HAPPY HALLOWEEN

Recall .0.1

Last time we began to consider the relationship between Lie groups and Lie algebras. One statement was that if $H \subseteq G$ are Lie groups, then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.

This holds because if V is an H-left-invariant vector field, then $W(g) := D(g \cdot -)_1 \cdot V(1)$ is a G-left-invariant vector field agreeing with V at points in H.

Furthermore, both the $[-,-]^G$ and $[-,-]^H$ are just brackets of vector fields (so they agree), and thus if $X, Y \in \mathfrak{h}$ then $[X, Y] \in \mathfrak{h}$.

Lemma .0.1

$[aF, bG] = aF(b)G - bG(a)F + ab[F, G]$.

Proof. Just expand as derivations to get

$$\begin{aligned} aF(bG) - bG(aF) &= aF(b)G + abFG - bG(a)F - baGF \\ &= aF(b)G - bG(a)F + ab(FG - GF) \\ &= aF(b)G - bG(a)F + ab[F, G]. \end{aligned}$$

Proof of the relationship of Lie groups/Lie algebras. We've just done the forward direction (easy exercise as well).

For the backwards direction, take $\mathfrak{h} \subseteq T_1 G$ (since it is G-left-invariant vector fields). Then we may take a distribution $V(g) = DL_g(\mathfrak{h})$ (where L_g is left translation by $g \in G$).

Claim

V is involutive, i.e. if X_1, X_2 are tangent to V then $[X_1, X_2]$ is tangent to V.

Let $z_1, \ldots, z_k$ form a basis of $\mathfrak{h}$ (which has dimension k). Thus z_i are left invariant vector fields with $[z_i, z_j] \in \mathfrak{h}$.

Then $X = \sum a_i z_i$ and $Y = \sum b_j z_j$, and by the lemma

$$[X, Y] = \sum_i f_i z_i + \sum_{i,j} g_{ij}[z_i, z_j]$$

for some functions f_i, g_{ij}, and this lies in V as desired.

Claim

Thus V is integrable.

When we did Frobenius we only did it locally...we need a global foliation. Take local charts $\mathcal{F}_{\text{loc}}(p)$, we must define a global foliation $\mathcal{F}$ (with global leafs).

Namely say $q \in \mathcal{F}_{\text{loc}}(p)$ we want $\mathcal{F}_{\text{loc}}(p) \cup \mathcal{F}_{\text{loc}}(q)$.

We need a quick lemma that if $q \in \mathcal{F}_{\text{loc}}(p)$ then for any neighborhood of U of q where $\mathcal{F}_{\text{loc}}(p)$ and $\mathcal{F}_{\text{loc}}(q)$ both are defined they must agree. This works because both are tangent to V.

Then we can consider $\mathcal{F}^1(p) = \mathcal{F}_{\text{loc}}(p)$ and

$$\mathcal{F}^{n+1}(p) = \bigcup_{q \in \mathcal{F}^n(p)} \mathcal{F}_{\text{loc}}(q),$$

and take $\mathcal{F}(p) = \bigcup \mathcal{F}^{n+1}(p)$. This construction gives a path-connected global leaf.

Frobenius then says $\mathfrak{h}$ is given by a foliation $\mathcal{F}$. Then we can set $H = \mathcal{F}(1)$. It remains to check H is a subgroup, since it is a smooth submanifold of G.

Suppose $h \in H$, then

$$\begin{aligned} h \cdot H = L_h H &= L_h(\mathcal{F}(1)) \\ &= \mathcal{F}(h \cdot 1) = \mathcal{F}(h) \\ &= \mathcal{F}(1) = H, \end{aligned}$$

where we have used that the vector field is left-invariant to get left-invariance of the foliation. If $h \in H$, is $h^{-1} \in H$? Well we know $h^{-1}h = 1$, so

$$h^{-1}\mathcal{F}(h) = \mathcal{F}(1),$$

but then $1 \in \mathcal{F}(h)$, so $h^{-1} \in \mathcal{F}(1) = H$.

By construction we know $T_1(H) = T_1(\mathcal{F}(1)) = V(1) = \mathfrak{h}$, as desired.

Facts: Know given a Lie group G, we can give a Lie algebra $\mathfrak{g}$. We can ask the converse question if we define a Lie algebra in general

Definition .0.1

A Lie algebra $\mathfrak{g}$ is a vector space over a field F equipped with a bilinear operation $[-,-]$ satisfying

- $[x, x] = 0$
- $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$. for all $x, y, z \in \mathfrak{g}$.

Then given any *finite-dimensional* Lie algebra $\mathfrak{g}$ is there a Lie group G with Lie algebra $\mathfrak{g}$ and how many? Answer:

- Yes you can find one
- No it is not unique
- But it's sort of unique. If G_1, G_2 both give rise to $\mathfrak{g}$ then the universal covers $\widetilde{G}_1$ and $\widetilde{G}_2$ coincide and both give rise to $\mathfrak{g}$.