MATH 465 Notes

Faye Jackson

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1 Announcement

- Today: Quiz §5.3 Partitions
- HW6 Due Tomorrow
- Thursday: Quiz. Ch. 7 Inclusion / Exclusion
- Tuesday: Quiz + Review (in groups)
- HW7 Due Next Wednesday
- Next Thursday: Exam (more info TBA)
- Office Hours
  - Thursday 11:30-1:00
  - No Friday
  - Monday 4:00-5:30
  - Tuesday 11:30-1:00

2 Integer Partitions: The Main Player Today

2.1 Recalling the Definition from Last Time

Definition 1. *A partition of $n \in \mathbb{Z}_{\geq 0}$ is a weakly decreasing finite sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of positive integers such that $\lambda_1 + \cdots + \lambda_\ell = n$.*

Then let $p(n) = \#$ of partitions of n. We also call $\lambda_1, \dots, \lambda_\ell$ is the parts of lambda and $\ell = \ell(\lambda)$ is the length of λ.

Then we have $\lambda(0) = 1, \lambda(1) = 1, \lambda(2) = 2, \lambda(3) = 3, \lambda(4) = 5, \lambda(5) = 7$.

2.2 Let's Get and Use a Generating Function

Theorem 1. *The generating function of the numbers $p(n)$ is given by:*

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Proof. The RHS is the infinite product:

$$(1+x+x^2+\cdots)(1+(x^2)^1+(x^2)^2+\cdots)(1+(x^3)^1+(x^3)^2+\cdots)\cdots$$

The coefficient of x^n is the number of ways to pick a term from each factor so that the sum of the exponents is n. We know all but finitely many such factors must be $x^0 = 1$. We want to think of think of these as follows:

$$(3,2,1,1) \leftrightarrow (x^3)^1(x^2)^1(x^1)^2$$

That gives an equivalence between our picks and partitions of n. Namely the power of x^i picked from the i-th factor is the number of i's in the partition.

One can also think of this as $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ we can group together the "like terms" where there are a_i i's, then we have:

$$1a_1 + 2a_2 + 3a_3 + \cdots = n$$

$\square$

Example. Note then that for partitions made up of 1's and 2's we'd have a generating function:

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} = (1+x^1+(x^1)^2+\cdots)(1+x^2+(x^2)^2+\cdots)$$

Where the coefficient of x^n gives the number of ways to pick $a, b \in \mathbb{Z}_{\geq 0}$ so that $(x^1)^a(x^2)^b$ such that $a + 2b = n$. But then this is exactly partitions made up of 1's and 2's

More Generally: Fix $I \subseteq \mathbb{Z}_{>0}$ and define

$$p_I(n) := \#\text{ of partitions of } n \text{ whose parts are elements of } I$$

Then:

$$\sum_{n\geq 0} p_I(n)x^n = \prod_{i\in I}\frac{1}{1-x^i}$$

Theorem 2. *The number of partitions of n into odd parts is equal to the number of partitons of n into distinct parts.*

Example. Odd parts where $n = 7$ we have:

$$(7), (3, 3, 1), (1, 1, 1, 1, 1, 1, 1), (5, 1, 1), (3, 1, 1, 1, 1)$$

And for distinct parts we have:

$$(7), (5, 2), (4, 3), (6, 1), (4, 2, 1)$$

Proof. The generating for partitions of n into odd parts is exactly $\prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}$ (using $I = \{1, 3, 5, \ldots\}$).

The generating function for partitions of n into distinct parts is $\prod_{i=1}^{\infty}(1 + x^i)$ because we can pick each positive integer 0 or 1 times.

It suffices to check that these two expressions are equal.

$$\prod_{k=1}^{\infty}(1+x^k) = \prod_{k=1}^{\infty}\left[(1+x^k)\cdot\frac{1-x^k}{1-x^k}\right] = \prod_{k=1}^{\infty}\frac{1-x^{2k}}{1-x^k}$$

Then observe:

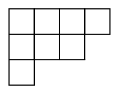
$$\prod_{k=1}^{\infty}\frac{1-x^{2k}}{1-x^k} = \frac{\cancel{1-x^2}}{1-x}\cdot\frac{1-x^4}{\cancel{1-x^2}}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{\cancel{1-x^4}}\cdots = \prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}\frac{1-x^{2i}}{1-x^{2i}} = \prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}$$

By cancelling lots of terms. $\square$

2.3 Young Diagrams

Definition 2. *The Young Diagram (Ferrer's shape) associated to a partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ *is a collection of unit squares on a rectangular grid which is made up of contiguous rows of lengths* $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ *(top-to-bottom). So that the left ends are aligned.*

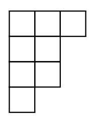
Example. Let $\lambda = (4, 3, 1)$ a partition of 8.

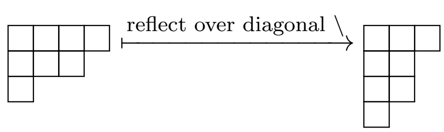


Definition 3. *If* λ *is a partition of* n *its conjugate partition is the partition of* n *whose parts are the number of boxes in the columns of* λ *in its Young Diagram.*

Notation: λ'

Example. $\lambda' = (3, 2, 2, 1)$





Observe that $(\lambda')' = \lambda$ so this defines a bijection $\{\text{partitions of n}\} \leftrightarrow \{\text{partitions of n}\}$

This shows us that partitions with a certain number of parts is equivalent to partitions with largest part equal to that certain number of parts.

Proposition 1. The number of partitions of n with the largest part equal to at most k is equal to the number of partitions of n with exactly at most k parts

Proof. Conjugation restricts to a bijection:

$$\{\text{partitions of } n \text{ with } \lambda_1 = k\} \leftrightarrow \{\text{partitions of } n \text{ with } \ell(\lambda) = k\}$$

The Red Gives inequalities in this bijection. $\square$

Corrolary 1. *The generating function for partitions of n with at most k parts is exactly equal to the generating function for partitions of n with largest part at most k, which is exactly:*

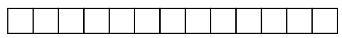
$$\prod_{i=1}^{k} \frac{1}{1-x^i}$$

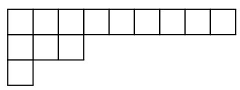
Taking $I = \{1, \ldots, k\}$.

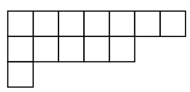
Proposition 2. The number of partitions of n into distinct odd parts is equal to the number of $\underbrace{\text{self-conjugate}}_{\lambda=\lambda'}$ partitions of n.

Example. Let $n = 13$ and consider this case.

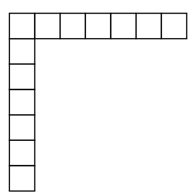
Distinct odd parts:

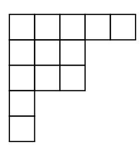
$(13) \mapsto$ 

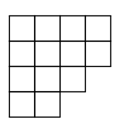
$(9, 3, 1) \mapsto$ 

$(7, 5, 1) \mapsto$ 

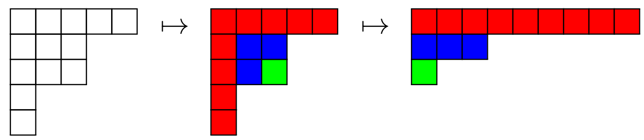
Self-conjugate:

$(7, 1, 1, 1, 1, 1, 1, 1) \mapsto$ 

$(5, 3, 3, 1, 1) \mapsto$ 

$(4, 4, 3, 2) \mapsto$ 

A bijection is given by “straightening out hooks.” In one case look at:



Proof. Define a function:

$$f : \{\text{self-conjugate partitions of } n\} \to \{\text{partitions of } n \text{ with distinct odd parts}\}$$
$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \overset{f}{\mapsto} (2\lambda_1 - 1, 2\lambda_2 - 3, \ldots, 2\lambda_\ell - (2\ell - 1)\}$$

These are in fact distinct odd integers in the right prder, and this is indeed a bijection, to see that, bend hooks at the midpoint. ☐