I. Vector Fields/Derivations/Lie Brackets

Definition I.0.1

If $E \xrightarrow{\pi} B$ is a fiber bundle then we call $\sigma : B \to E$ a section if $\pi \circ \sigma = \mathrm{Id}_B$.

Definition I.0.2

A vector field $V : M \to TM$ is a section (continuous, C^1, C^k, C^∞)of TM.

Example I.0.1

If $M = \mathbb{R}^n$, then we have very special vector fields which are constant at $e_1, \ldots, e_n$. We often denote these vector vields by $\frac{\partial}{\partial x_i}$, to specify that they are tangent vectors (and thus related to differentiation).

Then if $X : \mathbb{R}^n \to T\mathbb{R}^n$ is any vector field we may write for all $p \in \mathbb{R}^n$

$$X(p) = a_1(p)\frac{\partial}{\partial x_1}(p) + \cdots + a_n(p)\frac{\partial}{\partial x_n}(p),$$

where $a_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th coefficient function. X is C^r if and only if all the a_i are C^r.

How does one check $X : M \to TM$ is a differentiable vector field in practice? For convenience consider M is a C^∞ manifold. Take a chart $(U_\alpha, \varphi_\alpha)$ for M. Then $D\varphi_\alpha : TU_\alpha \to T(\varphi_\alpha(U_\alpha)) \subseteq T\mathbb{R}^n$ Thus we want to know that

$$z \mapsto D\varphi_\alpha(X(\varphi_\alpha^{-1}(z)) : \varphi_\alpha(U_\alpha) \to T(\varphi_\alpha(U_\alpha))$$

is differentiable.

Exercise I.0.2

If $f : M \to N$ is C^r then $Df : TM \to TN$ is C^{r-1}, where the obvious definition is

$$Df((p, v)) = (f(p), D_p f \cdot v).$$

Of course if f is C^∞ then Df is C^∞.

Check: Use coordinates. Take coordinates (U, φ) on M and (V, ψ) on N. Then $D\varphi, D\psi$ provide coordinates on TM, TN and so

$$D\psi \circ Df \circ D\varphi^{-1} = D(\psi \circ f \circ \varphi^{-1}).$$

By definition $\psi \circ f \circ \varphi^{-1}$ is C^r, so $D(\psi \circ f \circ \varphi^{-1})$ is C^{r-1}. Perfect!

Back to vector fields. To check smoothness (or do any calculation) write X in a chart. We have $e_1, \ldots, e_n$ on $\varphi(U)$. We can pull back $e_1, \ldots, e_n$ to get $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, so $\frac{\partial}{\partial x_i} = D_{\varphi(p)}(\varphi^{-1})(e_i)$. Then of course

$$X\big|_U = \sum_{i=1}^n a_i(p)\frac{\partial}{\partial x_i}(p)$$

where $a_i : U \to \mathbb{R}$ is the i-th coordinate function for the chart (U, φ).

Warning: $\frac{\partial}{\partial x_i}$ is a vector field. There will also be $\mathrm{d}x_i$, which are cotangent fields, which are differential 1-forms, that is local sections to $T^*M \coloneqq (TM)^*$. Then $\mathrm{d}x_i \in T^*U_\alpha$.

Definition I.0.3

Last time we learned how to take the dual space to a vector bundle. Thus we define the cotangent bundle of M to be $T^*M := (TM)^*$.

To summarize what we just did

Definition I.0.4

If (U, φ) is a coordinate chart on M we define local coordinates for TM via vector fields

$$\frac{\partial}{\partial x_i} : U \to TU$$
$$p \mapsto D_{\varphi(p)}(\varphi_\alpha^{-1})(e_i).$$

Then we define $\mathrm{d}x_i : U \to T^*U$ at each point $p \in M$ so that $\{\mathrm{d}x_i(p)\}_{i=1,\ldots,n}$ as the dual basis to $\left\{\frac{\partial}{\partial x_i}(p)\right\}$.

Example I.0.3

Consider the simplest nontrivial manifold, that is $M = S^1$. Consider two charts $\varphi^{-1}, \psi^{-1} : (-\pi, \pi), (0, 2\pi) \to S^1$ given by $\theta \mapsto e^{i\theta}$. These cover S^1.

We can then take

$$X(\varphi^{-1}(t)) = \sin(t)\frac{\partial}{\partial t}$$

and choose a compatible function for $X(\psi^{-1}(t))$.

Note: If $E \xrightarrow{\pi} M$ is a vector bundle and $\sigma_1, \sigma_2 : M \to E$ are sections, then for any functions $f_1, f_2 : M \to \mathbb{R}$ We can take

$$(f_1\sigma_1 + f_2\sigma_2)(p) = f_1(p)\sigma_1(p) + f_2(p)\sigma_2(p).$$

In algebraic terms, this means C^k-sections of a vector bundle form a module over $C^k(M)$, which is the ring of C^k functions $M \to \mathbb{R}$.

Definition I.0.5

A linear map $\partial : C^\infty(M) \to \mathbb{R}$ is called a derivation at p provided that for all $f, g \in C^\infty(M)$ we have

$$\partial(f \cdot g) = f(p)\partial(g) + \partial(f)g(p).$$

To spell out linearity we want for $c \in \mathbb{R}$ that

$$\partial(cf) = c\partial(f)$$
$$\partial(f + g) = \partial(f) + \partial(g).$$

For Non-Michigan students: 115/215 are single/multi-variable calculus.

Example I.0.4 (115/215 Example)

Take $M = \mathbb{R}$. The simplest derivation is $\partial(f) = f'(p)$.

For $f : \mathbb{R}^n \to \mathbb{R}$ we can take the directional derivative over a vector $v \in T_p\mathbb{R}^n$.

Likewise for $f : M \to \mathbb{R}$ where M is a C^∞ manifold for any $(p, v) \in T_pM$ we can take $\partial_v(f) := Df_p(v)$.

This gives us a derivation $\partial_v : C^\infty(M) \to \mathbb{R}$.

Remark I.0.1

One can alternatively frame tangent vectors in terms of derivations on a manifold. Professor Spatzier thinks this is beautiful and also useless. One must always eventually work with charts or use a Lie group structure.

Note for the very interested reader: for C^r manifolds where $r < \infty$ these two notions are not actually equivalent, see [NW56] and this mathoverflow post

https://mathoverflow.net/a/358273