Theorem .0.1
If there is a chain homotopy ψ from f_* to g_*, then the induced maps on homology are equal.

Proof. Let $\sigma \in A_n$ be an n-cycle, i.e. $\partial_n^A \sigma = 0$. Then we compute that:

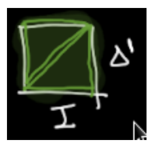
$$(f_n - g_n)(\sigma) = d_{n+1}^B(\psi_n(\sigma)) + \psi_{n-1}(d_n^A(\sigma)) = d_{n+1}^B(\psi_n(\sigma)) \in \operatorname{im} d_{n+1}^B$$

This tells us that $(f_n - g_n)(\sigma)$ is a boundary, and so $(f_n - g_n)(\sigma) = 0$ when considered as an element of the homology group. Thus $f_n(\sigma) = g_n(\sigma)$ in the homology group, and so f, g induce the same map as desired.

We now sketch the proof of **??** given in Hatcher. From this point in the course many of the theorems require much more algebraic work than we are interested in. We instead want to learn how to use the computational tools.

Proof idea. Suppose we have some homotopy $F : I \times X \to Y$ from f to g. The most difficulty in this proof is the combinatorial difficulty involved in the fact that the product of a simplex in X and I is not a simplex.

Key: Subdivide $\Delta^n \times I$ into $(n+1)$ dimensional subsimplices.



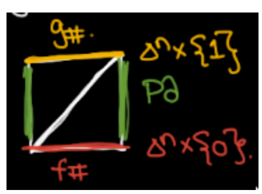
We define the prism operator:

$$P_n : C_n(X) \to C_{n+1}(Y)$$
$$[\sigma : \Delta^n \to X] \mapsto \begin{bmatrix} \text{alternating sums of restrictions} \\ \Delta^n \times I \xrightarrow{\sigma \times \mathrm{Id}} X \times I \xrightarrow{F} Y] \end{bmatrix}$$

We now need to check that

$$\partial_{n+1}^Y P_n = \boxed{g_\#} - \boxed{f_\#} - \boxed{P_{n-1}\partial_n^X}$$

We have the following diagram.



Thus P is a chain homotopy and we're done.

.1. Relative Homology

Definition .1.1 (Studied on Homework)
The reduced homology groups $\widetilde{H}_n(X) = H_n(X)$ when $n > 0$. When $n = 0$ we have that:

$$\widetilde{H}_0(X) \oplus \mathbb{Z} = H_0(X)$$

The usefulness of this is that for path-connected space X we have $\widetilde{H}_0(X) = 0$, and for contractible spaces X we have $\widetilde{H}_n(X) = 0$.

Definition .1.2
Let X be a space, and $A \subseteq X$. Then (X, A) is a good pair if A is closed and nonempty, and also it is a deformation retract of a neighborhood in X.

Example .1.1
If X is a CW complex and A is a subcomplex, then (X, A) is a good pair.
The proof is given in the Appendix of Hatcher and requires some point-set topology.

Non-Example .1.2
$(\text{Hawaiian earring}, \text{bad point})$ is a bad pair.

Theorem .1.1
If (X, A) is a good pair, then there exists a long exact sequence (exact at every n) on reduced homology groups given by:

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A)$$

$$\xrightarrow{\delta} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \xrightarrow{j_*} \widetilde{H}_{n-1}(X/A)$$

$$\xrightarrow{\delta} \cdots \xrightarrow{j_*} \widetilde{H}_0(X/A) \longrightarrow 0$$

Where $i : A \hookrightarrow X$ is the inclusion and $j : X \to X/A$ is the quotient map. We will define each δ in the proof. The fact that this sequence is exact often means that if we know the homology groups of two of the spaces we can compute the homology of the remaining space.