Recall: A linear representation of a group G is a homomorphism $\rho : G \to \mathrm{GL}(V)$ for some vector space V. We say ρ is irreducible if V has no subrepresentations except $\{0\}$ and V, where a subrepresentation is a subspace V of V such that $g \cdot W \subseteq W$ for all $g \in G$ (so that ρ induces a homomorphism $G \to \mathrm{GL}(W)$).

1-dimensional representations have the form $\rho : G \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$. But if G is finite then $\rho(G)$ is a finite subgroup of $\mathbb{C}^\times$, hence is cyclic ($|G|$-th roots of unity). So ρ is a homomorphism from G to a cyclic group.

Theorem .0.1 (Maschke's Theorem)

Every finite-dimensional complex representation of a finite group G can be written as a direct sum of irreducible subrepresentations.

That is: given $\rho : G \to \mathrm{GL}(V)$ we can write $V = W_1 \oplus \cdots \oplus W_k$ with W_i subspaces of V such that each (ρ, W_i) is an irreducible subrepresentation of (ρ, V).

This follows from the following by induction

Theorem .0.2

If $\rho : G \to \mathrm{GL}(V)$ is a finite-dimensional complex representation of a finite group G and W is a subrepresentation, then there is some subrepresentation W' of V such that $V = W \oplus W'$.

Remark .0.1

Same proof works over any field K such that $|G|$ is invertible in K.

Proof. Pick any "projection map" $\pi : V \to W$, meaning a linear transformation $V \to W$ which restricts to the identity map on W. This can be done by extending a basis of W to a basis on V, defining π to be the identity on the basis of W and anything in W on the other basis elements for V.

We want to be able to take the kernel of π, but this won't work because π is not a G-invariant map. We have to somehow "fix" π.

Define

$$\begin{aligned} \phi : V &\to W \\ v &\mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v). \end{aligned}$$

This should fix our problem

Claim

ϕ is a G-invariant projection map $V \to W$

Fix $w \in W$. Then $g^{-1} \cdot w \in W$ and we have:

$$\phi(w) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot w) = \frac{1}{|G|} \sum_{g \in G} g \cdot g^{-1} \cdot w = \frac{1}{|G|} \sum_{g \in G} w = w$$

It clearly maps into W. It is also linear since it is a linear combination of the linear transformations $v \mapsto g \cdot \pi(g^{-1} \cdot v)$.

We now check that ϕ is G-invariant. Let $h \in G$ and $v \in V$, then

$$h \cdot \phi(v) = \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot \pi(g^{-1} \cdot v))$$

$$= \frac{1}{|G|} \sum_{g' \in G} g' \cdot \pi((g')^{-1} h \cdot v)$$
$$= \phi(h \cdot v)$$

where we've made the subsitution $g' = hg$ (since $g \mapsto hg$ is a bijection $G \to G$).

This proves the claim. Now we need to show that $W' := \ker \phi$ satisfies the desired properties.

W' will clearly be a subrepresentation of V because ϕ is G-invariant. Then because ϕ is a projection map, $V = W \oplus W'$. Why? Well $v \in V$ has the form $\phi(v) + (v - \phi(v))$, $\phi(v) \in W$, and $v - \phi(v) \in W'$. This is a unique decomposition, as the intersection of W and W' is zero.

Great! This finishes the proof!

Theorem .0.3

If V is a finite-dimensional complex representation of a finite group G, then V can be written in exactly one way as an (internal) direct sum

$$V = V_1 \oplus \cdots \oplus V_k$$

where each V_i is itself a direct sum of (one or more) copies of an irreducible subrepresentation W_i and $W_i \ncong W_J$ for $i \neq j$.

This is a sort of generalization of eigenspaces. Said another way (more explicitly) if we write $V = U_1 \oplus \cdots \oplus U_\ell$ and $V = R_1 \oplus \cdots \oplus R_m$ with U_i, R_j irreducible subrepresentations, then they have the same length, for each i the number of U_j's isomorphic to U_i equals the number of R_j's isomorphic to U_i, and the direct sum of these U_j equals (not just isomorphic) the direct sum of these R_j.

Lemma .0.4

A homomorphism $\phi : V \to W$ between irreducible G-representations ie either zero or an isomorphism.

Proof. $\ker \phi$ is a subrepresentation of V. Thus $\ker \phi = 0$ or $\ker \phi = V$. If $\ker \phi = V$ then we're done.

$\operatorname{im} \phi$ is a subrepresentation of W. Thus $\operatorname{im} \phi = 0$ or $\operatorname{im} \phi = W$. If $\operatorname{im} \phi = 0$ we're done.

But if $\ker \phi = 0$ and $\operatorname{im} \phi = W$ then the function is bijective, and we're done.

Proof of Theorem .0.3. Now say $V = U_1 \oplus \cdots \oplus U_\ell = R_1 \oplus \cdots \oplus R_m$ with U_i, R_j irreducible subrepresetations of V.

Consider $U_i \hookrightarrow V \twoheadrightarrow R_j$ as inclusion then projection. This is a homomorphism of irreducible G-representations, and so it is either zero or an isomorphism by the lemma. However it can't be zero for all j, because $U_i \neq 0$ and $V = \bigoplus R_j$.

Thus there is some j such that $U_i \hookrightarrow V \twoheadrightarrow R_j$ is an isomorphism of G-representations. We get that the set of U_i's, up to $\cong$, equals the set of R_j's, up to $\cong$ (go the other way as well $R_j \to U_i$).

We may then write $V = U_1^{a_1} \oplus \cdots U_k^{a_k}$ and $V = R_1^{b_1} \oplus \cdots \oplus R_k^{b_k}$ where $a_i, b_i > 0$, $U_i \cong R_i$ irreducible, $U_i \ncong U_j$ for $i \neq j$.

Then consider that $U_1^{a_1} \hookrightarrow V \twoheadrightarrow R_2^{b_2} \oplus \cdots \oplus R_k^{b_k}$ is zero by the lemma. This shows $U_1^{a_1} \subseteq R_1^{b_1}$. Similarly $R_1^{b_1} \subseteq U_1^{a_1}$. Comparing dimensions gives $a_1 = b_1$. Can do similarly for the rest.