.1. Calculations with $\pi_1(S^n)$

Theorem .1.1
$\pi_1(S^1) \cong \mathbb{Z}$, and this identification is given by the following paths:

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))]$$

Intuitively this winds around S^1 n times. The key to this proof was to understand S^1 via the covering space $\mathbb{R} \to S^1$. We will talk about covering spaces more in class later.

Proof. See Homework

Theorem .1.2
There is a natural identification $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. The identfication is exactly:

$$[\gamma : I \to X \times Y] \leftrightarrow ([p_X \circ \gamma], [p_Y \circ \gamma])$$

Where p_X and p_Y are the projections.

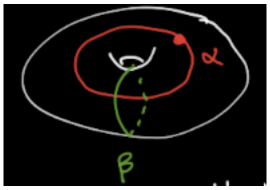
Exercise .1.1
Give a proof of ths result. The key is that a map:

$$Z \xrightarrow{f} X \times Y$$
$$z \overset{f}{\mapsto} (f_X(z), f_Y(z))$$

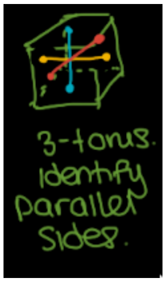
f is continuous if and only if f_X and f_Y are. The proof should go like:

- Apply this principle to paths $I \to X \times Y$
- Apply this principle again to homotopies of paths $I \times I \to X \times Y$.

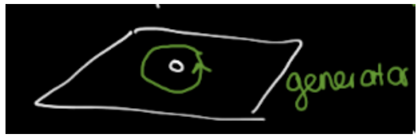
Corollary .1.3
The torus $T \cong S^1 \times S^1$ has fundamental group $\pi_1(T) \cong \mathbb{Z}^2$. This will in fact be generated by the loops around each of the factors:



Furthermore the n-torus $\underbrace{S^1 \times \cdots \times S^1}_{n}$ has fundamental group $\mathbb{Z}^n$. One way to think of the n-torus is as an n-dimensional cube with opposite $(n-1)$-dimensional faces identified by translation. We include a picture of the 3-torus with the generators:



Corollary .1.4
$\mathbb{R}^2 - \{0\} \cong S^1 \times \mathbb{R}$ must have fundamental group $0 \times \mathbb{Z} \cong \mathbb{Z}$. Intuitively the generators are just loops around the hole:



Theorem .1.5
$\pi_1(S^n) \cong 0$ for all $n \geq 2$. The picture for the 2-sphere is fairly simple:

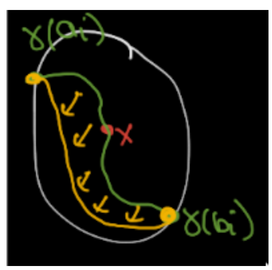
Great!

Proof. Let $\gamma : I \to S^n$ be loop based at x_0. Choose $x \neq x_0$.

Goal: Homotope γ (rel $\{0,1\}$) so that x is not in its image. Then we know $S^n \setminus \{x\} \cong \mathbb{R}^n$, which deformation retracts to x_0, and we can use this deformation retraction to push our loop to a point.

Strategy: Choose a small open ball B about x. Then we know that $\gamma^{-1}(B)$ is open in I, so it is a union of disjoint open intervals (a_i, b_i). The set $\gamma^{-1}(x)$ is compact, since it is a closed subset of I, so it is contained in finitely many (a_i, b_i)

The next observation is that by continuity of γ we have $\gamma([a_i, b_i]) \subseteq \overline{\gamma((a_i, b_i))} \subseteq \overline{B}$, and thus the endpoints a_i, b_i must map to ∂B. So then for all i for which $\gamma((a_i, b_i))$ passes through x choose a new path α_i in $\overline{B}$ from $\gamma(a_i)$ to $\gamma(b_i)$ that misses x:



We homotope $\gamma\big|_{[a_i, b_i]}$ to α_i rel $\{0,1\}$. We can always do this because D^n is contractible and has enough "room" to not pass through the point x for $n \geq 2$. We do this finitely many times by the compactness argument above, and so we're done! We have a path homotopic to γ that misses x.

Breakout Rooms: π_1 is a functor

Theorem .1.6
$\pi_1 : \underline{\text{Top}*} \to \underline{\text{Grp}}$ is a functor. Recalle the objects in $\underline{\text{Top}*}$ are spaces with a distinguished basepoint (X, x_0), and the maps are base point preserving maps.

The functor is defined on morphisms by considering that a map $f : X \to Y$ taking x_0 to y_0 induces:

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$[\gamma] \mapsto [f \circ \gamma]$$

Proof: In Breakout Groups. We need to check four things:

(1) f_* is well-defined on homotopy classes
(2) f_* is a group homomorphism
(3) $(\text{Id}_{(X,x_0)})_* = \text{Id}_{\pi_1(X,x_0)}$
(4) $(f \circ g)_* = f_* \circ g_*$.

Let's go in order!

(1) Take two homotopic paths γ, γ', then there's a relative homotopy between them $\Gamma : I \times I \to X$. Then we can postcompose this to get a homotopy $f \circ \Gamma$ from $f \circ \gamma$ to $f \circ \gamma'$. This will be a relative homotopy since f preserves basepoints.

(2) This is a pretty simple calculation. Fix two loops α, β at x_0. Then we see that:

$$\begin{aligned}(f \circ (\alpha \cdot \beta))(t) &= \begin{cases} f(\alpha(2t)) & \text{if } 0 \leq t \leq 1/2 \\ f(\beta(2t-1)) & \text{if } 1/2 \leq t \leq 1 \end{cases} \\ &= ((f \circ \alpha) \cdot (f \circ \beta))(t)\end{aligned}$$

Great! Thus this is a group homomorphism since:

$$f_*([\alpha][\beta]) = f_*([\alpha \cdot \beta]) = [f \circ (\alpha \cdot \beta)] = [(f \circ \alpha) \cdot (f \circ \beta] = f_*(\alpha) f_*(\beta)$$

(3) We see that for any loop γ at x_0:

$$(\mathrm{Id}_{(X,x_0)})_*([\gamma]) = [\mathrm{Id}_{(X,x_0)} \circ \gamma] = [\gamma]$$

And so this map is the identity on the group!

(4) The calculation here is similar to the quiz, using associativity of composition. Namely given a loop γ at x_0 we see that for $f : (Y, y_0) \to (Z, z_0)$ and $g : (X, x_0) \to (Y, y_0)$ that :

$$\begin{aligned}(f \circ g)_*([\gamma]) &= [(f \circ g) \circ \gamma] = [f \circ (g \circ \gamma)] \\ &= f_*([g \circ \gamma]) = f_*(g_*([\gamma])) = (f_* \circ g_*)([\gamma])\end{aligned}$$

Great!

With this established we've seen our first really cool example of a functor ☺.