MATH 465 Notes

Faye Jackson

30 January, 2020

1 Introduction

1.1 Stuff

- Quiz 6 Today
- HW4 due Wednesday
- Today: Stirling Numbers
- Next Time: Linear Recurrence
- No Offic Hours Monday, instead Friday 10:30-11:30 and 1:30-3.

2 Looking at Permutations

We want to view $\sigma \in S_n = \{\text{permutations of } [n]\}$ as a bijective function $\sigma : [n] \to [n]$, thinking that $\sigma(i)$ is the object in the ith place. We have "one-line" notation:

$$\sigma(1)\sigma(2)\ldots\sigma(n)$$

Suppose that $a \in [n]$ let k be the smallest positive integer for which $\sigma^k(a) = a$. [By the pigeonhole principle, there exists $i < j$ so that $\sigma^i(a) = \sigma^j(a)$ to get $\sigma^{j-i}(a)$].

We say that $(a, \sigma(a), \sigma^2(a), \ldots, \sigma^{k-1}(a))$ is a k-cycle of σ at a.

Example. Take $2561437 \in S_7$. Then:

$$\begin{aligned} \sigma(1) &= 2 & \sigma^2(1) &= 5 \\ \sigma^3(1) &= 4 & \sigma^4(1) &= 1 \end{aligned}$$

This is the cycle $(1,2,5,4) = (2,5,4,1)$. We also have sycles (7) and $(3,6)$.

So then $\sigma = (1254)(36)(7)$ in "cycle notation"

$$\begin{tikzcd} 1 \arrow[r, mapsto] & 2 \arrow[d, mapsto] \\ 4 \arrow[u, mapsto] & 5 \arrow[l, mapsto] \end{tikzcd}$$

Example. $n = 3$. For $\sigma \in S_n$, let $c(\sigma)$ be the number of cycles in σ's cycle notation.

σ (one-line)	"cycle notation"	$c(\sigma)$	$b(\sigma)$
123	(1)(2)(3)	3	$(*,*,*)$
132	(1)(32)	2	$(*,*,2)$
213	(12)(3)	2	$(*,1,*)$
231	(231)	1	$(*,1,2)$
312	(321)	1	$(*,1,1)$
321	(31)(2)	2	$(*,*,1)$

Now let $c(n,k) = \#\{\sigma \in S_n \mid c(\sigma = k)\}$. Then conventionally $c(0,0) = 1$, and $c(n,k) = 0$ unless $0 \leq k \leq n$

Example. For $n = 3$ we have $c(3,3) = 1$, $c(3,2) = 3$, and $c(3,1) = 2$. Note then that:

$$\sum_{k \geq 0} c(3,k)x^k = 2x^1 + 3x^2 + 1x^3 = x(x+1)(x+2)$$

Theorem 1. *Let $n \in \mathbb{Z}_{\geq 0}$ then the generating function for $c(n,0), c(n,1), c(n,2), \ldots$ is.*

$$\sum_{k \geq 0} c(n,k)x^k = \sum_{\sigma \in S_n} x^{c(\sigma)} = x(x+1)(x+2)\cdots(x+n-1)$$

Proof. We will build a permutation one step at a time keeping track of our cycles with some tuple $b(\sigma) = (b_1, b_2, \ldots, b_n)$ At the kth step for $1 \leq k \leq n$ we define b_k by either:

- Add (k) by a singleton cycle, record $b_k = *$
- Insert k into a cycle, say after b_k.

Refer back to the above table where we do this for $n = 3$. Remember for 231 we do $(1) \mapsto (12) \mapsto (123) = (231)$. For something like $312 = (321)$ it is:

$$(1) \mapsto (12) \mapsto (132) = (321)$$

Really this defines a bijection:

$$b : S_n \to \{*\} \times \{*, 1\} \times \{*, 1, 2\} \times \cdots \times \{*, 1, 2, \ldots, n-1\}$$

Like on Tuesday we need only show this is an injection or surjection since the number of elements is the same. Now think about weights.

On the left consider the weight $\sigma \overset{c}{\mapsto} c(\sigma)$ and on the right consider the weights of:

$$*, 1, 2, \ldots, n-1 \mapsto 1, 0, 0, \ldots, 0$$

Respectively. Since $(\sigma) = \#$ of $*$'s in $b(\sigma)$, this means b is weight preserving. By the multiplication principle of generating functions:

$$\sum_{\sigma \in S_n} x^{c(\sigma)} = \prod_{k=1}^{n} (1x^1 + (k-1)x^0) = x(x+1)\cdots(x+n-1)$$

□

Corrolary 1.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

Proof. Observe that:

$$\sum_{k \geq 0} c(n,k)x^k = (x+n-1)\sum_{k \geq 0} c(n-1,k)x^k$$

The coefficient of x^k is $c(n,k)$ on the LHS and the coefficient of x^k is what we would really like it to be (by reindexing): $c(n-1,k-1) + (n-1)c(n-1,k)$. □

Note, we know some nice things like $c(n,1) = (n-1)!$ and $c(n,n) = 1$.

3 Stirling Numbers???

3.1 These are Stirling Numbers???

Ok, so $c(n,k)$ is called a "signless Stirling number of the first kind," so what's the "Stirling number of the first kind?" Well:

$$s(n,k) = (-1)^{n-k}c(n,k)$$

Huh, we can make a triangle of $c(n,k)$ by the Corollary above.!

1 4

5

Corrolary 2. *Let $n \in \mathbb{Z}_{>0}$ The generating function for the stirling numbers of the first kind $s(n,0), s(n,1), s(n,2)\ldots$:*

$$\begin{aligned}\sum_{k=1}^{n} s(n,k)x^k &= x(x-1)(x-2)\cdots(x-n+1)\\ &= (x)_n = n!\binom{x}{n}\end{aligned}$$

Proof. Left as an exercise $\square$

3.2 New Stirling Numbers

A set partition of the set T is a set P of pairwise disjoint nonempty subsets such that $\bigcup_{B\in P} B = T$. The elements of P are called "blocks." Define the Stirling number of the Second Kind as:

$$S(n,k) = \#\text{ of partitons of } [n] \text{ with } k \text{ blocks}$$

Example. $n = 3$, $T = \{1,2,3\}$. The possible partitions are:

$$\begin{array}{cccc} & \{\{1,2,3\}\} & & S(3,1)=1\\ \{\{1,2\},\{3\}\} & \{\{1,3\},\{2\}\} & \{\{2,3\},\{1\}\} & S(3,2)=3\\ & \{\{1\},\{2\},\{3\}\} & & S(3,3)=1\end{array}$$

We can say that:

$$S(n,1) = 1 \qquad\qquad S(n,n) = 1$$

Also some conventions $S(0,0) = 1$ and $S(n,k) = 0$ unless $0 \le k \le n$.

Theorem 2. *For $n, k \in \mathbb{Z}_{\ge 0}$ such that $n \ge k$ we have that:*

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

Proof. A k-block partition of $[n]$ either has $\{n\}$ as a signleton block, there are $S(n-1,k-1)$ of these. Or it can be obtained by inserting n into one of the k blocks of a k-block partition fo $[n-1]$, that is $kS(n-1,k)$. $\square$

Theorem 3. *For $n \in \mathbb{Z}_{>0}$ we have that:*

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k = \sum_{k=1}^{n} S(n,k)k!\binom{x}{k}$$

Proof. We will give a combinatorial proof, note these are polynomials, so it suffices to prove this when $x = m$ is a positive integer by the Fundamental Theorem of Algebra. This is good, we can count things. There are m^n ways to color the set $[n]$ with m colors (a function $f : [n] \to [m]$).

OUT OF TIME ☹

…To be continued! $\square$