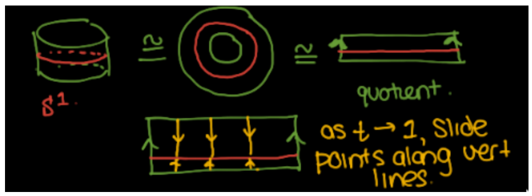
**Announcements**

- Quiz #1 is on Wednesday. Here are your hints!:
  - Know definition and examples from Lecture I on homotopies
  - You may assume the result from homework: "homotopic" is an equivalence relation.

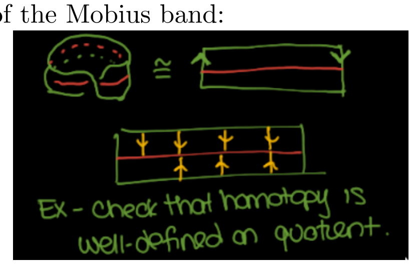
**Lecture Time!**

**Example .0.1**
The point $\{0\}$ is a deformation retract of $\mathbb{R}^n$ via the straight-line homotopy. (That is $\mathbb{R}^n$ deformation retracts onto $\{0\}$, to make things unambiguous)

**Example .0.2**
$S^1$ is the circle, and the circle is a deformation retract of the cylinder.




$S^1$ is also a deformation retract of the Mobius band:




Take-away: "Homotopy equivalence" does not respect orientability, since the cylinder is orientable but the Mobius band is not.

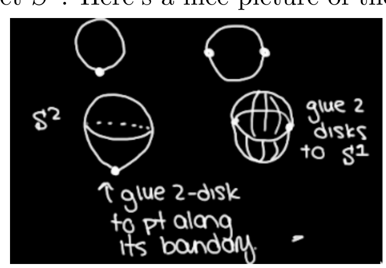
**Exercise .0.3**
Prove that any homotopy equivalence induces a bijection on path components, and thus the number of path components is a homotopy invariant. This is in a sense the most basic homotopy invariant, and much of our course is focused on building more of these invariants.

## .1. CW Complexes

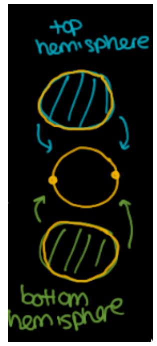
### .1.1. Examples of CW complexes

**Example .1.1** ($S^1$ and $S^2$)
We can take an interval and glue the two points of its boundary together to get $S^1$. Similarly we can cosnstruct $S^2$ by gluing the boundary of 2-disk together.

We could also take two intervals and glue them at their boundaries to make $S^1$, or two 2-disks and glue them at their boundaries to get $S^2$. Here's a nice picture of these four constructions:

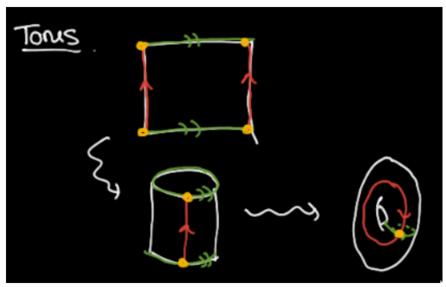



Here are some more explicit instructions for the $S^2$ construction, since it can be a little bit unclear:

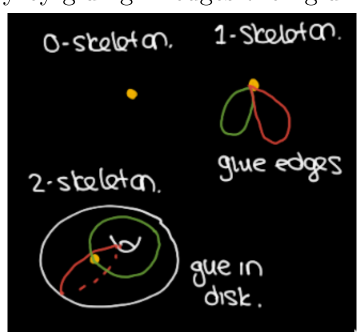



**Example .1.2** (Torus)

Here's the traditional method of building a torus as a quotient space. Notice that the four corners are identified:




We can also build a torus inductively by gluing in edges then gluing in disks:




We can now view the square above as giving us gluing instructions for gluing in the edges to the point in the 1-skeleton, and the disk to the edges in the 2-skeleton.

### Breakout Rooms

**Exercise .1.3**

Prove that if $X \simeq Y$ then $X$ is path connected if and only if $Y$ is.

*Solution.* Note that it suffices to prove that when $X \simeq Y$ and $X$ is path connected that $Y$ is path connected because homotopy equivalence is an equivalence relation. Let the homotopy equivalence be given by $f : X \to Y$ and $g : Y \to X$, and let $F : Y \times [0,1] \to Y$ be the homotopy from $f \circ g$ to $\mathrm{Id}_Y$. Note that for any $y \in Y$ this induces a path from $y$ to $f(g(y))$ by holding the first input to the homotopy fixed:

$$\phi_y(t) = F_t(y) \qquad \phi_y(0) = f(g(y)) \qquad \phi_y(1) = 1$$

Great!

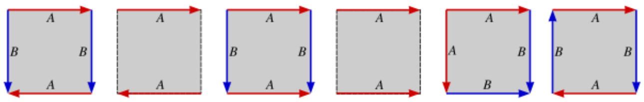
Now fix two points $y, z \in Y$. We know since $X$ is path connected that there is some path $p : [0,1] \to X$ from $g(y)$ to $g(z)$, and we can compose this with $f$ to get a path from $f(g(y))$ to $f(g(z))$ given by $f \circ p$. We then know that there is a path from $y$ to $f(g(y))$ and a path from $f(g(z))$ to $z$ given by the above, and pasting these paths appropriately we get a path from $y$ to $z$ as desired! Therefore $Y$ is path connected!

Awesome! We now have our first homotopy invariant!

**Exercise .1.4**

Identify these quotient spaces!

7. **(Quotient surfaces).** Identify among the following quotient spaces: a cylinder, a Möbius band, a sphere, a torus, real projective space, and a Klein bottle.




Let's go in order!

(1) Klein bottle
(2) Mobius band
(3) Torus
(4) Cylinder
(5) Sphere
(6) Real projective space

Great ☺

**Definition .1.1**

$D^n$ is the closed $n$-disk and $S^{n-1} = \partial D^n$

A 0-cell is a point, and a $n$-cell for $n \geq 1$ is the interior of $D^n$.

### .1.2. The CW Complex Definitions

**Definition .1.2**

A CW complex (cell complex) is a topological space constructed as follows:

- $X^0$ (0-skeleton) is a set of discrete points
- We build $X^n$ ($n$-skeleton) from $X^{n-1}$ by attaching $n$-cells $e^n_\alpha$ ($\alpha$ is an index). The instructions for how to "glue" $e^n_\alpha$ are given by the attaching map
$$\phi_\alpha : S^{n-1} \to X^{n-1}$$
This tells us where to glue the boudary of an $n$-disk $\partial D^n$, which will be a boundary of our $n$-cell $e^n_\alpha$. Formally, we take:
$$X^n = \left( X^{n-1} \coprod_\alpha D^n_\alpha \right) / \ (x \sim \phi_\alpha(x) \ \ \forall x \in \partial D^n_\alpha)$$
As a set then:
$$X^n = X^{n-1} \coprod_\alpha e^n_\alpha$$
- We define $X = \bigcup_{n \geq 0} X^n$ with the weak topology. That is $U$ is open in $X$ if and only if $U \cap X^n$ is open in $X^n$ for all $n \geq 0$.