Theorem .0.1 (HW)

If M is a smooth manifold (paracompact) then there is a smooth Riemannian metric (in fact many)

Proof Idea. Glue local solutions together using partition of unity.

Definition .0.1

A Lorentz metric is a nondegenerate inner product $\langle , \rangle_p$ on T_pM such that $p \mapsto \langle , \rangle_p$ is smooth. I.e. for all smooth vector fields X, Y on M we have $p \mapsto \langle X(p), Y(p)\rangle_p$ is smooth. Furthermore $\langle , \rangle_p$ has signature $(n-1, 1)$.

Given a nondegenerate $\langle , \rangle$ is a (nondegenerate) inner product on a finite dimensional verctor space V, $\dim V = n$. Then if this has signature $(k, n-k)$ then there is a basis $v_1, \dots, v_n$ such that

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \leq k \\ -1 & \text{if } i = j > k \end{cases}.$$

If $x = \sum x_i v_i, y = \sum y_j v_j$ then

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_k y_k - x_{k+1} y_{k+1} - \cdots - x_n y_n.$$

Every inner product has some signature.

Special relativity is Lorentz metrics on $\mathbb{R}^4$, and general relativity is the same spiel on a general manifold (that admits a Lorentz metric).

Theorem .0.2 (HW)

Not every smooth manifold supports a Lorentz metric.

Theorem .0.3

S^2, M where M is a compact connected orientable surface of genus > 1 does not admit a Lorentz metric.

Proof Idea. Look at S^2 and use that it does not admit a 1-dimensional distribution (follows from the fact that S^2 admits no nonvanishing vector field). Similarly for M where the genus > 1.

The fact that this follows is from covering space theory. Bad idea for finding distributions: $\{\langle v, v\rangle_p = 0\}$. Better idea: Use the standard Riemannian metric and grab the unit circle in T_pS^2 with respect to Euclidean metric on $\mathbb{R}^3$, call this $T_p^1 S^2$. Look at $\langle v, v\rangle_p$ restricted to $T_p^1 S^2$.

$\{\langle v, v\rangle_p \geq 0\}$.

In contrast, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ does since $T^2 = S^1 \times S^1$ and we can place $+, -$ on these respectively. Likewise $\mathbb{Z}_2$ preserves the $x_1^2 - x_2^2$ on $\mathbb{R}^2$, thus this descends.

For $\mathrm{SL}_2(\mathbb{R})$ there exists a left invariant $\langle , \rangle$ on $\mathrm{SL}(2, \mathbb{R})$ (in fact bi-invariant). Define it on X, Y given by for $X, Y \in \mathfrak{g}$,

$$\langle X, Y \rangle_1 = \operatorname{tr}(Z \mapsto [X, [Y, Z]]) = \operatorname{tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$$

where $(\operatorname{ad} X)(Z) \coloneqq [X, Z]$.

Aside on Lie groups. Let $T_1 G = \mathfrak{g}$. define

$$\begin{aligned} \operatorname{ad} X : \mathfrak{g} &\to \mathfrak{g} \\ Z &\mapsto [X, Z]. \end{aligned}$$

Then we can define the "Cartan-Killing form" of $\mathfrak{g}$ as

$$B(X, Y) = \operatorname{tr}(\operatorname{ad} X \circ \operatorname{ad} Y).$$

Exercise .0.1

Let X, Y lie in $\mathfrak{gl}(n, \mathbb{R}) = \operatorname{Lie} \mathrm{GL}(n, \mathbb{R})$. Then we must show $B(X, Y) = \operatorname{tr}(X \cdot Y)$ (up to some dimension factor.

Definition .0.2

Call $\mathfrak{g}$ semisimple if B is nondegenerate

Theorem .0.4

$\mathfrak{sl}(n, \mathbb{R})$ is in fact semisimple.

Note: If $\mathfrak{g}$ has a center. I.e. if there is $Z \in \mathfrak{g}$ so that for all $X \in \mathfrak{g}$ we have $[X, Z] = 0$, then B is degenerate. Why? Well $B(Z, X) = 0$.

Fact: If G is compact with no center, then B is positive definite and nondegenerate

Example .0.2

$\mathrm{SU}(n), \mathrm{SO}(n)$, etc.