Goal: Show that every rational # in the character table is an integer.

Definition .0.1

An algebraic integer is a complex number α which is a root of a monic polynomial in $\mathbb{Z}[x]$.

Lemma .0.1

The algebraic integers in $\mathbb{Q}$ are precisely $\mathbb{Z}$.

Proof. Suppose $\alpha \in \mathbb{Q}$ is a root of $x^n + c_1 x^{n-1} + \cdots + c_n$ with $c_i \in \mathbb{Z}$.

Write $\alpha = a/b$ for a, b coprime integers. Then we see that

$$0 = \frac{a^n}{b^n} + c_1 \frac{a^{n-1}}{b^{n-1}} + \cdots + c_n. \qquad = \frac{a^n + b\,(\text{some integer})}{b^n}$$
$$= \frac{(\text{integer coprime to b})}{b^n}.$$

But 0 is coprime to b if and only if $b = \pm 1$. Thus $\alpha \in \mathbb{Z}$

The other direction is trivial, if $z \in \mathbb{Z}$ consider the polynomial $x - z$.

Proposition .0.2

If α_1, α_2 are algebraic integers then $\alpha_1 + \alpha_2$ and $\alpha_1 \alpha_2$ are algebraic integers.

Lemma .0.3

For $\alpha \in \mathbb{C}$, α is an algebraic integer if and only if α is an eigenvalue of a square integer matrix.

Proof. If α is an eigenvalue of $A \in M_{n\times n}(\mathbb{Z})$ then α is a root of the characteristic polynomial of A, i.e., of $\det(x \operatorname{Id}_n - A)$, which is a monic polynomial in $\mathbb{Z}[x]$ with integer coefficients (since the determinant is a polynomial in the entries).

Conversely, let α be an algebraic integer, say α isa root of $x^n + c_1 x^{n-1} + \cdots + c_n$ with $c_i \in \mathbb{Z}$.

This polynomial is the characteristic polynomial of

$$\begin{bmatrix} 0 & & & \cdots & 0 & -c_n \\ 1 & 0 & & \cdots & 0 & -c_{n-1} \\ & 1 & 0 & \cdots & 0 & -c_{n-2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & -c_{n-1} \\ & & & & 1 & -c_n \end{bmatrix}$$

so α is an eigenvalue of this matrix.

Proof of Proposition .0.2. Let α_1, α_2 be eigenvalues of $A_1 \in M_{m\times m}(\mathbb{Z})$ and $A_2 \in M_{n\times n}(\mathbb{Z})$ respectively.

We can then consider $A_1 \otimes A_2$, defined by the following block form

$$A_1 \otimes A_2 = \begin{bmatrix} a_{11}A_2 & a_{12}A_2 & \cdots & a_{1m}A_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}A_2 & a_{m2}A_2 & \cdots & a_{mm}A_2 \end{bmatrix}$$

where $A_1 = (a_{ij})$. If $A_i \vec{v}_i = \alpha_i \vec{v}_i$ then write $\vec{v}_1 = (d_1, \ldots, d_m)^T$ then write $\vec{w} = (d_1 \vec{v}_2, \ldots, d_m \vec{v}_m)^T$.

Then of course $(A_1 \otimes A_2)\vec{w} = \alpha_1 \alpha_2 \vec{w}$ by explicit computation.

$\alpha_1 + \alpha_2$ is an eigenvalue of

$$(A_1 \otimes \mathrm{Id}_n) + (\mathrm{Id}_m \otimes A_2).$$

This can be computed explicitly, or via the tensor properties

$$\begin{aligned} ((A_1 \otimes \mathrm{Id}_n) + (\mathrm{Id}_m \otimes A_2))(v_1 \otimes v_2) &= A_1 v_1 \otimes v_2 + v_1 \otimes A_2 v_2 \\ &= \alpha_1 (v_1 \otimes v_2) + \alpha_2 (v_1 \otimes v_2) \\ &= (\alpha_1 + \alpha_2)(v_1 \otimes v_2). \end{aligned}$$

Fact: If χ is the character of an n-dimensional representation of C_k, then

$$\sum_{g = \text{generator of } C_k} |\chi(g)|^2 \geq \# \text{ of generatos of } C_k = \varphi(k).$$

where $\varphi(k)$ is the number of integers less than k which are coprime to k. UNLESS $\chi(g) = 0$ for all generators g of C_k.