Bonus Problems: Due 1 week after regular problems, if Gradescope complains let the professor know.

Idea for Problem 1 from Homework 1

Let M be a 1-dimensional, compact, connected manifold.

Choose charts U_x around each $x \in M$ homeomorphic to intervals. Choose a finite subcover $U_i \cong (a_i, b_i)$ and suppose this is a minimal such cover. Make a lemma that if $U_i \cup U_j$ is connected

Standards for what constitutes a proof will be determined over time by a conversation between the students, the grader, and the professor.

Deep Theorem

Theorem .0.1

There exist topological manifolds which do not admit any differentiable structure.

In fact: "Piecewise linear" manifolds of such type exist.

Next Question: Can two differentiable manifolds give the same topological manifolds?

Yes! Milnor in the late 50s constructed exotic S^7 i.e. $M_1, \ldots, M_{28}$ all homeomorphic but none of them diffeomorphic. There are infinitely many higher dimensional spheres exhibiting this behavior, though it does not occur in dimensions ≤ 6. There are even uncountably many differentiable structures on $\mathbb{R}^4$!!!

Definition .0.1

If Ω_1, Ω_2 are differentiable structures (aka atlases) on M, we say that Ω_1, Ω_2 are compatible if $\Omega_1 \cup \Omega_2$ is a differentiable structure.

A maximal atlas is the union of all compatible atlases.

Bourbakian method: Always use maximal atlas vs. Hands-on approach: find your atlas, work with it.

Proposition .0.2

Suppose M, N are both differentiable manifolds, then $M \times N$ is a differentiable manifold whose dimension is $\dim M + \dim N$.

Proof. Say $(U_\alpha, \varphi_\alpha)_\alpha$ is an atlas for M, $(V_\beta, \psi_\beta)_\beta$ an atlas for N. Then $\{(U_\alpha \times V_\beta, (\varphi_\alpha, \psi_\beta)\}_{\alpha,\beta}$ is an atlas for $M \times N$.

What to check?

- Clearly a cover of $M \times N$.
- $(\varphi_\alpha, \psi_\beta) : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$.
- Check compatibility of charts which is similarly clear.

Quotients on the other hand, are very ill-behaved. As you know, Hausdorff spaces can have non-Hausdorff quotients, and differentiable structure is also not respected.

Example .0.1

Take $S^1 \subseteq \mathbb{C}$, let $x \sim y$ provided that $xy^{-1} = e^{i2n\alpha\pi}$, where $\alpha \in \mathbb{R}$ is fixed, $n \in \mathbb{Z}$. In fact $S^1/\sim \cong S^1$ when $\alpha \in \mathbb{Q}$.

However when $\alpha \notin \mathbb{Q}$, the equivalence class of any x is dense in S^1, so $S^1/\sim$ is not Hausdorff. In fact

Claim

$\emptyset$ and $S^1/\sim$ are the only open sets.

Proof. Suppose $U \subseteq S^1/\sim$ is open. Assume U is nonempty, and let $V := \pi^{-1}(U)$ where $\pi : S^1 \to S^1/\sim$. Then fix $z \in S^1$, we see that $[z] \cap V$ is nonempty since $[z]$ is dense and V is a nonempty open set.

Thus z is equivalent to something in V, so $[z] \in U$. Thus $U = S^1/\sim$.

Note the quotient map is indeed open as well by density.

Definition .0.2

Let X be some space and $\sim$ an equivalence relation on X. We call $\sim$ open provided that $\pi : X \to X/\sim$ is an open map.

Equivalently, if $U \subseteq X$ is open, then $\{x \in X \mid \exists y \in U, x \sim y\} = \pi^{-1}(\pi(U))$ is open.

Fact: If X is second countable, then $\sim$ is an open equivalence relation if and only if $X \xrightarrow{\pi} X/\sim$ is an an