Announcements

- Quiz #3 Wednesday. Hints:
  - Know our calculation of π_1 for spheres and contractible spaces (don't need proof)
  - Know our result on π_1 of a product
  - Know the definition of a retraction
  - Understand solution to Homework #3, Assignment Questions 1a and 1b
- Midterm next week, February 18th 7-8pm

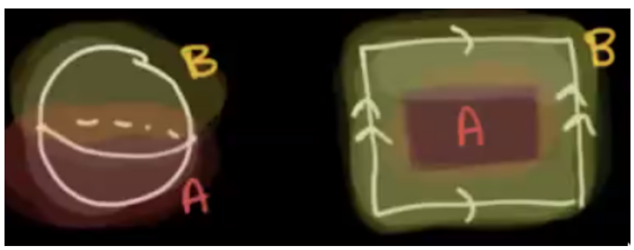
Definition .0.1 (From Homework)
Let X be a topological space, and $A \subseteq X$ a subspace. A retraction $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$. The subspace A is called a retract of X.

Lecture

Last time, we showed in breakout rooms that π_1 is a functor from Top$*$ to Grp. Note that parts (3) and (4) were very similar to the quiz!

.1. The Van Kampen Theorem

Goal: Compute $\pi_1(X)$ where $X = A \cup B$ using the data of $\pi_1(A)$, $\pi_1(B)$, $\pi_1(A \cap B)$.



Definition .1.1 (Free product of groups with amalgamation)
Given some collection of groups $\{G_\alpha\}_\alpha$, the free product $\underset{\alpha}{*} G_\alpha$ is a group:

- Elements are words $g_1 g_2 \cdots g_n$ where $g_i \in G_\alpha$ for some α. Modulo the equivalence relation generated by:
  - First we have
  $$w g_i g_j v \sim w(g_i g_j)v$$
  Whenever both $g_i, g_j \in G_\alpha$. And also:
  - We also want to deal with identities 1_α for $1_\alpha \in G_\alpha$ the identity element
  $$w 1_\alpha v = wv$$
  Great!
- Operation is concatenation of words.

If groups G_α and G_β have a common subgroup H (inclusion maps $i_\alpha : H \to G_\alpha$ and $i_\beta : H \to G_\beta$) then the free product with amalgamation ${}_\alpha *_H G_\alpha$ is defined as $\underset{\alpha}{*}$ modulo the subgroup generated by the words:
$$i_\alpha(h) i_\beta(h)^{-1}$$
Aka, $i_\alpha(h)$ and $i_\beta(h)$ will be identified in the quotient.

We can then write out words as such as $g_1 g_2 h g_3$ for $h \in H$, and view h as an element of G_α or G_β. In fact, we can do this construction even when i_α and i_β are not injective, though this means we are not working with a subgroup.

Exercise .1.1
Check that ${}_\alpha *_H G_\alpha$ is well-defined as a group under concatenation.

Exercise .1.2

${}_{\alpha}{*}_{H}\, G_\alpha$ contains each group G_α as a subgroup in a canonical way.

Exercise .1.3

$G_1 *_H G_2$ satisfies the universal property–which is called a pushout–meaning that whenever we have $f_1 : G_1 \to Z$ and $f_2 : G_2 \to Z$ such that $f_1 \circ i_1 = f_2 \circ i_2$ then there is a unique $u : G_1 *_H G_2$ making the diagram commute:

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d] \arrow[ddr, bend left, "f_1"] & \\ G_2 \arrow[r] \arrow[drr, bend right, "f_2"'] & G_1 *_H G_2 \arrow[dr, dashed, "\exists!\, u"'] & \\ & & Z \end{tikzcd}$$

Awesome!

Note on Notation: The α in ${}_{\alpha}{*}_{H}$ is the indexing set, and the amalgamating group is H, with maps $H \to G_\alpha$, $H \to G_\beta$ left implicit. This notation may only be standard for working with two groups.

Analogue: If we have sets A, B with common subset C (i.e. $A \cap B = C$), then we sometimes write $A \cup_C B = A \cup B$, then again we have this universal property:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & A \cup_C B \arrow[dr, dashed, "\exists!\, u"'] & \\ & & Z \end{tikzcd}$$

The universal property is actually a bit more general if we take any maps $C \to A$ and $C \to B$. In this case we have:

$$A \cup_C B = A \sqcup B / [f(c) \sim g(c)]$$

Theorem .1.1 (Van Kampen)

Here are the preconditions:

- Suppose we have a space X with base point x_0.
- We have $X = \bigcup_\alpha A_\alpha$
- A_α are each open, path-connected, and contain x_0
- $A_\alpha \cap A_\beta$ is path-connected.

Then there exists a surjective homomorphism $*_\alpha\, \pi_1(A_\alpha) \to \pi_1(X)$.

If we additionally assume that if $A_\alpha \cap A_\beta \cap A_\gamma$ are all path connected, then:

$$\pi_1(X) \cong {}_{\alpha}{*}_{\pi_1(A_\alpha \cap A_\beta)}\, \pi_1(A_\alpha)$$

associated to all maps $\pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha), \pi_1(A_\beta)$ induced by inclusions of spaces. I.e. $\pi_1(X)$ is a quotient of the free product $*_\alpha\, \pi_1(A_\alpha)$ where we have:

$$(i_{\alpha\beta})_* : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$$

which is induced by the inclusion $i_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$. We quotient by the normal subgroup generated by:

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_*(\gamma)^{-1} \mid \gamma \in \pi_1(A_\alpha \cap A_\beta)\}$$

We're often interested in the special case with two sets:

Theorem .1.2 (Van Kampen for two sets)
For $X = A \cup B$ and A, B open path connected sets containing x_0 with $A \cap B$ path connected, then:

$$\pi_1(X) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$