Last time: We defined $\int_M f \cdot \nu$ where M is a C^∞ manifold and ν is an n-form ("volume form"). This is well-defined

Definition .0.1

Let ν be an n-form on a C^∞-manifold M and let f be a function on M. If $\varphi_\alpha : U_\alpha \to V_\alpha$ is a chart we define

$$\int_{U_\alpha} f \cdot \nu := \int_{V_\alpha} (f \circ \varphi_\alpha)^{-1} \cdot (\varphi_\alpha^{-1})^*(\nu).$$

Then if $\{U_\alpha\}_\alpha$ is a collection of charts, take a partition of unity τ_α to U_α, and then set

$$\int_M f \cdot \nu = \sum_\alpha \int_{U_\alpha} \tau_\alpha f \cdot \nu.$$

Exercise .0.1

Show this is well-defined, and gives the sensible thing in general cases.

Difference to $\mathbb{R}^n$: no preferred volume form! On $\mathbb{R}^n$ we can look at $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

Some other good cases:

- If $M = G$ is a Lie group, take $X_1, \ldots, X_n$ a basis of $\mathfrak{g} = \operatorname{Lie} G$. Then turn these into a basis of left invariant vector fields.

  Let $\eta_1, \ldots, \eta_n$ be a dual basis at the identity. Make η_i left invariant so $\eta_i(X_j) = 1$ if $i = j$ and 0 if $i \neq j$. Then $\eta_1 \wedge \cdots \wedge \eta_n$ is left invariant.
- Can do the same thing for right invariant.

Proposition .0.1

If G is a Lie group then there exists a left invariant volume form ν_L unique up to scalar multiplication. Also there exists a unique (up to scalar) right invariant volume form ν_R.

Question: When is $\nu_L = \nu_R$.

Answer: Not always,

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \mid a \neq 0, b \right\}$$

The Lie algebra is

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & -B \end{pmatrix} \mid A, B \in \mathbb{R} \right\}.$$

But they are equal for

- Abelian groups
- nilpotent groups (e.g. heisenberg groups)
- $\mathrm{SL}_n(\mathbb{R})$.

Definition .0.2

If $\nu_L = \nu_R$ we call this group unimodular.

Compact groups are always unimodular. You can measure how unimodular something is by writing $\nu_R = \omega \cdot \nu_L$. Then one can prove $\omega(gh) = \omega(g)\omega(h)$ (check, Ralf thinks so). So measuring $\ker \omega$ tells you how unimodular it is.

Also if there is a $\Gamma \subseteq G$ discrete with G/Γ compact then G is unimodular.

Proposition .0.2

If M is a Riemannian manifold which is oriented, then the Riemannian metric induces a volume form.

The last case is suppose M has a (special) volume form ν and Γ acts on M properly discontinuously. Then M/Γ is a manifold.

Lemma .0.3

If ν a volume form on M is Γ-invariant, then ν descends to M/Γ.

Furthermore, if Γ is finite and orientation-preserving then one can always build such a Γ-invariant volume form from an arbitrary volume form on M.

Proof. Use that $\pi : M \to M/\Gamma$ is a submersion and a local diffeomorphism. Thus locally can pull back ν to $\overline{\nu}$ on M/Γ. Building it this way gives $\nu = \pi^*(\overline{\nu})$.

More explicitly. Let $\overline{p}, \overline{U}$ in M/Γ with diffeomorphisms $\gamma U \to \overline{U}$ for $\gamma \in \Gamma$.

Then ν on U we have $\nu = (\gamma^{-1})^*\nu$ on $\gamma\nu$. This commutes with the projection, and so ν defined from pushing ν on U down to $\overline{U}$ is the same as that defined from pushing ν on γU down to $\overline{U}$.

This allows one to paste it together into a preferred volume form! For the Γ finite case, just average!

Example .0.2

Suppose M^{2n} has a nonvanishing 2-form (symplectic form) α such that

$$\alpha \wedge \cdots \wedge \alpha$$

is nonvanishing, where we wedge n times.

More general integrals. Let $C : \Delta^k \to M$ be a smooth map from a k-dimensional simplex (sweeping under the rug–what does it mean to be smooth on the boundary?)

Let α be a k-form on M. Then $C^*(\alpha)$ is a k-form on Δ. Then

$$\int_{\Delta^k} C^*(\alpha) =: \int_C \alpha.$$

Note it depends on the map, which is why we write $\int_C$ instead of $\int_{C(\Delta^k)}$. This is a generalization of a line integral.

Example .0.3

When we're looking at the line integral, we're integrating vector fields over 1-simplices. The trick is

Definition .0.3

We'll call a smooth map $C : \Delta^k \to M$ a k-dimensional simplex in M.

These ideas are the brain-child of Poincaré, Elie Cartan, and de Rham. For now we'll leave them alone but we'll come back to them later.

.1. Exterior Derivatives

We now want to take α a k-form and associate to it $\mathrm{d}\alpha$, a $(k+1)$-form on M.

Example .1.1

For $F \in C^\infty(M)$ (aka a 0-form), we can take $\mathrm{d}F_p(v) = DF_p \cdot v$ (the directional derivative). This is a 1-form!

We'll use the notation $\Omega^k M$ for k-forms on M, and just Ω^k if M is clear. We want

$$\mathrm{d} : \Omega^k M \to \Omega^{k+1} M.$$

Recall that $\Omega^k(M)$ is zero for $k < 0, k > \dim M =: n$. So we get a chain

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

Here's what we want:

(1) d is d (defined above) on Ω^0.
(2) d is a linear map over $\mathbb{R}$ (not over $C^\infty(M)$!).
(3) $\mathrm{d} \circ \mathrm{d} = \mathrm{d}^2 = 0$.
(4) It works well with wedge product

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta,$$

where $\alpha \in \Omega^k(M)$.

Theorem .1.1

There exists a unique d satisfying Properties 1-4 above.

We'll prove this theorem in detail on Monday. Also Ralf Spatzier really likes the book Spivak, Calculus on Manifolds [Spi18]

Goal: Poincaré lemma. On $\mathbb{R}^n$, this will say that if α has $\mathrm{d}\alpha = 0$ (α is a "closed" form),then there exists a β so that $\alpha = \mathrm{d}\beta$, which is called being an "exact form" (notice the converse is always true). We'll be able to say something a bit more general... this exact statement doesn't always hold.

Definition .1.1

We can look at $\mathrm{Image}(\mathrm{d}|_{\Omega^{k-1}}) \subseteq \ker(\mathrm{d}|_{\Omega^k})$. By definition we have

$$H^k_{\mathrm{deRham}} M := \frac{\mathrm{Image}(\mathrm{d}|_{\Omega^{k-1}})}{\ker(\mathrm{d}|_{\Omega^k})}.$$

This is called the de Rham cohomology.

Miraculous–this is finite dimensional over $\mathbb{R}$. We'll abbreviate it H^k, though this is usually reserved for singular homology (see 592, they agree on manifolds). Instructive examples to compute

Example .1.2

$H^1(\mathbb{R})$, $H^1(S^1)$.