Theorem .0.1 (Excision)
Suppose we have subspace $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq \text{Int}(A)$. Then the inclusion:

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

induces isomorphisms:

$$H_n(X - Z, A - Z) \xrightarrow{\cong} H_n(X, A)$$

Exercise .0.1
Equivalently for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion:

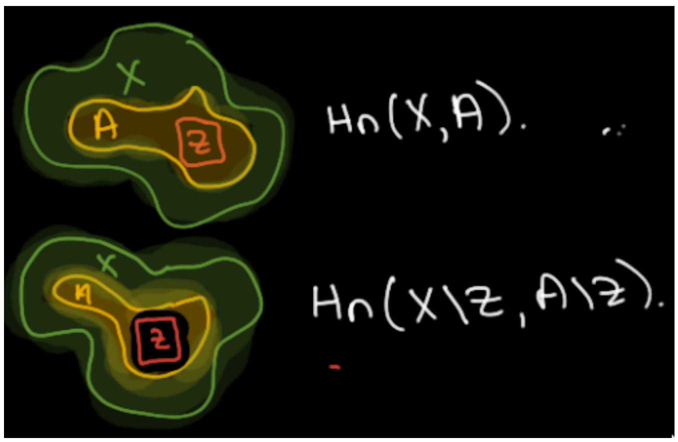
$$(B, A \cap B) \hookrightarrow (X, A)$$

induces an isomorphism:

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

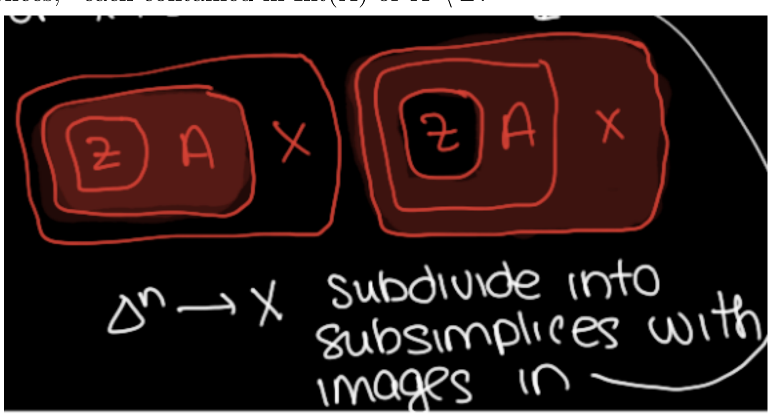
Hint: $B = Z \setminus Z$, $Z = X \setminus B$.

Picture!



Proof Sketch. We sketch the proof here, which is notorious for being hairy.

- Given a relative cycle x in (X, A), subdivide the simplices to make x a linear combination of chains on "smaller simplices," each contained in $\text{Int}(A)$ or $X \setminus Z$.



  This means x is homologous to sum of subsimplices with images in $\text{Int}(A)$ or $X \setminus Z$. One of the things we use is that simplices are compact, so this process takes finite time.

  Key: "Subdivision operator" is chain homotopic to the identity.

- Since we are working relative to A, the chains with image in A are zero. Thus we have a relative cycle homologous to x with all simplices contained in $X \setminus Z$.

Exercise .0.2
$H_*(Y, y_0) \cong \widetilde{H}(Y)$.

Theorem .0.2
Let (X, A) be a good pair. Then the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism:

$$H_n(X, A) \xrightarrow{\cong} H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$$

where the last equality is from the exercise.

Proof Outline. Let $A \subseteq V \subseteq X$ where V is a neighborhood of A that deformation retracts onto A. Using excision, we obtain a commutative diagram:

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\cong"] \arrow[d, "q_*"'] & H_n(X, V) & H_n(X - A, V - A) \arrow[l, "\cong"'] \arrow[d, "\cong"', "q_*"] \\
H_n(X/A, A/A) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "\cong"']
\end{tikzcd}$$

Done if we can prove all the colored isos.

- $\cong$ is an isomrphism by excision
- $\cong$ is an isomorphism by direct calculation (since q is a homeomorphism on the complement of A)
- $\cong$ on Homework, since V deformation retracts to A.