Continued proof of **??**. We now know there is a unique fixed point R of our isometry given by $\tau_Q \circ \rho_\theta$.

We then have that $\tau_{-R} \circ \tau_Q \circ \rho_\theta \circ \tau_R$ is a rotation about the origin. Why? It's an orientation-preserving isometry that fixes the origin, so the composition $\tau_{Q'} \circ \rho_{\theta'} \circ (\mathrm{Id} \text{ or } r)$ cannot have $Q' \neq 0$ or r.

Thus $\tau_Q \circ \rho_\theta$ is a rotation about R.

Now suppose the isometry reverses orientation, that is it equals $\tau_Q \circ \rho_\theta \circ r$.

Then all we need to understand is $\rho_\theta \circ r$, and show that this is a reflection through some line. Namely it's reflection through the line which passes through $(0, 0)$ and is $\rho_{\theta/2}(x - \text{axis})$.

Change coordinates to make this line be the x-axis, then we have $\tau_{Q'} \circ r$, which is a reflection through a horizontal line if Q' is on the y-axis and a glide reflection otherwise.

Theorem .0.1

Every finite group of isometries of $\mathbb{R}^2$ is cyclic or dihedral.

Proof. We do this in a few simple steps

Step a) There are no nonidentity translations and there are no nonidentity glide reflections, because these have infinite order.

Step b) All rotations in this group G have the same center. To show this, pick any point R_0. We may then form a new point R_1 via:

$$R_1 = \frac{1}{|G|} \sum_{g \in G} g(R)$$

We claim that R_1 is fixed by each $g' \in G$. If g' is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, then $g'(R_1) = R_1$ Why? Well:

$$g'(R_1) = \frac{1}{|G|} \sum_{g \in G} g'(g(R)) = \frac{1}{|G|} \sum_{g \in G} (g'g)(R) = \frac{1}{|G|} \sum_{h \in G} h(R) = R_1$$

The second to last equality is fundamental, and follows because $g \mapsto h = g'g$ is a bijection $G \to G$.

Also we have that τ_Q maps R_1 to $R_1 + Q$, why? Well:

$$\tau_Q(R_1) = Q + \frac{1}{|G|} \sum_{g \in G} g(R) = \frac{1}{|G|} \sum_{g \in G} [g(R) + Q] = \frac{1}{|G|} \sum_{g \in G} \tau_Q(g(R_1))$$

In fact, this means that every isometry maps the center of mass of a set of points to the center of mass of the images of these points.

Because the set $\{g(R) \mid g \in G\}$ and the image set $\{g'(g(R)) \mid g \in G\}$ are the same, this means that each $g' \in G$ fixes R_1. Rotations have a unique fixed point which is their center, and so we're done.

Step c) Suppose G consists solely of rotations. They all have a common fixed point, we may as well assume it is $(0, 0)$ without loss of generality. Say the rotations are by angles $0 = \theta_1 < \theta_2 < \ldots < \theta_k < 2\pi$.

We claim rotation by θ_2 generates the group. Well we know each $\theta_i = n_i \theta_2 + \delta_i$ for $n_i \in \mathbb{Z}$ and $0 \leq \delta < \theta_2$. But then this would imply that rotation by δ_i is in the group, showing that we must have $\delta_i = 0$ by minimality of θ_2.

This finishes this piece!

Step d) If we have a reflection, we can choose coordinates so it is through the x-axis, giving us the dihedral group. Why? Well, we generate the dihedral group, and any two r_1, r_2 composed give a rotation ρ_θ, so $r_1 = \rho_\theta \circ r_2^{-1}$, showing that r_1 must be in the dihedral group as well.