Announcements: Midterms

- Friday October 22nd, 6pm-8:30pm
- Thursday: December 9th, 6pm-8:30pm
- NO FINAL EXAM

Given a group G and a subgroup H we defined G/H to be the set of all (left)-cosets of H in G. Recall that:

- $[G : H] \coloneqq |G/H|$.
- If $H \trianglelefteq G$ then G/H is a group under the operation induced by G. That is $(aH)(bH) = (ab)H$. As a set this is exactly:

  $$\{ah_1bh_2 \mid h_1, h_2 \in H\}$$

  And so either way of interpreting $(aH)(bH)$ is correct!
- There is a quotient map $G \to G/H$ given by $g \mapsto gH$, which is clearly a surjective homomorphism. And in fact the kernel of this map is H.

Theorem .0.1 (Artin calls this the Correspondence Theorem)

Let $f : G \twoheadrightarrow G'$ be a surjective homomorphism with kernel K. There is then a natural bijection between subgroups of G containing K, and subgroups of G'. This is given by taking image/preimage under f.

$$H \mapsto f(H)$$
$$f^{-1}(J) \leftarrow\!\shortmid J$$

Proof. The images and preimages will in fact be subgroups (easy check). Furthermore if $J \leq G'$, then $f^{-1}(J)$ contains K because $1 \in J$, and every $k \in K$ maps to 1.

It's immediate that $f(f^{-1}(J)) = J$ because f is surjective. We then need to show for $K \leq H \leq G$ that $H = f^{-1}(f(H))$. From set theory we know that $H \subseteq f^{-1}(f(H))$. So we just need to show the other direction.

Let $x \in f^{-1}(f(H))$, so then $f(x) = f(h)$ for some $h \in H$. But then $f\left(xh^{-1}\right) = 1$, so $xh^{-1} \in K \subseteq H$. Thus $xh^{-1} \in H$ and we then know that $x = xh^{-1}h \in H$, and we are done!

Another proof looks like this. Preimages of points are exactly cosets of the kernel, so we have:

$$f^{-1}(f(H)) = \bigcup_{h \in H} f^{-1}(f(h)) = \bigcup_{h \in H} hK = H$$

Using the fact that $K \leq H$, so any element of hK lies in H.

Note if $K \leq H \leq G$ then $[G : H] = [G' : f(H)]$. Send the coset gH to $f(g)f(H)$, and this will be a bijection. The argument is standard from similar ideas to the above.

If G is a group and N is a normal subgroup, then the homomorphism $G \twoheadrightarrow G/H$ is called the quotient homomorphism/quotient map/canonical homomorphism. The correspondence theorem then tells us that subgroups of G/H are in bijection with the subgroups of G containing N.

Theorem .0.2 (The First Isomorphism Theorem)

Let $f : G \to \overline{G}$ be some surjective homomorphism with kernel K. Then $G/K \cong \overline{G}$. Precisely, let $\pi : G \twoheadrightarrow G/K$ be the quotient map. Then there is a unique isomorphism $\overline{f} : G/K \to \overline{G}$ which makes the following diagram commute:

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[dr, "\pi"'] & \overline{G} \\
& G/K \arrow[u, dashed, "\overline{f}"']
\end{tikzcd}$$

Proof. A function $\overline{f} : G/K \to \overline{G}$ such that $\overline{f} \circ \pi = f$ is completely determined by f because π is surjective. Namely for $gK \in G/K$, we know:

$$\overline{f}(gK) = \overline{f}(\pi(g)) = f(g)$$

We now just need to show that's a well-defined isomorphism.

Say that $gK = \widetilde{g}K$, then:

$$f(gK) = f(g)f(K) = f(g) = f(\widetilde{g}) = f(\widetilde{g})f(K) = f(\widetilde{g}K)$$

Great! Thus this function $\overline{f} : G/K \to \overline{G}$ is well-defined.

$\overline{f}$ is a homomorphism clearly because:

$$\overline{f}((g_1K)(g_2K)) = \overline{f}(g_1g_2K) = f(g_1g_2) = f(g_1)f(g_2) = \overline{f}(g_1K)\overline{f}(g_2K).$$

Furthermore $\overline{f}$ is surjective. Take $\overline{g} \in \overline{G}$, then $\overline{g} = f(g)$ for some $g \in G$, so:

$$\overline{f}(\pi(g)) = f(g) = \overline{g}$$

Finally, $\overline{f}$ is injective. To show this we show $\ker \overline{f} = 1$. Let $gK \in \ker \overline{f}$, then $f(g) = 1$, so $g \in \ker f = K$, and $gK = K$ and we're done!

Lets look at symmetries of a triangle:

- Rotation by 120 degrees around the center (order 3)
- Reflect through an angle bisector (order 2)

These generate S_3.

In general, a regular n-gon has $2n$ symmetries by rotations/reflections. These are generated by the reflections, as two reflections makes a rotation.

Definition .0.1

The dihedral group of order $2n$ is the largest group generated by x, y satisfying the relations

$$x^2 = y^2 = (xy)^n = 1$$

One can show that the elements are $z^i x^j$ for $0 \leq i < n$ and $0 \leq j \leq 1$.

This is a non-abelian group!