I. Midterm Review

There is a good list of review problems located at

http://www.math.kent.edu/ white/qual/list/group.pdf

For these review problems you should know the following definitions.

Definition I.0.1

For two elements $g, h \in G$, their commutator $[g, h] = ghg^{-1}h^{-1}$. If $[g, h] = 1$ then g, h commute. The commutator subgroup $[G, G] = G'$ of G is generated by the commutators. This is the smallest normal subgroup of G so that G/G' is abelian.

Theorem I.0.1 (Jordan-Hölder)

If G is a group and $N_0 = G \trianglerighteq N_1 \trianglerighteq N_2 \cdots \trianglerighteq N_k = 1$ is a chain so that N_{i+1} is a maximal normal subgroup in N_i, then the successive quotients N_i/N_{i+1} are simple. Furthermore, the sequence $N_0/N_1, N_1/N_2, \ldots, N_{k-1}/N_k = N_{k-1}$ depends only on the group G (up to reordering).

This should be thought of as an analogue of

Definition I.0.2

A group G is called solvable if the simple groups coming from the Jordan-Hölder Theorem are all cyclic groups of prime order.

I.1. Some Cool Stuff

Theorem I.1.1 (From Homework)

Let G be a finite group, then every minimal (nontrivial) normal subgroup of G is isomorphic to $L \times \cdots \times L$ for some simple group L.

A consequence: every minimal normal subgroup of a minimal normal subgroup of G is simple.

Fact: if a subgroup G of S_n is doubly transitive (i.e., G acts transitively on $\{(i, j) \mid i \neq j \; 1 \leq i, j \leq n\}$), then G has exactly one minimal normal subgroup N, which is either $(C_p)^k$ or a nonabelian simple group.

We have $\rho : G \to \mathrm{Aut}(N)$ by conjugation whose kernel K is a normal subgroup of G. If $K = 1$, then ρ is injective and it induces an isomorphism $G \to \rho(G) \leq \mathrm{Aut}(N)$, so we can understand G by understanding subgroups of $\mathrm{Aut}(N)$.

If N is nonabelian, then $K = 1$. Why? If $K \neq 1$, then K is a nontrivial normal subgroup of G, so it contains the minimal normal subgroup N. But then this would show N is abelian.

If G is doubly transitive then either $L \leq G \leq \mathrm{Aut}\, L$ for some nonabelian simple L, or $N \cong C_p^k$. In this case, $\mathbb{F}_p^k \leq G \leq \mathrm{AGL}_k(\mathbb{F}_p)$. Where

$$\mathrm{AGL}_k(\mathbb{F}_p) = \{\vec{x} \mapsto A\vec{x} + \vec{b} \mid \vec{A} \in \mathrm{GL}_k(\mathbb{F}_p), \vec{b} \in \mathbb{F}_p^k\}$$

I.2. Main Takeaways

Here are some of the main takeaways from the class.

- Subgroups of $\mathbb{Z}$ and cyclic groups **??**.
- Subgroups and Cosets. See **??** and **??**.
- Lagrange's Theorem: $H \leq G \implies |H|\,|G/H| = |G|$ (cosets). See **??**

- Normal subgroups + kernels are the same. Normal subgroups $\sim$ factors. See **??**, **??**
- Simple groups. See **??**.
- Quotient Groups, Correspondence Theorem, first isomorphism theorem. See **??** and **??**
- Actions and Orbit-Stabilizer (action by conjugation, action on the cosets, etc.). See **??**, **??**, **??**, and **??**
- Sylow's Theorems, including Cauchy's theorem (order p elements). See **??**
- Direct and semidirect products (see homework).
- Second and Third Isomorphism Theorem. See Piazza Post.

The idea of semi-direct products

- Internal: G is the internal semi-direct product of $N \trianglelefteq G$ by $H \leq G$ (written $G = N \rtimes H$) if N is normal in G, H is a subgroup of G, $N \cap H = 1$, and $G = \langle N, H \rangle$.

  In this case $G = NH$, and the action $\varphi : H \to \operatorname{Aut}(N)$ by conjugation provides all the information about how to multiply elements of N and H together.
- External: If H, N are groups and $\varphi : H \to \operatorname{Aut}(N)$ is homomorphism, then there is a group G (unique up to isomorphism) with subgroups $\overline{N} \cong N$, $\overline{H} \cong H$, such that $G = \overline{N} \rtimes \overline{H} = N \rtimes_\varphi H$ where $\overline{H} \to \operatorname{Aut}(\overline{N})$ by conjugation is identified with φ via $N \cong \overline{N}$, $H \cong \overline{H}$.
- Morally/How to Use: If $N \trianglelefteq G, H \leq G, N \cap H = 1, G = \langle N, H \rangle$, then we can understand G entirely by understanding the action $H \to \operatorname{Aut}(N)$ by conjugation. We may also understand it by understanding all homomorphisms $\varphi : H \to \operatorname{Aut}(N)$, since $G \cong N \rtimes_\varphi H$ (external) for some φ. This is often used to classify things.

Note: $N \rtimes_\varphi H \cong N \rtimes_\psi H$ (external) if $\operatorname{im} \varphi$ and $\operatorname{im} \psi$ are conjugates. Many people proved this on Homework #5 and Homework #6. This is fully citeable.

Idea of semi-direct products: If $N \trianglelefteq G$, $H \leq G$, $N \cap H = 1$, $G = \langle N, H \rangle$ then (internal semidirect product) $G = N \rtimes H = NH$. This tells us we can understand G by understanding the actions $H \to \operatorname{Aut}(N)$ (internally by conjugation, externally more generally, say when classifying).