.1. **Orientations on Manifolds**

Stuff:

- For HW8 #1, take a look at Spivak's Calculus. General Theorem that if $f : M \to N$ is C^1 and $\dim M \leq \dim N$, then the set of critical values have measure zero.
- For HW8 #3, Consider on U_i a local flow defined for time ε_i. Take a partition of unity f_i for U_i and then consider $\sum f_i \frac{1}{\varepsilon_i} X$.

Comment on Lie subgroups. Let $\varphi : H \to G$ be a smooth homomorphism. Instead of looking at $\mathfrak{h}, \mathfrak{g}$ as left invariant vector fields (which led us astray last time) look at the tangent space at the identity. Let $X \in T_1 H = \mathfrak{h}$, then $Y := D\varphi_1 \cdot X \in T_1 G = \mathfrak{g}$. Then Y defines a left invariant vector field on G, the claim is that X, Y are φ-related (good enough to justify brackets agree).

Cleanup from last time: It was said that k-forms measures area of an intersection. But instead it measures area of a projection.

Let V be a finite n-dimensional vector space. We know $\dim \Lambda^n V = 1$. We can't tell if a real number is positive or negative without placing an orientation on a line. But we *can* tell if they are positive multiples of each other (they have the same orientation)

Definition .1.1

Two n-forms $\alpha, \beta \neq 0$ on V have the same orientation if $\beta = c \cdot \alpha$ for $c > 0$. Otherwise they have the opposite orientation.

Definition .1.2

If M is an n-dimensional manifold (smooth). We let $(\Lambda^k M)_p := \Lambda^k T_p M$, which is a vector bundle $\Lambda^k M \to M$.

A k-form α is a (smooth) section of $\Lambda^k M \to M$.

Example .1.1

Consider $\mathbb{R}^n$, then $\alpha = \mathrm{d}x_{10^{44}}$ is a smooth 1-form (where $\mathrm{d}x_{10^{44}}$ is the dual vector to $\frac{\partial}{\partial x_{10^{44}}}$ which is a smooth vector field).

A two form could be something like $\mathrm{d}x_1 \wedge \mathrm{d}x_2$. We have to explain this though.

Question: We know that $\dim \Lambda^n T_p M = 1$ if $\dim M = n$. What would a section of $\Lambda^n M \to M$ (aka an n-form on M) tell us about M?

Definition .1.3

Any n-form τ (aka a section of $\Lambda^n M \to M$) such that for all $p \in M$ we have $\tau(p) \neq 0$ is called an orientation of M (a smooth manifold).

Given an orientation τ and another σ we say that σ, τ define the same orientation on M if there is a smooth map $f : M \to (0, \infty)$ so that $\sigma = f \cdot \tau$.

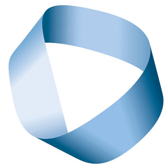
Also, if σ is an orientation, so is $-\sigma$, and these are NOT the same orientation.

Note: One can do orientation for topological manifolds but it requires Algebraic Topology and is harder.

Question: Do orientations always exist? No!!!

Example .1.2

We can look at the Möbius band, which is a strip glued in opposite directions



Definition .1.4

Call M orientable if it has an orientation. Also, an oriented manifold is a manifold M with a given orientation (M, σ).

Example .1.3

Observe, if we take the two caps of a sphere with natural orientations and glue them together to respect the orientation, we get S^n, which is orientable.

In contrast, if we look at $\mathbb{P}^n = S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on S^n by $x \mapsto -x$. Does this map preserve orientation?

Look at the simplest example for $S^1 \ldots$ then yes. For S^2 in fact no!

Proposition .1.1

Any Lie group G is orientable.

Proof. Pick $\sigma(1) \in \Lambda^n T_1 G$. Now make it left invariant by pushing it around.

Recall .1.4

$\mathbb{R}\mathrm{P}^3$ is diffeomorphic to $\mathrm{SO}(3)$, and this is double covered by $\mathrm{SU}(2)$. But then $\mathrm{SO}(3)$ is a group, so it is orientable.

Or: Stare at the antipodal map $A : S^n \to S^n$. If it preserves the orientation then just push it down to $\mathbb{R}\mathrm{P}^n$.

.2. The Wedge Product

If we have two multilinear maps f, g, then $f \otimes g$ is also multilinear (given as $(f \otimes g)(v, w) = f(v)g(w)$). But this may not be alternating even if f, g are alternating!!!

Given $\alpha \in \Lambda^k V, \beta \in \Lambda^\ell V$, then we wish to define $\alpha \wedge \beta \in \Lambda^{k+\ell} V$.

Definition .2.1

The wedge product $\alpha \wedge \beta$ of $\alpha \in \Lambda^k V, \beta \in \Lambda^\ell V$ is

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) = \frac{1}{k!\ell!} \cdot \sum_{\sigma \in S(k+\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}).$$

This is very similar to the definition of the determinant. Here $S(k+\ell)$ is the permutations of $\{1, \dots, k+\ell\}$ such that

$$\begin{gathered} \sigma(1) < \sigma(2) < \cdots < \sigma(k) \\ \sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+\ell). \end{gathered}$$

Thus it preserves the ordering on $\{1, \dots, k\}$ and on $\{k+1, \dots, k+\ell\}$ (but not necessarily both).