Idea of proof: somehow apply inverse function theorem

Summary of the proof of the regular value theorem in steps:

(Step 1) Use coordinate charts to reduce to a problem about $F : U \to \mathbb{R}^k$ so that $U \subseteq \mathbb{R}^n$ is open.

(Step 2) We have $F : U \to \mathbb{R}^n$ and $\overline{q}$ a regular value of F. We want to show $F^{-1}(\overline{q})$ is a submanifold of $\mathbb{R}^n$. We know that $\overline{p} \in F^{-1}(\overline{q})$ $DF_{\overline{p}}$ is surjective and

$$DF_{\overline{p}} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n} \end{pmatrix}.$$

Since $DF_{\overline{p}}$ has rank k, so it has k linearly independent row vectors. We may assume without loss of generality that they are the first k.

(Step 3) Take $G : \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ to be defined by $G(x_1, \ldots, x_n) = (F(x_1, \ldots, x_n), x_{k+1}, \ldots, x_n)$. Then we have

$$A := \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_k} & \vdots \\ \vdots & \ddots & \vdots & \\ \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_k} & \end{pmatrix}$$

$$DG_p = \begin{pmatrix} A & * \\ 0 & \mathrm{Id}. \end{pmatrix}$$

This is obviously invertible. Thus G is a local diffeomorphism, which means $G^{-1}(\{q\} \times \mathbb{R}^{n-k})$ willbe locally a submanifold.

(Step 4) $F^{-1}(q) = G^{-1}(\{q\} \times \mathbb{R}^{n-k})$. Thus $F^{-1}(q)$ is locally a submanifold.

Example .0.1

There is a canonical submersion given by $\mathbb{R}^n \to \mathbb{R}^k$ with $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$ for $k \leq n$.

Another way of stating the regular value theorem is

Theorem .0.1

If $f : M \to N$ is C^1 and q is a regular value, then for any $p \in f^{-1}(q)$ there exist coordinate charts $(U, \varphi), (V, \psi)$ such that

$$\begin{array}{ccc} \mathbb{R}^m & & \mathbb{R}^n \\ \cup & & \cup \\ \varphi(U) & \xrightarrow{\psi \circ f \circ \varphi^{-1}} & \psi(V) \end{array}$$

$$(x_1, \ldots, x_m) \longmapsto (x_1, \ldots, x_n).$$

with $m = \dim M, n = \dim N$. This is called the <u>normal form</u>.

What is the engine of the IVT? The contraction mapping fixed point theorem!!!

Drives many things in the subject, such as existence and uniqueness of solutions of ODEs. It also shows up in dynamics.

Example .0.2

Consider $O(n) = \{n \times n \text{ matrices with } AA^t = \mathrm{Id}\}$. We show it's a submanifold of $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ by looking at

$$\begin{aligned} F : M_n(\mathbb{R}) &\to \mathrm{Sym}(n \times n) \cong \mathbb{R}^{n(n+1)/2} \\ A &\mapsto AA^T, \end{aligned}$$

where $\mathrm{Sym}(n \times n)$ are the symmetric $n \times n$ matrices. We must show that the identity Id is a regular value.

We must calculate DF_g where $g \in F^{-1}(\mathrm{Id})$, so $gg^t = \mathrm{Id}$. We calculate $DF_g(v)$ using curves, where $v \in \mathbb{R}^{n^2}$. Consider

$$\begin{aligned} F(g+tv) &= (g+tv)(g+tv)^T = (g+tv)(g^T + tv^T) \\ &= \mathrm{Id} + t(gv^T) + t(vg^T) + t^2 \cdot * \\ \frac{\mathrm{d}}{\mathrm{d}t} F(g+tv)\Big|_{t=0} &= gv^T + vg^T. \end{aligned}$$

The claim is that any symmetric matrix has this form. Set $v = wg$, then

$$DF_g(v) = w^T + w.$$

If A is a symmetric matrix, then taking $w = \frac{A}{2}$ is sufficient.

Alternative approach: Compute the kernel $\ker(DF_g)$. If $v \in \ker(DF_g)$ then $gv^T = -vg^T$, so vg^T is skew-symmetric. The dimension of this is $n(n-1)/2$.

Then the dimension of the image of DF_g is

$$n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

But this is exactly the dimension of the symmetric matrices, and so we're done.

$\mathrm{SO}(n)$ is the connected component of the Id in $O(n)$ because $\det : O(n) \to \mathbb{R}$ takes values in $\{\pm 1\}$. Thus $\mathrm{SO}(n)$ is clopen in $O(n)$.

Fact: $\mathrm{SO}(n)$ is connected.