Announcements

- Midterm remains on Wednesday October 19th in class.

.1. **Flow on Vector Fields**

How do we flow on vector fields? That is how do we think of the vector field as a field of *force/acceleration* for a particle.

Well we wish to fill up a manifold M with curves and then differentiate them! That tells us the vector field at every point. However, we must avoid crossings so we can decide where to take the vector field

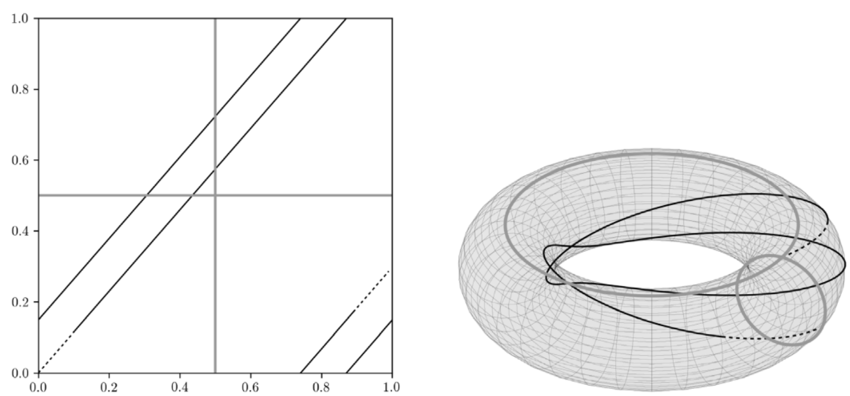
Recipe:

(1) Fill up M with disjoint differenitable curves c_i.
(2) Then take $X(p) = \dot{c}_{p_i}(p)$ for c_{p_i} a curve through p.
(3) What about $C^0, C^1, \ldots$?
(4) Along a C^∞-curve $c(t)$ the vector field is C^∞, "transversally" to the curves regularity is unclear. But if $c \mapsto c_{p_i}$ is sufficiently differentiable, then all is good.

Example .1.1

Strange example. Take an angle $\alpha \notin 2\pi\mathbb{Q}$ and take a line through the flat torus which forms an angle α.

For convenience here is a picture of curves in the flat and curved torus



(a) *Flat torus as square in $\mathbb{R}^2$ with edges identified.*

(b) *Curved torus embedded in $\mathbb{R}^3$.*

This picture is taken from [EHM18].

We want to go the opposite direction. Given a vector field, how do we produce a flow which incudes it?

Definition .1.1

Let X be a vector field on M. We call $c : (a, b) \to M$ a solution curve for X provided that for all $t_0 \in (a, b)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0} c(t) = X(c(t_0)).$$

In coordinates, for a C^∞-chart U take standard vector fields $\frac{\partial}{\partial x_i}$. Then we know

$$X\big|_U = \sum_{i=1}^n a_i(p)\frac{\partial}{\partial x_i}.$$

X is C^∞ if and only if a_i is C^∞.

Write $c(t) = (c_1(t), \ldots, c_n(t))$ in these coordinates. Then we have that

$$\dot{c}(t_0) = (\dot{c}_1(t_0), \ldots, \dot{c}_n(t_0)).$$

To require that $X(c(t_0))$ means to require that

$$\sum_{i=1}^n a_i(c(t_0))\frac{\partial}{\partial x_i}(c(t_0)) = \sum_{i=1}^n \dot{c}_i(t_0)\cdot\frac{\partial}{\partial x_i}(c(t_0)).$$

Therefore we must have $a_i(c(t_0)) = \dot{c}_i(t_0)$. We have that the a_i are given by the vector field. What's not given is the C's

"Bacid ODEs, Vague." For a C^1-manifold we can solve uniquely if a_i are Lipschitz functions then the solutions are unique.

Why vague? For what time t do we get a solution. Well something like

$$c_i : (-\varepsilon(p), \varepsilon(p)) \to \mathbb{R}^n$$

within coordinates. This occurs because the "speed" along which c_i goes on the vector field may escape to infinity, and then we don't know what to do at $\varepsilon(p)$.

More precise version. Let X be some C^∞ vector field. There exists an $\varepsilon > 0$ and a $\delta > 0$ such that for all $q \in B_\varepsilon(p)$ ther exists a solution to the ODE on the inversal $(-\delta, \delta)$.

This is called a local solution. We have existence and uniqueness of local solutions. We will not prove this because it is painful, it is an application of the Contraction Mapping Theorem.

Definition .1.2

Call M a C^∞ manifold. We say a vector field X on M is complete if solution curves exist through any point for all time.

Ad: Nearly impossible to actually calculate solutions to these curves (supercomputers can approximate), except in special cases (ex. linear ODEs). Actual computations is the Quantitative, explicit solutions, and would be called ODEs.

Dynamical systems would be considering the Qualititative study of vector fields! This goes back to Poincaré.

Lemma .1.1

If M is a compact C^∞ manifold and X is a C^K vector field for $k \geq 1$ then X is complete.

Proof. For short time, on a neighborhood U of $p \in M$ we have a solution curve $c_p : (-\varepsilon(p), \varepsilon(p)) \to U$

Then there are finitely many $p_1, \ldots, p_\ell$ with $\bigcup U_{p_\ell} = M$. Take $\varepsilon \coloneqq \min \varepsilon_{p_i}$.

For each $q \in U_{p_i}$ we can flow along the field for $(-\varepsilon, \varepsilon)$. Uniqueness of solutions on $(-\varepsilon, \varepsilon)$ implies that things will agree on the overlap. We can keep flowing in either direction forever!!! This finishes the proof.

Warning: The curve exists for all time but may have finite length! We may come to a stop at a stationary point on the vector field!!!

Definition .1.3

Let X be a complete vector field on M. Call a map $\Phi : \mathbb{R} \times M \to M$ so that $\phi(t, p)$ for fixed $p \in M$ and varying t is a solution curve at p the flow generated by X.

We define $\varphi_t(p) \coloneqq \Phi(t, p)$. We can call φ_t the (global) flow determined by X.

Next time: This gives you an action of the real numbers on M.