d}x_2) \left(\sum_i a_i \partial x_i, \sum_j b_j \partial x_j \right) \\
&= (\mathrm{d}x_1 \wedge \mathrm{d}x_2)\left(a_1 \partial x_1, b_2 \partial x_2\right) + (\mathrm{d}x_1 \wedge \mathrm{d}x_2)\left(a_2 \partial x_2, b_1 \partial x_1\right. \\
&= a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.
\end{aligned}$$

Recall .0.3

For $\alpha \in \Lambda^k V^*, \beta \in \Lambda^\ell V^*$ we defined

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) = \frac{1}{k!\ell!} \cdot \sum_{\sigma \in S(k+\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}).$$

where $\sigma \in S(k+\ell)$ provided that σ is a permutation of $k+\ell$ things so that $\sigma(i) < \sigma(j)$ for $1 \leq i < j \leq k$ and for $k+1 \leq i < j \leq k+\ell$.

Example .0.4

Let $k = \ell = 1$. Then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

Proposition .0.1

$\alpha \in \Lambda^k V^*, \beta \in \Lambda^\ell V^*$ implies $\alpha \wedge \beta \in \Lambda^{k+\ell} V^*$.

Proof. Check from the book, idea: if you transpose two things in $1, \ldots, k$ or in $k+1, \ldots, k+\ell$ it's just from α, β. If you transpose a thing between the two, then things are more complex.

This wedge product is super important. Why? Future: wedge product of forms, leading to Poincaré duality.

Important: $\wedge$ is a multiplicative operation.

Question: Let $\dim V = n, k + \ell = n, \alpha \in \Lambda^k V^*, \beta \in \Lambda^\ell V^*$. When multiplied we have

$$\alpha \wedge \beta \in \Lambda^{k+\ell} V^* = \Lambda^n V^* \cong \mathbb{R}.$$

The idea of Poincaré duality will be to associate to α a β so that $\alpha \wedge \beta$ is the determinant (a distinguished n-form, aka an orientation) on the nose.

If M is an n-dimensional manifold, we say it is oriented with orientation σ if there is an n-form, aka a section $\sigma : M \to \Lambda^n M$ so that σ never vanishes.

Definition .0.1 (Also Notation)

Call a σ like this a "volume form."

Lemma .0.2

If M is oriented then $\Lambda^n M = \{f : M \to \mathbb{R} \mid f \text{ smooth}\}$.

One can think of $\Lambda^0 M := \{f : M \to \mathbb{R} \mid f \text{smooth}\} = C^\infty(M)$. Thus this Lemma is Poincaré duality for n-forms and 0-forms.

Proof. If $\tau \in \Lambda^n M$ then $\tau(p) = f(p) \cdot \sigma(p)$ where $\sigma(p)$ is the volume form. Thus $\Lambda^n M \to \{f : M \to \mathbb{R} \mid f \text{ smooth}\}$. And the reverse also occurs.

More properties of wedge product:

(I) $\alpha \wedge \beta = (-1)^{k \cdot \ell} \beta \wedge \alpha$. Look at the formula and think about which things you have to switch.
Thus if k is odd, $\alpha \in \Lambda^k V^*$, then $\alpha \wedge \alpha = 0$.

(II) $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$, associative.

(III) Bilinearity.

An aside about orientability: Let M have coordinate charts $(U_\alpha, \varphi_\alpha)$.

Consider two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β). Then we get a transition map

$$T_{\alpha,\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).$$

Note $\mathbb{R}^n$ is orientable since

$$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \neq 0.$$

Thus on U_α we can take $\gamma^\alpha_{\boldsymbol{x}} := \mathrm{d}x^\alpha_1 \wedge \cdots \wedge \mathrm{d}x^\alpha_n$. We have a map down to $\gamma^\beta_{\boldsymbol{y}} := \mathrm{d}y^\beta_1 \wedge \cdots \mathrm{d}y^\beta_n$. What happens under the transition map $T_{\alpha\beta}$?

Recall .0.5

Smooth functions pullback k-forms. Given smooth $F : M \to N$ and $\alpha \in \Lambda^k N$ then $F^*\alpha \in \Lambda^k M$ as defined below

$$(F^*\alpha)_p(v_1, \ldots, v_k) = \alpha_p(DF_p \cdot v_1, \ldots, DF_p \cdot v_k).$$

We want to ask if $T^*_{\alpha\beta}\gamma^\beta_{\boldsymbol{y}}$ and $\gamma^\alpha_{\boldsymbol{x}}$ give the same orientation on $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$, can we build an orientation on M?

Can you put these forms together to make an orientation if things agree?