Announcements

- HW5 Deadline extended to 10/7 Friday 11pm

Definition .0.1

Suppose we have two vector-bundles $\mathcal{V}, \mathcal{W}$ over M, N respectively. A map of vector bundles consists of two maps $\Phi : \mathcal{V} \to \mathcal{W}$ and $\phi : M \to N$ (C^r for whatever r you want) such that the following diagram commutes

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & \mathcal{W} \arrow[d, "\tau"] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

and also

$$\Phi\big|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \tau^{-1}(\phi(p))$$

is linear. Call (Φ, ϕ) a vector bundle isomorphism if there are inverses (Ψ, ψ) to Φ and ϕ respectively. In this case we call $\mathcal{V}, \mathcal{W}$ equivalent.

We call $\mathcal{V}$ trivial if it is equivalent to a trivial bundle $\mathbb{R}^n \times M$, with $\mathbb{R}^n \cong \pi^{-1}(p)$.

For the trival bundle $\tau : \mathbb{R}^\ell \times M \to M$ we have lots of sections, say $\sigma_i(p) = (e_i, p)$, $e_i \in \mathbb{R}^\ell$ a basis of $\mathbb{R}^\ell$. The σ_i are smooth vector fields and $\{\sigma_i(p)\}$ are linearly independent and in fact a basis for $\tau^{-1}(p)$ for all $p \in M$.

Thus we get ℓ sections of $\tau : \mathbb{R}^\ell \times M \to M$ which are linearly independent at every point. The converse also holds!

Proposition .0.1

Let $\pi : \mathcal{V} \to M$ be a vector bundle of rank ℓ (that is $\pi^{-1}(p) \cong \mathbb{R}^\ell$). If there exist ℓ sections $\sigma_1, \ldots, \sigma_\ell$ which are linearly independent at every point then $\mathcal{V}$ is trivial (i.e., isomorphic to the trivial bundle of rank ℓ).

Proof. Consider the map

$$\mathbb{R}^\ell \times M \to \mathcal{V}$$

$$((a_1, \ldots, a_\ell), p) \mapsto \sum_{i=1}^{\ell} a_i \sigma_i(p).$$

Corollary .0.2

The tangent bundle of a differentiable manifold is trivial if and only if there exist $\dim M$ many vector fields which are linearly indepenent at every point.

Remark .0.1

Warning: There are two senses in which sections σ_i may be linearly independent. We can have that *in the space of sections*

$$\sum a_i \sigma_i = 0$$

implies $a_i = 0$. We can also have linear independence at every point, namely for every $p \in M$ we have

$$\sum a_i \sigma_i(p) = 0$$

implies $a_i = 0$. We'll call the latter notion linearly independent pointwise.

Consider $S^2 = M$. Then X has a non-zero vector field which is 0 somewhere but not 0 everywhere! X is linearly independent as a single vector in the space of sections, but not at every point.

Notice that

$$\dim\{\text{sections of } \mathcal{V} \to M\} = \infty$$

unless $\mathcal{V}$ or M is of dimension 0.

Last time: We looked briefly at derivations as point $p \in M$. We're going to continue to talk about them ☺

Example .0.1

Consider a smooth vector field X on M and we define

$$\Delta : C^\infty(M) \to C^\infty(M)$$
$$(\Delta f)(p) = \partial_{X(p)}(f)$$

where $\partial_{X(p)}$ is the directional derivative at p in the direction of $X(p)$ (see last time).

Then in fact we have

$$\Delta(f \cdot g) = f \cdot \Delta g + \Delta f \cdot g.$$

Example .0.2

Consider $X = y\frac{\partial}{\partial x}, Y = x\frac{\partial}{\partial y}$ on $\mathbb{R}^3$. Then we're going to look at

$$\begin{aligned}(X \circ Y) &= \left(y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial y}\right)\\ &= y \cdot \left(\frac{\partial}{\partial}x\right)\frac{\partial}{\partial y} + yx\frac{\partial}{\partial x}\frac{\partial}{\partial y}\\ &= y\frac{\partial}{\partial y} + yx\frac{\partial^2}{\partial x \partial y}.\end{aligned}$$

What is this??? It's not a vector field... What about the other way

$$\begin{aligned}(Y \circ X) &= \left(x\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x}\right)\\ &= x(\frac{\partial}{\partial y}y) \cdot \frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\frac{\partial}{\partial x}\\ &= x\frac{\partial}{\partial y} + xy\frac{\partial^2}{\partial y \partial x}.\end{aligned}$$

We can view X, Y as $\Delta_X, \Delta_Y : C^\infty(M) \to C^\infty(M)$. Then look at it as $\Delta_{X \circ Y} = \Delta_X \circ \Delta_Y$ and likewise.

Now we can consider

$$X \circ Y - Y \circ X = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}.$$

This is a vector field!

Theorem .0.3

Let M be a smooth manifold with smooth (C^1 is enough) vector fields X, Y. Then in fact $X \circ Y - Y \circ X$ is a derivation $C^\infty(M) \to C^\infty(M)$.

Because of this we'll call $[X, Y] \coloneqq X \circ Y - Y \circ X$, and we'll call it the Lie bracket of X and Y.

Proof. Linearity is immediate. We just need to check the product rule. Namely we must check

$$[X, Y](fg) = ([X, Y]f)g + f([X, Y]g).$$

Theorem .0.4

Every C^∞ derivation δ at p defines a tangent vector to p, i.e., there exists $v \in T_pM$ such that $\delta = \partial_v$.

Corollary .0.5

Every derivation $\Delta : C^\infty(M) \to C^\infty(M)$ defines a vector field.

Example .0.3

Take $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$. Then $[X, Y] = 0$, as the mixed partials are equal on smooth functions.

We don't see the geometry in these formulas. We need to see the geometry!!!