Theorem .0.1

Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. If $g \in G$ has conjugacy class C, where $|C|$ is coprime to $\dim \rho$, then either $\chi_\rho(g) = 0$ or $\rho(g)$ acts on V as $\lambda \cdot \mathrm{Id}_V$ for some $\lambda \in \mathbb{C}^\times$

Note: $\rho(g) = \lambda \cdot \mathrm{Id}_V$ if and only if $\overline{\rho}(g) = 1$, with

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[rr, bend right, dashed, "\overline{\rho}"'] & \mathrm{GL}(V) \arrow[r, two heads] & PGL(V). \end{tikzcd}$$

Where $PGL(V) = \mathrm{GL}(V)/\{\lambda \,\mathrm{Id}_V \mid \lambda \in \mathbb{C}^\times\}$.

We will use Theorem .0.1 to prove

Theorem .0.2

If $|G| = p^a q^b$ with p, q distinct primes and $a, b > 0$ then G is not simple.

Claim

If G is any nontrivial finite group, and $p \neq q$ are primes dividing $|G|$, then there exists $g \in G \setminus \{1\}$ and an irreducible nontirvial representation $\rho : G \to \mathrm{GL}(V)$ such that $\chi_\rho(g) \neq 0$ and p does not divide the conjugacy class of G and $q \nmid \dim \rho$.

Proof of Theorem .0.2. First find $g \neq 1$ such that p does not divide the conjugacy class of G.

If center of G is nontrivial, let $g \in Z(G) \setminus \{1\}$. If $Z(G) = 1$ then

$$|G| = \sum_C |C|$$
$$|G| - 1 = \sum_{C \neq 1} |C| \cong -1 \mod p$$

Thus there is some $C \neq 1$ so that $p \nmid |C|$. Fix $g \in C$.

Then the orthogonality of columns for 1 and g gives that

$$0 = \sum_{\chi \text{irr.}} \chi(g)\overline{\chi(1)} = \sum_\chi \chi(g) \dim \chi$$
$$-1 = \sum_{\chi \neq 1} \chi(g) \cdot \dim \chi$$
$$-\frac{1}{q} = \sum_{\chi \neq 1} \chi(g) \cdot \frac{\dim \chi}{q}$$

Thus there exists a $\chi \neq 1$ such that $\chi(g) \neq 0$ and $(\dim \chi)/q$ is not an algebraic integer. Why? Well $-1/q$ is not an algebraic integer, and $\chi(g)$ is always an algebraic integer, so we must have some non-algebraic integer part of the sum.

Since $(\dim \chi)/q \in \mathbb{Q}$, this means that $q \nmid \dim \chi$.

When $|G| = p^a q^b$ this implies that the size of the conjugacy class is coprime to $\dim \rho$. Then Theorem .0.1 implies that $g \in \ker \overline{\rho}$ (where $\overline{\rho} = \rho/\lambda$ for some $\lambda \in \mathbb{C}^\times$). Then $\ker \overline{\rho}$ is a nontrivial normal subgroup of G, and G is not simple unless $\ker \overline{\rho} = G$.

But then $\rho(h)$ acts as $\lambda \operatorname{Id}_V$ for fixed $\lambda \in \mathbb{C}^\times$, implying that $\dim V = 1$ because ρ is irreducible. Thus ρ is a homomorphism $G \to \mathbb{C}^\times$. We then have that

$$G/\ker\rho \cong \operatorname{im}\rho = \text{cyclic}$$

But then ρ is nontrivial, so $\ker\rho \neq G$. Thus if G is simple, $\ker\rho = 1$, so G is cyclic, and clearly then G is not simple.

It remains to prove Theorem .0.1. Use

Theorem .0.3

Let $c = \sum_{g\in G} c_g e_g$ in the group algebra $\mathbb{C}[G]$. Assume that c lies in the cneter of $\mathbb{C}[G]$, i.e. $c_g = c_h$ when g, h are conjugate.

Assume further that each c_g is an algebraic integer. Then c acts on any irreducible representation as scalar multiplication by an algebraic integer

The action is for $\rho : G \to \mathrm{GL}(V)$. c maps $V \to V$ via

$$v \mapsto \sum_g c_g (g \cdot v)$$

Since $ce_g = e_g c$ for all g, c is a homomorphism of representations from ρ to ρ, so c is a scalar multiple by Schur's Lemma (see homework).

In particular, for $g \in G$ and any irreducible representation $\rho : G \to \mathrm{GL}(V)$,

$$\frac{\chi_\rho(g) \cdot |C(g)|}{\dim\rho}$$

is an algebraic integer. This is given by setting $c = \sum_{h\in C(g)} e_h$.

Proof. The value of the scalar is

$$\omega_\rho(c) = \frac{1}{\dim\rho}\sum_g c_g \chi_\rho(g).$$

We see that

$$\sum_g c_g \chi(g) = \operatorname{tr}\left(\sum_g c_g \rho(g)\right).$$

We then compute this trace which must be $(\dim\rho)\omega_\rho(c)$ because c acts as a scalar.

Since this expresion in c respects addition and scalar multiplication, it suffices to prove $\omega_\rho(c)$ is an algebraic integer when $c = \sum_{g\in C} e_g$ for some conjugacy class C in G. That is we can assume each c_g is zero or one.

Let $e_\rho \in \mathbb{C}[G]$ induce projection of any representation $\theta : G \to \mathrm{GL}(V)$ onto its ρ-isotypic part. Then we see that

$$c \cdot e_\rho = \omega_\rho(c) \cdot e_\rho$$

so e_ρ is an eigenvector of the action on $\mathbb{C}[G]$, with eigenvalue $\omega_\rho(c)$. To see this explicitly

Consider the regular representation on $\mathbb{C}[G]$ given by θ, with $e_\rho = \sum_g e_{\rho,g} e_g$. Then necessarily

$$c \cdot e_\rho = \sum_g \sum_h c_g e_{\rho,h}(e_{gh}) = \sum_g \sum_h c_g e_{\rho,h}(\theta(g) e_h) = \sum_g c_g(\theta(g) \cdot e_\rho).$$

Writing $e_1 = \sum_i \vec{v}_i + \vec{w}$ where each $\vec{v}_i$ lies in a copy of $\mathbb{C}[G]$ isomorphic to ρ, and $\vec{w}$ lies in the complement of the ρ-isotypic part of $\mathbb{C}[G]$. Then

$$\begin{aligned} c \cdot e_\rho &= \sum_g c_g(\theta(g) \cdot e_\rho \cdot e_1) = \sum_g \sum_i c_g(\rho(g) \cdot e_\rho \cdot \vec{v}_i) \\ &= \sum_i (\omega_\rho(g) \vec{v}_i) = \omega_\rho(g) \cdot e_\rho \end{aligned}$$

because $\sum_i \vec{v}_i = e_\rho$ by definition. Perfect!

But $\mathbb{C}[G] \to \mathbb{C}[G]$ given by $x \mapsto cx$ can be represented as an <u>integer</u> matrix in terms of the basis e_g, because

$$\begin{aligned} ce_g &= \left(\sum_{h \in G} c_h e_h \right) e_g \\ &= \sum_{h \in G} c_h e_{hg} = \sum_{h' \in G} c_{h' g^{-1}} e_{h'} \end{aligned}$$

and each $c_{h' g^{-1}}$ is zero or one by assumption. The eigenvalues are the roots of the characteristic polynomial, and this then proves that $\omega_\rho(c)$ is an algebraic integer.

Proof of Theorem .0.1. Suppose $|C(g)|$ is coprime to $\dim \rho$ for an irreducible representation $\rho : G \to \mathrm{GL}(V)$.

Then let $c = \sum_{h \in C(g)} e_h$. We then know that

$$\omega_\rho(c) = \frac{\chi_\rho(g) \cdot |C(g)|}{\dim \rho}$$

is an algebraic integer. If $\chi_\rho(g) \neq 0$, then because $|C(g)|$ and $\dim \rho$ are coprime this implies that $\chi_\rho(g) = (\dim \rho) \cdot \lambda$ for some algebraic integer $\lambda \in \mathbb{C}$, as otherwise we will not be able to cancel the denominator of $\dim \rho$, and it will show up in any monic polynomial with integer coefficients (similar to the proof that if $\alpha \in \mathbb{Q}$ is an algebraic integer then $\alpha \in \mathbb{Z}$).

By using the relevant inequalities by which we showed that $\chi_\rho(g) = \dim \rho$ if and only if $\rho(g)$ is trivial, we can then derive that all the eigenvalues of $\rho(g)$ are equal, showing that $\rho(g)$ acts by scalar multiplication just as desired.

Theorem .0.4 ($\dim \rho \mid |G|$)
If $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation then $\dim \rho \mid |G|$.

Proof. We know by orthonormality that

$$\begin{aligned} |G| &= \sum_{g \in G} \chi_\rho(g) \overline{\chi_\rho(g)} \\ \frac{|G|}{\dim \rho} &= \sum_{g \in G} \frac{\chi_\rho(g)}{\dim \rho} \overline{\chi_\rho(g)} \end{aligned}$$

$$\frac{|G|}{\dim \rho} = \sum_{C(g)} \frac{\chi_\rho(g)\,|C|}{\dim \rho}\overline{\chi_\rho(g)}$$

where $\sum_{C(g)}$ is a sum over distinct conjugacy classes.

By **??** we know that the right hand side is an algebraic integer, and so because the left hand side is in $\mathbb{Q}$ we know that the left hand side lies in $\mathbb{Z}$ as desired.