MATH 465 Notes

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1 Trees! The First BlueJeans Lecture

Definition 1. *A graph $G = (V, E)$ is connected if any two vertices are connected by a walk.*

Definition 2. *A connected component of a graph $G = (V, E)$ is an induced subgraph on vertices $V' \subseteq V$ such that no vertex in V' is adjacent to a vertice in $V \setminus V'$*

Definition 3. *A tree is a connected acyclic graph, and an acyclic graph is a forest*

Note:

- Trees are forests are simple because no loops means no 1-cycles and it has no multiple edges because it cannot have 2-cycles.
- A tree is a connected forest and the connected components of a forest are trees.
- A graph is connected if and only if there is a path between any two vertices.

  ($\Rightarrow$) Given a walk we can construct a path by cutting out our repeated vertices.

  ($\Leftarrow$) A path is a walk.

Definition 4. *An edge e in a connected graph $G = (V, E)$ is a bridge provided that $G' = (V, E \setminus \{e\})$ is not connected.*

Proposition 1. An edge in a connected graph is a bridge if and only if it is not contained in a cycle.

Proof. Equivalently we prove that an edge in a connected graph is not a bridge if and only if it is in a cycle.

($\Leftarrow$) Suppose e is in a cycle, say:

$$b \overset{e_0}{\longrightarrow\!\!\!\!-} v_1 \text{———} \cdots \text{———} v_\ell \overset{e_\ell}{\text{———}} a \overset{e}{\text{———}} b$$

We need to show that $G' = (V, E \setminus \{e\})$ is connected. Let $u, v \in V$. Since G is connected there is a path in G from u to v. If this path doesn't use edge e, then it is a path from u to v in G'.

Otherwise, e is an edge in this path from u to v, say:

$$u \text{———} w_1 \text{———} b \overset{e}{- - -} a \text{———} w_2 \text{———} v$$

Replace the part with e with:

$$u \text{———} w_1 \text{———} b \overset{e_0}{\text{———}} v_1 \text{———} \cdots \text{———} v_\ell \overset{e_\ell}{\text{———}} a \text{———} w_2 \text{———} v$$

Generating some walk from u to v, which we can make into a path.

($\Rightarrow$) Suppose the edge e between a and b is not a bridge. So $G' = (V, E \setminus \{e\})$ is connected by definition. Thus there is a path from a to b in G', which obviously does not contain e. Adding the edge e to the end yields a cycle in G which contains e.

□

Corrolary 1. *A connected graph is a tree if and only if every edge is a bridge.*

Corrolary 2. *A graph is a tree if and only if there is a unique path between any two vertices.*

Characterization of trees via minimal number of edges

Definition 5. *A leaf is a vertex in a tree that has degree 1*

Proposition 2. A tree with at least two vertices has at least two leafs.

Proof. Consider a path of maximal length (this is possible because there are finitely many vertices and edges). Since there are at least two vertices in the tree, this path must have at least two vertices. Say:

$$v_0 \overset{e_1}{\text{———}} v_1 \overset{e_2}{\text{———}} \cdots \overset{e_\ell}{\text{———}} v_\ell$$

We claim that since the path is maximal, v_0 and v_ℓ are leaves (have degree 1). Suppose not, say v_ℓ is not a leaf. Thus there is some vertex u adjacent to v_ℓ, and $u \neq v_{\ell-1}$.

But then we can add the edge v_ℓ —— u to the end, we must get a walk because this path is maximal. Therefore we must have repeated a vertex. Therefore $u = v_i$ for some i. Then:

$$v_i \text{ —— } v_{i+1} \text{ —— } \cdots \text{ —— } v_\ell \text{ —— } u = v_i$$

Is a cycle, contradicting the fact that the graph is a tree. The proof that v_0 is a leaf is similar. Awesome! We have two leaves! $\square$

Proposition 3. A connected graph $G = (V, E)$ must have:

$$|E| \geq |V| - 1$$

and, G is a tree if and only if equality holds $|E| = |V| - 1$.

Proof. Step 1 If G is a tree, with n vertices, then it has $n - 1$ edges. We will induct on the number of vertices, that is we will induct on n.

- When $n = 1$ then my tree must be $\cdot$, which has $n - 1 = 0$ edges, since we can't have an edge from the one vertex to itself, since this would give us a cycle.

- Let n be a positive integer and assume that every tree with n vertices has $n-1$ edges. Let G be a tree with $n+1$ vertices. We know that $n+1 \geq 2$ since $n \in \mathbb{N}$. Thus G has at least one leaf v, by the above proposition. Consider removing the leaf v from G, that is let G' be the subgraph obtained by removing v and the edge incident to v.

  G' must be a tree on n vertices, because we cannot create cycles by removing edges, and furthermore since v was a leaf we know G' is still connected. Great! Thus G' has $n - 1$ edges. Then since G has one more vertice and one more edge, we know that G has n edges. Thus the inductive step holds. Awesome!

Step 2 Show that if G is connected and not a tree (that is it contains a cycle), then $|E| > |V| - 1$. With this and Step 1 the proposition is shown! Well, let's induct! Pick an edge in G which is not a bridge, and call it e, since it is in a cycle. Then $(V, E \setminus \{e\})$ is connected. If this graph also contains a cycle, remove another non-bridge.

Continue this process, since the graph is finite there is a finite nonempty set of edged E' that we must remove. Thus $G' = (V, E \setminus E')$ is connected and contains no cycles. Hence G' is a tree, and by step 1, $|E \setminus E'| = |V| - 1$. Note that $|E'| > 0$ since E' is finite and nonempty, so:

$$|E| = \left|E'\right| + \left|E \setminus E'\right| = |E|' + |V| - 1 > |V| - 1$$

This concludes Step 2, and so we win. □

Definition 6. *A* <u>*spanning tree*</u> *in a graph* $G = (V, E)$ *is a subgraph of* G *which contains all vertices of* G *and is also a tree.*

Theorem 1. *A graph is connected if and only if it has a spanning tree*

Proof. Clearly if G has a spanning tree then it is connected. Furthermore, Step 2 gives us a way of constructing a spanning tree using a connected graph. □