.1. Representations of Infinite Groups

We now look at finite-dimensional representations of infinite groups. Unfortunately, there are too many to be useful, as an example

Example .1.1

Consider the group $\mathbb{R}$ with addition, and we're going to look at its 1-dimensional representations. These are just the homomorphisms $\mathbb{R} \to \mathbb{C}^\times$. How many of these are there? We see that $\mathbb{R}$ contains a direct sum of uncountably many copies of $\mathbb{Z}$. One can map the generators of each copy of $\mathbb{Z}$ to an arbitrary element of $\mathbb{C}^\times$. This is larger than even the number of real numbers!

The moral of the story is that $\mathbb{R}$ has more structure than just being a group, and we should instead analyze representations that respect some other structure (for example the analytic structure).

Say we require that the maps $\mathbb{R} \to \mathbb{C}^\times$ are continuous.

Lemma .1.1

Every continuous group homomorphism $\chi : \mathbb{R} \to \mathbb{C}^\times$ is $\chi_s(t) = e^{st}$ for some fixed $s \in \mathbb{C}$.

Proof. If χ is differentiable then

$$\begin{aligned}
\chi'(t) &= \lim_{\Delta t \to 0} \frac{\chi(t + \Delta t) - \chi(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\chi(t)\chi(\Delta t) - \chi(t)}{\Delta t} \\
&= \chi(t) \cdot \lim_{\Delta t \to 0} \frac{\chi(\Delta t) - 1}{\Delta t} \\
&= \chi(t) \cdot \chi'(0)
\end{aligned}$$

Writing $s \coloneqq \chi'(0)$, this satisfies $\chi'(t) = s\chi(t)$, which implies that $\chi(t) = C \cdot e^{sx}$ for some constant C, but $\chi(0) = 1$, so $C = 1$.

We can justify this with the following manipulations. Letting $\psi(t) :- \chi(t)/e^{st}$, then ψ satisfies

$$\psi'(t) = \frac{\chi'(t) - s\chi(t)}{e^{st}} = 0$$

Thus ψ is a constant C.

It remains to show that every continous homomorphism $\chi : \mathbb{R} \to \mathbb{C}^\times$ is differentiable. Define $\psi(t) \coloneqq \int_0^t \chi(x)\,\mathrm{d}x$. Then $\psi'(t) = \chi(t)$. Then

$$\begin{aligned}
\psi(t + r) = \int_0^{t+r} \chi(x)\,\mathrm{d}x &= \int_0^x \chi(x)\,\mathrm{d}x + \int_t^{t+r} \chi(x)\,\mathrm{d}x \\
l = \psi(t) + \int_0^r \chi(t + u)\,\mathrm{d}u &= \psi(t) + \chi(t)\psi(r)
\end{aligned}$$

We know $\psi'(0) = \chi(0) = 1$ is nonzero, so ψ is not identically zero. Thus there is some r so that $\psi(r) \neq 0$. Fix one, then

$$\chi(t) = \frac{\psi(t + r) - \psi(t)}{\psi(r)}.$$

But then χ is a combination of differentiable functions, and so χ is differentiable.

Definition .1.1

A topological group is a group G which is also a topological space where the relevant maps are continuous

$$G \times G \longrightarrow G \qquad G \longrightarrow G$$
$$(g, h) \longmapsto gh \qquad g \longmapsto g^{-1}$$

A great example is $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$.

Definition .1.2

Direct sums and tensor pro A continuous represention of a topological group G is a continuous homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$.

Note: $\mathbb{C}^n$ has an inner product, so lets restrict to representations $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ where each $\rho(g)$ (with $g \in G$) preserves this inner product. That is

$$\langle v, u \rangle = \langle \rho(g)v, \rho(g)u \rangle$$

Definition .1.3

Say $\theta \in \mathrm{GL}_n(\mathbb{C})$ is unitary if θ preserves the inner product on $\mathbb{C}^n$.

Say a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is unitary if $\rho(g)$ is unitary for each $g \in G$.

Lemma .1.2

If G is a topological group, and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is a continuous unitary representation, then

- Any subrepresentation of ρ and any restriction of ρ to $H \leq G$ is continuous and unitary.
- Direct sums and tensor products of (continuous) unitary representations are (continuous) unitary.

Npote that if $\theta \in \mathrm{GL}_n(\mathbb{C})$ is unitary, then all eigenvalues of θ have absolute value 1. Why? Well if $\theta(v) = \lambda v$, then

$$|\lambda|^2 \|v\|^2 = \langle \theta v, \theta v \rangle = \langle v, v \rangle = \|v\|^2.$$

Great!

Proposition .1.3

Let $\rho : G \to \mathrm{GL}(V)$ be a unitary (continuous) representation of a topological group G.

Then any subrepresentation W has a complementary subrepresentation $W^\perp$ with $V = W \oplus W^\perp$.

Proof. Let $W^\perp$ be the orthogonal complement of W. Then we see that if $w^\perp \in W^\perp$ then

$$\langle g \cdot w^\perp, w \rangle = \langle w^\perp, g^{-1} \cdot w \rangle = 0.$$

For every $g \in G$ and $w \in W$ (because $g^{-1} \cdot w \in W$). Thus $g \cdot w^\perp \in W^\perp$, making $W^\perp$ a subrepresentation.