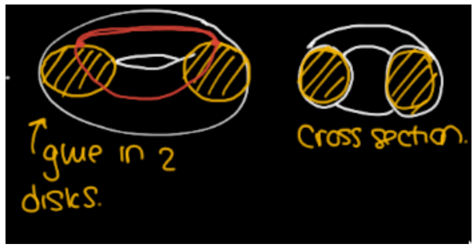
Announcements

- Midterm on next Thursday the 18th
- Fill out "when to meet" for midterm study groups
- Extra Office Hours 7pm-9pm Wednesday.

Back to Van Kampen!

Example .0.1

Start with a torus, and glue in two disks into the hollow inside:



We'll call this space X, and our goal is to find $\pi_1(X)$. We can place a CW complex structure on this space so that each disk is a subcomplex. Then by homework we can quotient eeach disk to a point without changing the homotopy type:

By the same property, we can expand one of these points into an interval, and then contract the red path:

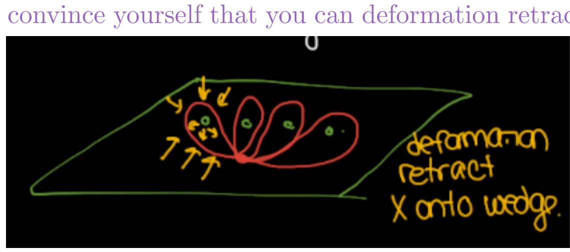
This is exactly $S^2 \vee S^2 \vee S^1$. OUr work with Van Kampen told us that:

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}$$

Exercise .0.2

Consider $\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}$, that is the plane punctured at n points. Then $X \simeq \bigvee_n S^1$, so then $\pi_1(X) \cong F_n$.

One way to do this is to convince yourself that you can deformation retract the plane onto this wedge:



.1. Group Presentations

Definition .1.1

A presentation $\langle S \mid R \rangle$ of a group G consists of:

- S, which is a generating set, generators
- R, a set of relations (words in generators) such that

Such that:

$$G \cong F_S/N$$

Where F_S is the free group on S, and N is the subgroup normally generated by the elements of R.

A presentation is finite if S, R are finite. G is finitely presented if it admits a finite presentation. One way to think about the relations is that if r is a word in R then $r = 1$, where 1 is the identity of G. People often do this.

Example .1.1

We have some nice examples!

Group	Presentation
F_2	$\langle a, b \mid \rangle$
$\mathbb{Z}^2$	$\langle a, b \mid aba^{-1}b^{-1} \rangle$
$\mathbb{Z}/3\mathbb{Z}$	$\langle a \mid a^3 \rangle$
$\mathrm{PSL}_2 \mathbb{Z}$	$\langle a, b \mid a^2, b^3 \rangle$
S_3	$\langle s, t \mid s^2, t^2, (st)^3 \rangle$

Theorem .1.1

Every group has a presentation

Proof. We'll give an outline:

- Choose generators $S \subseteq G$, we could even choose $S = G$
- There exists a surjective map $\varphi : F_S \to G$ which is given by $s \mapsto s$ for $s \in S$
- Choose R to be a generating set for $\ker \varphi$. By the first isomorphism theorem $G \cong F_S/\ker \varphi$.

Great

Advantages

Exercise .1.2

If $G = \langle S \mid R \rangle$ and we have a map $\varphi : S \to H$, then φ defines a group homomorphism $G \to H$ if and only if $\varphi(r) = 0$ for all $r \in R$. By this we mean something like if we have $G = \langle a, b \mid aba \rangle$, a map $\varphi : \{a, b\} \to H$ gives a group homomorphism if and only if:

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H$$

This essentially uses the universal property of quotients.

Exercise .1.3

Suppose all relations in R are commutators, so $R \subseteq [G, G]$. Then:

$$G^{ab} = (F_S)^{ab} = \bigoplus_S \mathbb{Z}$$

Disadvantages

Computationally very difficult.

Example .1.4

Show that $\langle a, b \mid aba^{-1}b^{-2}, a^{-2}b^{-1}ab \rangle$. This is a presentation of the trivial group, but this is entirely unclear.