Example .0.1 (Qual, May 2016)
Let X be a finite, connected CW complex. $\widetilde{X}$ is its universal cover, and $\widetilde{X}$ is compact.
Show that $\widetilde{X}$ cannot be contractible unless X is contractible.

Solution. By homework, we then know that, since $\widetilde{X}$ is contractible and $\widetilde{X}$ has finitely many sheets d over X:

$$1 = \chi(\widetilde{X}) = d \cdot \chi(X)$$

Therefore, $\chi(X) = d = 1$, and so $p : \widetilde{X} \to X$ is a 1-sheeted cover, so it is a homeomorphism. Therefore X is contractible. Perfect!

Solution. Since $\widetilde{X}$ is contractible, $\tau(f) = 1$ for all $f : \widetilde{X} \to \widetilde{X}$. Furthermore, because $\widetilde{X}$ is compact and covers a finite CW complex, it is a finite CW complex. Therefore the Lefschetz Fixed Point Theorem applies, so any such map has a fixed point. If f is a deck map, then that means that $f = \mathrm{Id}_{\widetilde{X}}$ from our covering space theory. Great!

We have proved then that $X \cong \widetilde{X}/G(\widetilde{X})$ because $p : \widetilde{X} \to X$ is normal, but then the deck group $G(\widetilde{X})$ is trivial, so $X \cong \widetilde{X}$, and we are done.

Exercise .0.2
A 1-sheeted cover is always injective and surjective. Furthermore, it's a local homeomorphism. This suffices to show that a 1-sheeted cover is a homeomorphism.

Theorem .0.1
If X is a finite CW complex, with cellular chain groups $H_n(X^n, X^{n-1})$. If we have a cellular map $f : X \to X$, so f induces maps $f_* : H_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1})$. Then:

$$\tau(f) = \sum_n (-1)^n \operatorname{tr}(f_* : H_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1}))$$

Proof. Do some algebra! This is a purely algebraic fact

Exercise .0.3
Given a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Then $\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$.

Using the exercise, the theorem follows by an argument analogous to the argument for Euler Characteristic on Homework 12.