Announcements

- Homework #2 posted – course website
- Quiz #2 Wednesday
  - Know definitions of category, functor
  - Review our examples of functors

Examples of Functors

Example .0.1
We have an identity functor $\mathscr{C} \to \mathscr{C}$ for any category $\mathscr{C}$.

Example .0.2 (Forgetful functors)
For example:

$$\begin{aligned} \mathscr{F} : \underline{\text{Grp}} &\to \underline{\text{Set}} \\ (G, \star) &\mapsto G \\ [f : (G, \star) \to (H, *)] &\mapsto \underbrace{[f : G \to H]}_{\text{same function}} \end{aligned}$$

There are lots of such examples. Consider:

$$\begin{aligned} \mathscr{F} : \underline{\text{Top}} &\to \underline{\text{Set}} \\ (X, \mathcal{T}) &\mapsto X \\ [f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)] &\mapsto \underbrace{[f : X \to Y]}_{\text{same function}} \end{aligned}$$

Similarly we have:

$$\begin{aligned} \mathscr{F} : \underline{\text{Top}^*} &\to \underline{\text{Top}} \\ (X, x_0) &\mapsto X \\ [f : (X, x_0) \to (Y, y_0)] &\mapsto [f : X \to Y] \end{aligned}$$

Example .0.3 (Free Functors)
For every ring R we have a free functor:

$$\begin{aligned} F : \underline{\text{Set}} &\to \underline{R\text{-mod}} \\ A &\mapsto F(A) \\ [f : A \to B] &\mapsto \text{map of } R\text{-modules} \\ &\qquad F(A) \to F(B) \text{ that extends the map} \\ &\qquad f : A \to B \text{ on their bases} \end{aligned}$$

We similarly get free group constructions:

$$\begin{aligned} F : \underline{\text{Set}} &\to \underline{\text{Grp}} \\ A &\mapsto \text{free group on } A \end{aligned}$$

Example .0.4
The dual space construction. Given a field k we have a contravariant functor:

$$\begin{aligned} \underline{k\text{-vect}} &\to \underline{k\text{-vect}} \\ V &\mapsto V^* = \text{Hom}_k(V, k) \\ [A : V \to W] &\mapsto [A^* : \text{Hom}_k(W, k) \to \text{Hom}_k(V, k)] \\ &\qquad [\phi : W \to k] \mapsto [\phi \circ A : V \to K] \end{aligned}$$

.1. Free Groups

Definition .1.1

Let S be a set. The free group is a group F_S equipped with a map $S \to F_S$ satisfying the following universal property.

If G is any group, and $f : S \to G$ is any map of sets, then f extends uniquely to a group homomorphism $\overline{f} : F_S \to G$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & G \\ F_S \arrow[ur, dashed, "\exists!\, \overline{f}"'] & \end{tikzcd}$$

Aside: This defines a (natural) bijection:

$$\mathrm{Hom}_{\underline{\mathrm{Set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\mathrm{Grp}}}(F_S, G)$$

Where $\mathscr{U}$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoints; specifically that the free functor is the left adjoint (appears on the left in the Hom's above).

Remark .1.1

Whenever we state a universal property (UP) for an object (+ map) in a category, an object (+ map) satisfying the UP may or may not eist.

However, if it does exist, the UP determines it "uniquely up to unique isomorphism." Sow we may take the UP to be the definition of the object (+ map)

Claim

The Universal Property determines F_S (+ map $S \to F(S)$) uniquely up to unique isomorphism

Proof. Fix S. Suppose $F_S, \widetilde{F}_S$ with maps $S \to F_S$ and $S \to \widetilde{F}_S$ which both satisfy the universal property.

There must exist unique maps filling in the bottom two diagrams by the universal property for $\widetilde{F}_S$ and F_S:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F_S \\ \widetilde{F}_S \arrow[ur, dashed, "\exists!\, f"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[d] & \widetilde{F}_S \\ F_S \arrow[ur, dashed, "\exists!\, g"'] & \end{tikzcd}$$

The goal is to show that f and g are inverses (and hence isomorphisms). The uniqueness follows from the condition that f and g are the only group homomorphisms making the above diagrams commute.

We now paste the above diagrams together in two different ways:

$$\begin{tikzcd} & \widetilde{F}_S \\ S \arrow[ur] \arrow[r] \arrow[dr] & F_S \arrow[u, dashed, "g"'] \\ & \widetilde{F}_S \arrow[u, dashed, "f"'] \arrow[uu, dashed, bend right=60, "g \circ f"'] \end{tikzcd} \qquad \begin{tikzcd} & F_S \\ S \arrow[ur] \arrow[r] \arrow[dr] & \widetilde{F}_S \arrow[u, dashed, "f"'] \\ & F_S \arrow[u, dashed, "g"'] \arrow[uu, dashed, bend right=60, "f \circ g"'] \end{tikzcd}$$

We then observe that the outer triangle in each case is a Universal Property diagram for $\widetilde{F}_S$ and F_S respectively. Since the identity makes these outer triangles commute, we can conclude from the commutativity of the above diagrams and uniqueness that these are isomorphisms, aka:

$$g \circ f = \mathrm{Id}_{\widetilde{F}_S} \qquad\qquad f \circ g = \mathrm{Id}_{F_S}$$