Announcements

- Bonus problem added to Homework 12 on orientability and homology
- Quiz on Wednesday
  - Calculation using the Mayer-Vietoris Long Exact Sequence

I. Lefschetz Fixed Point Theorem

I.1. Statement

Definition I.1.1

Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ be a group homomorphism, we may represent this with a matrix $A = \{a_{ij}\}$. The trace is the sum $a_{11} + \cdots + a_{nn}$.

For a group homomorphism $\varphi : M \to M$ where M is a finitely generated abelian group, we define the trace of φ to be the trace of the induced map $\overline{\varphi} : M/M_T \to M/M_T$, where M_T is the torsion subgroup of M.

Exercise I.1.1

We have

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

Thus, matrices related by a change of basis have the same trace.

Definition I.1.2

Let X be a space with the assumption that $\oplus_k H_k(X)$ is finitely generated. That is, each homology group is finitely generated, and there are finitely many nonzero homology groups. For example X could be a finite CW complex.

The Lefschetz number $\tau(f)$ of a map $f : X \to X$ is:

$$\tau(f) := \sum_k (-1)^k \operatorname{tr}(f_* : H_k(X) \to H_k(X))$$

Example I.1.2

When $f \simeq \mathrm{Id}_X$. Then $f_* = \mathrm{Id}_{H_k(X)}$ for all k. Then $\operatorname{tr}(f_* : H_k(X) \to H_k(X)) = \operatorname{rank}(H_k(X))$. Therefore:

$$\tau(f) = \sum_k \operatorname{rank}(H_k(X)) = \chi(X)$$

Where $\chi(X)$ is the Euler characteristic (see homework).

Theorem I.1.1 (Lefschetz Fixed Point Theorem)

Suppose X admits a finite triangulation (i.e. a finite simplicial complex structure). Or more generally, X is a retract of a finite simplicial complex.

Then if $f : X \to X$ is a map with $\tau(f) \neq 0$, then f has a fixed point. Note that the converse does not hold.

Theorem I.1.2 (Hatcher's Appendix A.7)

If X is a compact, locally contractible space that can embed in $\mathbb{R}^n$ for some n, then X is a retract of a finite simplicial complex.

This includes:

- Compact Manifolds
- Finite CW complexes

Definition I.1.3

Let $\mathbb{F}$ be a field, and let $H_k(X; \mathbb{F})$ be the k-th homology of X with coefficients in $\mathbb{F}$ (see homework).

THen $H_k(X;\mathbb{F})$ is always a vector space over $\mathbb{F}$. Define $\tau^{\mathbb{F}}(X)$ be:

$$\sum_k (-1)^k \operatorname{tr}(f_* : H_k(X;\mathbb{F}) \to H_k(X;\mathbb{F}))$$

The Lefschetz fixed point theorem still holds if we replace "$\tau(x) \neq 0$" with "$\tau^{\mathbb{F}} \neq 0$"

Example I.1.3

Let $f : S^n \to S^n$ be a degree d map. Then $\tau(f)$ is:

$$(-1)^0 \operatorname{tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \operatorname{tr}(f_* : H_n(S^n) \to H_n(S^n))$$

Then $f_* : H_0(S^n) \to H_0(S^n)$ is the identity, and $f_* : H_n(S^n) \to H_n(S^n)$ is given by the 1×1 matrix with entry d. And then we have:

$$\tau(f) = 1 + (-1)^n d$$

Corollary I.1.3

f has a fixed point whenever $1 + (-1)^n \neq 0$. Aka whenever $d \neq (-1)^{n+1}$. THat is f has a fixed point if its degree is not equal to the degree of the antipodal map.

Exercise I.1.4

If $f : X \to X$, then $\operatorname{tr}(f_* : H_0(X) \to H_0(X))$ is equal to the # of path-components of X mapped to themselves

Exercise I.1.5

If X is contractible, then its homology is concentrated in degree zero, so $\tau(f) = 1$.

If X is a compact manifold or finite CW-complex, every f has a fixed point (in particular, this recovers Brouwer's Fixed Point Theorem)

Example I.1.6

If we consider the map $f : \mathbb{R} \to \mathbb{R}$ given by translation by $x \neq 0$, then $\tau(f) = 1$, but f does not have a fixed point. The key here is that $\mathbb{R}$ is not compact.