Last time we defined degree (**??**). Now we list some of its properties

<u>Properties of Degree</u>:

(a) $\deg(\mathrm{Id}_{S^n}) = 1$ sinc $(\mathrm{Id}_{S_n})_* = \mathrm{Id}_{\mathbb{Z}}$.

(b) If $f : S^n \to S^n$ is not surjective, then $\deg(f) = 0$. To see this, we know that f_* factors as:

$$H_n(S^n) \longrightarrow H_n(S^n - \{*\}) = 0 \longrightarrow H_n(S^n)$$

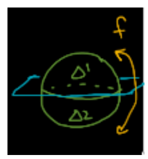
And since the middle group is zero, $f_* = 0$.

(c) If $f \simeq g$, then $f_* = g_*$, so $\deg(f) = \deg(g)$.
<u>Later</u>: The converse is true!

(d) $(f \circ g)_* = f_* \circ g_*$, and so $\deg(f \circ g) = \deg(f)\deg(g)$.
Consequently: If f is a homotopy equivalence then $\deg f = \pm 1$.

Exercise .0.1
It is possible to put a Δ-complex structure with 2 n-cells, Δ_1 and Δ_2 glued together along their boundary ($\cong S^{n-1}$), and $H_n(S^n) = \langle \Delta_1, \Delta_2 \rangle$.

(e) Consequences: If f is a reflection fixing the equator, and swapping the 2-cells, then $\deg f = -1$.



(f) We now have the following linear algebra exercise.

Exercise .0.2
The map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $x \mapsto -x$ is the composite of $(n+1)$ reflections.

So the antipodal map $S^n \to S^n$ given by $x \mapsto -x$ has degree which is the product of $n+1$ copies of (-1), and so it has degree $(-1)^{n+1}$.

(g) We again start with an exercise

Exercise .0.3
If f has no fixed points, then we can homotope f to the antipodal map via:

$$f_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$$

Therefore $\deg f = (-1)^{n+1}$.

Theorem .0.1 (Hairy Ball Theorem)
See the homework. This essentially says that there is no nonvanishing continuous tangent vector field on even-dimensional spheres.

Theorem .0.2 (Groups acting on S^{2n})
If G acts on S^{2n} freely, then $G = \mathbb{Z}/2\mathbb{Z}$ or $G = 1$

Corollary .0.3
S^{2n} is only the trivial cover $S^{2n} \to S^{2n}$ or degree 2 cover (for example, $S^{2n} \to \mathbb{RP}^{2n}$). This follows since any covering space action acts freely.

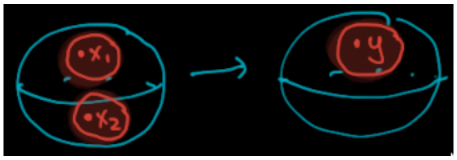
Proof. There exists a homomorphism given by:

$$\begin{aligned} G &\to \{\pm 1\} \\ g &\mapsto \deg(\tau_g) \end{aligned}$$

Where τ_g is the action of $g \in G$ on S^{2n} as a map $S^{2n} \to S^{2n}$. We know this map is well-defined since τ_g is invertible (simply take $\tau_{g^{-1}}$) for each $g \in G$. Our note on composites shows this is a homomorphism.

We want to show that the kernel is trivial, since then by the first isomorphism theorem $G \cong \text{im}$, and the image is either trivial or $\mathbb{Z}/2\mathbb{Z}$. Suppose that g is a nontrivial element of g, then since G acts freely we know that τ_g has no fixed points. With this in mind we have $\deg \tau_g = (-1)^{2n+1} = -1$. Thus $g \notin \ker$. Therefore the kernel is trivial as desired.

Definition .0.1
Let $f : S^n \to S^n$ $(n > 0)$. Suppose there exists $y \in S^n$ such that $f^{-1}(y)$ is finite, say, $\{x_1, \dots, x_m\}$. Then let $U_1, \dots, U_m$ be disjoint neighborhoods of $x_1, \dots, x_m$ that are mapped by f to some neighborhood V of y. In a picture



The local degree of f at x_i (denoted $\deg f\big|_{x_i}$) is the degree of the map

$$f_* : \mathbb{Z} \cong H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\}) \cong \mathbb{Z}$$

Theorem .0.4
Let $f : S^n \to S^n$ with $f^{-1}(y) = \{x_1, \dots, x_m\}$ as above, then:

$$\deg f = \sum_{i=1}^{m} \deg f\big|_{x_i}$$

Thus we can compute the degree of f by computing these degrees.