MATH 465 Notes

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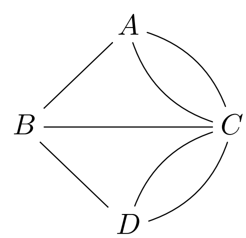
Eulerian Trails + Hamiltonian Cycles (Ch. 9)

Eulerian Trails

Example. Konigsberg Bridge Problem, how to cross all bridges between islands A, B, C, D exactly once, starting and ending at the same place. Is it possible?

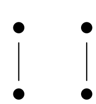


The answer is no, see theorem below.

Definition 1. *A Eulerian trail in a graph is a trail that traverses each edge exactly once. A graph is Eulerian if it has a closed Eulerian trail (AKA Eulerian circuit)*

Example. If $E = \emptyset$ then we win, pick a vertex and go nowhere:

Now what about a graph of the form:



This is not Eulerian, sice there are edges in more than one connected component. We will thus restrict out attention to connected graphs.

Theorem 1 (Euler)**.** *A connected graph has a closed Eulerian trail if and only if it has no vertices of odd degree.*

Proof. First assume that a connected graph G has a closed Eulerian trail. This means that at every time we visit vertex, we must use one edge to get there and one edge to leave (the start/end is an easy special case). Since we use every edge exactly once, this tells us that each vertex has even degree since we use two edges at each vertex.

To prove the other direction, we will proceed by strong induction on the number of edges $n = |E|$. The base case is not so bad, it is the first example above, if G has no edges, then since it is connected it consists of a single example v, and $\deg(v) = 0$. Furthermore, it has a closed Eulerian Trail (the trail v).

Now let n be a non-negative integer such that any connected graph with n or fewer edges and no vertieces of odd degree, it has a closed Eulerian trail.

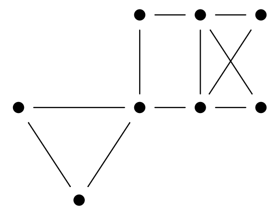
Let G be a connected graph with $n + 1$ edges and no vertex of odd degree. Because n is a nonnegative integer, we know G has at least one edge, and so G cannot be a tree, because trees with at least one edge have at least one leaf (which would have degree one).

Thus there is some cycle C in our graph, of length ℓ. Remove all the edges in the cycle C. This gives us a subgraph G' of G. Thus G' has $n + 1 - \ell$ edges, and $\ell \geq 1$, so $n + 1 - \ell \leq \ell$. We cannot directly apply the inductive hypothesis to G' since G' is not necessarily connected.

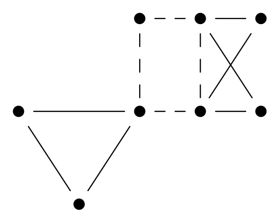
Instead we will consider each of its connected components, since each of its connected components is a connected graph with $\leq n$ edges and no vertex of odd degree. This holds because if the vertex is not in the cycle then $\deg_{G'}(v) = \deg_G(v)$, or if the vertex is in the cycle, then by removing the cycle we remove 2 edges from V, so $\deg_{G'}(v) = \deg_G(v) - 2$.

By induction, each connected component has a closed Eulerian Trail. We can sew them together using C: as we walk around the cycle, each time we encounter a vertex in a connected component of G' that we have not visited yet, we walk around the connected component using the closed Eulerian Trail in it. Then we continue.

Example.



Remove the cycle with 4 vertices in the middle to get:



It is easy to find closed Eulerian Trails in each connected component. The strategy is to walk between connected components by following the cycle, and each time you land in a connected component follow a closed Eulerian trail in it.

□

Corrolary 1. *A connected graph G has an Eulerian trail if and only if G has at most 2 vertices of odd degree.*

Proof. Suppose we have an Eulerian Trail from v to w in the connected graph G. If $v = w$ then we are done, since this is a closed Eulerian trail so the previous theorem applie.

If not, add an edge between v and w. This gives us a closed Eulerian trail in a bigger graph $G' = (V, E \cup \{ v - w \})$. Therefore by the theorem G' has no vertices of odd degree. v and w are the only vertices in G allowed to have odd degree, because they are the only ones changed by adding in the edge.

Conversely, assume a connected graph G has at most 2 vertices of odd degree. If there are none, then the result follows from the theorem, because a closed Eulerian trail is in fact a Eulerian trail.

It is not possible to have one vertex of odd degree, because when we started graph theory we showed that there are always an even number of vertices of odd degree

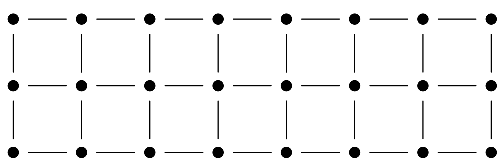
So suppose there are 2 vertices of odd degree; say v and w. Then add the edge $v - w$ and consider the graph $G' = (V, E \cup \{ v - w \})$. By the theorem, since G' has no vertices odd degree, there is a closed Eulerian trail in G'. Removing the edge $v - w$ yields a Eulerian trail in G. $\square$

Hamiltonian Cycles

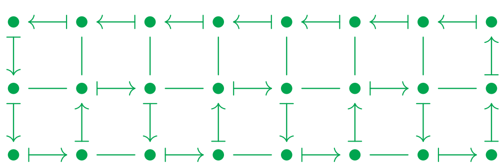
Definition 2. *A* ***Hamiltonian cycle (or path)*** *is a cycle (resp. path) that visits each vertex of a graph exactly once (unless the start and end are the same, that's allowed to be 2).*

A graph is ***Hamiltonian*** *if it contains a Hamiltonian cycle. There is no simple theorem that tells us when a graph is Hamiltonian like we have for closed Eulerian trails.*

Example. Let $m, n \geq 2$. Let $G_{m,n}$ be the graph on mn vertices on an $m \times n$ grid with vertical & horizontal edges: Let $m = 3$ and $n = 8$.



We draw a Hamiltonian Cycle as :



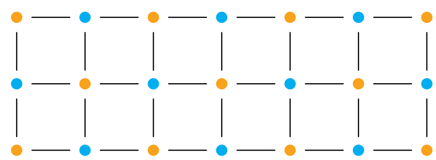
Proposition 1. $G_{m,n}$ is hamiltonian if and only if mn is even.

Proof. If nm is even, WLOG, assume n is even. Use the same technique as above. Start in the upper left corner, go all the way down the first column, up the second column to the second row, all the way down the third column, etc, up last column to the top, then left along the first row. This is a Hamiltonian cycle.

Now assume nm is odd. I.E. both n & m are odd. Color the vertices red and blue, alternating, as below, so that no two red are adjacent and no two blue are adjacent. If there was a Hamiltonian cycle, then the vertices in the cycle would have to alternate color:

$$R — B — R \cdots — R$$

But this isn't possible with an odd number of vertices. If we start at a red vertex, then the mn-th vertex would also be red and there is no edge from this vertex to the starting point. We lose!



Note that no two blue vertices are adjacent to each other and no two orange vertices are adjacent. But a Hamiltonian cycle has to alternate color. $OBOBOB \cdots O$. This can't happen because there is an odd number of vertices. $\square$

Example. The Peterson Graph is not Hamiltonian, you can use the fact that it has vertices of odd degree.