Last time we showed that the 1-dimensional continuous representations of $\mathbb{R}$ are

$$\begin{aligned}\rho_s : \mathbb{R} &\to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times \\ x &\mapsto e^{sx}\end{aligned}$$

for all $s \in \mathbb{C}$.

A 2-dimensional continuous representation of $\mathbb{R}$

$$\begin{aligned}\mathbb{R} &\to \mathrm{GL}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{C}) \\ x &\mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

The vector $(1,0)^T$ is fixed by these matrices, so its span $W := \mathrm{span}((1,0)^T)$ is an isomorphic copy of the trivial representation.

But $\mathbb{R}^2$ (or $\mathbb{C}^2$) is not $W \oplus W'$ for any subrepresentation W' of $\mathbb{R}$. This means that Maschke's Theorem fails for this representation. The problem is that the matrices in the image are not unitary.

Last time: if $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is a continuous representation of a topological space whose image $\rho(G)$ is contained in the set of unitary matrices in $\mathrm{GL}_n(\mathbb{C})$, then Maschke's Theorem holds.

Observation: From any 1-dimensional representation of

$$\theta : S^1 \cong \mathbb{R}/\mathbb{Z} \to \mathbb{C}^\times$$

we get a representation of $\mathbb{R}$

$$\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{\theta} \mathbb{C}^\times$$

Which representations $\rho_s : \mathbb{R} \to \mathbb{C}^\times$ arise in this way? But we see that

$$\begin{tikzcd} \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & \mathbb{C}^\times \\ 1 \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 1 \end{tikzcd}$$

and thus $e^s = 1$. Therefore $s = (2\pi i)n$ for some $n \in \mathbb{Z}$.

Schur's Lemma didn't require G to be finite (and most of it works over any field). Namely if $L : V \to W$ is a homomorphism of representations between two irreducible representations, then L is either zero or invertible.

Lemma .0.1 (Schur's Lemma)

Suppose $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(W)$ are two irreducible representations (even infinite-dimensional), and let $L : V \to W$ be a homomorphism of G-representations.

That is the following commutes for all $g \in G$

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "L"'] & V \arrow[d, "L"] \\ W \arrow[r, "\rho'(g)"'] & W \end{tikzcd}.$$

Then either L is an isomorphism or zero.

Proof. $\ker L$ is a subrepresentation of ρ, so $\ker L = 0$ or $\ker L = V$ by irreducibility. Thus L is injective or $L = 0$.

$\operatorname{im} L$ is a subrepresentation of ρ'. Thus ρ' is irreducible, and either $\operatorname{im} L = 0$ or $\operatorname{im} L = W$. Thus $L = 0$ or L is surjective.

Therefore if $L \neq 0$ then L is bijective. We may then check that its inverse on the level of sets is an isomorphism

If we work over a finite-dimensional vector space over $\mathbb{C}$ (or any algebraically closed field), it is easy to then derive that if $L : V \to V$ then $L = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{C}$ (by finding an eigenvalue).

Proof. Let $\vec{v}$ be an eigenvector for L with $L\vec{v} = \lambda\vec{v}$ for some λ.

Then $\vec{v} \in \ker(L - \lambda \operatorname{Id}_V)$ is also a subrepresentation of V.

Since V is irreducible, $\ker(L - \lambda \operatorname{Id}_V) = V$, and so $L = \lambda \operatorname{Id}_V$.

Corollary .0.2

If G is abelian any (finite-dimensional) irreducible representation of G (over $\mathbb{C}$) is 1-dimensional.

Proof. For all $g \in G$, $\rho(g)$ is an invertible linear map $V \to V$. We claim that it is a homomorphism of G-representations. This is precisely the statement that for any $g' \in G$ and $v \in V$

$$g \cdot (g' \cdot v) = g' \cdot (g \cdot v).$$

Clearly this holds when G is abelian. By Schur's Lemma (Lemma .0.1), we know for all $g \in G$ there is a λ such that $\rho(g) = \lambda \operatorname{Id}_V$.

We then know that $\rho(g)$ maps every 1-dimensional subspace of V to itself. So each such subspace is a subrepresentation. Because ρ is irreducible, this implies any such subspace must be all of V.

.1. Compact Groups (namely $S^1 \cong \mathbb{R}/\mathbb{Z}$)

Let $G = \{x \in \mathbb{C}^\times \mid |x| = 1\} = S^1$. Then G is a compact topological group. Furthermore

$$\begin{aligned} \mathbb{R}/\mathbb{Z} &\xrightarrow{\cong} G \\ x &\mapsto e^{2\pi i x} \end{aligned}$$

is an isomorphism of topological groups. G is abelian, so the irreducible representations of G are 1-dimensional. Earlier, we showed that the 1-dimensional representations are

$$\begin{aligned} \rho : \mathbb{R}/\mathbb{Z} &\to \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times \\ x &\mapsto e^{2\pi i n x} \end{aligned}$$

for some $n \in \mathbb{Z}$, and these are of course unitary.

Thus the finite-dimensional <u>unitary</u> representations of these are the direct sums of copies of the above representations.

Decomposing a representation into irreducibles turns into the problem of writing a function as a combination of these ρ's.

For any integrable function φ, the fourier series of φ is

$$\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x} = \sum_{n \in \mathbb{Z}} c_n \rho_n(x)$$

such that $c_n \in \mathbb{R}$. Furthermore the c_n is given in terms of an integral.

$$c_n = \int_{\mathbb{R}/\mathbb{Z}} \varphi(x) e^{2\pi i n x} \, \mathrm{d}x$$