Stuff:

- HW 3 is due next time, as we haven't covered enougn to make it tractable. Also Ralf is not sure if it is true.
- For HW 2d, if you want to show $\mathrm{GL}(n, \mathbb{C})$ is connected, you might want to look at $\mathbb{C}^* \times \mathrm{SL}(n, \mathbb{C})$. Then

$$\mathrm{SL}(n, \mathbb{C}) = \mathrm{SU}(n) \cdot \text{ upper triangular matrix with any complex numbers.}$$

  Alternately, realize $\mathrm{GL}(1, \mathbb{C})$ is orientation preserving and so $\mathbb{C}$ carries a natural orientation.

General principle: If Γ acts on M, and M/Γ sia manifold and M carries a "structure" invariant by Γ which is invariant, it induces this structure on M/Γ.

Example .0.1

$\mathbb{R}\mathrm{P}^n = S^N/\mathbb{Z}_2$ and $\mathbb{C}\mathrm{P}^n = S^{2n-1}/S^1$.

Example .0.2

Suppose M is $\mathbb{C}$-differentiable and Γ acts by $\mathbb{C}$-differentiable maps, then M/Γ is $\mathbb{C}$-differentiable.

Example .0.3

If M has a Riemannian metric and Γ acts by isometries, then M/Γ carries a Riemannian metric. By acting by isometries we mean that for $\gamma \in \Gamma$

$$\langle D\gamma_p \cdot v, D\gamma_p \cdot w \rangle = \langle v, w \rangle.$$

Last time: $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with coordinates $y_1, \ldots, y_n$ in the domain and $x_1, \ldots, x_n$ in the domain. Then

$$\varphi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = (\det D\varphi)(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n).$$

Orientability Theorem, atlas $\implies$ volume form. If $x_1, \ldots, x_n$ are coordinates on U_β, and $y_1, \ldots, y_n$ are coordinates on U_α. Then

$$T_{\alpha\beta}^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = (\det DT_{\alpha\beta})(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n).$$

Thus these are related by a positive number. Pick a partition of unity $\{\tau_i\}$ subordinate to U_α. On U_α we get an n-form σ_α given by pulling back $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. Then

$$\sum \tau_\alpha \sigma_\alpha$$

will define an n-form on M which is nonvanishing.

Orientability Theorem, volume form $\implies$ atlas. Call φ_α positive if the pullback form on U_α given by $\varphi_\alpha^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)$ (for coordinates $x_1, \ldots, x_n$) is positive with respect to σ (a fixed orientation on M). That is it equals $f \cdot \sigma$ for $f > 0$.

If all φ_α are $+$ then get coordinate charts are compatible with orientation. If not all φ_α are $+$ then "flip" the negative ones. I..e replace the coordinates $x_1, \ldots, x_n$ with $-x_1, x_2, \ldots, x_n$.

.1. **Defining Integrals**

Why bother with orientation? If $f : M \to \mathbb{R}$ is smooth, then how do we define $\int_M f$??? On $\mathbb{R}^n$ we just use Lebesgue integration (or Riemannian integration). Main thing is we know the volume of a cube.

In contrast, there is no preferred way to measure volume on a manifold! You would need a Riemannian metric. Similarly, an n-form can tell you the volume... Maybe if we have an n-form we can do something!!!

So $\int_M f$ NO IDEA. If $\int_M f\tau$ where τ is a volume form we have an idea. How to actually do it? In a chart U_α, with $\varphi_\alpha : U_\alpha \to V_\alpha$ we consider

$$\int_{V_\alpha \subseteq \mathbb{R}^n} (f \circ \varphi_\alpha^{-1}) \cdot \varphi_\alpha^*(\tau).$$

This is $g_\alpha \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ for some coordinates. Why is this well-defined? If we have a change of variables on $\mathbb{R}^n$ called T, then

$$\int_B (h \circ T) \det DT \, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n = \int_A h \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n,$$

where $A = T \cdot B$. Thus integrals agree on the overlaps of charts! Namely, the forms transform according to $T^*_{\alpha\beta}$ which acts via the determinant of the jacobian matrix from the work we've done above