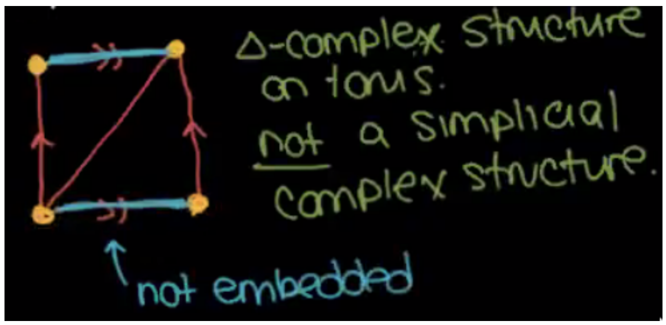
Announcements

- Fill our the Office Hour / Review Session scheduling survey
- No quiz next week
- No more homeworks
- Final Exam–April 28 1:30pm ET

Definition .0.1 (Simplicial Complexes)
A simplicial complex is a Δ-complex with the conditions that:

- Each simplex is embedded
- Intersection of simplices $\sigma_1 \cap \sigma_2$ must be $\emptyset$ or a single subsimplex o both σ_1 and σ_2

Let's look at our Δ-complex structure on the torus:

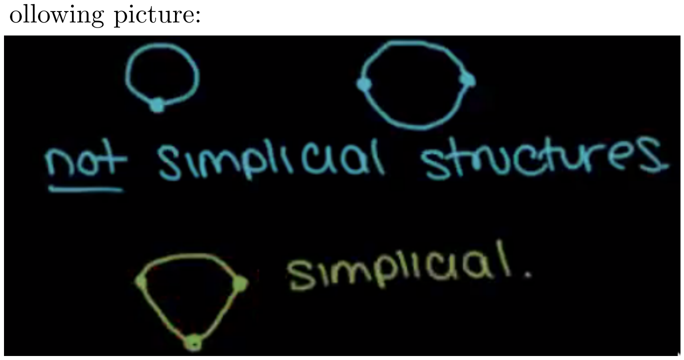


A simplicial complex structure on the torus requires at least fourteen triangles.
To specify a simplicial complex, we can do the following

- Start with X^0 (discrete set)
- Indicate which subsets of X^0 span a simplex.

This completely specifies the data of a simplicial complex.

Example .0.1
For S^1 we have the ollowing picture:



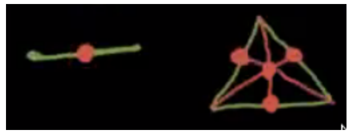
Definition .0.2
A *simplicial map* $f : K \to L$ is a continuous map that sends each simplex of K to a (possibly smaller dimensional) simplex of L by a linear maps as follows

$$\sum t_i v_i \mapsto \sum t_i f(v_i)$$

A simplicial map is completely determined by its restriction to the vertex set.

Theorem .0.1 (Simplicial Approximation)
Given any continuous map $f : K \to L$ where K is a finite simplicial complex and L is any simplicial complex. Then f is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K

Here is barycentric subdivision in pictures:



That is, we add a new vertex to the center of every subsimplex, filling things in like the above. For an n-simplex we end up with $(n+1)!$ simplices which replace it.

Proof Outline of Lefschetz Fixed Point Theorem. We now prove **??**. Fix a space X which is a finitee simplicial complex (or a retract of a fintie simplicial complex) and a map $f : X \to X$.

(Step 1) Reduce to the case of a finite simplicial complex X. Suppose K is a finite simplicial complex, with $r : K \to X$ a retraction. First notice that the following composite of maps

$$K \xrightarrow{r} X \xrightarrow{f} X \xrightarrow{\iota} K$$

has the same fixed points as f.

Exercise .0.2
$r_* : H_n(K) \to H_n(X)$ is split surjective (see ι_*), and so it has to be a projection onto a direct summand

Exercise .0.3
It follows that $\operatorname{tr}(\iota_* \circ f * \circ r_*) = \operatorname{tr}(f_*)$ on degree n homology.

This implies that $\tau(f) = \tau(\iota \circ f \circ r)$. Therefore if we can prove the result for a simplicial complex then we are done.

(Step 2) Let X be a finite simplicial complex. We show that if $f : X \to X$ has no fixed points then $\tau(f) = 0$.
Goal: Find subdivisions K, L of X and $g : K \to L$ so that:
- g is simplicial
- $g \simeq f$, $\tau(f) = \tau(g)$
- $g(\sigma) \cap \sigma = \emptyset$ for all simplices σ.

So this becomes a few steps, none of which we'll justify too formally:
- Choose a metric d on X
- Since X is compact, and f has no fixed point, then $d(x, f(x))$ has some minimum value $\varepsilon > 0$.
- Subdivide all simplices of X until simplices have diameter smaller than $\frac{\varepsilon}{53}$. Call this subdivision L.
- Use the simplicial approximation theorem to obtain a map $g : K \to L$, where K is a subdivision of L, $g \simeq f$
- By proof of simplicial approximation theorem, we can construct g so that for all simplices σ, $g(\sigma)$ is not too far from $f(\sigma)$. We can then conclude that $g(\sigma) \cap \sigma = \emptyset$.
- So then g is a cellular map $K \to K$ that moves every cell. We can then check that:

$$\tau(f) = \tau(g) = \sum (-1)^n \operatorname{tr}(g_* : \text{cellular } n\text{-chains} \to \text{cellular } n\text{-chains}) = 0$$

Because each g_* has vanishing diagonal entries.

Then we're done! Great ☺