Definition .0.1

An isometry $f : M \to N$ between two metric spaces M, N is a function such that

$$d(x,y) = d(f(x), f(y)).$$

Lemma .0.1

Every isometry $\mathbb{R}^2 \to \mathbb{R}^2$ is invertible, and the inverse is an isometry.

Lemma .0.1 is a consequence of

Lemma .0.2

The isometries of the plane are precisely translations composed with rotations about the origin through some angle composed with either the identity or a reflection about the x-axis.

Proof. It's clear that these are isometries (and are invertible).

Conversely, given any isometry f, we peel off each piece in layers. Write $f_1(x) = f(x) - f(0)$. Then:

$$d(f_1(x), f_1(y)) = d(f(x) - f(0), f(y) - f(0)) = d(f(x), f(y)) = d(x, y)$$

This is then an isometry so that $f_1(0) = 0$. We prove these isometries which fix 0 are rotations about the origin composed with either the identity or a reflection about the x-axis

Now consider $\|f_1(1,0)\| = \|(1,0)\| = 1$ must lie on the unit circle. Thus $f_1(1,0) = (\cos\theta, \sin\theta)$ for some $\theta \in [0, 2\pi)$. Simply set f_2 to be f_1 composed on the left with a rotation by $-\theta$ about the origin, to undo this. Then $f_2(1,0) = f_2(1,0)$ and $f_2(0) = 0$. Furthermore this is an isometry, we prove isometries fixing the origin and $(1,0)$ are either the identity or reflection through the x-axis.

The proof then goes by saying that $(0,1)$ is distance 1 from $(0,0)$ and distance $\sqrt{2}$ from $(1,0)$. Thus $f_2(0,1)$ lies on circles of distance 1 from $(0,0)$ and distance $\sqrt{2}$ from $(1,0)$. Circles only ever intersect at at most two points, and so $f_2(0,1) = (0,1)$ or $f_2(0,1) = (0,-1)$.

Thus write $f_3 = f_2$ or $f_3 = \text{reflect} \circ f_2$. Thus f_3 is an isometry fixing $(0,0)$, $(1,0)$, $(0,1)$. We show f_3 is the identity.

First we show the x-axis and y-axis are fixed. $(x,0)$ is the only point in $\mathbb{R}^2$ with distance $|x|$ from $(0,0)$ and distance $(|x-1|$ from $(1,0)$. Similarly, the y-axis is fixed. Then (x,y) is the unique point with distance $|y|$ from $(x,0)$, distance $|x|$ from $(y,0)$ and distance $\sqrt{x^2+y^2}$ from $(0,0)$.

Theorem .0.3

The isometries are precisely

(1) Translations

(1) Rotation about some point through some angle

(1) Reflection through some line

(1) "Glide reflection," reflect through a line ℓ and then translate by a nonzero vector along ℓ.

Translations and rotations are orientation-preserving, and the reflections and glide reflections are orientation-reversing. If we envision these as symmetries of a plane lying in $\mathbb{R}^3$ this is exactly talking about preserving the "top" of the plane.

Proof. Suppose $f = \tau_Q \circ \rho_\theta$, a translation by a point Q and rotation counterclockwise about $(0,0)$ by $0 \le \theta < 2\pi$. If $\rho_\theta = \text{Id}$ we 're done, so assume $\rho_\theta \neq \text{Id}$.

We must show that f is a rotation about some point. Our first goal is to show f has one and only one fixed point R (the point which it rotates about), and then to translate that point back to the origin via a conjugation (by a translation). We then show that $\tau_R^{-1} f \tau_R = \rho_\psi$ is rotation about the origin, proving the claim that $f = \tau_R \rho_\psi \tau_R^{-1}$ is rotation about R (as the translation τ_R just changes coordinates).

We want $f(R) = R$, so that means we want $(\rho_\theta - \text{Id})R = -Q$ for one and only one R. Thus we show $\rho_\theta - \text{Id}$ is an invertible linear transformation. Well:

$$\det(\rho_\theta - I) = \begin{vmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{vmatrix} = 2 - 2\cos\theta \neq 0$$

Because $\rho_\theta \neq \text{Id}$, so $\cos\theta \neq 1$.

The rest of the proof is similar in flavor, and will be completed Thursday.