Announcements

Quiz 8 Wednesday

- Compute cellular homology of a CW complex
- Similar to examples in today's lecture

Corollary .0.1 (of **??**)

We get a good bit of mileage out of this theorem:

- $H_n(X) = 0$ if X has a CW-complex structure with no n-cells.
- If X has a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements.
- If $H_n(X)$ is a group with a minimum of k generators, then any CW complex structure on X must have at least k n-cells.
- If X has a CW complex with no cells in conscecutive dimensions, then its homology is free abelian on its n-cells. For example $S^n, n \geq 2$ or $\mathbb{C}\mathrm{P}^n$.

Example .0.1

S^n with $n \geq 2$, using the CW coplex structure of e^n attached to a single point x_0. The cellular chain complex is given as:

$$0 \longrightarrow 0 \longrightarrow \langle e^n \rangle \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \langle x_0 \rangle$$

So then all the boundary maps are zero and we see that:

$$H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Exercise .0.2

Redo this calculation with other CW complex structure on S^n, e.g. glue 2 n-cells onto S^{n-1} and proceed inductively.

Example .0.3

Let's do this with the torus

The chain complex looks like:

$$0 \longrightarrow \langle D \rangle \xrightarrow{\partial_2} \langle a, b \rangle \xrightarrow{\partial_1} \langle x \rangle \longrightarrow 0$$

Note that $a \mapsto x - x = 0$ and $b \mapsto x - x = 0$ and so $\partial_1 = 0$. Now D is glued along $aba^{-1}b^{-1}$, so we look at the composed up map

We wind forwards then backwards around a, so the degree is zero. The same thing happens for b so:

$$\partial_2 D = 0 \cdot a + 0 \cdot b = 0$$

This gives a nice principle: If a 2-cell D is glued down via some word w (this only makes sense for 2-cells), then the coefficient to a letter b in $\partial_2 D$ is the sum of the exponents of b in w.

Great! Now we just have that the homology groups are equal to the chain groups because the boundary maps are all zero.

Example .0.4

A genus g surface Σ_g has the CW complex strucutre:

- 1 0-cell x
- 2g 1-cells $a_1, b_1, a_2, b_2, \ldots$
- 1 2-cell D glued along $[a_1, b_2][a_2, b_2] \cdots [a_g, b_g]$ (a product of commutators)

We obtain the result that:

$$\partial_1(a_i) = \partial_1(b_i) = x - x = 0$$

Furthermore by the principle discussed above, we know that every 1-cell appears once in the word, and its inverse appears once, so all the coefficients of 1-cells in $\partial_2(D)$ are zero, so $\partial_2(D) = 0$. This means we have a chain complex:

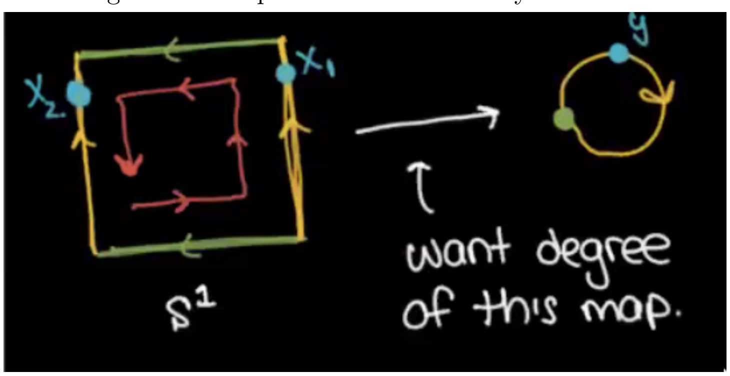
$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

And so then we have that:

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example .0.5 (Torus example: ∂_2 in more detail)

We're going to work through this example a bit more carefully.



Let's zoom in on these two preimage points and use local homology to compute this: