We'll assume that $\widetilde{X}$ is connected for now.

Definition .0.1

A covering space $p : \widetilde{X} \to X$ is normal or regular if for every $x_0 \in X$ and every pair of lifts $\widetilde{x}_1, \widetilde{x}_2 \in p^{-1}(x_0)$, there exists a Deck transformation mapping $\widetilde{x}_1$ to $\widetilde{x}_2$.

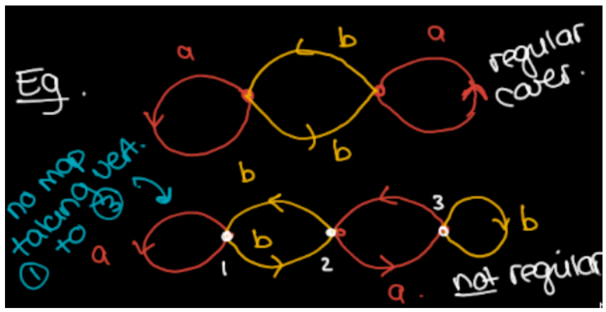
Slogan: A regular cover is "as symmetric as possible"

Exercise .0.1

"Regular" means that the group $G(\widetilde{X})$ acts transitively on $p^{-1}(x_0)$. Explain why we cannot ask for more than this–$G(\widetilde{X})$ cannot (eg) induce the full symmetric group on $p^{-1}(x_0)$ (key: uniqueness) provided that $\left|p^{-1}(x_0)\right| > 2$.

Example .0.2

Lets go back to the wedge of two circles!



Exercise .0.3

A Deck transformation of covers of $S^1 \vee S^1$ is precisely a graph automorphism that preserves the labels / directed edges.

Definition .0.2

If G is a group and H is a subgroup, then the normalizer of H is:

$$N(H) = \{g \in G \mid gH = Hg\}$$

Exercise .0.4

Check that:

- $N(H)$ is a subgroup containing H
- H is normal in $N(H)$.
- $N(H)$ is the largest subgroup of G which contains H in which H is normal.

Theorem .0.1

Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map. We assume that $\widetilde{X}, X$ are path-connected, and locally path-connected. For convenience, let $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$. Then:

(a) p is a normal cover if and only if H is normal in $\pi_1(X, x_0)$.

(b) The group of Deck Transformations $G(\widetilde{X}) \cong N(H)/H$, where the normalizer of H is taken in $\pi_1(X, x_0)$.

Corollary .0.2

If p is a normal covering, then $G(\widetilde{X}) \cong \pi_1(X, x_0)/H$.

Corollary .0.3

If $\widetilde{X}$ is the universal cover, then $G(\widetilde{X}) \cong \pi_1(X, x_0)$

Exercise .0.5

Consider $\mathbb{R}^2 \to T^2$ and $\mathbb{R} \to S^1$.