Resume Breakout Rooms

Last time we were working on the following problem:

Exercise .0.1
How could you prove that two maps are not homotopic

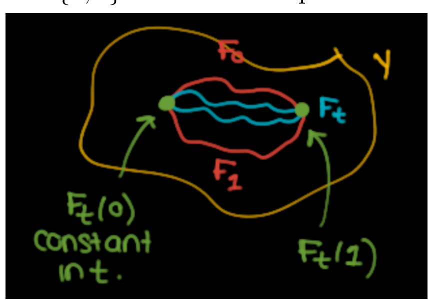
Solution. We had a few ideas for the example maps from last time:

- We can embed S^1 in a larger space like $\mathbb{R}^2$ and show that any homotopy between the two maps will land outside the circle at some point
- We can use the fact that one map is orientation-preserving and one map is orientation-reversing and show that homotopic maps will have the same behavior on orientation.
- Show that any homotopy in S^1 will become a bad homotopy in $\mathbb{R}$ when we lift it back by using the fact that $S^1 = \mathbb{R}/\mathbb{Z}$.

Resume Lecture

Definition .0.1
Let X, Y be spaces and let $B \subseteq X$ be a subspace. A homotopy $F_t(x) : X \times [0,1] \to Y$ is called a homotopy relative to B ("rel B") if $F_t(b)$ is independent of t for every $b \in B$.

Example .0.2
Homotopies of paths $[0,1] \to Y \operatorname{rel} \{0,1\}$. Here's a nice picture courtesy of Jenny!



.1. Homotopy Equivalence

Definition .1.1
A map $f : X \to Y$ is a homotopy equivalence if there exists a $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. You can say these are inverses "up to homotopy"

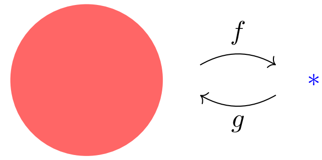
X and Y are called homotopy equivalent and we say they have the same homotopy type provided that there exists a homotopy equivalence between them. We write $X \simeq Y$.

Exercise .1.1 (Homework)
You will prove that "homotopy equivalent" is an equivalence relation

Example .1.2
If we look at D^n (the closed n-ball), then we see that $D^n \simeq *$, aka it is homotopy eqivalent to a point.

The map $f : D^n \to *$ is trivial, and we can choose any map $g : * \to D^n$, but for simplicitly we'll choose the map $g : * \mapsto 0$. Then $f \circ g = \mathrm{Id}_*$ so that's easy. On the other side $g \circ f : D^n \to D^n$ is the constant map which maps the entire disk to the origin.

Note that by using the straight line homotopy we can see that $g \circ f \simeq \mathrm{Id}_{D^n}$, and so D^n is homotopic to a point. Here's a picture



Definition .1.2

A space X is contractible if it is homotopy equivalent to a point.

Example .1.3

$\mathbb{R}^n \simeq *$, using a proof similar to the above.

Take-aways:

- $\mathbb{R}^n \simeq \mathbb{R}^m \simeq *$ for all n, m
- Homotopy equivalence does not preserve dimension
- It does not preserve compactness since $D^2 \simeq * \simeq \mathbb{R}^2$

Example .1.4

The inclusion $S^1 \hookrightarrow D^2$ is not a homotopy equivalence. Right now this would be fairly hard to prove.

Definition .1.3

Given a space X and a subspace $B \subseteq X$. A (strong) deformation retraction F_t is a homotopy rel B to a map with image in B from Id_X.

In plain terms:

- $F_0(x) = x$ for all $x \in X$
- $F_1(x) \in B$ for all $x \in X$
- $F_t(b) = b$ for all $b \in B$ and $t \in [0, 1]$.

Exercise .1.5

When X deformation retracts to a subspace B, X is homotopy equivalent to the subspact B.