Proof of **??**. We now prove that topological manifolds are paracompact. We first find an exhaustion by compact sets, $K_1 \subsetneq K_2 \subsetneq \cdots$ with $\bigcup K_j = M$. We set $V_j \coloneqq K_{j+1} \setminus (\operatorname{Int} K_j)$. Now let $W_j \coloneqq \operatorname{Int} K_{j+2} \setminus K_{j-1}$.

Note that V_j, W_j are compact/open respectively. Consider an open cover χ, given $x \in M$, let $\chi_x \in \chi$ be a set containing x. Take $\mathcal{B}$ a countable basis, and find $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq \chi_x$.

The V_j are compact, so there are finitely many B_{x_i} which cover V_j. Thus $\{B_{x_\ell}\}$ are a refinement of χ. We can also require $B_{x_\ell} \subseteq W_j$. This will immediately imply locally finite.

Deep Fact from 100 years ago which we will not prove right now. Namely if $\varphi : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^\ell$ is a homeomorphism and U, V are open then $n = \ell$. This is called "invariance of domain theorem." A special case is $\varphi : (a, b) \to U \subseteq \mathbb{R}^\ell$, then $\ell = 1$ (disconnect (a, b) by removing a single point). In 592 (algebraic topology) you can generalize this argument using loops or homology.

Corollary .0.1

Dimension of a connected topological manifold is well-defined. Namely $\dim M = \ell$ if for every point p, there is a neighborhood of p which is homeomorphic to $\mathbb{R}^\ell$.

Convention: on any connected component the dimension is well-defined, and we assume the dimesnion is constant across connected components in this class.

Proposition .0.2

Let M be a topological manifold, then M is connected if and only if it is path-connected.

Proof. Forward direction is the difficult piece. Fix $p \in M$, take $X = \{y \in M \mid \text{there exists a path from } p \text{ to } y\}$. We will prove X is clopen, so then since $p \in X$, $X = M$.

Take $y \in X$, then taking a neighborhood of y which is homeomorphic to $\mathbb{R}^n$, we see that within this neighborhood we're path-connected, so X is open. Likewise if $z \in \overline{X}$, then take a neighborhood of z homeomorphic to $\mathbb{R}^n$, this intersects X, and so $z \in X$. Thus X is closed.

.1. Definition of a Differentiable Manifold

Definition .1.1

Suppose M is a topological manifold of dimension n, we call it a differentiable manifold if it has a differentiable (C^k, $k = 1, \ldots, \infty, \omega$ [analytic behaves differently]) structure.

Namely, we require that there exists a cover by open sets U_i and homeomorphisms $\varphi_i : U_i \to V_i \subseteq \mathbb{R}^n$ of M such that for each i, j the map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \circ U_j) \to \varphi_j(U_i \circ U_j)$ between open subsets of $\mathbb{R}^n$ is differentiable (C^k). These maps are called transition maps, and this data $\{(U_j, \varphi_j)\}_j$ is called an atlas.

We often call C^∞ manifolds smooth manifolds.

Example .1.1

For spheres S^n you can take enlargened hemispheres and do stereographic projection. In fact we can take $S^n \setminus \{P\}$ and $S^n \setminus \{Q\}$ where P, Q are the north,south poles. The transition map is algebraic and well-defined, so it's differentiable (for $n = 2$ it's $z \mapsto 1/z$).