I. Applications of Group Theory

I.1. Polynomials

If $f(x) \in \mathbb{C}[x]$ has degree n, then the function $f : \mathbb{C} \to \mathbb{C}$ is n-to-1 over all but finitely many values (since $f(x) - c$ has n distinct roots unless c is a critical value of $f(x)$).

For each critical value c, let $E_f(c)$ be the collection (multiset) of multiplicities of the roots of $f(x) - c$. I.e.,

$$f(x) - c = \alpha \cdot \prod_{i=1}^{k} (x - \gamma_i)^{e_i}$$

with $\gamma_i \neq \gamma_j$, $e_i > 0$, then $E_f(c) = [e_1, e_2, \ldots, e_k]$. Note that $\sum_i e_i = n$.

Thus $E_f(c)$ is a partition of n, and $E_f(c) \neq [1, 1, \ldots, 1]$ if and only if c is a critical value of f.

Questions:

(1) For a degree n $f(x) \in \mathbb{C}[x]$, what are the possibilities for the collection of pairs

$$(c_1, E_f(c_1)), \ldots, (c_\ell, E_f(c_\ell))$$

where $c_1, \ldots, c_\ell$ are the critical values of f.

There is no known algebraic proof of this.

(2) For a given choice of this data (the collection of pairs), how many corresponding f's are there.

(3) The analogous questions for rational functions are open.

Definition I.1.1

For $f(x) \in \mathbb{C}[x] \setminus \mathbb{C}$, and $a \in \mathbb{C}$. Define $m_a(f)$ (the "multiplicity of a as a root of $f(x)$") to be the largest integer $k \geq 0$ such that $(x - a)^k$ divides $f(x)$.

Equivalently this says that $m_a(f)$ is the largest k such that $f(a), f'(a), \ldots, f^{(k-1)}(x) = 0$, or equivalently that this is the smallest $k \geq 0$ such that $f^{(k)}(a) \neq 0$.

Theorem I.1.1 (Riemann-Hurwitz)

If $f(x) \in \mathbb{C}[x]$ has degree n, then $n - 1 = \sum_{c \in \mathbb{C}} (n - |E_f(c)|) = \sum_{c \in \mathbb{C}} (n - |f^{-1}(c)|)$.

Proof. We count in two ways

$$\begin{aligned} n - 1 = \deg(f'(x)) &= \sum_{a \in \mathbb{C}} m_a(f'(x)) \\ &= \sum_{a \in \mathbb{C}} (m_a(f(x) - f(c)) - 1) \\ &= \sum_{c \in \mathbb{C}} \sum_{a \in f^{-1}(c)} (m_a(f(x) - c) - 1) \\ &= \sum_{c \in \mathbb{C}} (\deg(f(x) - c) - |f^{-1}(c)|) = \sum_{c \in \mathbb{C}} (n - |f^{-1}(c)|). \end{aligned}$$

Great!

Answers to questions **??**.

(1) Answered by Thom. Exactly the collections $(c_1, P_1), \ldots, (c_\ell, P_\ell)$ where $c_1, \ldots, c_\ell \in \mathbb{C}$ are distinct, $P_1, \ldots, P_\ell$ are partitions of n, $P_1 \neq [1, 1, \ldots, 1]$ for all i such that

$$n - 1 = \sum_{i=1}^{\ell} (n - |P_i|).$$

(2) Given distinct $c_1, \ldots, c_\ell \in \mathbb{C}$ and partitions $P_1, \ldots, P_\ell$ of n satisfying $P_i \neq [1, 1, \ldots, 1]$ and $n - 1 = \sum_{i=1}^{\ell}(n - |P_i|)$, then the # of degree n $f(x) \in \mathbb{C}[x]$ with $E_f(c_i) = P_i$, up to $f(x) \sim f(ax + b)$ ($a \in \mathbb{C}^\times, b \in \mathbb{C}$), that is up to linear changes of variable, is

the # of equivalence claseses of tuples $(g_1, \ldots, g_\ell)$ of elements o S_n such that P_i is the collection of cycle lengths of g_i and $g_1 g_2 \cdots g_\ell$ is an n-cycle, where $(g_1, \ldots, g_\ell) \sim (\sigma g_1 \sigma^{-1}, \ldots, \sigma g_\ell \sigma^{-1})$ for $\sigma \in S_n$.

For rational functions if $f(x) \in \mathbb{C}(x)$ has degree n then

$$2n - 2 = \sum_{c \in \mathbb{C}_\infty} (n - |E_f(c)|) = \sum_{c \in \mathbb{C}_\infty} \left(n - \left|f^{-1}(c)\right|\right)$$

People believe that this is the main constraint, but it is not true that it is the <u>only</u> contraint.

But it's not true that $P_1, \ldots, P_\ell$ are partitions of n such that $\sum_{i=1}^{\ell}(n - |P_i|) = 2n - 2$ then $\exists\, f(x)$ such that $E_f(c_i) = P_i$.

e.g. $[2, 2], [2, 2], [1, 3]$ doesn't occur.

<u>Fact</u>: Given distinct $c_1, \ldots, c_\ell \in \mathbb{C}_\infty$ and partitions $P_1, \ldots, P_\ell$ of n such that

$$2n - 2 = \sum_{i=1}^{\ell} (n - |P_i|),$$

Then the # of $f(x) \in \mathbb{C}(x)$ such that $E_f(c_i) = P_i$ for all i modulo the equivalence relation $f \sim f \circ \mu$, $\deg \mu = 1$ is exactly the # of $(g_1, \ldots, g_\ell)$ elements of S_n such that

- P_i is the collection of cycle lengths of g_i.
- $g_1 \cdots g_\ell = 1$.
- The group generated by the g_i is transitive.