I. Wrap-Up

I.1. Representations of C_n

Let $C_n = \langle g \rangle$. Then

$$\rho : C_n \to \mathrm{GL}_k(\mathbb{C})$$
$$g \mapsto \rho(g) = M$$

where $M^n = \mathrm{Id}_k$.

What are the subrepresentations? Well M is diagonalizable

Theorem I.1.1

General Fact: A $k \times k$ matrix A (over F) is diagonalizable over a field F if and only if $h(A) = 0$ for some monic degree-d $h(x) \in F[x]$ which has d distinct roots in F.

Theorem I.1.2 (Cayley-Hamilton)

A matrix A satisfies its characteristic polynomial.

Because M is diagonalizable, $\mathbb{C}^k = V_1 \oplus \cdots \oplus V_r$ where V_i are eigenspaces fro M with eigenvalue λ_i (where λ_i are pairwise distinct element of $\mathbb{C}^\times$). What are all subrepresentations? They're all $W_1 + \oplus + W_r$ with W_i a subspace of V_i.

Theorem I.1.3 (Brouwer's Theorem)

Every (complex finite-dimensional) character of every finite group G is a $\mathbb{Z}$-linear combination of characters that are induced from degree-1 characters of "elementary" subgroups.

Elementary subgroups are direct products of cyclic groups with p-groups, $C_m \times P$, where P is a p-group for some prime p.

I.2. Products of Conjugacy Classes

Suppose G is a finite group and $C_1, \ldots, C_k$ are conjugacy classes in G. What can you say about the multiset $C_1 C_2 \cdots C_k$? It's a $(\mathbb{Z}_{\geq 0})$-linear combination of conjugacy classes.

$$\sum_{\substack{\text{conj. class} \\ C}} n_C C$$

where $n_C \in \mathbb{Z}_{\geq 0}$ are called the "structure constants of G."

In terms of the group algebra, define $e_C \coloneqq \sum_{g \in C} e_g$. Then we are examining

$$e_{C_1} e_{C_2} \cdots e_{c_K} = \sum_{\substack{\text{conj. class} \\ C}} n_C e_C$$

We define

$$\mathcal{N}(C_1, \ldots, C_k) \coloneqq \#\text{ of } (g_1, \ldots, g_k) \in C_1 \times \cdots \times C_k \text{ s.t. } g_1 \cdots g_k = 1$$

If $k = 1$, then

$$\mathcal{N}(C_1) = \begin{cases} 1 & \text{if } C_1 = \{1\} \\ 0 & \text{otherwise} \end{cases}.$$

For $k = 2$, we have

$$\mathcal{N}(C_1, C_2) = \begin{cases} |C_1| & \text{if } C_1 = C_2^{-1} \\ 0 & \text{otherwise} \end{cases}$$

secretly this is the column orthogonality relation for characters.

For $k = 3$, we have

$$\mathcal{N}(C_1, C_2, C_3) = \#\{(g_1, g_2) \in C_1 \times C_2 \mid g_1 g_2 \in C_3^{-1}\}$$

this doesn't tell us much...

The answer! Representation Theory!

Theorem I.2.1 (Frobenius's Theorem)

We have that

$$\mathcal{N}(C_1, \dots, C_k) = \frac{|C_1| \cdots |C_k|}{|G|} \sum_{\substack{\text{irr.} \\ \chi}} \frac{\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k-2}}$$

We should verify it for small k. If $k = 1$, this reads as

$$\begin{aligned}
\mathcal{N}(C_1) &= \frac{|C_1|}{|G|} \sum_{\chi} \frac{\chi(C_1)}{\chi(1)^{-1}} \\
&= \frac{|C_1|}{|G|} \sum_{\chi} \chi(C) \overline{\chi(1)} \\
&= \begin{cases} 1 & \text{if } C = \{1\} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

by the column orthogonality relation. For $k = 2$, this reads as

$$\begin{aligned}
\mathcal{N}(C_1, C_2) &= \frac{|C_1|\,|C_2|}{|G|} \sum_{\chi} \frac{\chi(C_1)\chi(C_2)}{\chi(1)^0} \\
&= |C_1| \cdot \frac{|C_2^{-1}|}{|G|} \sum_{\chi} \chi(C_1) \overline{\chi(C_2^{-1})} \\
&= \begin{cases} |C_1| & \text{if } C_1 = C_2^{-1} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Proof of Theorem I.2.1. If C is a conjugacy class of G, define

$$e_C := \sum_{g \in C} e_g \in \mathbb{C}[G]$$

for all representations $\rho : G \to \mathrm{GL}(V)$, any element $f \in \mathbb{C}[G]$ of the group algebra acts on V. Formally if $f = \sum_g f_g e_g$ we have L_f given by

$$\begin{aligned} L_f : V &\to V \\ v &\mapsto \sum_{g \in G} f_g(\rho(g)v) \end{aligned}$$

which is a $\mathbb{C}$-linear map. We simplify notation by writing $L_C = L_{e_C}$. It turns out that L_C is a representation of homomorphisms

$$L_C \rho(h) v = \sum_{g \in C} \rho(gh) v = \sum_{g \in C} \rho(hgh^{-1}h) v = \sum_{g \in C} \rho(hg) v = \rho(h) L_C v$$

If ρ is irreducible, then Schur's Lemma implies that L_C is a scalar multiple by some constant $\omega_\rho(C)$. Taking traces, we see that

$$\omega_\rho(C) \cdot \dim \rho = \mathrm{tr}(L_C) = \mathrm{tr}\left(\sum_{g \in C} \rho(g)\right) = \sum_{g \in C} \chi(g) = |C|\,\chi(C) = |C|\,\omega_\rho(C).$$

Therefore

$$\omega_\rho(C) = \frac{|C|\,\chi(C)}{\chi(1)}$$

We now compute the action $e_{C_1} \cdots e_{C_k}$ on $\mathbb{C}[G] = \bigoplus_{\substack{\text{irr.} \\ V_i}} (\dim V_i) V_i$ by the regular representation. On one hand we have

$$\begin{aligned} e_{C_1} \cdots e_{C_k} &= \sum_{g_i \in C_i} e_{g_1 \cdots g_k} \\ \mathrm{tr}(e_{C_1} \cdots e_{C_k}) &= \sum_{g_i \in C_i} \begin{cases} |G| & \text{if } g_1 g_2 \cdots g_k = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= |G|\,\mathcal{N}(C_1, \ldots, C_k) \end{aligned}$$

But also $e_{C_1} \cdots e_{C_k}$ acts on V_i as scalar multiplication by $\omega_{\rho_i}(C_1) \cdots \omega_{\rho_i}(C_k)$. Then

$$\mathrm{tr}(e_{C_1} \cdots e_{C_k}) = \sum_i n_i^2 \omega_{\rho_i}(C_1) \cdots \omega_\rho(C_k)$$

where $n_i = \dim V_i = \chi_i(1)$ where χ_i is the character of ρ_i. We may then just substitute

$$\begin{aligned} \mathrm{tr}(e_{C_1} \cdots e_{C_k}) &= \sum_i n_i^2 \prod_{j=1}^{k} \frac{|C_j|\,\chi_i(C_j)}{\chi_i(1)} \\ &= \sum_i \prod_{j=1}^{k} \frac{|C_j|\,\chi_j(C_j)}{\chi_i(1)^{k-2}} \end{aligned}$$