Announcements

- Math 695
  - Cohomology
  - Poincare duality
  - Spectral Sequences
  - “Modern Perspectives”
  - Homotopy groups.
- Evaluations: Please give your feedback!
- Today: Some proof outlines

Proof outline for local degree computations. If $f : S^n \to S^n$ and we have some $y \in S^n$ with $f^{-1}(\{y\}) = \{x_1, \ldots, x_m\}$ then:

$$\deg f = \sum_i \deg f\big|_{x_i}$$

We have a nice commutative diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[rr] \arrow[d] & & H_n(S^n) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x_1, \ldots, x_m\}) & & H_n(S^n, S^n \setminus \{y\}) \\
H_n\left(\bigsqcup_i U_i, \bigsqcup U_i \setminus \{x_i\}\right) \arrow[u, "\cong"] & & \\
\bigoplus_i H_n(U_i, U_i \setminus \{x_i\}) \arrow[u, "\cong"] \arrow[rr] & & H_n(V, V \setminus \{y\}) \arrow[uu, "\cong"']
\end{tikzcd}$$

Where we have the isomorphisms $\cong$ by excision and maps / isomorphisms blue by the LES of a pair. And we also have $\cong$ from the homology of a disjoint union.

But then tracing around the outside of the diagram we get:

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[d, mapsto] & \deg f \arrow[d, mapsto] \\
(1, \ldots, 1) \arrow[r, mapsto] & \deg f = \sum_i \deg f\big|_{x_i}
\end{tikzcd}$$

And this proves the result. Perfect! ☺

We’re now going to work towards proving that cellular homology agrees with singular homology. First we need some nontrivial preliminaries:

(1) $H_k(X^n, X^{n-1}) = \begin{cases} 0 & \text{if } k \neq n \\ \langle n - \text{cells} \rangle & \text{if } k = n \end{cases}$

(2) $H_k(X^n) = 0$ for all $k > n$. If X is finite dimensional, then $H_k(X^n) = 0$ for all $k > \dim X$.

(3) The inclusion $X^n \hookrightarrow X$ induces $H_k(X^n) \to H_k(X)$. Then this map is
   - An isomorphism for $k < n$.
   - Surjective for $k = n$
   - Zero for $k > n$.

Exercise .0.1

Check (2) and (3) directly in the case that the CW-complex structure is a Δ-complex structure using simplicial chains

Proof of (2) + (3). We consider the Long Exact Sequence of a pair for fixed n:

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \longrightarrow \widetilde{H}_k(X^{n-1}) \longrightarrow \widetilde{H}_k(X^n) \longrightarrow H_k(X^n, X^{n-1}) \longrightarrow \cdots$$

When $k+1 < n$ or $k > n$ then $H_{k+1}(X^n, X^{n-1}) = 0$ and $H_k(X^n, X^{n-1}) = 0$, so the above map $\widetilde{H}_k(X^{n-1}) \to \widetilde{H}_k(X^n)$ is an isomorphism. We also get sequences telling us the injective and surjective maps when $k = n$ or $k = n-1$:

$$0 = H_{n+1}(X^n, X^{n-1}) \longrightarrow \widetilde{H}_n(X^{n-1}) \longrightarrow \widetilde{H}_n(X^n) \longrightarrow H_n(X^n, X^{n-1})$$

$$\xrightarrow{\cong} \widetilde{H}_{n-1}(X^{n-1}) \longrightarrow \widetilde{H}_{n-1}(X^n) \longrightarrow H_{n-1}(X^n, X^{n-1}) = 0$$

So the maps $\widetilde{H}_n(X^{n-1}) \to \widetilde{H}_n(X^n)$ is injective, and the map $\widetilde{H}_{n-1}(X^{n-1}) \to \widetilde{H}_{n-1}(X^n)$ is surjective.

Fix k, then we a pile of maps induced by the inclusions $X^n \hookrightarrow X^{n+1}$:

$$\widetilde{H}_k(X^0) \xrightarrow{\cong} \widetilde{H}_k(X^1) \xrightarrow{\cong} \widetilde{H}_k(X^2) \xrightarrow{\cong} \cdots$$

$$\cdots \longrightarrow \widetilde{H}_k(X^{k-1}) \xrightarrow{\text{inj.}} \widetilde{H}_k(X^k) \xrightarrow{\text{surj.}} \widetilde{H}_k(X^{k+1})$$

$$\xrightarrow[\cong]{} \widetilde{H}_k(X^{k+2}) \xrightarrow[\cong]{} \widetilde{H}_k(X^{k+3}) \xrightarrow[\cong]{} \cdots$$

Note: This sequence is not exact. Descriptions of maps (in red) follow from our analysis of the LES of a pair above.

To prove (2):

- $k = 0$, we do by hand
- $k \geq 1$, then $\widetilde{H}_k(X^0) = 0$, so we have that $\widetilde{H}_k(X^0), \ldots, \widetilde{H}_k(X^{k-1})$ are all zero from the isomorphisms above. That is the k-th homology $\widetilde{H}_k(X^n) = H_k(X^n)$ is zero for every n-skeleton where $n < k$, just as desired.

We also have the following collection of maps for fixed k:

$$H_k(X^k) \xrightarrow{surj.} H_k(X^{k+1}) \xrightarrow{\cong} H_k(X^{k+2}) \xrightarrow{\cong} \cdots$$

This implies (3) when X is finite dimensional. For general X, we use the fact that every simplex has image contained in some finite skeleton (since image is compact).