Proof of Frobenius, in general. Let $X_i = (D\pi)^{-1}\left(\frac{\partial}{\partial x_i}(\pi(q))\right)$ (for an adapted chart). If $Y_i = \frac{\partial}{\partial x_i}(p)$, the these are π-related, and so the X_i commute. Namely we know $[X_i, X_j]$ is tangent to the chart V, and then

$$D\pi([X_i, X_j]) = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0,$$

thus $[X_i, X_j] = 0$, showing these commute.

Great! Then the special case implies the general theorem. This proof is only local, but we can do this globally as well, which we will do in the next section.

I. Lie Groups/Lie Algebras

Let G be a C^∞ Lie group. We want to look at left invariant vector fields. I.e. we have $V(g) = DL_g(V(1))$, and this is clearly a vector space. Its dimension is $\dim G$.

Definition I.0.1

Let G be a C^∞ Lie group. We define

$$\operatorname{Lie} G \coloneqq \mathfrak{g} \coloneqq \{\text{ left invariant vector fields}\} \cong T_1 G \cong \{\text{ right invariant vector fields}\},$$

which is a vector space often called the Lie algebra of G. Its dimension is $\dim G$ as mentioned above.

This comes with extra structure, since if X, Y are left invariant, then $[X,Y](g) = DL_g([X,Y](1))$. Well we know for any diffeomorphism φ that

$$D\varphi([X,Y]) = [D\varphi X, D\varphi Y].$$

This is the algebra structure.

Recall that $[X,Y] = -[Y,X]$ via the derivation definition.

Lemma I.0.1 (Jacobi Identity)

We have for X, Y, Z vector fields that

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Proof. Expand with the definition

We can define a general Lie algebra as a vector space which is equipped with an anticommutative bilinear form which satisfies the Jacobi identity).

Let G be a Lie group, $H \subseteq G$ a Lie subgroup, i.e. an immersed submanifold. We know that the set $\{gH\}_{g\in G}$ is a foliation of G. We know $V(g) = T_g(g \cdot H)$ which is a left invariant distribution.

We can then look a the left invariant vector fields tangent to $V(g)$ (i.e, tangent to gH). This defines $\mathfrak{h} \subseteq \mathfrak{g}$. And in fact, if $X, Y \in \mathfrak{h}$ then $[X,Y] \in \mathfrak{h}$ via the Frobenius theorem (since the $V(g)$ is integrable).

Definition I.0.2

Given a Lie algebra $\mathfrak{g}$, a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a vector subspace such that for any $X, Y \in \mathfrak{h}$ we have $[X,Y] \in \mathfrak{h}$.

Theorem I.0.2 (Lie Groups/Lie Algebras)

If H is a Lie subgroup of G, then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra. i.e, $[\mathfrak{h}, \mathfrak{h}] \in \mathfrak{h}$.

Conversely, if $\mathfrak{h} \subseteq \mathfrak{g}$, and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, then there exists $H \subseteq G$ a connected Lie subgroup such that $\mathfrak{h} = T_g(gH)$ (i.e., $\mathfrak{h}$ is left-invariant vector fields tangent to gH).

This gives a bijective correspondence between *connected* Lie subgroups H of G and Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$.

Example I.0.1

Here is an example that you need the connected statement. Take $\mathbb{Z} \subseteq \mathbb{R}$, then

$$\mathrm{Lie}(\mathbb{Z}) = \{0\} = \mathrm{Lie}(\{0\}).$$