Proof that Cellular Homology ≅ Singular Homology. We get some exact sequences from our preliminaries last time:

$$0 = H_{n+1}(X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1})$$

$$H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) = 0$$

These come from the long exact sequences of a pair combined with the things we've deduced in the preliminaries. We can paste these together into a diagram:

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Hatcher tells us this diagram commutes, and what we've done here tells us that the two red diagonal pieces crossing at $H_n(X^n)$ are exact. We also have exactness of the bottom right diagonal by just going down a degree.

Then this has to at least be a chain complex. Why? Well the diagram commutes because of Hatcher. We then know that:

$$d_{n+1} \circ d_n = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0$$

By exactness, we know that if $\iota_* : H_n(X^n) \to H_n(X^{n+1})$ then using the first isomorphism theorem:

$$H_n(X) \cong H_n(X^{n+1}) = \operatorname{im} \iota_* \cong \frac{H_n(X^n)}{\ker \iota_*} = \frac{H_n(X^n)}{\operatorname{im} \partial_{n+1}}$$

Since j_n injects by exactness,

$$j_n : H_n(X^n) \xrightarrow{\cong} j_n(H_n(X^n))$$

$$\operatorname{im} \partial_{n+1} \xrightarrow{\cong} \operatorname{im}(j_n \circ \partial_{n+1}) = \operatorname{im} d_{n+1}$$

j_{n-1} must also inject by exactness, and so applying exactness:

$$\ker d_n = \ker \partial_n = \operatorname{im} j_n$$

Then we just do some group theory, the n-th cellular homology group is:

$$\frac{\ker d_n}{\operatorname{im} d_{n+1}} \cong \frac{\operatorname{im} j_n}{\operatorname{im}(j_n \circ \partial_{n+1})} \cong \frac{H_n(X^n)}{\operatorname{im} \partial_{n+1}} \cong H_n(X)$$

There is one thing left to show, namely commutativity of this map. That is

> **Claim**
> The differentials $d_n = j_n \circ \partial_{n+1}$ satisfy the formula (in terms of degree) that we stated. This is done by direct analysis of definitions of maps; details in Hatcher.

.1. The Formal Viewpoint: Eilenberg-Steenrod axioms

Definition .1.1

Given two functors $F, G : C \to D$, a natural transformation $\eta : F \to G$ is a collection of maps $\eta_X : F(X) \to G(X)$ lying in D for every $X \in C$ so that for any map $f : X \to Y$ we have a commutative diagram:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

Definition .1.2

A homology theory is a sequence of functors:

$$H_n : \text{pairs } (X, A) \text{ of spaces} \to \text{abelian groups}$$

Equipped with natural transformations $\partial : H_n(X, A) \to H_{n-1}(A)$, where $H_{n-1}(A) := H_{n-1}(A, \emptyset)$ called the boundary map. Naturality here means that for any map $f : (X, A) \to (Y, B)$ we have a commutative diagram:

$$\begin{tikzcd} H_n(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[d, "f_*"] \\ H_n(Y, B) \arrow[r, "\partial"] & H_{n-1}(B) \end{tikzcd}$$

These must satisfy these axioms:

1. (Homotopy) If $f, g : (X, A) \to (Y, B)$ and $f \simeq g$, then $f_* = g_*$
2. (Excision) If $U \subseteq A \subseteq X$ so that $\overline{U} \subseteq \mathrm{Int}(A)$ then $\iota : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces isomorphisms on homology
3. (Dimension) $H_n(*) = 0$ for all $n \neq 0$, where $*$ denotes some arbitrary point
4. (Additivity) $H_n\left(\bigsqcup_\alpha X_\alpha\right) = \bigoplus_\alpha H_n(X_\alpha)$.
5. (Exactness) If we have an inclusion $\iota : A \hookrightarrow$ and $j : X \to (X, A)$ induces a long exact sequence on homology:

$$\cdots \longrightarrow H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

If H_* satisfies all axioms but dimension, it is called an extraordinary homology theory

Example .1.1

Topological K-theory and cobordism.

Theorem .1.1

If H_n : CW pairs $\to$ Ab is a homology theory and $H_0(*) = \mathbb{Z}$, then H_n are exactly the singular homology functors up to a natural isomorphism of functors

More generally, without the assumption that $H_0(*) = \mathbb{Z}$, then H_n are exactly the singular homology functors with coefficients in the abelian group $H_0(*)$.

Proof. Reconstruct the cellular homology groups using the axioms. The exact same argument we did today follows. We then check that the cellular homology groups we just constructed satisfies the degree formula as in our last step. This is a bit more difficult, but we won't get into it.