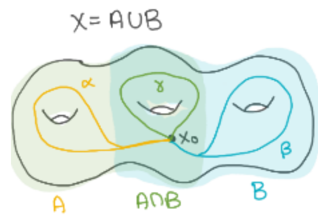
Here's a great visualization of the Van Kampen Theorem:



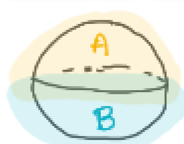
Intuitively we see the fundamental group of X–which is built by gluing A and B along their intersection–as the fundamental group of A and B glued along the fundamental group of their intersection. In essence, $\pi_1(X, x_0)$ is the quotient of $\pi_1(A) * \pi_1(B)$ by relations to impose the condition that loops like γ lying in $A \cap B$ can be viewed as elements of either $\pi_1(A)$ or $\pi_1(B)$.

Announcements

- Midterm – Thursday February 18th
- Shorter homework next week on Van Kampen to review for the exam
- No quiz next week.
- Extra Office Hours next Wednesday 17th February from 7pm-9pm (Midterm review)

Back to Van Kampen

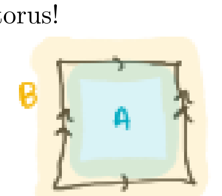
Example .0.1
Lets compute the fundamental group of S^2 again using Van Kampen.



$\pi_1(S^2)$ must be a quotient of $\pi_1(A) * \pi_1(B)$, but since $A, B \simeq D^2$ we know $\pi_1(A)$ and $\pi_1(B)$ are both zero groups. Thus $\pi_1(A) * \pi_1(B)$ is the zero group, and $\pi_1(S^2)$ is the zero group.

Remark .0.1
Something to note, the inclusion of $A \cap B \to A$ induces the zero map $\pi_1(A \cap B) \to \pi_1(A)$, which cannot be an injection. In fact we know that $\pi_1(A \cap B) \cong \mathbb{Z}$ since $A \cap B \simeq S^1$.

Example .0.2
Now let's do the same thing with the torus!



Now note that $A \simeq D^2$ and $B \simeq S^1 \vee S^1$, since it is a thickening of the two loops around the torus in both ways. This suggests the question of how do we find $\pi_1(B)$? We grab a bit of knowledge from Van Kampen before we continue.

Exercise .0.3
Suppose we have path connected spaces (X_α, x_α) and we take their wedge sum $\bigvee_\alpha X_\alpha$ by identifying the points x_α to a single point x. We also suppose a mild condition for all α, the point x_α is a deformation retract of some neighborhood of x_α.

For example, this doesn't work if we choose the "bad point" on the Hawaiian earring. Then we can use Van Kampen to show that:

$$\pi_1\left(\bigvee_\alpha X_\alpha, x\right) = \mathop{*}_\alpha \pi_1(X_\alpha, x_\alpha)$$

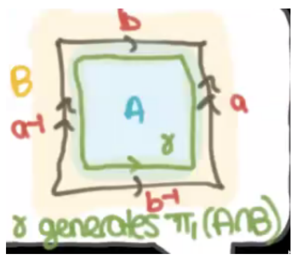
Proof idea. Take $A_\alpha = X_\alpha \bigcup_\beta U_\beta \simeq X_\alpha$ where U_β is a neighborhood of x_β which deformation retracts to x_β. This makes A_α open as desired.

Corollary .0.1
The wedge sum of circles $\pi_1\left(\bigvee_{\alpha\in A} S^1\right) = *_\alpha \mathbb{Z}$ is the free group on A. In particular when A is finite, the fundamental group of a bouquet of circles is the free group on $|A|$ generators

Returning to Example .0.2 we have that:

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$.

We know that $\pi_1(A \cap B) \to \pi_1(A)$ is the zero map. We need to understand $\pi_1(A \cap B) \to \pi_1(B)$. To do so we need to understand how we're able to identify $\pi_1(S^1 \vee S^1$ with F_2 and how we identify $\pi_1(S^1)$ with $\mathbb{Z}$. We update our picture to talk about this



From picture we have that:

$$\pi_1(A \cap B) \to \pi_1(B) \cong F_{a,b}$$
$$\gamma \mapsto aba^{-1}b^{-1}$$

By Van Kampen: identify the image of γ in $\pi_1(B)$ $[aba^{-1}b^{-1}]$ with its image in $\pi_1(A)$ (trivial). Therefore:

$$\pi_1(T^2) = F_{a,b}/\langle aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2$$