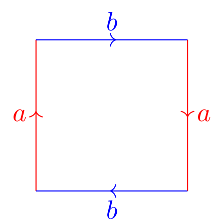
Announcements

- Midterm II on Thursday.
- Extra Office Hours Wendesday: 7-9pm ET
- Student study group: Tuesday 4pm-6pm ET
- Review package posted
- HW #8 warm-up + Problem 1 is exam-relevant

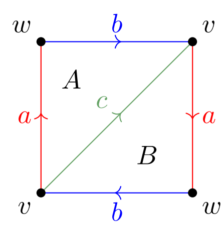
.1. Computing Simplicial Homology

Example .1.1
$X = \mathbb{R}\mathrm{P}^2$. Goal: Compute simiplical homology groups.

Consider the fundamental polygon given below for $\mathbb{R}\mathrm{P}^2$:



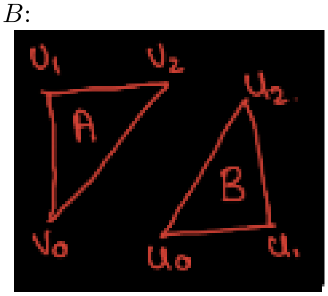
Now make this into a Δ-complex structure as below:



So now we have three nonzero chain groups, where we use $\langle f_1, \ldots, f_n \rangle$ to denote the free abelian group on n generators:

$$C_0(\mathbb{R}\mathrm{P}^2) = \langle v, w \rangle$$
$$C_1(\mathbb{R}\mathrm{P}^2) = \langle a, b, c \rangle$$
$$C_2(\mathbb{R}\mathrm{P}^2) = \langle A, B \rangle$$

We must choose orientations on A and B:



Let $A = [v_0, v_1, v_2]$ and take $B = [v_0, v_1, v_2]$ and so then:

$$\partial_2 A = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = b - c + a$$
$$\partial_2 B = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = -a - c - b$$

Now for the boundaries on our edges:

$$\partial_1 a = w - v \qquad \partial_1 b = v - w \qquad \partial_1 c = v - v = 0$$

Now that we've computed this we want to look at our chain complex:

$$0 \xrightarrow{\partial_3} C_2(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_2} C_1(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_1} C_0(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_0} 0$$

We know the following images and kernels:

$$\begin{aligned} \operatorname{im}\partial_3 &= 0 \\ \operatorname{im}\partial_2 &= \langle b-c+a, -a-b-c\rangle \\ \operatorname{im}\partial_1 &= \langle v-w\rangle \\ \ker\partial_0 &= C_0 = \langle v, w\rangle \end{aligned}$$

Now to compute $\ker\partial_2$ we see that:

$$\begin{aligned} \partial_2(mA+nB) &= 0 \\ m(b-c+a)+n(-a-b-c) &= 0 \\ a(m-n)+b(m-n)-c(m+n) &= 0 \end{aligned}$$

And so $m-n=0$ and $m+n=0$. This means that we need to have $m=n=0$, and so:

$$\ker\partial_2 = 0$$

We can also check what the kernel of ∂_1 is as below:

$$\begin{aligned} \partial_1(\alpha a+\beta b+\gamma c) &= 0 \\ \alpha(w-v)+\beta(v-w) &= 0 \\ (\beta-\alpha)v+(\alpha-\beta)w &= 0 \end{aligned}$$

And so to have this we need to have $\alpha-\beta=\beta-\alpha=0$, this happens when $\alpha=\beta$ and we have no conditions on γ, and therefore:

$$\ker\partial_1 = \langle c, a+b\rangle$$

Now all we need to do is take the quotients to get the homology groups.

$$\begin{aligned} H_2(\mathbb{R}\mathrm{P}^2) &= \frac{\ker\partial_2}{\operatorname{im}\partial_3} = \frac{0}{0} = 0 \\ H_1(\mathbb{R}\mathrm{P}^2) &= \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle c, a+b\rangle}{\langle a+b-c, -a-b-c\rangle} \\ H_0(\mathbb{R}\mathrm{P}^2) &= \frac{\ker\partial_0}{\operatorname{im}\partial_1} = \frac{\langle v, w\rangle}{\langle v-w\rangle} \cong \mathbb{Z} \end{aligned}$$

Lets think about how to compute the quotient in $H_1(\mathbb{R}\mathrm{P}^2)$. We can use row operations from linear algebra (ways to change from one basis to another) to get that:

$$H_1(\mathbb{R}\mathrm{P}^2) = \frac{\langle c, a+b\rangle}{\langle a+b-c, -a-b-c\rangle} = \frac{\langle c, a+b-c\rangle}{\langle a+b-c, -2c\rangle} \cong \mathbb{Z}/2\mathbb{Z}$$

Key: Given a basis for a free abelian group $\langle b_1, \ldots, b_n\rangle$ we can replace b_i with

$$b_i \pm m_1 b_1 \pm \cdots \pm \widehat{m_i b_i} \pm \cdots \pm m_n b_n$$

Exercise .1.2

If b_1, b_2 is a basis for $A \subseteq \mathbb{Z}^n$, then $b_1 - b_2, b_1 + b_2$ is not a basis, it is an index-2 subgroup. The key to this is that $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ has determinant 2 (not unit in $\mathbb{Z}$).

Principle

We can transform a basis for a free group into a different basis by applying a matrix of determinant ± 1. If we apply a matrix of determinant D we will obtain generators for a subgroup of index $|D|$..

$$\begin{bmatrix}
1 & 0 & \cdots & 0 & \pm m_1 & 0 & \cdots & 0 \\
0 & 1 & \cdots 0 & \pm m_2 & 0 & \cdots & 0 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \pm m_{i-1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \pm m_{i+1} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \pm m_n & 0 & \cdots & 1
\end{bmatrix}$$

Summary of Procedure

(1) Choose Δ-complex structure on X. (Later: We will prove $H_*(X)$ is independent of the choice of Δ-complex structure)
(2) Choose orientations on each simplex (Any choice is okay but you must commit to a choice or you will make a sign error!)
(3) For each n-simplex σ compute $\partial_n(\sigma)$ (careful with signs!)
(4) $\operatorname{im}\partial_n = \langle \partial_n(\sigma) \mid \sigma \text{ an } n\text{-simplex}\rangle$. Use linear algebra to compute $\ker(\partial_n)$
(5) For each n compute $H_n(X) = \frac{\ker \partial_n}{\operatorname{im}\partial_{n+1}}$. Be careful that any change-of-variables map you apply is invertible over $\mathbb{Z}$.