Corollary .0.1

Submersions are open maps

Proof. The local normal form is a projection, which is an open map.

Now lets look at examples of submersions.

Example .0.1

If F, M are manifolds, then we can look at the projection $M \times F \to M$. Then this is obviously a submersion!

We call this type of submersion a trivial bundle

I. Fiber/Vector Bundles

Definition I.0.1

A submersion $\pi : M \to N$ is a fiber bundle provided that

- π is surjective.
- We equip N with a covering by open sets $\{V_\alpha\}$ such that $\pi^{-1}(V_\alpha)$ is diffeomorphic by φ_α to $V_\alpha \times F$ for some fixed manifold F. These are called local trivializations
- For each V_α the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & V_\alpha \times F \arrow[dl, dashed] \\ & V & \end{tikzcd}$$

The manifold F is called the fiber of the bundle.

Example I.0.1

$N \times F \to N$, and for M the Möbius band, $M \to S^1$, with $F = (-1, 1)$.

Note that the Möbius band is not diffeomorphic to $S^1 \times (-1, 1)$.

If $N \subseteq \mathbb{R}^L$ is an embedded submanifold, we can consider the unit tangent bundle

$$S(N) = \{v \in TpN \mid p \in N, \|v\| = 1\}.$$

For $N = S^2$ is sort of complicated. For $N = S^1$, we get

$$N(S^1) = S^1 \times \{0, 1\}.$$

Fact: $S(S^2)$ is not a trivial fiber bundle.

HW: $S(S^3)$ is a trivial bundle. Hint: It's a group.

Definition I.0.2

Let M be an abstract differentiable manifold. As a set the tangent bundle of M is

$$TM := \coprod_{p \in M} T_p M.$$

Claim: This is a fiber bundle, in fact it is a vector bundle

Definition I.0.3

A vector bundle $\pi : M \to N$ is a fiber bundle with fiber F a vector space such that $\pi^{-1}(z_0)$ is intrinsically a vector space and for any local trivialization $(U_\alpha, \varphi_\alpha)$ induces a linear map $\varphi_\alpha : \pi^{-1}(z_0) \to \{z_0\} \times F$ for all $z_0 \in U_\alpha$.

Equivalently, if $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ are any two trivializations, then

$$\begin{tikzcd} \pi^{-1}(U_\alpha \times U_\beta) \arrow[r, bend left, "\varphi_\alpha"] \arrow[r, bend right, "\varphi_\beta"'] & (U_\alpha \cap U_\beta) \times F \end{tikzcd}$$

has $\varphi_\beta^{-1} \circ \varphi_\alpha$ (called is a linear map (and therefore a linear isomorphism). This allows one to place a canonical vector space structure on $\pi^{-1}(z_0)$ for any $z_0 \in N$.