I. Representation Theory

We will study groups via their actions on vector spaces over $\mathbb{C}$. Namely via group homomorphisms $G \to \mathrm{GL}(V)$, where V is some vector space over $\mathbb{C}$.

I.1. Review of Linear Algebra (over $\mathbb{C}$)

Definition I.1.1

A vector space V is an abelian group (under $+$) with an operation $(c, v) \mapsto cv$ for $c \in \mathbb{C}$ and $v \in V$ such that

$$c(dv) = (cd)v$$
$$c(v + w) = cv + cw$$
$$(c + d)v = cv + dv$$
$$1v = v$$

Example I.1.1

$V = \mathbb{C}^n$ under the standard rules of the game.

Definition I.1.2

A subspace of V is a subset of V which is a vector space under the induced $+, \cdot$. That is a subgroup of $(V, +)$ which is preserved by multiplication by $\mathbb{C}$.

Definition I.1.3

A sequence $v_1, \ldots, v_n$ of vectors in a vector space V is linearly independent if

$$c_1 v_1 + \cdots + c_n v_n = 0 \iff c_1, \ldots, c_n = 0$$

Definition I.1.4

A sequence $v_1, \ldots, v_n$ of vectors in V spans V provided that

$$V = \{c_1 v_1 + \cdots + c_n v_n \mid c_1, \ldots, c_n \in \mathbb{C}\}.$$

Definition I.1.5

$v_1, \ldots, v_n$ is a basis of V if $v_1, \ldots, v_n$ is linearly independent and spans V (i.e., every $v \in V$ can be written as a linearly combination of $v_1, \ldots, v_n$ in exactly one way).

Proposition I.1.1

Any two bases of V have the same size which we call the "dimension of V" and write $\dim V$. Moreover any linearly independent sequence in V can be extended to yield a basis of V. Likewise any spanning sequence then some subsequence is a basis.

Definition I.1.6

A linear transformation $T : V \to W$ between two vector spaces V, W is a homomorphism of additive groups which respects scalar multiplication. That is for $v, w \in V$ and $c \in \mathbb{C}$ we have

$$T(cv) = cT(v)$$
$$T(v + w) = T(v) + T(w).$$

Naturally we have notions of kernel and image, these turn out to be vector subspaces (as they should be).

Definition I.1.7

$\operatorname{nullity} T \coloneqq \dim \ker(T)$ (i.e., the nullity of T) and $\operatorname{rank}(T) \coloneqq \dim \operatorname{im}(T)$ (i.e., the rank of T)

Theorem I.1.2 (Rank-Nullity)

We have for any linear transformation $T : V \to W$ that

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V.$$

A linear transformation $T : \mathbb{C}^n \to \mathbb{C}^m$ has the form $v \mapsto Av$ for some $m \times n$ matrix A in $\mathbb{C}$. This holds because a linear transformation is exactly determined by the values it takes on a basis, and $\mathbb{C}^n$ has a standard basis

$$(e_i)_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

That is e_i has a 1 in the i-th position and zeroes elsewhere

$$e_2 = (0, 1, 0, \ldots, 0).$$

In fact we can do a similar thing for any linear transformation $T : V \to W$ given finite bases $\mathbb{v}$, $\mathbb{w}$ of the domain and codomain.

If $v_1, \ldots, v_n$ is a basis of $\mathbb{C}^n$, and $T : v \mapsto Av$ is a linear transformation $\mathbb{C}^n \to \mathbb{C}^n$ then the matrix for T with respect to the basis $v_1, \ldots, v_n$ is $C^{-1}AC$ where $C = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.

usual coords $\quad Cv \xrightarrow{A} ACv$ $\mathbb{v}$ coords $\quad v \xrightarrow[C^{-1}Ac]{} Tv$

This is referred to as "change-of-basis."

Definition I.1.8

If A is an $m \times n$ matrix then the tranpose A^T of A is a $n \times m$ matrix defined by

$$(A^T)_{ij} = A_{ji}.$$

Definition I.1.9

There is a unique function $\det : \{n \times n \text{ matrices}\} \to \mathbb{C}$ such that

- $\det(\mathrm{Id}) = 1$
- det is linear in each individual row of the matrix. (aka multilinear in the rows).
- $\det(A) = 0$ if two adjacent rows of A are equal.

This is called the determinant.

This has the following key properties

$$\det(AB) = (\det A)(\det B)$$

if A, B are $n \times n$ matrices. It follows that

$$\det(C^{-1}AC) = \det A$$

if C is invertible. Therefore $\det T$ for $T : V \to W$ is well-defined for a linear transformation $T : V \to W$. A useful property is that

$$\det \begin{bmatrix} a_{11} & * & * & * \\ 0 & a_{22} & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = a_{11}a_{22}\cdots a_{nn}$$

We also have the cofactor expansion formula, which says that if

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

then

$$AC^T = (\det A) \cdot \mathrm{Id}_n\,.$$

where C is the cofactor matrix and C^T is its transpose. C's ij-th entry is $(-1)^{i+j} \cdot \det M_{ij}$ with M_{ij} a matrix gotten from A by removing the i-th row and j-th column.

This shows us that A is invertible if and only if $\det A \neq 0$.

Definition I.1.10

If $T : V \to V$ is a linear transformation then an eigenvector for T is some nonzero $v \in V$ such that

$$Tv = \lambda v$$

for some $\lambda \in \mathbb{C}$. Then λ is called an eigenvalue.

Eigenvectors with distinct eigenvalues are automatically linearly independent.

If $\mathbb{v} = (v_1, \ldots, v_n)$ is a basis of V then the matrix of $T : V \to V$ with respect to this basis $\mathbb{v}$ is

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

if and only if $T(v_i) = \lambda_i v_i$ (i.e., each v_i is an eigenvector with eigenvalue λ_i). This is called an eigenbasis of V for T.

Definition I.1.11

The characteristic polynomial of a linear transformation $T : V \to V$ is $\det(T - \lambda\,\mathrm{Id}_V)$ for $\lambda \in \mathbb{C}$.

The eigenvalues are precisely the roots of this polynomial. There are n roots counting with multiplicity by the Fundamental Theorem of Algebra, as this will be a degree n polynomial.

Theorem I.1.3 (Cayley-Hamilton)

A linear transformation $T : V \to V$ (likewise an $n \times n$ matrix A) satsifies its own characteristic polynomial (aka yields zero as a linear transformation [or $n \times n$ matrix]).

If the characteristic polynomial of A (an $n \times n$ matrix) has n distinct roots, then there is an eigenbasis for A.

Thus there exists an invertible $n \times n$ matrix C such that $C^{-1}AC$ is a diagonal matrix.

note also that if $Av = \lambda v$ then $A^k v = \lambda^k v$.