Announcements

- Quiz 1 now graded - Gradescope, and solutions are posted on the webpage
- Homework #1 due 8pm tonight on Gradescope, be sure to select the pages
- Office hours 2:30-4:30 in "Lounge Zoom"

Back to Lecture!

.1. Intro to Category Theory

.1.1. Our Definitions

Definition .1.1
A category $\mathscr{C}$ is three pieces of data with two conditions. Here's the data first:

- A class of objects, $\mathrm{Ob}(\mathscr{C})$.
- For all $X, Y \in \mathrm{Ob}(\mathscr{C})$ we have a class of morphisms (or arrows) denoted $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- For every $X, y, Z \in \mathrm{Ob}(\mathscr{C})$ we have a composition law, that is a map:

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$$
$$(f, g) \mapsto g \circ f$$

All of this satisfies the following two conditions:

- Associativity of composition:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

  And this holds for all f, g, h such that these compositions make sense.
- For every object $X \in \mathrm{Ob}(\mathscr{C})$ there should exist some $\mathrm{Id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that;

$$f \circ \mathrm{Id}_X = f \qquad \mathrm{Id}_X \circ g = g$$

  For all f, g such that these compositions are defined.

Example .1.1
Let's get some examples down.

$\mathscr{C}$	$\mathrm{Ob}(\mathscr{C})$	$\mathrm{Mor}(\mathscr{C})$
Set	Sets	Functions
Grp	Groups	Group Homomorphisms
Ab	Abelian Groups	Group Homomorphisms
k-Vect	Vector spaces over k	k-linear maps
Rng	Rings	Ring Homomorphisms
Top	Top. Spaces	continuous maps
Haus	Hausdorff Spaces	continuous maps
hTop	Top Spaces	homotopy classes of continuous maps
Top*	Top spaces with a distinguished basepoint	Continuous maps that preserve the basepoint

Example .1.2
Any "diagram" with a composition law defines a category. Just consider:

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{Id}_A"] \arrow[r, "f"] & B \arrow[loop right, "\mathrm{Id}_B"] \end{tikzcd}$$

Definition .1.2
A morphism $f : M \to N$ in a category $\mathscr{C}$ is monic if for all $g_1, g_2 : X \to M$ with the same domain:

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_1"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, f is epic if for all $g_1, g_2 : N \to X$ we have:

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_1"'] & \bullet \end{tikzcd}$$

Breakout Rooms

Lemma .1.1
In **Set**, **Ab**, **Top** a map is monic if and only if it is injective.

Solution. We'll do the problem in abelian groups. We go in both directions!

($\Rightarrow$) Suppose that $f : G \to H$ is a monic map of abelian groups. We wish to show that $f : G \to H$ is injective. Fix any two elements $a, b \in G$, and construct morphisms $\phi_a : \mathbb{Z} \to G$ and $\phi_b : \mathbb{Z} \to G$ as follows:

$$\phi_a(n) = n \cdot a \qquad \qquad \phi_b(n) = n \cdot b$$

Since we know that abelian groups are $\mathbb{Z}$-modules. Now suppose that $f(a) = f(b)$. Then consider that:

$$(f \circ \phi_a)(n) = f(n \cdot a) = n \cdot f(a) = n \cdot f(b) = f(n \cdot b) = (f \circ \phi_b)(n)$$

And therefore $f \circ \phi_a = f \circ \phi_b$. This shows since f is monic that $\phi_a = \phi_b$. However then we're in business since:

$$a = \phi_a(1) = \phi_b(1) = b$$

And so $a = b$. This shows that f is injective. Awesome ☺

($\Leftarrow$) Suppose that the map $f : G \to H$ is injective. We will show that f is monic. To do so, fix two maps $g_1, g_2 : A \to G$ where A is an abelian group, and suppose that $f \circ g_1 = f \circ g_2$. Then for any $a \in A$ we know that $f(g_1(a)) = f(g_2(a))$, giving us since f is injective that $g_1(a) = g_2(a)$. Since this holds for arbitrary $a \in A$ we know that $g_1 = g_2$!!! Great! ☺

With this we've finished the problem

.1.2. Functors

Definition .1.3
For $\mathscr{C}, \mathscr{D}$ categories a (covariant) functor $F : \mathscr{C} \to \mathscr{D}$ is

(a) A map of objects:

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X) \end{aligned}$$

(b) A map of morphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ f &\mapsto F(f) \end{aligned}$$

In another way we can say that if we have $f : X \to Y$ in $\mathscr{C}$ then we get a new map lying in $\mathscr{D}$, $F(f) : F(X) \to F(Y)$.

With the extra conditions that:

(1) $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for all X in C
(2) $F(f \circ g) = F(f) \circ F(g)$ for all maps f, g in $\mathscr{C}$ for which the composition makes sense

For a contravariant functor we replace some conditions:

(b)' A map of morphisms:

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(Y),F(X))$$
$$f \mapsto F(f)$$

In another way we can say that if we have $f : X \to Y$ in $\mathscr{C}$ then we get a new map lying in $\mathscr{D}$, $F(f) : F(Y) \to F(X)$.

(2)' $F(f \circ g) = F(g) \circ F(f)$ for all maps f, g in $\mathscr{C}$ for which the composition makes sense