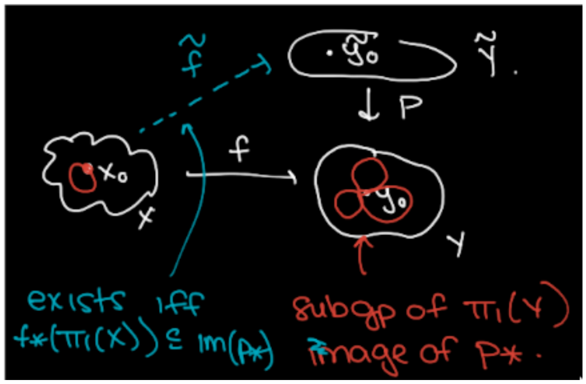
Proposition .0.1
Suppose we have a covering map $p : (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$, a continuous function $f : (X, x_0) \to (Y, y_0)$, with X path-connected and locally path-connected. Then there exists a lift $\widetilde{f} : (X, x_0) \to (\widetilde{Y}, \widetilde{y}_0)$ if and only if $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(\widetilde{Y}, \widetilde{y}_0))$. In a picture:

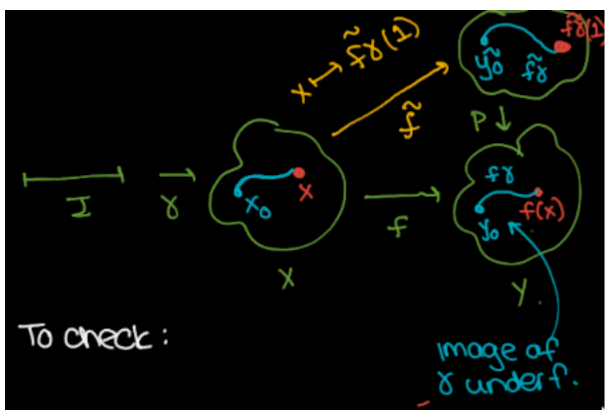


Proof. The "only if" portion is straightforward from the factorization $f_* = p_* \circ \widetilde{f}_*$ due to functoriality.

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "p"] \\ X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(\widetilde{Y}, \widetilde{y}_0) \arrow[d, "p_*"] \\ \pi_1(X, x_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(Y, y_0) \end{tikzcd}$$

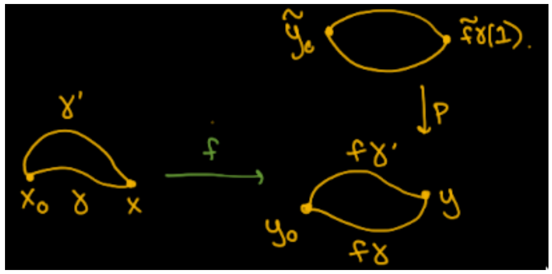
Then we have that $f_*(\pi_1(X, x_0)) = p_*(\widetilde{f}_*(\pi_1(X, x_0))) \subseteq p_*(\pi_1(\widetilde{Y}, \widetilde{y}_0))$.

Now lets do the "if" portion. Let $x \in X$. Choose a path γ from x_0 to x. The path $f \circ \gamma$ has a unique lift starting at $\widetilde{y}_0$. Define $\widetilde{f}(x) = \widetilde{f\gamma}(1)$. Consider the following picture:



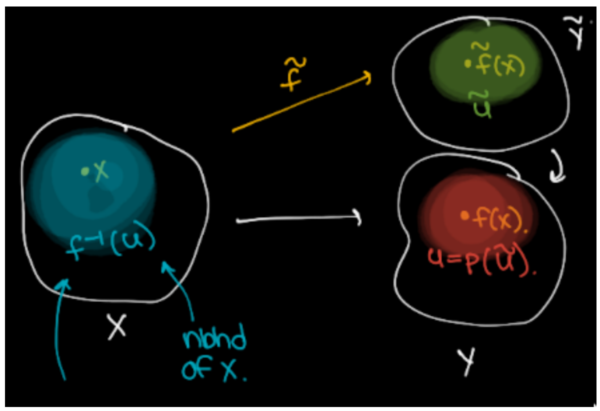
We must show that $\widetilde{f}$ is well-defined and that $\widetilde{f}$ is continuous:

(1) Let γ' be some other path from x_0 to x. We want to show that $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. Since $\gamma \cdot \overline{\gamma'}$ is a loop in X at x_0, we know that $f\gamma \cdot \overline{f\gamma'}$ is a loop in Y in $\mathrm{im}(f_*)$. Thus this loop is in $\mathrm{im}(p_*)$ by assumption, so it must form a loop when lifted there. Here's the picture:



But wait! By uniqueness of lifts, the loop lifting $(f\gamma)\cdot\overline{(f\gamma')}$ to $\widetilde{Y}$ must be equal to the lifts $\widetilde{f\gamma}\cdot\widetilde{\overline{f\gamma'}}$ with a common value at $t = 1/2$. And then $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$ as desired. We'll leave the details of this use of uniqueness as an exercise.

(2) Choose $x \in X$ and choose a neighborhood $\widetilde{U}$ of $\widetilde{f}(x)$ in $\widetilde{Y}$. We may shrink $\widetilde{U}$ so that $p|_{\widetilde{U}}$ is a homeomorphism to $p(\widetilde{U}) = U$, via the definition of a covering space. We know that $f^{-1}(U)$ is an open neighborhood of x. It suffices to show that $f^{-1}(U) \subseteq \widetilde{f}^{-1}(\widetilde{U})$, and so we may show $\widetilde{f}(f^{-1}(U)) \subseteq \widetilde{U}$, and we actually pass to a smaller neighborhood V and show that $\widetilde{f}(V) \subseteq \widetilde{U}$. Here's the picture:



Replace $f^{-1}(U)$ with a p(ossibly smaller) path-connected open neighborhood $V \subseteq f^{-1}(U)$ using the fact that X is locally path-connected. Now for any $x' \in V$ choose a path α from x to x'. If γ is some path from x_0 to x, then we get a path $\gamma\cdot\alpha$ from x_0 to x'. Now $f\gamma\cdot f\alpha$ in Y has a lift $\widetilde{f\gamma}\cdot\widetilde{f\alpha}$ where $\widetilde{f\alpha} = p^{-1}(f\alpha)$, since $f\alpha$ is contained entirely in U, and so p is invertible here. But then necessarily, $\widetilde{f}(x') = \widetilde{f\alpha}(1) \in \widetilde{U}$. But this is exactly what we wanted! $\widetilde{f}(V) \subseteq \widetilde{U}$.

Exercise .0.1

On Homework we had that $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2\mathbb{Z}$. Prove that every map $\mathbb{R}\mathrm{P}^2 \to S^1$ is nullhomotopic.

Solution. Consider the cover $\mathbb{R} \xrightarrow{p} S^1$ given on previous homework. For convenience, choose a presentation $\pi_1(\mathbb{R}\mathrm{P}^2) = \langle a \mid a^2\rangle$. Now for any function $f : \mathbb{R}\mathrm{P}^2 \to S^1$, we know that $f_*(a) + f_*(a) = f_*(a^2) = 0$, but necessarily because we are working in $\mathbb{Z}$ this means $f_*(a) = 0$. Therefore $f_*(\pi_1(\mathbb{R}\mathrm{P}^2))$ is trivial because f_* sends the generators to 0, and this must be contained in the trivial image of $\pi_1(\mathbb{R})$ in $\pi_1(S^1)$.

Therefore, by the proposition (Proposition .0.1), we know that f extends to some map $\widetilde{f} : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}$. Any such map into $\mathbb{R}$ is nullhomotopic to some constant map c because $\mathbb{R}$ is contractible. Thus $f = p\circ\widetilde{f}$ is nullhomotopic, because composition respects homotopies, so $f = p\circ\widetilde{f} \simeq p\circ c$, and $p\circ c$ is a constant map.