Announcements

- Quiz on Wednesday
  - Know definition and basic properties (simply connected) of the universal cover
  - Know lifting properties (existence / uniqueness) for covers.

Back to Math!

Proposition .0.1
Let $p : \widetilde{Y} \to Y$ be a covering map with X a connected space, then if two lifts $\widetilde{f}_1, \widetilde{f}_2$ of the same map f agree at a single point then they agree everywhere.

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "p"] \\ X \arrow[ur, bend left, "\widetilde{f}_1"] \arrow[ur, "\widetilde{f}_2"'] \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. Let $S = \{x \in X \mid \widetilde{f}_1(x) = \widetilde{f}_2(x)\}$. Our goal is to show that S is both open and closed. Since X is connected, the only sets that are open and closed are X and $\emptyset$, and S is nonempty by hypothesis.

Choose $x \in X$ and let U be a neighborhood of $f(x)$ so that $p^{-1}(U)$ is a disjoint union of open subsets $\{U_\alpha\}$ each mapped homeomorphically to U by p. (Aside: we say that U is evenly covered by p, and that each open subset U_α is a slice of the preimage).

Now since $f = p \circ \widetilde{f}_1 = p \circ \widetilde{f}_2$ we must have that $\widetilde{f}_1(x), \widetilde{f}_2(x) \in p^{-1}(f(x))$. Let $\widetilde{f}_1(x) \in U_1$ and $\widetilde{f}_2(x) \in U_2$.

Exercise .0.1
Since $\widetilde{f}_1, \widetilde{f}_2$ are continuous there exists a neighborhood $N \subseteq X$ so that $\widetilde{f}_1(x) \in \widetilde{f}_1(N) \subseteq U_1$ and $\widetilde{f}_2(x) \in \widetilde{f}_2(N) \subseteq U_2$.

There are two cases:

- Suppose that $\widetilde{f}_1(x) \neq \widetilde{f}_2(x)$. Then U_1 and U_2 are disjoint because each U_α contains only one preimage of $f(x)$, so $\widetilde{f}_1$ and $\widetilde{f}_2$ must differ on every point of N. Therefore $X \setminus S$ is open (aka S is closed) because $x \in N \subseteq X \setminus S$.
- Suppose that $\widetilde{f}_1(x) = \widetilde{f}_2(x)$, that is $x \in S$. Then $U_1 = U_2$, so for all $n \in N$ we have:
  $$\widetilde{f}_1(n) = p^{-1}(f(n)) = \widetilde{f}_2(n)$$
  Where we're using p^{-1} here to mean $p^{-1} : U \to U_1 = U_2$, the inverse of the restriction $p\big|_{U_1} : U_1 = U_2 \to U$. This shows that S is open since $x \in N \subseteq S$.

.1. Deck Transformations

Definition .1.1
Given covering maps $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$, then an isomorphism of covers is a homeomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_1 = p_2 \circ f$:

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

We can actually talk about a category of covering spaces, but we won't delve into that too much.

Exercise .1.1
This defines an equivalence relation on covers.

Definition .1.2

Fix a cover $p : \widetilde{X} \to X$. The isomorphims of the cover $\widetilde{X} \to \widetilde{X}$ are called deck transformations. We'll let $G(\widetilde{X})$ be the set of deck transformations. Notice that we've suppressed the data of p in the notation, but this data is essential to what a deck transformation is, when this is unclear we write $G(\widetilde{X}, p)$.

Exercise .1.2

Deck transformations $G(\widetilde{X})$ are a subgroup of the group of homeomorphisms of $\widetilde{X}$.

Example .1.3

Consider the cover $p : \mathbb{R} \to S^1$, then $G(\mathbb{R}) \cong \mathbb{Z}$, and $n \in \mathbb{Z}$ acts on $\mathbb{R}$ by translating n units.

Example .1.4

There are covers $p_n : S^1 \to S^1$ where we "wind n times." Then $G(S^1, p_n) \cong \mathbb{Z}/n\mathbb{Z}$ which acts by rotation.

Exercise .1.5

Notice that a deck transformation $\tau : \widetilde{X} \to \widetilde{X}$ is a lift of $p : \widetilde{X} \to X$:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ \widetilde{X} \arrow[r, "p"'] \arrow[ur, dashed, "\tau"] & X \end{tikzcd}$$

Then by the unique lifting property (Proposition .0.1), τ is determined by the image of a single point when X is connected.

Corollary .1.1

If a deck transformation has a fixed point, it is the identity transformation.

Exercise .1.6

Let X be connected. Given a deck transformation $\tau : \widetilde{X} \to \widetilde{X}$, and $x_0 \in X$, τ defines a permutation of $p^{-1}(x_0)$. If this permutation has a fixed point, then it is the identity.