Final: Thursday December 15th, 4-6pm.

Question: What are the continuous homomorphisms $\varphi : S^1 \to \mathbb{R}$. We know $\varphi(S^1)$ is compact, and so is bounded. If we ahve $a \in \varphi(S)$ then $n \cdot a \in \varphi(S)$, which cannot be bounded unless $a = 0$. Thus φ is the constant map at 0.

BUT! $\mathrm{Lie}(S^1) = \mathbb{R}$, and $\mathrm{Lie}(\mathbb{R}) = \mathbb{R}$. Of course we have a map $\mathrm{Lie}(S^1) \to \mathrm{Lie}(\mathbb{R})$ given by the identity (this is a Lie algebra homomorphism since all brackets are zero...).

Thus a homomorphism between Lie algebras does not give rise to a smooth Lie group homomorphism. Thus what we did last time is **wrong**. But it is almost true!

But it is almost true. If $\widetilde{G}_1$ is the universal cover of G_1 (see 592, algebraic topology), then given a Lie algebra homomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ there is a Lie group homomorphism $\widetilde{G}_1 \to G_2$.

Universal Cover: If M is a reasonable space (e.g. a manifold) then we can find a space $\widetilde{M} \xrightarrow{\pi} M$ which is a submersion, nad $\widetilde{M}$ has the following property for all $p \in M$

$$\pi_1(\widetilde{M}, p) = 1.$$

$\pi_1(X, p)$ for $p \in X$ is defined as a set by

$$\{f : S^1 \to X \mid f \text{ continuous}, f(1) = p\}$$

where $f \sim g$ if there is an $F : X \times [0,1] \to X$ with $F(1,t) = p$ and $f(z) = F(z,0), g(z) = F(z,1)$. In fact this has a group structure. Again see algebraic topology (592). There are notes at

http://www-personal.umich.edu/ alephnil/notes/MATH-592-notes.pdf

We have $\widetilde{S^1} = \mathbb{R}$, where $\mathbb{R} \to S^1$ is given by $\exp(2\pi i t)$.

Fact: G is a Lie group implies $\widetilde{G}$ is also a Lie group. For a small enough neighborhood U of 1 the cover in $\widetilde{G}$ is given by $\widetilde{U} = \pi^{-1}(U)$ and the restriction of π to $\widetilde{U}$ is a homeorphism. Then if a, b are in some open subset of U, their product lies in U. We can then define the product in the covering space by lifting this product.

Note: The Lie algebra corresponding to $\widetilde{G}$ is the same as the Lie algebra of G.

Theorem .0.1

There is a bijective correspondence between Lie group homomorphisms $\widetilde{G}_1 \to G_2$ and Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$.

Idea of Proof. Let $H \subseteq G_1 \times G_2$ be defined as last time. This is a subgroup induced by the Lie algebra $\mathfrak{h} = \mathrm{graph}\,\Phi$, where $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$.

We know $\dim H = \dim \mathfrak{h} = \dim \mathfrak{g}_1$. We then have maps

$$\begin{tikzcd}
 & H \arrow[d] \arrow[ddl, bend right, "\pi_1\big|_H"'] & \\
 & G_1 \times G_2 \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
G_1 & & G_2
\end{tikzcd}$$

In fact $\pi_1\big|_H$ is a local diffeomorphism This holds because $D(\pi_1\big|_H) = \pi_1\big|_{\mathfrak{h}}$. We would like $\pi_1\big|_H$ to be injective. It turns out $H \cap \pi^{-1}(1)$ is discrete. This cannot be if we pass to the universal cover, but this requires work from 592.

Now let G be a Lie group and M a smooth manifold.. Suppose G acts on M smoothly. Let $\mathfrak{g}$ be a Lie algebra of G. Then there is a Lie algebra homomorphism $\mathfrak{g} \to \{\text{smooth vector fields on } M\}$.

To see this, take $X \in \mathfrak{g}$ a left-invariant vector field on G. Now write $g \cdot p =: E(g, p) =: E_p(g)$ where $p \in M$. We can push forward the vector field X to M using the map E_p (must check smoothness in p.

We can also take $g_t \coloneqq \exp(t \cdot X)$ (a one-parameter-subgroup) and consider $Y(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_t \cdot p$. If we move transversely to a point $q \in M$ near p, it's conceivable $Y(p)$ and $Y(q)$ does not vary smoothly. But this is possible to check.

Converse also holds, but is harder. If $\mathfrak{g} \to \{\text{smooth vector fields on } M\}$ then there is a smooth local action of G on M. This requires a bit of work.

I. Differential Forms and Integration on Manifolds

I.1. Partitions of Unity

Theorem I.1.1

Let M be compact (can do this all when M is not compact, but it's more painful) and $\{U_\alpha\}$ is a cover of M. Then we have a finite subcover $V_1, \ldots, V_\ell$ where for each i there is an α with $V_i \subseteq U_\alpha$. Furthermore, we can give smooth functions

$$\varphi_\beta : V_i \to [0, 1]$$

such that for all $x \in M$ we have $\sum_i \varphi_i(x) = 1$.

Proof. For all $p \in M$ find a neighborhood W_p and a smooth bump function $\psi_p : W_p \to [0, 1]$ with $\psi_p \equiv 1$ on a neighborhood of p. Then there's an $R_p \subseteq W_p$ so that $\psi_p \equiv 0$ outside R_p.

We can wlog that $W_{p_1}, \ldots, W_{p_k}$ cover M and are contained in the V_i. Then we look at

$$\varphi_j = \frac{1}{\sum_{i=0}^k \psi_{p_i}} \cdot \psi_{p_j}.$$

Then $\sum \varphi_j = 1$ as desired.

In the general case one uses paracompactness.