Note: Everything from last time works over any ring.

Definition .0.1

A ring R is an abelian group under $+$ equipped with a multiplication $\cdot$ which is associative, has an identity, and distributes over addition.

Example .0.1

Given any abelian group G, then the set of endomorphisms $\mathrm{End}(G) \coloneqq \mathrm{Hom}(G, G)$ is naturally a group under addition and becomes a ring when equipped with composition.

Rings will be the first thing we study next semester.

Definition .0.2

An algebra over a field K is a vector space A over K with the structure of a ring such that for vectors $a, b \in A$ and scalars $c, d \in K$ we have

$$(ca) \cdot (db) = (cd)(a \cdot b).$$

This product is bilinear.

Definition .0.3

Let G be a finite group. Then $\mathbb{C}[G]$ is the group algebra.

As a vector space this is $\mathbb{C}$-linear combinations of a basis $\{e_g\}_{g \in G}$. For convenience we identify $c \in \mathbb{C}$ with ce_1. Another way to see this as as $\{\text{functions } G \to \mathbb{C}, g \mapsto c_g\}$.

Recall that G acts on $\mathbb{C}[G]$ by the regular representation

$$h \cdot \left(\sum_{g \in G} c_g e_g \right) = \sum_{g \in G} c_g e_{hg}.$$

Define then a ring structure on $\mathbb{C}[G]$ by the following for all $g, h \in G$ and $c \in \mathbb{C}$

$$e_h e_g = e_{hg} \qquad\qquad ce_g = e_g c.$$

Secretly the above formula is

$$ce_g = e_g(ce_1).$$

What is the center of $\mathbb{C}[G]$ (under multiplication)? This is the set of all $\theta \in \mathbb{C}[G]$ with $\theta x = x\theta$ for all $x \in \mathbb{C}[G]$. Equivalently $\theta e_g = e_g \theta$ for all $g \in G$.

Write $\theta = \sum_{h \in G} c_h e_h$. Then what this means is

$$\theta e_g = \sum_{hinG} c_h e_{hg} = \sum_{h \in G} c_h e_{gh} = e_g \theta.$$

Reindexing then gives

$$\sum_{h \in G} c_h e_{hg} = \sum_{h \in G} c_h e_{(ghg^{-1})g} = \sum_{h' \in G} c_{g^{-1}h'g} e_{h'g}$$

Therefore $c_h = c_{g^{-1}hg}$ for every $h \in G$. This means that

$$\begin{aligned}
\text{center of } \mathbb{C}[G] &= \left\{ \sum_{h \in G} c_h e_h \mid c_h = c_{g^{-1}hg} \ \ \forall\, g \in G \right\} \\
&= \left\{ \sum_{h \in G} c_h e_h \mid c_{_} : G \to \mathbb{C} \text{ is a class function} \right\} \\
&= \text{the class functions on } G \\
&= \left\{ \mathbb{C}\text{-linear combinations of } \sum_{h \in C} e_h \ \ \forall \text{ conjugacy classes } C \subseteq G \right\}
\end{aligned}$$

Definition .0.4

If $\rho : G \to \mathrm{GL}(V)$ is a representaiton and π is an irreducible representation, then we can decompose V by Maschke's Theorem into a direct sum of subspaces on which ρ acts isomorphically to irreducibles.

Collecting all the subspaces on which ρ acts as π into a direct sum gives the π-isotypic part of V. This is well-defined by Machke's Theorem.

Recall .0.2

If $\rho : G \to \mathrm{GL}(V)$ is a representation, and π is an irreducible representation of G, then the projection of V onto its π-isotypic part (aka a direct sum of things isomorphic to π) is

$$v \mapsto \frac{\dim \pi}{|G|} \sum_{g \in G} \chi_\pi(g^{-1})(g \cdot v).$$

In group algebra language (when ρ is the regular representation), this projection is multiplication of each element in $\mathbb{C}[G]$ by

$$e_\pi := \frac{\dim \pi}{|G|} \sum_{g \in G} \chi_\pi(g^{-1}) e_g.$$

Lemma .0.1

Let π be an irreducible representation of G. Let $s = \sum_g c_g e_g$ lie in the center of $\mathbb{C}[G]$. Define

$$\omega_\pi(s) := \frac{1}{\dim \pi} \sum_{g \in G} c_g \chi_\pi(g).$$

Then $s \mapsto \omega_\pi(s)$ is a homomorphism (linearly, and multiplicatively) from $\mathrm{Center}(\mathbb{C}[G]) \to \mathbb{C}$.

Theorem .0.2 (Burnside)

Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of a finite group G. If $g \in G$ and the size of the conjugacy class of g is coprime to $\dim V$ then either $\chi(g) = 0$ or g is in the kernel of $G \xrightarrow{\rho} \mathrm{GL}(V) \twoheadrightarrow \mathrm{PGL}(V)$, where $\mathrm{PGL}(V) = \mathrm{GL}(V)/\{c \cdot \mathrm{Id}_V \mid c \neq 0\}$.

That is either $\chi(g) = 0$ or $\rho(g) = c\,\mathrm{Id}_V$ for some $c \neq 0$.

Theorem .0.3 (Burnside)

Let $s = \sum_{h \in G} c_h e_h \in \mathbb{C}[G]$ where each c_h is an algebraic integer. If s is in the center of $\mathbb{C}[G]$ then

> s acts on any irreducible representation $\rho : G \to \mathrm{GL}(V)$ as multiplication by a scalar $\omega_\rho(s)$, and even better, this scalar is also an algebraic integer.
>
> In particular, for any $g \in G$,
> $$\frac{\chi_\rho(g) \cdot (\text{size of conjugacy class of } g)}{\dim \rho}$$
> is an algebraic integer.

Proof of very last part. Apply the first part to $s = \sum_{h \in C} e_h$, where C is the conjugacy class of g. Then

$$\omega_\rho(s) = \frac{1}{\dim \rho} \sum_{h \in C} \chi_\rho(h) = \frac{\chi_\rho(g) \cdot |C|}{\dim \rho}.$$

This is then an algebraic integer.

Next time: Proofs!!!