Announcements

- Midterm tomorrow!
- Extended Office Hours tonight 7pm-9pm

.1. Singular Homology

Definition .1.1
A singular n-simplex in a space X is a continuous map $\sigma : \Delta^n \to X$.

Definition .1.2
Let $C_n(X)$ be the free group on singular n-simplices in X. The singular n-chains with boundary maps:

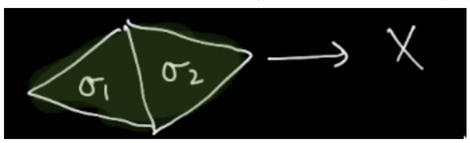
$$\partial_n : C_n(X) \to C_{n-1}(X)$$
$$\sigma \xrightarrow{\sum_{i=1}^{n}} (-1)^i \sigma\big|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_n]}$$

This gives us a singular chain complex

Definition .1.3
The singular homology groups are the homology groups of this singular chain complex given as $H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$.

Since the generating sets for $C_n(X)$ are almost always hugely uncountable, it's almost impossible to compute with these. However it does give us a definition that does not depend on any other structure than the topology of X, making it useful for developing theory ☺.

Heuristic: Interpret a chain $\sigma_1 \pm \sigma_2 \pm \cdots \pm \sigma_k$ as a map from a Δ-complex to X. For example with $\sigma_1 + \sigma_2$.



Where we've glued $[v_1, v_2]$ of σ_1 to $[v_0, v_2]$ of σ_2 if $\sigma_1\big|_{[v_1,v_2]}$ and $\sigma_{[v_0,v_2]}$ are the same singular n-chain with opposite signs.

Goals:

- Singular homology is a homotopy invariant
- Singular and simplicial homology groups are isomorphic.

Exercise .1.1
Check that if X has path components $\{X_\alpha\}$ then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$.

Exercise .1.2
If $X = *$ then

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

Exercise .1.3
If X is path-connected, then $H_0(X) \cong \mathbb{Z}$

.2. Functoriality and Homotopy Invariance

Definition .2.1
For a given continuous map $f : X \to Y$ we can consider the following map:

$$f_\# : C_n(X) \to C_n(Y)$$
$$[\sigma : \Delta^n \to X] \mapsto [f \circ \sigma : \Delta^n \to Y]$$

Definition .2.2

Given two chain complexes (C_*, ∂_*) and (D_*, δ_*), a chain map between them is a collection of group homomorphisms $g_n : C_n \to D_n$ such that the below diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "g_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r, "\delta_{n+2}"] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots
\end{tikzcd}$$

i.e. we have that $\delta_n \circ f_n = f_{n-1} \circ \partial_n$.

Exercise .2.1

We have that $f_\# \partial = \partial f_\#$. In other words, $f_\#$ is a chain map. Thus by the homework $f_\#$ induces a group homomorphism on the homology groups. We write this as $f_* : H_n(X) \to H_n(Y)$ for all n.

Exercise .2.2

We have functoriality, i.e. $(f \circ g)_* = f_* \circ g_*$. Also we have that $(\mathrm{Id}_X)_* = \mathrm{Id}_{H_n(X)}$.

Theorem .2.1

The n-th homology group $H_n : X \mapsto H_n(X)$ gives a functor from Top to Ab. This follows from the two exercises above.

Theorem .2.2

If $f, g : X \to Y$ are homotopic, then they will induce the same map on homology $f_* = g_* : H_n(X) \to H_n(Y)$.

Exercise .2.3

These two theorems imply that H_n is a homotopy invariant.

To prove the second theorem, we introduce some homological algebra.

Definition .2.3

Given chain complexes (A_*, d_*^A) and (B_*, d_*^B) and chain maps $f_*, g_* : A_* \to B_*$. A chain homotopy from f to g is a sequence of group homomorphisms $\psi_n : A_n \to B_{n+1}$ such that:

$$f_n - g_n = d_{n+1}^B \circ \psi_n + \psi_{n-1} d_n^A$$

In a diagram, letting $h_n = f_n - g_n$:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}^A"] & A_{n+1} \arrow[r, "d_{n+1}^A"] \arrow[d, "h_{n+1}"] & A_n \arrow[r, "d_n^A"] \arrow[d, "h_n"] \arrow[dl, "\psi_n"] & A_{n-1} \arrow[r, "d_{n-1}^A"] \arrow[d, "h_{n-1}"] \arrow[dl, "\psi_{n-1}"] & \cdots \\
\cdots \arrow[r, "d_{n+2}^B"] & B_{n+1} \arrow[r, "d_{n+1}^B"] & B_n \arrow[r, "d_n^B"] & B_{n-1} \arrow[r, "d_{n-1}^B"] & \cdots
\end{tikzcd}$$

This diagram does not commute, but it shows everything that is going on. However the **red** map is the sum of the **green** maps composed up.