Announcements

- No class Wednesday, Office Hour moved to 8pm on Thursday
- Homework #5 corrected.

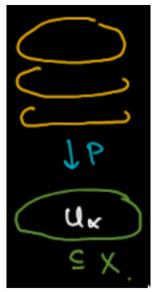
Resume Math!

I. Covering Spaces

I.1. Definitions and Lifting Properties

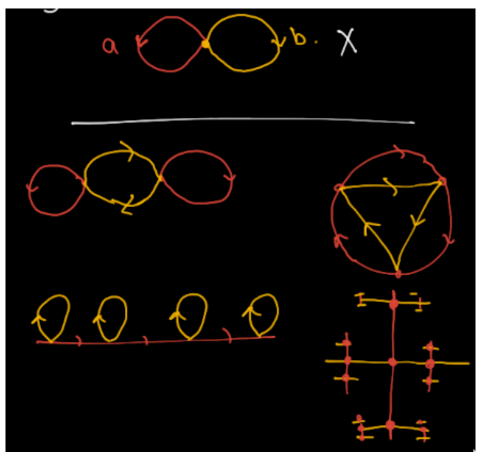
Definition I.1.1

A covering space $\widetilde{X}$ of X is a space $\widetilde{X}$ equipped with a map $p : \widetilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X so that for all α, $p^{-1}(U_\alpha)$ is a disjoint union (possibly empty) of open subsets in $\widetilde{X}$, each of which is mapped homeomorphically by p to U_α. Here's the picture:



Example I.1.1

Covers of $S^1 \vee S^1$, lifted from Hatcher:



Proposition I.1.1

Covering spaces (say $\widetilde{Y}$ over Y) satisfy the homotopy lifting property. That is, we may fill in diagrams in the following way:

$$\begin{tikzcd}
X \times \{0\} \cong X \arrow[r, "\widetilde{F}_0"] \arrow[d] & \widetilde{Y} \arrow[d] \\
X \times I \arrow[r, "F_t"'] \arrow[ur, dashed, "\widetilde{F}_t", "\exists!"'] & Y
\end{tikzcd}$$

That is given a lift $\widetilde{F}_0$ of F_0, there is a unique lift $\widetilde{F}_t$ of F_t extending $\widetilde{F}_0$.

Corollary I.1.2

Covering spaces satisf the path-lifting property:

For each path $I \xrightarrow{\gamma} Y$ and for each preimage $\widetilde{y}_0$ of $\gamma(0) = y_0$ there exists a unique path $I \xrightarrow{\widetilde{\gamma}} \widetilde{Y}$ lifting γ and starting at $\widetilde{y}_0$.

Proof. See Homework.

Proposition I.1.3

Suppose that $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map (i.e. p is a covering map and $p(\widetilde{x}_0) = x_0$). Then we have the following relationships between the fundamental groups:

(i) $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective

(ii) $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$ picks out the subset:

$$\{[\gamma] \mid \text{Lift } \widetilde{\gamma} \text{ of } \gamma \text{ starting at } \widetilde{x}_0 \text{ is a loop}\}$$

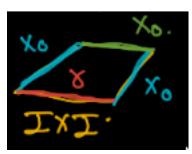
Proof. Suppose that $\widetilde{\gamma} \in \ker p_*$. Then $\gamma = p \circ \widetilde{\gamma}$. Let γ_t be a null-homotopy from γ to the constant loop c_{x_0} rel $\{0, 1\}$. Then we can lift γ_t to $\widetilde{\gamma}_t$ where $\widetilde{\gamma}_0 = \widetilde{\gamma}$. We then claim that, using a similar proof as in Homework 2:

- $\widetilde{\gamma}$ is a homotopy rel $\{0, 1\}$
- $\widetilde{\gamma}_1$ is the constant loop $c_{\widetilde{x}_0}$.

In diagrams and pictures:

$$\begin{tikzcd} & \widetilde{X} \arrow[d] \\ I \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{X} \arrow[d] \\ I \times I \arrow[ur, "\widetilde{\gamma}_t"] \arrow[r, "\gamma_t"'] & X \end{tikzcd}$$

This picture provides a proof of the first claim, we know that the left and right edge of $I \times I$ maps to x_0 under γ_t, and $c_{\widetilde{x}_0}$ lifts this, so by uniqueness $t \mapsto \widetilde{\gamma}_t(0)$ and $t \mapsto \widetilde{\gamma}_t(1)$ must be constant paths at $\widetilde{x}_0$ as desired.



This shows that $\ker p_*$ is trivial. Proving part (i). We leave part (ii) as an exercise. The proof uses similar ideas.