Example .0.1 (Examples of Lie Groups)

$\mathbb{R}, S^1, \mathbb{R}^n, T^n = S^1 \times \cdots \times S^1, \mathrm{GL}(n, \mathbb{R})$.

Note: Famously, S^2 is not a Lie group. In fact S^0, S^1, S^3 are the only spheres which are Lie groups. These correspond to unit norm in the real numbers, complex numbers, and quaternions.

Reason: Euler characteristic $\chi(S^2) = 2$, and there is a theorem

Theorem .0.1

If a manifold M has a nonvanishing vector field (to be defined later) then $\chi(M) = 0$.

Aside: There exists an exotic $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$

Definition .0.1

G is a group (possibly topological, Lie) acts (possibly continuously, smoothly) on a space X (possibly topological, smooth manifold) provided there exists a map (possibly continuous, smooth)

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

such that

$$1 \cdot x = x \qquad\qquad (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x).$$

Notation: One might say differentiable to mean each $X \to X$ induced by $x \mapsto g \cdot x$ is differentiable, and use *jointly* differentiable to mean $G \times X \to X$ is differentiable.

Example .0.2

S^1 acts on S^1 by multiplication, $\mathbb{R}^n$ acts on $\mathbb{R}^n$ by addition, and importantly $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplicationu.

All examples given of group actions are jointly C^∞ (smooth).

Theorem .0.2

If G is a compact topological group, G acts continuously on X, X is compact Hausdorff, then X/G is Hausdorff.

Proof. We must check $\sim$ is open and graph $\sim$ is closed. Let $\pi : X \to X/G$ be the quotient map.

Take $U \subseteq X$ open, then we see that

$$\begin{aligned} \pi^{-1}(\pi(U)) &= \{y \in X \mid y \sim x\} = \{g \cdot u \mid u \in U, g \in G\} \\ &= \bigcup_{g \in G} g \cdot U \end{aligned}$$

is open because $g \cdot U = (g^{-1})^{-1}(U)$ is a preimage of a continuous map.

Now we must show Γ is closed. Look at $\varphi : G \times X \to X \times X$ given by $(g, x) \mapsto (x, gx)$. Note that $\operatorname{im} \varphi = \Gamma$. But wait! $G \times X$ is compact, so Γ is compact, so Γ is closed since $X \times X$ is Hausdorff.

Example .0.3

Take $X = S^n, G = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 = \{1, A\}$. Then G acts on S^n, where $A \cdot x = -x$ for $x \in S^n$.

Then $S^n/G = \mathbb{R}\mathrm{P}^n$.

Example .0.4

Consider $S^{2n-1} \subseteq \mathbb{C}^n$ with the action of S^1 on $\mathbb{C}^n$ via

$$e^{i\alpha} \cdot (z_1, \ldots, z_n) = (e^{i\alpha} z_1, \ldots, e^{i\alpha} z_n).$$

This is a continuous action on S^{2n-1} by S^1. Therefore

$$S^{2n-1}/S^1 = \mathbb{C}\mathrm{P}^{n-1}$$

Example .0.5 (Very General Example)

Suppose H is a Hausdorff topological group and $G \subseteq H$ is a compact subgroup. Then G acts on H by $(g, h) \mapsto gh$. Then H/G is compact Hausdorff if H is compact. Spaces of the form H/G (even when G is not compact) are called *homogeneous spaces* so long as H/G is Hausdroff. These spaces are extremely important.

Addendum: homogeneous spaces are important because

(1) You can calculate

(2) "Systems" with symmetry are typically homogeneous

(3) $\mathrm{GL}(n, \mathbb{R})/\,\mathrm{GL}(n, \mathbb{Z})$ shows up in number theory everywhere.

For those doubting since $\mathrm{GL}(n, \mathbb{Z})$ is not compact, look at $\mathbb{Z}$ acting on $\mathbb{R}$. We claim $\mathbb{R}/\mathbb{Z}$ is nice, check the graph is closed!