Aside: maps of constant rank. A submersion gives rise to level sets. For $f : M \to N$ a C^k map consider

$$p \mapsto \operatorname{rank} Df_p \coloneqq \dim \operatorname{Image} Df_p.$$

f is a submersion if and only if this map is constant and the rank of f is always $\dim N$.

Definition .0.1

f has constant rank if $p \mapsto \operatorname{rank} Df_p$ is constant in p.

Theorem .0.1 (Constant Rank Theorem)

If f has constant rank, then $f^{-1}(q)$ is a C^k=submanifold

Idea of Proof. Locally we can take a projection g from $f(M)$ to $\operatorname{Image} Df_p$, which is a linear subspace in coordinates. This is a submersion, and then we use local submersion theorem.

Consider C^∞ vector fields X, Y on a manifold M. We know

$$[X, Y] \coloneqq X \circ Y - Y \circ X$$

defines a vecto field, where we view these as derivations. Now consider a special situation, where X, Y don't vanish on M (or some $U \subseteq M$ open).

Proposition .0.2

Assume $[X, Y] = 0$. Then there exist coordinates on a chart such that $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$.

More generally, if $X_1, \ldots, X_k$ are vector fields so that $\{X_i(p)\}$ are linearly independent and $[X_i, X_j] = 0$ for all i, j, then there exist coordinate charts about p so that $X_i = \frac{\partial}{\partial x_i}$.

Idea of Proof. If $k = 1$, then $X \rightsquigarrow \varphi_t$ a local flow of X. Let $\overline{U} \subseteq U$ so that the flow is defined.

Pick T a submanifold of dimension $m - 1$. Then $T = \{(0, y_2, \ldots, y_n)\}$ in some coordinates. We may also pick T so that it is transversal to $X(p)$. Give coordinates on $\overline{U}$ as

$$\Phi : (t, y_2, \ldots, y_n) \mapsto \varphi_t(0, y_2, \ldots, y_n).$$

It suffices to check $D\Phi_{(0,\ldots,0)}$ is a local diffeomorphism at p (which is 0 in local coordinates), and then Φ is a chart. We compute

$$D\Phi_{(0,\ldots,0)} = \begin{pmatrix} * & \cdots & 0 & \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1. \end{pmatrix}$$

(invertibility and this computation comes from transversality.

If $k = 2$, Let $[X, Y]$. Without loss of generality, $X = \frac{\partial}{\partial x}$ in local coordinates. We'll cheat and look at $\dim M = 2$ (it will be clear how to generalize).

We may then let $Y = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$. We compute that

$$\begin{aligned}[X,Y] &= \frac{\partial}{\partial x}\left(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}\right) - a(x,y)\frac{\partial^2}{\partial x^2} - b(x,y)\frac{\partial}{\partial x \partial y} \\ &= \frac{\partial a}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b}{\partial x}(x,y)\frac{\partial}{\partial y}.\end{aligned}$$

Thus $a(x,y) = a(y), b(x,y) = b(y)$. Let ψ_t be a local flow for y. We know

$$\Psi(x,t) = \psi_t(x,0).$$

Then we can use this as a coordinate chart. If $\dim M > 2$, take a submanifold (local) through p of codimension which is tranversal to both X, Y. Then apply the same trick as when $k = 1$. We also take a flow ϕ_t and set

$$\Psi(s,t,z) = \Psi_t \circ \varphi_s(z).$$

You can then do it for any number of vector fields.

For k and $\dim M$ arbitrary. Find a transversal submaniofld to $X_1, \ldots, X_k$ at p. We then compose flows just as above

$$(t_1, \ldots, t_n, z) \mapsto \varphi^1_{t_1} \circ \cdots \circ \varphi^n_{t_n}(z).$$

Because the flows commute (Lie bracket zero) this will give exactly what we need. We can move $\phi^i_{t_i}$ to the front. This is a corollary of the discussion for $k = 2$ (where we take the derivatives explicitly).

One can always find tranversals because we're only working locally, and so in coordinates we can just take a linear subspace tranversal to $X_i(p)$ for all i.

Corollary .0.3

If $[X,Y] = 0$ then their local flows commute.

Proof. Look at the case $k = 2$ above.

Consider any X, Y to be C^∞ vector fields. Let φ, ψ be local flows for X, Y. Then we can consider $C(t)$ to be defined as

$$C(t) = \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}}(p)$$

for $t > 0$. This is in Spivak's text on differentiable manifolds. We define $C(-t)$ similarly but flowing in the opposite direction.

Theorem .0.4

Let X, Y be C^∞ vector fields. Then $C(t)$ is differentiable and $C'(0) = [X,Y](p)$.