Clarification: 5a is still part of HW6, 5 b,c are the extra credit parts.

.1. deRham Cohomology

Now, lets compute $H^*_{\text{deRham}}(S^1)$. Well we know that

$$H^0(S^1) = \mathbb{R}^{\#\text{ connected components}} = \mathbb{R}.$$

How do we compute $H^1(S^1)$? Well recall, we showed that $H^1(\mathbb{R}) = 0$ by showing that given a closed 1-form, i.e. all 1-forms, then

$$\alpha = \mathrm{d}\beta$$

where β is a 0-form on $\mathbb{R}$ defined by

$$\beta(x) = \int_0^x f(t)\,\mathrm{d}t$$

where $\alpha = f\,\mathrm{d}x$.

Now think of S^1 as $[0,1]/(0 \sim q)$. Then if α is a 1-fom it looks like $f\,\mathrm{d}x$ where $\mathrm{d}x$ makes sense on $S^1 = \mathbb{R}/\mathbb{Z}$ since it is invariant under $x \mapsto x + a$ for all $a \in \mathbb{R}$ (we only need in $\mathbb{Z}$, but this is better).

Moreover, if $\alpha \in \Omega^1(S^1)$ then $\alpha = f\,\mathrm{d}x$ where $f : S^1 \to \mathbb{R}$ is smooth. Then we should of course take $\beta : S^1 \to \mathbb{R}$, so take

$$\beta(x) = \int_0^x f(t)\,\mathrm{d}t\ldots$$

BUT WAIT! We need to know if $\beta(0) = \beta(1)$! This gives us a condition

$$\alpha \text{ is exact} \iff \int_0^1 f(t)\,\mathrm{d}t = 0.$$

We want to look for $H^1(S^1) = \{\text{closed}\}/\{\text{exact}\}$. The closed one-forms are just all of them since $\Omega^2(S^1) = 0$. Now let $\alpha = f(x)\,\mathrm{d}x$ be a closed 1-form. Let $A \coloneqq \int_0^1 f(t)\,\mathrm{d}t$. Then consider $\alpha - A \cdot \mathrm{d}x$. Then

$$\int_0^1 (f(t) - A)\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t - A = 0.$$

Thus there exists $\beta \in \Omega^0(S^1)$ such that $\alpha - A\,\mathrm{d}x = \mathrm{d}\beta$. Thus $[\alpha] = [A \cdot \mathrm{d}x]$, which we can think of as $\mathbb{R}$ since there is one parameter.

Moral: The way the coordinate charts are put together to give you a manifold determines $H^*_{\mathrm{dR}}(M)$.

Crucial to put all this together:

Lemma .1.1 (Poincaré Lemma)

If $A \subseteq \mathbb{R}^n$ is an open, star-shaped set, then any closed k-form on A is exact.

Definition .1.1

A set $A \subseteq \mathbb{R}^n$ is called star-shaped provided there exists a point $p_0 \in A$ (called an observer) such that for any $p \in A$, the line segment $[p_0, p] \in A$.

Motivation for the Proof:

- This is really a vast generalization of the fundamental theorem of calculus. It is a long calculation.
- In dimension 1 we look at $g(x) = \int_0^x f(t)\,\mathrm{d}t$ and it turned out $\mathrm{d}g = f(t)\,\mathrm{d}t$.

- Idea for star-shaped: integrate along segments (i.e. "radially").

Proof. We'll actually define the following in this proof, called a chain homotopy, the middle maps

$$\begin{tikzcd}
\Omega^{k-1}(A) \arrow[r, "\mathrm{d}"] & \Omega^k(A) \arrow[r, "\mathrm{d}"] & \Omega^{k+1}(A) \\
\Omega^{k-1}(A) \arrow[r, "\mathrm{d}"'] & \Omega^k(A) \arrow[r, "\mathrm{d}"'] \arrow[ul, "I_k"] & \Omega^{k+1}(A) \arrow[ul, "I_{k+1}"]
\end{tikzcd}$$

What we want: I is a linear map,

$$\mathrm{d}_{k-1} \circ I_k + I_{k+1} \circ \mathrm{d}k = \mathrm{Id}\,.$$

Consequence: If $\mathrm{d}\alpha = 0$ for $\alpha \in \Omega^k(A)$. Then

$$\alpha = I\,\mathrm{d}\alpha + \mathrm{d}(I\alpha) = I(0) + \mathrm{d}(I\alpha) = \mathrm{d}(I\alpha).$$

Thus we'll have $H^*(A) = 0$. In the one-dimensional case I was simply integration from 0 to x. We'll define $I_\ell : \Omega^\ell(A) \to \Omega^{\ell-1}(A)$. We'll have

$$\omega = \sum_{i_1 < i_2 < \cdots < i_\ell} \omega_I \,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}$$

where ω_I is a smooth function on A (this works since we're in $\mathbb{R}^n$, so this is true globally, here I is the index set). We now set

$$(I\omega)(x) := \sum_{i_1<i_2<\cdots<i_\ell} \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left(\int_0^1 t^{\ell-1}\omega_I(tx)\,\mathrm{d}t\right) \cdot x_{i_\alpha} \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_\alpha}} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \in \Omega^{\ell-1}(A)$$

Without loss of generality here we've assumed $p_0 = 0$ to make things easier to write down. How do we prove this works? I.e. that $\mathrm{d} \circ I + I \circ \mathrm{d} = \mathrm{Id}$. Well, you just write it out. . .

$$\begin{aligned}
\mathrm{d}(I\omega) = \ell \sum_{i_1<\cdots<i_\ell} &\left(\int_0^1 t^{\ell-1}\omega_I(tx)\,\mathrm{d}t\right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \\
&+ \sum_{i_1<\cdots<i_\ell} \sum_{\alpha=1}^{\ell} \sum_{j=1}^{n} (-1)^{\alpha-1} \left(\int_0^1 t^\ell \frac{\partial\omega}{\partial x_j}(tx)\right) x_{i_\alpha} \wedge \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_\alpha}} \wedge \mathrm{d}x_{i_\ell}.
\end{aligned}$$

The first bit is from $\frac{\partial}{\partial x_{i_\alpha}}$ (and the second bit of product rule), and the $(-1)^{\alpha-1}$ disappears because we've switched it to put it in the right place. The second bit is from $\frac{\partial}{\partial x_j}$ for any j, and uses differentiation under the integral sign.

Then we look at $\mathrm{d}\alpha$. Before we do this, note by linearity it suffices to check equality in a fixed term $i_1, \ldots, i_\ell$. We'll suppose $i_1 = n - \ell + 1, \ldots, i_\ell = n$. So we'll omit the sum over $i_1 < \ldots < i_\ell$.

Well this is

$$\begin{aligned}
\mathrm{d}\omega &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\omega_I \,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \\
I(\mathrm{d}\omega) &= \sum_{j=1}^{n} \left(\int_0^1 t^\ell \frac{\partial}{\partial x_j}(\omega_1, \ldots, i_\ell\,\mathrm{d}t\right) x_j \widehat{\mathrm{d}x_j} \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}
\end{aligned}$$

$$- \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left(\int_0^1 t^\ell \frac{\partial}{\partial x_j} \omega_I(tx)\,\mathrm{d}t \right) x_{i_\alpha}\,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_\alpha}} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}.$$

We'll pick up this proof on Friday[1].

The messy terms then cancel, and we add the other terms

$$\begin{aligned}
&\ell \cdot \left(\int_0^1 t^{\ell-1} \omega_I(tx)\,\mathrm{d}t \right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} + \sum_{j=1}^{n} \left(\int_0^1 t^\ell \frac{\partial}{\partial x_j} \omega_I(tx)\,\mathrm{d}t \right) x_j \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} (t^\ell \omega_I(tx))\,\mathrm{d}t\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \\
&= \omega_I(x)\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}
\end{aligned}$$

[1]The rest of this was technically done on December 2nd