**Lemma .0.1**

In a compact Hausdorff space, any two disjoint closed sets can be separated by open neighborhoods which contain these closed sets.

*Proof.* First we prove that if $C$ is closed and $x \notin C$ then we can separate $x, C$. We know $C$ is compact since $X$ is compact and Hausdorff. For each $y \in C$, let $U_y, V_y$ separate $x, y$. Then $\{V_y\}$ covers $C$, so we can take a finite subcover $V_{y_i}$. Let $V \coloneqq \bigcup_i V_{y_i}$ and $U \coloneqq \bigcap_i U_{y_i}$. $U, V$ clearly separate $x, C$ and are open.

Now take $C, C'$ which are disjoint closed sets. For each $x \in C$, take neighborhoods $U_x, V_x$ separating $x, C'$. Then by compactness take a finite subcover $U_{x_i}$. As before union the $U_{x_i}$ and intersect the $V_{x_i}$ to separate $C, C'$.

**Theorem .0.2**

Take $X$ as a topological space with $\sim$ open on $X$. Then

$$\text{graph} \sim \coloneqq \Gamma \coloneqq \{(x, y) \in X \times X \mid x \sim y\}$$

Then $X/\sim$ is Hausdorff if and only if $\Gamma \subseteq X \times X$ is a closed subset.

*Proof.* Proof of $\implies$ left as an exercise. Suppose $[x] \neq [y]$ within $X/\sim$. Then $(x, y) \in \Gamma^c$, so using the basis for the product topology, there are $U, V$ open so that $(x, y) \in U \times V$ and $U \times V \subseteq \Gamma^c$.

Thus $\pi(U) \cap \pi(V)$ is empty, where $\pi : X \to X/\sim$. Furthermore, $\pi(U), \pi(V)$ are open, and so they separate $[x], [y]$.

**Proposition .0.3**

If $\sim$ is open, $X$ is second countable, then $X/\sim$ is second countable

*Proof Idea.* Take a countable basis of $X$ and take their images. These are open, and it's easy to check this is a basis.

Aside: There is an interesting non-Hausdorff topology. Namely, the closed sets in $\mathbb{R}^n, \mathbb{C}^n$ (some algebraic variety) are given by zeros of polynomials.

Fact: This is compact.

## .1. Topological Groups/Homogeneous Spaces

**Definition .1.1**

We say $G$ is a topological group provided that $G$ is a group equipped with a topology such that the maps

$$(g, h) \mapsto gh$$
$$g \mapsto g^{-1}$$

are continous as maps $G \times G \to G$ and $G \to G$.

**Example .1.1**

$\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n, \mathbb{Z}, \mathbb{Z}^k$, and any discrete group.

Also $\mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{C})$ which are invertible matrices. $O(n)$, the orthogonal group which is the matrices so $A \cdot A^t = I$.

$\mathrm{SO}(n) = \{A \in O(n) \mid \det A = 1\}$, the rotations. Then $\mathrm{SL}_n(\mathbb{R})$ which are the matrices of determinant one.

The circle $S^1 = \mathrm{SO}(2)$. We may also consider $T^n = S^1 \times \cdots \times S^1$, $n$ times and $T^\infty = S^1 \times \cdots \times S^1 \times \cdots$. $T^\infty$ is compact (Tychonoff).

**Example .1.2**

$\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the $p$-adic norm $\|\cdot\|_p$. Write $a = p^k c, b = p^\ell d$ where $c, d$ are coprime to $p$, then

$$\left\|\frac{a}{b}\right\|_p = p^{\ell - k}.$$

Note $p^n \to 0$ as $n \to \infty$. Furthermore $\mathbb{Q}_p$ is a topological field.

We may look at $\mathrm{SL}(n, \mathbb{Q}_p), \mathrm{GL}(n, \mathbb{Q}_p)$ which have dimension 0.

**Definition .1.2**

A topological group $G$ is called a Lie group if it is equipped with a smooth differentiable structure such that the maps

$$(g, h) \mapsto gh \qquad\qquad g \mapsto g^{-1}$$

are smooth.

**Definition .1.3**

Let $M, N$ are differentiable manifolds and $f : M \to N$ is some continuous map. We call $f$ differentiable ($C^1, C^k, C^\infty$ aka smooth) provided that for any coordinate chart $(U_\alpha, \varphi_\alpha)$ around $x \in M$ and any coordinate chart $(V_\beta, \psi_\beta)$ around $f(x) \in N$ we have

$$f_{\alpha,\beta} \coloneqq \psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi(U_\alpha \cap f^{-1}(V_\beta)) \subseteq \mathbb{R}^{\dim M} \to \psi_\beta(f(U_\alpha) \cap V_\beta) \subseteq \mathbb{R}^{\dim N}$$

is differentiable ($C^1, C^k, C^\infty$) for all $\alpha, \beta$. Notice these sets are nonempty as $x, f(x)$ respectively lie in each of them.

**Lemma .1.1**

This is well-defined. That is, it is independent of compatible atlases.