MATH 465 Notes

Faye Jackson

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Ramsey Theory §13

Proposition 1. Suppose that the edges of K_6 have been colored two colors, then there are three vertices such that all edges connecting them are the same color [see HW1 #5]

Proof. Let v be a vertex in K_6, note $\deg(v) = 5$. By PHP three of the edges incident to v have the same color, say red. Call the other endpoints of these edges v_1, v_2, v_3. If there is a red edge connecting any two of these vertices then along with v this forms a red triangle. If not, then the three edges connecting v_1, v_2, v_3 form a blue triangle. $\square$

Example. This is not true for K_5. Color the outside C_5 red and the inner pentagram blue.

Definition. *Let a, b be positive integers. The* ***Ramsey number R(a, b)*** *is the smallest n such that any coloring of the edges of K_n in two colors (say red and blue) must contain either a red K_a or a blue K_b.*

Example. The proposition $\iff$ $R(3,3) \leq 6$, the example coloring of K_5 shows $R(3,3) > 5$, and so $R(3,3) = 6$.

How do we know that there is a complete graph so that this is true? How do we prove that the Ramsey Number's even exist???

Convention: $R(a,1) = R(1,b) = 1$.

Example. For $n \geq 2$ what is $R(n,2)$?

Well $R(n,2) \leq n$, because K_n will either have a blue edge (a blue K_2), or it will be a red K_n. And it's greater than $n-1$, just color K_{n-1} all red. So by symmetry $R(n,2) = R(2,n) = n$, noting that $R(a,b) = R(b,a)$.

These numbers are hard to compute, for example all we know about $R(5,5)$ is that it is somewhere between 43 and 48. These are ridiculously hard, let's prove they even exist!!

Theorem 1. *For all positive integers a, b, $R(a,b)$ exists and:*

$$R(a,b) \leq \binom{a+b-2}{a-1}$$

Lemma 1. *Let $a, b \geq 2$ and assume $R(a-1,b)$ and $R(a,b-1)$ exist. Then $R(a,b)$ exists and:*

$$R(a,b) \leq R(a-1,b) + R(a,b-1)$$

Proof. Let $p = R(a-1,b)$ and $q = R(a,b-1)$. We need to show that any red/blue coloring of the edges of K_{p+q} has either a red K_a or a blue K_b.

Pick a vertex v in K_{p+q}, which has degree $p+q-1$. Among the edges incident to v there are either at least p red edges or at least q blue edges by PHP.

If we have p red edges incident to v, consider the induced subgraph on the other endpoints of those edges, it is isomorphic to K_p. Then since $p = R(a-1,b)$ we have either a red K_{a-1} or a blue K_b. If we have a blue K_b then we're done. If we have a red K_{a-1} we just add the vertex v to obtain a red K_a.

If we have q blue edges incident to v, consider the induced subgraph on the other endpoints of those edges, it is isomorphic to K_q. Then since $q = R(a,b-1)$ we have either a red K_a or a blue K_{b-1}. If we have a red K_a we are done. Then if we have a blue K_{b-1} we can add in v to obtain a blue K_b. $\square$

Theorem 2. *For all positive integers a, b, $R(a,b)$ exists and:*

$$R(a,b) \leq \binom{a+b-2}{a-1}$$

Proof by double induction. First, if $a = 1$ then $R(1,b) = 1$, and if $b = 1$ then $R(a,1) = 1$ and we get the equality:

$$R(1,b) = 1 \leq \binom{b-1}{0} \qquad\qquad R(a,1) = 1 \leq \binom{a-1}{a-1}$$

Now, let $a, b \geq 2$ and assume $R(a-1,b)$ and $R(a,b-1)$ exist. Then also assume:

$$R(a-1,b) \leq \binom{a+b-3}{a-2}$$
$$R(a,b-1) \leq \binom{a+b-3}{a-1}$$

By the lemma, $R(a,b)$ must also exist, and $R(a,b) \leq R(a-1,b) + R(a,b-1)$. But then we can write:

$$\begin{aligned} R(a,b) &\leq R(a-1,b) + R(a,b-1) \\ &\leq \binom{a+b-3}{a-2} + \binom{a+b-3}{a-1} \\ &= \binom{a+b-2}{a-1} \end{aligned}$$

Using Pascal's Recurrence. We win! □

Example. What is $R(4,3)$? By lemma $R(4,3) \leq R(3,3) + R(4,2) = 6 + 4 = 10$.

And by Theorem:

$$R(4,3) \leq \binom{4+3-2}{4-1} = \binom{5}{3} = 10$$

But actually $R(4,3) \leq 9$.
Color the edges of K_9 by red and blue,

Claim. *There exists a vertex that does not have exactly* 5 *red edges incident to it.*

Otherwise the subgraph of red edges would have $\frac{9 \cdot 5}{2}$ edges, and this is impossible!

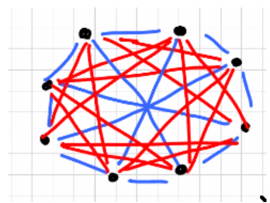
So let v be a vertex such that there are not exactly 5 red edges incident to it. There are two cases:

- If v is incident to ≤ 4 red edges, then since it has degree 8 there are at least 4 blue edges incident to it.

  Consider the edges connecting the other four endpoints. If they are all red we have a red K_4 and we're done. If not then we can form a blue triangle with v and a blue edge.

- In case two, v is incident to ≥ 6 red edges. Look at the other six endpoints of these six edges, they form a K_6, and K_6 always has a monochromatic triangle. If it's a blue triangle then we're done. If it's a red triangle then we can form a K_4 by adjoining v to the red triangle.

In fact $R(4,3) > 8$. There is a coloring of K_8 that has no blue K_3 and no red K_4. See lecture notes because drawing graphs is hard.



One can show in the book that $R(4,4) = 18$.

Let's look at something interesting

Definition. *Let $G = (V, E)$ be a simple graph. $I \subseteq V$ is* ***independent*** *if the induced subgraph on I has no edges.*

A set $Q \subseteq V$ is a ***clique*** *if the induced subgraph on Q is a complete graph.*

Definition. *$R(a, b)$ is the smallest n for which any simple graph on n vertices contains either a clique of size a or an independent set of b vertices.*

<u>Note</u>: Simple graphs on n vertices may be viewed as subgraphs of K_n, that means it can also be seen as a red-blue coloring of K_n. With this in mind think about why the above definition is equivalent to the one we gave before.

Theorem 3 (A Multicolor Ramsey Theorem)**.** *Let $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ then there is a smallest positive integer $N = R(n_1, \ldots, n_k)$ such that if we color the edges of K_N by k (fixed) colors, there is a K_{n_i}-subgraph whose edges are colored by i.*