.1. The Basics

Definition .1.1

A linear representation of a group G on a vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ (where $\mathrm{GL}(V)$ is the group of invertible linear transformations $V \to V$).

This is also sometimes called a G-representation.

Definition .1.2

If $\rho : G \to \mathrm{GL}(V)$ is a linear representation then $\deg \rho \coloneqq \dim \rho \coloneqq \dim V$.

Example .1.1

Here are a few simple examples

- The trivial representation $g \overset{\rho}{\mapsto} \mathrm{Id}_V$.
- Representation of C_3 on $V = \mathbb{C}^3$ mapping
$$(123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$
- For any action of G on a finite set S, let V be a vector space with basis in bijection with S. Say the basis is e_s $(s \in S)$. Where $\rho(g)$ maps $e_s \mapsto e_{g \cdot s}$. For example $S_3 \to \mathrm{GL}_3(\mathbb{C})$ via the action of S_3 on $\{1, 2, 3\}$. As another example, $D_4 \to \mathrm{GL}_4(\mathbb{C})$ via the action of D_4 on the vertices of a square.
- $D_4 \to \mathrm{GL}_2(\mathbb{R})$ via the action of D_4 on a square geometrically (center the square at $(0, 0)$), as reflections about a line through the origin and rotations about the origin are linear transformations.
- The sign representation of S_n is $\rho : S_n \to \mathrm{GL}_1(\mathbb{C})$ given by $\sigma \mapsto \mathrm{sgn}(\sigma)$.

Definition .1.3

The regular representation is the representation associated to the action of G on itself by left multiplication (dimension is $|G|$).

Definition .1.4

If V is a G-representation then a sub-representation of V is a subspace W of V which is G-invariant, that is $g \cdot W \subseteq W$.

A subspace W of V is a subrepresentation if and only if $\rho : G \to \mathrm{GL}(V)$ factors as

$$\begin{tikzcd} G \arrow[rr, "\rho"] \arrow[dr, dashed] & & \mathrm{GL}(V) \\ & \{\psi \in \mathrm{GL}(V) \mid \psi(W) \subseteq W\} \arrow[ur, hook] & \end{tikzcd}$$

Example .1.2

The following

- The trivial representation of G on V acts as the trivial representation of G on V acts as the trivial representation on every subspace of V
- The action of D_4 on $\mathbb{R}^2$ via rotations, but there is no 1-dimensional subspace of $\mathbb{R}^2$ which is D_4-invariant because of rotations.
- If G acts on a set S, then the linear representation of G on some V with basis S has a 1-dimensional invariant subspace

$$V_1 := \mathbb{C}\left(\sum_s e_s\right).$$

It induces the trivial representation on this subspace. There is also a $(|S|-1)$-dimensional invariant subspace given by

$$V_{|S|-1} := V_1^\perp = \left\{\vec{v} \in V \mid \sum_s v_s e_s = 0\right\}.$$

Note that the direct sum of these is all of V. All of the interesting behavior happens in $V_1^\perp$.

- If $G = D_4$, $S = \{1,2,3,4\}$, and $V = \mathbb{C}^4$. Then V_3 has a G-invariant subspace $W = \operatorname{span}(e_1 - e_2 + e_3 - e_4)$. We can take the orthogonal complement of W in V_3 which is

$$W^\perp := \left\{c_1 e_1 + \cdots + c_4 e_4 \mid \sum c_i = 0,\ c_1 - c_2 + c_3 - c_4 = 0\right\}$$

This $W^\perp$ has no 1-dimensional G-invariant subspace (check!).

Definition .1.5

If V, W are G-representations then so is $V \oplus W$ via $g \cdot (v, w) = (g \cdot v, g \cdot w)$. In terms of matrices, for $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$ then

$$g \mapsto \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$$

.

Example .1.3

We have

$$\begin{aligned} \rho_1 : C_2 &\to \mathbb{C}^\times \\ g &\mapsto 1 \\ \rho_{-1} : C_2 &\to \mathbb{C}^\times \\ g &\mapsto \operatorname{sgn}(g). \end{aligned}$$

Then $\rho_1 \oplus \rho_2$ is also a linear representation

$$\rho_1 \oplus \rho_{-1} : g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(g) \end{bmatrix}$$

Example .1.4

For any n-th root of unity $\zeta \in \mathbb{C}$ (that is $\zeta^n = 1$), we have a representation $\rho_\zeta : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ given by $i \mapsto \zeta^i$.

Definition .1.6

A representation is irreducible if it has no proper positive dimensional subrepresentations.

Definition .1.7

Two representations $\rho_V : G \to \mathrm{GL}(V), \rho_W : G \to \mathrm{GL}(W)$ are called isomorphic provided that there is an isomorphism $T : V \to W$ of vector spaces making the following diagram commute

$$\begin{tikzcd} & G \arrow[dl, "\rho_V"'] \arrow[dr, "\rho_W"] & \\ \mathrm{GL}(V) \arrow[rr, "T\circ - \circ T^{-1}"'] & & \mathrm{GL}(W) \end{tikzcd}$$

Put another way for every $g \in G$ we have

$$g \cdot T(v) = T(g \cdot v).$$

Or in other words a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\rho_V(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\rho_W(g)"'] & W \end{tikzcd}$$