MATH 465 Notes

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1 The Catalan Numbers

These counts super important things, and they count a lot of things. They're super cool.

Example. Let $c_0 = 1$ and let $c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}$ for $n \geq 0$.

Consider the generating function of this thing:

$$\mathbb{C}(x) = \sum_{n \geq 0} c_n x^n$$

Now let's consider:

$$\begin{aligned}
[\mathbb{C}(x)]^2 &= \left(\sum_{n \geq 0} c_n x^n\right)\left(\sum_{n \geq 0} c_n x^n\right) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} c_i c_{n-i}\right) x^n \\
&= \sum_{n \geq 0} c_{n+1} x^n
\end{aligned}$$

So then we have:

$$\begin{aligned}
x\mathbb{C}(x)^2 &= \sum_{n \geq 0} c_{n+1} x^{n+1} = \mathbb{C}(x) - c_0 x^0 = \mathbb{C}(x) - 1 \\
x\mathbb{C}(x)^2 &= \mathbb{C}(x) - 1
\end{aligned}$$

We solve this by completing the square:

$$\begin{aligned}
x\mathbb{C}(x)^2 - \mathbb{C}(x) + 1 &= 0 \\
x^2\mathbb{C}(x)^2 - x\mathbb{C}(x) + x &= 0 \\
\left(x\mathbb{C}(x) - \frac{1}{2}\right)^2 - \frac{1}{4} + x &= 0 \\
\left(x\mathbb{C}(x) - \frac{1}{2}\right)^2 &= \frac{1}{4} - x \\
x\mathbb{C}(x) - \frac{1}{2} &= \pm\frac{\sqrt{1-4x}}{2} \\
&= -\frac{\sqrt{1-4x}}{2} \\
\mathbb{C}(x) &= \frac{1-\sqrt{1-4x}}{2x}
\end{aligned}$$

Note that we get the $-$ from the $\pm$ by plugging in 0 to line five. By homework we know that:

$$\mathbb{C}(x) = \sum_{n\geq 0} \binom{2n}{n} \frac{x^n}{n+1}$$

These are the Catalan Numbers $c_n = \binom{2n}{n} \cdot \frac{1}{n+1}$

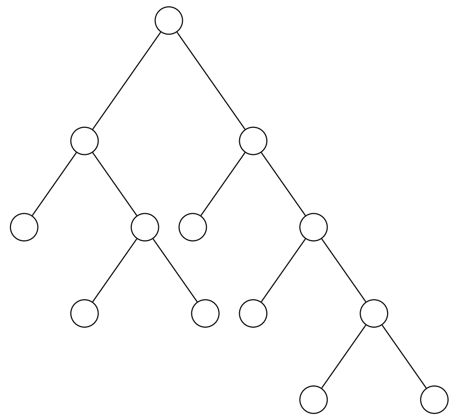
If we write this out in factorials it looks like:

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{2n+1}\binom{2n+1}{n}$$

n	c_n
0	1
1	1
2	2
3	5
4	14
5	42
6	132
7	429
8	1430
9	4862

2 Catalan Numbers: What Do They Count?

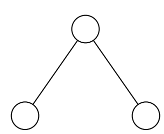
2.1 Answer #1: Rooted Binary Trees



Every node is either a branch, if it has children, and a leaf if it doesn't. Further every branch has two children. The root is always drawn at the top.

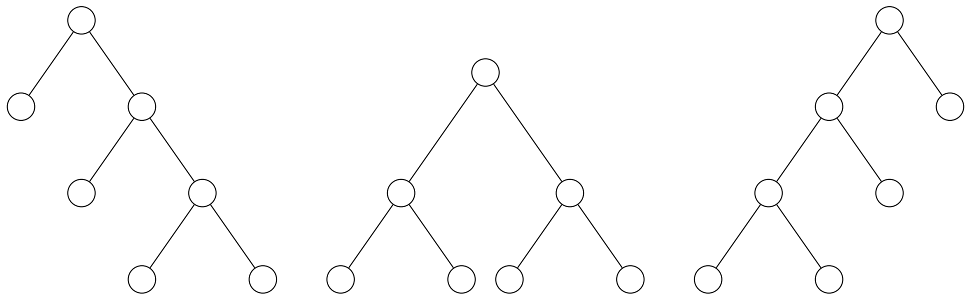
These look similar to binary choices. Let a_n be the number of binary rooted trees with n branches. Checking ti for Some values we have

Example. Note $n = 0$ has $a_0 = 1$ because it's just a root. $n = 1$, $a_1 = 1$.



$n = 2$. $a_2 = 2$.

$n = 3$, $a_3 = 5$.

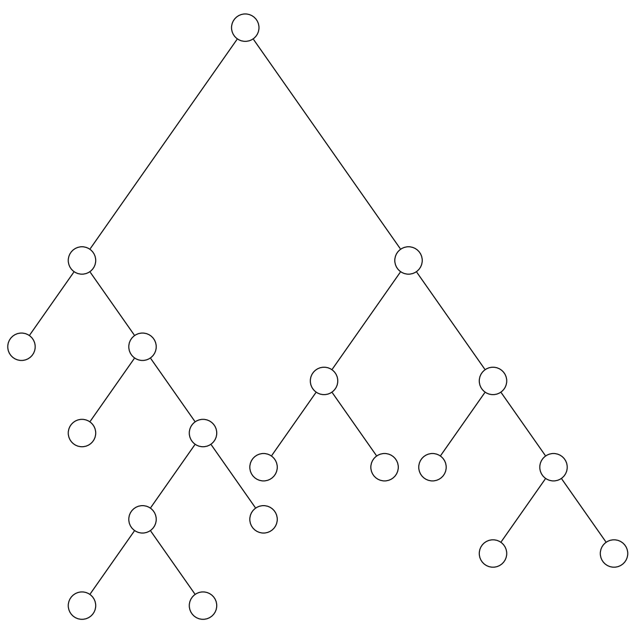


We want to show that these satisfy the linear recurrence:

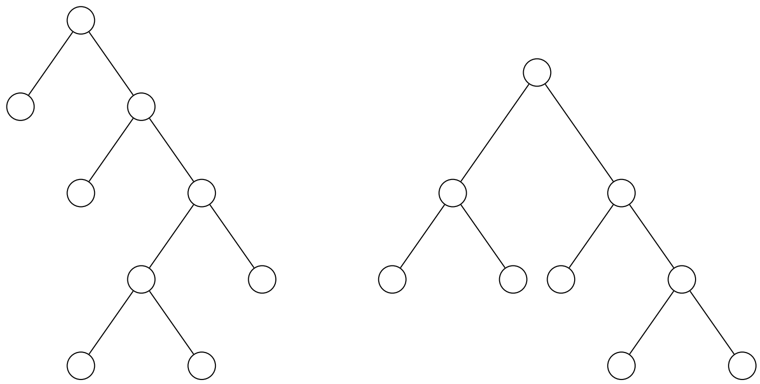
$$a_{n+1} = \sum_{i=0}^{n} a_i a_{n-i}$$

That is we want to make big trees out of two smaller trees. Well, chop the tree at the root. If we have a tree with n branches, then the left tree will have i branches and the right tree will have j branches, with $i + j = n + 1$.

Consider this as:



Splits into:



This gives a bijection between binary trees with $n+1$ branches and ordered pairs of (T, U) of binary trees with T and U together having n branches. Then we clearly have the recurrence and so:

Theorem 1. $a_n = c_n$

2.2 Answer #2: Polygon Triangulations

Theorem 2. *The number of triangulations of a convex $(n+2)$-gon is c_n.*

Let D_n be the number of such triangulations of an $(n+2)$-gon

Example. For $n = 0$ we set $D_0 = 1$ by convention.

For $n = 1$ we have $D_1 = 1$ trivially since it's just a triangle.

For $n = 2$ we can go along either diagonal of the square so $D_2 = 2$.

For $n = 3$ we actually have 5 triangulations so $D_3 = 5$.

In fact for $n = 4$ we actually have $D_4 = 14$ for a hexagon... whoo. This is hard to Latex

Proof. Consider tiling a $(n+3)$-gon, from D_{n+1}.

Well we can choose an edge, and then that edge must be in some triangle. But if we delete that edge, then the rest of the triangulation corresponds to a triangulation of an $(i+2)$-gon and a $(n-i+2)$-gon, because we count rotations differently!

Thus there's a bijection from triangulations of an $(n+3)$-gon and ordered pairs of triangulations of a $(i+2)$-gon and na $(n-i+2)$-gon for all i. The bijection in the other direction comes from gluing together two triangulations at a point and then filling in the "special" wedge that's missing with one edge.

So:

$$D_{n+1} = \sum_{n \geq 0} D_i D_{n-i}$$

And thus $c_n = D_n$ for all $n \in \mathbb{N} \cup \{0\}$ since they agree at 0. $\square$

In fact there's a bijection between binary rooted trees and triangulations. The idea is to view each triangle as rooms with "doors," and draw a node of the tree inside each triangle, where edges represent sharing a boundary, and except at the root we can "go outside" the "palace." We distinguish a root because orientation matters in these triangulations.

This bijection also tells us something about what it would mean to "rotate" binary trees since the triangulations have a natural operation of rotation.

2.3 Answer #3: Dyck Paths

Corresponds to a walk in the Cartesian Plane from $(0,0)$, but only in two directions, one unit to the right or one unit up, ending at (n,n) and that never go strictly above $y = x$.

Let P_n denote those walks.

Example. For $n = 0$, there is only one path so $P_0 = 1$.

For $n = 1$ we have to go from $(0,0)$ to $(1,1)$ so $P_1 = 1$.

For $n = 2$ we have to go from $(0,0)$ to $(2,2)$. There are two ways to do this, "stay away from crocodiles" or "get up close", so $P_2 = 2$

For $n = 3$ we have to go from $(0,0)$ to $(3,3)$. It turns out there are five such paths so $P_3 = 5$.

Proof. OUT OF TIME. $\square$