Announcements

- Homework #1 feedback is on gradescope
- Homework #2 due 8pm Friday
- Office hours 8pmm-9pm today

.0.1. Construction of free groups

Proposition .0.1
The free group defined via the universal property before exists. We will give a construction below.

Definition .0.1
Fix a set S. A word is a sequence (possibly empty) of formal symbols $\{s, s^{-1} \mid s \in S\}$.

Proof. Fix S, F_S is equivalence classes of words:

$$vss^{-1}w \sim vw$$
$$vs^{-1}sw \sim vw$$

For every words v, w. The group operation is concatenation of words.

Example .0.1
Given words ab^{-1}, bba their product is:

$$ab^{-1} \cdot bba = ab^{-1}bba = aba$$

Exercise .0.2
This product is well-defined on equivalence classes.

Exercise .0.3
Every equivalence class of words has a unique "reduced form."

Exercise .0.4
F_S satisfies the Universal Property with respect to the map:

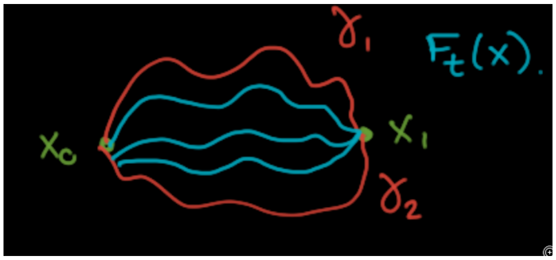
$$S \to F_S$$
$$s \mapsto s$$

I. The Fundamental Group π_1

I.1. Basic Definitions

Definition I.1.1
A path in a space X is a continuous map $\gamma : I \to X$
γ is a loop if $\gamma(0) = \gamma(1)$

Definition I.1.2
A homotopy of paths γ_1, γ_2 is a homotopy from γ_1 to γ_2 rel $\{0, 1\}$.



Definition I.1.3

For paths α, β with $\alpha(1) = \beta(0)$, the composition, product, or concatenation of these paths is defined as follows:

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le 1/2 \\ \beta(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

By the pasting lemma this is continuous. Thus $\alpha \cdot \beta$ is a path from $\alpha(0)$ to $\beta(1)$.

Definition I.1.4

A reparameterization of γ is a path:

$$I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

Such that $\varphi(0) = 0$ and $\varphi(1) = 1$, and $\varphi : I \to I$ is continuous

Exercise I.1.1 (Homework)

A path γ is homotopic rel $\{0, 1\}$ to all of its reparameterizations.

Exercise I.1.2

Fix $x_0, x_1 \in X$. "Homotopy of paths" (relative $\{0, 1\}$) is an equivalence relation on paths from x_0 to x_1.

Definition I.1.5

Let X be a space, and $x_0 \in X$. The fundamental group of X based at x_0 (denoted $\pi_1(X, x_0)$) is a group:

- Elements are homotopy classes rel $\{0, 1\}$ of loops γ with $\gamma(0) = \gamma(1) = x_0$ (we say γ is based at x_0)
- The operation is composition of paths
- The identity is the constant loop at x_0

- The inverse $[\gamma]^{-1}$ is represented by the loop $\overline{\gamma}(t) = \gamma(1-t)$:

The proof that this is a group is Homework

Exercise I.1.3
Composition of paths is well-defined on homotopy classes rel $\{0, 1\}$.

Theorem I.1.1 (Homework)
If X is path-connected then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for every $x_0, x_1 \in X$. So we can write $\pi_1(X)$ up to isomorphism.

Exercise I.1.4
If X is a contractible space, then X is path connected and $\pi_1(X)$ is trivial.