MATH 465 Notes

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April 9, 2020

Meow

Definition. *A **partially ordered set (poset)** $(P, \leq)$ is a set P endowed with a binary relation which satisfies for all $x, y, z \in P$ that:*

$$x \leq x \qquad \text{(reflexive)}$$

$$x \leq y \text{ and } y \leq x \implies x = y \qquad \text{(antisymmetric)}$$

$$x \leq y \text{ and } y \leq z \implies x \leq z \qquad \text{(transitive)}$$

If $x \leq y$ and $x \neq y$ write $x < y$. And $x \geq y$ means $y \leq x$.

*For any two distinct $x, y \in P$ exactly one of the following is true: $x \leq y$ or $y \leq x$ or x and y are **incomparable***

*If any two elements of P are comparable, say that P is **linearly, or totally ordered***

Example. $(\mathbb{R}, \leq)$ is linearly ordered. Also n-letter words in (ordered) alphabet is totally ordered, see dictionaries.

Example. But subsets of $[n]$ with $\subseteq$, this is a poset but it is not totally ordered.

From here on we assume our posets are finite.

Example. Let P be a finite set of positive integers ordered by divisibility. For example $P = \{1, 2, 3\}$, and we have $1 \leq 2$ and $1 \leq 3$ but 2 and 3 are incomparable.

Example. Consider P to be n-tuples ordered componentwise, so that:

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \iff x_i \leq y_i \forall i$$

Definition. *In a poset (P, S) we say y **covers** x $(x \prec y)$ if $x < y$ and there is no $z \in P$ so that $x < z < y$. Note that $x < y$ if and only if there is $x_1, \ldots, x_k$ so that $x \prec x_1 \prec \cdots \prec x_k = y$.*

*A **Hasse Diagram** is a pictorial representation of $(P, \leq)$ by a graph with $V = P$ with an edge $x \text{ --- } y$ whenever $x \prec y$ (or $y \prec x$) drawn so taht if x, y then y is higher than x. One can also envision a directed graph.*

Example. Consider 2-letter words in alphabet $\{0, 1\}$ ordered lexicographically (with $0 < 1$). Then $00 < 01 < 10 < 11$, so the diagram is just:

$$\begin{tikzcd}
11 \arrow[d, dash] \\
10 \arrow[d, dash] \\
01 \arrow[d, dash] \\
00
\end{tikzcd}$$

Example. $P = \{1, 2, 3, 4, 5, 6\}$ ordered by divisibility.

$$\begin{tikzcd}
4 \arrow[dr, dash] & & 6 \arrow[dl, dash] \arrow[d, dash] & \\
& 2 \arrow[dr, dash] & 3 \arrow[d, dash] & 5 \arrow[dl, dash] \\
& & 1 &
\end{tikzcd}$$

Example. Let $G = (V, E)$ be an acyclic (no directed cycles) directed graph. Consider the set V as a poset with the order $u \leq v$ if and only if there is a directed path from u to v.

Definition. *In a poset $(P, \leq)$ we say that an element $m \in P$ is **maximal** if for all $x \in P$, $m \leq x \implies m = x$. In the Hasse diagram this corresponds to having* $\text{outdeg} = 0$.

*And m is a **maximum** if for all $x \in P$, $x \leq m$. We can define the corresponding ideas of **minimal** and **minimum** minimal corresponds to having* $\text{indeg} = 0$.

Note: P has maximum if and only if it has a unique maximal element.

Example. If P is a poset where no two elements are comparable, its Hasse diagram is a set of isolated vertices. Furthermore every element is both maximal and minimal. This is called an anti-chain.

Definition. *A **chain** in a poset is a subset in which any two elements are comparable, and an **antichain** is a subset where no two distinct elements are comparable A chain is called maximal provided that it is not contained in a larger chain.*

Definition. *The **Boolean lattice** B_n is subsets of $[n]$ ordered by the subset relation. Furthermore, maximal chains in B_n are in bijection with S_n.*

Definition. *The **height** of a poset is the largest size of a chain. The **width** of a poset is the largest size of an antichain.*

Example. B_3 above has width 3 and height 4.

Example. If P is totally ordered it has height $|P|$ and width 1. If P is totally unordered (i.e. an antichain), then it has height 1 and width $|P|$.

Idea: height/width roughly measure how close P is to being totally oredered or totally unordered.

Definition. *A **chain partition** of a poset P is a partition of the set P where each block is a chain, likewise we have a definition for an **antichain partition***

Theorem 1 (Dilworth)**.** *In a finite poset, the $\overbrace{\textit{maximum size of an antichain}}^{\textit{width}}$ is the minimum number of blocks in a chain partition*

Theorem 2 (Mirsky)**.** *In a finite poset, the $\overbrace{\textit{maximum size of a chain}}^{\textit{height}}$ is the minimum number of blocks in an antichain partition*

Lemma 1. *Let P be a finite poset. If $\{C_1, \ldots, C_n\}$ is a chain partition of P, then every antichain has at most n elements. Likewise if $\{A_1, \ldots, A_m\}$ is an antichain partition of P, then every chain has at most m elements.*

Proof. Let A be an antichain and $C_1, \ldots, C_n$ a chain partition of P. Every element of A is in some block C_i, but we can't have more than one element of A in the same block, since this would imply two elements were both comparable and incomparable. This defines an injective function $A \to [n]$, and so $|A| \leq n$.

The second argument is similar!!! Do it yourself as practice!!! $\square$