The character table of G determines

- Recover the position of 1 because $\sum_\chi \chi(1)^2$ is maximal among $\sum_\chi |\chi(g)|^2$, and $\{\chi(1)\}_\chi$ contains only integers.
- Size of the group, $|G| = \sum_\chi \chi(1)^2$
- Sizes of conjugacy classes of G $|G|/C(g) = \sum_\chi |\chi(g)|^2$.
- Sizes of the normal subgroups of G (and their intersections).

If $\rho : G \to \mathrm{GL}(V)$ is a representation, then

$$\ker(\rho) = \{g \in G \mid \rho(g) = \mathrm{Id}_V\} = \{g \in G \mid \chi(g) = \dim V\} = \{g \in G \mid \chi(g) = \chi(1)\}$$

where χ is a character of ρ by diagonalization. The $\supseteq$ inclusion follows from the fact that a sum of n roots of unity which is literally n implies that each root of unity is 1.

Further $\ker(\rho_1 \oplus \rho_2) = \ker \rho_1 \cap \ker \rho_2$. Thus the kernels of all representations come from intersecting kernels of irreducible characters (which we can read off the character tables).

For any normal subgroup $N \trianglelefteq G$, N is the kernel of the homomorphism $G \twoheadrightarrow G/N$. Let $\rho : G \to \mathrm{GL}(\mathbb{C}^{|G/N|})$ be the associated linear representation (by left multiplication). The kernel of this representation is N. Well

$$\begin{aligned} \ker \rho &= \{g \in G e_{ghN} = e_{hN} \ \forall h \in G\} \\ &= \{g \in G \mid ghN = hN \ \forall h \in G\} = N. \end{aligned}$$

By setting $h = 1$ for $\subseteq$, and by simple algebra for $\supseteq$.

For an example of a fabulous representation theory result

Theorem .0.1 (Gowers, Nikolav-Pyber)

Let G be a nontrivial finite group. Let r be the smallest dimension of a nontrivial irreducible representation of G.

For any subsets A, B, C of G such that $\frac{|A||B||C|}{|G|^3} > \frac{1}{r}$, then we have $G = ABC$ (as sets).

Corollary .0.2

For $G = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ for an odd prime p we have that $r = (p-1)/2$.

Thus for A a subset of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ with

$$\frac{|A|}{|\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})|} > \left(\frac{2}{p-1}\right)^{1/3}$$

Then for all $g \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ we have $a, b, c \in A$ with $g = abc$.

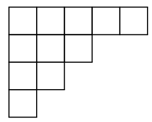
.1. Representations of S_n

Conjugacy classes of S_n are in bijection with partitions of n (i.e., expresions $n = \lambda_1 + \cdots + \lambda_k$, $\lambda_i \in \mathbb{Z}$, $\lambda_1 \leq \cdots \leq \lambda_k$) via a correspondence consisting of all $g \in S_n$ whose cycle lengths are $\lambda_1, \ldots, \lambda_n$.

Call the number of such partitons $p(n)$. A result of Ramanujan tells us that

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\frac{2\pi}{\sqrt{6}}\sqrt{n}} \text{ as } n \to \infty$$

Now: produce an irreducible representation of S_n from a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n. Given λ, first maake the "Young diagram," which looks like this for $\lambda = (1, 2, 3, 5)$.



Next make a λ-tableau by filling in each box with a # in $\{1, \ldots, n\}$ with no repetitions. So for all λ there are $n!$ such λ-tableaux.

Say two λ-tableaux are equivalent if, $\forall i$, the i-th row of one tableau is a permutation of the i-th row of the other tableau.

A λ-tabloid is an equivalence class of a λ-tableaux.

The # of λ-tabloids is

$$\binom{n}{\lambda_1, \ldots, \lambda_k} = \frac{n!}{\lambda_1! \cdots \lambda_k!}.$$

S_n acts on the λ-tableaux in the natural way, and this descends to an action on the set of λ-tabloids. This yields a linear representation of the above dimension. Consider the subrepresentation on the subspace generated by the following, where t is a λ-tableaux

$$e_t = \sum_{\substack{\sigma \in S_n \\ \sigma \text{ permutes the columns of } t}} = \operatorname{sgn}(\sigma)(\sigma \cdot t).$$

Theorem .1.1

This is an irreducible representation, and these are all of the irreducible representaitons of S_n.