I. Foundations

This week: Some foundations.

- Homotopies of maps
- Homotopy Equivalence of spaces. A coarser notion of equivalence of spaces than homeomorphism
- CW Complexes. A class of topological spaces that is "the right setting" to do algebraic topology. They are more general than manifolds but still very well-behaved and also combinatorial.

I.1. Homotopies

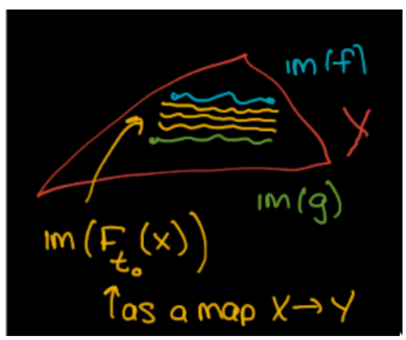
I.1.1. Basic Definitions

Definition I.1.1
Let X, Y be topological spaces and f, g be continuous maps $X \to Y$. By definiton a homotopy from f to g is a continuous 1-parameter family of maps that we can view as continuously deforming the map f to the map g.
Concretely, a homotopy from f to g it is a map $F : X \times I \to Y$, where $I = [0, 1]$ is a closed interval, subject to the conditions that for all $x \in X$:

$$F(x, 0) = f(x) \qquad\qquad F(x, 1) = g(x)$$

We often write $F_t(x)$ for $F(x, t)$.

We should think of t as a time parameter, and the map F as giving a deformation of the map f into a map g. In other words, this is a family of maps $X \to Y$ interpolating between f and g. In pictures, this looks like:

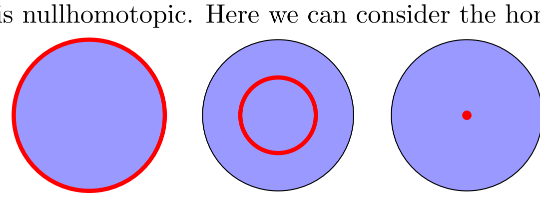


Definition I.1.2
If a homotopy exists from f to g, we say that f and g are homotopic and write $f \simeq g$.
If f is homotopic to a constant map, then we write $f \simeq *$ and we call f nullhomotopic

Example I.1.1
Any two maps $f, g : X \to \mathbb{R}$ are homotopic. We can deform f to g by the "straight line homotopy":

$$F_t(x) = tf(x) + (1 - t)g(x)$$

Example I.1.2
Let S^1 be the unit circle in $\mathbb{R}^2$ and let D^2 be the closed unit ball in $\mathbb{R}^2$.
The inclusion $S^1 \hookrightarrow D^2$ is nullhomotopic. Here we can consider the homotopy $F_t(x) = (1 - t)f(x)$



$t = 0$ $t = 1/2$ $t = 1$

Example I.1.3
The maps:

$$\begin{aligned} S^1 &\to S^1 \\ \theta &\mapsto \theta \end{aligned} \qquad\qquad \begin{aligned} S^1 &\to S^1 \\ \theta &\mapsto -\theta \end{aligned}$$

are not homotopic.

Exercise I.1.4
On homework, you will prove that “homotopic” is an equivalence relation on maps $X \to Y$

Breakout Rooms

Exercise I.1.5
A subset of $\mathbb{R}^n$ is called star-shaped if there exists some $x_0 \in S$ so that for all $x \in S$, the line segment from x to x_0 is contained in S. Show that any map from a space to S is nullhomotopic.

Solution. We will show that any map $f : X \to S$ is homotopic to the constant map $x_0 : X \to S$. This is given by the straight line homotopy:

$$F_t(x) = (1-t)f(x) + tx_0$$

We know that $f(x) \in S$, so this straight line is contained in S because S is star-shaped. Furthermore this is continuous since it is a convex combination of continuous functions. Of course $F_0(x) = f(x)$ and $F_1(x) = x_0$, and so f is nullhomotopic.

Exercise I.1.6
Suppose that we have the following maps:

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

And further that $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then show that $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Solution. Write $F : X \times [0,1] \to Y$ and $G : Y \times [0,1] \to Z$ as the homotopies from f_0 to f_1 and g_0 to g_1 respectively. Then consider the map:

$$H_t(x) = G_t(F_t(x))$$

By writing out this map more explicitly we can show that it is continuous:

$$H(x,t) = G(F(x,t),t)$$

Note then this is a composition of the continuous maps given as:

$$(x,t) \mapsto (F(x,t),t) \mapsto G(F(x,t),t)$$

Of course, note that since the map $(x,t) \mapsto (F(x,t),t)$ is continuous in each component it is continuous overall. Therefore H is continuous since it is a composition of continuous functions. Thus, H gives a homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$ since F and G are homotopies and we know:

$$\begin{aligned} H_0(x) &= G_0(F_0(x)) = g_0(f_0(x)) = (g_0 \circ f_0)(x) \\ H_1(x) &= G_1(F_1(x)) = g_1(f_1(x)) = (g_1 \circ f_1)(x) \end{aligned}$$

Therefore $g_0 \circ f_0 \simeq g_1 \circ f_1$ just as desired.

Exercise I.1.7
How could you prove that two maps are not homotopic