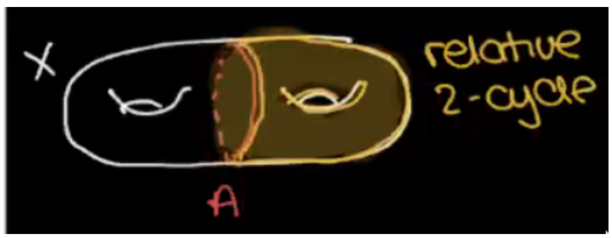
Definition .0.1
Let X be a space and let $A \subseteq X$ be a subspace. Then we define the relative chain complex

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}$$

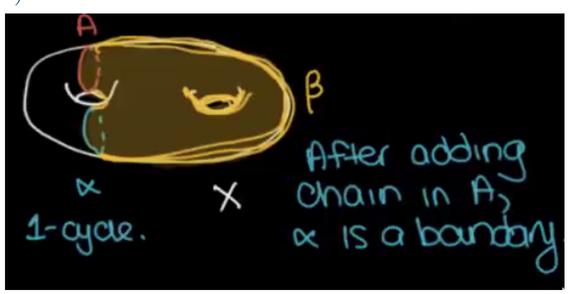
Exercise .0.1
The boundary map $\partial : C_n(X) \to C_{n-1}(X)$ induces a well-defined map $\partial : C_n(X, A) \to C_{n-1}(X, A)$. Since $\partial^2 = 0$ we can conclude that these groups will in fact form a chain complex $(C_*(X, A), \partial)$.

Definition .0.2
The homology groups of $(C_*(X, A), \partial)$ are denoted by $H_n(X, A)$, and they are called relative homology groups

Elements in $\ker \partial_n$ are called relative n-cycles. These are elements $\alpha \in C_n(X)$ such that $\partial_n \alpha \in C_{n-1}(A)$.



Likewise elements in $\operatorname{im} \partial_{n+1}$ are called relative n-boundaries. This means that $\alpha = \partial\beta + \gamma$ where $\beta \in C_n(X)$ and $\gamma \in C_{n-1}(A)$.



Theorem .0.1 (LES of a pair)
Let $A \subseteq X$ be spaces, then there exists a long exact sequence

$$\longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow H_n(X, A)$$

$$\longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow H_0(X, A) \longrightarrow 0$$

Later: We will prove that when (X, A) is a good pair, then $H_n(X, A) = \widetilde{H}_n(X/A)$. Then **??** is a special case of Theorem .0.1. The key to the proof of Theorem .0.1 above is the following slogan.

Remark .0.1
Slogan A short exact sequence of chain complexes gives rise to a long exact sequence of homology groups. This will be proved on homework. Then Theorem .0.1 will follow from a short exact sequence:

$$0 \longrightarrow \widetilde{C}_*(A) \longrightarrow \widetilde{C}_*(X) \longrightarrow C_*(X, A) \longrightarrow 0$$

where $\widetilde{C}_*$ denotes the augmented chain complex (the one with $\mathbb{Z}$ after it).

Exercise .0.2
If A is a single point in X, then $H_n(X, A) = \widetilde{H}_n(X/A) = \widetilde{H}_n(X)$.