Announcements

- Midterm Thursday 7pm ET
- Study Groups
  - 7pm tonight ET
  - 6pm Tuesday ET
- Midterm review package posted on webpage under "Exams"
- Midterm
  - You are responsible for material up to / including today
  - You are responsible for material on homework.
  - Two questions, many parts
  - True and Counterexample Questions
  - Spaces and presentations for the fundamental groups.

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> **Exercise .0.1**
> Consider $G_1 = \langle S_1 \mid R_1\rangle$ and $G_2 = \langle S_2 \mid R_2\rangle$. Then we have
> - $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2\rangle$
> - $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\}\rangle$
> - $G_1 *_H G_2$ where $f_1 : H \to G_1$ and $f_2 : H \to G_2$. Then we have
>
> $$G_1 *_H G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{f_1(h)f_2(h)^{-1} \mid h \in H\}\rangle$$
>
> This is super useful!

.1. Presentations for π_1 of CW Complexes

Outline: For X a CW complex:

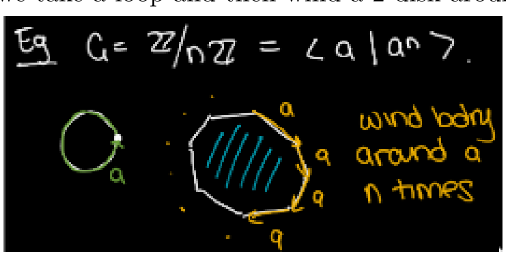
(a) A 1-dimensional CW complex has free π_1 (call its generators $a_1, \ldots, a_n$)

(b) Gluing a 2-disk by its boundary along a word w in the generators "kills" w in π_1
We then get a presentation for $\pi_1(X^2)$ given by:

$$\langle a_1, \ldots, a_n \mid w \text{ for each 2-cell in } X^2\rangle$$

(c) Gluing in any higher dimensional cells along their boundary will not change π_1. That is in a CW complex we have $\pi_1(X) = \pi_1(X^2)$.

> **Example .1.1**
> $G = \mathbb{Z}/n\mathbb{Z} = \langle a, a^n\rangle$, then we take a loop and then wind a 2-disk around the loop a n times.



Consequence: Given a group G with presentation $\langle S \mid R\rangle$ one can construct a 2-dimensional CW complex with $\pi_1 = G$:

- Set $X^1 = \bigvee_{s\in S} S^1$
- For each relation $r \in R$ glue in a 2-disk along loops specified by the word r.

Every group is then π_1 of some space.
This theorem will give us part c)

> **Theorem .1.1** (From Homework)
> If X is a CW complex and $\iota_1 : X^1 \hookrightarrow X$ and $\iota_2 : X^2 \hookrightarrow X$, then $(i_1)_*$ surjects onto π_1 and $(i_2)_*$ is an iso on π_1.

Definition .1.1

We import some topological definitions of graph theoeretic concepts:

- A graph is a 1-dimensional CW complex.
- A subgraph is a subcomplex
- A tree is a contractible graph.
- A tree in a graph X (necessarily a subgraph) is maximal or spanning if it contains all the vertices.

Theorem .1.2

Every connected graph has a maximal tree. Every tree is contained in a maximal tree.

Corollary .1.3

Suppose X is a connected graph with basepoint x_0. Then $\pi_1(X, x_0)$ is a free group.

Furthermore, we can give a presentation for $\pi_1(X, x_0)$ by finding a spanning tree T in X. The generators of π_1 will be indexed by cells $e_\alpha \in X - T$.

e_α will correspond to a loop that passes through T, traverses e_α once, then returns to the basepoint x_0 through T.

Idea: X is homotopy equivalent to X/T. via previous work on the homework. T contains all the vertices, so the quotient has a single vertex. Thus it is a wedge of circles, and each e_α projects to a loop in X/T. Here's a picture illustrating the process: