Definition .0.1

We call a group action of G on a set X transitive provided that X is one G-orbit, i.e., for every $p, q \in X$ there is a $g \in G$ so that $g \cdot p = q$.

Example .0.1

The action of $\mathrm{SO}(n+1)$ on $S^n \subseteq \mathbb{R}^{n+1}$ is transitive. Take $v \in S^n$, and extend it to an orthonormal basis and make these the columns of $g \in \mathrm{SO}(n+1)$. We see that $g \cdot \left(\begin{smallmatrix} 1 & 0 & \vdots & 0 \end{smallmatrix}\right) = v$.

Definition .0.2

Suppose G acts on X, $p \in X$, the stabilizer of p in G or isotropy group of p in G if

$$G_p := \{g \in G \mid g \cdot p = p\}.$$

Lemma .0.1

The stabilizer G_p is a subgroup of G. If G is a topological group, $G_p \subseteq G$ is a closed subgroup (i.e. it is a closed set).

Proof. Showing it's a subgroup if trivial. We can quickly show it is closed. Why? Well if $g_n \to g$ and $g_n \in G_p$ then $p = g_n \cdot p \to g \cdot p$ by continuity, so $g \cdot p = p$, so $g \in G_p$.

Future: We will prove that if G is a Lie group, $H \subseteq G$ is a closed subgroup, then H has a Lie group structure.

Example .0.2

Consider $\mathrm{SO}(n+1)$ acting on S^n. The stabilizer of the $N := (1, 0, \ldots, 0) \in S^n$ is

$$\mathrm{SO}(n+1)_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \mid * \in \mathrm{SO}(n) \right\}.$$

This shows $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$.

Lemma .0.2

If G acts on M transitively and $p \in M$, then there is a bijective continuous map $G/G_p \to M$. If G/G_p is compact and M is Hausdorff, then this has a continuous inverse, so $G/G_p \cong M$.

Proof. We have a surjective map $f : G \to M$ given by $g \xmapsto{f} g \cdot p$ since the action is transitive. By the universal property of quotients since for $x \in G_p$ we have $f(gx) = g \cdot x \cdot p = f(g)$ this map descends as

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & M \\ G/G_p \arrow[ur, dashed, "\widetilde{f}"] & \end{tikzcd}$$

to a continuous map. To show it is one-to-one we see that if $\widetilde{f}([g]) = \widetilde{f}([h])$ then $g \cdot p = h \cdot p$, so $h^{-1}g \cdot p = p$, so $h^{-1}g \in G_p$.

Example .0.3

Consider $\mathrm{GL}(2, \mathbb{R})$ acting on $\mathbb{R}^2$. This is not transitive since $A \cdot 0 = 0$. But it is transitive on $\mathbb{R}^2 \setminus \{0\}$.

Now consider

$$P := \mathrm{GL}(2,\mathbb{R})_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = \left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix} \mid c \in \mathbb{R}, d \neq 0 \right\}.$$

We then see that

$$\mathrm{GL}(2,\mathbb{R})/P \cong \mathbb{R}^2 \setminus \{0\}.$$

(in this case we're not compact, but it still works)

Example .0.4

$G := \mathrm{GL}(n+1,\mathbb{R})$ acts on $\mathbb{R}^{n+1}$ and transitively on $\mathbb{R}^{n+1} \setminus \{0\}$. This descends to a transitive action on $\mathbb{R}\mathrm{P}^n$.

Consider $\ell_1 = \mathbb{R}^\times \cdot e_1$, then

$$G_{\ell_1} = \left\{ \begin{pmatrix} \lambda & *_1 \\ 0 & *_2 \end{pmatrix} \mid \lambda \in \mathbb{R}^\times, \det *_2 \neq 0 \right\}.$$

Goal: G/G_{ℓ_1} is differentiable.

Wanted: $T \subseteq G$ "transversal" to G_{ℓ_1} of dimension $\dim \mathbb{R}\mathrm{P}^n = n$. We can look at

$$T = \left\{ \begin{pmatrix} 1 & 0 \\ v & \mathrm{Id} \end{pmatrix} \mid v \in \mathbb{R}^n \right\}.$$

Then $\mathbb{R}^n \cong T \cdot \ell_1$ so we have a chart!

Recipe:

(1) Suppose a Lie group G acts transitively on M. We want to endow M with a differentiable $(C^1, C^k, C^\infty, C^\omega)$ structure.
(2) Take $p \in M$, $G/G_p \cong M$.
(3) If you can find a "transversal" "subspace" of G to G_p, say T
(4) Try coordinate charts $T \to T \cdot p, t \mapsto t \cdot p$.

Back to the Future: If G is a Lie group, H is a closed sugroup, then G/H is always a smooth manifold.