Theorem .0.1

Let M be a smooth manifold then the following are equivalent

(a) M is orientable.

(b) M has an atlas of coordinate charts so that the transition maps are orientation preserving. In other words $\det DT_{\alpha\beta} > 0$ when it is defined for each transition map $T_{\alpha,\beta}$

Lemma .0.2

Let $\mu \in \Lambda^n V^*$ and $\{e_1, \dots, e_n\}$ be a basis of V with $(a_{ij}) = A$ an $n \times n$ matrix so that

$$f_i = \sum_{j=1}^{n} a_{ij} e_j,$$

then

$$\mu(f_1, \dots, f_n) = (\det A)\mu(e_1, \dots, e_n).$$

Proof. We see that

$$\mu\left(\sum_{j=1}^{n} a_{1j}e_j, \dots, \sum_{j=1}^{n} a_{nj}e_j\right) = \sum_{j_1,\dots,j_n} a_{1j_1} \cdots a_{nj_n} \mu(e_{j_1}, \dots, e_{j_n}).$$

If any two of $j_1, \dots, j_n$ are the same then the term corresponding to this choice is zero. We can then rewrite this as

$$\begin{aligned}\mu\left(\sum_{j=1}^{n} a_{1j}e_j, \dots, \sum_{j=1}^{n} a_{nj}e_j\right) &= \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \mu(e_{\sigma(1)}, \dots, e_{\sigma(n)}) \\ &= \left(\sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}\right) \mu(e_1, \dots, e_n) = \det(A)\mu(e_1, \dots, e_n).\end{aligned}$$

For forms we have if $F : W \to V$ is linear then

$$F^* : \Lambda^k V^* \to \Lambda^k W^*.$$

Then of course

$$(F^*\alpha)(w_1, \dots, w_k) = \alpha(F(w_1), \dots, F(w_k)).$$

Lemma .0.3

Suppose $\dim V = \dim W = n$, with V, W vector spaces. Further, let $F : W \to V$ be linear, $e_1, \dots, e_n$ be a basis of V, $f_1, \dots, f_n$ be a basis of W, $\varepsilon_1, \dots, \varepsilon_n$ the dual basis of V^*, $\phi_1, \dots, \phi_n$ the dual basis of W^*.

Let A be the matrix with respect to this basis. We know

$$F(f_i) = \sum_{j=1}^{n} a_{ij} e_j.$$

In this case we have

$$F^*(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) = (\det A)(\phi_1 \wedge \cdots \wedge \phi_n).$$

Proof. Apply the above lemma. Namely evaluate the left hand side at $f_1, \ldots, f_n$ and show you get $\det A$ using previous lemma (which is enough since $\dim \Lambda^n W^* = 1$).

Definition .0.1

Suppose M, N are manifolds and $\Phi : M \to N$ is a smooth map. We define

$$\Phi^* \Lambda^k N \to \Lambda^k M$$

with $\alpha \in \Lambda^k N$, $p \in M$, $v_1, \ldots, v_n \in T_p M$ via

$$(\phi^*\alpha)_p(v_1, \ldots, v_n) = \alpha_{\Phi(p)}(D\Phi_p \cdot v_1, \ldots, D\Phi_p \cdot v_n).$$

This is called the pullback of a differential form.

Lemma .0.4

Let $T : U \to V$ be a diffeomorphism where $U, V \subseteq \mathbb{R}^n$. Let $x_1, \ldots, x_N$ be coordinates in V, $x_1, \ldots, x_n$ coordinates for U. Here $(\mathrm{d}x_i)_q$ is dual to $\left(\frac{\partial}{\partial x_i}\right)_q$.

We then have that

$$(T^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n))_p = (\det DT_p)(\mathrm{d}y_1 \wedge \cdots \mathrm{d}y_n)_p$$

Proof. Apply previous lemma.