I. Introduction to Category Theory

I.1. The Motivation

Category Theory as a subject grows out of a need to study the relationships between different areas of mathematics. Often this comes in the form of associating to every object in a certain area some object in another area according to some rules. A classic example is the fundamental group, which associates a group to every topological space (for more about algebraic topology, see [**hatcher**]).

To be able to formalize these types of mappings and their properties, we need a general setting for objects and also for maps between them. These will be our categories.

In the process, we will be able to give nice descriptions of many familiar objects in more abstract settings. The technique for doing so uses what are called universal properties. The advantage of these is that we can prove many results about things like free groups, tensor products, cartesian products, direct sums, and many more in extremely general settings. Such settings occur all throughout modern mathematics wherever groups might not be enough structure.

Most importantly though, we will develop a new way of looking at mathematics and of looking at definitions. This method of looking at things is sometimes appropriate and sometimes not. But it's a crucial tool in my mathematical toolbox, and one of the most elegant.

For my standard reference on this material see [**ctContext**]. For a more algebraic perspective see [**aluffi**]

I.2. The Basic Definitions

Lets go ahead and jump right into things!!!

Definition I.2.1

A category $\mathscr{C}$ has the following data:

- A class of objects $\mathrm{Ob}(\mathscr{C})$
- For any two objects $X, Y \in \mathscr{C}$ a class of arrows (aka morphisms aka maps, lots of names) $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. We often write $f : X \to Y$ when the ambient category is clear to mean that $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Sometimes one writes $\mathrm{Mor}_{\mathscr{C}}(X, Y)$ in place of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- For any three objects X, Y, Z, a function $\circ : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$.

and it has the following structure:

- For every object X in $\mathscr{C}$, there is an arrow $\mathrm{Id}_X : X \to X$ so that for all $f : X \to Y$ and $g : Z \to X$ we have

$$f \circ \mathrm{Id}_X = f \qquad\qquad \mathrm{Id}_X \circ g = g$$

- Composition is associative. That is for $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

We say a category is **small** if it only has a set's worth of arrows in total (note this implies it has a set's worth of objects as well)

A category is **locally small** if it only has a set's worth of arrows between any two objects. We will mostly work with locally small categories.

One might ask what we can do that's interesting with such a broad collection of objects. For those wondering, remember how abstract groups are and how much structure they contain. Categories are not quite so well-behaved, but they are an extremely good setting for defining many many many well-behaved and beautiful things.

Example I.2.1

With this in mind, lets see some examples of categories. Many of these will be familiar to you!

Category	Objects	Morphisms
Set	sets	functions
Grp	groups	homomorphisms
Ab	abelian groups	homomorphisms
Vect_F	vector spaces over F	F-linear maps
R-Mod	modules over R	R-linear maps
Top	spaces	continuous maps
Haus	Hausdorff spaces	continuous maps
SmoothMan	smooth manifolds	smooth maps
Nat	natural numbers	ordering (a unique arrow $a \to b$ if $a \leq b$)

Notice that the collection of objects can be huge. This is why I specified a class of objects in the definition.

Exercise I.2.2

Show that these are all categories.

We can also make some suggestive definitions which give us a whole class of examples.

Definition I.2.2

We say that an arrow $f : X \to Y$ in a category is **invertible** (or is an **isomorphism**) provided there are arrows $g, h : Y \to X$ so that

$$g \circ f = \mathrm{Id}_X \qquad\qquad f \circ h = \mathrm{Id}_Y$$

In this case we may in fact show $g = h$ and that g is unique (exercise...). When only g exists, g is called a left inverse, and when only h exists, h is called a right inverse. We also say that X and Y are **isomorphic** via the isomorphism f, which may be written as $X \cong Y$ or more specifically $X \overset{f}{\cong} Y$.

We call a category $\mathscr{C}$ a **groupoid** provided that all of its morphisms are invertible.

Example I.2.3

To give an idea of how useful the idea of an isomorphism is, we list here the different isomorphisms in the above categories:

Category	Isomorphisms
Set	bijections
Grp	isomorphisms
Ab	isomorphisms
F-Vect	F-linear isomorphisms
R-Mod	R-linear isomorphisms
Top	homeomorphisms
Haus	homeomorphisms
SmoothMan	diffeomorphisms
Nat	equality of naturals

As one should expect, groupoids get their name for a reason! Which we now verify.

Exercise I.2.4

Show that groups and groupoids with one object are exactly the same.

Definition I.2.3

There are a variety of nice names for particular types of morphisms. We list them here

- An **endomorphism** is an arrow $f : X \to X$
- An **automorphism** is an invertible endomorphism
- A **monomorphism** is a morphism $f : X \to Y$ such that for all morphisms $g, h : Z \to X$ we have
$$f \circ g = f \circ h \implies g = h$$
- An **epimorphism** is a morphism $f : X \to Y$ such that for all morphisms $g, h : Y \to Z$ we have
$$g \circ f = h \circ f \implies g = h$$

We can describe a morphism as being **endo** (**auto, mono, epi**) as shorthand.

Example I.2.5

Category	Monomorphisms	Epimorphisms
Set	injections	surjections
Haus	continuous injection	continuous maps with dense image
Nat	any arrow	any arrow

Note that in the category Haus there are arrows which are both mono and epi but which are not isomorphisms. Consider the inclusion $A \hookrightarrow X$ of a dense subspace A in a space X.

We also make some convenient notation for talking about categorical concepts. Namely, we specify what a commutative diagram is at an informal level. Later we will make this formal in order to talk about other categorical concepts.

Definition I.2.4

A **commutative diagram** consists of drawn arrows and objects, and we specify that any way to get between two objects by composing morphisms are the same. A **diagram** simply removes the condition that any composition of arrows is equivalent.

This is best explained via many examples. As the simplest example, saying the left diagram commutes says that $g \circ f = h$, and saying that the right diagram commutes specifies that $p_2 \circ q_1 = q_2 \circ p_1$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "p_1"] \arrow[d, "q_1"'] & B \arrow[d, "q_2"] \\ C \arrow[r, "p_2"'] & D \end{tikzcd}$$

An often useful concept in category theory is *dualization*. Formally, this consists of replacing a category $\mathscr{C}$ by its "opposite" category $\mathscr{C}^{\mathrm{op}}$

Definition I.2.5

Let $\mathscr{C}$ be some category. We define $\mathscr{C}^{\mathrm{op}}$ by $\mathrm{Ob}\,\mathscr{C}^{\mathrm{op}} := \mathrm{Ob}\,\mathscr{C}^{\mathrm{op}}$, and $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$. The composition is then defined for $f : X \xrightarrow{\mathrm{op}} Y, g : Y \xrightarrow{\mathrm{op}} Z$ by

$$g \circ_{\mathrm{op}} f = f \circ_{\mathrm{op}} g$$

Identities remain the same as they are in the original category.

I.3. Functors and Natural Transformations

The natural question to ask in algebraic or categorical subjects when given a collection of objects is whether they form a category, that is what is the appropriate notion of a "morphism" between such objects. This of course extends to categories themselves.

Definition I.3.1

Given two categories $\mathscr{C}, \mathscr{D}$, a **functor** $F : \mathscr{C} \to \mathscr{D}$ consists of the following data

- For every object $X \in \mathrm{Ob}\,\mathscr{C}$ a unique object $F(X) \in \mathscr{D}$
- For every arrow $f : X \to Y$ in $\mathscr{C}$ a unique arrow $F(f) : F(X) \to F(Y)$ in $\mathscr{D}$

satisfying the functoriality laws

- $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$
- For $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$ we have

$$F(g \circ f) = F(g) \circ F(f).$$

A functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ might be called a **contravariant functor** from $\mathscr{C}$ to $\mathscr{D}$, whereas $F : \mathscr{C} \to \mathscr{D}$ is called **covariant**. A contravariant functor satisfies for $f : X \to Y$, $g : Y \to Z$ in $\mathscr{C}$ that

$$F(g \circ f) = F(f) \circ F(g).$$

Example I.3.1

Say $\mathscr{C} = \mathrm{Grp}$ and $\mathscr{D} = \mathrm{Set}$. Then there is a functor from $\mathscr{C}$ to $\mathscr{D}$ given by taking a group G and "forgetting" the group structure to obtain a mere set G. The action on group homomorphisms is to "forget" that they respect the group operation.

There is also a functor Set $\to$ Grp, which associates a set S to the "free group" on S. Formally, this consists of all words in the language $S \cup S^{-1}$ (where S^{-1} is a formal copy of S, where we take $s^{-1} \in S^{-1}$ if $s \in S$). Two words are considered equivalent via the reduction rule which deletes pairs $ss^{-1}, s^{-1}s$, and the operation on words is concatenation. (The empty word being the identity element)

These two functors are intimately related, and we will discover they are "adjoint" in **??**

Example I.3.2

There is a functor $\pi_0 : \mathrm{Top} \to \mathrm{Set}$ given on objects by taking a topological space X and mapping it to the set of path components of X (that is the largest subspaces of X which are path-connected).

Given a continuous map $f : X \to Y$, $\pi_0(f)$ is given by consdering some path component U of X, then $f(U)$ is path-connected and non-empty, so it belongs to a unique path component V of Y. We then set $[\pi_0(f)](U) = V$.

Generally, there are many techniques to associate sets, groups, rings, and other algebraic structures to spaces. This is the realm of algebraic topology, and almost always these associations are functorial. In fact the notation π_0 suggests the corresponding $\pi_1, \pi_2, \ldots$. In this case $\pi_1 : \mathrm{Top} \to \mathrm{Grp}$, and for $n > 1$ we have $\pi_n : \mathrm{Top} \to \mathrm{Ab}$.

For more on this subject, [**hatcher**] is the standard reference, and [**may**] is a more concise and modern treatment.

Example I.3.3

Given two categories $\mathscr{C}, \mathscr{D}$ one can form the product category $\mathscr{C} \times \mathscr{D}$ in the natural way. Then for locally small categories there is a functor

$$\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Set}\,.$$

On objects this agrees with the notation we have previously established, so that $\mathrm{Hom}(X, Y)$ is the set of arrows from X to Y in $\mathscr{C}$. On arrows, if we have $f : X' \to X$ and $g : Y \to Y'$ in $\mathscr{C}$ (seeing that $f^{\mathrm{op}} : X \to X'$ in $\mathscr{C}^{\mathrm{op}}$) we have the function

$$\begin{aligned} \mathrm{Hom}(f, g) : \mathrm{Hom}(X, Y) &\to \mathrm{Hom}(X', Y') \\ h &\mapsto g \circ h \circ f \end{aligned}$$

Functoriality may be easily verified. As we should expect, the Hom functor carries a lot of the information about $\mathscr{C}$, as it encodes composition in the category.

We also have for fixed $X \in \mathrm{Ob}\,\mathscr{C}$ that $\mathrm{Hom}(X, -), \mathrm{Hom}(-, X)$ are contravariant/covariant functors respectively from $\mathscr{C} \to \mathrm{Set}$, as one should expect.

Exercise I.3.4

Prove that Hom is functorial.

Exercise I.3.5

There is a functor $\mathrm{Vect}^{\mathrm{op}} \to \mathrm{Vect}$ given by taking a vector space V to its dual V^*.

Work out the details of how this functor acts on linear maps and why it is functorial.

Exercise I.3.6

Show that, informally (that is without regards to set-theoretic size issues), define the category of all categories Cat.

The next natural question to ask is what are the arrows between functors themselves?

Definition I.3.2

A **natural transformation** $\eta : F \Rightarrow G$ between two functors $F, G : \mathscr{C} \to \mathscr{D}$ is a collection of maps $\eta_X : F(X) \to G(X)$ for each $X \in \operatorname{Ob} C$ satisfying the following commutative diagram for each arrow $f : X \to Y$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

This is called the **naturality condition** or **naturality square**.

Exercise I.3.7

For fixed categories $\mathscr{C}, \mathscr{D}$, define a category $[\mathscr{C}, \mathscr{D}]$ whose objects are functors $\mathscr{C} \to \mathscr{D}$ and whose arrows are natural transformations.

Exercise I.3.8

Show that the "double dual" functor taking V to $(V^*)^*$ from Vect $\to$ Vect is naturally isomorphic (that is isomorphic in $[\text{Vect}, \text{Vect}]$) to the identity functor Id_{Vect}.

Fortunately, this marks the "end of the line" for standard category theory. At higher levels of category theory, we can define higher morphisms, but for most mathematical purposes this level is sufficient.

Exercise I.3.9

Try to come up with a cohesive definition of arrows between two natural transformations $\eta, \mu : F \Rightarrow G$.

Exercise I.3.10

Given $\eta : F \Rightarrow G$, where $F, G : \mathscr{C} \to \mathscr{D}$ and $\mu : F' \Rightarrow G'$ where $F', G' : \mathscr{D} \to \mathscr{E}$ define

$$\eta \cdot \mu : F' \circ F \Rightarrow G' \circ G$$

this is called the "horizontal composition" of natural transformations, whereas the other composition is called the "vertical composition" and written $\eta \circ \eta'$. Show that where it makes sense we have

$$(\eta \circ \eta') \cdot (\mu \circ \mu') = (\eta \cdot \mu) \circ (\eta' \cdot \mu').$$

This is called the **interchange law**.

Functors / Natural Transformations

I.4. Presheaves and the Yoneda Lemma

I.5. Adjoint Functors

Write a Category Theory Appendix with Good References

TODOS: