Let's grab some intuition. What really is local homology?

Well, by excision, there is an isomorphism $H_n(S^n, S^n \setminus \{x_i\}) \cong H_n(U, U \setminus \{x_i\})$ for any open neighborhood U of x_i.

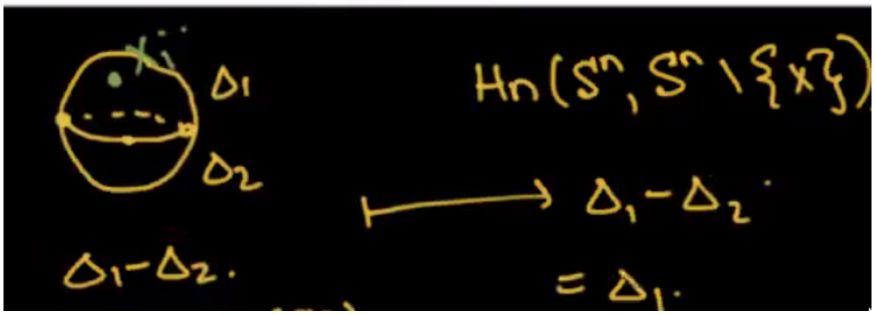
The long exact sequence of a pair also gives us:

$$\cdots \longrightarrow H_k(S^n \setminus \{x_i\}) \longrightarrow H_k(S^n) \xrightarrow{j_*} H_k(S^n, S^n \setminus \{x_i\}) \longrightarrow H_{k-1}(S^n \setminus \{x_i\}) \longrightarrow \cdots$$

Since $S^n \setminus \{x_i\}$ is homeomorphic to an open n-ball, we see that $H_k(S^n \setminus \{x_i\}) = H_{k-1}(S^n \setminus \{x_i\}) = 0$. With this in mind, j_* is an isomorphism.

We want to think about what j_* does when $k = n$, aka when this is an isomorphism $\mathbb{Z} \cong H_n(S^n) \to H_n(S^n, S^n \setminus \{x_i\}) \cong \mathbb{Z}$.

We see that $\Delta_1 - \Delta_2$ generate $H_n(S^n)$, where Δ_1, Δ_2 are the top and bottom hemisphere indicated here:



We then understand that $j_*(\Delta_1 - \Delta_2) = \Delta_1 - \Delta_2 = \Delta_1$¡ since $\Delta_2 = 0$ in $C_n(S^n)/C_n(S^n \setminus \{x_i\})$.

Upshot: $H_n(S^n, S^n \setminus \{x\})$ is generated by an n-simplex with x in its interior.

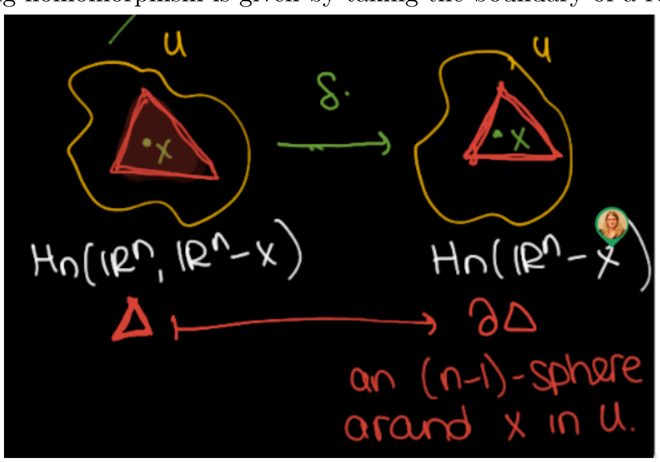
Suppose M is an n-manifold. Then $H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$, where U is a small ball around x. Because U is a ball homeomrphic to $\mathbb{R}^n$, we see that:

$$H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$

By the long exact sequence of a pair:

$$0 = H_n(\mathbb{R}^n) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \xrightarrow{\cong} H_{n-1}(\mathbb{R}^n \setminus \{x\}) \longrightarrow H_{n-1}(\mathbb{R}^n) = 0$$

And since $\mathbb{R}^n \setminus \{x\}$ is homotopy equivalent to an $n-1$ sphere, this means that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$. By homework, this connecting homomorphism is given by taking the boundary of a relative cycle as below:



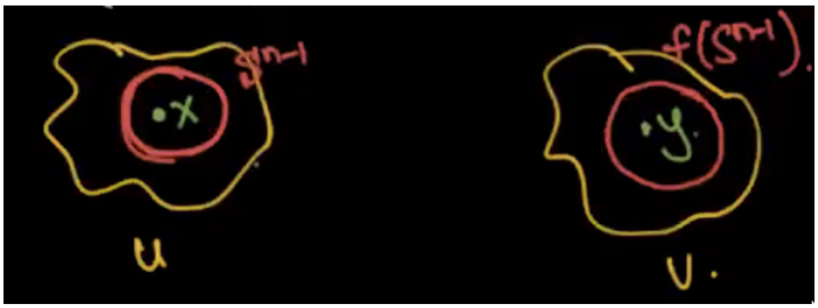
We intuitively want to use this idea to compute degree using this idea. We use naturality of the long exact sequence, namely the fact that where $f : (U_i, U_i \setminus \{x_i\}) \to (V, y)$ is a map of pairs, then the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(U_i, U_i \setminus \{x_i\}) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(U_i, U_i \setminus \{x_i\}) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(V, V \setminus \{y\}) \arrow[r] & H_{n-1}(V, V \setminus \{y\}) \arrow[r] & \cdots
\end{tikzcd}$$

By naturality of the LES and the isomorphism discussed above, we can compute the local degree of a map $S^n \to S^n$ at a point x by computing the degree of the map:

$$H_{n-1}(U \setminus \{x\}) \longrightarrow H_{n-1}(V - \{y\})$$

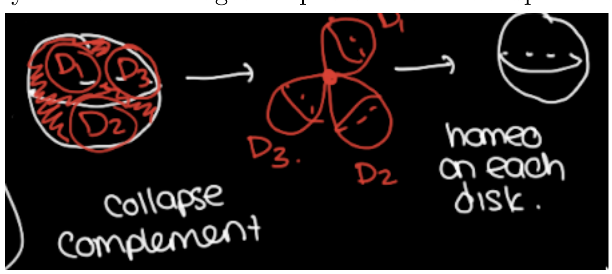
In fact the local degree will be the degree restricted to a small S^{n-1} in the neighborhood U.



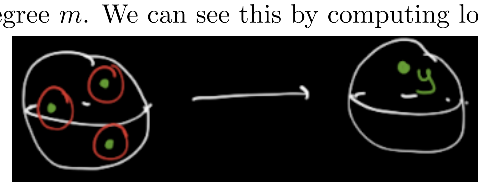
Let's work with some examples for our edification

Example .0.1

Consider S^n and choose m disks in S^n. Namely we first collapse the complement of the m disks to a point, and then we identify each of the wedged n-spheres with the n-sphere itself



The result will be a map of degree m. We can see this by computing local degree:



By choosing a good point in the codomain, we get one point for each disk in the preimage, and the map is a local homeomorphism around these points which is orientation preserving. Perfect! We could likewise compose the maps to S^n from the wedge with a reflection to construct a map of degree $-m$.

Example .0.2

Consider the composition of the quotient maps below $S^n \to \mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n$. We want to compute the degree of this map.

Note that this restricts to a homeomorphism on each component of $S^n \setminus$ equator as a map to $\mathbb{RP}^n \setminus \mathbb{RP}^{n-1}$. Suppose we've oriented our copies of S^n in such a way that the homeomorphism on the top hemisphere is orientation-preserving. The homeomorphism on the bottom hemisphere is given by taking the antipodal map and composing with the homeomorphism of the top hemisphere

$$\deg = \deg(\mathrm{Id}) = \deg(\text{antipodal}) = 1 + (-1)^{n+1} = \begin{cases} 0 & \text{if } n \text{ even} \\ 2 & \text{if } n \text{ odd} \end{cases}$$

.1. Cellular Homology

Suppose that X is a CW complex. Then (X^n, X^{n-1}) is a good pair for all $n > 1$, and X^n/X^{n-1} is a wedge of n-spheres, one for each n-cell e^n_α. Hence:

$$H_k(X^n, X^{n-1}) \cong \begin{cases} 0 & \text{if } k \neq n \\ \langle e^n_\alpha \mid e^n_\alpha \text{ is an } n\text{-cell} \rangle & \text{if } k = n \end{cases}$$

Definition .1.1

The cellular chain complex of X has chain groups $H_n(X^n, X^{n-1})$ with $X^{-1} = \emptyset$.

The boundary maps are given as:

$$\begin{aligned} d_1 : H_1(X^1, X^0) &\to H_0(X^0) \\ \langle \text{1-cells} \rangle &\to \langle \text{0-cells} \rangle \end{aligned}$$

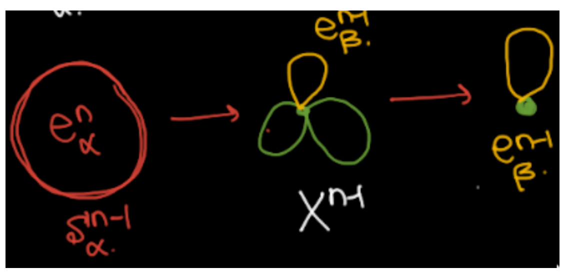
is the usual simplicial boundary map. FOr $n > 1$, the boundayr map:

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

where $d_{\alpha\beta}$ is the degree of the map:

$$\partial e^n_\alpha = S^{n-1}_\alpha \xrightarrow{\text{attaching map}} X^{n-1} \xrightarrow{\text{quotient by } X^{n-1} \setminus e^{n-1}_\beta} S^{n-1}_\beta$$

In pictures, this is given as:



Theorem .1.1

The homology groups of the cellular chain complex (cellular homology groups) coincide with the singular homology groups.