Note first that if G is an abelian group, then every subgroup of G is normal.

Definition .0.1

A **simple group** G is a nontrivial group whose only normal subgroups are G and 1.

Example .0.1

The only abelian simple groups are $\mathbb{Z}/p\mathbb{Z}$ for p a prime (isomorphic to cyclic of prime order).

Theorem .0.1 (Feit-Thompson)

If G is a simple group of odd order, then it is cyclic of prime order.

Proof. A 200 page book for a world-class expert in the subject.

Theorem .0.2

The finite simple groups are known, and understood very very well.

Proof. The experts in group theory classified all the simple groups around the 1960s resulting in an approximately 15,000 page book.

Fact: There is a surjective homomorphism $S_m \to S_n$ if and only if $m = n$ or $n = 1$ or $n = 2$ or ($m = 4$ and $n = 3$).

Definition .0.2

A map $f : G \to \widetilde{G}$ is an **isomorphism** provided that f is a bijective homomorphism.

In this case we say that $G \cong \widetilde{G}$, and that G and $\widetilde{G}$ are **isomorphic**

Lemma .0.3

Let $f : G \to \widetilde{G}$ is an isomorphism, then the inverse function $f^{-1} : \widetilde{G} \to G$ is an isomorphism.

Proof. Let $x = f(a)$ and $y = f(b)$. Then $f(ab) = f(a)f(b) = xy$. Thus

$$f^{-1}(xy) = ab = f^{-1}(x)f^{-1}(y)$$

Perfect!

Example .0.2

The Klein 4-group $V_4 = C_2 \times C_2 = \left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\right\}$.

In S_4, there is a normal subgroup $\{1, (12)(34), (13)(24), (14)(23)\}$. This is not cyclic since everything has order two, and so it is isomorphic to V_4.

There is another copy of V_4 in S_4, namely $\{1, (12), (34), (12)(34)\}$, and this is **not** normal.

This means we need to be careful about talking about isomorphisms between subgroups of a given group.

Definition .0.3

An **automorphism** of a group G is an isomorphism from G to itself.

The set of automorphisms form a group under composition, called $\mathrm{Aut}(G)$.

Definition .0.4

For $g \in G$, define a function

$$\begin{aligned}\varphi_g : G &\to G \\ h &\mapsto ghg^{-1}\end{aligned}$$

called **conjugation by** g. This is an automorphism, and these are called the **inner automorphisms**

Proof. Well we see that

$$(gh_1g^{-1})(gh_2g^{-1}) = gh_1g^{-1}gh_2g^{-1} = g(h_1h_2)g^{-1}$$

So this is a group homomorphism. Furthermore it has inverse $h \mapsto g^{-1}hg$, as

$$h \mapsto ghg^{-1} \mapsto g^{-1}ghg^{-1}g = h$$
$$h \mapsto g^{-1}hg \mapsto gg^{-1}hgg^{-1} = h$$

Perfect!

We can also note that

$$\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 g_2}$$

Which we can simply compute

$$h \mapsto g_2hg_2^{-1} \mapsto g_1g_2hg_2^{-1}g_1^{-1} = (g_1g_2)h(g_1g_2)^{-1}$$

Thus $g \mapsto \varphi_g$ is a homomorphism $G \to \mathrm{Aut}(G)$. We can ask what is the kernel of this homomorphism? Well

$$\ker = \{g \in G \mid \varphi_g = \mathrm{Id}_G\} = \{g \in G \mid \varphi_g(h) = h \ \ \forall h \in G\} = \{g \in G \mid gh = hg \ \ \forall h \in G\}$$

And this is a normal subgroup.

Definition .0.5

This kernel of the conjugation homomorphism $G \to \mathrm{Aut}(G)$ is called the **center** of the group G and is denoted $Z(G)$.

$$Z(G) := \{g \in G \mid gh = hg \ \ \forall h \in G\}.$$

This is of course a normal subgroup.

Lets think about conjugation in S_n. For example consider

$$(123)(12)(132) = (23)$$

More concretely we can write this compositon as

$$\begin{tikzcd}
1 & 3 \arrow[l] & 3 \arrow[l] & 1 \arrow[l] \\
3 & 2 \arrow[l] & 2 \arrow[dl] & 3 \arrow[l] \\
2 & 1 \arrow[l] & 1 \arrow[ul] & 2 \arrow[l]
\end{tikzcd}$$

In general

$$\sigma \circ \theta \circ \sigma^{-1}$$

is gotten from θ by applying σ to all elements in all cycles of θ, when written in cycle notation.

$$\sigma(i) \xmapsto{\sigma^{-1}} i \xmapsto{\theta} \theta(i) \xmapsto{\sigma} \sigma(\theta(i))$$

Proposition .0.4

Let G be a group and H be a subgroup, with G/H the set of all (left-)cosets gH for $g \in G$.

Then let $\eta_g : xH \mapsto gxH$ be a map, then

$$\begin{aligned} \eta : G &\to \mathrm{Sym}(G/H) = \{\text{permiutations of } G/H\} \\ g &\mapsto \eta_g \end{aligned}$$

is a homomorphism.

Proof. η_g is in $\mathrm{Sym}(G/H)$ because $\eta_{g^{-1}}$ is an inverse by an easy computation.

η is a homomorphism because

$$\eta_{g_1} \circ \eta_{g_2} : xH \xmapsto{\eta_{g_2}} g_2 xH \xmapsto{\eta_{g_1}} (g_1 g_2)xH = \eta_{g_1 g_2}(xH)$$

Perfect!

In case $H = 1$, this homomorphism has trivial kernel, and so its an injective homomorphism $G \hookrightarrow \mathrm{Sym}(G)$. It then induces an isomorphism $G \xrightarrow{\cong} \eta(G) \subseteq \mathrm{Sym}(G)$. This says that every group is isomorphic to a subgroup of permutations.