Stuff

- For HW8 #3, try to use partitions of unity to slow down the vector field.
- Fact we might prove later / on homework: For any two points p, q lying in a compact manifold M, there is a diffeomorphism $M \to M$ taking p to q.

.1. Embedding of Manifolds into $\mathbb{R}^N$

Theorem .1.1 (Whitney Embedding Theorem)

If M is a manifold, then for some N there exists an $f : M \to \mathbb{R}^N$ which is an embedding.

Proof when M is compact. Let $n = \dim M$. Then if $(U_\alpha, \varphi_\alpha)$ are the coordinate charts (balls around 0) then we can map

$$M \to \prod_\alpha \mathbb{R}^n =: \mathbb{R}^N$$

where $N = n \cdot \#\{\alpha\}$ (we can take finitely many charts since M is compact. If $x \in U_{\alpha_1}, \ldots, U_{\alpha_j}$ then we can map x to have zeros for all β not an α_i, and $\varphi_{\alpha_i}(x)$ for all those included.

This is a *BAD* mapping. Make this construction smooth by tampering with a partition of unity of $\{U_\alpha\}$. Call this partition of unity τ_α. Then we replace $\varphi_{\alpha_i}(x)$ with $\tau_{\alpha_i}(x) \cdot \varphi_{\alpha_i}(x)$.

Also, $\tau_{\alpha_i} \equiv 1$ on $V_{\alpha_i} \subseteq U_{\alpha_i}$. We should make sure we get a finite covering of the manifolds by V_{α_i} (and then we'll be done.

Proof Idea in General. Look at $\prod_{\alpha_i} \mathbb{R}^n$ which is infinite, and project to a large dimensional $\mathbb{R}^N$.

.2. Multilinear Algebra

Definition .2.1

Let V be a vector space, then we define the exterior product $\Lambda^k(V)$ to be

$$\Lambda^k(V) := \{k - \text{multilinear alternating functionals}\},$$

i.e. $\lambda \in \Lambda^k(V)$ is a multilinear function $\lambda : \underbrace{V \times \cdots \times V}_{k \text{ times}}$ where for all i, j we have

$$\lambda(\ldots, v_i, \ldots, v_j, \ldots) = -\lambda(\ldots, v_j, \ldots, v_i, \ldots).$$

Note: $\lambda(\ldots, v, \ldots, v, \ldots) = 0$ (any one repetition gives us 0).

Theorem .2.1

$\dim \Lambda^{\dim V} V = 1$.

Proof Idea. Choose an isomorphism of V with $\mathbb{R}^n$, and work there. The dimension is ≥ 1 because we can construct the determinant function. It is *difficult* to show the determinant exists.

The dimension is ≤ 1 part is pretty easy.

Why is this important to us? It's not just algebraic garbage (Ralf's words). There's a geometric interpretation of the determinant!

$|\det|$ **is VOLUME**

Example .2.1

$\Lambda^1 V = V^*$ (the dual of V).

We can see that $\dim \Lambda^k V \le \binom{\dim V}{k}$. Explicitly when $k = 2$, let $\lambda \in \Lambda^2 V$, and e_i a basis of V. Then let $v = \sum_i \alpha_i e_i, w = \sum_j \beta_j e_j$. Then

$$\lambda(v, w) = \sum_{ij} \alpha_i \beta_j \lambda(e_i, e_j) = \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i) \lambda(e_i, e_j).$$

With this in mind we can define $e_i \wedge e_j$ as the element of $\Lambda^2 V$ which acts as

$$(e_i \wedge e_j) \left(\sum_i \alpha_i e_i, \sum_j \beta_j e_j \right) = \alpha_i \beta_j - \alpha_j \beta_i.$$

ChecK: This is an alternating multilinear form. This give s $e_i \wedge e_j$ for $i < j$ as a basis of $\Lambda^2 V$.

Similarly we can get a basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$ where $i_1 < \cdots < i_k$ as a basis of $\Lambda^k V$.

MEANING: Lets go to $\mathbb{R}^3$. We see that

$$(e_1 \wedge e_2) \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = 1.$$

$e_1 \wedge e_2$ is giving the signed area of a square... But which square?

$$(e_1 \wedge e_2) \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = 0.$$

The area is the 2-dimensional area of the shape projected to a 2-dimensional slice of a plane!

Philosophy: $\lambda \in \Lambda^k \mathbb{R}^n$ "measures" the k-dimensional area of a parallelipiped with respect to a particlar fixed k-dimensional subspace.