Notes on MATH 591 (Differential Topology)

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Contents

I. Introduction/Administration

- Professor: Ralf Spatzier
- Office Hours:
  - Monday 11-12
  - Tuesday 5-6, EH 4088
  - Friday 11-12
  - By appointment.
- HW: Due Wednesdays

II. Definitions and Building Blocks

Definition II.0.1

M is called locally euclidean provided that for all $p \in M$ there exists a neighborhood U of p and a homeomorphism $U \to \mathbb{R}^n$ for some n.

Lemma II.0.1

It is good enough that for all $p \in M$ there exists a neighborhood V of p such that V is homeomorphic to an open subset in $\mathbb{R}^n$.

Proof. Take $V \xrightarrow{\phi} V^* \subseteq \mathbb{R}^n$ with V^* open. Then there is an open ball $U^* \subseteq V^*$ containing p, and so we can take $U = \phi^{-1}(U^*) \cong U^*$. It is clear from real analysis that $U^* \cong \mathbb{R}^n$.

II.1. Paracompactness

Definition II.1.1

Consider a collection of subsets χ of M. χ is called locally finite provided that each point $p \in M$ has a neighborhood U intersecting only finitely many $C \in \chi$.

Definition II.1.2

A topological space M is called paracompact if every open cover χ of M admits a locally finite open subcover.

Recall II.1.1

A cover of M is a collection of χ such that $\bigcup_{C \in \chi} C = M$. A subcover of a cover χ of M is a collection $\chi^* \subseteq \chi$ such that every $C^* \in \chi^*$ is contained in some $C \in \chi$.

χ^* is also called a refinement. A cover χ is open if every element of χ is an open set.

Definition II.1.3

M is called locally compact if every point $p \in M$ and neighborhood U of p there exists a neighborhood $V \subseteq U$ such that $\overline{V} \subseteq U$ (the closure) is compact.

Lemma II.1.1

Topological manifolds are locally compact.

Proof. They are locally euclidean and $\mathbb{R}^n$ is locally compact.

Theorem II.1.2

Topological manifolds are paracompact.

Proposition II.1.3

A 2nd countable locally compact Hausdorff space admits an exhaustion by compact sets.

Definition II.1.4

An exhaustion is a sequence of sets $K_n \subseteq K_{n+1}$ with $\bigcup K_n = M$.

Proof of Proposition II.1.3. In the appendix of Lee [**lee**]. We repeat it here. There is a basis of precompact open sets since M is locally compact. We should extract countably many precompact open sets $\{U_i\}_{i\in\mathbb{N}}$ such that $\bigcup U_i = M$.

By second countability, let $\{W_j\}_{j\in\mathbb{N}}$. Then taking $p \in M$, we know $p \in W_j$. There then exists a precompact neighborhood $\overline{V} \subseteq W_j$ which is compact. Take sets $\{W_{V_i}\}$ whose union contains $\overline{V}$. Then take finitely many such precompact open sets $W_{V_1}, \dots, W_{V_N}$. It is possible to make the previous argument happen in some neighborhood $\mathcal{O}$ which is precompact. Thus we have $W_{V_i} \subseteq \mathcal{O}$, so the $\overline{W_{V_i}} \subseteq \overline{\mathcal{O}}$ are compact (by Hausdorffness).

Let's define the exhaustion by compact sets. First let $\{U_i\}_{i\in\mathbb{N}}$ be precompact open sets covering M. Let $K_m = \bigcup_{i=1}^{m} \overline{U_i}$.

Exercise II.1.2

Think of how to define a differentiable manifold.

The idea is as follows.

Definition II.1.5

If $p \in M$ and $p \in U \xrightarrow{\varphi} U^* \subseteq \mathbb{R}^n$ where U^* is open is a homeomorphism, then (U, φ) is called a coordinate chart at U.

Definition II.1.6

Call charts (U, φ) and (V, ψ) compatible if $\varphi \circ \psi^{-1}\big|_{U^* \cap V^*}$ is differentiable

Proof of Theorem II.1.2. We now prove that topological manifolds are paracompact. We first find an exhaustion by compact sets, $K_1 \subsetneq K_2 \subsetneq \cdots$ with $\bigcup K_j = M$. We set $V_j \coloneqq K_{j+1} \setminus (\operatorname{Int} K_j)$. Now let $W_j \coloneqq \operatorname{Int} K_{j+2} \setminus K_{j-1}$.

Note that V_j, W_j are compact/open respectively. Consider an open cover χ, given $x \in M$, let $\chi_x \in \chi$ be a set containing x. Take $\mathcal{B}$ a countable basis, and find $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq \chi_x$.

The V_j are compact, so there are finitely many B_{x_i} which cover V_j. Thus $\{B_{x_\ell}\}$ are a refinement of χ. We can also require $B_{x_\ell} \subseteq W_j$. This will immediately imply locally finite.

Deep Fact from 100 years ago which we will not prove right now. Namely if $\varphi : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^\ell$ is a homeomorphism and U, V are open then $n = \ell$. This is called "invariance of domain theorem." A special case is $\varphi : (a,b) \to U \subseteq \mathbb{R}^\ell$, then $\ell = 1$ (disconnect (a,b) by removing a single point). In 592 (algebraic topology) you can generalize this argument using loops or homology.

Corollary II.1.4

Dimension of a connected topological manifold is well-defined. Namely $\dim M = \ell$ if for every point p, there is a neighborhood of p which is homeomorphic to $\mathbb{R}^\ell$.

Convention: on any connected component the dimension is well-defined, and we assume the dimesnion is constant across connected components in this class.

Proposition II.1.5

Let M be a topological manifold, then M is connected if and only if it is path-connected.

Proof. Forward direction is the difficult piece. Fix $p \in M$, take $X = \{y \in M \mid \text{there exists a path from } p \text{ to } y\}$. We will prove X is clopen, so then since $p \in X$, $X = M$.

Take $y \in X$, then taking a neighborhood of y which is homeomorphic to $\mathbb{R}^n$, we see that within this neighborhood we're path-connected, so X is open. Likewise if $z \in \overline{X}$, then take a neighborhood of z homeomorphic to $\mathbb{R}^n$, this intersects X, and so $z \in X$. Thus X is closed.

II.2. Definition of a Differentiable Manifold

Definition II.2.1

Suppose M is a topological manifold of dimension n, we call it a <u>differentiable manifold</u> if it has a differentiable (C^k, $k = 1, \ldots, \infty, \omega$ [analytic behaves differently]) structure.

Namely, we require that there exists a cover by open sets U_i and homeomorphisms $\varphi_i : U_i \to V_i \subseteq \mathbb{R}^n$ of M such that for each i, j the map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \circ U_j) \to \varphi_j(U_i \circ U_j)$ between open subsets of $\mathbb{R}^n$ is differentiable (C^k). These maps are called <u>transition maps</u>, and this data $\{(U_j, \varphi_j)\}_j$ is called an <u>atlas</u>.

We often call C^∞ manifolds <u>smooth manifolds</u>.

Example II.2.1

For spheres S^n you can take enlargened hemispheres and do stereographic projection. In fact we can take $S^n \setminus \{P\}$ and $S^n \setminus \{Q\}$ where P, Q are the north,south poles. The transition map is algebraic and well-defined, so it's differentiable (for $n = 2$ it's $z \mapsto 1/z$).

Bonus Problems: Due 1 week after regular problems, if Gradescope complains let the professor know.

Idea for Problem 1 from Homework 1

Let M be a 1-dimensional, compact, connected manifold.

Choose charts U_x around each $x \in M$ homeomorphic to intervals. Choose a finite subcover $U_i \cong (a_i, b_i)$ and suppose this is a <u>minimal</u> such cover. Make a lemma that if $U_i \cup U_j$ is connected

Standards for what constitutes a proof will be determined over time by a conversation between the students, the grader, and the professor.

Deep Theorem

Theorem II.2.1

There exist topological manifolds which do not admit any differentiable structure.

In fact: "Piecewise linear" manifolds of such type exist.

Next Question: Can two differentiable manifolds give the same topological manifolds?

Yes! Milnor in the late 50s constructed exotic S^7 i.e. $M_1, \dots, M_{28}$ all homeomorphic but none of them diffeomorphic. There are infinitely many higher dimensional spheres exhibiting this behavior, though it does not occur in dimensions ≤ 6. There are even uncountably many differentiable structures on $\mathbb{R}^4$!!!

Definition II.2.2

If Ω_1, Ω_2 are differentiable structures (aka atlases) on M, we say that Ω_1, Ω_2 are compatible if $\Omega_1 \cup \Omega_2$ is a differentiable structure.

A maximal atlas is the union of all compatible atlases.

Bourbakian method: Always use maximal atlas vs. Hands-on approach: find your atlas, work with it.

Proposition II.2.2

Suppose M, N are both differentiable manifolds, then $M \times N$ is a differentiable manifold whose dimension is $\dim M + \dim N$.

Proof. Say $(U_\alpha, \varphi_\alpha)_\alpha$ is an atlas for M, $(V_\beta, \psi_\beta)_\beta$ an atlas for N. Then $\{(U_\alpha \times V_\beta, (\varphi_\alpha, \psi_\beta)\}_{\alpha,\beta}$ is an atlas for $M \times N$.

What to check?

- Clearly a cover of $M \times N$.
- $(\varphi_\alpha, \psi_\beta) : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$.
- Check compatibility of charts which is similarly clear.

Quotients on the other hand, are very ill-behaved. As you know, Hausdorff spaces can have non-Hausdorff quotients, and differentiable structure is also not respected.

Example II.2.2

Take $S^1 \subseteq \mathbb{C}$, let $x \sim y$ provided that $xy^{-1} = e^{i2n\alpha\pi}$, where $\alpha \in \mathbb{R}$ is fixed, $n \in \mathbb{Z}$. In fact $S^1/\sim \cong S^1$ when $\alpha \in \mathbb{Q}$.

However when $\alpha \notin \mathbb{Q}$, the equivalence class of any x is dense in S^1, so $S^1/\sim$ is not Hausdorff. In fact

Claim

$\emptyset$ and $S^1/\sim$ are the only open sets.

Proof. Suppose $U \subseteq S^1/\sim$ is open. Assume U is nonempty, and let $V := \pi^{-1}(U)$ where $\pi : S^1 \to S^1/\sim$. Then fix $z \in S^1$, we see that $[z] \cap V$ is nonempty since $[z]$ is dense and V is a nonempty open set.

Thus z is equivalent to something in V, so $[z] \in U$. Thus $U = S^1/\sim$.

Note the quotient map is indeed open as well by density.

Definition II.2.3

Let X be some space and $\sim$ an equivalence relation on X. We call $\sim$ open provided that $\pi : X \to X/\sim$ is an open map.

Equivalently, if $U \subseteq X$ is open, then $\{x \in X \mid \exists y \in U, x \sim y\} = \pi^{-1}(\pi(U))$ is open.

Fact: If X is second countable, then $\sim$ is an open equivalence relation if and only if $X \xrightarrow{\pi} X/\sim$ is an

Lemma II.2.3

In a compact Hausdorff space, any two disjoint closed sets can be separated by open neighborhoods which contain these closed sets.

Proof. First we prove that if C is closed and $x \notin C$ then we can separate x, C. We know C is compact since X is compact and Hausdorff. For each $y \in C$, let U_y, V_y separate x, y. Then $\{V_y\}$ covers C, so we can take a finite subcover V_{y_i}. Let $V := \bigcup_i V_{y_i}$ and $U := \bigcap_i U_{y_i}$. U, V clearly separate x, C and are open.

Now take C, C' which are disjoint closed sets. For each $x \in C$, take neighborhoods U_x, V_x separating x, C'. Then by compactness take a finite subcover U_{x_i}. As before union the U_{x_i} and intersect the V_{x_i} to separate C, C'.

Theorem II.2.4

Take X as a topological space with $\sim$ open on X. Then

$$\text{graph} \sim := \Gamma := \{(x, y) \in X \times X \mid x \sim y\}$$

Then $X/\sim$ is Hausdorff if and only if $\Gamma \subseteq X \times X$ is a closed subset.

Proof. Proof of $\implies$ left as an exercise. Suppose $[x] \neq [y]$ within $X/\sim$. Then $(x, y) \in \Gamma^c$, so using the basis for the product topology, there are U, V open so that $(x, y) \in U \times V$ and $U \times V \subseteq \Gamma^c$.

Thus $\pi(U) \cap \pi(V)$ is empty, where $\pi : X \to X/\sim$. Furthermore, $\pi(U), \pi(V)$ are open, and so they separate $[x], [y]$.

Proposition II.2.5

If $\sim$ is open, X is second countable, then $X/\sim$ is second countable

Proof Idea. Take a countable basis of X and take their images. These are open, and it's easy to check this is a basis.

Aside: There is an interesting non-Hausdorff topology. Namely, the closed sets in $\mathbb{R}^n, \mathbb{C}^n$ (some algebraic variety) are given by zeros of polynomials.

Fact: This is compact.

II.3. Topological Groups/Homogeneous Spaces

Definition II.3.1

We say G is a topological group provided that G is a group equipped with a topology such that the maps

$$(g, h) \mapsto gh$$
$$g \mapsto g^{-1}$$

are continous as maps $G \times G \to G$ and $G \to G$.

Example II.3.1

$\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n, \mathbb{Z}, \mathbb{Z}^k$, and any discrete group.

Also $\mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{C})$ which are invertible matrices. $O(n)$, the orthogonal group which is the matrices so $A \cdot A^t = I$.

$\mathrm{SO}(n) = \{A \in O(n) \mid \det A = 1\}$, the rotations. Then $\mathrm{SL}_n(\mathbb{R})$ which are the matrices of determinant one.

The circle $S^1 = \mathrm{SO}(2)$. We may also consider $T^n = S^1 \times \cdots \times S^1$, n times and $T^\infty = S^1 \times \cdots \times S^1 \times \cdots$. T^∞ is compact (Tychonoff).

Example II.3.2

$\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm $\|\cdot\|_p$. Write $a = p^k c, b = p^\ell d$ where c, d are coprime to p, then

$$\left\|\frac{a}{b}\right\|_p = p^{\ell - k}.$$

Note $p^n \to 0$ as $n \to \infty$. Furthermore $\mathbb{Q}_p$ is a topological field.

We may look at $\mathrm{SL}(n, \mathbb{Q}_p), \mathrm{GL}(n, \mathbb{Q}_p)$ which have dimension 0.

Definition II.3.2

A topological group G is called a Lie group if it is equipped with a smooth differentiable structure such that the maps

$$(g, h) \mapsto gh \qquad\qquad g \mapsto g^{-1}$$

are smooth.

Definition II.3.3

Let M, N are differentiable manifolds and $f : M \to N$ is some continuous map. We call f differentiable (C^1, C^k, C^∞ aka smooth) provided that for any coordinate chart $(U_\alpha, \varphi_\alpha)$ around $x \in M$ and any coordinate chart (V_β, ψ_β) around $f(x) \in N$ we have

$$f_{\alpha,\beta} \coloneqq \psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi(U_\alpha \cap f^{-1}(V_\beta)) \subseteq \mathbb{R}^{\dim M} \to \psi_\beta(f(U_\alpha) \cap V_\beta) \subseteq \mathbb{R}^{\dim N}$$

is differentiable (C^1, C^k, C^∞) for all α, β. Notice these sets are nonempty as $x, f(x)$ respectively lie in each of them.

Lemma II.3.1

This is well-defined. That is, it is independent of compatible atlases.

Example II.3.3 (Examples of Lie Groups)

$\mathbb{R}, S^1, \mathbb{R}^n, T^n = S^1 \times \cdots \times S^1, \mathrm{GL}(n, \mathbb{R})$.

Note: Famously, S^2 is not a Lie group. In fact S^0, S^1, S^3 are the only spheres which are Lie groups. These correspond to unit norm in the real numbers, complex numbers, and quaternions.

Reason: Euler characteristic $\chi(S^2) = 2$, and there is a theorem

Theorem II.3.2

If a manifold M has a nonvanishing vector field (to be defined later) then $\chi(M) = 0$.

Aside: There exists an exotic $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$

Definition II.3.4

G is a group (possibly topological, Lie) acts (possibly continuously, smoothly) on a space X (possibly topological, smooth manifold) provided there exists a map (possibly continuous, smooth)

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot x$$

such that

$$1 \cdot x = x \qquad\qquad (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x).$$

Notation: One might say differentiable to mean each $X \to X$ induced by $x \mapsto g \cdot x$ is differentiable, and use *jointly* differentiable to mean $G \times X \to X$ is differentiable.

Example II.3.4

S^1 acts on S^1 by multiplication, $\mathbb{R}^n$ acts on $\mathbb{R}^n$ by addition, and importantly $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplicationu.

All examples given of group actions are jointly C^∞ (smooth).

Theorem II.3.3

If G is a compact topological group, G acts continuously on X, X is compact Hausdorff, then X/G is Hausdorff.

Proof. We must check $\sim$ is open and graph $\sim$ is closed. Let $\pi : X \to X/G$ be the quotient map.

Take $U \subseteq X$ open, then we see that

$$\pi^{-1}(\pi(U)) = \{y \in X \mid y \sim x\} = \{g \cdot u \mid u \in U, g \in G\}$$
$$= \bigcup_{g \in G} g \cdot U$$

is open because $g \cdot U = (g^{-1})^{-1}(U)$ is a preimage of a continuous map.

Now we must show Γ is closed. Look at $\varphi : G \times X \to X \times X$ given by $(g, x) \mapsto (x, gx)$. Note that $\operatorname{im} \varphi = \Gamma$. But wait! $G \times X$ is compact, so Γ is compact, so Γ is closed since $X \times X$ is Hausdorff.

Example II.3.5

Take $X = S^n, G = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 = \{1, A\}$. Then G acts on S^n, where $A \cdot x = -x$ for $x \in S^n$.

Then $S^n/G = \mathbb{R}\mathrm{P}^n$.

Example II.3.6

Consider $S^{2n-1} \subseteq \mathbb{C}^n$ with the action of S^1 on $\mathbb{C}^n$ via

$$e^{i\alpha} \cdot (z_1, \ldots, z_n) = (e^{i\alpha} z_1, \ldots, e^{i\alpha} z_n).$$

This is a continuous action on S^{2n-1} by S^1. Therefore

$$S^{2n-1}/S^1 = \mathbb{C}\mathrm{P}^{n-1}$$

Example II.3.7 (Very General Example)

Suppose H is a Hausdorff topological group and $G \subseteq H$ is a compact subgroup. Then G acts on H by $(g, h) \mapsto gh$. Then H/G is compact Hausdorff if H is compact. Spaces of the form H/G (even when G is not compact) are called *homogeneous spaces* so long as H/G is Hausdroff. These spaces are extremely important.

Addendum: homogeneous spaces are important because

(1) You can calculate
(2) "Systems" with symmetry are typically homogeneous
(3) $\mathrm{GL}(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{Z})$ shows up in number theory everywhere.

For those doubting since $\mathrm{GL}(n, \mathbb{Z})$ is not compact, look at $\mathbb{Z}$ acting on $\mathbb{R}$. We claim $\mathbb{R}/\mathbb{Z}$ is nice, check the graph is closed!

Definition II.3.5

We call a group action of G on a set X transitive provided that X is one G-orbit, i.e., for every $p, q \in X$ there is a $g \in G$ so that $g \cdot p = q$.

Example II.3.8

The action of $\mathrm{SO}(n+1)$ on $S^n \subseteq \mathbb{R}^{n+1}$ is transitive. Take $v \in S^n$, and extend it to an orthonormal basis and make these the columns of $g \in \mathrm{SO}(n+1)$. We see that $g \cdot \left(\begin{smallmatrix} 1 & 0 & \vdots & 0 \end{smallmatrix}\right) = v$.

Definition II.3.6

Suppose G acts on X, $p \in X$, the stabilizer of p in G or isotropy group of p in G if

$$G_p \coloneqq \{g \in G \mid g \cdot p = p\}.$$

Lemma II.3.4

The stabilizer G_p is a subgroup of G. If G is a topological group, $G_p \subseteq G$ is a closed subgroup (i.e. it is a closed set).

Proof. Showing it's a subgroup if trivial. We can quickly show it is closed. Why? Well if $g_n \to g$ and $g_n \in G_p$ then $p = g_n \cdot p \to g \cdot p$ by continuity, so $g \cdot p = p$, so $g \in G_p$.

Future: We will prove that if G is a Lie group, $H \subseteq G$ is a closed subgroup, then H has a Lie group structure.

Example II.3.9

Consider $\mathrm{SO}(n+1)$ acting on S^n. The stabilizer of the $N \coloneqq (1, 0, \dots, 0) \in S^n$ is

$$\mathrm{SO}(n+1)_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \mid * \in \mathrm{SO}(n) \right\}.$$

This shows $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$.

Lemma II.3.5

If G acts on M transitively and $p \in M$, then there is a bijective continuous map $G/G_p \to M$. If G/G_p is compact and M is Hausdorff, then this has a continuous inverse, so $G/G_p \cong M$.

Proof. We have a surjective map $f : G \to M$ given by $g \overset{f}{\mapsto} g \cdot p$ since the action is transitive. By the universal property of quotients since for $x \in G_p$ we have $f(gx) = g \cdot x \cdot p = f(g)$ this map descends as

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & M \\ G/G_p \arrow[ur, dashed, "\widetilde{f}"] & \end{tikzcd}$$

to a continuous map. To show it is one-to-one we see that if $\widetilde{f}([g]) = \widetilde{f}([h])$ then $g \cdot p = h \cdot p$, so $h^{-1}g \cdot p = p$, so $h^{-1}g \in G_p$.

Example II.3.10

Consider $\mathrm{GL}(2, \mathbb{R})$ acting on $\mathbb{R}^2$. This is not transitive since $A \cdot 0 = 0$. But it is transitive on $\mathbb{R}^2 \setminus \{0\}$. Now consider

$$P \coloneqq \mathrm{GL}(2, \mathbb{R})_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = \left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix} \mid c \in \mathbb{R}, d \neq 0 \right\}.$$

We then see that

$$\mathrm{GL}(2, \mathbb{R})/P \cong \mathbb{R}^2 \setminus \{0\}.$$

(in this case we're not compact, but it still works)

Example II.3.11

$G \coloneqq \mathrm{GL}(n+1, \mathbb{R})$ acts on $\mathbb{R}^{n+1}$ and transitively on $\mathbb{R}^{n+1} \setminus \{0\}$. This descends to a transitive action on $\mathbb{RP}^n$.

Consider $\ell_1 = \mathbb{R}^\times \cdot e_1$, then

$$G_{\ell_1} = \left\{ \begin{pmatrix} \lambda & *_1 \\ 0 & *_2 \end{pmatrix} \mid \lambda \in \mathbb{R}^\times, \det *_2 \neq 0 \right\}.$$

Goal: G/G_{ℓ_1} is differentiable.

Wanted: $T \subseteq G$ "transversal" to G_{ℓ_1} of dimension $\dim \mathbb{RP}^n = n$. We can look at

$$T = \left\{ \begin{pmatrix} 1 & 0 \\ v & \mathrm{Id} \end{pmatrix} \mid v \in \mathbb{R}^n \right\}.$$

Then $\mathbb{R}^n \cong T \cdot \ell_1$ so we have a chart!

Recipe:

(1) Suppose a Lie group G acts transitively on M. We want to endow M with a differentiable ($C^1, C^k, C^\infty, C^\omega$) structure.
(2) Take $p \in M$, $G/G_p \cong M$.
(3) If you can find a "transversal" "subspace" of G to G_p, say T
(4) Try coordinate charts $T \to T \cdot p, t \mapsto t \cdot p$.

Back to the Future: If G is a Lie group, H is a closed sugroup, then G/H is always a smooth manifold.

Example II.3.12

We'll give one more example of a homogeneous space. We want a Lie group G and a closed subgroup $H \subseteq G$, and $M = G/H$. This is the same as G acting transitively on M, and $H = G_p$ for some $p \in M$. Note: If G_p does not depend on p, then?

So we're going to look at Grassmannian of k-planes in n-planes ($\mathbb{R}^n$). We call this $\mathrm{Gr}_{k,n}(\mathbb{R})$. Recall that

$$\begin{aligned}\mathrm{Gr}_{1,n} = \mathbb{R}\mathrm{P}^{n-1} &= \mathrm{GL}(n,\mathbb{R})/\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\ &= \mathrm{SO}(n)/O(n-1) = (\underbrace{\mathrm{SO}(n)/\mathrm{SO}(n-1)}_{S^{n-1}})/\mathbb{Z}_2\end{aligned}$$

where $O(n-1)$ is embedded in $\mathrm{SO}(n)$ as

$$A \mapsto \begin{pmatrix} \det A & 0 \\ 0 & A \end{pmatrix}.$$

In the general case take $e_1, \dots, e_n$ as a basis for $\mathbb{R}^n$, $p \coloneqq \langle e_1, \dots, e_k \rangle$ is a k-dimensional subspace. Then we can define an action by

$$\begin{aligned}\mathrm{GL}(n,\mathbb{R}) \times \mathrm{Gr}_{k,n} &\to \mathrm{Gr}_{k,n} \\ A \cdot V &= \{A \cdot v \mid v \in V\}.\end{aligned}$$

This is also transitive. If $V = \langle v_1, \dots, v_k \rangle$ is a k-dimensional subspace, then $A = (v_1, \dots, v_k, ?, \dots, ?)$ where we have extended to a basis maps p to V. Thus $\mathrm{Gr}_{k,n} = \mathrm{GL}(n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})_p$. We see that

$$A \in \mathrm{GL}(n,\mathbb{R})_p \iff A \cdot p = p \iff A \cdot e_i \in p \iff A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

where the blocks are $k \times k, k \times (n-k), (n-k) \times k, (n-k) \times (n-k)$.

Then the transversal subspace making this a manifold is an $(n-k) \times k$ block of anything in the lower left hand block, 1s on the diagonal, and then 0s elsewhere.

Lemma II.3.6

If $q = g \cdot p$, then $G_q = gG_pg^{-1}$. If we have a transversal T_p we should try $T_q = gT_pg^{-1}$.

Exercise II.3.13

$\mathrm{SO}(n)$ also acts transitively on $\mathrm{Gr}_{k,n}$, so one can do the same work here.

Example II.3.14

$S^1 = \mathbb{R}/\mathbb{Z}, T^n = S^1 \times \cdots \times S^1 = \mathbb{R}^n/\mathbb{Z}^n$. In these cases everything we said works although $\mathbb{Z}^n$ is not compact.

In contrast we have the bad (interesting) example given by $\mathbb{Z}$ acting on S^1 by irrational rotation.

Definition II.3.7

Let Γ be a discrete group acting on a topological space X. We say the action is properly discontinuous provided that $\Gamma \cdot x$ can be taken to be "separate." We make this precise via

- Namely, for any compact set $K \subseteq X$, we have $(\Gamma \cdot x) \cap K$ is finite. In other words,

$$\Gamma \times X \to X$$

  is a proper map.
- If for some $x \in X, \gamma \in \Gamma$, we have $\gamma \cdot p = p$, then $\gamma = 1$. (This is also said as Γ acts freely, and is only included by some author).

Exercise II.3.15

Suppose Γ acts on a manifold M properly discontinuously, then M/Γ is a manifold.

The same holds for differentiable (C^k) structure so long as Γ acts via differentiable (C^k) maps.

Example II.3.16

$\mathrm{SL}(n, \mathbb{R}) / \mathrm{SL}(n, \mathbb{Z})$ is the most famous example of this type. This is in fact the space of lattices in $\mathbb{R}^n$ of volume 1. This has a deep connection to number theory.

Particularly the case $n = 2$ is important because the tori carry complex analytic structure.

III. Tangent Vectors/Differentiation

Take M to be a differentiable manifold. How can we define a "tangent vector" on it. Well a tangent vector for $V \subseteq \mathbb{R}^n$ is just a choice (p, h) where $p \in V, h \in \mathbb{R}^n$.

So what if we just work chart-wise for charts $(\varphi_\alpha, U_\alpha)$? Well then for a $p \in M$, we can look at a tangent vector $(\varphi_\alpha(p), h) \in T_{\varphi_\alpha(p)} V_\alpha$. But how do we look between charts??? Aka what does $(\varphi_\alpha(p), h)$ look like in V_β?

Well we can look at the transition map $T_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$. Then we can define an equivalence relation

$$(\varphi_\alpha(p), h) \sim (\varphi_\beta(p), h') \iff \mathrm{d}T_{\alpha,\beta}(h) = h'.$$

Definition III.0.1

We define the tangent space

$$T_pM = \{[v, (U_\alpha, \varphi_\alpha)] \mid [v, (U_\alpha, \varphi_\alpha)] \text{ is an equivalence class of tangent vectors in charts}\}$$

Nice interpretation, $p \in \mathbb{R}^n$, $w \in T_pM$. Take $c(t)$ differentiable for $t \in (-\varepsilon, \varepsilon)$, $c(0) = p$, then we can look at $c'(0)$.

We can talk about two different curves then say c_1, c_2 are equivalent as tangent vectors if $c_1'(0) = c_2'(0)$.

For $p \in M$ a differentiable manifold, and a differentiable curve $c(t)$ through that point at $t = 0$, then $[c]_p$ is a tangent vector defined at charts as $c'(0)$ upon appropraite choice of coordinates.

Definition III.0.2

Let $f : M \to N$ be a differentiable map at $p \in M$. We define

$$\mathrm{d}f_p : T_pM \to T_{f(p)}N.$$

Take some differentiable curve c representing our tangent vector in T_pM. We can then take $\mathrm{d}f_p([c]_p) = [f \circ c]_{f(p)} \in T_{f(p)}N$.

Also $T_pM, T_{f(p)}N$ have vector space structures inherited from the case in $\mathbb{R}^n$, and as before for multivariable calculus, $\mathrm{d}f_p$ is a linear map.

Recall III.0.1

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. be a differentiable map. Recall from multivariable calculus that if $p \in U$ then we define $\mathrm{d}f_p$ to be the best linear approximation of f at p, that is we require

$$\lim_{\varepsilon \to 0} \frac{f(p+\varepsilon) - f(p) - \mathrm{d}f_p(\varepsilon)}{\|\varepsilon\|} = 0.$$

We can compute that in the standard coordinates

$$\mathrm{d}f_p = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Lets make the definition of tangent vectors with curves more explicit

Definition III.0.3

Call two differentiable curves c, d in M with $c(0), d(0) = p$ equivalent if for one (and hence every chart in the atlast) U_α, φ_α we have

$$(\varphi_\alpha \circ c)'(0) = (\varphi_\alpha \circ d)'(0).$$

Then

$$T_pM = \{[c] \mid c : (-\varepsilon, \varepsilon) \to M \text{ differentiable at } 0, c(0) = p\}.$$

Claim

T_pM has a vector space vector of dimension $\dim M$.

Proof. Use the coordinate chart definition. Take $A, B \in T_pM$. Then in coordinates A, B correspond to $v_\alpha, w_\alpha \in T_{\varphi_\alpha(p)}\mathbb{R}^n$ for some chart $(U_\alpha, \varphi_\alpha)$.

Then $v_\alpha + w_\alpha \in T_{\varphi_\alpha(p)}\mathbb{R}^n$, take $A + B = [v_\alpha + w_\alpha]$. We should check that it doesn't matter where we do the addition. Well let v_β, w_β represent A, B in $T_{\varphi_\beta(p)}\mathbb{R}^n$.

We check that

$$(\mathrm{d}T_{\beta\alpha})_{\varphi_\beta(p)}(v_\beta + w_\beta) = (\mathrm{d}T_{\beta\alpha})_{\varphi_\beta(p)}(v_\beta) + (\mathrm{d}T_{\beta\alpha})_{\varphi_\beta(p)}(w_\beta) = v_\alpha + w_\alpha.$$

Scalar multiplication is quite similar.

Recall III.0.2

Recall Definition III.0.2 of the derivative of a map $f : M \to N$ which is differentiable at p.

Note on notation first: We can do all of

$$(f_*)_p = D_p f = Df_p = (\mathrm{d}f_p = \mathrm{d}_p f)$$

But we **really** shouldn't be using $\mathrm{d}f_p = \mathrm{d}_p f$, as later it will confuse us with differential forms.

Now we give the definition in terms of charts. Take a chart $(U_\alpha, \varphi_\alpha)$ about p and take $A \in T_pM$, then A is represented by some $v_\alpha \in T_{\varphi_\alpha(p)}\mathbb{R}^n$.

Take some other chart (V_γ, ψ_γ) about $f(p)$ in N. Then we take $Df_p(A)$ to be represented by

$$D(\psi_\gamma \circ f \circ \varphi_\alpha^{-1})_{\varphi_\alpha(p)}(v_\alpha).$$

Theorem III.0.1

If $f : U \to V$ is differentiable at $p \in U$ where $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^\ell$ are open, and Df_p is onto, then $f^{-1}(f(p))$ (aka a level set) is a manifold.

Might be nice on homework...

Convention: All manifolds are assumed to be differentiable (in fact C^1). Later we will prove that

Theorem III.0.2

If M has a C^1 structure then it has a compatible C^∞ structure.

Proposition III.0.3 (Chain Rule)

Let $f : M \to N, g : N \to O$ be differentiable maps, then $g \circ f$ is differentiable and

$$D_p(g \circ f) = D_{f(p)}g \circ D_p f.$$

Writing this diagramatically

$$\begin{tikzcd}
T_pM \arrow[rr, "D_p(g\circ f)"] \arrow[dr, "D_pf"'] & & T_{(g\circ f)(p)}O \\
& T_{f(p)}N \arrow[ur, "D_{f(p)}g"'] &
\end{tikzcd}$$

Proof. Use curves (aka hide the coordinate charts in the equivalence of curves with charts)! To do this, let c be a curve then

$$D_{f(p)}g \circ D_p f([c]) = D_{f(p)}g([f \circ c]) = [g \circ f \circ c] = [(g \circ f) \circ c] = D_p(g \circ f).$$

What is a diffeomorphism?

Definition III.0.4

A differentiable map $f : M \to N$ is called a diffeomorphism provided that it is bijective and its inverse $g : N \to M$ is differentiable.

A map $f : M \to N$ is a local diffeomorphism at $p \in M$ if there exists open neighborhoods U of p and V of $f(p)$ usch that $f : U \to V$ is a diffeomorphism.

Remark III.0.1

In this case we have

$$\mathrm{Id} = D_p(\mathrm{Id}) = D_p(g \circ f) = D_{f(p)}g \circ D_p f.$$

That is $D_p f$ has an inverse map.

Furthermore, coordinate charts $\varphi : U \to \widetilde{U} \subseteq \mathbb{R}^n$ are always invertible.

Corollary III.0.4

If f is a local diffeomorphism at p, then $D_p f : T_p M \to T_{f(p)} N$ has an inverse. This implies that $\dim M = \dim N$ about p.

Theorem III.0.5 (Inverse Function Theorem from Real Analysis)

If $f : U \to V$ with $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ has an invertible derivative $D_p f$ at p, then there is a neighborhood $U' \subseteq U$ such that $f : U' \to f(U')$ is a diffeomorphism.

Theorem III.0.6 (Inverse Function Theorem for Manifolds)

Suppose $f : M \to N$ is C^1 and suppose $Df_p : T_p M \to T_{f(p)} N$ is invertible (as a linear map), then f is a local diffeomorphism at p.

Proof. Fix charts $(U, \varphi), (V, \psi)$ about $p, f(p)$, with $U \subseteq \mathbb{R}^n$, so that $f(U) \subseteq V \subseteq \mathbb{R}^m$ (this requires minor yoga). Call $T = \psi \circ f \circ \varphi^{-1}$.

By the chain rule, $D_{\varphi(p)}T$ is invertible, and so $n = m$. By the inverse funct

Definition III.0.5

Suppose M is a C^1 manifold, we say that $S \subseteq M$ is called a embedded submanifold of M provided that for all $p \in S$, there exists a coordinate chart (U, φ) about M such that $\varphi\big|_S : S \cap U \to \mathbb{R}^k \subseteq \mathbb{R}^n$. That is

$$S = \{q \in U \mid \varphi(q) = (*, \dots, *, 0, 0, \dots, 0).$$

We call such a thing an adapted chart (adapted to S).

Note: S is a C^1-manifold in its own right.

Example III.0.3

$\emptyset, M \subseteq M$, $\mathbb{R}^\ell \subseteq \mathbb{R}^n$, $S^\ell \subseteq S^n \subseteq \mathbb{R}^{n+1}$, $\mathbb{R}\mathrm{P}^\ell \subseteq \mathbb{R}\mathrm{P}^n$.

Example III.0.4

Consider $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \subseteq S^1$, and let $p : \mathbb{R}^2 \to T^2$. Let $\ell \subseteq \mathbb{R}^2$ be a line forming an angle of α to the origin. If $\alpha = 0$ this is $S^1 \times \{1\}$. If $\alpha \notin \pi\mathbb{Q}$ then $p(\ell)$ is dense in T^2.

This means that $p(\ell)$ will not be a submanifold when $\alpha \notin \pi\mathbb{Q}$. This doesn't cross over itself, but the density prevents you from taking a small open chart making the rest a line.

Definition III.0.6

Suppose $f : M \to N$ is a C^1 map. Then f is called an immersion provided that for all $p \in M$ we have that $D_p f$ is injective.

This will imply that $f(M)$ is "locally" a submanifold.

Definition III.0.7

We call $S \subseteq M$ an immersed submanifold if S is a manifold, and $\iota : S \hookrightarrow M$ is an immersion.

Definition III.0.8

Call a map $f : M \to N$ which is C^1 a submersion provided that every value $y \in N$ is a regular value. That is, for every $x \in M$ such that $f(x) = y$ we have $D_x f : T_x M \to T_y N$ is surjective.

Convention: If $f : M \to N$ and $y \notin f(M)$ then y is a regular value of f.

Note: If $M \to N$ is a submersion, then $\dim M = \dim T_x M \geq \dim T_y N = \dim N$.

Question: Why is $\dim M = \dim T_x M$ for all $x \in M$? Well it is clear that if φ is a chart we have

$$\dim T_x M = \dim T_{\varphi(x)} \mathbb{R}^{\dim M} = \dim M.$$

Note that once we pick a coordinate chart φ, it induces an isomorphism of vector spaces

$$T_x M \cong T_{\varphi(x)} \mathbb{R}^{\dim M} \cong \mathbb{R}^{\dim M}$$

This isomorphism is given by taking representatives of the equivalence classes by which $T_x M$ is defined. The isomorphism is intimately related to φ.

Theorem III.0.7

Suppose $f : M \to N$ and $q = f(p)$ is a regular value, then $f^{-1}(q)$ is an embedded submanifold of M. In fact $f^{-1}(q)$ has dimension $\dim M - \dim N$.

Proof Idea. Really, work with a coordinate chart for M at $p \in M$. Select a chart (W_β, ψ_β) about $q = f(p)$ to $\mathbb{R}^n$ with $\dim N = n$. Now take a chart $(U_\alpha, \varphi_\alpha)$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ with $\dim M = m$ and $U_\alpha \subseteq f^{-1}(W_\beta)$. For convenience let $\boldsymbol{p} = \varphi_\alpha(p)$ and $\boldsymbol{q} = \psi_\beta(q)$.

Now consider the map $F_{\alpha\beta} = \psi_\beta \circ f \circ \varphi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$. It now suffices to check the claim for the coordinate map $F_{\alpha\beta}$. We want $F^{-1}(\boldsymbol{q})$ to be a submanifold. Well, we know that $D_{\boldsymbol{p}} F : T_{\boldsymbol{p}} \mathbb{R}^m \to T_{\boldsymbol{q}} \mathbb{R}^n$ is surjective.

This means $m \geq n$ and $\ker D_{\boldsymbol{p}} F \subseteq \mathbb{R}^m$ has dimension $m - n$. Put this kernel into $\mathbb{R}^m$ as the last $m - n$ coordinates, to do this use an invertible linear map B with $B^{-1}(\ker D_{\boldsymbol{p}} F) = \mathbb{R}^{m-n}$.

We may then precompose to get $\overline{F} = F \circ B$.We know $D_{B^{-1}(\boldsymbol{p})} \overline{F} = D_{\boldsymbol{p}} F \circ B$, and so this is surjective with kernel $\mathbb{R}^{m-n}$. We define an extended map

$$G : R^m \to R^n \times R^{m-n}$$
$$G(x_1, \ldots, x_m) = (\overline{F}(x_1, \ldots, x_m), x + m - n + 1, \ldots, x_m),$$

then $D_{B^{-1}(\boldsymbol{p})}(G)$ is an isomorphism. Why? Well it has the form $(D_{B^{-1}(\boldsymbol{p})} \overline{F}, \mathrm{Id}_{\mathbb{R}^{m-n}})$. This is clearly surjective with zero kernel.

Now use inverse function theorem on G. G is a local diffeomorphism, so $G^{-1}(q) \to \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$. We then use G to get an adapted coordinate chart!

Example III.0.5

$\mathrm{SL}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$ is an embedded submanifold (seen on HW). To show this, we proved $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ has $D_p \det$ is surjective for any $p \in \mathrm{SL}(n,\mathbb{R})$, and so $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$ is a submanifold.

Idea of proof: somehow apply inverse function theorem

Summary of the proof of the regular value theorem in steps:

(Step 1) Use coordinate charts to reduce to a problem about $F : U \to \mathbb{R}^k$ so that $U \subseteq \mathbb{R}^n$ is open.

(Step 2) We have $F : U \to \mathbb{R}^n$ and $\overline{q}$ a regular value of F. We want to show $F^{-1}(\overline{q})$ is a submanifold of $\mathbb{R}^n$. We know that $\overline{p} \in F^{-1}(\overline{q})$ $DF_{\overline{p}}$ is surjective and

$$DF_{\overline{p}} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n} \end{pmatrix}.$$

Since $DF_{\overline{p}}$ has rank k, so it has k linearly independent row vectors. We may assume without loss of generality that they are the first k.

(Step 3) Take $G : \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ to be defined by $G(x_1, \dots, x_n) = (F(x_1, \dots, x_n), x_{k+1}, \dots, x_n)$. Then we have

$$A \coloneqq \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_k} & \vdots \\ \vdots & \ddots & \vdots & \\ \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_k} & \end{pmatrix}$$

$$DG_p = \begin{pmatrix} A & * \\ 0 & \mathrm{Id}. \end{pmatrix}$$

This is obviously invertible. Thus G is a local diffeomorphism, which means $G^{-1}(\{q\} \times \mathbb{R}^{n-k})$ willbe locally a submanifold.

(Step 4) $F^{-1}(q) = G^{-1}(\{q\} \times \mathbb{R}^{n-k})$. Thus $F^{-1}(q)$ is locally a submanifold.

Example III.0.6

There is a canonical submersion given by $\mathbb{R}^n \to \mathbb{R}^k$ with $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k)$ for $k \leq n$.

Another way of stating the regular value theorem is

Theorem III.0.8

If $f : M \to N$ is C^1 and q is a regular value, then for any $p \in f^{-1}(q)$ there exist coordinate charts $(U, \varphi), (V, \psi)$ such that

$$\begin{array}{ccc} \mathbb{R}^m & & \mathbb{R}^n \\ \cup & & \cup \\ \varphi(U) & \xrightarrow{\psi \circ f \circ \varphi^{-1}} & \psi(V) \end{array}$$

$$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_n).$$

with $m = \dim M, n = \dim N$. This is called the <u>normal form</u>.

What is the engine of the IVT? The contraction mapping fixed point theorem!!!

Drives many things in the subject, such as existence and uniqueness of solutions of ODEs. It also shows up in dynamics.

Example III.0.7

Consider $O(n) = \{n \times n \text{ matrices with } AA^t = \mathrm{Id}\}$. We show it's a submanifold of $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ by looking at

$$\begin{aligned} F : M_n(\mathbb{R}) &\to \mathrm{Sym}(n \times n) \cong \mathbb{R}^{n(n+1)/2} \\ A &\mapsto AA^T, \end{aligned}$$

where $\mathrm{Sym}(n \times n)$ are the symmetric $n \times n$ matrices. We must show that the identity Id is a regular value.

We must calculate DF_g where $g \in F^{-1}(\mathrm{Id})$, so $gg^t = \mathrm{Id}$. We calculate $DF_g(v)$ using curves, where $v \in \mathbb{R}^{n^2}$. Consider

$$\begin{aligned} F(g + tv) &= (g + tv)(g + tv)^T = (g + tv)(g^T + tv^T) \\ &= \mathrm{Id} + t(gv^T) + t(vg^T) + t^2 \cdot * \\ \frac{\mathrm{d}}{\mathrm{d}t} F(g + tv)\Big|_{t=0} &= gv^T + vg^T. \end{aligned}$$

The claim is that any symmetric matrix has this form. Set $v = wg$, then

$$DF_g(v) = w^T + w.$$

If A is a symmetric matrix, then taking $w = \frac{A}{2}$ is sufficient.

Alternative approach: Compute the kernel $\ker(DF_g)$. If $v \in \ker(DF_g)$ then $gv^T = -vg^T$, so vg^T is skew-symmetric. The dimension of this is $n(n-1)/2$.

Then the dimension of the image of DF_g is

$$n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

But this is exactly the dimension of the symmetric matrices, and so we're done.

$\mathrm{SO}(n)$ is the connected component of the Id in $O(n)$ because $\det : O(n) \to \mathbb{R}$ takes values in $\{\pm 1\}$. Thus $\mathrm{SO}(n)$ is clopen in $O(n)$.

Fact: $\mathrm{SO}(n)$ is connected.

Corollary III.0.9

Submersions are open maps

Proof. The local normal form is a projection, which is an open map.

Now lets look at examples of submersions.

Example III.0.8

If F, M are manifolds, then we can look at the projection $M \times F \to M$. Then this is obviously a submersion!

We call this type of submersion a trivial bundle

IV. Fiber/Vector Bundles

Definition IV.0.1

A submersion $\pi : M \to N$ is a fiber bundle provided that

- π is surjective.
- We equip N with a covering by open sets $\{V_\alpha\}$ such that $\pi^{-1}(V_\alpha)$ is diffeomorphic by φ_α to $V_\alpha \times F$ for some fixed manifold F. These are called local trivializations
- For each V_α the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & V_\alpha \times F \arrow[dl, dashed] \\ & V & \end{tikzcd}$$

The manifold F is called the fiber of the bundle.

Example IV.0.1

$N \times F \to N$, and for M the Möbius band, $M \to S^1$, with $F = (-1, 1)$.

Note that the Möbius band is not diffeomorphic to $S^1 \times (-1, 1)$.

If $N \subseteq \mathbb{R}^L$ is an embedded submanifold, we can consider the unit tangent bundle

$$S(N) = \{v \in TpN \mid p \in N, \|v\| = 1\}.$$

For $N = S^2$ is sort of complicated. For $N = S^1$, we get

$$N(S^1) = S^1 \times \{0, 1\}.$$

Fact: $S(S^2)$ is not a trivial fiber bundle.

HW: $S(S^3)$ is a trivial bundle. Hint: It's a group.

Definition IV.0.2

Let M be an abstract differentiable manifold. As a set the tangent bundle of M is

$$TM := \coprod_{p \in M} T_p M.$$

Claim: This is a fiber bundle, in fact it is a vector bundle

Definition IV.0.3

A vector bundle $\pi : M \to N$ is a fiber bundle with fiber F a vector space such that $\pi^{-1}(z_0)$ is intrinsically a vector space and for any local trivialization $(U_\alpha, \varphi_\alpha)$ induces a linear map $\varphi_\alpha : \pi^{-1}(z_0) \to \{z_0\} \times F$ for all $z_0 \in U_\alpha$.

Equivalently, if $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ are any two trivializations, then

$$\begin{tikzcd} \pi^{-1}(U_\alpha \times U_\beta) \arrow[r, bend left, "\varphi_\alpha"] \arrow[r, bend right, "\varphi_\beta"'] & (U_\alpha \cap U_\beta) \times F \end{tikzcd}$$

has $\varphi_\beta^{-1} \circ \varphi_\alpha$ (called is a linear map (and therefore a linear isomorphism). This allows one to place a canonical vector space structure on $\pi^{-1}(z_0)$ for any $z_0 \in N$.

Recall the claim from last time: TM is a vector bundle over M. To do this we need to show that TM carries a manifold structure. It turns out we lose regularity (summed up below)

Remark IV.0.1

If M has a C^r-manifold structure, then TM has a C^{r-1}-manifold structure. Thus M must have at least C^2-structure to get TM with C^1-structure (that is a manifold).

Note: In the end this is not a problem, as we will later show that every C^1 manifold has a C^∞ structure

Proposition IV.0.1

If M is a C^r-manifold then $TM \coloneqq \coprod_p T_pM$ is a C^{r-1}-manifold which is a vector bundle over M.

Proof the Tangent Bundle is a Vector Bundle. Take M to be an abstract differentiable manifold, and $\pi : TM \to M$ the obvious map. Call $m \coloneqq \dim M$, then we'll take $F \coloneqq \mathbb{R}^m$. We must do two things

(a) Give TM a manifold structure.

(b) Show it can be endowed with a vector bundle structure.

We take a covering of M by charts $(U_\alpha, \varphi_\alpha)$, $\varphi_\alpha : U_\alpha \to W_\alpha \subseteq \mathbb{R}^m$. Note φ_α is in fact a diffeomorphism, by how we've set up the definition of differentiability on manifolds.

Then we'll do each of the above

(a) We know that $TW_\alpha \subseteq \mathbb{R}^{2m}$ is an open subset, as $TW_\alpha = W_\alpha \times \mathbb{R}^m$.

Then

$$TU_\alpha = \coprod_{p \in U_\alpha} T_pU_\alpha \xrightarrow{D\varphi_\alpha} \coprod_{p \in U_\alpha} T_{\varphi_\alpha(p)}W_\alpha = TW_\alpha.$$

We take this as a coordinate chart on TM. Namely, take TM to have a topological strcture with basis the open sets $\{TU_\alpha\}$.

Then we claim $(TU_\alpha, D\varphi_\alpha)$ is an atlas. We must look at the transition map, that is

$$\begin{tikzcd} & TU_\alpha \cap TU_\beta \arrow[dl, "D\varphi_\alpha"'] \arrow[dr, "D\varphi_\beta"] & \\ D\varphi_\alpha(TU_\alpha \cap TU_\beta) \arrow[rr, "D\varphi_\beta \circ D\varphi_\alpha^{-1}"'] & & D\varphi_\beta(TU_\alpha \cap TU_\beta) \end{tikzcd}$$

By the chain rule this is $D(\varphi_\beta \circ \varphi_\alpha^{-1})$, which is C^{r-1} by assumption. Thus this is a C^{r-1}-atlas.

(b) We now will show this is a vector bundle. Note that

$$\pi^{-1}(U_\alpha) = \coprod_{p \in U_\alpha} T_pM.$$

We then have $U_\alpha \xrightarrow{\varphi_\alpha} W_\alpha \subseteq \mathbb{R}^m$. Note then

$$\pi^{-1}(U_\alpha) \xrightarrow{D\varphi_\alpha} TW_\alpha = W_\alpha \times \mathbb{R}^m.$$

This is nearly our trivialization. Follow up with $(\varphi_\alpha^{-1}, \mathrm{Id})$ to get

$$\pi^{-1}(U_\alpha) \xrightarrow{(\varphi_\alpha^{-1}, \mathrm{Id}) \circ D\varphi_\alpha} U_\alpha \times \mathbb{R}^m.$$

Call this ψ_α. Clearly $\mathrm{proj} \circ \psi_\alpha = \pi$, so this is a fiber bundle.

Last thing to check is that ψ_α is linear on fibers. This comes from the fact that T_pU_α inherits its linear structure from $T_{\varphi_\alpha(p)}W_\alpha$.

That is

$$\psi_\alpha : T_pU_\alpha = T_pM \xrightarrow{(\varphi_\alpha^{-1}, \mathrm{Id})\circ D_p\varphi_\alpha} \{p\} \times \mathbb{R}^m$$

is linear because $D_p\varphi_\alpha$ is linear by construction of the linear structure on T_pM.

We'll now do constructions with vector bundles! Take $\mathcal{V}, \mathcal{W}$ to be vector bundles over M with maps π_1, π_2 to M.

- HW5: Define $\mathcal{V} \oplus \mathcal{W} \to M$, and if $V_p = \pi_1^{-1}(p)$ and W_p are similar then the fiber over p should be $V_p \oplus W_p$.
- We can take $\mathcal{V}^*$ with fibers V_p^*, where $*$ denotes the dual space. For this one looking fiber by fiber

$$\phi_\alpha : \pi^{-1}(p) \to \{p\} \times V$$

is linear, and we have

$$\phi_\alpha^* : \{p\} \times V^* \to (\pi^{-1}(p))^*.$$

This goes in the opposite direction as desired, but ϕ_α is invertible! Thus we can look at

$$(\phi_\alpha^*)^{-1} : (\pi^{-1}(p))^* \to \{p\} \times V^*.$$

- If $\mathcal{V}, \mathcal{W}$ are two vector bundles you can look at $\mathcal{V} \otimes \mathcal{W}$.
- Important one down the road: Given one specific vector space V. We cal look at the k-fold tensor product $\underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}}$. Hiding inside of this is something important, the k-alternating linear forms $\bigwedge^k V$.

We can of course do this with $\mathcal{V}$ as a vector bundle $\bigwedge^k \mathcal{V}$. Later then $\bigwedge^k TM$ will be differential k-forms, which will lead to de-Rham cohomology at the end.

Recall IV.0.2

A multilinear form on $V_1, \ldots, V_k$ is a map

$$V_1 \oplus V_2 \oplus \cdots \oplus V_k \to \mathbb{R}$$

if $\lambda(v_1, \ldots, v_k)$ is linear in each coordinate. If $V = V_1 = \cdots = V_k$ then λ is called alternating when

$$\lambda(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\lambda(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Prime example: Determinant, $\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$.

V. Vector Fields/Derivations/Lie Brackets

Definition V.0.1

If $E \xrightarrow{\pi} B$ is a fiber bundle then we call $\sigma : B \to E$ a section if $\pi \circ \sigma = \mathrm{Id}_B$.

Definition V.0.2

A vector field $V : M \to TM$ is a section (continuous, C^1, C^k, C^∞)of TM.

Example V.0.1

If $M = \mathbb{R}^n$, then we have very special vector fields which are constant at $e_1, \dots, e_n$. We often denote these vector vields by $\frac{\partial}{\partial x_i}$, to specify that they are tangent vectors (and thus related to differentiation).

Then if $X : \mathbb{R}^n \to T\mathbb{R}^n$ is any vector field we may write for all $p \in \mathbb{R}^n$

$$X(p) = a_1(p)\frac{\partial}{\partial x_1}(p) + \cdots + a_n(p)\frac{\partial}{\partial x_n}(p),$$

where $a_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th coefficient function. X is C^r if and only if all the a_i are C^r.

How does one check $X : M \to TM$ is a differentiable vector field in practice? For convenience consider M is a C^∞ manifold. Take a chart $(U_\alpha, \varphi_\alpha)$ for M. Then $D\varphi_\alpha : TU_\alpha \to T(\varphi_\alpha(U_\alpha)) \subseteq T\mathbb{R}^n$ Thus we want to know that

$$z \mapsto D\varphi_\alpha(X(\varphi_\alpha^{-1}(z)) : \varphi_\alpha(U_\alpha) \to T(\varphi_\alpha(U_\alpha))$$

is differentiable.

Exercise V.0.2

If $f : M \to N$ is C^r then $Df : TM \to TN$ is C^{r-1}, where the obvious definition is

$$Df((p, v)) = (f(p), D_pf \cdot v).$$

Of course if f is C^∞ then Df is C^∞.

Check: Use coordinates. Take coordinates (U, φ) on M and (V, ψ) on N. Then $D\varphi, D\psi$ provide coordinates on TM, TN and so

$$D\psi \circ Df \circ D\varphi^{-1} = D(\psi \circ f \circ \varphi^{-1}).$$

By definition $\psi \circ f \circ \varphi^{-1}$ is C^r, so $D(\psi \circ f \circ \varphi^{-1})$ is C^{r-1}. Perfect!

Back to vector fields. To check smoothness (or do any calculation) write X in a chart. We have $e_1, \dots, e_n$ on $\varphi(U)$. We can pull back $e_1, \dots, e_n$ to get $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$, so $\frac{\partial}{\partial x_i} = D_{\varphi(p)}(\varphi^{-1})(e_i)$. Then of course

$$X\big|_U = \sum_{i=1}^n a_i(p)\frac{\partial}{\partial x_i}(p)$$

where $a_i : U \to \mathbb{R}$ is the i-th coordinate function for the chart (U, φ).

Warning: $\frac{\partial}{\partial x_i}$ is a vector field. There will also be $\mathrm{d}x_i$, which are cotangent fields, which are differential 1-forms, that is local sections to $T^*M \coloneqq (TM)^*$. Then $\mathrm{d}x_i \in T^*U_\alpha$.

Definition V.0.3

Last time we learned how to take the dual space to a vector bundle. Thus we define the cotangent bundle of M to be $T^*M \coloneqq (TM)^*$.

To summarize what we just did

Definition V.0.4

If (U, φ) is a coordinate chart on M we define local coordinates for TM via vector fields

$$\begin{aligned} \frac{\partial}{\partial x_i} &: U \to TU \\ p &\mapsto D_{\varphi(p)}(\varphi_\alpha^{-1})(e_i). \end{aligned}$$

Then we define $\mathrm{d}x_i : U \to T^*U$ at each point $p \in M$ so that $\{\mathrm{d}x_i(p)\}_{i=1,\dots,n}$ as the dual basis to $\left\{\frac{\partial}{\partial x_i}(p)\right\}$.

Example V.0.3

Consider the simplest nontrivial manifold, that is $M = S^1$. Consider two charts $\varphi^{-1}, \psi^{-1} : (-\pi, \pi), (0, 2\pi) \to S^1$ given by $\theta \mapsto e^{i\theta}$. These cover S^1.

We can then take

$$X(\varphi^{-1}(t)) = \sin(t)\frac{\partial}{\partial t}$$

and choose a compatible function for $X(\psi^{-1}(t))$.

Note: If $E \xrightarrow{\pi} M$ is a vector bundle and $\sigma_1, \sigma_2 : M \to E$ are sections, then for any functions $f_1, f_2 : M \to \mathbb{R}$ We can take

$$(f_1\sigma_1 + f_2\sigma_2)(p) = f_1(p)\sigma_1(p) + f_2(p)\sigma_2(p).$$

In algebraic terms, this means C^k-sections of a vector bundle form a module over $C^k(M)$, which is the ring of C^k functions $M \to \mathbb{R}$.

Definition V.0.5

A linear map $\partial : C^\infty(M) \to \mathbb{R}$ is called a derivation at p provided that for all $f, g \in C^\infty(M)$ we have

$$\partial(f \cdot g) = f(p)\partial(g) + \partial(f)g(p).$$

To spell out linearity we want for $c \in \mathbb{R}$ that

$$\begin{aligned} \partial(cf) &= c\partial(f) \\ \partial(f + g) &= \partial(f) + \partial(g). \end{aligned}$$

For Non-Michigan students: 115/215 are single/multi-variable calculus.

Example V.0.4 (115/215 Example)

Take $M = \mathbb{R}$. The simplest derivation is $\partial(f) = f'(p)$.

For $f : \mathbb{R}^n \to \mathbb{R}$ we can take the directional derivative over a vector $v \in T_p\mathbb{R}^n$.

Likewise for $f : M \to \mathbb{R}$ where M is a C^∞ manifold for any $(p, v) \in T_pM$ we can take $\partial_v(f) \coloneqq Df_p(v)$.

This gives us a derivation $\partial_v : C^\infty(M) \to \mathbb{R}$.

Remark V.0.1

One can alternatively frame tangent vectors in terms of derivations on a manifold. Professor Spatzier thinks this is beautiful and also useless. One must always eventually work with charts or use a Lie group structure.

Note for the very interested reader: for C^r manifolds where $r < \infty$ these two notions are not actually equivalent, see [**tangentPlanes**] and this mathoverflow post

https://mathoverflow.net/a/358273

Announcements

- HW5 Deadline extended to 10/7 Friday 11pm

Definition V.0.6

Suppose we have two vector-bundles $\mathcal{V}, \mathcal{W}$ over M, N respectively. A map of vector bundles consists of two maps $\Phi : \mathcal{V} \to \mathcal{W}$ and $\phi : M \to N$ (C^r for whatever r you want) such that the following diagram commutes

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & \mathcal{W} \arrow[d, "\tau"] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

and also

$$\Phi\big|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \tau^{-1}(\phi(p))$$

is linear. Call (Φ, ϕ) a vector bundle isomorphism if there are inverses (Ψ, ψ) to Φ and ϕ respectively. In this case we call $\mathcal{V}, \mathcal{W}$ equivalent.

We call $\mathcal{V}$ trivial if it is equivalent to a trivial bundle $\mathbb{R}^n \times M$, with $\mathbb{R}^n \cong \pi^{-1}(p)$.

For the trival bundle $\tau : \mathbb{R}^\ell \times M \to M$ we have lots of sections, say $\sigma_i(p) = (e_i, p)$, $e_i \in \mathbb{R}^\ell$ a basis of $\mathbb{R}^\ell$. The σ_i are smooth vector fields and $\{\sigma_i(p)\}$ are linearly independent and in fact a basis for $\tau^{-1}(p)$ for all $p \in M$.

Thus we get ℓ sections of $\tau : \mathbb{R}^\ell \times M \to M$ which are linearly independent at every point. The converse also holds!

Proposition V.0.1

Let $\pi : \mathcal{V} \to M$ be a vector bundle of rank ℓ (that is $\pi^{-1}(p) \cong \mathbb{R}^\ell$). If there exist ℓ sections $\sigma_1, \ldots, \sigma_\ell$ which are linearly independent at every point then $\mathcal{V}$ is trivial (i.e., isomorphic to the trivial bundle of rank ℓ).

Proof. Consider the map

$$\mathbb{R}^\ell \times M \to \mathcal{V}$$

$$((a_1, \ldots, a_\ell), p) \mapsto \sum_{i=1}^{\ell} a_i \sigma_i(p).$$

Corollary V.0.2

The tangent bundle of a differentiable manifold is trivial if and only if there exist $\dim M$ many vector fields which are linearly indepenent at every point.

Remark V.0.2

Warning: There are two senses in which sections σ_i may be linearly independent. We can have that *in the space of sections*

$$\sum a_i \sigma_i = 0$$

implies $a_i = 0$. We can also have linear independence at every point, namely for every $p \in M$ we have

$$\sum a_i \sigma_i(p) = 0$$

implies $a_i = 0$. We'll call the latter notion linearly independent pointwise.

Consider $S^2 = M$. Then X has a non-zero vector field which is 0 somewhere but not 0 everywhere! X is linearly independent as a single vector in the space of sections, but not at every point.

Notice that

$$\dim\{\text{sections of } \mathcal{V} \to M\} = \infty$$

unless $\mathcal{V}$ or M is of dimension 0.

Last time: We looked briefly at derivations as point $p \in M$. We're going to continue to talk about them ☺

Example V.0.5

Consider a smooth vector field X on M and we define

$$\begin{aligned} \Delta : C^\infty(M) &\to C^\infty(M) \\ (\Delta f)(p) &= \partial_{X(p)}(f) \end{aligned}$$

where $\partial_{X(p)}$ is the directional derivative at p in the direction of $X(p)$ (see last time).

Then in fact we have

$$\Delta(f \cdot g) = f \cdot \Delta g + \Delta f \cdot g.$$

Example V.0.6

Consider $X = y\frac{\partial}{\partial x}, Y = x\frac{\partial}{\partial y}$ on $\mathbb{R}^3$. Then we're going to look at

$$\begin{aligned} (X \circ Y) &= \left(y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial y}\right) \\ &= y \cdot \left(\frac{\partial}{\partial} x\right)\frac{\partial}{\partial y} + yx\frac{\partial}{\partial x}\frac{\partial}{\partial y} \\ &= y\frac{\partial}{\partial y} + yx\frac{\partial^2}{\partial x \partial y}. \end{aligned}$$

What is this??? It's not a vector field... What about the other way

$$\begin{aligned} (Y \circ X) &= \left(x\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x}\right) \\ &= x(\frac{\partial}{\partial y}y) \cdot \frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\frac{\partial}{\partial x} \\ &= x\frac{\partial}{\partial y} + xy\frac{\partial^2}{\partial y \partial x}. \end{aligned}$$

We can view X, Y as $\Delta_X, \Delta_Y : C^\infty(M) \to C^\infty(M)$. Then look at it as $\Delta_{X \circ Y} = \Delta_X \circ \Delta_Y$ and likewise.

Now we can consider

$$X \circ Y - Y \circ X = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}.$$

This is a vector field!

Theorem V.0.3

Let M be a smooth manifold with smooth (C^1 is enough) vector fields X, Y. Then in fact $X \circ Y - Y \circ X$ is a derivation $C^\infty(M) \to C^\infty(M)$.

Because of this we'll call $[X, Y] \coloneqq X \circ Y - Y \circ X$, and we'll call it the Lie bracket of X and Y.

Proof. Linearity is immediate. We just need to check the product rule. Namely we must check

$$[X, Y](fg) = ([X, Y]f)g + f([X, Y]g).$$

Theorem V.0.4

Every C^∞ derivation δ at p defines a tangent vector to p, i.e., there exists $v \in T_pM$ such that $\delta = \partial_v$.

Corollary V.0.5

Every derivation $\Delta : C^\infty(M) \to C^\infty(M)$ defines a vector field.

Example V.0.7

Take $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$. Then $[X, Y] = 0$, as the mixed partials are equal on smooth functions.

We don't see the geometry in these formulas. We need to see the geometry!!!

Lemma V.0.6

If $v \in T_pM$, then $\partial_v(f) \coloneqq D_pf \cdot v$ is a derivation. Moreover if $\partial_v = \partial_w$, then $v = w$, where $v, w \in T_pM$.

Proof Idea. We've already seen the first property. For the second, take a coordinate chart (U_ρ, ρ). We can take $D\rho_p(v) = \frac{\partial}{\partial x_1}$ and $D\rho_p(w) = \frac{\partial}{\partial x_1}$. We can do this unless $w = a \cdot v$. This works because we have linearly independent vectors $D\rho_p(v), D\rho_p(w)$ and take a linear map A taking these to e_1, e_2. Replace ρ by $A \circ \rho$.

Look at $x_1 : \rho(U_\alpha) \to \mathbb{R}$, which is the coordinate (projection) map to the first coordinate. We see that

$$\frac{\partial}{\partial x_1}x_1 = 1$$
$$\frac{\partial}{\partial x_2}x_1 = 0.$$

This will show ∂_v, ∂_w disagree on this function. . . but wait! x_1 is only defined on a small neighborhood U_ρ of p.

We need to understand the relationship between $C^\infty(M)$ and $C^\infty(U)$ for U a neighborhood of some $p \in M$. We know one map

$$C^\infty(M) \longrightarrow C^\infty(U)$$

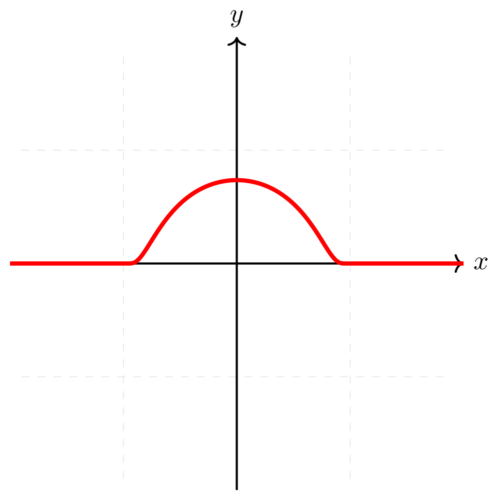
$$f \longmapsto f\big|_U$$

$$\overline{g} \xleftarrow[?]{} g$$

so that $\overline{g}\big|_U = g$.

Need: Bump functions

- Ad: This comes up a lot
- On $\mathbb{R}$ we want a function $f(x)$ that looks like



  Warning: Cannot do in $C^\omega(\mathbb{R})$.
- We take
$$\psi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } x \in (-1,1) \\ 0 & \text{if } x \notin (-1,1) \end{cases}.$$
- Likewise for $\overline{\psi} : \mathbb{R}^n \to \mathbb{R}$, where we want $\overline{\psi} \equiv 0$ outside $B_1(0)$ and $\overline{\psi}$ is C^∞ on $\mathbb{R}^n$, and $\neq 0$ on $B_1(0)$. We take $\overline{\psi}(x) = \psi(|x|^2)$.
- We can generalize this. Want $\phi \equiv 1$ on $B_{1/2}(0)$, $\phi \equiv 0$ outside $B_1(0)$. We pick something like $\overline{\psi}(x) \cdot \psi\left(\frac{1}{2|x|^2}\right)$.
- We can use these to construct our $\overline{g}$.

We can prove Lemma V.0.6 by taking everything locally, and using bump functions

Corollary V.0.7

There exists n linearly independent derivations since $T_pM \hookrightarrow \{\text{derivations at } p\}$.

Lemma V.0.8

Derivations at p form an n-dimensional vector space.

Proof. It is at least n-dimensional since $\{\delta_v \mid v \in T_pM\}$ is n-dimensional.

Now for $f \in C^\infty(M)$, we can use bump functions to look locally at $C^\infty(U_\rho)$, and work locally around 0 in $\mathbb{R}^n$. Consider

$$\mathcal{I}_p = \{f \in C^\infty \text{ near } p \mid f(p) = 0\}$$
$$\mathcal{I}_p^2 = \{\sum f_i g_i \mid f_i g_i \in \mathcal{I}_p\}$$

Then we see that

$$\delta(f \cdot g) = \delta(f) \cdot g(p) + f(p) \cdot \delta(g) = \delta(f) \cdot 0 + 0 \cdot \delta(g) = 0.$$

Therefore, the derivations vanish on $\mathcal{I}_p^2$. Thus the derivations embed into $(\mathcal{I}_p/\mathcal{I}_p^2)^*$ (which is a vector space.

We prove another result Corollary V.0.12 to finish this proof.

Corollary V.0.9

We have that

$$\{\text{derivations at } p\} = \{\partial_v \mid v \in T_pM\} = (\mathcal{I}_p/\mathcal{I}_p^2)^*$$

by equality of dimensions.

Lemma V.0.10

Suppose $f : U \to \mathbb{R}$, C^∞ with $0 \in U$ open in $\mathbb{R}^n$, then there exist C^∞ functions f_i on U such that

$$f(x) = f(0) + x_i f_i,$$

with $f_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof. By Fundamnetal theorem of calculus, we have that

$$\begin{aligned} f(x) &= f(0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tx)\,\mathrm{d}t \\ &= f(0) + \int_0^1 \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx)\,\mathrm{d}t \\ &= f(0) + \sum_{i=1}^n x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\,\mathrm{d}t}_{f_i \in C^\infty}. \end{aligned}$$

Lemma V.0.11

Suppose f is C^∞ in $\mathbb{R}^n$ near 0. Then there exist C^∞ functions f_{ij} on $\mathbb{R}^n$ near 0.

$$f(x) = f(0) + \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j=1}^n x_i x_j f_{ij}(x).$$

Proof. Apply last lemma to $f_i(x)$.

Corollary V.0.12

$\dim(\mathcal{I}_p/\mathcal{I}_p^2) = n$.

Proof. Apply the lemma just above. For any $\delta \in (\mathcal{I}_p/\mathcal{I}_p^2)^*$ we can take locally

$$\delta(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0) \cdot \delta(x_i),$$

since δ vanishes on $\mathcal{I}_p^2$, so we can say it vanishes on $x_i x_j f_{ij}(x)$ above. The proof in general is similar.

Dealing with germs of functions. Fix $p \in M$ a C^∞ manifold. Suppose $f \in C^\infty$ is defined in an open neighborhood U_f of p, and $g \in C^\infty$ is defined on $U_g \ni p$. We say f, g define the same germ if

$$f\big|_{U_f \cap U_g} = g\big|_{U_f \cap U_g}.$$

Note: f_1, f_2 having the same germ $[f]$ and g_1, g_2 having the same germ h, then $f_1 + g_1, f_2 + g_2$ define the same germ. Really f_1, f_2 having the same germ defines an equivalence relation.

Partial derivatives are well-defined on germs. They're somewhere "between local and infinitesimal." So note, we have

(1) Globally defined functions $f \in C^\infty(M)$
(2) Locally defined functions f on some open neighborhood of p.
(3) Germs at p
(4) Partial derivatives at p.

where the order reflects closer and closer to infinitesimal information. Note that $\mathcal{I}_p, \mathcal{I}_p^2$ make sense for germs. Furthermore our above discussion tells us

$$\dim \mathcal{I}_{p,\mathrm{germ}}/\mathcal{I}_{p,\mathrm{germ}}^2 = n.$$

It also has the falling property. Given $\varphi : M \to N$, and a germ $[f]$ on N. Then $[f \circ \varphi]$ in fact defines a germ on N. In representatives this takes $f : N \supseteq U \to \mathbb{R}$ and compose $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$. This does not depend on the representatives. We then get a map

$$\begin{aligned} \varphi^* : \mathcal{I}_{\varphi(p)}/\mathcal{I}_{\varphi(p)}^2 &\to \mathcal{I}_p/\mathcal{I}_p^2 \\ [f] &\mapsto [f \circ \varphi]. \end{aligned}$$

We have a duality! In diagrams we have

$$\begin{tikzcd} T_p M \arrow[r, "D\varphi = \varphi_*"] \arrow[d, leftrightarrow] & T_{\varphi(p)} N \arrow[d, leftrightarrow] \\ (\mathcal{I}_p/\mathcal{I}_p^2)^* \arrow[r, "(\varphi^*)^*"] & (\mathcal{I}_{\varphi(p)}/\mathcal{I}_{\varphi(p)}^2)^* \end{tikzcd}$$

$$\mathcal{I}_p/\mathcal{I}_p^2 \xleftarrow[\varphi^*]{} \mathcal{I}_{\varphi(p)}/\mathcal{I}_{\varphi(p)}^2$$

because of our discussion above concerning identifying $T_p M$ with derivations and derivations with the middle row. Then we can think of all this as

$$T_p^* M = \mathcal{I}_p/\mathcal{I}_p^2$$

which is the cotangent space at p. Then we have a duality

$$\varphi^* : T^*N \to T^*M$$
$$\varphi_* : T_*M \to T_*N.$$

Announcements

- Move midterm by 1 or 2 weeks.
- Bonus 5 + HW6 Due Friday 11PM

Last time: Take a chart $\varphi : U \to \mathbb{R}^n$ which takes p to 0. We really want to take a smooth f on $\varphi^{-1}(B_\varepsilon(0))$ to a smooth f which is 0 outside $\varphi^{-1}(B_{2\varepsilon}(0))$.

For this, we take a bump function $\rho : \mathbb{R}^n \to \mathbb{R}$ which is 0 outside $B_{2\varepsilon}(0)$ and which is 1 inside $B_\varepsilon(0)$.

Now back to immersions, submersions, and regular values.

Recall V.0.8

Let M, N be C^∞ manifolds, $q \in N$ is a regular value of $f : M \to N$ if $Df_p : T_pM \to T_qN$ is surjective for all $p \in f^{-1}(q)$. Note that if $q \notin \text{Image}(f)$ then q is regular

We recall a bit of measure theory. Let $A \subseteq \mathbb{R}^n$ and define $\text{vol}(A) = \int_A 1 \,\mathrm{d}x$. Lebesgue measure is preferable, but we say A mas measure zero if $\text{vol}(A) = 0$.

Definition V.0.7

Suppose M is a C^1 manifold. Consider $B \subseteq U$ some chart. We say B has measure zero if $\varphi(B)$ has measure zero.

Note this is well-defined since the transition maps are C^1, which gives for two charts φ, ψ that

$$\begin{aligned}\text{vol}(\psi(B)) &= \int_{\psi(B)} 1 \,\mathrm{d}x \\ &= \int_{\varphi(B)} \det(D(\psi \circ \varphi^{-1})_x \,\mathrm{d}x\end{aligned}$$

Note then that volume changes across charts; but zero volume is well-defined.

Say $B \subseteq M$ has zero measure if for all charts (U, φ) we have $\text{vol}(\varphi(B \cap U)) = 0$.

We say $A \subseteq M$ has full measure if $M \setminus A$ has 0 measure.

Theorem V.0.13 (Sard's Theorem)

Let M, N be C^∞ manifolds and $f : M \to N$ be smooth, then the set of regular values has full measure.

Warning: needs C^∞ (at least some C^k for k large enough).

Proof Idea. Approximate f by a linear map $Df = L$. Actually lol Professor Spatzier doesn't know

Example V.0.9

Here's an application. To show $\text{SL}_n(\mathbb{R})$ is a manifold it suffices to show 1 is a regular value of det since $\text{SL}_n(\mathbb{R}) = \det^{-1}(1)$.

Note $\det(\lambda A) = \lambda^n \det(A)$ for $A \in \text{GL}_n(\mathbb{R})$.

Claim

If 1 is not a regular value, then neither is λ^n for $\lambda \neq 0$.

If λ is nonzero, then $A \xmapsto{m_\lambda} \lambda \cdot A$ is invertible. Suppose λ^n is a regular value. Then fix A so that $\det(A) = 1$. Then we see that $D_A \det$ is surjective if and only if

$$D_A(\det \circ m_\lambda) = D_{\lambda A} \det \circ D_A(m_\lambda)$$

is surjective, which follows by regularity and invertibility of m_λ.

Now by Sard, since any $\{\lambda^n \mid \lambda \neq 0\}$ doe snot have measure zero, 1 must be a regular value.

The simplest immersion is given for $k \leq n$ as

$$\mathbb{R}^k \hookrightarrow \mathbb{R}^n$$
$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0).$$

Proposition V.0.14

Suppose $f : M \to N$ where $k = \dim M, n = \dim N$ is an immersion at p, so that Df_p is injective. Then there exist charts (U, φ) at p and (V, ψ) at $f(p)$ so that $\psi \circ f \circ \varphi^{-1}$ has the form given above.

Proof. Fix arbitrary charts (U, φ) and (V, ψ) as well. We'll work on the charts, and this is good enough. From now on conflate f with its coordinate map.

We know Df_p is injective and $n \times k$ so we can look at

$$Df_p = A = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_k \\ \vdots \end{pmatrix} =: \widehat{A}.$$

We know that the rank of A is k, so there exists k linearly independent rows. We can compose with an inverse to these rows to get

$$\begin{pmatrix} \mathrm{Id} & * \\ * & *. \end{pmatrix}$$

Then $\widehat{A} \circ F : \mathbb{R}^k \to \mathbb{R}^n$ extends to $\mathbb{R}^n \to \mathbb{R}^n$.

This gives us for the derivative

$$\begin{pmatrix} \mathrm{Id} & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

We can then get a local diffeomorphism to define coordinates in $\mathbb{R}^n$. By construction in this chart f has the desired form...

Announcements

- Midterm remains on Wednesday October 19th in class.

V.1. **Flow on Vector Fields**

How do we flow on vector fields? That is how do we think of the vector field as a field of *force/acceleration* for a particle.

Well we wish to fill up a manifold M with curves and then differentiate them! That tells us the vector field at every point. However, we must avoid crossings so we can decide where to take the vector field

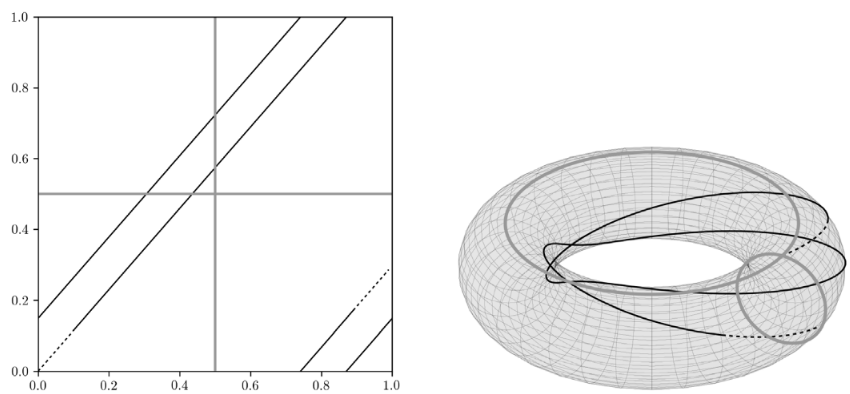
Recipe:

(1) Fill up M with disjoint differenitable curves c_i.
(2) Then take $X(p) = \dot{c}_{p_i}(p)$ for c_{p_i} a curve through p.
(3) What about $C^0, C^1, \ldots$?
(4) Along a C^∞-curve $c(t)$ the vector field is C^∞, "transversally" to the curves regularity is unclear. But if $c \mapsto c_{p_i}$ is sufficiently differentiable, then all is good.

Example V.1.1

Strange example. Take an angle $\alpha \notin 2\pi\mathbb{Q}$ and take a line through the flat torus which forms an angle α.

For convenience here is a picture of curves in the flat and curved torus



(a) *Flat torus as square in* $\mathbb{R}^2$ *with edges identified.*

(b) *Curved torus embedded in* $\mathbb{R}^3$.

1

This picture is taken from [**eltzner**].

We want to go the opposite direction. Given a vector field, how do we produce a flow which incudes it?

Definition V.1.1

Let X be a vector field on M. We call $c : (a, b) \to M$ a solution curve for X provided that for all $t_0 \in (a, b)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} c(t) = X(c(t_0)).$$

In coordinates, for a C^∞-chart U take standard vector fields $\frac{\partial}{\partial x_i}$. Then we know

$$X\big|_U = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}.$$

X is C^∞ if and only if a_i is C^∞.

Write $c(t) = (c_1(t), \dots, c_n(t))$ in these coordinates. Then we have that

$$\dot{c}(t_0) = (\dot{c}_1(t_0), \dots, \dot{c}_n(t_0)).$$

To require that $X(c(t_0))$ means to require that

$$\sum_{i=1}^{n} a_i(c(t_0)) \frac{\partial}{\partial x_i}(c(t_0)) = \sum_{i=1}^{n} \dot{c}_i(t_0) \cdot \frac{\partial}{\partial x_i}(c(t_0)).$$

Therefore we must have $a_i(c(t_0)) = \dot{c}_i(t_0)$. We have that the a_i are given by the vector field. What's not given is the C's

"Bacid ODEs, Vague." For a C^1-manifold we can solve uniquely if a_i are Lipschitz functions then the solutions are unique.

Why vague? For what time t do we get a solution. Well something like

$$c_i : (-\varepsilon(p), \varepsilon(p)) \to \mathbb{R}^n$$

within coordinates. This occurs because the "speed" along which c_i goes on the vector field may escape to infinity, and then we don't know what to do at $\varepsilon(p)$.

More precise version. Let X be some C^∞ vector field. There exists an $\varepsilon > 0$ and a $\delta > 0$ such that for all $q \in B_\varepsilon(p)$ ther exists a solution to the ODE on the inversal $(-\delta, \delta)$.

This is called a local solution. We have existence and uniqueness of local solutions. We will not prove this because it is painful, it is an application of the Contraction Mapping Theorem.

Definition V.1.2

Call M a C^∞ manifold. We say a vector field X on M is <u>complete</u> if solution curves exist through any point for all time.

Ad: Nearly impossible to actually calculate solutions to these curves (supercomputers can approximate), except in special cases (ex. linear ODEs). Actual computations is the Quantitative, explicit solutions, and would be called ODEs.

Dynamical systems would be considering the Qualititative study of vector fields! This goes back to Poincaré.

Lemma V.1.1

If M is a compact C^∞ manifold and X is a C^K vector field for $k \geq 1$ then X is complete.

Proof. For short time, on a neighborhood U of $p \in M$ we have a solution curve $c_p : (-\varepsilon(p), \varepsilon(p)) \to U$

Then there are finitely many $p_1, \dots, p_\ell$ with $\bigcup U_{p_\ell} = M$. Take $\varepsilon := \min \varepsilon_{p_i}$.

For each $q \in U_{p_i}$ we can flow along the field for $(-\varepsilon, \varepsilon)$. Uniqueness of solutions on $(-\varepsilon, \varepsilon)$ implies that things will agree on the overlap. We can keep flowing in either direction forever!!! This finishes the proof.

Warning: The curve exists for all time but may have finite length! We may come to a stop at a stationary point on the vector field!!!

Definition V.1.3

Let X be a complete vector field on M. Call a map $\Phi : \mathbb{R} \times M \to M$ so that $\phi(t, p)$ for fixed $p \in M$ and varying t is a solution curve at p the <u>flow generated by X</u>.

We define $\varphi_t(p) := \Phi(t,p)$. We can call φ_t the (global) flow determined by X.

Next time: This gives you an action of the real numbers on M.

Midterm in class on Wednesday October. Things you should know:

- Charts
- Tangent Stuff: T_pM, TM, T^*M.
- Vector bundles and sections
- Basic Examples/Counterexamples
- Constructions
  - Products
  - Group Actions
  - Level sets via Regular Value Theorem (Remember: Sard's Theorem, easy proof that SL_n is a manifold)

Midterm: 50 minutes, ≈4-5 questions, should be able to answer questions in ≈10-15 minutes.

CONTENT FOR MIDTERM I STOPS HERE
(does not include flows)

Recall that uniqueness of ODEs tells us that if $q = \varphi_{t_0}(p)$ then

$$\varphi_s(q) = \varphi_{t_0+s}(p)$$

by uniqueness of ODEs. Therefore

$$\varphi_{s+t_0}(p) = \varphi_s(\varphi_{t_0}(p))$$
$$\varphi_{s+t_0} = \varphi_s \circ \varphi_{t_0},$$

where φ_{s+t_0} is defined. If X is complete, we get an $\mathbb{R}$-action on X, commonly called a flow on X.

Definition V.1.4

Now suppose X is a vector field and $F : M \to N$ is a diffeomorphism. Then we can define the pushforward of X by F.

$$(F_*(X))(p) := D_qF(X(q)).$$

where $p = F(q)$. Then $F_*(X)$ is a vector field on M.

If $Y = F_*(X)$ we say that X, Y are F-related (still makes sense for local diffeomorphisms). We also say X and Y are $C^?$-conjugate if F is $C^?$.

Let φ_t, ψ_t be flows for X, Y (vector fields on M, N). If $F : M \to N$ is a diffeomorphism and $Y = F_*(X)$, what can we say about the flows?

Fix $q \in M$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(F(\varphi_t(q)))\Big|_{t=t_0} &= DF_{\varphi_{t_0}(q)} \cdot X(\varphi_{t_0}(q)). \\ &= Y(F(\varphi_{t_0}(q)))\end{aligned}$$

From this and the uniqueness of ODEs we can see that

$$\psi_t = F \circ \varphi_t \circ F^{-1}.$$

Namely, by the chain rule again we have for $p \in N$ that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\varphi_t(F^{-1}(p)))\Big|_{t=t_0} = Y(F(\varphi_{t_0}(F^{-1}(p)))).$$

More generally: one might have a map $\pi : M \to N$ and flows φ_t, ψ_t on M, N where $\psi_t \circ \pi = \pi \circ \varphi_t$. Something like this would be called a "quotient of φ_t or a factor of φ_t."

Suppose X, Y are vector fields on M. Recall that a vector field X can be thought of as a derivation $X : C^\infty(M) \to C^\infty(M)$. We had an unproved lemma from last time we discussed commutators

Lemma V.1.2

$[X, Y] := Y \circ X - X \circ Y$ is a vector field, that is the Lie bracket of two vector fields is a vector field.

Proof. We must verify the product rule, since linearity of $[X, Y]$ is clearly. Thus we compute

$$\begin{aligned}
(X \circ Y)(fg) &= X(Y(f)g + fY(g)) \\
&= (X \circ Y)(f)g + Y(f)X(g) + X(f)Y(g) + f(X \circ Y)(g) \\
(Y \circ X)(fg) &= (Y \circ X)(f)g + X(f)Y(g) + Y(f)X(g) + f(Y \circ X)(g) \\
[X, Y](fg) &= [X, Y](f) \cdot g + f \cdot [X, Y](g).
\end{aligned}$$

Perfect!

Message: $[X, Y]$ measures how much X, Y do not commute. What does it mean in terms of vector fields?

Suppose φ_t, ψ_s are local flows of X, Y respectively. Consider the following sequence of moves starting at $p \in M$

$$\begin{aligned}
&\varphi_t(p) \\
&\psi_t(\varphi_t(p)) \\
&\varphi_{-t}(\psi_t(\varphi_t(p))) \\
&\psi_{-t}(\varphi_{-t}(\psi_t(\varphi_t(p)))).
\end{aligned}$$

As $t \to 0$ this goes to p by a continuity argument. But what about the *derivative* at $t = 0$.

Example V.1.2

For $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$ this commutator is zero.

Aside: maps of constant rank. A submersion gives rise to level sets. For $f : M \to N$ a C^k map consider

$$p \mapsto \operatorname{rank} Df_p := \dim \operatorname{Image} Df_p.$$

f is a submersion if and only if this map is constant and the rank of f is always $\dim N$.

Definition V.1.5

f has constant rank if $p \mapsto \operatorname{rank} Df_p$ is constant in p.

Theorem V.1.3 (Constant Rank Theorem)

If f has constant rank, then $f^{-1}(q)$ is a C^k=submanifold

Idea of Proof. Locally we can take a projection g from $f(M)$ to $\operatorname{Image} Df_p$, which is a linear subspace in coordinates. This is a submersion, and then we use local submersion theorem.

Consider C^∞ vector fields X, Y on a manifold M. We know

$$[X, Y] \coloneqq X \circ Y - Y \circ X$$

defines a vecto field, where we view these as derivations. Now consider a special situation, where X, Y don't vanish on M (or some $U \subseteq M$ open).

Proposition V.1.4

Assume $[X, Y] = 0$. Then there exist coordinates on a chart such that $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$.

More generally, if $X_1, \ldots, X_k$ are vector fields so that $\{X_i(p)\}$ are linearly independent and $[X_i, X_j] = 0$ for all i, j, then there exist coordinate charts about p so that $X_i = \frac{\partial}{\partial x_i}$.

Idea of Proof. If $k = 1$, then $X \rightsquigarrow \varphi_t$ a local flow of X. Let $\overline{U} \subseteq U$ so that the flow is defined.

Pick T a submanifold of dimension $m - 1$. Then $T = \{(0, y_2, \ldots, y_n)\}$ in some coordinates. We may also pick T so that it is transversal to $X(p)$. Give coordinates on $\overline{U}$ as

$$\Phi : (t, y_2, \ldots, y_n) \mapsto \varphi_t(0, y_2, \ldots, y_n).$$

It suffices to check $D\Phi_{(0,\ldots,0)}$ is a local diffeomorphism at p (which is 0 in local coordinates), and then Φ is a chart. We compute

$$D\Phi_{(0,\ldots,0)} = \begin{pmatrix} * & \cdots & 0 & \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1. \end{pmatrix}$$

(invertibility and this computation comes from transversality.

If $k = 2$, Let $[X, Y]$. Without loss of generality, $X = \frac{\partial}{\partial x}$ in local coordinates. We'll cheat and look at $\dim M = 2$ (it will be clear how to generalize).

We may then let $Y = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$. We compute that

$$\begin{aligned} [X, Y] &= \frac{\partial}{\partial x}\left(a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}\right) - a(x, y)\frac{\partial^2}{\partial x^2} - b(x, y)\frac{\partial}{\partial x \partial y} \\ &= \frac{\partial a}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b}{\partial x}(x, y)\frac{\partial}{\partial y}. \end{aligned}$$

Thus $a(x, y) = a(y), b(x, y) = b(y)$. Let ψ_t be a local flow for y. We know

$$\Psi(x, t) = \psi_t(x, 0).$$

Then we can use this as a coordinate chart. If $\dim M > 2$, take a submanifold (local) through p of codimension which is tranversal to both X, Y. Then apply the same trick as when $k = 1$. We also take a flow ϕ_t and set

$$\Psi(s, t, z) = \Psi_t \circ \varphi_s(z).$$

You can then do it for any number of vector fields.

For k and $\dim M$ arbitrary. Find a transversal submaniofld to $X_1, \ldots, X_k$ at p. We then compose flows just as above

$$(t_1, \ldots, t_n, z) \mapsto \varphi^1_{t_1} \circ \cdots \circ \varphi^n_{t_n}(z).$$

Because the flows commute (Lie bracket zero) this will give exactly what we need. We can move $\phi^i_{t_i}$ to the front. This is a corollary of the discussion for $k = 2$ (where we take the derivatives explicitly).

One can always find tranversals because we're only working locally, and so in coordinates we can just take a linear subspace tranversal to $X_i(p)$ for all i.

Corollary V.1.5

If $[X, Y] = 0$ then their local flows commute.

Proof. Look at the case $k = 2$ above.

Consider any X, Y to be C^∞ vector fields. Let φ, ψ be local flows for X, Y. Then we can consider $C(t)$ to be defined as

$$C(t) = \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}}(p)$$

for $t > 0$. This is in Spivak's text on differentiable manifolds. We define $C(-t)$ similarly but flowing in the opposite direction.

Theorem V.1.6

Let X, Y be C^∞ vector fields. Then $C(t)$ is differentiable and $C'(0) = [X, Y](p)$.

Midterm Announcements:

- Graded–will get it back today
- 120 points possible out of 100 because Problem 2 had an error
- Median was around 100. Very good job
- Midterm given back in last 5 minutes of class.

Recall V.1.3

If $F : M \to N$ is C^∞, and X, Y vector fields on M, N respectively then we call X, Y F-related if

$$\mathrm{d}F_p(X(p)) = Y(F(p)) \in T_{F(p)}N.$$

Call this $X \sim Y$. If $X_2 \sim Y_2$ are well, then $[X, X_2] \sim [Y, Y_2]$.

To show this, it's convenient to know the flow of $[X, X_2]$. Letting φ, ψ be local flows for these respectively, we claim the flow for $[X, X_2]$ is given by

$$C_t = \psi_{\sqrt{t}} \circ \varphi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}}.$$

Well consider

$$G(s,t) = \psi_{-s} \circ \varphi_{-t} \circ \psi_s \circ \varphi_t(p).$$

It is then clear that

$$\frac{\partial}{\partial s}(G(s,0))\Big|_{s=c} = \frac{\partial}{\partial s}\Big|_{s=c}(p) = 0.$$

One must then use Taylor Expansion up to order s^2, t^2, st in order to derive the result.

If X has solution curve $\varphi_t(p)$, then $F(\varphi_t(p))$ is a solution curve for Y if X, Y are F-related. Then by the characterization of the flow $[X,Y]$

V.2. **Distributions**

This means way too many things in math. We might also call them k-plane fields.

Consider a manifold M which is C^∞, cosndier taking T_pM to $\mathrm{Gr}_k(T_pM)$, which si k-dimensional vector subspaces of T_pM. Fancy: Make a fiber bundle out of

$$\mathrm{Gr}_k(M) = \coprod_{p \in M} \mathrm{Gr}_k(T_pM).$$

Make this a smooth manifold using the local product structure of TM. In fact

$$\mathrm{Gr}_{k,n} \to \mathrm{Gr}_k(M) \xrightarrow{\pi} M$$

is a fiber bundle, where $n \coloneqq \dim M$.

Definition V.2.1

A distribution is a smooth section of this fiber bundle. In down to earth terms, $D(p) \subseteq T_pM$ is a k-dimensional subspace, spanned by say $\langle v_1, \dots, v_k \rangle$. Do this for every point.

Locally we get $v_1(q), \dots, v_k(q)$ where q is in a neighborhood of p. We require that the $v_i(q)$ are smooth vector fields on this neighborhood p. We could do stupid things, like making $v_i(q)$ be changed by a linear transformation at rational points... so instead we just require there is a choice.

Thus smooth distributions of dimension k are given by the following data

- For all $p \in M$, $D(p) \subseteq T_pM$ is a k-dimensional subspace.
- To define smoothness of D, it suffices to do it locally. I.e., for all $p \in M$, there exists a neighborhood U of p and there exist smooth vector fields $v_1, \dots, v_k$ on U so that
  (1) For all i,q, $v_i(q) \in D(q)$
  (2) For all i, q, $v_1(q), \dots, v_k(q)$ are linearly independent.

  Equivalently the span of $v_1(q), \dots, v_k(q)$ is $D(q)$ for all $q \in U$.

There are two types of distributions, the boring ones (which are most important), and the exciting ones (which are not used very much).

Example V.2.1

Let $\mathbb{R}^n = M$, and for each $p \in \mathbb{R}^n$ let $D(p) = \mathbb{R}^k = \{(x_1, \dots, x_k, 0, \dots, 0)\}$. This is spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$.

Example V.2.2

Suppose G is a Lie group that acts on some manifold M. Suppose for all $p \in M$, we have G_p is discrete. Then $\{G \cdot p\}$ for $p \in M$ will cut up M into submanifolds (we haven't shown this formally).

The distribution will then be given by $D(p) = T_p(G \cdot p)$.

Example V.2.3

Take V a nonvanishing vector field on $\mathbb{R}^n$. We can take $D(p) = V(p)^\perp$.

Consider $M = \mathbb{R}^n \setminus \{0\}$, and take $V(p) = p$. This is exactly the tangent spaces to spheres of certain radii. This is actually a Lie group example–it's $\mathrm{SO}(n)$ acting on M. To see explicitly the vector fields, one can think of polar coordinates + the angles.

Even more explicitly one can look at one coordinate being nonzero and then take a radial vector field there.

Example V.2.4

Consider the Heisenberg group

$$\mathrm{Heis} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Consider tangent vectors

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which are tangent vector fields at the identity. Then $\overline{A}_1, \overline{A}_2$ are left-invariant vector fields. We can consider $D = \mathbb{R} \cdot \overline{A}_1 + \mathbb{R} \cdot \overline{A}_2$.

This is not a Lie group vector field, and so is much more complicated.

Stuff:

- Problem 3.2 is incorrect as stated. If $S \subseteq M$ is a submanifold and X is a vector field on M which is tangent to S, then if X is tangent to S then for all $p \in S$, the integral curve $\theta^{(p)}(t)$ (the flow) of X is contained in S for *small* values of t.

  The problem stated was for all values of t, obviously false.
- Hint for Problem 2: If $M + v, N$ intersect, then $v = y - x$ for some $y \in N, x \in M$. Consider the map $F : M \times N \to \mathbb{R}^n$ given by $F(x, y) = y - x$ and apply Sard's Theorem.
- Last time: distributions (k-plane fields), $V(p) \subseteq T_pM$ a k-dimensional subspace..

There are two kinds of distributions

- integrable (tractable)
- non-integrable (more fun)

Definition V.2.2

We call a k-plane V integrable provided that for all $p \in M$ there exists a coordinate chart (U, φ) such that for all $x \in M$,

$$\{(x_1, \ldots, x_k, 0, \ldots, 0) \mid x_i \in \mathbb{R}\} = D\varphi(V(x)) \subseteq T_{\varphi(x)}\mathbb{R}^n \cong \mathbb{R}^n.$$

Example V.2.5

Take $M = \mathbb{R}^n$, and $V(p) = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle$.

This is in particular an example of a "foliation," which we will define now. Namely the foliation is given by the partition of $\mathbb{R}^2$ as

$$\mathbb{R}^2 = \bigcup_{x \in \mathbb{R}} \mathbb{R} \times \{x\}.$$

Non-Example V.2.6

The Heisenberg group from last time, namely if

$$\text{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Then if we take

$$V(1) = \left\langle \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

We can of course set $V(g) = DL_g(V(1))$.

Definition V.2.3

Let M be a C^∞ manifold. A foliation is a partition $\mathcal{F}$ of M such that for all $x \in M$ we have F_x is an immersed submanifold of M, and

$$F_x \cap F_y \neq \emptyset \iff F_x = F_y.$$

(i.e., F defines an equivalence relation).

Furthermore, we require that for all $p \in M$ there exists a coordinate chart (U, φ) such that for each $x \in U$, if V_x is the connected component of $F_x \cap U$ in U, then $\varphi(V)$ locally looks like $\mathbb{R}^k \times \{0\} + \varphi(x)$ (aka looks locally like the above example).

We take the connected component in case F_x "loops back" into U.

If $\mathcal{F}$ is a foliation, then call F_x the leaf of $\mathcal{F}$ through x. Then we can define a distribution $V(p) = T_pF_p$, which is a k-dimensional distribution, C^∞.

Consider: Let X, Y be vector fields on M such that for all $p \in M$, we have $X(p), Y(p) \in V(p) := T_pF_p$ for some foliation F. By the Homework 7 Problem 3c we know that $[X, Y](p) \in V(p)$.

Fact: If $V(p)$ is the tangent distribution to a foliation $\mathcal{F}$ (i.e., $V(p) = T_pF_p$), then for any two vector fields X, Y with $X(p), Y(p) \in V(p)$ for all p, we have $[X, Y](p) \in V(p)$.

Definition V.2.4

Given any smooth k-dimensional distribution V on a C^∞-manifold M, we call V involutive if for any two vector fields X, Y with $X(p), Y(p) \in V(p)$ (tangent to V) for all p we have $[X, Y](p) \in V(p)$ for all p.

Theorem V.2.1 (Frobenius Theorem)

A distribution is involutive if and only if it is integrable (defined with charts).

Proof. This is in [**lee**], p490. The $\Longleftarrow$ direction we just did with HW 7 Problem 3

Example V.2.7

Give $p + \mathbb{R}^2$ as a foliation on $\mathbb{R}^3$, with V its corresponding distribution. Now quotient out by $\mathbb{R}^3$, so then $\overline{V}(\overline{p}) = D\pi_p \cdot V(p)$ where $\pi : \mathbb{R}^3 \to \mathbb{R}^3/\mathbb{Z}^3 =: \mathbb{T}^3$.

This gives us 2-tori foliating $\mathbb{T}$.

Mess it up a little, rotate $V(p)$ by an angle irratioanl with the embedded $\mathbb{R}^2$. Namely consider a foliation $\mathbb{R} \cdot v_1 + \mathbb{R} \cdot v_2$ where v_1, v_2 are irrational with respect to $\mathbb{Z}^2$.

We may then push this down to $\mathbb{T}^3$ as before (check this is well-defined...). Then $\mathbb{T}^3$ is foliated by "planes" (they cannot close up) densely.

Proof of Frobenius, in special case. Suppose for all p there exists a U neighborhood of p with vector fields $X_1, \ldots, X_k$ with $\langle X_1(q), \ldots, X_k(q) \rangle = V(q)$ such that for all i, j we have $[X_i, X_j] = 0$.

Then Frobenius holds. By last Friday, local flows φ_i associated with X_i commute. We can then build an immersion

$$(t_1, \ldots, t_k) \mapsto \varphi_k(t_k) \circ \cdots \circ \varphi_2(t_2) \circ \varphi_1(t_1) \cdot p.$$

Proof of Frobenius, in general. Let $X_i = (D\pi)^{-1}\left(\frac{\partial}{\partial x_i}(\pi(q))\right)$ (for an adapted chart). If $Y_i = \frac{\partial}{\partial x_i}(p)$, the these are π-related, and so the X_i commute. Namely we know $[X_i, X_j]$ is tangent to the chart V, and then

$$D\pi([X_i, X_j]) = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0,$$

thus $[X_i, X_j] = 0$, showing these commute.

Great! Then the special case implies the general theorem. This proof is only local, but we can do this globally as well, which we will do in the next section.

VI. Lie Groups/Lie Algebras

Let G be a C^∞ Lie group. We want to look at left invariant vector fields. I.e. we have $V(g) = DL_g(V(1))$, and this is clearly a vector space. Its dimension is $\dim G$.

Definition VI.0.1

Let G be a C^∞ Lie group. We define

$$\operatorname{Lie} G := \mathfrak{g} := \{\text{ left invariant vector fields}\} \cong T_1 G \cong \{\text{ right invariant vector fields}\},$$

which is a vector space often called the Lie algebra of G. Its dimension is $\dim G$ as mentioned above.

This comes with extra structure, since if X, Y are left invariant, then $[X, Y](g) = DL_g([X, Y](1))$. Well we know for any diffeomorphism φ that

$$D\varphi([X, Y]) = [D\varphi X, D\varphi Y].$$

This is the algebra structure.

Recall that $[X, Y] = -[Y, X]$ via the derivation definition.

Lemma VI.0.1 (Jacobi Identity)

We have for X, Y, Z vector fields that

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Proof. Expand with the definition

We can define a general Lie algebra as a vector space which is equipped with an anticommutative bilinear form which satisfies the Jacobi identity).

Let G be a Lie group, $H \subseteq G$ a Lie subgroup, i.e. an immersed submanifold. We know that the set $\{gH\}_{g \in G}$ is a foliation of G. We know $V(g) = T_g(g \cdot H)$ which is a left invariant distribution.

We can then look a the left invariant vector fields tangent to $V(g)$ (i.e, tangent to gH). This defines $\mathfrak{h} \subseteq \mathfrak{g}$. And in fact, if $X, Y \in \mathfrak{h}$ then $[X, Y] \in \mathfrak{h}$ via the Frobenius theorem (since the $V(g)$ is integrable).

Definition VI.0.2

Given a Lie algebra $\mathfrak{g}$, a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a vector subspace such that for any $X, Y \in \mathfrak{h}$ we have $[X, Y] \in \mathfrak{h}$.

Theorem VI.0.2 (Lie Groups/Lie Algebras)

If H is a Lie subgroup of G, then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra. i.e, $[\mathfrak{h}, \mathfrak{h}] \in \mathfrak{h}$.

Conversely, if $\mathfrak{h} \subseteq \mathfrak{g}$, and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, then there exists $H \subseteq G$ a connected Lie subgroup such that $\mathfrak{h} = T_g(gH)$ (i.e., $\mathfrak{h}$ is left-invariant vector fields tangent to gH).

This gives a bijective correspondence between *connected* Lie subgroups H of G and Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$.

Example VI.0.1

Here is an example that you need the connected statement. Take $\mathbb{Z} \subseteq \mathbb{R}$, then

$$\mathrm{Lie}(\mathbb{Z}) = \{0\} = \mathrm{Lie}(\{0\}).$$

HAPPY HALLOWEEN

Recall VI.0.2

Last time we began to consider the relationship between Lie groups and Lie algebras. One statement was that if $H \subseteq G$ are Lie groups, then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.

This holds because if V is an H-left-invariant vector field, then $W(g) := D(g \cdot -)_1 \cdot V(1)$ is a G-left-invariant vector field agreeing with V at points in H.

Furthermore, both the $[-,-]^G$ and $[-,-]^H$ are just brackets of vector fields (so they agree), and thus if $X, Y \in \mathfrak{h}$ then $[X, Y] \in \mathfrak{h}$.

Lemma VI.0.3

$[aF, bG] = aF(b)G - bG(a)F + ab[F, G]$.

Proof. Just expand as derivations to get

$$\begin{aligned} aF(bG) - bG(aF) &= aF(b)G + abFG - bG(a)F - baGF \\ &= aF(b)G - bG(a)F + ab(FG - GF) \\ &= aF(b)G - bG(a)F + ab[F, G]. \end{aligned}$$

Proof of the relationship of Lie groups/Lie algebras. We've just done the forward direction (easy exercise as well).

For the backwards direction, take $\mathfrak{h} \subseteq T_1G$ (since it is G-left-invariant vector fields). Then we may take a distribution $V(g) = DL_g(\mathfrak{h})$ (where L_g is left translation by $g \in G$).

Claim

V is involutive, i.e. if X_1, X_2 are tangent to V then $[X_1, X_2]$ is tangent to V.

Let $z_1, \ldots, z_k$ form a basis of $\mathfrak{h}$ (which has dimension k). Thus z_i are left invariant vector fields with $[z_i, z_j] \in \mathfrak{h}$.

Then $X = \sum a_i z_i$ and $Y = \sum b_j z_j$, and by the lemma

$$[X, Y] = \sum_i f_i z_i + \sum_{i,j} g_{ij}[z_i, z_j]$$

for some functions f_i, g_{ij}, and this lies in V as desired.

Claim

Thus V is integrable.

When we did Frobenius we only did it locally... we need a global foliation. Take local charts $\mathcal{F}_{\text{loc}}(p)$, we must define a global foliation $\mathcal{F}$ (with global leafs).

Namely say $q \in \mathcal{F}_{\text{loc}}(p)$ we want $\mathcal{F}_{\text{loc}}(p) \cup \mathcal{F}_{\text{loc}}(q)$.

We need a quick lemma that if $q \in \mathcal{F}_{\text{loc}}(p)$ then for any neighborhood of U of q where $\mathcal{F}_{\text{loc}}(p)$ and $\mathcal{F}_{\text{loc}}(q)$ both are defined they must agree. This works because both are tangent to V.

Then we can consider $\mathcal{F}^1(p) = \mathcal{F}_{\text{loc}}(p)$ and

$$\mathcal{F}^{n+1}(p) = \bigcup_{q \in \mathcal{F}^n(p)} \mathcal{F}_{\text{loc}}(q),$$

and take $\mathcal{F}(p) = \bigcup \mathcal{F}^{n+1}(p)$. This construction gives a path-connected global leaf.

Frobenius then says $\mathfrak{h}$ is given by a foliation $\mathcal{F}$. Then we can set $H = \mathcal{F}(1)$. It remains to check H is a subgroup, since it is a smooth submanifold of G.

Suppose $h \in H$, then

$$h \cdot H = L_h H = L_h(\mathcal{F}(1))$$

$$= \mathcal{F}(h \cdot 1) = \mathcal{F}(h)$$
$$= \mathcal{F}(1) = H,$$

where we have used that the vector field is left-invariant to get left-invariance of the foliation. If $h \in H$, is $h^{-1} \in H$? Well we know $h^{-1}h = 1$, so

$$h^{-1}\mathcal{F}(h) = \mathcal{F}(1),$$

but then $1 \in \mathcal{F}(h)$, so $h^{-1} \in \mathcal{F}(1) = H$.

By construction we know $T_1(H) = T_1(\mathcal{F}(1)) = V(1) = \mathfrak{h}$, as desired.

Facts: Know given a Lie group G, we can give a Lie algebra $\mathfrak{g}$. We can ask the converse question if we define a Lie algebra in general

Definition VI.0.3

A Lie algebra $\mathfrak{g}$ is a vector space over a field F equipped with a bilinear operation $[-,-]$ satisfying

- $[x, x] = 0$
- $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$. for all $x, y, z \in \mathfrak{g}$.

Then given any *finite-dimensional* Lie algebra $\mathfrak{g}$ is there a Lie group G with Lie algebra $\mathfrak{g}$ and how many? Answer:

- Yes you can find one
- No it is not unique
- But it's sort of unique. If G_1, G_2 both give rise to $\mathfrak{g}$ then the universal covers $\widetilde{G}_1$ and $\widetilde{G}_2$ coincide and both give rise to $\mathfrak{g}$.

Consider a group homomorphism $G_1 \xrightarrow{\varphi} G_2$. There is trouble: there exists a $\varphi : \mathbb{R} \to \mathbb{R}$ which is a homomorphism but which is not differentiable. You can construct nice ones with Galois theory, but a simpler one is given by writing $\mathbb{R}$ as a $\mathbb{Q}$-vector space with an uncountable basis and mapping the basis around in a strange way.

Definition VI.0.4

If $\varphi : G_1 \to G_2$ between Lie groups is a group homomorphism and C^∞, then we call it a Lie group homomorphism

Remark VI.0.1

It is good enough to assume φ is measurable (more strongly, continuous). When we say measurable we mean to be with respect to charts. This is very very important, although we won't use it. The thought: it's generally much easier to prove something is measurable than to prove something is differentiable.

This induces a Lie algebra homomorphism

$$D\varphi_1 : \mathrm{Lie}(G_1) = T_1G_1 \to T_1G_2 = \mathrm{Lie}(G_2).$$

Call these $\mathfrak{g}_1, \mathfrak{g}_2$. We want this to respect the bracket.

Let $X \in \mathfrak{g}_1$ be some left-invariant vector field on G_1. $D\varphi_1 X(1) \in \mathfrak{g}_2$, and corresponds to some left invariant vector field Y on $\mathfrak{g}_2$.

Claim

X, Y are φ-related.

Proof. Take some $g \in G_1$. We must show that

$$D_g\varphi \cdot X(g) = Y(\varphi(g)).$$

Well, we know that

$$\begin{aligned} D_g\varphi \cdot X(g) &= D_g\varphi \cdot D_1 L_g \cdot X(1) \\ Y(\varphi(g)) &= D_1 L_{\varphi(g)} \cdot Y(1) = D_1 L_{\varphi(g)} \cdot D_1\varphi \cdot X(1). \end{aligned}$$

The result then follows since $L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$ since this is a group homomorphism.

THE FOLLOWING THEOREM IS WRONG, CORRECTED NEXT TIME

Theorem VI.0.4

There is a bijective correspondence between Lie group homomorphisms $G_1 \to G_2$ and Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$.

Proof. The forward direction we just did (modulo 1-1 business). For the converse we need a trick. Namely if $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism we see that

$$\operatorname{graph}\psi = \{(X, \psi(X)) \mid X \in \mathfrak{g}_1\}$$

is in fact Lie subalgebra of $\mathfrak{g}_1 \times \mathfrak{g}_2$, which is a Lie algebra with bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2]).$$

We then see that

$$[(X_1, \psi(X_1)), (X_2, \psi(X_2))] = ([X_1, X_2], [\psi(X_1), \psi(X_2)]) = ([X_1, X_2], \psi([X_1, X_2])).$$

We now have a Lie subalgebra, so by the main result last time there is a Lie subgroup $H \subseteq G_1 \times G_2$ (note: $\mathrm{Lie}(G_1 \times G_2) = \mathrm{Lie}(G_1) \times \mathrm{Lie}(G_2)$).

Claim

H is the graph of a homomorphism $\Psi : G_1 \to G_2$.

I.e., $\Psi(g_1) = g_2$ if $(g_1, g_2) \in H$. One must check that this is well-defined and a homomorphism

Exercise VI.0.3

Check well-definedness and homomorphism. We'll come back to it later.

VI.1. Exponential Map

Let $X \in \mathfrak{g}$, where G is a Lie group with Lie algebra $\mathfrak{g}$. Then $\{tX \mid t \in \mathbb{R}\}$ is a Lie subalgebra since

$$[sX, tX] = st[X, X] = 0.$$

Thus there exists a connected Lie subgroup of G corresponding to $X \in \mathfrak{g}$.

This is extremely abstract. Lets get down to Earth again. Le t $X \in \mathfrak{g}$ be a left invariant vector field. This gives us a local flow φ_t on G.

We can consider $1 \in G$ and define $g_t \coloneqq \varphi_t(1)$. Then

$$g_t \cdot g_s = \varphi_t \cdot \varphi_s(1) = \varphi_{t+s}(1) = g_{t+s}.$$

We also have

Claim

φ_t is a global flow, i.e. defined for all t.

Proof. Appeal to the subgroups argument. Or more simply, we know the local flow of X through g is simply

$$L_g(\varphi_t(1)) = g \cdot g_t.$$

Thus if local flow at 1 is defined on $(-\varepsilon, \varepsilon)$ so is it at g. We can then define it globally, around each point in $(-\varepsilon, \varepsilon)$ the flow is defined in $(-\varepsilon, \varepsilon)$ about it, and then we can continue, defining the flow on $(-2\varepsilon, 2\varepsilon)\ldots$

Since $\varepsilon > 0$ is fixed this gives us a global flow.

Example VI.1.1

We want to look at this very concretely. Prime Example is $G = \mathrm{GL}_n(\mathbb{R})$. We see that

$$\mathfrak{gl}_n(\mathbb{R}) = T_1 \mathrm{GL}_n(\mathbb{R}) = M_{n,n}.$$

If $X \in M_{n,n}$ then what is φ_t, well

$$e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!},$$

converges sicne

$$\left\| \sum_{n=0}^{\infty} \frac{(tX)^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{|t|^n \|X\|^n}{n!} = e^{|t| \cdot \|X\|},$$

where $\|\cdot\|$ is the operator norm, and it is easy to check that $\|AB\| \leq \|A\| \cdot \|B\|$, whcih gives $\|X^n\| \leq \|X\|^n$.

Finally note that $\frac{\mathrm{d}}{\mathrm{d}t}(e^{tX}) = Xe^{tX}$. We also must show $e^{tX} \in \mathrm{GL}(n, \mathbb{R})$. This will be because if A, B commute then $e^A e^B = e^{A+B}$, so $e^{tX} e^{-tX} = \mathrm{Id}$.

Example VI.1.2

This also works for any subgroups of $H \subseteq \mathrm{GL}_n(\mathbb{R})$, namely if we have a flow for $X \in T_1 H$ lying in $\mathrm{GL}_n(\mathbb{R})$, then of course the flow lies in H.

Final: Thursday December 15th, 4-6pm.

Question: What are the continuous homomorphisms $\varphi : S^1 \to \mathbb{R}$. We know $\varphi(S^1)$ is compact, and so is bounded. If we ahve $a \in \varphi(S)$ then $n \cdot a \in \varphi(S)$, which cannot be bounded unless $a = 0$. Thus φ is the constant map at 0.

BUT! $\mathrm{Lie}(S^1) = \mathbb{R}$, and $\mathrm{Lie}(\mathbb{R}) = \mathbb{R}$. Of course we have a map $\mathrm{Lie}(S^1) \to \mathrm{Lie}(\mathbb{R})$ given by the identity (this is a Lie algebra homomorphism since all brackets are zero...).

Thus a homomorphism between Lie algebras does not give rise to a smooth Lie group homomorphism. Thus what we did last time is **wrong**. But it is almost true!

But it is almost true. If $\widetilde{G}_1$ is the universal cover of G_1 (see 592, algebraic topology), then given a Lie algebra homomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ there is a Lie group homomorphism $\widetilde{G}_1 \to G_2$.

Universal Cover: If M is a reasonable space (e.g. a manifold) then we can find a space $\widetilde{M} \xrightarrow{\pi} M$ which is a submersion, nad $\widetilde{M}$ has the following property for all $p \in M$

$$\pi_1(\widetilde{M}, p) = 1.$$

$\pi_1(X, p)$ for $p \in X$ is defined as a set by

$$\{f : S^1 \to X \mid f \text{ continuous}, f(1) = p\}$$

where $f \sim g$ if there is an $F : X \times [0,1] \to X$ with $F(1,t) = p$ and $f(z) = F(z,0), g(z) = F(z,1)$. In fact this has a group structure. Again see algebraic topology (592). There are notes at

http://www-personal.umich.edu/ alephnil/notes/MATH-592-notes.pdf

We have $\widetilde{S^1} = \mathbb{R}$, where $\mathbb{R} \to S^1$ is given by $\exp(2\pi i t)$.

Fact: G is a Lie group implies $\widetilde{G}$ is also a Lie group. For a small enough neighborhood U of 1 the cover in $\widetilde{G}$ is given by $\widetilde{U} = \pi^{-1}(U)$ and the restriction of π to $\widetilde{U}$ is a homeorphism. Then if a, b are in some open subset of U, their product lies in U. We can then define the product in the covering space by lifting this product.

Note: The Lie algebra corresponding to $\widetilde{G}$ is the same as the Lie algebra of G.

Theorem VI.1.1

There is a bijective correspondence between Lie group homomorphisms $\widetilde{G}_1 \to G_2$ and Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$.

Idea of Proof. Let $H \subseteq G_1 \times G_2$ be defined as last time. This is a subgroup induced by the Lie algebra $\mathfrak{h} = \operatorname{graph} \Phi$, where $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$.

We know $\dim H = \dim \mathfrak{h} = \dim \mathfrak{g}_1$. We then have maps

$$\begin{tikzcd}
& H \arrow[d] \arrow[ddl, bend right, "\pi_1\big|_H"'] & \\
& G_1 \times G_2 \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
G_1 & & G_2
\end{tikzcd}$$

In fact $\pi_1\big|_H$ is a local diffeomorphism This holds because $D(\pi_1\big|_H) = \pi_1\big|_{\mathfrak{h}}$. We would like $\pi_1\big|_H$ to be injective. It turns out $H \cap \pi^{-1}(1)$ is discrete. This cannot be if we pass to the universal cover, but this requires work from 592.

Now let G be a Lie group and M a smooth manifold.. Suppose G acts on M smoothly. Let $\mathfrak{g}$ be a Lie algebra of G. Then there is a Lie algebra homomorphism $\mathfrak{g} \to \{\text{smooth vector fields on } M\}$.

To see this, take $X \in \mathfrak{g}$ a left-invariant vector field on G. Now write $g \cdot p =: E(g,p) =: E_p(g)$ where $p \in M$. We can push forward the vector field X to M using the map E_p (must check smoothness in p.

We can also take $g_t \coloneqq \exp(t \cdot X)$ (a one-parameter-subgroup) and consider $Y(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_t \cdot p$. If we move transversely to a point $q \in M$ near p, it's conceivable $Y(p)$ and $Y(q)$ does not vary smoothly. But this is possible to check.

Converse also holds, but is harder. If $\mathfrak{g} \to \{\text{smooth vector fields on } M\}$ then there is a smooth local action of G on M. This requires a bit of work.

VII. Differential Forms and Integration on Manifolds

VII.1. Partitions of Unity

Theorem VII.1.1

Let M be compact (can do this all when M is not compact, but it's more painful) and $\{U_\alpha\}$ is a cover of M. Then we have a finite subcover $V_1, \ldots, V_\ell$ where for each i there is an α with $V_i \subseteq U_\alpha$. Furthermore, we can give smooth functions

$$\varphi_\beta : V_i \to [0, 1]$$

such that for all $x \in M$ we have $\sum_i \varphi_i(x) = 1$.

Proof. For all $p \in M$ find a neighborhood W_p and a smooth bump function $\psi_p : W_p \to [0, 1]$ with $\psi_p \equiv 1$ on a neighborhood of p. Then there's an $R_p \subseteq W_p$ so that $\psi_p \equiv 0$ outside R_p.

We can wlog that $W_{p_1}, \ldots, W_{p_k}$ cover M and are contained in the V_i. Then we look at

$$\varphi_j = \frac{1}{\sum_{i=0}^k \psi_{p_i}} \cdot \psi_{p_j}.$$

Then $\sum \varphi_j = 1$ as desired.

In the general case one uses paracompactness.

Stuff

- For HW8 #3, try to use partitions of unity to slow down the vector field.
- Fact we might prove later / on homework: For any two points p, q lying in a compact manifold M, there is a diffeomorphism $M \to M$ taking p to q.

VII.2. Embedding of Manifolds into $\mathbb{R}^N$

Theorem VII.2.1 (Whitney Embedding Theorem)

If M is a manifold, then for some N there exists an $f : M \to \mathbb{R}^N$ which is an embedding.

Proof when M is compact. Let $n = \dim M$. Then if $(U_\alpha, \varphi_\alpha)$ are the coordinate charts (balls around 0) then we can map

$$M \to \prod_\alpha \mathbb{R}^n =: \mathbb{R}^N$$

where $N = n \cdot \#\{\alpha\}$ (we can take finitely many charts since M is compact. If $x \in U_{\alpha_1}, \ldots, U_{\alpha_j}$ then we can map x to have zeros for all β not an α_i, and $\varphi_{\alpha_i}(x)$ for all those included.

This is a *BAD* mapping. Make this construction smooth by tampering with a partition of unity of $\{U_\alpha\}$. Call this partition of unity τ_α. Then we replace $\varphi_{\alpha_i}(x)$ with $\tau_{\alpha_i}(x) \cdot \varphi_{\alpha_i}(x)$.

Also, $\tau_{\alpha_i} \equiv 1$ on $V_{\alpha_i} \subseteq U_{\alpha_i}$. We should make sure we get a finite covering of the manifolds by V_{α_i} (and then we'll be done.

Proof Idea in General. Look at $\prod_{\alpha_i} \mathbb{R}^n$ which is infinite, and project to a large dimensional $\mathbb{R}^N$.

VII.3. Multilinear Algebra

Definition VII.3.1

Let V be a vector space, then we define the exterior product $\Lambda^k(V)$ to be

$$\Lambda^k(V) \coloneqq \{k - \text{multilinear alternating functionals}\},$$

i.e. $\lambda \in \Lambda^k(V)$ is a multilinear function $\lambda : \underbrace{V \times \cdots \times V}_{k \text{ times}}$ where for all i, j we have

$$\lambda(\ldots, v_i, \ldots, v_j, \ldots) = -\lambda(\ldots, v_j, \ldots, v_i, \ldots).$$

Note: $\lambda(\ldots, v, \ldots, v, \ldots) = 0$ (any one repetition gives us 0).

Theorem VII.3.1

$\dim \Lambda^{\dim V} V = 1$.

Proof Idea. Choose an isomorphism of V with $\mathbb{R}^n$, and work there. The dimension is ≥ 1 because we can construct the determinant function. It is *difficult* to show the determinant exists.

The dimension is ≤ 1 part is pretty easy.

Why is this important to us? It's not just algebraic garbage (Ralf's words). There's a geometric interpretation of the determinant!

$|\det|$ **is VOLUME**

Example VII.3.1

$\Lambda^1 V = V^*$ (the dual of V).

We can see that $\dim \Lambda^k V \leq \binom{\dim V}{k}$. Explicitly when $k = 2$, let $\lambda \in \Lambda^2 V$, and e_i a basis of V. Then let $v = \sum_i \alpha_i e_i, w = \sum_j \beta_j e_j$. Then

$$\lambda(v, w) = \sum_{ij} \alpha_i \beta_j \lambda(e_i, e_j) = \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i) \lambda(e_i, e_j).$$

With this in mind we can define $e_i \wedge e_j$ as the element of $\Lambda^2 V$ which acts as

$$(e_i \wedge e_j) \left(\sum_i \alpha_i e_i, \sum_j \beta_j e_j \right) = \alpha_i \beta_j - \alpha_j \beta_i.$$

ChecK: This is an alternating multilinear form. This give s $e_i \wedge e_j$ for $i < j$ as a basis of $\Lambda^2 V$.

Similarly we can get a basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$ where $i_1 < \cdots < i_k$ as a basis of $\Lambda^k V$.

MEANING: Lets go to $\mathbb{R}^3$. We see that

$$(e_1 \wedge e_2)\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right) = 1.$$

$e_1 \wedge e_2$ is giving the signed area of a square... But which square?

$$(e_1 \wedge e_2)\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 0.$$

The area is the 2-dimensional area of the shape projected to a 2-dimensional slice of a plane!

Philosophy: $\lambda \in \Lambda^k \mathbb{R}^n$ "measures" the k-dimensional area of a parallelipiped with respect to a particlar fixed k-dimensional subspace.

VII.4. **Orientations on Manifolds**

Stuff:

- For HW8 #1, take a look at Spivak's Calculus. General Theorem that if $f : M \to N$ is C^1 and $\dim M \leq \dim N$, then the set of critical values have measure zero.
- For HW8 #3, Consider on U_i a local flow defined for time ε_i. Take a partition of unity f_i for U_i and then consider $\sum f_i \frac{1}{\varepsilon_i} X$.

Comment on Lie subgroups. Let $\varphi : H \to G$ be a smooth homomorphism. Instead of looking at $\mathfrak{h}, \mathfrak{g}$ as left invariant vector fields (which led us astray last time) look at the tangent space at the identity. Let $X \in T_1 H = \mathfrak{h}$, then $Y := D\varphi_1 \cdot X \in T_1 G = \mathfrak{g}$. Then Y defines a left invariant vector field on G, the claim is that X, Y are φ-related (good enough to justify brackets agree).

Cleanup from last time: It was said that k-forms measures area of an intersection. But instead it measures area of a projection.

Let V be a finite n-dimensional vector space. We know $\dim \Lambda^n V = 1$. We can't tell if a real number is positive or negative without placing an orientation on a line. But we *can* tell if they are positive multiples of each other (they have the same orientation)

Definition VII.4.1

Two n-forms $\alpha, \beta \neq 0$ on V have the same orientation if $\beta = c \cdot \alpha$ for $c > 0$. Otherwise they have the opposite orientation.

Definition VII.4.2

If M is an n-dimensional manifold (smooth). We let $(\Lambda^k M)_p := \Lambda^k T_p M$, which is a vector bundle $\Lambda^k M \to M$.

A k-form α is a (smooth) section of $\Lambda^k M \to M$.

Example VII.4.1

Consider $\mathbb{R}^n$, then $\alpha = \mathrm{d}x_{10^{44}}$ is a smooth 1-form (where $\mathrm{d}x_{10^{44}}$ is the dual vector to $\frac{\partial}{\partial x_{10^{44}}}$ which is a smooth vector field).

A two form could be something like $\mathrm{d}x_1 \wedge \mathrm{d}x_2$. We have to explain this though.

Question: We know that $\dim \Lambda^n T_p M = 1$ if $\dim M = n$. What would a section of $\Lambda^n M \to M$ (aka an n-form on M) tell us about M?

Definition VII.4.3

Any n-form τ (aka a section of $\Lambda^n M \to M$) such that for all $p \in M$ we have $\tau(p) \neq 0$ is called an orientation of M (a smooth manifold).

Given an orientation τ and another σ we say that σ, τ define the same orientation on M if there is a smooth map $f : M \to (0, \infty)$ so that $\sigma = f \cdot \tau$.

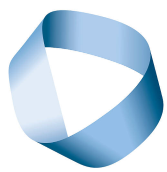
Also, if σ is an orientation, so is $-\sigma$, and these are NOT the same orientation.

Note: One can do orientation for topological manifolds but it requires Algebraic Topology and is harder.

Question: Do orientations always exist? No!!!

Example VII.4.2

We can look at the Möbius band, which is a strip glued in opposite directions



Definition VII.4.4

Call M orientable if it has an orientation. Also, an oriented manifold is a manifold M with a given orientation (M, σ).

Example VII.4.3

Observe, if we take the two caps of a sphere with natural orientations and glue them together to respect the orientation, we get S^n, which is orientable.

In contrast, if we look at $\mathbb{P}^n = S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on S^n by $x \mapsto -x$. Does this map preserve orientation?

Look at the simplest example for $S^1 \ldots$ then yes. For S^2 in fact no!

Proposition VII.4.1

Any Lie group G is orientable.

Proof. Pick $\sigma(1) \in \Lambda^n T_1 G$. Now make it left invariant by pushing it around.

Recall VII.4.4

$\mathbb{RP}^3$ is diffeomorphic to $\mathrm{SO}(3)$, and this is double covered by $\mathrm{SU}(2)$. But then $\mathrm{SO}(3)$ is a group, so it is orientable.

Or: Stare at the antipodal map $A : S^n \to S^n$. If it preserves the orientation then just push it down to $\mathbb{RP}^n$.

VII.5. The Wedge Product

If we have two multilinear maps f, g, then $f \otimes g$ is also multilinear (given as $(f \otimes g)(v, w) = f(v)g(w)$). But this may not be alternating even if f, g are alternating!!!

Given $\alpha \in \Lambda^k V, \beta \in \Lambda^\ell V$, then we wish to define $\alpha \wedge \beta \in \Lambda^{k+\ell} V$.

Definition VII.5.1

The wedge product $\alpha \wedge \beta$ of $\alpha \in \Lambda^k V, \beta \in \Lambda^\ell V$ is

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\ell!} \cdot \sum_{\sigma \in S(k+\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

This is very similar to the definition of the determinant. Here $S(k+\ell)$ is the permutations of $\{1, \dots, k+\ell\}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k)$$
$$\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+\ell).$$

Thus it preserves the ordering on $\{1, \dots, k\}$ and on $\{k+1, \dots, k+\ell\}$ (but not necessarily both).

Stuff:

- HW due Thursday 11pm, November 17th
- The book uses the notation $\Lambda^k V^*$ to refer to the alternating k-multilinear maps on V. We've been using $\Lambda^k V$ to refer to the same thing. Ditto for $\Lambda^k M$ (our notation) versus $\Lambda^n T^* M$ (the book's notation. Make sure to keep this in mind.

  We will try to use the book's notation from now on, but remember that we will always be talking about alternating k-multilinear maps (k-forms).

Recall VII.5.1

A k-form α is smooth if either

- $X_1, \dots, X_k$ are smooth vector fields, and then the function

  $$p \mapsto \alpha_p(X_1(p), \dots, X_k(p))$$

  is smooth.
- if locally we can write in terms of smooth coordinates $\alpha = \sum \alpha_i (\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k})$ with $\alpha_i : M \to \mathbb{R}$

Example VII.5.2

We'll look at

$$\begin{aligned}
&(\mathrm{d}x_1 \wedge \mathrm{d}x_2)(\partial x_i, \partial x_j) \\
&= \mathrm{d}x_1(\partial x_i) \cdot \mathrm{d}x_2(\partial x_i) - \mathrm{d}x_1(\partial x_j) \cdot \mathrm{d}x_2(\partial x_i) \\
&= \begin{cases} 0 & \text{if } \{i, j\} \neq \{1, 2\} \\ 1 & \text{if } i = 1, j = 2 \\ -1 & \text{if } i = 2, j = 1 \end{cases}.
\end{aligned}$$

More generally, we can look at

$$\begin{aligned}
&(\mathrm{d}x_1 \wedge \mathrm{d}x_2)\left(\sum_i a_i \partial x_i, \sum_j b_j \partial x_j\right) \\
&= (\mathrm{d}x_1 \wedge \mathrm{d}x_2)(a_1\partial x_1, b_2 \partial x_2) + (\mathrm{d}x_1 \wedge \mathrm{d}x_2)(a_2 \partial x_2, b_1 \partial x_1 \\
&= a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.
\end{aligned}$$

Recall VII.5.3

For $\alpha \in \Lambda^k V^*, \beta \in \Lambda^\ell V^*$ we defined

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\ell!} \cdot \sum_{\sigma \in S(k+\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

where $\sigma \in S(k+\ell)$ provided that σ is a permutation of $k+\ell$ things so that $\sigma(i) < \sigma(j)$ for $1 \leq i < j \leq k$ and for $k+1 \leq i < j \leq k+\ell$.

Example VII.5.4

Let $k = \ell = 1$. Then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

Proposition VII.5.1

$\alpha \in \Lambda^k V^*, \beta \in \Lambda^\ell V^*$ implies $\alpha \wedge \beta \in \Lambda^{k+\ell} V^*$.

Proof. Check from the book, idea: if you transpose two things in $1, \dots, k$ or in $k+1, \dots, k+\ell$ it's just from α, β. If you transpose a thing between the two, then things are more complex.

This wedge product is super important. Why? Future: wedge product of forms, leading to Poincaré duality.

Important: $\wedge$ is a multiplicative operation.

Question: Let $\dim V = n, k+\ell = n, \alpha \in \Lambda^k V^*, \beta \in \Lambda^\ell V^*$. When multiplied we have

$$\alpha \wedge \beta \in \Lambda^{k+\ell} V^* = \Lambda^n V^* \cong \mathbb{R}.$$

The idea of Poincaré duality will be to associate to α a β so that $\alpha \wedge \beta$ is the determinant (a distinguished n-form, aka an orientation) on the nose.

If M is an n-dimensional manifold, we say it is oriented with orientation σ if there is an n-form, aka a section $\sigma : M \to \Lambda^n M$ so that σ never vanishes.

Definition VII.5.2 (Also Notation)

Call a σ like this a "volume form."

Lemma VII.5.2

If M is oriented then $\Lambda^n M = \{f : M \to \mathbb{R} \mid f \text{ smooth}\}$.

One can think of $\Lambda^0 M \coloneqq \{f : M \to \mathbb{R} \mid f\text{smooth}\} = C^\infty(M)$. Thus this Lemma is Poincaré duality for n-forms and 0-forms.

Proof. If $\tau \in \Lambda^n M$ then $\tau(p) = f(p) \cdot \sigma(p)$ where $\sigma(p)$ is the volume form. Thus $\Lambda^n M \to \{f : M \to \mathbb{R} \mid f \text{ smooth}\}$. And the reverse also occurs.

More properties of wedge product:

(I) $\alpha \wedge \beta = (-1)^{k \cdot \ell} \beta \wedge \alpha$. Look at the formula and think about which things you have to switch.
Thus if k is odd, $\alpha \in \Lambda^k V^*$, then $\alpha \wedge \alpha = 0$.

(II) $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$, associative.

(III) Bilinearity.

An aside about orientability: Let M have coordinate charts $(U_\alpha, \varphi_\alpha)$.

Consider two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β). Then we get a transition map

$$T_{\alpha,\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).$$

Note $\mathbb{R}^n$ is orientable since

$$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \neq 0.$$

Thus on U_α we can take $\gamma^\alpha_{\boldsymbol{x}} := \mathrm{d}x^\alpha_1 \wedge \cdots \wedge \mathrm{d}x^\alpha_n$. We have a map down to $\gamma^\beta_{\boldsymbol{y}} := \mathrm{d}y^\beta_1 \wedge \cdots \mathrm{d}y^\beta_n$. What happens under the transition map $T_{\alpha\beta}$?

Recall VII.5.5

Smooth functions pullback k-forms. Given smooth $F : M \to N$ and $\alpha \in \Lambda^k N$ then $F^*\alpha \in \Lambda^k M$ as defined below

$$(F^*\alpha)_p(v_1, \ldots, v_k) = \alpha_p(DF_p \cdot v_1, \ldots, DF_p \cdot v_k).$$

We want to ask if $T^*_{\alpha\beta}\gamma^\beta_{\boldsymbol{y}}$ and $\gamma^\alpha_{\boldsymbol{x}}$ give the same orientation on $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$, can we build an orientation on M?

Can you put these forms together to make an orientation if things agree?

Theorem VII.5.3

Let M be a smooth manifold then the following are equivalent

(a) M is orientable.

(b) M has an atlas of coordinate charts so that the transition maps are orientation preserving. In other words $\det DT_{\alpha\beta} > 0$ when it is defined for each transition map $T_{\alpha,\beta}$

Lemma VII.5.4

Let $\mu \in \Lambda^n V^*$ and $\{e_1, \ldots, e_n\}$ be a basis of V with $(a_{ij}) = A$ an $n \times n$ matrix so that

$$f_i = \sum_{j=1}^n a_{ij} e_j,$$

then

$$\mu(f_1, \ldots, f_n) = (\det A)\mu(e_1, \ldots, e_n).$$

Proof. We see that

$$\mu\left(\sum_{j=1}^n a_{1j}e_j, \ldots, \sum_{j=1}^n a_{nj}e_j\right) = \sum_{j_1,\ldots,j_n} a_{1j_1}\cdots a_{nj_n}\mu(e_{j_1},\ldots,e_{j_n}).$$

If any two of $j_1,\ldots,j_n$ are the same then the term corresponding to this choice is zero. We can then rewrite this as

$$\begin{aligned}\mu\left(\sum_{j=1}^n a_{1j}e_j, \ldots, \sum_{j=1}^n a_{nj}e_j\right) &= \sum_{\sigma\in S_n} a_{1\sigma(1)}\cdots a_{n\sigma(n)}\mu(e_{\sigma(1)},\ldots,e_{\sigma(n)}) \\ &= \left(\sum_{\sigma\in S_n}(-1)^\sigma\prod_{i=1}^n a_{i\sigma(i)}\right)\mu(e_1,\ldots,e_n) = \det(A)\mu(e_1,\ldots,e_n).\end{aligned}$$

For forms we have if $F : W \to V$ is linear then

$$F^* : \Lambda^k V^* \to \Lambda^k W^*.$$

Then of course

$$(F^*\alpha)(w_1,\ldots,w_k) = \alpha(F(w_1),\ldots,F(w_k)).$$

Lemma VII.5.5

Suppose $\dim V = \dim W = n$, with V, W vector spaces. Further, let $F : W \to V$ be linear, $e_1,\ldots,e_n$ be a basis of V, $f_1,\ldots,f_n$ be a basis of W, $\varepsilon_1,\ldots,\varepsilon_n$ the dual basis of V^*, $\phi_1,\ldots,\phi_n$ the dual basis of W^*.

Let A be the matrix with respect to this basis. We know

$$F(f_i) = \sum_{j=1}^n a_{ij}e_j.$$

In this case we have

$$F^*(\varepsilon_1\wedge\cdots\wedge\varepsilon_n) = (\det A)(\phi_1\wedge\cdots\wedge\phi_n).$$

Proof. Apply the above lemma. Namely evaluate the left hand side at $f_1,\ldots,f_n$ and show you get $\det A$ using previous lemma (which is enough since $\dim\Lambda^n W^* = 1$).

Definition VII.5.3

Suppose M, N are manifolds and $\Phi : M \to N$ is a smooth map. We define

$$\Phi^*\Lambda^k N \to \Lambda^k M$$

with $\alpha \in \Lambda^k N$, $p \in M$, $v_1,\ldots,v_n \in T_pM$ via

$$(\phi^*\alpha)_p(v_1,\ldots,v_n) = \alpha_{\Phi(p)}(D\Phi_p\cdot v_1,\ldots,D\Phi_p\cdot v_n).$$

This is called the pullback of a differential form.

Lemma VII.5.6

Let $T : U \to V$ be a diffeomorphism where $U, V \subseteq \mathbb{R}^n$. Let $x_1, \ldots, x_N$ be coordinates in V, $x_1, \ldots, x_n$ coordinates for U. Here $(\mathrm{d}x_i)_q$ is dual to $\left(\frac{\partial}{\partial x_i}\right)_q$.

We then have that

$$(T^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n))_p = (\det DT_p)(\mathrm{d}y_1 \wedge \cdots \mathrm{d}y_n)_p$$

Proof. Apply previous lemma.

Stuff:

- HW 3 is due next time, as we haven't covered enougn to make it tractable. Also Ralf is not sure if it is true.
- For HW 2d, if you want to show $\mathrm{GL}(n, \mathbb{C})$ is connected, you might want to look at $\mathbb{C}^* \times \mathrm{SL}(n, \mathbb{C})$. Then

$$\mathrm{SL}(n, \mathbb{C}) = \mathrm{SU}(n) \cdot \text{ upper triangular matrix with any complex numbers.}$$

Alternately, realize $\mathrm{GL}(1, \mathbb{C})$ is orientation preserving and so $\mathbb{C}$ carries a natural orientation.

General principle: If Γ acts on M, and M/Γ sia manifold and M carries a "structure" invariant by Γ which is invariant, it induces this structure on M/Γ.

Example VII.5.6

$\mathbb{RP}^n = S^N/\mathbb{Z}_2$ and $\mathbb{CP}^n = S^{2n-1}/S^1$.

Example VII.5.7

Suppose M is $\mathbb{C}$-differentiable and Γ acts by $\mathbb{C}$-differentiable maps, then M/Γ is $\mathbb{C}$-differentiable.

Example VII.5.8

If M has a Riemannian metric and Γ acts by isometries, then M/Γ carries a Riemannian metric. By acting by isometries we mean that for $\gamma \in \Gamma$

$$\langle D\gamma_p \cdot v, D\gamma_p \cdot w \rangle = \langle v, w \rangle.$$

Last time: $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with coordinates $y_1, \ldots, y_n$ in the domain and $x_1, \ldots, x_n$ in the domain. Then

$$\varphi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = (\det D\varphi)(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n).$$

Orientability Theorem, atlas $\implies$ volume form. If $x_1, \ldots, x_n$ are coordinates on U_β, and $y_1, \ldots, y_n$ are coordinates on U_α. Then

$$T^*_{\alpha\beta}(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = (\det DT_{\alpha\beta})(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n).$$

Thus these are related by a positive number. Pick a partition of unity $\{\tau_i\}$ subordinate to U_α. On U_α we get an n-form σ_α given by pulling back $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. Then

$$\sum \tau_\alpha \sigma_\alpha$$

will define an n-form on M which is nonvanishing.

Orientability Theorem, volume form $\implies$ atlas. Call φ_α positive if the pullback form on U_α given by $\varphi_\alpha^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)$ (for coordinates $x_1, \ldots, x_n$) is positive with respect to σ (a fixed orientation on M). That is it equals $f \cdot \sigma$ for $f > 0$.

If all φ_α are $+$ then get coordinate charts are compatible with orientation. If not all φ_α are $+$ then "flip" the negative ones. I..e replace the coordinates $x_1, \ldots, x_n$ with $-x_1, x_2, \ldots, x_n$.

VII.6. **Defining Integrals**

Why bother with orientation? If $f : M \to \mathbb{R}$ is smooth, then how do we define $\int_M f$??? On $\mathbb{R}^n$ we just use Lebesgue integration (or Riemannian integration). Main thing is we know the volume of a cube.

In contrast, there is no preferred way to measure volume on a manifold! You would need a Riemannian metric. Similarly, an n-form can tell you the volume. . . Maybe if we have an n-form we can do something!!!

So $\int_M f$ NO IDEA. If $\int_M f\tau$ where τ is a volume form we have an idea. How to actually do it? In a chart U_α, with $\varphi_\alpha : U_\alpha \to V_\alpha$ we consider

$$\int_{V_\alpha \subseteq \mathbb{R}^n} (f \circ \varphi_\alpha^{-1}) \cdot \varphi_\alpha^*(\tau).$$

This is $g_\alpha \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ for some coordinates. Why is this well-defined? If we have a change of variables on $\mathbb{R}^n$ called T, then

$$\int_B (h \circ T) \det DT \, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n = \int_A h \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n,$$

where $A = T \cdot B$. Thus integrals agree on the overlaps of charts! Namely, the forms transform according to $T_{\alpha\beta}^*$ which acts via the determinant of the jacobian matrix from the work we've done above

Last time: We defined $\int_M f \cdot \nu$ where M is a C^∞ manifold and ν is an n-form ("volume form"). This is well-defined

Definition VII.6.1

Let ν be an n-form on a C^∞-manifold M and let f be a function on M. If $\varphi_\alpha : U_\alpha \to V_\alpha$ is a chart we define

$$\int_{U_\alpha} f \cdot \nu := \int_{V_\alpha} (f \circ \varphi_\alpha)^{-1} \cdot (\varphi_\alpha^{-1})^*(\nu).$$

Then if $\{U_\alpha\}_\alpha$ is a collection of charts, take a partition of unity τ_α to U_α, and then set

$$\int_M f \cdot \nu = \sum_\alpha \int_{U_\alpha} \tau_\alpha f \cdot \nu.$$

Exercise VII.6.1

Show this is well-defined, and gives the sensible thing in general cases.

Difference to $\mathbb{R}^n$: no preferred volume form! On $\mathbb{R}^n$ we can look at $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

Some other good cases:

- If $M = G$ is a Lie group, take $X_1, \ldots, X_n$ a basis of $\mathfrak{g} = \operatorname{Lie} G$. Then turn these into a basis of left invariant vector fields.

  Let $\eta_1, \ldots, \eta_n$ be a dual basis at the identity. Make η_i left invariant so $\eta_i(X_j) = 1$ if $i = j$ and 0 if $i \neq j$. Then $\eta_1 \wedge \cdots \wedge \eta_n$ is left invariant.

- Can do the same thing for right invariant.

Proposition VII.6.1

If G is a Lie group then there exists a left invariant volume form ν_L unique up to scalar multiplication.

Also there exists a unique (up to scalar) right invariant volume form ν_R.

Question: When is $\nu_L = \nu_R$.

Answer: Not always,

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \mid a \neq 0, b \right\}$$

The Lie algebra is

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & -B \end{pmatrix} \mid A, B \in \mathbb{R} \right\}.$$

But they are equal for

- Abelian groups
- nilpotent groups (e.g. heisenberg groups)
- $\mathrm{SL}_n(\mathbb{R})$.

Definition VII.6.2

If $\nu_L = \nu_R$ we call this group unimodular.

Compact groups are always unimodular. You can measure how unimodular something is by writing $\nu_R = \omega \cdot \nu_L$. Then one can prove $\omega(gh) = \omega(g)\omega(h)$ (check, Ralf thinks so). So measuring $\ker \omega$ tells you how unimodular it is.

Also if there is a $\Gamma \subseteq G$ discrete with G/Γ compact then G is unimodular.

Proposition VII.6.2

If M is a Riemannian manifold which is oriented, then the Riemannian metric induces a volume form.

The last case is suppose M has a (special) volume form ν and Γ acts on M properly discontinuously. Then M/Γ is a manifold.

Lemma VII.6.3

If ν a volume form on M is Γ-invariant, then ν descends to M/Γ.

Furthermore, if Γ is finite and orientation-preserving then one can always build such a Γ-invariant volume form from an arbitrary volume form on M.

Proof. Use that $\pi : M \to M/\Gamma$ is a submersion and a local diffeomorphism. Thus locally can pull back ν to $\overline{\nu}$ on M/Γ. Building it this way gives $\nu = \pi^*(\overline{\nu})$.

More explicitly. Let $\overline{p}, \overline{U}$ in M/Γ with diffeomorphisms $\gamma U \to \overline{U}$ for $\gamma \in \Gamma$.

Then ν on U we have $\nu = (\gamma^{-1})^* \nu$ on $\gamma\nu$. This commutes with the projection, and so ν defined from pushing ν on U down to $\overline{U}$ is the same as that defined from pushing ν on γU down to $\overline{U}$.

This allows one to paste it together into a preferred volume form! For the Γ finite case, just average!

Example VII.6.2

Suppose M^{2n} has a nonvanishing 2-form (symplectic form) α such that

$$\alpha \wedge \cdots \wedge \alpha$$

is nonvanishing, where we wedge n times.

More general integrals. Let $C : \Delta^k \to M$ be a smooth map from a k-dimensional simplex (sweeping under the rug–what does it mean to be smooth on the boundary?)

Let α be a k-form on M. Then $C^*(\alpha)$ is a k-form on Δ. Then

$$\int_{\Delta^k} C^*(\alpha) =: \int_C \alpha.$$

Note it depends on the map, which is why we write $\int_C$ instead of $\int_{C(\Delta^k)}$. This is a generalization of a line integral.

Example VII.6.3

When we're looking at the line integral, we're integrating vector fields over 1-simplices. The trick is

Definition VII.6.3

We'll call a smooth map $C : \Delta^k \to M$ a k-dimensional simplex in M.

These ideas are the brain-child of Poincaré, Elie Cartan, and de Rham. For now we'll leave them alone but we'll come back to them later.

VII.7. **Exterior Derivatives**

We now want to take α a k-form and associate to it $\mathrm{d}\alpha$, a $(k+1)$-form on M.

Example VII.7.1

For $F \in C^\infty(M)$ (aka a 0-form), we can take $\mathrm{d}F_p(v) = DF_p \cdot v$ (the directional derivative). This is a 1-form!

We'll use the notation $\Omega^k M$ for k-forms on M, and just Ω^k if M is clear. We want

$$\mathrm{d} : \Omega^k M \to \Omega^{k+1} M.$$

Recall that $\Omega^k(M)$ is zero for $k < 0, k > \dim M =: n$. So we get a chain

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

Here's what we want:

(1) d is d (defined above) on Ω^0.
(2) d is a linear map over $\mathbb{R}$ (not over $C^\infty(M)$!).
(3) $\mathrm{d} \circ \mathrm{d} = \mathrm{d}^2 = 0$.
(4) It works well with wedge product

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta,$$

where $\alpha \in \Omega^k(M)$.

Theorem VII.7.1

There exists a unique d satisfying Properties 1-4 above.

We'll prove this theorem in detail on Monday. Also Ralf Spatzier really likes the book Spivak, Calculus on Manifolds [**spivak**]

Goal: Poincaré lemma. On $\mathbb{R}^n$, this will say that if α has $\mathrm{d}\alpha = 0$ (α is a "closed" form),then there exists a β so that $\alpha = \mathrm{d}\beta$, which is called being an "exact form" (notice the converse is always true). We'll be able to say something a bit more general...this exact statement doesn't always hold.

Definition VII.7.1

We can look at $\mathrm{Image}(\mathrm{d}\big|_{\Omega^{k-1}}) \subseteq \ker(\mathrm{d}\big|_{\Omega^k})$. By definition we have

$$H^k_{\mathrm{deRham}} M \coloneqq \frac{\mathrm{Image}(\mathrm{d}\big|_{\Omega^{k-1}})}{\ker(\mathrm{d}\big|_{\Omega^k})}.$$

This is called the de Rham cohomology.

Miraculous–this is finite dimensional over $\mathbb{R}$. We'll abbreviate it H^k, though this is usually reserved for singular homology (see 592, they agree on manifolds). Instructive examples to compute

Example VII.7.2

$H^1(\mathbb{R})$, $H^1(S^1)$.

Theorem VII.7.2 (HW)

If M is a smooth manifold (paracompact) then there is a smooth Riemannian metric (in fact many)

Proof Idea. Glue local solutions together using partition of unity.

Definition VII.7.2

A Lorentz metric is a nondegenerate inner product $\langle , \rangle_p$ on T_pM such that $p \mapsto \langle , \rangle_p$ is smooth. I.e. for all smooth vector fields X, Y on M we have $p \mapsto \langle X(p), Y(p) \rangle_p$ is smooth. Furthermore $\langle , \rangle_p$ has signature $(n-1, 1)$.

Given a nondegenerate $\langle , \rangle$ is a (nondegenerate) inner product on a finite dimensional verctor space V, $\dim V = n$. Then if this has signature $(k, n-k)$ then there is a basis $v_1, \ldots, v_n$ such that

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \leq k \\ -1 & \text{if } i = j > k \end{cases}.$$

If $x = \sum x_i v_i, y = \sum y_j v_j$ then

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_k y_k - x_{k+1} y_{k+1} - \cdots - x_n y_n.$$

Every inner product has some signature.

Special relativity is Lorentz metrics on $\mathbb{R}^4$, and general relativity is the same spiel on a general manifold (that admits a Lorentz metric).

Theorem VII.7.3 (HW)

Not every smooth manifold supports a Lorentz metric.

Theorem VII.7.4

S^2, M where M is a compact connected orientable surface of genus > 1 does not admit a Lorentz metric.

Proof Idea. Look at S^2 and use that it does not admit a 1-dimensional distribution (follows from the fact that S^2 admits no nonvanishing vector field). Similarly for M where the genus > 1.

The fact that this follows is from covering space theory. Bad idea for finding distributions: $\{\langle v, v\rangle_p = 0\}$. Better idea: Use the standard Riemannian metric and grab the unit circle in T_pS^2 with respect to Euclidean metric on $\mathbb{R}^3$, call this $T_p^1S^2$. Look at $\langle v, v\rangle_p$ restricted to $T_p^1S^2$.

$\{\langle v, v\rangle_p \geq 0\}$.

In contrast, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ does since $T^2 = S^1 \times S^1$ and we can place $+, -$ on these respectively. Likewise $\mathbb{Z}_2$ preserves the $x_1^2 - x_2^2$ on $\mathbb{R}^2$, thus this descends.

For $\mathrm{SL}_2(\mathbb{R})$ there exists a left invariant $\langle , \rangle$ on $\mathrm{SL}(2, \mathbb{R})$ (in fact bi-invariant). Define it on X, Y given by for $X, Y \in \mathfrak{g}$,

$$\langle X, Y\rangle_1 = \mathrm{tr}(Z \mapsto [X, [Y, Z]]) = \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$$

where $(\mathrm{ad}\, X)(Z) := [X, Z]$.

Aside on Lie groups. Let $T_1G = \mathfrak{g}$. define

$$\begin{aligned} \mathrm{ad}\, X : \mathfrak{g} &\to \mathfrak{g} \\ Z &\mapsto [X, Z]. \end{aligned}$$

Then we can define the "Cartan-Killing form" of $\mathfrak{g}$ as

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y).$$

Exercise VII.7.3

Let X, Y lie in $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{Lie}\, \mathrm{GL}(n, \mathbb{R})$. Then we must show $B(X, Y) = \mathrm{tr}(X \cdot Y)$ (up to some dimension factor.

Definition VII.7.3

Call $\mathfrak{g}$ semisimple if B is nondegenerate

Theorem VII.7.5

$\mathfrak{sl}(n, \mathbb{R})$ is in fact semisimple.

Note: If $\mathfrak{g}$ has a center. I.e. if there is $Z \in \mathfrak{g}$ so that for all $X \in \mathfrak{g}$ we have $[X, Z] = 0$, then B is degenerate. Why? Well $B(Z, X) = 0$.

Fact: If G is compact with no center, then B is positive definite and nondegenerate

Example VII.7.4

$\mathrm{SU}(n), \mathrm{SO}(n)$, etc.

Clarification: 5a is still part of HW6, 5 b,c are the extra credit parts.

VII.8. deRham Cohomology

Now, lets compute $H^*_{\text{deRham}}(S^1)$. Well we know that

$$H^0(S^1) = \mathbb{R}^{\#\text{ connected components}} = \mathbb{R}.$$

How do we compute $H^1(S^1)$? Well recall, we showed that $H^1(\mathbb{R}) = 0$ by showing that given a closed 1-form, i.e. all 1-forms, then

$$\alpha = \mathrm{d}\beta$$

where β is a 0-form on $\mathbb{R}$ defined by

$$\beta(x) = \int_0^x f(t)\,\mathrm{d}t$$

where $\alpha = f\,\mathrm{d}x$.

Now think of S^1 as $[0,1]/(0 \sim q)$. Then if α is a 1-fom it looks like $f\,\mathrm{d}x$ where $\mathrm{d}x$ makes sense on $S^1 = \mathbb{R}/\mathbb{Z}$ since it is invariant under $x \mapsto x + a$ for all $a \in \mathbb{R}$ (we only need in $\mathbb{Z}$, but this is better).

Moreover, if $\alpha \in \Omega^1(S^1)$ then $\alpha = f\,\mathrm{d}x$ where $f : S^1 \to \mathbb{R}$ is smooth. Then we should of course take $\beta : S^1 \to \mathbb{R}$, so take

$$\beta(x) = \int_0^x f(t)\,\mathrm{d}t \ldots$$

BUT WAIT! We need to know if $\beta(0) = \beta(1)$! This gives us a condition

$$\alpha \text{ is exact} \iff \int_0^1 f(t)\,\mathrm{d}t = 0.$$

We want to look for $H^1(S^1) = \{\text{closed}\}/\{\text{exact}\}$. The closed one-forms are just all of them since $\Omega^2(S^1) = 0$. Now let $\alpha = f(x)\,\mathrm{d}x$ be a closed 1-form. Let $A \coloneqq \int_0^1 f(t)\,\mathrm{d}t$. Then consider $\alpha - A \cdot \mathrm{d}x$. Then

$$\int_0^1 (f(t) - A)\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t - A = 0.$$

Thus there exists $\beta \in \Omega^0(S^1)$ such that $\alpha - A\,\mathrm{d}x = \mathrm{d}\beta$. Thus $[\alpha] = [A \cdot \mathrm{d}x]$, which we can think of as $\mathbb{R}$ since there is one parameter.

Moral: The way the coordinate charts are put together to give you a manifold determines $H^*_{\text{dR}}(M)$.

Crucial to put all this together:

Lemma VII.8.1 (Poincaré Lemma)

If $A \subseteq \mathbb{R}^n$ is an open, star-shaped set, then any closed k-form on A is exact.

Definition VII.8.1

A set $A \subseteq \mathbb{R}^n$ is called star-shaped provided there exists a point $p_0 \in A$ (called an observer) such that for any $p \in A$, the line segment $[p_0, p] \in A$.

Motivation for the Proof:

- This is really a vast generalization of the fundamental theorem of calculus. It is a long calculation.
- In dimension 1 we look at $g(x) = \int_0^x f(t)\,\mathrm{d}t$ and it turned out $\mathrm{d}g = f(t)\,\mathrm{d}t$.
- Idea for star-shaped: integrate along segments (i.e. "radially").

Proof. We'll actually define the following in this proof, called a chain homotopy, the middle maps

$$\begin{tikzcd} \Omega^{k-1}(A) \arrow[r, "\mathrm{d}"] & \Omega^k(A) \arrow[r, "\mathrm{d}"] & \Omega^{k+1}(A) \\ \Omega^{k-1}(A) \arrow[r, "\mathrm{d}"'] & \Omega^k(A) \arrow[r, "\mathrm{d}"'] \arrow[ul, "I_k"] & \Omega^{k+1}(A) \arrow[ul, "I_{k+1}"] \end{tikzcd}$$

What we want: I is a linear map,

$$\mathrm{d}_{k-1} \circ I_k + I_{k+1} \circ \mathrm{d}k = \mathrm{Id}\,.$$

Consequence: If $\mathrm{d}\alpha = 0$ for $\alpha \in \Omega^k(A)$. Then

$$\alpha = I\,\mathrm{d}\alpha + \mathrm{d}(I\alpha) = I(0) + \mathrm{d}(I\alpha) = \mathrm{d}(I\alpha).$$

Thus we'll have $H^*(A) = 0$. In the one-dimensional case I was simply integration from 0 to x. We'll define $I_\ell : \Omega^\ell(A) \to \Omega^{\ell-1}(A)$. We'll have

$$\omega = \sum_{i_1 < i_2 < \cdots < i_\ell} \omega_I \,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}$$

where ω_I is a smooth function on A (this works since we're in $\mathbb{R}^n$, so this is true globally, here I is the index set). We now set

$$(I\omega)(x) := \sum_{i_1 < i_2 < \cdots < i_\ell} \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left(\int_0^1 t^{\ell-1} \omega_I(tx)\,\mathrm{d}t\right) \cdot x_{i_\alpha} \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_\alpha}} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \in \Omega^{\ell-1}(A)$$

Without loss of generality here we've assumed $p_0 = 0$ to make things easier to write down. How do we prove this works? I.e. that $\mathrm{d} \circ I + I \circ \mathrm{d} = \mathrm{Id}$. Well, you just write it out...

$$\begin{aligned} \mathrm{d}(I\omega) &= \ell \sum_{i_1 < \cdots < i_\ell} \left(\int_0^1 t^{\ell-1} \omega_I(tx)\,\mathrm{d}t\right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \\ &\quad + \sum_{i_1 < \cdots < i_\ell} \sum_{\alpha=1}^{\ell} \sum_{j=1}^{n} (-1)^{\alpha-1} \left(\int_0^1 t^\ell \frac{\partial \omega}{\partial x_j}(tx)\right) x_{i_\alpha} \wedge \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_\alpha}} \wedge \mathrm{d}x_{i_\ell}. \end{aligned}$$

The first bit is from $\frac{\partial}{\partial x_{i_\alpha}}$ (and the second bit of product rule), and the $(-1)^{\alpha-1}$ disappears because we've switched it to put it in the right place. The second bit is from $\frac{\partial}{\partial x_j}$ for any j, and uses differentiation under the integral sign.

Then we look at $\mathrm{d}\alpha$. Before we do this, note by linearity it suffices to check equality in a fixed term $i_1, \ldots, i_\ell$. We'll suppose $i_1 = n - \ell + 1, \ldots, i_\ell = n$. So we'll omit the sum over $i_1 < \ldots < i_\ell$.

Well this is

$$\begin{aligned} \mathrm{d}\omega &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \omega_I \,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \\ I(\mathrm{d}\omega) &= \sum_{j=1}^{n} \left(\int_0^1 t^\ell \frac{\partial}{\partial x_j}(\omega_1, \ldots, i_\ell\,\mathrm{d}t\right) x_j \widehat{\mathrm{d}x_j} \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \\ &\quad - \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left(\int_0^1 t^\ell \frac{\partial}{\partial x_j} \omega_I(tx)\,\mathrm{d}t\right) x_{i_\alpha}\,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_\alpha}} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}. \end{aligned}$$

We'll pick up this proof on Friday[1].

The messy terms then cancel, and we add the other terms

$$\begin{aligned}&\ell \cdot \left(\int_0^1 t^{\ell-1}\omega_I(tx)\,\mathrm{d}t\right)\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_\ell} + \sum_{j=1}^n \left(\int_0^1 t^\ell \frac{\partial}{\partial x_j}\omega_I(tx)\,\mathrm{d}t\right) x_j\cdot \mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_\ell}\\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(t^\ell\omega_I(tx))\,\mathrm{d}t\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_\ell}\\ &= \omega_I(x)\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_\ell}\end{aligned}$$

VIII. Stoke's Theorem

VIII.1. Manifolds with Boundary

Stuff:

- For users of the notes, the rest of the proof from Wednesdaywas added to the November 30th notes.
- The bonus contains some stuff about computing cohomology (Mayer-Vietoris).
- For the next few days we'll discuss Stoke's Theorem. You are free to use it on the homework now.

Definition VIII.1.1

For convenience call $\mathbb{H}^n = \{(x_1,\ldots,x_n)\in\mathbb{R}^n \mid x_n \geq 0\}$. A topological manifold with boundary is a paracompact, Hausdorff, second countable space M with a cover of M by $\{U_\alpha\}_{\alpha\in I}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{H}^n$.

We can then require the transition maps to be smooth to get a smooth manifold with boundary.

Example VIII.1.1

$\mathbb{H}^n$ is a manifold with boundary. So is $\mathbb{R}^n$ (empty boundary). You can have things like intervals with endpoints, or taking a standard genus g surface and slicing it in half.

Definition VIII.1.2

We define the boundary of M to be $\{x\in M \mid \varphi_\alpha \in \partial\mathbb{H}^n\}$ where $\partial\mathbb{H}^n := \{x\in\mathbb{H}^n \mid x_n = 0\}$.

Lemma VIII.1.1

∂M is well defined, i.e. independent of the chart. I.e., a diffeomorphism between $\mathbb{H}^n$ and itself preserves the boundary.

Furthermore, ∂M is a manifold (without boundary) of dimension $\dim M - 1$.

Exercise VIII.1.2

Prove this lemma above.

Lemma VIII.1.2

Let $M, \partial M$ be a manifold with boundary. Suppose M is oriented. Then ∂M is also oriented.

Proof. Take the situation in $\mathbb{H}^n$. If $p\in\partial\mathbb{H}^n$. How do we tell if $v_1,\ldots,v_{n-1}\in T_p\partial\mathbb{H}^n$ is a positively oriented basis?

[1]The rest of this was technically done on December 2nd

Well take $u = (u_1, \ldots, u_n)$ an outward normal, aka so that $u_n < 0$. We then call $v_1, \ldots, v_{n-1}$ positively oriented for $\partial\mathbb{H}^n$ if $u, v_1, \ldots, v_{n-1}$ are positively oriented for $\mathbb{R}^n$. This does not depend on the particular u chosen, bc it can be taken to $(0, \ldots, 0, u_n)$ by a linear combination with $v_1, \ldots, v_{n-1}$.

For M a manifold with boundary, we endow M with the pullback orientation from $\partial\mathbb{H}^n$. One must check this is well-defined, and one checks that transition maps preserve outward normals.

Theorem VIII.1.3 (Stoke's Theorem)

Let M be an oriented manifold with boundary ∂M (under the induced orientation).

Given $\omega \in \Omega^{n-1}M$, we have that

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Example VIII.1.3

Pick $M = [0,1]$ and pick the left to right orientation. Take $\omega \in \Omega^0([0,1])$, aka a smooth function $f : [0,1] \to \mathbb{R}$.

Then we have

$$\int_{[0,1]} \mathrm{d}\omega = \int_0^1 \omega'(t)\,\mathrm{d}t = \omega(1) - \omega(0)$$
$$\int_{\partial[0,1]} \omega = \int_{\{0,1\}} \omega = \omega(1) - \omega(0),$$

where the $-$ comes from the orientation. Thus we should think of Stoke's Theorem as a generalization of the fundamental.

Corollary VIII.1.4

If M is a manifold (without boundary) and $\omega \in \Omega^{n-1}(M)$ then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega = 0.$$

This means the integral of exact forms over manifolds are zero.

What do we need to do to prove this thing? The Idea: look at differentiable cubes, aka smooth maps $C : I^k \to M$. Then taking some $\omega_k(M)$ we look at an integral

$$\int_C \omega = \omega_{I^k} C^*(\omega).$$

Then we'll cover M by cubes, take a partition of unity, and reduce the whole problem to something about integrating around cubes.

Final Stuff:

- Extra Office Hours: Next Monday 2pm, Next Tuesday 4pm, plus by appointment/drop by.
- Material will concentrate on things since the midterms, but of course mathematics is largely cumulative
- Old QR problems are good practice. As are old HW problems.
- Each Question will be worth 20pts.

Last time we began discussing singular cubes, i.e. smooth maps $I^k \coloneqq I^k \to M$.

Definition VIII.1.3

A smooth map $C : I^k \to M$ is called a singular k-cube in M.

New idea, do something crazy: look at formal sums of singular k-cubes. Say

$$\mathcal{C}_k \coloneqq \left\{ \sum_{i=1}^{m} a_i c_i \mid a_i \in \mathbb{R}, m \in \mathbb{N}, c_i k\text{-cubes} \right\}.$$

This is then the free $\mathbb{R}$-module (i.e. vector space) with basis $\{c \mid ck\text{-cube}\}$. This space is infinite-dimensional.

Definition VIII.1.4

An form sum $\sum_{i=1}^m a_i c_i$ as above is called a singular k-chain in M.

Goal: manifolds to algebra. We need some sort of map between these things.

Note: $\mathcal{C}_0$ is formal sums of points in M, as 0-cubes are points in M. Now, we have a map $\partial : \mathcal{C}_k \to \mathcal{C}_{k-1}$. What is it? It's the "boundary" operation (with signs)!

Want: $\partial^2 = 0$ (this whole thing is "dual" to what we do with forms).

We'll think of I^n as both $[0,1]^n$ and $I^n = \mathrm{Id} : [0,1]^n \to [0,1]^n$. We now take

$$I_{i,0}^n(x) \coloneqq I(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$
$$I_{i,1}^n(x) \coloneqq I(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}).$$

If $i = 0$ we kick the 0 to the last coordinate (in some sense this is all modulo n). where $x = (x_1, \ldots, x_{n-1})$. Well

Example VIII.1.4

$I^1_{(1,0)}() = (0)$. Then $I^2_{(1,0)}(x_1) = (0, x_1)$ and $I^2_{(1,1)} = (1, x_1)$. Likewise

$$I^2_{(0,0)}(x_1) = (x_1, 0)$$
$$I^2_{(0,1)}(x_1) = (x_1, 1).$$

Crucial Fact: Each $(n-2)$-dimensional face of I^n is the $(n-2)$-face of two $(n-1)$-faces of I^n. Must figure out a combinatorial way of assigning opposite signs to get $\partial^2 = 0$.

Definition VIII.1.5 (Formal Definition of ∂)

We define

$$\partial I^n \coloneqq \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}.$$

This works generally. For $C : I^n \to M$ define

$$\partial C \coloneqq C \circ \partial I^n \coloneqq \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} (C \circ I^n_{(i,\alpha)}),$$

(this is essentially defined by pushing forward ∂I^n from I^n to M along C).

Proposition VIII.1.5

$\partial^2 I^n = 0$.

Proof. More formally, suppose $i \leq j$. Let $x = (x_1, \dots, x_{n-2})$. We compute

$$\begin{aligned}(I^n_{i,\alpha})_{j,\beta}(x) &= I^n_{(i,\alpha)}(x_1, \dots, x_{j-1}, \beta, x_j, \dots, x_{n-2}) \\ &= (x_1, \dots, x_{i-1}, \alpha, x_i, \dots, x_{j-1}, \beta, x_j, \dots, x_{n-2}).\end{aligned}$$

We likewise compute that

$$\begin{aligned}(I^n_{j+1,\beta})_{i,\alpha}(x) &= I^n_{j+1,\beta}(x_1, \dots, x_{i-1}, \alpha, x_i, \dots, x_{n-2}) \\ &= (x_1, \dots, x_{i-1}, \alpha, x_i, \dots, x_{j-1}, \beta, x_j, \dots, x_{n-2}).\end{aligned}$$

Thus these maps are equal! But the signs associated to them, i.e. $(-1)^{i+\alpha}(-1)^{j+\beta}$ and $(-1)^{j+\beta+1}(-1)^{i+\alpha}$ are opposite! Thus these will cancel in $\partial^2 I^n$.

Then extend ∂ linearly to $\mathcal{C}_k$ to $\mathcal{C}_{k-1}$.

Lemma VIII.1.6

$\partial^2 = 0$.

Proof. $\partial^2(I^k) = 0$. Then $\partial^2(C) = C \circ \partial^2(I^k) = C \circ 0 = 0$. Then including sums gives zero.

We now want to integrate over singular k-chains. The setup: if $C : I^k \to M$ is a k-cube, $\omega \in \Omega^k M$ a k-form, then

$$\int_C \omega \coloneqq \int_{I^k} C^*(\omega).$$

We know $C^*(\omega) = f \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. We can then just take

$$\int_{I^k} C^*(\omega) = \int_{I^k} f \, \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

(In fact: you can integrate on the interior of I^k, since this is a Lebesgue integral and the boundary has measure zero). For chains $\sum_{i=1}^m \alpha_i c_i$ we take

$$\int_{\sum_{i=1}^m \alpha_i c_i} \omega = \sum_{i=1}^m \alpha_i \int_{c_i} \omega$$

Theorem VIII.1.7 (Basic Stokes)

Suppose $\omega \in \Omega^{k-1}(A)$ where $A \subseteq \mathbb{R}^n$ is open, then if C is a k-chain then

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega.$$

Definition VIII.1.6

A manifold M is called <u>closed</u> if it is compact and has no boundary.

For HW11 #4, assume M is oriented and closed. We want to integrate $F_t^*(\mu)$ over $\partial(M \times [0,1])$. But then how do we apply stokes, i.e. how do we differentiate $\mathrm{d}F_t^*(\mu)$. The instinct is to use commutativity to get $F_t^*(\mathrm{d}\mu) = 0$. However d here lives in $M \times [0,1]$, not in M, so this is not quite immediate.

Stokes says that for C a k-chain, α a $(k-1)$-form.,

$$\int_C \mathrm{d}\alpha = \int_{\partial C} \alpha.$$

Corollary is stokes for manifolds themselves (cover with k-chain), regular stokes (curl), divergence (div), and Green's theorem

Proof of Stokes for simplicial k-chains. Good enough to check for singular k-cubes since

$$\int_C \mathrm{d}\alpha = \int_{[0,1]^k} C^*(\mathrm{d}\alpha) = \int_{[0,1]^k} \mathrm{d}(C^*\alpha)$$

$$\int_{\partial C} \alpha = \sum_{i=1}^{k} \sum_{\beta=0,1} (-1)^{\beta+i} \int_{I^{k-1}} C^*_{1,i}(\alpha).$$

Now we see that α is

$$\alpha = \sum_{i=1}^{k} f_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k.$$

Then it is good enough to check on $\alpha = f\,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \mathrm{d}x_n$.

Claim

$\int_{I^k} \mathrm{d}\alpha = \int_{\partial I^k} \alpha$

Note that

$$\int_{[0,1]^{k-1}} I^*_{j,\beta}(f\,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k) = \begin{cases} 0 & \text{if } j \neq i \\ \int_{[0,1]}^{k-1} f(x_1, \ldots, \beta, x_i, \ldots, x_k) & \text{if } j = i \end{cases}$$

because $\mathrm{d}x_j$ restricted is just 0, and when $j \neq i$ it shows up. The other one is a bit harder but not too bad. and

$$\int_{\partial I^k} \alpha = \sum_{j=1}^{k} \sum_{\beta=0,1} (-1)^{j+\beta} \int_{[0,1]^{k-1}} I^*_{(j,\beta)}(\alpha).$$

Thus the right hand side of the about equation is

$$\sum_{i=1}^{k} (-1)^{i+1} \int_{[0,1]^{k-1}} f(x_1, \ldots, 1, \ldots, x_k) + (-1)^i \int_{[0,1]^{k-1}} f(x_1, \ldots, 0, \ldots, x_k).$$

On the other hand we have that

$$\int_{I^k} \mathrm{d}(f\,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k) = \int_{I^k} \mathrm{d}f \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \mathrm{d}x_k.$$

Then $\mathrm{d}f = \sum_{j=1}^k \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j$, and so the wedge is nonzero only when $j = i$ so we have, switching things to be in standard order that this is

$$\int_{I^k} (-1)^{i-1} \frac{\partial f}{\partial x_i}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k.$$

Then we apply Fubini's Theorem to get

$$
\begin{aligned}
&(-1)^{i-1} \int_0^1 \int_0^1 \cdots \left(\int_0^1 \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \right) \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k \\
&= \int_0^1 \int_0^1 \cdots (f(x_1, \ldots, 1, \ldots, x_k) - f(x_1, \ldots, 0, \ldots, x_k)) \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_k
\end{aligned}
$$

via the fundamnetal theorem of calculus.

Via the above reductions we win!

Stuff:

- Office Hours moved to Monday 5pm and Tuesday 4pm.
- Final: Thursday December 15th from 4pm to 6pm

Last Time: We proved for C a singular $(k-1)$-chain and α a $(k-1)$-form that

$$\int_C \mathrm{d}\alpha = \int_{\partial C} \alpha.$$

Corollary VIII.1.8 (Manifold Stokes)

$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha$ for α an $(n-1)$-form, where $n = \dim M$.

But how is $\int_M \omega$ defined for an n-form ω? We want M to be oriented, compact, and we'll let $n \coloneqq \dim M$.

Well take an open cover U_i such that each $U_i \subseteq \mathrm{Image}(C_i)$ for C_i an orientation-preserving singular n-cube. We may also take a partition of unity subordinate to U_i. Write $\omega = \sum f_i \omega$ where $f_i \omega$ is supported on U_i. Then we define

$$\int_M \omega \coloneqq \sum_i \int_{C_i} f_i \omega.$$

Lemma VIII.1.9

This is in fact well-defined, i.e. does not depend on U_i, C_i, f_i.

Proof. Essentially the change of variables formula.

Proof Idea of Manifold Stokes. In the definition of $\int_M \omega$ use C_i such that only one face of C_i lies in ∂M. Write $\alpha = \sum_i f_i \alpha$. Then

$$\mathrm{d}\alpha = \sum_i (\mathrm{d}f_i) \wedge \alpha + \sum_i f_i \, \mathrm{d}\alpha.$$

Then we see by how we wrote the

$$
\begin{aligned}
\int_{\partial M} \alpha &= \sum_i \int_{\partial C_i} f_i \alpha \\
&= \sum_i \int_{C_i}
\end{aligned}
$$

Degree and $H^n_{\mathrm{dR}}(M)$.

Theorem VIII.1.10

If M is a compact, oriented, manifold then $H^n_{\mathrm{dR}}(M) = \mathbb{R}$.

Idea of Proof. We see that $\mathbb{R} \subseteq H^n_{\mathrm{dR}}(M)$ by orientation.

We then have a map precisely $H^n_{\mathrm{dR}}(M) \to \mathbb{R}$ given by $\omega \mapsto \int_M \omega$. Since M is oriented, if ν is a volume form then $\int_M \nu > 0$. Thus the map is onto.

Now suppose $\int_M \omega = 0$. The claim is that $\omega = \mathrm{d}\beta$. To prove this claim, you cover M by open sets U_i contained in some singular n-cubes C_i. We do this in such a way that

$$(U_1 \cup \cdots \cup U_k) \cap U_{k+1} \neq \emptyset.$$

Call $M_k = U_1 \cup \cdots \cup U_k$. We know $M = M_k$ for some k since M is compact.

It suffices to prove that if ω on M_k has zero integral then $\omega = \mathrm{d}\eta$ for some η defined on M_k. We prove this by induction on k.

For $k = 1$, we're on a chart so this is just the Poincare Lemma. Suppose the result holds for k. We see that

$$\int_{M_{k+1}} \omega = 0,$$

Let θ be a form supported in $M_k \cap U_{k+1}$ such that $\int_{M_{k+1}} \theta = 1$. Let $\{\varphi, \psi\}$ be a partition of unity subordinate to $\{M_k, U_{k+1}\}$ and let $c := \int_{M_{k+1}} \varphi\omega$.

We see that $\varphi\omega - c\theta$ is zero on M_k, thus $\mathrm{d}\alpha = \varphi\omega - c\theta$ for some α. Likewise $\psi\omega + c\theta$ has integral zero on U_{k+1} so is $\mathrm{d}\beta = \psi\omega + c\theta$ (it must have this integral since ω has integral zero on M_{k+1}).

Then we see that $\mathrm{d}(\alpha + \beta) = \omega$!

Suppose M, N are dimension n, oriented and compact. Let ν be a volume form on N.

Definition VIII.1.7

The degree of a map $f : M \to N$ is some map then the degree of f is defined by

$$\int_M f^*\nu = (\deg f) \int_N \nu,$$

since this number is uniquely defined

Example VIII.1.5

The map $z \mapsto z^k$ on $S^1 \to S^1$ has degree k. Furthermore, if $f : S^1 \to S^1$ is an orientation preserving diffeomorphism then it has degree 1.

Proposition VIII.1.11

$\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Theorem VIII.1.12

Brouwer's Fixed Point theorem. Let D^n be a closed ball in $\mathbb{R}^n$. Then if $f : D^n \to D^n$ is continuous, then f has a fixed point.

Proof. Only prove for f smooth.

Claim

It is enough to prove for f smooth.

Approximate f continuous by smooth maps homotopic to it. Then see Lee.

A homotopy between f, g is a map $F : X \times [0,1] \to X$ so that $F(x,0) = f(x), F(x,1) = g(x)$. It turns out the action on de Rham cohomology for a continuous map can be defined by approximating with a smooth map this way, and is independent of the approximation chosen.

Suppose $f : D \to D$ is smooth and has no fixed points. Then define

$$G(x) = \frac{x - f(x)}{\|x - f(x)\|}.$$

This map is then well-defined and continuous. G is a map from $D \to \partial D = S^{n-1}$. We can also let $H(t,x)$ on S^{n-1} be given by

$$H_t(x) = \frac{x - tf(x)}{\|x - tf(x)\|}.$$

We see that if $0 \le t < 1$ then $\|x\| = 1$, $\|tf(x)\| \le t < 1$, so this is well-defined. Likewise we know $x - f(x) \neq 0$ so it's well-defined for $t = 1$ as well.

Then H_0 is the identity map, so $\deg H_0 = 1$. On the other hand, if ν is the volume form on S^{n-1}, then

$$\int_{S^{n-1}} H_0^* \nu = \int_{S^{n-1}} H_1^* \nu$$

But then since $H_1 = g$, $H_1^* \nu$ can be defined over D as $G^* \nu$. But wait! Applying Stokes yields

$$\begin{aligned}\int_{S^{n-1}} g^* \nu = \int_{S^{n-1}} G^* \nu &= \int_D \mathrm{d}G^* \nu \\ &= \int_D G^*(\mathrm{d}\nu) = \int_D 0 = 0.\end{aligned}$$

Thus the degree is zero! Contradiction ☺.