**Lemma .0.1**

If $v \in T_pM$, then $\partial_v(f) \coloneqq D_p f \cdot v$ is a derivation. Moreover if $\partial_v = \partial_w$, then $v = w$, where $v, w \in T_pM$.

*Proof Idea.* We've already seen the first property. For the second, take a coordinate chart $(U_\rho, \rho)$. We can take $D\rho_p(v) = \frac{\partial}{\partial x_1}$ and $D\rho_p(w) = \frac{\partial}{\partial x_1}$. We can do this unless $w = a \cdot v$. This works because we have linearly independent vectors $D\rho_p(v), D\rho_p(w)$ and take a linear map $A$ taking these to $e_1, e_2$. Replace $\rho$ by $A \circ \rho$.

Look at $x_1 : \rho(U_\alpha) \to \mathbb{R}$, which is the coordinate (projection) map to the first coordinate. We see that

$$\frac{\partial}{\partial x_1} x_1 = 1$$
$$\frac{\partial}{\partial x_2} x_1 = 0.$$

This will show $\partial_v, \partial_w$ disagree on this function... but wait! $x_1$ is only defined on a small neighborhood $U_\rho$ of $p$.

We need to understand the relationship between $C^\infty(M)$ and $C^\infty(U)$ for $U$ a neighborhood of some $p \in M$. We know one map

$$C^\infty(M) \longrightarrow C^\infty(U)$$

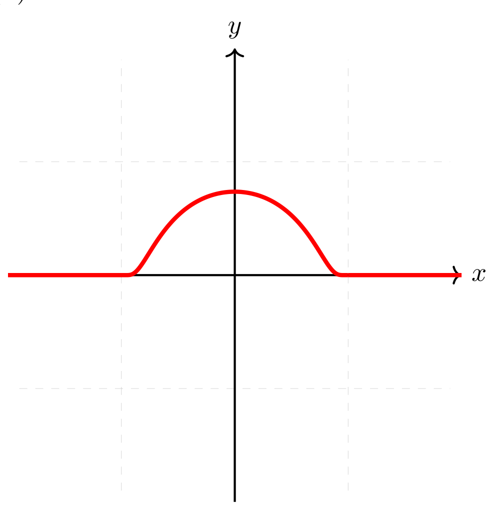
$$f \longmapsto f\big|_U$$

$$\overline{g} \xleftarrow[?]{} g$$

so that $\overline{g}\big|_U = g$.

Need: Bump functions

- Ad: This comes up a lot
- On $\mathbb{R}$ we want a function $f(x)$ that looks like




Warning: Cannot do in $C^\omega(\mathbb{R})$.

- We take

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } x \in (-1,1) \\ 0 & \text{if } x \notin (-1,1) \end{cases}.$$

- Likewise for $\overline{\psi} : \mathbb{R}^n \to \mathbb{R}$, where we want $\overline{\psi} \equiv 0$ outside $B_1(0)$ and $\overline{\psi}$ is $C^\infty$ on $\mathbb{R}^n$, and $\neq 0$ on $B_1(0)$. We take $\overline{\psi}(x) = \psi(|x|^2)$.
- We can generalize this. Want $\phi \equiv 1$ on $B_{1/2}(0)$, $\phi \equiv 0$ outside $B_1(0)$. We pick something like $\overline{\psi}(x) \cdot \psi\left(\frac{1}{2|x|^2}\right)$.
- We can use these to construct our $\overline{g}$.

We can prove Lemma .0.1 by taking everything locally, and using bump functions

**Corollary .0.2**

There exists $n$ linearly independent derivations since $T_pM \hookrightarrow \{\text{derivations at } p\}$.

**Lemma .0.3**

Derivations at $p$ form an $n$-dimensional vector space.

*Proof.* It is at least $n$-dimensional since $\{\delta_v \mid v \in T_pM\}$ is $n$-dimensional.

Now for $f \in C^\infty(M)$, we can use bump functions to look locally at $C^\infty(U_\rho)$, and work locally around 0 in $\mathbb{R}^n$. Consider

$$\mathcal{I}_p = \{f \in C^\infty \text{ near } p \mid f(p) = 0\}$$
$$\mathcal{I}_p^2 = \{\sum f_i g_i \mid f_i g_i \in \mathcal{I}_p\}$$

Then we see that

$$\delta(f \cdot g) = \delta(f) \cdot g(p) + f(p) \cdot \delta(g) = \delta(f) \cdot 0 + 0 \cdot \delta(g) = 0.$$

Therefore, the derivations vanish on $\mathcal{I}_p^2$. Thus the derivations embed into $(\mathcal{I}_p/\mathcal{I}_p^2)^*$ (which is a vector space.

We prove another result Corollary .0.7 to finish this proof.

**Corollary .0.4**

We have that

$$\{\text{derivations at } p\} = \{\partial_v \mid v \in T_pM\} = (\mathcal{I}_p/\mathcal{I}_p^2)^*$$

by equality of dimensions.

**Lemma .0.5**

Suppose $f : U \to \mathbb{R}$, $C^\infty$ with $0 \in U$ open in $\mathbb{R}^n$, then there exist $C^\infty$ functions $f_i$ on $U$ such that

$$f(x) = f(0) + x_i f_i,$$

with $f_i(0) = \frac{\partial f}{\partial x_i}(0)$.

*Proof.* By Fundamnetal theorem of calculus, we have that

$$\begin{aligned} f(x) &= f(0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tx)\,\mathrm{d}t \\ &= f(0) + \int_0^1 \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx)\,\mathrm{d}t \\ &= f(0) + \sum_{i=1}^n x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\,\mathrm{d}t}_{f_i \in C^\infty}. \end{aligned}$$

**Lemma .0.6**

Suppose $f$ is $C^\infty$ in $\mathbb{R}^n$ near 0. Then there exist $C^\infty$ functions $f_{ij}$ on $\mathbb{R}^n$ near 0.

$$f(x) = f(0) + \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j=1}^n x_i x_j f_{ij}(x).$$

*Proof.* Apply last lemma to $f_i(x)$.

**Corollary .0.7**

$\dim(\mathcal{I}_p/\mathcal{I}_p^2) = n$.

*Proof.* Apply the lemma just above. For any $\delta \in (\mathcal{I}_p/\mathcal{I}_p^2)^*$ we can take locally

$$\delta(f) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(0) \cdot \delta(x_i),$$

since $\delta$ vanishes on $\mathcal{I}_p^2$, so we can say it vanishes on $x_i x_j f_{ij}(x)$ above. The proof in general is similar.

Dealing with germs of functions. Fix $p \in M$ a $C^\infty$ manifold. Suppose $f \in C^\infty$ is defined in an open neighborhood $U_f$ of $p$, and $g \in C^\infty$ is defined on $U_g \ni p$. We say $f, g$ define the same germ if

$$f\big|_{U_f \cap U_g} = g\big|_{U_f \cap U_g}.$$

Note: $f_1, f_2$ having the same germ $[f]$ and $g_1, g_2$ having the same germ $h$, then $f_1 + g_1, f_2 + g_2$ define the same germ. Really $f_1, f_2$ having the same germ defines an equivalence relation.

Partial derivatives are well-defined on germs. They're somewhere "between local and infinitesimal." So note, we have

(1) Globally defined functions $f \in C^\infty(M)$
(2) Locally defined functions $f$ on some open neighborhood of $p$.
(3) Germs at $p$
(4) Partial derivatives at $p$.

where the order reflects closer and closer to infinitesimal information. Note that $\mathcal{I}_p, \mathcal{I}_p^2$ make sense for germs. Furthermore our above discussion tells us

$$\dim \mathcal{I}_{p,\text{germ}}/\mathcal{I}_{p,\text{germ}}^2 = n.$$

It also has the falling property. Given $\varphi : M \to N$, and a germ $[f]$ on $N$. Then $[f \circ \varphi]$ in fact defines a germ on $N$. In representatives this takes $f : N \supseteq U \to \mathbb{R}$ and compose $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$. This does not depend on the representatives. We then get a map

$$\begin{aligned} \varphi^* : \mathcal{I}_{\varphi(p)}/\mathcal{I}_{\varphi(p)}^2 &\to \mathcal{I}_p/\mathcal{I}_p^2 \\ [f] &\mapsto [f \circ \varphi]. \end{aligned}$$

We have a duality! In diagrams we have

$$\begin{tikzcd} T_pM \arrow[r, "D\varphi=\varphi_*"] \arrow[d, leftrightarrow] & T_{\varphi(p)}N \arrow[d, leftrightarrow] \\ (\mathcal{I}_p/\mathcal{I}_p^2)^* \arrow[r, "(\varphi^*)^*"] & (\mathcal{I}_{\varphi(p)}/\mathcal{I}_{\varphi(p)}^2)^* \\ \mathcal{I}_p/\mathcal{I}_p^2 & \mathcal{I}_{\varphi(p)}/\mathcal{I}_{\varphi(p)}^2 \arrow[l, "\varphi^*"'] \end{tikzcd}$$

because of our discussion above concerning identifying $T_pM$ with derivations and derivations with the middle row. Then we can think of all this as

$$T_p^*M = \mathcal{I}_p/\mathcal{I}_p^2$$

which is the cotangent space at $p$. Then we have a duality

$$\varphi^* : T^*N \to T^*M$$

$$\varphi_* : T_*M \to T_*N.$$