Example .0.1

We'll give one more example of a homogeneous space. We want a Lie group G and a closed subgroup $H \subseteq G$, and $M = G/H$. This is the same as G acting transitively on M, and $H = G_p$ for some $p \in M$. Note: If G_p does not depend on p, then?

So we're going to look at Grassmannian of k-planes in n-planes ($\mathbb{R}^n$). We call this $\mathrm{Gr}_{k,n}(\mathbb{R})$. Recall that

$$\begin{aligned} \mathrm{Gr}_{1,n} = \mathbb{R}\mathrm{P}^{n-1} &= \mathrm{GL}(n,\mathbb{R}) / \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\ &= \mathrm{SO}(n)/O(n-1) = (\underbrace{\mathrm{SO}(n)/\mathrm{SO}(n-1)}_{S^{n-1}})/\mathbb{Z}_2 \end{aligned}$$

where $O(n-1)$ is embedded in $\mathrm{SO}(n)$ as

$$A \mapsto \begin{pmatrix} \det A & 0 \\ 0 & A \end{pmatrix}.$$

In the general case take $e_1, \ldots, e_n$ as a basis for $\mathbb{R}^n$, $p \coloneqq \langle e_1, \ldots, e_k \rangle$ is a k-dimensional subspace. Then we can define an action by

$$\begin{aligned} \mathrm{GL}(n,\mathbb{R}) \times \mathrm{Gr}_{k,n} &\to \mathrm{Gr}_{k,n} \\ A \cdot V &= \{A \cdot v \mid v \in V\}. \end{aligned}$$

This is also transitive. If $V = \langle v_1, \ldots, v_k \rangle$ is a k-dimensional subspace, then $A = (v_1, \ldots, v_k, ?, \ldots, ?)$ where we have extended to a basis maps p to V. Thus $\mathrm{Gr}_{k,n} = \mathrm{GL}(n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})_p$. We see that

$$A \in \mathrm{GL}(n,\mathbb{R})_p \iff A \cdot p = p \iff A \cdot e_i \in p \iff A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

where the blocks are $k \times k, k \times (n-k), (n-k) \times k, (n-k) \times (n-k)$.

Then the transversal subspace making this a manifold is an $(n-k) \times k$ block of anything in the lower left hand block, 1s on the diagonal, and then 0s elsewhere.

Lemma .0.1

If $q = g \cdot p$, then $G_q = gG_pg^{-1}$. If we have a transversal T_p we should try $T_q = gT_pg^{-1}$.

Exercise .0.2

$\mathrm{SO}(n)$ also acts transitively on $\mathrm{Gr}_{k,n}$, so one can do the same work here.

Example .0.3

$S^1 = \mathbb{R}/\mathbb{Z}, T^n = S^1 \times \cdots \times S^1 = \mathbb{R}^n/\mathbb{Z}^n$. In these cases everything we said works although $\mathbb{Z}^n$ is not compact.

In contrast we have the bad (interesting) example given by $\mathbb{Z}$ acting on S^1 by irrational rotation.

Definition .0.1

Let Γ be a discrete group acting on a topological space X. We say the action is properly discontinuous provided that $\Gamma \cdot x$ can be taken to be "separate." We make this precise via

- Namely, for any compact set $K \subseteq X$, we have $(\Gamma \cdot x) \cap K$ is finite. In other words,

$$\Gamma \times X \to X$$

  is a proper map.
- If for some $x \in X, \gamma \in \Gamma$, we have $\gamma \cdot p = p$, then $\gamma = 1$. (This is also said as Γ acts freely, and is only included by some author).

Exercise .0.4

Suppose Γ acts on a manifold M properly discontinuously, then M/Γ is a manifold.

The same holds for differentiable (C^k) structure so long as Γ acts via differentiable (C^k) maps.

Example .0.5

$\mathrm{SL}(n, \mathbb{R})/\,\mathrm{SL}(n, \mathbb{Z})$ is the most famous example of this type. This is in fact the space of lattices in $\mathbb{R}^n$ of volume 1. This has a deep connection to number theory.

Particularly the case $n = 2$ is important because the tori carry complex analytic structure.

I. Tangent Vectors/Differentiation

Take M to be a differentiable manifold. How can we define a "tangent vector" on it. Well a tangent vector for $V \subseteq \mathbb{R}^n$ is just a choice (p, h) where $p \in V, h \in \mathbb{R}^n$.

So what if we just work chart-wise for charts $(\varphi_\alpha, U_\alpha)$? Well then for a $p \in M$, we can look at a tangent vector $(\varphi_\alpha(p), h) \in T_{\varphi_\alpha(p)} V_\alpha$. But how do we look between charts??? Aka what does $(\varphi_\alpha(p), h)$ look like in V_β?

Well we can look at the transition map $T_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$. Then we can define an equivalence relation

$$(\varphi_\alpha(p), h) \sim (\varphi_\beta(p), h') \iff \mathrm{d}T_{\alpha,\beta}(h) = h'.$$

Definition I.0.1

We define the tangent space

$$T_pM = \{[v, (U_\alpha, \varphi_\alpha)] \mid [v, (U_\alpha, \varphi_\alpha)] \text{ is an equivalence class of tangent vectors in charts}\}$$

Nice interpretation, $p \in \mathbb{R}^n$, $w \in T_pM$. Take $c(t)$ differentiable for $t \in (-\varepsilon, \varepsilon)$, $c(0) = p$, then we can look at $c'(0)$.

We can talk about two different curves then say c_1, c_2 are equivalent as tangent vectors if $c_1'(0) = c_2'(0)$.

For $p \in M$ a differentiable manifold, and a differentiable curve $c(t)$ through that point at $t = 0$, then $[c]_p$ is a tangent vector defined at charts as $c'(0)$ upon appropraite choice of coordinates.

Definition I.0.2

Let $f : M \to N$ be a differentiable map at $p \in M$. We define

$$\mathrm{d}f_p : T_pM \to T_{f(p)}N.$$

Take some differentiable curve c representing our tangent vector in T_pM. We can then take $\mathrm{d}f_p([c]_p) = [f \circ c]_{f(p)} \in T_{f(p)}N$.

Also $T_pM, T_{f(p)}N$ have vector space structures inherited from the case in $\mathbb{R}^n$, and as before for multivariable calculus, $\mathrm{d}f_p$ is a linear map.