.1. Characters: The Power of the Trace

First a **warning**

For the remainder of representation theory we will work almost always with finite-dimensional complex representations over a finite group G

unless otherwise specified, this is assumed.

Definition .1.1

If $A = n \times n$ matrix then the trace of A (denoted $\mathrm{tr}(A)$) is the sum of the diagonal entries of A.

Key properties

- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- $\mathrm{tr}(C^{-1}AC) = \mathrm{tr}(ACC^{-1}) = \mathrm{tr}(A)$.
- $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.

Thus the trace of a linear map $V \to V$ is defined.

Definition .1.2

Given a representation $\rho : G \to \mathrm{GL}(V)$, its character $\chi = \mathrm{tr} \circ \rho$, that is

$$\begin{aligned} \chi : G &\to \mathbb{C} \\ g &\mapsto \mathrm{tr}(\rho(g)). \end{aligned}$$

An irreducible character is a character of an irreducible representation

Fact: $\chi(hgh^{-1}) = \chi(g)$ because

$$\chi(hgh^{-1}) = \mathrm{tr}(\rho(h)\rho(G)\rho(h)^{-1}) = \mathrm{tr}(\rho(g)) = \chi(g).$$

. Thus χ is a "class function," meaning a function $G \to \mathbb{C}$ which is constant on each conjugacy class of G.

Further, $\rho_i : G \to \mathrm{GL}(V_i)$ $(i = 1, 2)$ then the character of $\rho_1 \oplus \rho_2$ is exactly $\chi_1 + \chi_2$. Thus the character of any finite-dimensional representation is the sum of the characters of finitely many irreducable characters.

Amazing Fact: We lose no information by replacing a finite-dimensional complex representation $\rho : G \to \mathrm{GL}(V)$ of a finite group G with its character $\chi : G \to \mathbb{C}$. Formally

Proposition .1.1

Two representations ρ_1, ρ_2 are isomorphic if and only if their characters are equal (as functions $G \to \mathbb{C}$).

Great Fact: The irreducable characters of a finite group G form a basis for the space of class functions on G. This implies that the number of irreducible representations of G (up to $\cong$) equals the # of conjugacy classes on G.

Definition .1.3

We can define an inner product on functions $\varphi, \psi : G \to \mathbb{C}$ via

$$\langle \varphi, \psi \rangle \coloneqq \frac{1}{|G|} \cdot \sum_{g \in G} \varphi(g)\overline{\psi(g)}$$

where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. This is linear in the first component and antilinear in the second component as desired.

Furthermore $\langle \varphi, \varphi \rangle \in \mathbb{R}_{\geq 0}$ and $\langle \varphi, \varphi \rangle = 0 \iff \varphi = 0$.

Greater Fact: This basis of irreduciable characters for the space of class functions is orthonormal with respect to the above inner product.

Before we prove these facts we'll do some applications and examples

.1.1. Applications + Examples of Characters

If G acts on a finite set S, then the corresponding linear representation $\rho : G \to \mathrm{GL}(\mathbb{C}^{|S|})$ has character χ where

$$\chi(g) = \text{The \# of fixed points of } g \text{ on } S.$$

The character of the regular representation (that is G acting on G by left multiplication) is exactly

$$\chi(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G| & \text{if } g = 1 \end{cases}$$

We will prove that every irreducible character occurs in the decomposition of the regular representation, and that the multiplicity says something about the dimension.

Example .1.1

The irreducible representations/characters of S_3 are

- The trivial representation $G \to \mathbb{C}^\times$ mapping $g \mapsto 1$ has $\chi_0 = 1$.
- The sign representation $G \to \mathbb{C}^\times$ given by $g \mapsto \mathrm{sgn}(g)$. Then

$$\begin{aligned} \chi_s : (123) &\mapsto 1 \\ (12) &\mapsto -1 \\ (1) &\mapsto 1. \end{aligned}$$

- A two-dimensional representation $G \to \mathrm{GL}(V)$ for $V = \{(a, b, c) \in \mathbb{C}^3 \mid a + b + c = 0\}$, where G permutes the coordinates in V.

  This character χ satisfies $\chi + 1 = \chi_\sigma$, where χ_σ is the permutation representation of S_3 via the action on $\{1, 2, 3\}$. Thus

$$\begin{aligned} \chi : (123) &\mapsto -1 \\ (12) &\mapsto 0 \\ (1) &\mapsto 5 = |S_3| - 1 \end{aligned}$$

We may then check that

$$\begin{aligned}
\langle \chi_0, \chi_0 \rangle &= \frac{1}{|G|} \sum_{g \in G} 1 = 1 \\
\langle \chi, \chi \rangle &= \frac{1}{6}(2(-1 \cdot \overline{-1}) + 3(0) + 1(4)) = 1 \\
\langle \chi_0, \chi \rangle &= \frac{1}{|6|}(2(-1) + 3(0) + 1(2)) = 0 \\
\langle \chi_s, \chi_s \rangle &= \frac{1}{6}(2(1) + 3(-1 \cdot \overline{-1}) + 1(1)) = 1 \\
\langle \chi_s, \chi \rangle &= \frac{1}{6}(2(-1) + 3(0) + 1(2)) = 0
\end{aligned}$$