There was a file system error erasing my notes for this day. Before I retype them, I do still have the pdf. Here it is!

Lemma .0.1

If $\rho : G \to \mathrm{GL}(V)$ is a finite-dimensional $\mathbb{C}$-representation of a finite group G and χ is the character of ρ, then the multiplicity of the trivial representation in any decomposition of ρ as the sum of irreducible representations is $(\chi_{\mathrm{triv}}, \chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g)$

Using this, if ρ, χ is nontrivial/irreducible then $\sum_{g\in G}\chi(g) = 0$.

Proof. Let $V^G = \{v \in V \mid g \cdot v = v \ \forall g \in G\}$. This is the subspace of V on which ρ acts as the trivial representation.

Consider the G-equivariant projection $\pi : V \to V^G$ given by

$$v \mapsto \frac{1}{|G|}\sum_{g\in G} g \cdot v.$$

This is a G-equivariant linear projection $V \to V^G$, by the reindexing trick.

Thus $V = (\ker \pi) \oplus V^G$. We see that $\mathrm{tr}(\pi) = \dim V^G$ by block matrices. We can also compute the trace in terms of characters

$$\mathrm{tr}(\pi) = \frac{1}{|G|}\sum_{g\in G}\chi(g).$$

Theorem .0.2

If $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(W)$ are irreducible representations of G with characters χ and χ', then

$$(\chi, \chi') = \begin{cases} 1 & \text{if } \rho \cong \rho' \\ 0 & \text{otherwise} \end{cases}$$

Proof. This says that

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi'(g)} = \begin{cases} 1 & \text{if } \rho \cong \rho' \\ 0 & \text{otherwise} \end{cases}$$

Note from homework that $\chi\overline{\chi'}$ is the character of the induced representation on $\mathrm{Hom}(V, W)$. Thus we are looking for the number of copies of the trivial representation present in the induced representation on $\mathrm{Hom}(V, W)$. Namely

$$v \xmapsto{g\cdot\varphi} g \cdot \varphi(g^{-1} \cdot v)$$

φ is fixed by G when for all $g \in G$ we have $g \cdot \varphi(g^{-1} \cdot v) = \varphi(v)$. That is $\varphi(g^{-1} \cdot v) = g^{-1} \cdot \varphi(v)$. That is φ is fixed by G exactly when φ is a homomorphism of G-representations.

From homework, we know that because V, W are irreducible φ is either zero or an isomorphism (in which case it is a scalar times the identity). If $\rho \ncong \rho'$ then $\varphi = 0$ so $(\chi, \chi') = 0$ as desired. If $\rho \cong \rho'$, then $\dim(\text{space of } \varphi) = 1$ because we are only varying the scalar.

This proves the claim!

Lemma .0.3

Let $f : G \to \mathbb{C}$ be a class function (i.e. a function which is constant on each conjugacy class of G. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Let $\varphi : V \to V$ be $\varphi = \sum_{g \in G} f(g)\rho(g)$.

If ρ is irreducible of $\deg n$ with character χ then φ is always scaling by $\frac{1}{n}\sum_{g\in G} f(g)\chi(g) = \frac{|G|}{n}(f, \overline{\chi})$.

Proof. φ is $\mathscr{C} - linear$ and G-invariant. Clearly φ is linear. To show invariance Consider that

$$\rho(g_\star) \circ \varphi = \sum_{g\in G} f(g)\rho(g_\star g) = \sum_{h \in G} f(g_\star^{-1} h)\rho(h)$$
$$\varphi \circ \rho(g_\star) = \sum_{g\in G} f(g)\rho(g g_\star) = \sum_{h \in G} f(h g_\star^{-1})\rho(h)$$

Because $f(g_\star^{-1}h) = f(hg_\star^{-1})$ we have G-equivariance.

Thus φ is a homomorphism of G-representations from $V \to V$ so it must be scaling by some constant α. We then see that

$$\operatorname{tr}(\varphi) = \alpha \dim V$$
$$\operatorname{tr}(\varphi) = \sum_{g\in G} f(g)\operatorname{tr}(\rho(g)) = \sum_{g \in G} f(g)\chi(g) = |G|\,(\chi, f).$$

Thus $\alpha = \frac{|G|}{n}(\chi, f)$ as desired.

Theorem .0.4

The characters $\chi_1, \ldots, \chi_n$ of the non-isomorphic irreducible representations of G form an orthonormal basis of the space of class functions on G.

Proof. Just need to show that χ_i's span the space of class functions by previous work. Pick any class function f.

We can replace f by $f - \sum_{i=1}^n (f, \chi_i)\chi_i$ to asume f is orthogonal to every χ_i. Then we wish to show $f = 0$.

By the lemma, for all i the φ_i corresponding to χ_i is zero. By Maschke's theorem, for every representation ρ the φ coming from ρ is zero.

We now apply this to the regular representation. Let $\{v_g\}_{g\in G}$ be a basis for the regular representation. Then we have that

$$\varphi(v_1) = \sum_{g\in G} f(g) v_{g\cdot 1} = 0.$$

Therefore $f(g) = 0$ for all g. This finishes the problem!

Proposition .0.5

For $g \in G$, let $C(G)$ be the size of the conjugacy class of G. Then if $\chi_1, \ldots, \chi_n$ are the irreducible characters of G then

$$\sum_{i=1}^n \overline{\chi_i(g)}\chi_i(g) = \frac{|G|}{C(g)} = |Z_G(g)|\,.$$

where $Z_G(g)$ is the centralizer of $g \in G$. Furthermore if $g' \in G$ is not conjugate to g

$$\sum_{i=1}^{n} \overline{\chi_i(g)}\chi_i(g') = 0$$

Proof. Let $f : G \to \mathbb{C}$ be the indicator function for the conjugacy class $C(g)$ (that is 1 on this conjugacy class, and 0 elsewhere).

Then we have that

$$f = \sum_{i=1}^{n} (f, \chi_i)\chi_i$$

$$(\chi_i, f) = \frac{1}{|G|} \sum_{g' \in G} \overline{\chi_i(g')f(g')} = \frac{1}{|G|} \sum_{g' \in C(g)} \overline{\chi_i(g')} = \frac{|C(g)|}{|G|} G\overline{\chi_i(g)}$$

$$f = \sum_{i=1}^{n} \frac{|C(g)|}{|G|} \sum_{i=1}^{n} \overline{\chi_i}(g)\chi_i$$

$$f(g') = 1 = \frac{|C(g)|}{|G|} \sum_{i=1}^{n} \overline{\chi_i}(g)\chi_i(g') \qquad (g' \in C(g))$$

$$f(g') = 0 = \frac{|C(g)|}{|G|} \sum_{i=1}^{n} \overline{\chi_i}(g)\chi_i(g') \qquad (g' \notin C(g))$$

This proves the result.

Example .0.1

Consider the following "character table" of S_3, considering representatives $(1), (12), (123)$ of each conjugacy class with size 1,3,2 respectively

	1	3	2
	(1)	(12)	(123)
χ_1	1	1	1
χ_{sgn}	1	-1	1
χ	2	0	-1