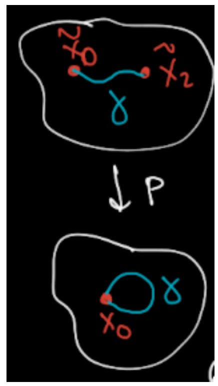
Proof of Theorem. Notation: Let (X, x_0) be the base space and $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$ where $p : \widetilde{X} \to X$ is a covering map. Further let $H := p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

On Homework: (X, x_0), $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$, if we change the basepoint from $\pi_1(\widetilde{X}, \widetilde{x}_0)$ to $\pi_1(\widetilde{X}, \widetilde{x}_1)$. Then we have the the induced subgroups of the base space's fundamental group are conjugate by some loop $[\gamma] \in \pi_1(X, x_0)$, that is:

$$p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = [\gamma] \cdot p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \cdot [\gamma]^{-1}$$

Where γ lifts to a path from $\widetilde{x}_0$ to $\widetilde{x}_1$.



Therefore $[\gamma] \in N(H)$ if and only if $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, and this holds if and only if there is a deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$ by the classification of based covering spaces in the homework (alternatively use the lifting criterion).

Therefore p is a normal cover if and only if H is normal, proving (a).

We then define a map to help us out:

$$\Phi : N(H) \to G(\widetilde{X})[\gamma] \qquad\qquad \mapsto \tau$$

Where τ lifts to a path $\widetilde{x}_0$ to $\widetilde{x}_1$ and τ is a deck transformation mapping $\widetilde{x}_0$ to $\widetilde{x}_1$, which will be uniquely defined by uniquness of lifts with specified base points. We need to check some things

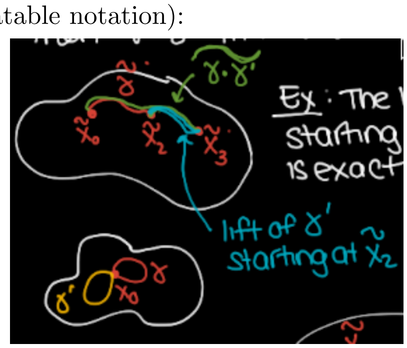
(i) Check that Φ is a group homomorphism
(ii) Φ is surjective
(iii) $\ker(\Phi) = H$

If we can prove these things, then the first isomorphism theorem gives us the desired result.

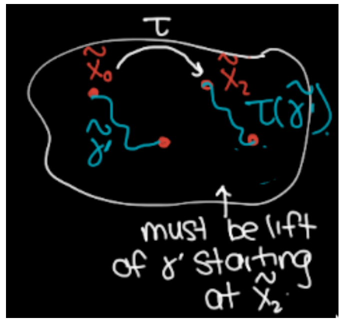
(ii) We've proved that Φ is surjective before in our work above.
(iii) $\Phi([\gamma])$ is the identity if and only if τ sends $\widetilde{x}_0$ to $\widetilde{x}_0$, meaning that $[\gamma]$ lifts to a loop, well then by our characterization of the fundamental group downstairs:

$$\ker(\Phi) = \{[\gamma] \mid \gamma \text{ lifts to a loop}\} = H$$

(i) Suppose we have loops $[\gamma_1] \overset{\Phi}{\mapsto} \tau_1$ and $[\gamma_2] \overset{\Phi}{\mapsto} \tau_2$. We claim that $\gamma_1 \cdot \gamma_2$ lifts to $\widetilde{\gamma}_1 \cdot \tau(\widetilde{\gamma}_2)$. Here's our motivating picture (with translatable notation):



It's an exercise to check that the lift of γ_2 starting at $\widetilde{x}_1$ is exactly $\tau_1(\widetilde{\gamma}_2)$, where $\widetilde{\gamma}_2$ is a lift starting at $\widetilde{x}_0$. The picture of the claim is below:



The idea is that by uniqueness of lifts we'll have the desired claim. We then just observe that this path $\widetilde{\gamma}_1 \cdot \tau_1(\widetilde{\gamma}_2)$ is a path from $\widetilde{x}_0$ to $\tau_1(\widetilde{\gamma}_2(1)) = \tau_1(\tau_2(\widetilde{x}_0))$, so the image must be a deck transformation sending $\widetilde{x}_0$ to $\tau_1(\tau_2(\widetilde{x}_0))$. But then $\tau_1 \circ \tau_2$ maps $\widetilde{x}_0$ to this same point, and since deck transformations are determined by where they send a single point, we're done ☺.