Midterm Announcements:

- Graded–will get it back today
- 120 points possible out of 100 because Problem 2 had an error
- Median was around 100. Very good job
- Midterm given back in last 5 minutes of class.

Recall .0.1

If $F : M \to N$ is C^∞, and X, Y vector fields on M, N respectively then we call X, Y F-related if

$$\mathrm{d}F_p(X(p)) = Y(F(p)) \in T_{F(p)}N.$$

Call this $X \sim Y$. If $X_2 \sim Y_2$ are well, then $[X, X_2] \sim [Y, Y_2]$.

To show this, it's convenient to know the flow of $[X, X_2]$. Letting φ, ψ be local flows for these respectively, we claim the flow for $[X, X_2]$ is given by

$$C_t = \psi_{\sqrt{t}} \circ \varphi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}}.$$

Well consider

$$G(s, t) = \psi_{-s} \circ \varphi_{-t} \circ \psi_s \circ \varphi_t(p).$$

It is then clear that

$$\frac{\partial}{\partial s}(G(s, 0))\Big|_{s=c} = \frac{\partial}{\partial s}\Big|_{s=c}(p) = 0.$$

One must then use Taylor Expansion up to order s^2, t^2, st in order to derive the result.

If X has solution curve $\varphi_t(p)$, then $F(\varphi_t(p))$ is a solution curve for Y if X, Y are F-related. Then by the characterization of the flow $[X, Y]$

.1. Distributions

This means way too many things in math. We might also call them k-plane fields.

Consider a manifold M which is C^∞, cosndier taking T_pM to $\mathrm{Gr}_k(T_pM)$, which si k-dimensional vector subspaces of T_pM. Fancy: Make a fiber bundle out of

$$\mathrm{Gr}_k(M) = \coprod_{p \in M} \mathrm{Gr}_k(T_pM).$$

Make this a smooth manifold using the local product structure of TM. In fact

$$\mathrm{Gr}_{k,n} \to \mathrm{Gr}_k(M) \xrightarrow{\pi} M$$

is a fiber bundle, where $n := \dim M$.

Definition .1.1

A distribution is a smooth section of this fiber bundle. In down to earth terms, $D(p) \subseteq T_pM$ is a k-dimensional subspace, spanned by say $\langle v_1, \ldots, v_k \rangle$. Do this for every point.

Locally we get $v_1(q), \ldots, v_k(q)$ where q is in a neighborhood of p. We require that the $v_i(q)$ are smooth vector fields on this neighborhood p. We could do stupid things, like making $v_i(q)$ be changed by a linear transformation at rational points. . . so instead we just require there is a choice.

Thus smooth distributions of dimension k are given by the following data

- For all $p \in M$, $D(p) \subseteq T_pM$ is a k-dimensional subspace.
- To define smoothness of D, it suffices to do it locally. I.e., for all $p \in M$, there exists a neighborhood U of p and there exist smooth vector fields $v_1, \ldots, v_k$ on U so that
  (1) For all i,q, $v_i(q) \in D(q)$
  (2) For all i, q, $v_1(q), \ldots, v_k(q)$ are linearly independent.

  Equivalently the span of $v_1(q), \ldots, v_k(q)$ is $D(q)$ for all $q \in U$.

There are two types of distributions, the boring ones (which are most important), and the exciting ones (which are not used very much).

Example .1.1

Let $\mathbb{R}^n = M$, and for each $p \in \mathbb{R}^n$ let $D(p) = \mathbb{R}^k = \{(x_1, \ldots, x_k, 0, \ldots, 0)\}$. This is spanned by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$.

Example .1.2

Suppose G is a Lie group that acts on some manifold M. Suppose for all $p \in M$, we have G_p is discrete. Then $\{G \cdot p\}$ for $p \in M$ will cut up M into submanifolds (we haven't shown this formally).

The distribution will then be given by $D(p) = T_p(G \cdot p)$.

Example .1.3

Take V a nonvanishing vector field on $\mathbb{R}^n$. We can take $D(p) = V(p)^\perp$.

Consider $M = \mathbb{R}^n \setminus \{0\}$, and take $V(p) = p$. This is exactly the tangent spaces to spheres of certain radii. This is actually a Lie group example–it's $\mathrm{SO}(n)$ acting on M. To see explicitly the vector fields, one can think of polar coordinates + the angles.

Even more explicitly one can look at one coordinate being nonzero and then take a radial vector field there.

Example .1.4

Consider the Heisenberg group

$$\mathrm{Heis} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Consider tangent vectors

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which are tangent vector fields at the identity. Then $\overline{A}_1, \overline{A}_2$ are left-invariant vector fields. We can consider $D = \mathbb{R} \cdot \overline{A}_1 + \mathbb{R} \cdot \overline{A}_2$.

This is not a Lie group vector field, and so is much more complicated.