Instead of the higher homotopy groups π_n, we will study "higher-dimensional holes" in our space using homology groups.

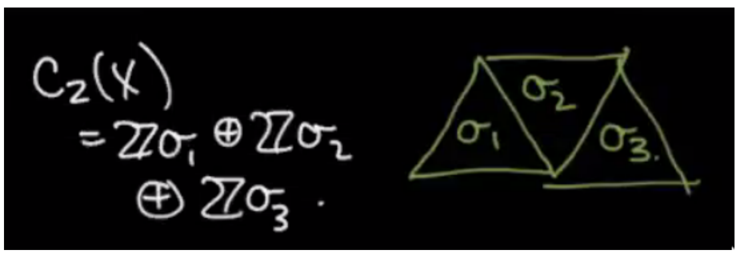
Homology Pros	Homology Cons
Homotopy invariants (like π_n)	Definition (at first) seems less natural
Functorial (like π_n)	
Abelian (like π_n, $n > 1$)	
No basepoints	
Lots of computational tools	
Can compute from cell structure on X	
Good properties like $H_n = 0$ if $n > \dim X$	

Idea for the Homology Definition

Fix a space X, which is a Δ-complex. We define $C_n(X)$ to be the free abelian group on the n-simplices of X. That is:

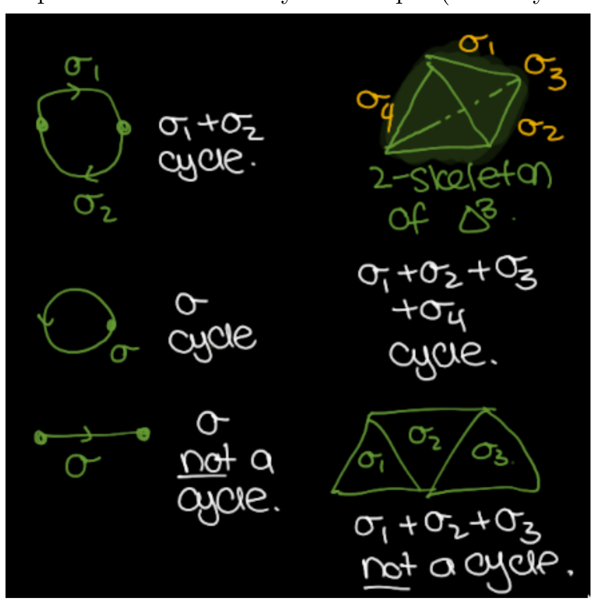
$$C_n(X) = \{\text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha : \Delta^n \to X\}$$

In a picture:

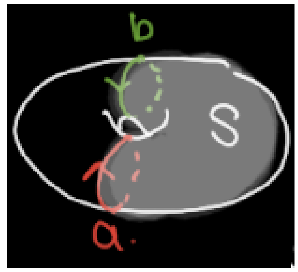


The n-th homology group will be a subquotient of $C_n(X)$. The Heuristic / imprecise idea is:

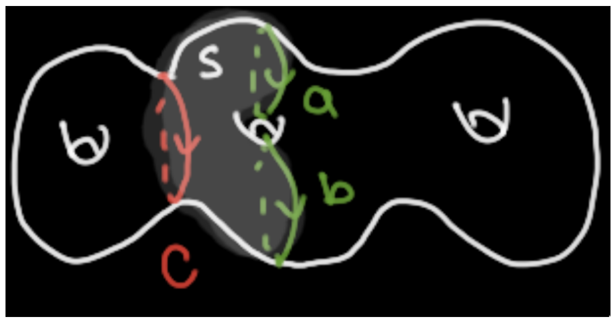
- Take subgroup of C_n of "cycles." These are sums of simplices satisfying a combinatorial condition on the boundary gluing maps to ensure that they "close up." (i.e. they have no boundary)



- To take the quotient, we consider two cycles to be equivalent if their difference is a boundary. For example, in this picture of the torus, a is homologous to b since $a - b$ is the boundary of the shaded subsurface S.



In fact, a and b are homotopic (which will imply they're homologous), but two loops do not need to be homotopic to be homologous. For example:



$a + b$ is homologous to c, since $a + b - c$ is the boundary of S ($a + b$ [which isn't even a loop] and c are not homotopic).

Formal Definition

For the duration, take X with a Δ-complex structure.

Definition .0.1

We define the chain group $C_n(X)$ of order n to be the free abelian group on the n-simplices of X. Formally:

$$C_n(X) = \{\text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha : \Delta^n \to X\}$$

Definition .0.2

We now define the boundary homomorphism, which will be a map $\partial_n : C_n(X) \to C_{n-1}(X)$. We'll first give this in lower dimensions to motivate the general definition:

$$\partial_1 : C_1(X) \to C_0(X)$$
$$[\sigma_\alpha : [v_0, v_1] \to X] \mapsto \sigma_\alpha\big|_{[v_1]} - \sigma_\alpha\big|_{[v_0]}$$
$$\partial_2 : C_2(X) \to C_1(X)$$
$$[\sigma_\alpha : [v_0, v_1, v_2] \to X] \mapsto \sigma_\alpha\big|_{[v_1, v_2]} - \sigma_\alpha\big|_{[v_0, v_2]} + \sigma_\alpha\big|_{[v_0, v_1]}$$

So in general what we have is:

$$\partial_n : C_n(X) \to C_{n-1}(X)$$
$$[\sigma_\alpha] \mapsto \sum_{i=1}^{n} (-1)^i \sigma_\alpha\big|_{[v_0, \ldots, \widehat{v}_i, \ldots, v_n]}$$

And this defines the map on the basis, and we extend linearly ☺.

Lemma .0.1

For any $n \geq 2$ we have that:

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n"] \arrow[rr, bend right, "\partial_{n-1} \circ \partial_n = 0"'] & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] & C_{n-2}(X) \end{tikzcd}$$

Definition .0.3

A chain complex (C_*, d_*) is a collection of maps:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of abelian groups and group homomorphism such that $d_{n-1} \circ d_n = 0$. We call C_n the n-th chain group and d_n the n-th differential.

This means that $\ker(d_n)$ contains $\mathrm{im}(d_{n+1})$, since $d_n \circ d_{n+1} = 0$.

The sequence is exact at C_n provided that $\ker(d_n) = \mathrm{im}(d_{n+1})$. A chain complex is exact if it is exact at each point. The previous lemma guarantees that our simplicial chain groups form a chain complex.

Definition .0.4

The n-th homology group of a chain complex (C_*, d_*) is written H_n or $H_n(C_*)$. It is the quotient:

$$H_n = \frac{\ker(d_n)}{\mathrm{im}(d_{n+1})}$$

It measures how far the chain complex is from being exact at C_n.

Definition .0.5

This means that we may now define the homology groups of spaces X with a Δ-complex structure. Namely $\ker(\partial_n)$ is the subgroup of cycles in $C_n(X)$, and $\mathrm{im}(\partial_{n+1})$ is the subgroup of boundaries in $C_n(X)$. We then set:

$$H_n(X) = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})} = \frac{\text{cycles}}{\text{boundaries}}$$

I.e., it is the homology of our chain complex:

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

Where we take it to be 0 in all negative indices.

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

Elements of $H_n(X)$ are called homology classes