.1. Applications of Sylow's Theorems

Proposition .1.1

We have the following applications of Sylow's theorems

(1) If p, q are distinct primes with $p > q$, then no group of order $p^a q$ can be simple ($a > 0$).

(2) There are no simple groups of order 12

(3) If $|G| = 28$ and G has a normal Sylow 2-subgroup then G is abelian.

(4) There are no simple groups G of order 120.

Proof of (1). Let H be a Sylow p-subgroup, so $|H| = p^a$. The number of Sylow p-subgroups is $\equiv 1 \mod p$ and divides q. Because $p > q$ this implies that the number of Sylow p-subgroups is one.

Thus H is normal in this group, and it is not simple.

Proof of (2). Consider the # of Sylow 3-subgroups divides 4 and is $1 \mod 3$. Thus it's 1 or 4. If there is 1 then the Sylow 3-subgroup is normal and G is not simple.

If there are 4 Sylow 3-subgroups, then since these groups have prime order and have trivial intersection, G has $4 \cdot 2 = 8$ order three elemenets. This leaves three elements of G having order not 1 or 3. But any Sylow 2-subgroup of G contains 3 elements having order not 1 or 3. Thus there is exactly on Sylow 2-subgroup, and it is normal.

Proof of (3). The # of Sylow 7-subgroups divides 4 and is $1 \mod 7$, so there is only one Sylow 7-subgroup. Thus we have a normal Sylow 2-subgroup N and a normal Sylow 7-subgroup H. Their orders are coprime so $N \cap H = 1$. Thus from homework

$$G = \langle N, H \rangle = NH \cong N \times H.$$

All groups of order 4 and order 7 are abelian, and direct products of abelian groups are abelian. Thus G is abelian.

Proof of (4). The # of Sylow 5-subgroups divides 24 and is $1 \mod 5$, so it's 1 or 6. If it's one then G is not simple.

So assume the # of Sylow 5-subgroups is 6. G acts transitively by conjugation on these 6 Sylow 5-subgroups. This yields a homomorphism $\varphi G \to S_6$. Under the assumption that G is simple, the kernel must be trivial (as G doesn't fix the Sylow 5-subgroups).

Thus $G \cong \operatorname{im} \varphi$. Since $A_6 \trianglelefteq S_6$ we know that $\varphi(G) \cap A_6 \trianglelefteq \varphi(G)$. But then we have

$$[\varphi(G) : \varphi(G) \cap A_6] \leq 2.$$

Thus $\varphi(G) \subseteq A_6$, or else we would have that $\varphi(G) \cap A_6$ is a nontrivial normal subgroup.

But then by comparing sizes $[A_6 : \varphi(G)] = 3$. Then A_6 acts by left multiplication on $A_6/\varphi(G)$, which gives a homomorphism $A_6 \to S_3$. Because A_6 is bigger than S_3, this has a nontrivial kernel. Thus the kernel must be A_6 beause A_6 is simple. But the left-multiplicaiton action is transitive, so it can't be trivial.

Definition .1.1

Say a subgroup G of S_n is k-transitive (for $k \leq n$) if G acts transitively on the set of k-tuples of pairwise distinct elements of $\{1, \ldots, n\}$.

For example, if G is 2-transitive, this means G is transitive on the set of pairs $\{(i,j) \in \{1, \ldots, n\}^2 \mid i \neq j\}$. I.e. for all $i, j, k, \ell \in \{1, \ldots, n\}$ with $i \neq j$ and $k \neq \ell$ there exists a $g \in G$ with $g \cdot i = k$ and $g \cdot j = \ell$.

Here are some theorems about k-transitivity, all of which rely on the classification of finite simple groups.

Theorem .1.2

If $G \leq S_n$ is 6-transitive (or k-transitive for any $k \geq 6$), then $G = A_n$ $(n \geq 8)$ or S_n $(n \geq 6)$.

Note that A_n is $(n-2)$-transitive and S_n is n-transitive.

If $G \leq S_n$ is 5-transitive then $G = A_n$ $(n \geq 7)$, or S_n $(n \geq 5)$, or M_{23} $(n = 23)$, or M_{11} $(n = 11)$.

These M_{-} are some sporadic simple groups discovered by Matthew.

If $G \leq S_n$ is 4-transitive then $G = A_n$ $(n \geq 6)$, S_n $(n \geq 4)$, and four small groups in low degree.

For 3-transitive and 2-transitive groups there are infinite families but the list is small enough to be tractable. Namely for 2-transitive groups we have if $G \leq S_n$ is 2-transitive

- $G = A_n$ $(n \geq 4)$
- $G = S_n$ $(n \geq 2)$
- $\mathrm{PSL}_d(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_d(q))$ acting on $\mathbb{P}^d(\mathbb{F}_q)$ for $n = \frac{q^d-1}{q-1}$
- Similar description with $\mathrm{PSU}_3(q0$ for $n = q^3 + 1$.
- Similar description with $\mathrm{PSp}_{2k}(2)$ for $n = 2^{2k-1} \pm 2^{k-1}$
- Two other small families.
- Seven sporadic small simple groups.
- A few others

If G is a transitive subgroup of S_p (for p prime) then $G = S_p$ or A_p or

$$G \leq \mathrm{AGL}_1(\mathrm{p}) = \{x \mapsto ax + b \mid a, b \in \mathbb{Z}/p\mathbb{Z}, a \neq 0\}$$

unless $p = 11, 23$, $\frac{q^d-1}{q-1}$ for $d \geq 2$ or q a prime power.

Then also the doubly transitive ones.