Announcements

- Midterm Exams graded
- Correction to Homework Problem # 1
- Quiz Wednesday
  - Know definition of singular homology

Application of the quotient Long Exact Sequence.

Proposition .0.1
We have that:

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Proof. Some facts we need:

- $(D^n, \partial D^n)$ is a good pair (since it is a CW complex and a subcomplex)
- $D^n/\partial D^n \cong S^n$ (previous homework)
- $\widetilde{H}_n(D^n) = 0$ for all n since D^n is contractible
- $\partial D^n \cong S^{n-1}$

We then proceed by induction on n.

Exercise .0.1
Verify the theorem in the case $n = 0$, so $S^0 = 2$ points.

Now using the long exact sequence, we have:

$$\cdots \longrightarrow \widetilde{H}_n(\partial D^n) \xrightarrow{i_*} \widetilde{H}_n(D^n) \xrightarrow{j_*} \widetilde{H}_n(S^n)$$

$$\xrightarrow{\delta} \widetilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{j_*} \widetilde{H}_{n-1}(S^n)$$

$$\xrightarrow{\delta} \cdots \xrightarrow{j_*} \widetilde{H}_0(S^n) \longrightarrow 0$$

By induction we can fill in some of these groups as follows:

$$\cdots \longrightarrow 0 \xrightarrow{i_*} 0 \xrightarrow{j_*} \widetilde{H}_n(S^n)$$

$$\xrightarrow{\delta} \mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{j_*} \widetilde{H}_{n-1}(S^n)$$

$$\xrightarrow{\delta} \cdots \xrightarrow{j_*} \widetilde{H}_0(S^n) \longrightarrow 0$$

Summary: We have an exact sequence:

$$0 \longrightarrow \widetilde{H}_n(S^n) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0$$

By exactness, δ is an isomorphism, thus $\widetilde{H}_n(S^n) \cong \mathbb{Z}$. Now we must verify $\widetilde{H}_i(S^n) = 0$ when $i \neq n$. In that case the exact sequence looks like:

$$\longrightarrow \widetilde{H}_i(D^n) \longrightarrow \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_{i-1}(\partial D^n)$$

$$\longrightarrow 0 \longrightarrow \widetilde{H}_i(S^n) \longrightarrow 0$$

Exactness then tells us that $\widetilde{H}_i(S^n) = 0$.

Theorem .0.2 (Brouwer's Fixed Point Theorem)
∂D^n is not a retract of D^n. Hence every continuous map $f : D^n \to D^n$ has a fixed point.

Proof. If $r : D^n \to \partial D^n$ were a retraction, then by definition this would give us that:

$$\begin{tikzcd} \partial D^n \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{Id}_{\partial D^n}"'] & D^n \arrow[r, "r"] & \partial D^n \end{tikzcd}$$

Functoriality of homology tells us that:

$$\begin{tikzcd} \widetilde{H}_{n-1}(\partial D^n) \arrow[r, "i_*"] \arrow[rr, bend right, "\mathrm{Id}"'] & \widetilde{H}_{n-1}(D^n) \arrow[r, "r_*"] & \widetilde{H}_{n-1}(\partial D^n) \end{tikzcd}$$

So then:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "\mathrm{Id}"'] & 0 \arrow[r, "r_*"] & 0 \end{tikzcd}$$

This is impossible.

Exercise .0.2
As with D^2, if $f : D^n \to D^n$ had no fixed point, we could build a retraction.

Tool for proving Theorem: diagram chase

Lemma .0.3 (The Short Five Lemma)
Suppose we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\psi'"'] & B' \arrow[r, "\varphi'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

So that the rows are exact. Then:

(1) If α, γ are injective then β is injective.
(2) If α, γ are surjective then β is surjective.
(3) If α, γ are isomorphisms then β is an isomorphism

Proof. (1) and (2) imply (3). We leave (2) as an exercise. We fix $b \in B$ such that $\beta(b) = 0$. We want to show that $\beta = 0$. Well, we draw a diagram chase:

$$\begin{tikzcd} 0 \arrow[r, mapsto] & \bullet \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\beta"] & \varphi(b) \arrow[r, mapsto] \arrow[d, mapsto, "\gamma"] & 0 \\ 0 \arrow[r, mapsto] & \bullet \arrow[r, mapsto, "\psi'"'] & 0 \arrow[r, mapsto, "\varphi'"'] & 0 \arrow[r, mapsto] & 0 \end{tikzcd}$$

And thus by injectivity of γ we know $\varphi(b) = 0$. By exactness, $b \in \operatorname{im} \psi$. We then may write for some $a \in A$:

$$\begin{tikzcd} 0 \arrow[r, mapsto] & a \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\beta"] & 0 \arrow[r, mapsto] \arrow[d, mapsto, "\gamma"] & 0 \\ 0 \arrow[r, mapsto] & \alpha(a) \arrow[r, mapsto, "\psi'"'] & 0 \arrow[r, mapsto, "\varphi'"'] & 0 \arrow[r, mapsto] & 0 \end{tikzcd}$$

Therefore $\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) = 0$ by commutativity. By exactness of the bottom row we know that ψ' is an injection. Thus $\alpha(a) = 0$, so since α is injective, $a = 0$. With this $b = \psi(a) = \psi(0) = 0$. Great! With this $\ker(\beta) = 0$, and β injects. This ends the proof! ☺