I. Group Actions

Groups most often arise in other fields of mathematics via the automorphisms of certain objects. As such, it makes sense to study groups by looking in the opposite direction. Namely, for a group G, we can study homomorphisms $G \to \mathrm{Aut}(W)$ for some automorphism group of some structure W. We call these maps *representations* of a group, and we say that G *acts* on W.

The two most most common objects to consider for a group to act on are sets and vector spaces. These give *permutation representations* and *linear representations* respectively, and these are given as homomorphisms $G \to \mathrm{Sym}(S)$ and $G \to \mathrm{GL}(V)$ respectively.

For notational reasons, we take the name *group action* to mean a permutation representation, and *representation* by itself to mean a linear representation.

I.1. Permutation Representations

Definition I.1.1

Suppose G is a group and S is a set. We say that a group action of G on S is a homomorphism $\rho : G \to \mathrm{Sym}(S)$

Carrying around this homomorphism can clutter notation, so we often use the following equivalent definition

Definition I.1.2

Suppose G is a group and S is a set. We say that a group action is a map $G \times S \to S$, written by $g \cdot s$, such that for all $g_1, g_2 \in G$ and $s \in S$

$$g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s$$
$$1 \cdot s = s$$

We leave the fact that these are equivalent definitions as a simple exercise.

Definition I.1.3

Suppose G is group acting on a set S. We say that the kernel K of the action is the kernel of the associated group homomorphism ρ, equivalently

$$K \coloneqq \{g \in G \mid \forall s \in S, g \cdot s = s\}$$

The kernel is then a normal subgroup of G

Definition I.1.4

Suppose G is a group acting on a set S, and that $s \in S$. We say that the stabilizer of s is the set

$$\mathrm{Stab}_G(s) = \{g \in G \mid g \cdot s = s\}$$

We sometimes also denote the stabilizer of s by G_s

Note that for G acting on S throug the homomorphism ρ, if H is the subgroup of $\mathrm{Sym}(S)$ which fixes s, then $G_s = \rho^{-1}(H)$. In this way, we immediately see that G_s is a subgroup of G.

Definition I.1.5

Suppose G is a group acting on a set S, and let $s \in S$. We say that the orbit of s is the set

$$\mathrm{Orb}_G(s) = \{g \cdot s \mid g \in G\} = \{t \in S \mid \exists g \in G \text{ s.t. } g \cdot t = s\}$$

We sometimes also denote the orbit of s by $\mathcal{O}_s$.

Lemma I.1.1 (Orbits partition)

Suppose G is a group acting on a set S. Then the set $\{\mathcal{O}_s\}_{s \in S}$ is a partition of S.

Proof. We see that the orbits cover S as for any $s \in S$ we know $s \in \mathcal{O}_s$. Thus we just need to show these are disjoint.

Fix $r \in \mathcal{O}_s \cap \mathcal{O}_t$. Now pick an arbitrary $x \in \mathcal{O}_s$. Then we know that there is some $a, g, h \in G$ such that $r = g \cdot s = h \cdot t$ and $x = a \cdot s$. Then

$$a \cdot s = ag^{-1} \cdot r = ag^{-1}h \cdot t$$

Thus $x \in \mathcal{O}_t$, and $\mathcal{O}_s \subseteq \mathcal{O}_t$. By symmetry, $\mathcal{O}_t \subseteq \mathcal{O}_s$. We then see that these are equal sets, and we're done.

Sometimes we will pick a specific representative of an orbit to work with. However, this is really arbitrary in nature, and we should understand what effect this choice has on us.

Lemma I.1.2

Suppose G is a group acting on a set S. Then for $s \in S$ and $g \in G$ we have

$$G_{g \cdot s} = gG_sg^{-1}$$

Proof. This is simple logic

$$\begin{aligned}
h \in G_{g\cdot s} &\iff hg \cdot s = g \cdot s \\
&\iff g^{-1}hg \cdot s = g^{-1}g \cdot s \\
&\iff g^{-1}hg \cdot s = s \\
&\iff g^{-1}hg \in G_s \\
\iff h \in gG_sg^{-1}
\end{aligned}$$

The intuition is that if h fixes s, and if we relabel $g \cdot s$ to s, h will then fix $g \cdot s$, and we do this relabeling via conjugation.

Definition I.1.6

Suppose G is a group acting on a set S, we say that the action is transitive provided that there is only one orbit $\mathcal{O}_s = S$ for some $s \in S$.

Theorem I.1.3 (Orbit Stabilizer)

Supppose G is a group acting on the set S. Then, for arbitrary $s \in S$

$$[G : G_s] = |\mathcal{O}_s|$$

Proof. Fix arbitrary $s \in S$. Consider the map $f : G \to \mathcal{O}_s$ given by $f : g \mapsto g \cdot s$. We see that f is surjective by the definition of an orbit.

The structure of the theorem is the following

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[dr, "\pi"'] & \mathcal{O}_s \\
& G/G_s \arrow[u, dashed, "\overline{f}"']
\end{tikzcd}$$

In this case, as G_s need not be a normal subgroup of G, π is not a homomorphism, and G/G_s is only a set.

The function $\overline{f}$ which makes the diagram commute is essentially already defined for us. Why? Well $\overline{f} \circ \pi = f$ if and only if for all $gG_s \in G/G_s$ we have

$$\overline{f}(gG_s) = \overline{f}(\pi(g)) = f(g) = g \cdot s$$

This is well defined because if $gG_s = hG_s$ then $g^{-1}h \in G_s$ and

$$\overline{f}(gG_s) = g \cdot s = g(g^{-1}h) \cdot s = h \cdot s = \overline{f}(hG_s)$$

We see that $\overline{f}$ is injective as if $\overline{f}(gG_s) = \overline{f}(hG_s)$ we conclude that $g \cdot s = h \cdot s$, so $g^{-1}h \in G_s$, and then $gG_s = hG_s$.

Finally, we see that $\overline{f}$ is surjective by surjectivity of f, $f = \overline{f} \circ \pi$.