Announcements

- Midterm 1 is over!!! Here are the statistics (grades were out of 20)
  - Ranged from 13.5 to 20
  - Median: 15.5
  - Average: 16.11
- The exam was made more difficult in order to prevent searchability, since algebraic topology is very searchable. Grades will be interpreted accordingly
- The grades for homework this week may come back a bit late due to grading of the midterm taking precedence.

.1. Proof of the Van Kampen Theorem

Van Kampen: Proof Outline. Let $X = \bigcup_\alpha A_\alpha$ where the A_α are open, path-connected, and contain the basepoint x_0. We also must guarantee that $A_\alpha \cap A_\beta$ is path-connected.

Step 1) We have a map induced by the inclusions:

$$\Phi : \mathop{*}_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

We want to show that Φ surjects. Take some $\gamma : I \to X$. You use compactness of the interval I to show that you can partition I into pieces, each of which is mapped completely into one A_α. In particular lets partition I with $s_1 < \cdots < s_n$ so that $\gamma\big|_{s_i, s_{i+1}} =: \gamma_i$ has image in A_{α_i} for some α_i. We'll leave the full point-set argument as an exercise, but as some hints:

- A_α is open for all α
- I is compact

Fora ll i, we choose a path h_i from x_0 to $\gamma(s_i)$ in $A_{\alpha_{i-1}} \cap A_{\alpha_i}$, using path-connectedness of the pairwise intersections. Now take γ and write it as follows:

$$\gamma = (\gamma_1 \cdot \overline{h_1}) \cdot (h_1 \cdot \gamma_2) \cdots (\gamma_{n-1} \cdot \overline{h_{n-1}}) \cdot (h_{n-1} \cdot \gamma_n)$$

Great! Each of these paths is fully contained in A_{α_i}, and so this shows that $\gamma \in \mathrm{im}(\Phi)$. Therefore Φ surjects.

Step 2) For the next step, showing the second part of Van Kampen, we assume that our triple intersections are path connected.

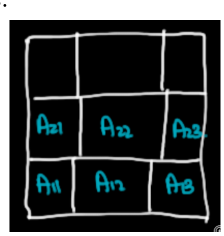
We want to show that $\ker(\Phi)$ is generated by $(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)^{-1}$, where $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ for all loops $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$.

> **Definition .1.1**
> A factorization of a homotopy class $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1][f_2]\cdots[f_\ell]$ with $[f_i] \in \pi_1(A_{\alpha_i}, x_0)$, such that $f \simeq f_1 \cdot f_2 \cdots f_\ell$.

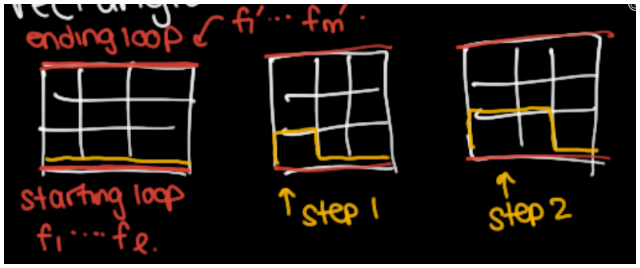
We showed that every $[f]$ has a factorization in Step 1. Now we want to show that two facotrizations $[f_1]\cdots[f_\ell]$ and $[f_1']\cdots[f_m']$ of $[f]$ must be related by two moves:

- $[f_i] \cdot [f_{i+1}] = [f_i \cdot f_{i+1}]$ if $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$. Aka, the relation defining the free product of groups.
- $[f_i]$ can be viewed as an element of $\pi_1(A_\alpha, x_0)$ or $\pi_1(A_\beta, x_0)$ whenever $[f_i] \in \pi_1(A_\alpha \cap A_\beta, x_0)$. This is the relation defining the amalgamated free product.

Now let $F_t : I \times I \to X$ be a homotopy from $f_1 \cdots f_\ell$ to $f_1' \cdots f_m'$, since they both represent $[f]$. We subdivide $I \times I$ into rectangles R_{ij} so that $F(R_{ij}) \subseteq A_{\alpha_{ij}} =: A_{ij}$ for some α_{ij}, using compactness. We also argue that we can perturb the corners of the squares so that a corner lies in only three of the A_α's indexed by adjacent rectangles:

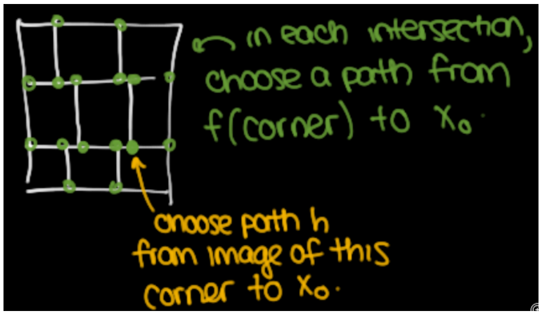


We also argue that we can set up our subdivision so that the partition of the top and bottom intervals must correspond with the two factorizations of $[f]$. We then perform our homotopy one rectangle at a time:



Idea: Argue that homotoping over a single rectangle has the effect of using allowable moves to modify the factorization.

At each triple intersection choose a path from $f(\text{corner})$ to x_0 which lies in the triple intersection, so we use the assumption that the triple intersections are path connected.



Along the top and bottom we make choices compatible with the two factorizations. It's now an exercise to check that these choices result in homotoping across a rectangle gives a new factorization related by an allowable move.