I. Stoke's Theorem

I.1. Manifolds with Boundary

Stuff:

- For users of the notes, the rest of the proof from Wednesdaywas added to the November 30th notes.
- The bonus contains some stuff about computing cohomology (Mayer-Vietoris).
- For the next few days we'll discuss Stoke's Theorem. You are free to use it on the homework now.

Definition I.1.1

For convenience call $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. A topological manifold with boundary is a paracompact, Hausdorff, second countable space M with a cover of M by $\{U_\alpha\}_{\alpha \in I}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{H}^n$.

We can then require the transition maps to be smooth to get a smooth manifold with boundary.

Example I.1.1

$\mathbb{H}^n$ is a manifold with boundary. So is $\mathbb{R}^n$ (empty boundary). You can have things like intervals with endpoints, or taking a standard genus g surface and slicing it in half.

Definition I.1.2

We define the boundary of M to be $\{x \in M \mid \varphi_\alpha \in \partial\mathbb{H}^n\}$ where $\partial\mathbb{H}^n := \{x \in \mathbb{H}^n \mid x_n = 0\}$.

Lemma I.1.1

∂M is well defined, i.e. independent of the chart. I.e., a diffeomorphism between $\mathbb{H}^n$ and itself preserves the boundary.

Furthermore, ∂M is a manifold (without boundary) of dimension $\dim M - 1$.

Exercise I.1.2

Prove this lemma above.

Lemma I.1.2

Let $M, \partial M$ be a manifold with boundary. Suppose M is oriented. Then ∂M is also oriented.

Proof. Take the situation in $\mathbb{H}^n$. If $p \in \partial\mathbb{H}^n$. How do we tell if $v_1, \ldots, v_{n-1} \in T_p\partial\mathbb{H}^n$ is a positively oriented basis?

Well take $u = (u_1, \ldots, u_n)$ an outward normal, aka so that $u_n < 0$. We then call $v_1, \ldots, v_{n-1}$ positively oriented for $\partial\mathbb{H}^n$ if $u, v_1, \ldots, v_{n-1}$ are positively oriented for $\mathbb{R}^n$. This does not depend on the particular u chosen, bc it can be taken to $(0, \ldots, 0, u_n)$ by a linear combination with $v_1, \ldots, v_{n-1}$.

For M a manifold with boundary, we endow M with the pullback orientation from $\partial\mathbb{H}^n$. One must check this is well-defined, and one checks that transition maps preserve outward normals.

Theorem I.1.3 (Stoke's Theorem)

Let M be an oriented manifold with boundary ∂M (under the induced orientation).

Given $\omega \in \Omega^{n-1}M$, we have that

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Example I.1.3

Pick $M = [0,1]$ and pick the left to right orientation. Take $\omega \in \Omega^0([0,1])$, aka a smooth function $f : [0,1] \to \mathbb{R}$.

Then we have

$$\int_{[0,1]} \mathrm{d}\omega = \int_0^1 \omega'(t)\,\mathrm{d}t = \omega(1) - \omega(0)$$
$$\int_{\partial[0,1]} \omega = \int_{\{0,1\}} \omega = \omega(1) - \omega(0),$$

where the $-$ comes from the orientation. Thus we should think of Stoke's Theorem as a generalization of the fundamental.

Corollary I.1.4

If M is a manifold (without boundary) and $\omega \in \Omega^{n-1}(M)$ then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega = 0.$$

This means the integral of exact forms over manifolds are zero.

What do we need to do to prove this thing? The Idea: look at differentiable cubes, aka smooth maps $C : I^k \to M$. Then taking some $\omega_k(M)$ we look at an integral

$$\int_C \omega = \omega_{I^k} C^*(\omega).$$

Then we'll cover M by cubes, take a partition of unity, and reduce the whole problem to something about integrating around cubes.