MATH 465 Notes

Faye Jackson

30 January, 2020

1 Introduction

1.1 Stuff

- Substitute: Al(exander) Gawer
- HW5 Due Wednesday

1.2 Last Time, Homogeneous Linear Recurrences

Theorem 1. *Let k be a positive integer $r_1, \ldots, r_k \in \mathbb{R}$. Suppose $q_1, \ldots, q_d$ are all the distinct roots of:*

$$q^k + r_1 q^{k-1} + \cdots + r_{k-1} q + r_k = 0$$

with multiplicities $m_1, \ldots, m_d$. Then any sequence (a_n) satisfying:

$$a_n + r_1 a_{n-1} + \cdots r_k a_{n-k} = 0$$

is of the form:

$$a_n = \sum_{i=1}^{d} \sum_{j=0}^{m_i - 1} c_{i,j} n^j q_i^n$$

For some constants $c_{i,j}$.

Theorem 2 (Generalized Binomial Theorem)**.** *Let $d \in \mathbb{Z}_{\geq 0}$:*

$$\frac{1}{(1-x)^{d+1}} = \sum_{n=0}^{\infty} \binom{n+d}{d} x^n$$

2 Some Cool Examples

2.1 Homogeneous Linear Recurrences

Example. Suppose we have:

$$\sum_{n=0}^{\infty} a_n x^n = \frac{2}{1-3x} + \frac{1}{(1+x)^3}$$

We want to determine what is (a_n)?

$$\begin{aligned}
\frac{2}{1-3x} + \frac{1}{(1+x)^3} &= 2 \cdot \sum_{n=0}^{\infty} \binom{n+0}{0} (3x)^n + \sum_{n=0}^{\infty} \binom{n+2}{2} (-x)^n \\
&= 2 \cdot \sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} \binom{n+2}{2} (-1)^n x^n \\
&= \sum_{n=0}^{\infty} \left(2 \cdot 3^n + \binom{n+2}{2} (-1)^n \right) x^n
\end{aligned}$$

Therefore for each $n \geq 0$:

$$a_n = 2 \cdot 3^n + (-1)^n \binom{n+2}{2}$$

Awesome!

Example. Find (a_n) where we have $a_0 = 1$, $a_1 = -2$,

$$a_n = 5a_{n-1} - 6a_{n-2}$$

We have two techniques

- The characteristic polynomial is:

$$\begin{aligned}
q^2 - 5q^1 + 6 &= 0 \\
(q-2)(q-3) &= 0
\end{aligned}$$

  And so $q_1 = 2$ and $q_2 = 3$, with multiplicities of 1. Thus for any n we have

$$a_n = c_1 2^n + c_2 3^n$$

Using the initial values we have:

$$\begin{aligned}
1 &= a_0 = c_1 + c_2 \\
2 &= a_2 = c_1 \cdot 2 + c_2 \cdot 3
\end{aligned}$$

We could solve for c_1 and c_2

- Set

$$\begin{aligned}
A(x) = \sum_{n\geq 0} a_n x^n &= a_0 + a_1 x + \sum_{n\geq 2} a_n x^n \\
&= a_0 + a_1 x + \sum_{n\geq 2} (5a_{n-1} - 6a_{n-2}) x^n \\
&= a_0 + a_1 x + 5 \cdot \sum_{n\geq 2} a_{n-1} x^n - 6 \cdot \sum_{n\geq 2} a_{n-2} x^n \\
&= a_0 + a_1 x + 5x \cdot \sum_{n\geq 2} a_{n-1} x^{n-1} - 6x^2 \cdot \sum_{n\geq 2} a_{n-2} x^{n-2} \\
&= a_0 + a_1 x + 5x \cdot \sum_{n\geq 2} a_{n-1} x^{n-1} - 6x^2 A(x) \\
&= a_0 + a_1 x + 5x[A(x) - a_0] - 6x^2 A(x)
\end{aligned}$$

Collect all the $A(x)$ terms to one side:

$$\begin{aligned}
A(x) - 5x \cdot A(x) + 6x^2 \cdot A(x) &= a_0 + a_1 x - 5x a_0 \\
(1 - 5x + 6x^2) \cdot A(x) &= 1 - 7x \\
A(x) &= \frac{1 - 7x}{(1-2x)(1-3x)}
\end{aligned}$$

Then we will use partial fractions:

$$\begin{aligned}
\frac{1 - 7x}{(1-2x)(1-3x)} &= \frac{C_1}{1-2x} + \frac{C_2}{1-3x} \\
1 - 7x &= c_1(1-3x) + c_2(1-2x) \\
&= c_1 + c_2 - (3c_1 - 2c_2)x
\end{aligned}$$

$$\begin{aligned}
1 &= c_1 + c_2 \\
-7 &= -3c_1 - 2c_2
\end{aligned}$$

Then we would solve the linear equation to get $c_1 = 5, c_2 = -4$. Thus:

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \frac{5}{1-2x} + \frac{-4}{1-3x} \\
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} 5(2x)^n + \sum_{n\geq 0} -4(3x)^n \\
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} (5\cdot 2^n - 4\cdot 3^n)x^n \\
a_n &= 5\cdot 2^n - 4\cdot 3^n
\end{aligned}$$

Example. $a_0, a_1 = 1$, $a_2 = 4$. And $a_n = 3a_{n-2} + 2a_{n-3}$.

Let's do this with generating functions (BOO ☹). Goal is to find $A(x) = \sum_{n\geq 0} a_n x^n$:

$$\begin{aligned}
\sum_{n\geq 3} a_n x^n &= \sum_{n\geq 3} 3a_{n-2}x^n + \sum_{n\geq 3} 2a_{n-3}x^n \\
&= 3\cdot \sum_{n\geq 3} a_{n-2}x^n + 2\cdot \sum_{n\geq 3} a_{n-3}x^n \\
&= 3x^2 \cdot \sum_{n\geq 3} a_{n-2}x^{n-2} + 2x^3 \cdot \sum_{n\geq 3} a_{n-3}x^{n-3} \\
\sum_{n\geq 3} a_n x^n &= 3x^2\cdot [A(x) - a_0] + 2x^3 \cdot A(x) \\
A(x) - a_2x^2 - a_1x - a_0 &= 3x^2A(x) - 3x^2a_0 + 2x^3A(x) \\
A(x) - 3x^2A(x) - 2x^3A(x) &= a_2x^2 + a_1x + a_0 - 3x^2a_0 \\
A(x)(1 - 3x^2 - 2x^3) &= 4x^2 + x + 3 - 9x^2 \\
A(x) &= \frac{5x^2 - x - 3}{2x^3 + 3x^2 - 1} = \frac{5x^2 - x - 3}{(1+x)^2(1-2x)}
\end{aligned}$$

We look at it in terms of partial fractions:

$$\begin{aligned}
A(x) &= \frac{5x^2 - x - 3}{2x^3 + 3x^2 - 1} = \frac{5x^2 - x - 3}{(1+x)^2(1-2x)} \\
&= \frac{c_1}{1+x} + \frac{c_2}{(1+x^2)} + \frac{c_3}{(1-2x)} \\
3 + x - 5x^2 &= c_1(1-2x)(1+x) + c_2(1-2x) + c_3(x+1)^2 \\
5 &= -2c_1 + c_3 \\
1 &= -c_1 - 2c_2 + 2c_3 \\
3 &= c_1 + c_2 + c_3
\end{aligned}$$

If you did solve these you would get $c_1 = 3, c_2 = -1, c_3 = 1$.

$$\begin{aligned}
A(x) &= \frac{3}{1+x} - \frac{1}{(1+x)^2} + \frac{1}{1-2x} \\
&= \sum_{n\geq 0} 3(-x)^n - \sum_{n\geq 0} (n+1)(-x)^n + \sum_{n\geq 0} (2x)^n \\
&= \sum_{n\geq 0} [(-1)^n \cdot 3 - (-1)^n(n+1) + 2^n]x^n
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
a_n &= 3(-1)^n + (-1)^{n+1}(n+1) + 2^n \\
&= (3-1)(-1)^n - n(-1)^n + 2^n \\
&= (2-n)(-1)^n + 2^n
\end{aligned}$$

2.2 Nonhomogeneous Linear Recurrence

Example. Let $a_0 = 3, a_1 = 9$ and:

$$a_n + 4n = a_{n-1} + 2a_{n-2}$$

Set $A(x) = \sum_{n\geq 0} a_n$. Then we have that:

$$\begin{aligned}
\sum_{n\geq 2} a_n x^n &= \sum_{n\geq 2} (a_{n-1} + 2a_{n-2} - 4n)x^n \\
&= x \cdot \sum_{n\geq 2} a_{n-1}x^{n-1} + 2x^2 \cdot \sum_{n\geq 2} a_{n-2}x^{n-2} - 4 \cdot \sum_{n\geq 2} n \cdot x^n \\
&= x \cdot [A(x) - a_0] + 2x^2 A(x) - 4x \cdot \sum_{n\geq 2} nx^{n-1} \\
&= x \cdot [A(x) - a_0] + 2x^2 A(x) - 4x \cdot \sum_{n\geq 1} (n+1)x^n \\
&= x \cdot [A(x) - a_0] + 2x^2 A(x) - 4x \left[\frac{1}{(1-x)^2} - 1\right]
\end{aligned}$$

So then we get the following:

$$\begin{aligned}
A(x) - 9x - 3 &= x \cdot A(x) - 3x + 2x^2 \cdot A(x) - \frac{4x}{(1-x)^2} + 4x \\
A(x) \cdot \left[1 - x - 2x^2\right] &= 10x + 3 - \frac{4x}{(1-x)^2} \\
A(x) &= \frac{10x+3}{1-x-2x^2} - \frac{4x}{(1-x)^2(1-x-2x^2)} \\
&= \frac{c_1}{1-2x} + \frac{c_2}{1+x} + \frac{c_3}{1-x} + \frac{c_4}{(1-x)^2}
\end{aligned}$$

Then $c_1 = 0$, $c_2 = -2$, $c_3 = 3$, and $c_4 = 2$. With these you can simplify and:

$$a_n = -2(-1)^n + 3 + 2(n+1) = -2(-1)^n + 2n + 5$$

2.3 Non-Linear Recurrence

Example. $c_0 = 1$, $c_1 = 1$, and:

$$c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}$$

Call $C(x)$ the generating function, $\sum_{n\geq 0} c_n x^n$. Then:

$$(C(x))^2 = \sum_{n\geq 0} \left(\sum_{i=0}^{n} c_i c_{n-i}\right) x^n = \sum_{n\geq 0} c_{n+1}x^n$$

So then we have:

$$\begin{aligned}
x(C(x))^2 &= \sum_{n\geq 0} c_{n+1}x^{n+1} = C(x) - c_0 \\
x(C(x))^2 - C(x) + 1 &= 0 \\
x^2(C(x))^2 - xC(x) + x &= 0 \\
x^2(C(x))^2 - xC(x) &= -x \\
x^2(C(x))^2 - x2\left(\frac{1}{2}\right)C(x) + \left(\frac{1}{2}\right)^2 &= \frac{1}{4} - x \\
\left(xC(x) - \frac{1}{2}\right)^2 &= \frac{1-4x}{4} \\
C(x) &= \frac{\sqrt{1-4x}}{2x} + \frac{1}{2x}
\end{aligned}$$

Then we can get a formula for c_n.

OUT OF TIME ♡