Announcements

- Quiz #7 Wednesday
  - Know the long exact sequence of a good pair, applications like Homework 9 #6
  - Know homology/reduced homology of basic spaces. (contractible space, spheres S^n, discrete set of points, $\mathbb{R}\mathrm{P}^2$, torus, ..., spaces homotopy equivalent to any of the above).
  - The following fact below

Fact: If M is a smooth manifold and N is an embedded smooth closed submanifold, then (M, N) is a good pair. Why? Well this follows from the tubular neighborhood theorem, which should be proven in a course like 591. We will only use the result in obvious cases, and simply assert that certain pairs are good pairs.

Upshot: With pairs like $(\mathbb{R}^{n+1}, S^n)$, you can just assert that this is a good pair (and do not need to prove that S^n is a smooth submanifold of $\mathbb{R}^{n+1}$. Another good example is manifolds and their boundary always form a good pair.

Theorem .0.1

Let X be a Δ-complex. We use $\Delta_n(X)$ to represent the simplicial chain groups on X, and $C_n(X)$ to denote the singular chain groups. Likewise $\Delta_n(X, A) = \Delta_n(X)/\Delta_n(A)$ and $C_n(X, A) = C_n(X)/C_n(A)$.

With this notation, we claim that the inclusion $\Delta_n(X, A) \hookrightarrow C_n(X, A)$ given by:

$$[\sigma : \Delta^n \to X] \mapsto [\sigma : \Delta^n \to X]$$

induces isomorphisms on homology.

$$H_n^{\Delta}(X, A) \cong H_n(X, A)$$

If we consider the case that $A = \emptyset$, we recover the case of absolute homology.

$$H_n^{\Delta}(X) \cong H_n(X)$$

Comment: This says, given a singular homology class x, we can assume x is represented by a simplicial n-cycle.

The Proof uses the Five Lemma:

Lemma .0.2 (The Five Lemma)

If I have a commutative diagram with exact rows:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "\ell"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \arrow[r, "k'"'] & D' \arrow[r, "\ell'"'] & E'
\end{tikzcd}$$

If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

Proof. Diagram chase!

Proof Sketch of the Theorem. Here's the idea

- We can use the long exact sequence of a pair and the five lemma to reduce to proving the result for absolute homology groups (and we will recover the general result).
- Because the image $\Delta^n \to X$ is compact, it is contained in some finite skeleton X^k. Use this ot reduce the proof to the finite skeleta X^k of X

From the LES of a pair we get:

$$\begin{tikzcd}
H_{n+1}^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "\alpha"] & H_n^{\Delta}(X^{k-1}) \arrow[r] \arrow[d, "\beta"] & H_n^{\Delta}(X^k) \arrow[r] \arrow[d, "\gamma"] & H_n^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "\delta"] & H_{n-1}^{\Delta}(X^{k-1}) \arrow[d, "\varepsilon"] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

The Goal is to prove γ is an isomorphism using the 5-lemma.

We assume that β, ε are isomorphisms by induction, checking the case manually for X^0 (which will be a discrete set of points). It remains to show that α, δ are isomorphisms.

We know then that:

$$\begin{aligned}\Delta_n(X^k, X^{k-1}) &= \begin{cases} \mathbb{Z}[k-\text{simplices}] & \text{if } k = n \\ 0 & \text{otherwise} \end{cases} \\ &= H_n^{\Delta}(X^k, X^{k-1})\end{aligned}$$

Claim: $H_n(X^k, X^{k-1})$ are also free abelian on the singular k-simplices defined by the characteristic maps $\Delta^k \to X^k$ when $n = k$, and 0 otherwise. Consider the map:

$$\Phi : \coprod_{\alpha} (\Delta^k_\alpha, \partial \Delta^k_\alpha) \to (X^k, X^{k-1})$$

Defined by the characteristic map. This induces an isomorphism on homology since:

$$\frac{\coprod_\alpha \Delta^k_\alpha}{\coprod \partial \Delta^k_\alpha} \xrightarrow{\cong} \frac{X^k}{X^{k-1}}$$

This reduces to checking that:

$$H_n(\Delta^k, \partial \Delta^k) = \begin{cases} 0 & \text{if } n \neq k \\ \mathbb{Z} & \text{if } n = k \end{cases}$$

generated by the identity map $\Delta^k \to \Delta^k$.

.1. Degree

Definition .1.1

Let $f : S^n \to S^n$. Then $f_* : \mathbb{Z} \cong H_n(S^n) \to H_n(S^n) \cong \mathbb{Z}$. From group theory, this map must be multiplication by some integer $d \in \mathbb{Z}$, which we call the degree $\deg(f)$ of f