I. **Review for Midterm II**

If you have a finite group G and any inner product $\langle -, - \rangle_{\text{bad}}$ you can upgrade it to a G-invariant inner product via

$$\langle v, w \rangle \coloneqq \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle_{\text{bad}}$$

The big theorems for finite-dimensional representations of finite groups over $\mathbb{C}$

(1) Every (finite-dimensional complex) representation of a finite group is the direct sum of irreducible subrepresentations [**thm:maschke-exist**].

(2) If $V = V_1 \oplus \cdots \oplus V_k = W_1 \oplus \cdots \oplus W_\ell$ (internal direct sums) where V is a G-representation, and V_i's, W_j's are irreducible then $k = \ell$ and after relabeling the W_i's we can make $V_i \cong W_i$ for all i. Furthermore, for any irreducible representation ψ of G,

$$\bigoplus_{V_i \cong \psi} V_i = \bigoplus_{W_j \cong \psi} W_j$$

See [**thm:maschke-unique**]. This is called the ψ-isotypic part of V.

(3) Given a representation $\rho : G \to \mathrm{GL}(V), \rho' : G \to \mathrm{GL}(W)$, get representations of G acting on $V^*, \mathrm{Hom}(V, W), V \oplus W, V \otimes W$. This also has a nice action on characters
- $\chi_{V^*} = \overline{\chi_V}$
- $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \chi_W$
- $\chi_{V \otimes W} = \chi_V \chi_W$
- $\chi_{V \oplus W} = \chi_V + \chi_W$.

(4) Given a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$, the character is

$$\begin{aligned} \chi : G &\to \mathbb{C} \\ g &\mapsto \mathrm{tr}(\rho(g)). \end{aligned}$$

Properties of characters

- Two represention have the same character $\iff$ they're $\cong$.
- Characters are class functions, that is $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$. If C is a conjugacy class then we can unambiguously write $\chi(C) \coloneqq \chi(g)$ for any $g \in C$.
- The irreducible characters form an orthonormal basis for the space of class functions under the usual inner product on $\mathbb{C}^G$

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta(g)}$$

This means that every class function $f : G \to \mathbb{C}$ satisfies

$$f = \sum_{\substack{\text{irr char} \\ \chi}} (f, \chi) \chi$$

It also tells you that the # of irreducible characters equals the # of conjugacy classes of G.

- If ρ is a representation and ρ_{irr} is an irreducible representation then

$$\langle \chi_{\text{irr}}, \chi_\rho \rangle$$

is the number of copies of ρ_{irr} in the decomposition of ρ into irreducibles.

- We also have orthogonality of columns, that is given two distinct conjugacy classes C, C'

$$\frac{1}{|G|} \sum_{\substack{\text{irr char} \\ \chi}} \chi(C)\overline{\chi(C')} = 0$$

$$\frac{1}{|G|} \sum_{\substack{\text{irr char} \\ \chi}} \chi(C)\overline{\chi(C)} = \frac{1}{|C|}$$

Here is a proof of this fact

If C is a conjugacy class of h in G, then let

$$f_C : G \to \mathbb{C}$$

$$g \mapsto \begin{cases} 1 & \text{if } g \in C \\ 0 & \text{otherwise} \end{cases}$$

thus $f_C = \sum_{\text{irr } \chi} (f_C, \chi)\chi$, but we know (f_C, χ). Namely

$$(f_C, \chi) = \frac{1}{|G|} \sum_{g \in C} \overline{\chi(g)} = \frac{|C|\,\overline{\chi(C)}}{|G|}.$$

Therefore

$$f_C = \sum_{\text{irr } \chi} \frac{|C|}{|G|} \cdot \overline{\chi(C)} \cdot \chi.$$

Evaluating at some g we see that if $g \in C$ then

$$1 = \frac{|C|}{|G|} \cdot \sum_{\text{irr } \chi} \overline{\chi(g)}\chi(g)$$

and if $g \notin C$ then

$$0 = \frac{|C|}{|G|} \cdot \sum_{\text{irr } \chi} \overline{\chi(C)}\chi(g)$$

- If $\rho : G \to \mathrm{GL}(V)$ is a representation, where $n \coloneqq \dim V$, then $\chi_\rho(g)$ is a sum of n (order of g)-th roots of unity. This can be useful for finding the order of elements from a character table.
- If $\rho_1, \ldots, \rho_n$ are the irreducible representations, with $\chi_1, \ldots, \chi_n$ their characters, then

$$|G| = \sum_{i=1}^{n} (\dim \rho_i)^2 = \sum_{i=1}^{n} \chi_i(1).$$

- If ρ is an irreducible representation with character χ then

$$\ker \rho = \{g \in G \mid \chi(g) = \chi(1) = \dim \rho\}$$

is a normal subgroup of G. Furthermore, every normal subgroup is an intersection of subgroups of this form.

- Fact: If ρ is an irreducible representation of G then $\dim \rho \mid |G|$.

  Note: if $|G|$ is odd, then $\dim \rho$ is odd, so because an odd number squared is $1 \mod 8$, we have

$$|G| \equiv (\# \text{ conjugacy classes of G}) \mod 8$$

- If $\rho = \sum_{i=1}^{k} e_i \rho_i$ with ρ_i non-isomorphic irreducibles and $e_i \in \mathbb{Z}_{>0}$ then

$$(\chi_\rho, \chi_\rho) = \sum_{i=1}^{k} e_i^2$$

Information we should be able to recover from a character table.

- Orders of elements from a conjugacy class
- Normal subgroups as unions of conjugacy classes based on kernels of the irreducible representations (and then their intersections)
- Be able to fill in a partial character table

Some representations you should know

- Representations of C_n, D_5, S_3.
- Representations of group actions

$$\text{character of regular representation} = \sum_{\text{irr. } \chi} (\dim \chi) \chi$$