MATH 465 Notes

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January 9, 2020

Today: Pigeonhole Principle (Ch. 1) Next Time: Basic Counting Principles (Ch. 3)

Theorem 1 (The Pigeonhole Principle, PHP)**.** *If we have two positive integers with $m > n$, and if we have m pigeons to put into n pigeonholes, then at least one pigeonhole contains more than one pigeon.*

In other words, if you place m objects into n boxes, then at least one box has more than one object in it.

In rigourous terms. Let M and N be finite sets with $|M| > |N|$. Then any function $f : M \to N$ cannot be injective.

Example. Suppose that we select 11 different integers from the set $\{1, 2, 3 \ldots 20\}$. Prove that there will always be two among the selected integers, whose difference is two.

Proof. We construct ten boxes of the form:

$$\begin{array}{ccccc} \{1,3\} & \{5,7\} & \{9,11\} & \{13,15\} & \{17,19\} \\ \{2,4\} & \{6,8\} & \{10,12\} & \{14,16\} & \{18,20\} \end{array}$$

Place the 11 integers into the 10 boxes above. By the Pigeonhole Principle there is a box with two selected integers in it. By construction, these integers have a difference of two. □

Theorem 2 (The Generalized Pigeonhole Principle, GPHP)**.** *Let m, n, r be positive integers with $m > nr$. If we place m objects into n boxes, then at least one box has more than r objects.*

The Pigeonhole Principle is in the case where $r = 1$

Proof. Done by contradiction. We will assume the contrary, that is assume that we can place m objects into n boxes such that every box contains at most r objects.

Then the total number of objects (m) is at most $r+r+r\ldots+r$, n times, that is nr. And so, $m \leq nr$. This contradicts the assumption that $m > nr$, and so we are done. □

Example. The Michigan stadium can hold up to 107,601 spectators. Prove that, during a sell-out crowd, there is a group of 294 spectators with the same birthday.

Proof. There are 366 days on which someone could have a birthday. These are our boxes. Place the spectators into each of these boxes. We know that $107,601 > 107238 = 366 * 293$. Therefore, by the The Generalized Pigeonhole Principle, there is a box with more than 293 people in it. That is, there is a group of at least 294 people with the same birthday ☺ □

Example. What is the minimum number of spectators necessary to ensure 100

$99 \cdot 366 + 1 = 36235$. The proof is a quick generalization.

Example. Let k be a positive integer which is not divisible by 2 or 5. Prove that k divides a number of the form $99\ldots9 = 10^N - 1$, for some N.

Proof. Consider the remainders of:

$$10^1 - 1, 10^2 - 1, \ldots 10^{k+1} - 1 \qquad \text{(Objects)}$$

after division by k. There are k possible remainders $(0, 1\ldots, k-1)$ [Boxes], and there are $k+1$ numbers.

Therefore there are two such $1 \le m \le n \le k+1$ such that $10^n - 1$ and $10^m - 1$ have the same remainder upon dividing by k by the Pigeonhole Principle.

Then k divides $10^n - 10^m = 10^m(10^{n-m} - 1)$. Since k is not divisible by 2 or 5, $10^m = 2^m 5^m$ and k share no prime factors. Therefore k divides $10^{n-m} - 1$. $\square$

Theorem 3 (Erdős-Szekeres)**.** *Let n, m be positive integers. Any sequence of $nm+1$ distinct real numbers contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$.*

Example. Let $n = 2$ and $m = 3$,

$\underline{5}, \underline{6}, 3, 4, 1, 2, \underline{7}$

This is 7 numbers. The underlined numbers are an increasing subsequence of length $2+1$.

$\underline{5}, 6, \underline{3}, 4, \underline{1}, 2, \underline{0}$

This is 7 numbers. The underlined numbers are a decreasing subsequence of length $3+1$

$5, 6, 3, 4, 1, 2$

This is six numbers with no increasing or decreasing subsequence of the correct length

Proof. Let $a_1, a_2, \ldots a_{nm+1}$ be our sequence of distinct real numbers. For each $1 \le k \le nm+1$ define t_k to be the length of the longest increasing subsequence starting at a_k.

If there is a k with $t_k > n$, then we have found an increasing subsequence of the right length and we are done.

So suppose not, that is assume $t_k \le n$ for all k. So $t_k \in \{1, 2, \ldots n\}$. By the The Generalized Pigeonhole Principle, there are $nm+1 > nm$ of these numbers, so there is a value attained by at least $m+1$ of these numbers. With the following:

$$\{t_1, t_2, \ldots, t_{nm+1}\} \qquad \text{(Objects)}$$

$$\{1, 2, \ldots, n\} \qquad \text{(Boxes)}$$

That is we have a subsequence $a_{k_1}, a_{k_2}, \dots, a_{k_{m+1}}$ such that $t_{k_i} = t_{k_j}$ for every $1 \le i, j \le m+1$. We want to show that:

$$a_{k_1} > a_{k_2} > \dots > a_{k_{m+1}}$$

This follows from the claim given since these numbers are distinct.

Claim. *If $1 \le k < \ell \le nm+1$ and $t_k = t_\ell$, then $a_k > a_\ell$.*

To prove the claim, suppose otherwise, that is assume $a_k < a_\ell$ [not $\le$ because these are distinct].

Then we could add a_k to the beginning of the longest increasing subsequence starting at a_ℓ, and we would have a subsequence of length $t_\ell + 1$ starting at a_k. This is impossible, because the longest subsequence starting at a_k is of length $t_k = t_\ell$. Oops ☺. □

Example. 51 of the 100 squares on a 10×10 checkerboard are marked. Prove that there exists three marked squares which form three corners of a 2×2 square.

Proof. So begin by tiling the checkerboard with squares that are 2×2, place each of the marked squares in one of these boxes. Then since $51 > 2 * 25$, where 25 is the number of boxes. Then there is a box with more than 2 marked squares in it. There are then at least three marked tiles and so we are done! ☺. □

MATH 465 Notes

Faye Jackson

14 January, 2020

1 Stuff

- If on waitlist: please let registered students sit at desks
- Quiz 1 today (PHP)
- HW1 due tomorrow (see Canvas)
- Quiz 2 Thursdady
- Today: Basic Counting Principles
- Next Time: More Counting (Ch. 3, §5.1)

2 The Golden Rule of Counting

Golden Rule of counting: Count everything exactly once.

Things we might count:

(1) Elements of a set (Ex. How many math majors are in this room?)

(2) Choices or possibilities (Ex. How many ways could we arrange ourselves into seats)

The Additive Principle: If a finite set S can be divided into k $\boxed{\text{pairwise disjoint}}$ $S_i \cap S_j = \emptyset$ for $i \neq j$ subsets, $S_1, S_2, \dots, S_k$, then one has:

$$|S| = |S_1| + |S_2| + \dots + |S_k|$$

Golden: Everything in S is counted exactly once

In terms of choices, view the sets as cases or mutually exclusive events.

The Multiplicative Principle: If a finite set S can be divided into k pairwise disjoint subsets, each with n elements, then $|S| = nk$.

This has the most power when we iterate it. Think: sequences of choices

Example. Suppose we are organizing a panel discussion and we need to arrange the four panel members in a row. Two are graduate students, two are undergrads.

(a) How many possible ways could the panel members be arranged.

Well $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, choose the first person out of 4, the second person out of 3, and so on. Definite multiplicative principle.

(b) How many ways. . . so that grad students are next to each other.

Treat grad students as one person, but in two cases, then $2 \cdot 3! = 12$ because we're now arranging three people.

(c) How many ways. . . so that grad students are NOT next to each other

Well $24 - 12 = 12$. . . Also can consider things like this:

$$GUUG \to 4$$
$$GUGU \to 4$$
$$UGUG \to 4$$

Subtraction is like $\{(a)\} = \{(b)\} \cup \{(c)\}$, and the sets $\{(b)\}$ and $\{(c)\}$ are disjoint. So additive principle.

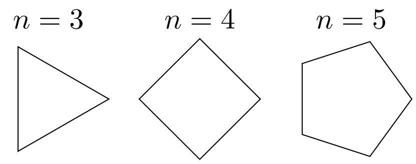
Subtraction Principle

If S is a finite set and $T \subset S$ then $|S \setminus T| = |S| - |T|$. This follows from Addition Principle easily. [Idea: Count irrelevant things]

The Division Principle If we count every possibility k times, then we may divide by k to obtain the number of distinct possibilities.

Example. Let $n \neq 3$. How many diagonals are there in a convex n-gon:

$3 \mapsto \triangle$	$4 \mapsto \square$	$5 \mapsto$ ⬠	$6 \mapsto$ ⬡
0	2	5	9



For each of the n vertices there are $n-3$ other vertices it can connect to via a diagonal. So we'd expect $n(n-3)$ diagonals, but this counts every diagonal exactly twice, so there should be $\boxed{\frac{n(n-3)}{2}}$

Example. (a) How many ways can 4 people sit in a row? $4 \cdot 3 \cdot 3 \cdot 1 = 24$

(b) How many ways can 4 people sit in the 4 seats of a car? The same! 24.

(c) How about 5 people in 5 seats. $5 \cdot 24 = 120$

(d) How do I arrange n people into n seats?

(e) A <u>permutation</u> of a finite set is a linear arrangement of its elements.

Theorem 1. *Let n be a positive integer. The number of permutations of an n-element set is $n!$ that is:*

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1 \qquad \text{n factorial}$$

As convention, we let $0! = 1$. Linear arrangement of the $\emptyset$.

Proof. When $n = 1$ we have 1 permutation of a 1-element set, and $1! = 1$.

Let n be a positive integer and assume there are $n!$ permutations of <u>any</u> n-element set.

Now consider an $(n+1)$-element set. To arrange this set in a line, we have $n+1$ choices for the first element; after it is chosen, we have $n!$ ways to arrange the remaining n elements by the Inductive Hypothesis. Then by the multiplication priciple, there are $(n+1)n! = (n+1)!$ permutations of an $(n+1)$-element set. $\square$

Example. Let $n \neq 2$. How many ways are there to arrange n distinct objects in a circle, up to notation?

There should be $(n-1)!$ ways to do so. Count all the linear arrangements, which can be glued together to give a circular arrangement, then there are n rotations which give the same circular arrangement. Thus we have counted each circle n times, so we have $\frac{n!}{n} = (n-1)!$.

Example. How many ways can 4 people sit in the 6 seats of a car?

Fill in all the seats including empty ones with "phantom people", well then there are two extra seats, and we can permute those two empty seats without changing things, so we can do this:

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2}$$

Theorem 2. *Let n, k be positive integers with $n \geq k$. The number of permutations of k-element subsets of an n-element set is:*

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Falling factorial $(n)_k$

Proof. First, permute the entire n-element set, then remove the last $n-k$ elements. There is one way to remove them, but they show up in $(n-k)!$ permutations. By the division principle we should have that the number of permutations of k-element subsets of an n-element set is $\frac{n!}{(n-k)!}$ □

Definition 1. *A word in an alphabet is any string of characters (i.e. elements of alphabet).*

Idea: A permutation is a word where letters are not repeated.

Example. Let your alphabet be $\{0, 1\}$. Three letter words are like:

000 010 111 110 011 101

We have $2 \cdot 2 \cdot 2$, 3-letter binary words.

Theorem 3. *Let n, k be positive integers. There are n^k k-lettter words in an a-letter alphabet.*

Idea: There are n ways to choose each letter, and we choose k times. It is slicker to use induction on k and the multiplication principle.

Convention: There is $n^0 = 1$ "empty word" (word with no letters).

MATH 465 Notes

Faye Jackson

16 January, 2020

1 Stuff!

- Quiz 2 Today
- HW2 due Wednesday
- Quiz 3 Tuesday
- Today: More Counting (Finish Ch. 3 and start Ch. 4 + 5.1)
- Next Time: Binomial Theorem and Combinatorial Proofs (Ch. 4)
- No Office Hours tomorrow. Instead: 4-6pm Monday or email.

2 Bijective Proof

Proposition 1.

(a) If there is an $\boxed{\text{injection}}$ $f : M \to N$ then $|M| \leq |N|$ by the Pigeonhole Principle contrapositive

$f(a) = f(b) \implies a = b$ for all $a, b \in M$.

(b) If there is a $\boxed{\text{surjection}}$ $f : M \to N$ then $|M| \geq |N|$

for all $n \in \mathbb{N}$ there is some $m \in M$ such that $f(m) = n$

(c) If there is a $\boxed{\text{bijection}}$ $f : M \to N$ then $|M| = |N|$. Both inj. and surj.

By Pigeonhole Principle, also the converses hold (for some reason...). (c) is the heart of a bijective proof

Theorem 1. *Let $n \in \mathbb{Z}_{\geq 0}$. The number of subsets of an n-element set is 2^n.*

Proof. In order to show this we will prove there is a bijection between our subsets and the set of binary words (that is words in the alphabet $\{0, 1\}$).

Let $S = \{x_1, x_2, \ldots, x_n\}$ be our n-element set. Consider binary words. Define a function:

$$\beta : \{\text{subsets of } S\} \to \{n\text{-letter binary word}\}$$

Example. Let $n = 3$ and $S = \{1, 2, 3\}$, we want to do something like:

$$\{1,2,3\} \leftrightarrow 111 \qquad \{1,2\} \leftrightarrow 110 \qquad \{1,3\} \leftrightarrow 101 \qquad \{2\} \leftrightarrow 010 \qquad \emptyset \leftrightarrow 000$$

Fix $T \subset S$, let $\beta(T)$ be the n-letter binary word where the ith letter is 1 if $x_i \in T$ and 0 if $x_i \notin T$.

To prove that β is injective: Suppose $T_1, T_2 \subset S$ such that $\beta(T_1) = \beta(T_2)$. This means that for each i, the i-th letter of $\beta(T_1)$ is equal to the i-th letter of $\beta(T_2)$.

Thus for each i, either $x_i \in T_1$ and $x_i \in T_2$, or $x_i \notin T_1$ and $x_i \notin T_2$. That is $T_1 = T_2$.

To prove that β is surjective: Let $w = a_1 a_2 \ldots a_n$ be an n-letter binary word. Define $T = \{x_i \mid a_i = 1\}$. $T \subset S$ and $\beta(T) = w$.

Since β is a bijection, these two sets have the same number of elements, so since there are 2^n n-letter binary words, there are 2^n subsets of S. $\square$

3 Multinomials and Binomials

Definition 1. *A multiset is a set whose elements are not necessarily distinct.*

Example. How many anagrams are there of the word "STATISTICS" There is 1A, 1C, 2I, 3T, 3S. If all the letters were distinct there'd be 10! permutations, but this overcounts by some factor.

STATISTICS
STATISTICS

This overcounting factor is 3!3!2! and so the total number is $\frac{10!}{3!3!2!}$

Theorem 2. *Let $\{A_1, A_2, \ldots, A_m\}$ be a set and consider a multiset S wich contains k_i copies of A_i, for each i. There are $\frac{(k_1+k_2+\ldots+k_m)!}{k_1!k_2!\cdots k_m!}$ permutations of the multiset S.*

Notationally if $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 0}$ and $n = k_1 + \ldots + k_m$, then:

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \ldots k_m!}$$

These are called "multinomial coefficients"

Corrolary 1. *The number of ways to divide an n-element set S into pairwise disjoint subsets $S_1, S_2, \ldots S_m$ of size $k_1, k_2, k_3 \ldots k_m$ is $\binom{n}{k_1, k_2, \ldots k_m}$.*

In the case of $m = 2$ we simplify to:

$$\binom{n}{k} := \binom{n}{k, n-k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \qquad \text{"binomial coefficient"}$$

Divides our set into subsets, one of size k and one of size $n - k$, this is equivalent to choosing a k-element subset of an n-element set.

Corrolary 2. *There are $\binom{n}{k}$ k-element subsets of an n-element set. There are then $\binom{n}{k}$ n-letter binary words which have k 1's (and $n - k$ 0's), [refine bijection β from earlier]*

3.1 Binomial Applications

Definition 2. *Let $n, k \in \mathbb{Z}_{>0}$. A composition of n with k parts is a positive integer solution to the equation:*

$$x_1 + x_2 + \ldots + x_k = n$$

A weak composition allows zero, so a nonzero integer solution to the above.

A solution is ordered and need not all be distinct.

Example. $n = 5$ and $k = 3$.

compositions:

$$1 + 2 + 2 = 5 \qquad\qquad 1 + 1 + 3 = 5$$

All of this form but permuted. So 6 compositions

weak compositions

$$1+2+2 \qquad 1+1+3 \qquad 0+1+4 \qquad 0+3+2 \qquad 0+0+5$$

All of this form but permuted. So $12 + 3 + 6 = 21$ weak compositions.

Another way to think of this is handing n identical cookies to k distinct kids (either so they all get one or so that they don't). This is like stars and bars:

$$\star|\star\star|\star\star$$
$$1+2+2=5$$

These are 7-letter words in the alphabet $\{\star, |\}$ with 5 $\star$'s and 2 $|$'s. So then:

$\{$weak composition of n with k parts$\} \leftrightarrow \{(n+k-1)$-letter binary words with n $\star$'s and $k-1$ $|$'s$\}$

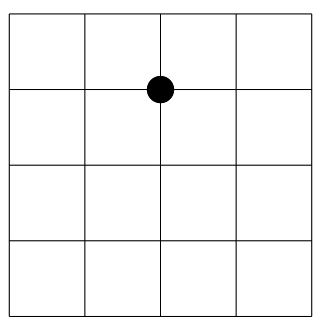
Theorem 3. *There are $\binom{n+k-1}{n}$ weak compositions of n with k parts. For compositions, first put a $\star$ into each bin, and use any weak composition of $n-k$ into k parts. So the number of compositions is $\binom{n-1}{k-1} = \binom{n-1}{(n-1)-(k-1)}$.*

A NE lattice path is awalk on the grid of points with integer coordinates that uses the steps $(1,0)$ and $(0,1)$:

$$E = ((x,y) \mapsto (x+1,y)) \qquad\qquad N = ((x,y) \mapsto (x,y+1))$$

Let $k, \ell \in \mathbb{Z}_{\geq 0}$. How many NE lattice paths are there from $(0,0)$ to (k,ℓ).

Look at lattice paths from $(0,0)$ to $(2,3)$.



Some possible paths are EENNN, NNNEE, ENENN, NENEN

It is clear that there is a bijection:

$\{$NE lattice paths from $(0,0)$ to $(k,\ell)\} \leftrightarrow \{(k+\ell)$-letter word in alphabet N, E with k E's, ℓ N's$\}$

There are $\binom{k+\ell}{k}$ NE lattice paths from $(0,0)$ to (k,ℓ).

A NE lattice path from $(0,0)$ to (k,ℓ) can be broken into exactly one of these two cases:

1. A NE lattice path from $(0,0)$ to $(k-1,\ell)$ followed by an E step, so there are $\binom{k-1+\ell}{\ell}$ or $\binom{k-1+\ell}{k-1}$
2. A NE lattice path from $(0,0)$ to $(k,\ell-1)$ followed by a N step, so there are $\binom{k-1+\ell}{k}$.

Let $n = k+\ell$, By Addition Principle:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{(Pascal's Recurrence)}$$

MATH 465 Notes

Faye Jackson

21 January, 2020

1 Today

1.1 Stuff

- Substitute Anna (Bibby is out)
- Quiz (Ch. 3 and 5.1)
- HW2 due Wednesday

1.2 Topics

- Binomial Theorem
- Combinatorial Proofs

2 Lets look at Binomial Coefficients again

2.1 4.2 Pascal's Triangle

Recall. The binomial coefficient is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (n, k \in \mathbb{Z}_{\geq 0})$$

We'll add the convention that $\binom{n}{k} = 0$ if $k < 0$ or $k > n$

Proposition 1. Let $n, k \in \mathbb{Z}_{>0}$. Then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

We've already shown this one way, now lets do a combinatorial proof, i.e. we will show that they count the same set.

Combinatorial Proof. Let S be an n-element set and fix $x \in S$. THe Left Hand Side counts the number of k-element subsets of S.

The Right Hand Side: By the addition principle, the number of k-element subsets of S is the same as the number of k-element subsets which don't contain x plus the number of k-element subsets which do contain x. Lets count these:

- How many k-element subsets of S contain x. Well we are choosing from $n-1$ things ($S \setminus \{x\}$) and we need $k-1$ of them, that is $\binom{n-1}{k-1}$
- How many k-element subsets of S don't contain x. Well we are choosing from $n-1$ things ($S \setminus \{x\}$) and we need k of them, that is $\binom{n-1}{k}$.

Thus we are done. $\square$

The triangle looks like:

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \end{array} \qquad \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Some patterns:

- Symmetry gives us $\binom{n}{k} = \binom{n}{n-k}$ (we've already proved this)
- Sum of the n-th row is 2^n
- Alternating sums of rows are always 0

Proposition 2. For $n, k \in \mathbb{Z}_{\geq 0}$ we have the symmetry $\binom{n}{k} = \binom{n}{n-k}$ and the sum of each row is:

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

Proof. We observed symmetry on Thursday. We will use a combinatorial proof again for the second claim.

We know that 2^n counts subsets of an n-element set S. We will break this down:

$$2^n = |\{T \subset S\}| = \sum_{i=0}^{n} |\{T \subset S \mid |S| = i\}| = \sum_{i=0}^{n} \binom{n}{i}$$

□

Proposition 3. If $n > 0$ then:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Equivalently:

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$$

Bijective Proof. Let S be an n-element set. Notice that a subset of S has either an even or odd cardinality, but not both.

Fix $x > 0$ and for an $T \in S$ define a piecewise function:

$$f(T) = \begin{cases} T \cup \{x\} & \text{if } x \notin T \\ T \setminus \{x\} & \text{if } x \in T \end{cases}$$

Note that T is even if and only if $f(T)$ is odd:

$$f : \{T \subset S \mid |T| \text{ is even}\} \to \{T \subset S \mid |T| \text{ is odd}\}$$
$$f : \{T \subset S \mid |T| \text{ is odd}\} \to \{T \subset S \mid |T| \text{ is even}\}$$

Omitting the restrictions. It is enough to show that $f(f(T)) = T$. Time for two quick cases:

- Suppose $x \in T$. Then $f(f(T)) = f(T \setminus \{x\}) = (T \setminus \{x\}) \cup \{x\} = T$ This works since $x \in T$.

- Suppose $x \notin T$. Compute $f(f(T))$:

  $$f(f(T)) = f(T \cup \{x\}) = (T \cup \{x\}) \setminus \{x\} = T$$

  This works since $x \notin T$.

The result follows easily since f is now a bijection. $\square$

Theorem 1 (The Binomial Theorem)**.** *Let n be a non-negative integer and x, y be "variables" (or complex numbers... or more). Then:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example.

$$\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^1 &= xx + xy + yx + yy \\
&= x^2 + 2xy + y^2 \\
(x+y)^3 &= xxx + xxy + xyx + yxx + xyy + yxy + yyx + yyy \\
&= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}$$

Idea: If we expand $(x+y)^n$, each term in this expansion corresponds to an n-letter word in the alphabet $\{x, y\}$. The coefficient of $x^k y^{n-k}$ is the number of n-letter words with k x's and $n-k$ y's. This is counted by $\binom{n}{k}$. This is a combinatorial proof.

Proof by Pascal's Recurrence and Induction. In the base case, $n = 0$, the formula works since:

$$(x+y)^0 = 1 = \binom{0}{0} x^0 y^0$$

Let $n \in \mathbb{Z}_{\geq 0}$. Assume that:

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
(x+y)^{n+1} &= (x+y)\left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right) \\
(x+y)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k+1}
\end{aligned}$$

Lets now do some clever reindexing:

$$\begin{aligned}
(x+y)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k+1} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} x^{k} y^{n-k+1} + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k+1} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^{k} y^{n-k+1} + \sum_{k=1}^{n} \binom{n}{k} x^{k+1} y^{n-k+1} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} - \binom{n}{k} \right) x^{k} y^{n-k+1} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{k} y^{(n+1)-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{k} y^{(n+1)-k}
\end{aligned}$$

Thus the binomial theorem holds for $n+1$ and so by induction we're done. $\square$

Set $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Set $x = -1$ and $y = 1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Proposition 4.

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m}{\ell - k} = \binom{n+m}{\ell}$$

Proof 1. Note that $(x+1)^n (x+1)^m = (x+1)^{n+m}$, and then look at the coefficient of x^ℓ.

We know then that:

$$\left(\sum_{k=0}^{n} \binom{n}{k} x^k \right) \left(\sum_{i=0}^{n} \binom{m}{i} x^i \right) = \sum_{\ell=0}^{m+n} \binom{n+m}{\ell} x^\ell$$

Look for ways that $k + i = \ell$, and so $i = \ell - k$. $\square$

Combinatorial Proof. Take disjoint sets S, T such that $|S| = n$ and $|T| = m$.

Count the ℓ-element subsets of the set $S \cup T$.

- The Right Hand Side surely counts this from thursday, $\binom{n+m}{\ell}$
- The Left Hand Side breaks into cases, let $0 \leq k \leq n$. Let A_k be the set of ways to select a k-element subset of S and an $\ell - k$-element subset of T.

  These are separate events so by the multiplication principle so $|A_k| = \binom{n}{k}\binom{n}{\ell-k}$

  Every ℓ-element subset of $S \cup T$ will be in exactly one of these, and so the total number of ℓ-element subsets of $S \cup T$ by the addition principle is:

$$\sum_{k=0}^{n} |A_k| = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{\ell - k} = \binom{n+m}{\ell}$$

□

Proposition 5.

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Proof. Set $\ell = m = n$, by the previous proposition:

$$\sum_{k=0}^{k} \binom{n}{k}\binom{n}{n-k} = \binom{n+n}{n}$$

By symmetry:

$$\sum_{k=0}^{k} \binom{n}{k}^2 = \binom{2n}{n}$$

□

Proposition 6. $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

Proof 1. Use the binomial theorem:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Take the derivative with respect to x.

$$n(x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

Then set $x = 1$

$$n2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k$$

$\square$

Combo Proof. Count ways to choose a subset of an n-element set with a distinguished element (Pick a committee and its chair).

On the Left Hand Side we will split into cases based on the size of the subset: There are $\binom{n}{k}$ ways to choose the committee, and then k choices for the chairperson. So there are $k\binom{n}{k}$ to choose a k-element subset with a distinguished element.

On the Right Hand Side we will pick our chairperson first. There are n ways to pick our chairperson, then we must choose from an $n-1$-element set to fill out the rest of our committee. There are 2^{n-1} ways to fill out the committee. So this gives us $n2^{n-1}$.

And so we are done. $\square$

MATH 465 Notes

Faye Jackson

23 January, 2020

1 Introduction

1.1 Stuff

- Quiz 4 Today
- Quiz 5 Tuesday
- HW3 due Wednesday

1.2 Today

- Finish Chapter 4
- Intro to Generating Functions (8.1)
- Next Time: Ch. 6 Permutations

1.3 Announcements

- No OH Monday
- Instead: 12-3 tomorrow or email
- Final Exam Thursday 4/30, 1:30-3:30pm
- Don't Use Words like "Obviously" or "clearly" in Homework. Justify everything. Also remember to assign pages/problems

2 Multinomial Theorem

Theorem 1 (Multinomial)**.** *For $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{N}$ we have:*

$$(x_1 + \ldots + x_m)^n = \sum_{\substack{k_i \in \mathbb{Z}_{\geq 0} \\ \sum k_i = n}} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$$

Proof. You can prove by induction on m and in the inductive step you will use the binomial theorem □

Proposition 1. For $n \in \mathbb{Z}_{\geq 0}$:

$$\sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ a+b+c=n}} \binom{n}{a,b,c} = 3^n$$

Proof. Use multinomial Theorem with $m = 3$ and $x_1 = x_2 = x_3 = 1$:

$$3^n = (1+1+1)^n = \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ a+b+c=n}} \binom{n}{a,b,c} 1^a 1^b 1^c = \text{LHS}$$

□

Combinatorial Proof. There are 3^n letter words in the alphabet $\{A, B, C\}$. We can also divide these n-letter words into categories where there are a A's, b B's, and c C's so that $a + b + c = n$. In each category there are $\binom{n}{a,b,c}$ since this is the number of n-letter words using a A's, b B's, and c C's. Thus we are done by addition principle. □

Recall. We defined:

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)(n-k)(n-k-1)\ldots}{k!(n-k)!} \\ &= \frac{n(n-1)\ldots(n-k+1)}{k!}\end{aligned}$$

So for $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$ define:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$$

Further if $k > n$ and $n \in \mathbb{Z}_{\geq 0}$ let $\binom{n}{k} = 0$.

Look at the binomial theorem with $y = 1$:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

Well what happens if we change n to $\alpha \ldots$

Theorem 2 (Generalized Binomial Theorem)**.** *Let $\alpha \in \mathbb{C}$.*

$$(x+1)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

when it all makes sense. Holds as a "formal power series"

Proof. Omitted □

Example. Let $\alpha = -n$ for $n \in \mathbb{Z}_{>0}$. Then:

$$\begin{aligned}
\binom{-n}{k} &= \frac{-n(-n-1)(-n-2)\ldots(-n-k+1)}{k!} \\
&= \frac{(-1)^k \cdot n(n+1)(n+2)\ldots(n+k-1)}{k!} \\
&= (-1)^k \frac{(n+k-1)!}{k!(n-1)!} \\
\binom{-n}{k} &= (-1)^k \binom{n+k-1}{n-1}
\end{aligned}$$

What the Generalized Binomial Theorem then tells us $\implies$

$$\begin{aligned}
(1-x)^{-n} &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} (-x)^k \\
(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k
\end{aligned}$$

In particular when $n = 1$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^k$$

Which you know as a geometric series.

3 The (ordinary) Generating Functions of Sequences

Let $a_0, a_1, a_2, \ldots$ be a sequence. The generating function is the formal power series:

$$a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

(x is a "symbolic variable," we will not worry about convergence)

Really this is just a tool for counting.

Example. $\frac{1}{1-x}$ is the generating function of $1, 1, 1, \ldots$

And $\frac{1}{(1-x)^n}$ is the generating function of $\binom{n-1}{n-1}, \binom{n}{n-1}, \binom{n+1}{n-1}, \ldots$. Note that these count weak compositions of k with n parts, $\binom{n+k-1}{n-1}$

For a finite sequence the generating function is a polynomial:

Example. $(1+x)^n$ is the generating function for $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$

Formal power series behave just like polynomials with addition and multiplication:

$$\sum_{k \geq 0} a_k x^k + \sum_{k \geq 0} b_k x^k = \sum_{k \geq 0} (a_k + b_k) x^k$$

$$\left(\sum_{k \geq 0} a_k x^k \right) \left(\sum_{k \geq 0} b_k x^k \right) = \sum_{k \geq 0} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k$$

These form a ring if you know what that is. Also, we will relate these to the addition and multiplication principles respectively.

Now lets do some stuff:

$$\begin{aligned} \frac{1}{(1-x)^n} &= \left(\frac{1}{1-x} \right)^n \\ &= (1 + x + x^2 + x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots) \cdots (1 + x + x^2 + \ldots) \\ &= \sum_{k \geq 0} (?) x^k \end{aligned}$$

Well (?) is just how many ways to pick n nonnegative integers which add up to k. That is weak compositions of k with n parts. And so:

$$\frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n+k-1}{n-1} x^k$$

Example. Lets look at triangular numbers. Define T_k:

$$T_k = 1 + 2 + \ldots + k$$

for $k \in \mathbb{Z}_{>0}$ and let $T_0 = 0$. We know by induction:

$$T_k = \frac{k(k+1)}{2} = \binom{k+1}{2}$$

Note that we know:

$$\frac{1}{(1-x)^3} = \sum_{k \geq 0} \binom{k+2}{2} x^k = 1 + 3x + 6x^2 + 10x^3$$

So then:

$$\sum_{k \geq 0} \binom{k+1}{2} x^k = 0 + x + 3x^2 + 6x^3 + 10x^4 + \ldots$$

Take the following:

$$\begin{aligned}
\sum_{k \geq 0} T_k x^k &= \sum_{k \geq 0} \binom{k+1}{2} x^k = \sum_{k \geq 1} \binom{k+1}{2} x^k \\
&= \sum_{\ell \geq 0} \binom{\ell+2}{2} x^{\ell+1} = x \sum_{\ell \geq 0} \binom{\ell+2}{2} x^\ell \\
\sum_{k \geq 0} T_k x^k &= \frac{x}{(1-x)^3}
\end{aligned}$$

Example. Note then by the same reasoning:

$$\frac{x^2}{(1-x)^3} = \sum_{k \geq 0} \binom{k}{2} x^k$$

Example. Find a closed formula for the generating function for $a_k = k^2$. Well:

$$2T_k - k = (k+1)k - k = k^2$$

We know the generating function of T_k. Let's find the generating function of k:

$$\frac{\mathrm{d}\left[\frac{1}{1-x}\right]}{\mathrm{d}x} = \sum_{k=0}^{\infty} k x^{k-1}$$

So then actually taking this derivative and multiplying by x:

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$$

So lets look at this as:

$$\begin{aligned}
\sum_{k\geq 0} k^2x^k &= \sum_{k\geq 0}(2T_k - k)x^k \\
&= \frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2} \\
&= \frac{2x}{(1-x)^3} - \frac{x(1-x)}{(1-x)^3} \\
&= \frac{2x - x + x^2}{(1-x)^3} = \frac{x^2+x}{(1-x)^3}
\end{aligned}$$

We can also look at $T_k + T_{k-1} = k^2$. This is interesting:

$$\begin{aligned}
\sum_{k\geq 0} k^2x^k &= \sum_{k\geq 0}(T_k + T_{k-1})x^k \\
&= \frac{x}{(1-x)^3} + \sum_{k\geq 0} T_{k-1}x^k \\
&= \frac{x}{(1-x)^3} + \sum_{\ell\geq 0} T_\ell x^{\ell+1} \\
&= \frac{x}{1-x^2}
\end{aligned}$$

MATH 465 Notes

Faye Jackson

28 January, 2020

1 Introduction

1.1 Stuff

- Quiz 5 Tuesday
- Substitute

2 More Generating Functions

2.1 Some Examples

Recall.

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$$

Question: Let a_k be the number of k-card hands dealt from 2 standard 52-card decks. What is a_k.

If $k = 5$ there's not much that can happen, three possibilities, how many duplicates do you have (none, one pair, two pairs):

$$a_5 = \binom{52}{47, 5, 0} + \binom{52}{48, 3, 1} + \binom{52}{49, 1, 2} = 3,748,160$$

Generating Function:

$$\sum_{k \geq 0} a_k x^k = \left(1 + x + x^2\right)^{52}$$

This is because each card can show up 0 times, 1 time, or 2 times.

Question: How many ways can we give change for a dollar with dimes and quarters?

Either 0 quarters and 10 dimes, or 2 quarters with 5 dimes, or 4 quarters with 0 dimes. So the answer is 3.

Harder Question: Define $a_k = \#$ of integer solutions to $10d + 25q = k$ (where d is dimes and q is quarters.

Solution $(d, q) \longleftrightarrow x^{10d}x^{25q} = x^k$

$$\begin{aligned} \sum_{k \geq 0} a_k x^k &= (1 + x^{10} + x^{20} + \ldots)(1 + x^{25} + x^{50} + \ldots) \\ &= \frac{1}{(1 - x^{10})(1 - x^{25})} \end{aligned}$$

2.2 The Multiplication Principle

Proposition 1. [Multiplication Principle]
Let A, B, C be finite sets with a bijection $A \xrightarrow{f} B \times C$, with weight functions:

$$w_A : A \to \mathbb{Z}_{\geq 0} \qquad w_B : B \to \mathbb{Z}_{\geq 0} \qquad w_C : C \to \mathbb{Z}_{\geq 0}$$

Such that for $\alpha \in A \overset{f}{\mapsto} (\beta, \gamma) \in B \times C$, we have:

$$w_A(\alpha) = w_B(\beta) + w_C(\gamma)$$

Let:

$$a_k = |\{\alpha \in A \mid w_A(\alpha) = k\}| \quad b_k = |\{\beta \in B \mid w_B(\beta) = k\}| \quad c_k = |\{\gamma \in C \mid w_C(\gamma) = k\}|$$

Then:

$$\begin{aligned} \sum_{k \geq 0} a_k x^k &= \left(\sum b_k x^k\right)\left(\sum c_k x^k\right) \\ \sum_{\alpha \in A} x^{w_A(\alpha)} &= \left(\sum_{\beta \in B} x^{w_B(\beta)}\right)\left(\sum_{\gamma \in C} x^{w_C(\gamma)}\right) \end{aligned}$$

Example.

$$
\begin{aligned}
B &= \{\text{collections of dimes}\}\\
C &= \{\text{collections of quarters}\}\\
A &= \{\text{collection of both kinds of coins}\}
\end{aligned}
$$

Example. How many ways to change k cents into pennies, nickels, dimes, and quarters? Call this a_k. Then:

$$
\begin{aligned}
\sum_{k\geq 0} a_k x^k &= (1+x+x^2+\ldots)(1+x^5+x^{10}+\ldots)(1+x^{10}+x^{20}+\ldots)(1+x^{25}+x^{50}+\ldots)\\
\sum_{k\geq 0} a_k x^k &= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}
\end{aligned}
$$

2.3 What about Permutations

Some Notation for Convenience:

$$
\begin{aligned}
[n] &= \{1,2,\ldots,n\}\\
S_n &= \text{ set of permutations of } [n] = \text{ The Symmetric Group}
\end{aligned}
$$

One line notation $2314 = 2,3,1,4 \in S_4$

For $\sigma \in S_n$, let σ_i be the ith object

Definition 1. *An inversion of σ is a pair (i,j) with $i<j$ but $\sigma_i > \sigma_j$.*

An inversion tells you how "out of order" you are.

Goal: Count permutations by the # of inversions = $\mathrm{inv}(\sigma)$.

Example. S_3:

σ	$\mathrm{inv}(\sigma)$	$\mathrm{code}(\sigma)$
123	0	(0, 0, 0)
132	1	(0, 0, 1)
213	1	(0, 1, 0)
231	2	(0, 1, 1)
312	2	(0, 0, 2)
321	3	(0, 1, 2)

Generating function:

$$1 + 2x^1 + 2x^2 + x^3 = 1 \cdot (1 + x) \cdot (1 + x + x^2)$$

Definition 2. *The (inv) code of $\sigma \in S_n$ is a sequence $c = (c_1, \ldots, c_n)$ with c_k is the # of inversions (i, j) with $\sigma_i = k$.*

Note: $\mathrm{inv}(\sigma) = \sum \mathrm{code}(\sigma)$

Theorem 1. *Let a_k be the # of $\sigma \in S_n$ with $\mathrm{inv}(\sigma) = k$. THen:*

$$\begin{aligned}\sum_{k \geq 0} a_k x^k &= \sum_{\sigma \in S_n} x^{\mathrm{inv}(\sigma)} \\ &= 1(1+x)(1+x+x^2)(1+x+x^2+x^3)\cdots(1+x+\cdots+x^{n-1}) \\ &= \prod_{k=1}^{n} \frac{1-x^k}{1-x}\end{aligned}$$

Proof.

$$\mathrm{code} : S_n \to \{0\} \times \{0,1\} \times \cdots \times \{0, \ldots, n-1\} = c_n$$

This is actually a bijection! Note that $|S_n| = n! = |c_n|$, it is enough to show either code is an injection or surjection. We will show it is a surjection. By example, cosnider $c = (0, 1, 0, 2, 2, 1) \in c_n$.

We want to form a permutation $\sigma \in S_n$ with $\mathrm{code}(\sigma) = c$. We build it up piecewisely:

$$1 \mapsto 21 \mapsto 213 \mapsto 2413 \mapsto 24513 \mapsto 245163$$

We use $\mathrm{inv} : S_n \to \mathbb{Z}_{\geq 0}$ as weight function, and $\mathrm{weight}(c_i) = c_i$. With these weights is a weight preserving bijection, apply the multiplication principle and the result falls out.

TODO $\square$

MATH 465 Notes

Faye Jackson

30 January, 2020

1 Introduction

1.1 Stuff

- Quiz 6 Today
- HW4 due Wednesday
- Today: Stirling Numbers
- Next Time: Linear Recurrence
- No Offic Hours Monday, instead Friday 10:30-11:30 and 1:30-3.

2 Looking at Permutations

We want to view $\sigma \in S_n = \{\text{permutations of } [n]\}$ as a bijective function $\sigma : [n] \to [n]$, thinking that $\sigma(i)$ is the object in the ith place. We have "one-line" notation:

$$\sigma(1)\sigma(2)\dots\sigma(n)$$

Suppose that $a \in [n]$ let k be the smallest positive integer for which $\sigma^k(a) = a$. [By the pigeonhole principle, there exists $i < j$ so that $\sigma^i(a) = \sigma^j(a)$ to get $\sigma^{j-i}(a)$].

We say that $(a, \sigma(a), \sigma^2(a), \dots, \sigma^{k-1}(a))$ is a k-cycle of σ at a.

Example. Take $2561437 \in S_7$. Then:

$$\begin{aligned} \sigma(1) &= 2 & \sigma^2(1) &= 5 \\ \sigma^3(1) &= 4 & \sigma^4(1) &= 1 \end{aligned}$$

This is the cycle $(1, 2, 5, 4) = (2, 5, 4, 1)$. We also have sycles (7) and $(3, 6)$.

So then $\sigma = (1254)(36)(7)$ in "cycle notation"

$$\begin{tikzcd} 1 \arrow[r, mapsto] & 2 \arrow[d, mapsto] \\ 4 \arrow[u, mapsto] & 5 \arrow[l, mapsto] \end{tikzcd}$$

Example. $n = 3$. For $\sigma \in S_n$, let $c(\sigma)$ be the number of cycles in σ's cycle notation.

σ (one-line)	"cycle notation"	$c(\sigma)$	$b(\sigma)$
123	(1)(2)(3)	3	$(*, *, *)$
132	(1)(32)	2	$(*, *, 2)$
213	(12)(3)	2	$(*, 1, *)$
231	(231)	1	$(*, 1, 2)$
312	(321)	1	$(*, 1, 1)$
321	(31)(2)	2	$(*, *, 1)$

Now let $c(n, k) = \#\{\sigma \in S_n \mid c(\sigma = k)\}$. Then conventionally $c(0, 0) = 1$, and $c(n, k) = 0$ unless $0 \leq k \leq n$

Example. For $n = 3$ we have $c(3, 3) = 1$, $c(3, 2) = 3$, and $c(3, 1) = 2$. Note then that:

$$\sum_{k \geq 0} c(3, k)x^k = 2x^1 + 3x^2 + 1x^3 = x(x + 1)(x + 2)$$

Theorem 1. *Let $n \in \mathbb{Z}_{\geq 0}$ then the generating function for $c(n, 0), c(n, 1), c(n, 2), \ldots$ is.*

$$\sum_{k \geq 0} c(n, k)x^k = \sum_{\sigma \in S_n} x^{c(\sigma)} = x(x + 1)(x + 2) \cdots (x + n - 1)$$

Proof. We will build a permutation one step at a time keeping track of our cycles with some tuple $b(\sigma) = (b_1, b_2, \ldots, b_n)$ At the kth step for $1 \leq k \leq n$ we define b_k by either:

- Add (k) by a singleton cycle, record $b_k = *$
- Insert k into a cycle, say after b_k.

Refer back to the above table where we do this for $n = 3$. Remember for 231 we do $(1) \mapsto (12) \mapsto (123) = (231)$. For something like $312 = (321)$ it is:

$$(1) \mapsto (12) \mapsto (132) = (321)$$

Really this defines a bijection:

$$b : S_n \to \{*\} \times \{*, 1\} \times \{*, 1, 2\} \times \cdots \times \{*, 1, 2, \ldots, n-1\}$$

Like on Tuesday we need only show this is an injection or surjection since the number of elements is the same. Now think about weights.

On the left consider the weight $\sigma \overset{c}{\mapsto} c(\sigma)$ and on the right consider the weights of:

$$*, 1, 2, \ldots, n-1 \mapsto 1, 0, 0, \ldots, 0$$

Respectively. Since $(\sigma) = \#$ of $*$'s in $b(\sigma)$, this means b is weight preserving. By the multiplication principle of generating functions:

$$\sum_{\sigma \in S_n} x^{c(\sigma)} = \prod_{k=1}^{n} (1x^1 + (k-1)x^0) = x(x+1)\cdots(x+n-1)$$

□

Corrolary 1.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

Proof. Observe that:

$$\sum_{k \geq 0} c(n,k)x^k = (x+n-1) \sum_{k \geq 0} c(n-1,k)x^k$$

The coefficient of x^k is $c(n,k)$ on the LHS and the coefficient of x^k is what we would really like it to be (by reindexing): $c(n-1,k-1) + (n-1)c(n-1,k)$. □

Note, we know some nice things like $c(n,1) = (n-1)!$ and $c(n,n) = 1$.

3 Stirling Numbers???

3.1 These are Stirling Numbers???

Ok, so $c(n,k)$ is called a "signless Stirling number of the first kind," so what's the "Stirling number of the first kind?" Well:

$$s(n,k) = (-1)^{n-k} c(n,k)$$

Huh, we can make a triangle of $c(n,k)$ by the Corollary above.!

1 4

5

Corrolary 2. *Let $n \in \mathbb{Z}_{>0}$ The generating function for the stirling numbers of the first kind $s(n,0), s(n,1), s(n,2)\dots$:*

$$\sum_{k=1}^{n} s(n,k)x^k = x(x-1)(x-2)\cdots(x-n+1)$$
$$= (x)_n = n!\binom{x}{n}$$

Proof. Left as an exercise □

3.2 New Stirling Numbers

A set partition of the set T is a set P of pairwise disjoint nonempty subsets such that $\bigcup_{B\in P} B = T$. The elements of P are called "blocks." Define the Stirling number of the Second Kind as:

$$S(n,k) = \#\text{ of partitons of } [n] \text{ with } k \text{ blocks}$$

Example. $n = 3$, $T = \{1,2,3\}$. The possible partitions are:

$$\begin{array}{cccc} & \{\{1,2,3\}\} & & S(3,1)=1 \\ \{\{1,2\},\{3\}\} & \{\{1,3\},\{2\}\} & \{\{2,3\},\{1\}\} & S(3,2)=3 \\ & \{\{1\},\{2\},\{3\}\} & & S(3,3)=1 \end{array}$$

We can say that:

$$S(n,1) = 1 \qquad\qquad S(n,n) = 1$$

Also some conventions $S(0,0) = 1$ and $S(n,k) = 0$ unless $0 \le k \le n$.

Theorem 2. *For $n,k \in \mathbb{Z}_{\ge 0}$ such that $n \ge k$ we have that:*

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

Proof. A k-block partition of $[n]$ either has $\{n\}$ as a signleton block, there are $S(n-1, k-1)$ of these. Or it can be obtained by inserting n into one of the k blocks of a k-block partition fo $[n-1]$, that is $kS(n-1, k)$. □

Theorem 3. *For $n \in \mathbb{Z}_{>0}$ we have that:*

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k = \sum_{k=1}^{n} S(n,k)k!\binom{x}{k}$$

Proof. We will give a combinatorial proof, note these are polynomials, so it suffices to prove this when $x = m$ is a positive integer by the Fundamental Theorem of Algebra. This is good, we can count things. There are m^n ways to color the set $[n]$ with m colors (a function $f : [n] \to [m]$).

OUT OF TIME ☹

…To be continued! □

MATH 465 Notes

Faye Jackson

30 January, 2020

1 Introduction

1.1 Stuff

- Quiz (5 min.) Remember to justify
- Today is Linear Recurrences
- HW4 due Wednesday

2 Linear Recurrencese (Homogeneous / Non-Homogeneous) (8.2-8.5)

2.1 Fibonacci Numbers

Example. Suppose you have an infinite supply of two types of tile: squares and dominoes (dominoes are twice as long as squares).

Question: How many ways are there to arrange these tiles into a row of some fixed length?

Answer: Let R_n be the number of ways to arrange these tiles into a row of length $n \in \mathbb{N}$. Let's look at examples

$$\begin{aligned} R_0 &= 1 & R_1 &= 1 \\ R_2 &= 2 & R_3 &= 3 \\ R_4 &= 5 & R_5 &= 8 \end{aligned}$$

In general if R_n is the number of ways to tile a row of length $n \in \mathbb{N}$,

$$R_n = R_{n-1} + R_{n-2} \qquad \text{for } n \geq 2$$

A tiling of length n is a tiling of length $n-1$ with a square added or a tiling of length $n-2$ with a domino added.

Theorem 1 (Binet's Formula)**.**

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

We let $F_0 = 0$ and $F_n = R_{n-1}$, then this satisfies $F_n = F_{n-1} + F_{n-2}$.

Note a combinatorial proof of this is quite impossible. We have to use other tools. Let's build them!

2.2 Linear Recurrence in General, a Definition

Definition 1. *A sequence of numbers $a_0, a_1, a_2, \ldots$ satisfies a <u>homogeneous linear recurrence</u> if it satisfies:*

$$a_n + r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} = 0$$

For k fixed and $n \geq k$. The r_i are constants, and $r_k \neq 0$.

A <u>non-homogeneous linear recurrence</u> satisfies:

$$a_n + r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} = h(n)$$

For some constant funtion $h : \mathbb{N} \to \mathbb{N}$.

<u>Goal:</u> Find a Formula for a_n:

Example. $F_n - F_{n-1} - F_{n-2} = 0$ with $k = 2$, $r_1 = r_2 = -1$.

Example. Define a sequence $a_0, a_1, a_2, \ldots$ by setting $a_0 = 1, a_1 = 1, a_2 = 4$, and:

$$\begin{aligned} a_n &= 3a_{n-2} + 2a_{n-3} \qquad \text{for } n \geq 3 \\ a_3 &= 3a_1 + 2a_0 = 5 \\ a_4 &= 3a_2 + 2a_1 = 12 + 2 = 14 \end{aligned}$$

Write this as:

$$a_n - 0a_{n-1} - 3a_{n-2} - 2a_{n-3} = 0$$

Gives this as a homogeneous linear recurrence with $k = 3$, $r_1 = 0$, $r_2 = -3$, $r_3 = -2$.

Example. Let $a_0 = 3, a_1 = 9$, and:

$$a_n = a_{n-1} + 2a_{n-2} - 4n$$

This is a clear non-homogeneous linear recurrence with $h(n) = -4n$.

Non-Example. Let $a_0 = 1$ and $a_n = na_{n-1}$. Well this is:

$$\begin{aligned} a_0 &= 1 & a_1 &= 1 \\ a_2 &= 2 & a_3 &= 3 \cdot 2 \\ a_4 &= 4 \cdot 3 \cdot 2 & a_5 &= 5 \cdot 4 \cdot 3 \cdot 2 \end{aligned}$$

We have a guess that $a_n = n!$. We can prove it using a counting argument and induction. We know $a_0 = 1 = 0!$. Assume $a_n = n!$, we must show $a_{n+1} = (n+1)!$, but this is simply by definition $a_{n+1} = (n+1)a_n = (n+1)n! = (n+1)!$.

In general we need stronger tools to study non-linear recurrence

2.3 The Characteristic Equation Approach

Definition 2. *Given a homogeneous linear recurrence:*

$$a_n + r_1 a_{n-1} + \cdots + r_k a_{n-k} = 0$$

We write a polynomial associated to it called the characteristic polynomial:

$$q^k + r_1 q^{k-1} + \cdots + r_k = 0$$

Theorem 2. *Let k be a positive integer, and let the $r_1, \ldots, r_k \in \mathbb{R}$, let $q_1, q_2, \ldots, q_k$ be k distinct roots of the characteristic polynomial.*

Then for any $a_n + r_1 a_{n-1} + \cdots + r_k a_{n-k} = 0$. We have that the n-th term has the following form:

$$a_n = \sum_{i=1}^{k} c_i q_i^n \qquad \text{for some constants } c_i$$

Sketch of Proof. Morally this is like a generalized antiderivative: i.e. if we know the initial conditions $a_0, a_1, \ldots, a_{k-1}$, then we could solve for all of the c_i. Like the $+C$ in an antiderivative

(1) Consider the set V of all real-valued sequences $(a_0, a_1, a_2, \ldots)$ which satisfy the recurrence. BUT not necessarily the same initial conditions.

$$V = \{(a_0, a_1, \ldots) \mid a_n + r_1 a_{n-1} + \cdots r_k a_{n-k} = 0 \;\; \forall n \geq k\}$$

- Because of homogeneity note then that V is a vector space over $\mathbb{R}$.
- There's a map

$$T : V \to \mathbb{R}^k$$

$$(a_0, a_1, \ldots) \underset{\sim}{\overset{T}{\longmapsto}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix}$$

  Furthermore this map is an isomorphism, We can uniquely recover the sequence $(a_0, a_1, \ldots)$ from the first k terms using the linear recurrence.

- This tells us the dimension of our vector space V, k.

(2) If $q \neq 0$ then the geometric sequence defined by $a_n = q^n$ satisfies the linear recurrence if and only if q is a root of the characteristic polynomial.

In particular, consider k distinct roots $q_1, \ldots, q_k$ of the characteristic polynomial (which are nonzero since $r_k \neq 0$). Then the geometric sequence:

$$s_1 = (1, q_1, q_1^2, \dots)$$
$$\vdots$$
$$s_k = (1, q_k, q_k^2, \dots)$$

Are solutions of the linear recurrence. Thus for each $1 \leq i \leq k$ we have $s_i \in V$.

(3) We want to make $s_1, s_2, \dots, s_k$ a basis for V.

Proof. We just need them to be linearly independent. Suppose they were not. Then we would have:

$$c_1 s_1 + c_2 s_2 + \cdots + c_k s_k = 0$$

with $c_i \neq 0$ for some i. We can just look at the first k terms:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix}$$

On HW1 Problem 3 we proved that this matrix is invertible, so we proved that we cannot have:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Unless $(c_1, c_2, \dots, c_k) = (0, 0, \dots, 0)$. $\square$

$\square$

Example. [Fibonacci Redux] The characteristic polynomial for the Fibonacci recurrence is $q^2 - q - 1 = 0$. It has roots:

$$q_1 = \frac{1+\sqrt{5}}{2}$$
$$q_2 = \frac{1-\sqrt{5}}{2}$$

We then know that:

$$F_n = c_1 q_1^n + c_2 q_2^n$$

For some $c_1, c_2 \in \mathbb{R}$ by $F_0 = 0 = c_1 + c_2$ and $F_1 = 1 = c_1 q_1 + c_2 q_2$. So then we must have that:

$$\begin{aligned} 1 &= c_1(q_1 - q_2) \\ c_1 &= \frac{1}{q_1 - q_2} = \frac{1}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} = \frac{1}{\sqrt{5}} \\ c_2 &= -\frac{1}{\sqrt{5}} \end{aligned}$$

Which gives that:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example. Say we have:

$$a_n + 0a_{n-1} - 3a_{n-2} - 2a_{n-3} = 0$$

Which gives the associated charateristic polynomial:

$$\begin{aligned} q^3 - 3q - 2 &= 0 \\ (q+1)^2(q-2) &= 0 \end{aligned}$$

But now we have a root with multiplicity! How do we deal with it.

Theorem 3. *Let k be a positive integer, and let the $r_1, \ldots, r_k \in \mathbb{R}$, let $q_1, q_2, \ldots, q_d$ be distinct roots of the characteristic polynomial with $d \leq k$ and multiplicities $m_1, m_2, \ldots, m_d$.*

$$q^k + r_1 q^{k-1} + \cdots = r_{k-1} q + r_k = 0$$

Then for any recurrence of the form $a_n + r_1 a_{n-1} + \cdots + r_k a_{n-k} = 0$. We have that the n-th term has the following form:

$$a_n = \sum_{i=1}^{d} \sum_{j=1}^{m_i} c_{i,j} n^{j-1} q_i^n \qquad \text{for some constants } c_{i,j}$$

This has to do with derivatives.

Proof Omitted. The idea is a root with multiplicity is a root of the derivatives. □

MATH 465 Notes

Faye Jackson

30 January, 2020

1 Introduction

1.1 Stuff

- Substitute: Al(exander) Gawer
- HW5 Due Wednesday

1.2 Last Time, Homogeneous Linear Recurrences

Theorem 1. *Let k be a positive integer $r_1, \dots, r_k \in \mathbb{R}$. Suppose $q_1, \dots, q_d$ are all the distinct roots of:*

$$q^k + r_1 q^{k-1} + \cdots + r_{k-1} q + r_k = 0$$

with multiplicities $m_1, \dots, m_d$. Then any sequence (a_n) satisfying:

$$a_n + r_1 a_{n-1} + \cdots r_k a_{n-k} = 0$$

is of the form:

$$a_n = \sum_{i=1}^{d} \sum_{j=0}^{m_i - 1} c_{i,j} n^j q_i^n$$

For some constants $c_{i,j}$.

Theorem 2 (Generalized Binomial Theorem)**.** *Let $d \in \mathbb{Z}_{\geq 0}$:*

$$\frac{1}{(1-x)^{d+1}} = \sum_{n=0}^{\infty} \binom{n+d}{d} x^n$$

2 Some Cool Examples

2.1 Homogeneous Linear Recurrences

Example. Suppose we have:

$$\sum_{n=0}^{\infty} a_n x^n = \frac{2}{1-3x} + \frac{1}{(1+x)^3}$$

We want to determine what is (a_n)?

$$\begin{aligned}
\frac{2}{1-3x} + \frac{1}{(1+x)^3} &= 2 \cdot \sum_{n=0}^{\infty} \binom{n+0}{0} (3x)^n + \sum_{n=0}^{\infty} \binom{n+2}{2} (-x)^n \\
&= 2 \cdot \sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} \binom{n+2}{2} (-1)^n x^n \\
&= \sum_{n=0}^{\infty} \left(2 \cdot 3^n + \binom{n+2}{2} (-1)^n \right) x^n
\end{aligned}$$

Therefore for each $n \geq 0$:

$$a_n = 2 \cdot 3^n + (-1)^n \binom{n+2}{2}$$

Awesome!

Example. Find (a_n) where we have $a_0 = 1$, $a_1 = -2$,

$$a_n = 5a_{n-1} - 6a_{n-2}$$

We have two techniques

- The characteristic polynomial is:

$$\begin{aligned}
q^2 - 5q^1 + 6 &= 0 \\
(q-2)(q-3) &= 0
\end{aligned}$$

  And so $q_1 = 2$ and $q_2 = 3$, with multiplicities of 1. Thus for any n we have

$$a_n = c_1 2^n + c_2 3^n$$

Using the initial values we have:

$$
\begin{aligned}
1 &= a_0 = c_1 + c_2 \\
2 &= a_2 = c_1 \cdot 2 + c_2 \cdot 3
\end{aligned}
$$

We could solve for c_1 and c_2

- Set

$$
\begin{aligned}
A(x) = \sum_{n\geq 0} a_n x^n &= a_0 + a_1 x + \sum_{n\geq 2} a_n x^n \\
&= a_0 + a_1 x + \sum_{n\geq 2} (5a_{n-1} - 6a_{n-2}) x^n \\
&= a_0 + a_1 x + 5 \cdot \sum_{n\geq 2} a_{n-1} x^n - 6 \cdot \sum_{n\geq 2} a_{n-2} x^n \\
&= a_0 + a_1 x + 5x \cdot \sum_{n\geq 2} a_{n-1} x^{n-1} - 6x^2 \cdot \sum_{n\geq 2} a_{n-2} x^{n-2} \\
&= a_0 + a_1 x + 5x \cdot \sum_{n\geq 2} a_{n-1} x^{n-1} - 6x^2 A(x) \\
&= a_0 + a_1 x + 5x[A(x) - a_0] - 6x^2 A(x)
\end{aligned}
$$

Collect all the $A(x)$ terms to one side:

$$
\begin{aligned}
A(x) - 5x \cdot A(x) + 6x^2 \cdot A(x) &= a_0 + a_1 x - 5x a_0 \\
(1 - 5x + 6x^2) \cdot A(x) &= 1 - 7x \\
A(x) &= \frac{1 - 7x}{(1 - 2x)(1 - 3x)}
\end{aligned}
$$

Then we will use partial fractions:

$$
\begin{aligned}
\frac{1 - 7x}{(1 - 2x)(1 - 3x)} &= \frac{C_1}{1 - 2x} + \frac{C_2}{1 - 3x} \\
1 - 7x &= c_1(1 - 3x) + c_2(1 - 2x) \\
&= c_1 + c_2 - (3c_1 - 2c_2)x
\end{aligned}
$$

$$
\begin{aligned}
1 &= c_1 + c_2 \\
-7 &= -3c_1 - 2c_2
\end{aligned}
$$

Then we would solve the linear equation to get $c_1 = 5, c_2 = -4$. Thus:

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \frac{5}{1-2x} + \frac{-4}{1-3x} \\
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} 5(2x)^n + \sum_{n\geq 0} -4(3x)^n \\
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} (5\cdot 2^n - 4\cdot 3^n)x^n \\
a_n &= 5\cdot 2^n - 4\cdot 3^n
\end{aligned}$$

Example. $a_0, a_1 = 1$, $a_2 = 4$. And $a_n = 3a_{n-2} + 2a_{n-3}$.

Let's do this with generating functions (BOO ☹). Goal is to find $A(x) = \sum_{n\geq 0} a_n x^n$:

$$\begin{aligned}
\sum_{n\geq 3} a_n x^n &= \sum_{n\geq 3} 3a_{n-2}x^n + \sum_{n\geq 3} 2a_{n-3}x^n \\
&= 3\cdot \sum_{n\geq 3} a_{n-2}x^n + 2\cdot \sum_{n\geq 3} a_{n-3}x^n \\
&= 3x^2\cdot \sum_{n\geq 3} a_{n-2}x^{n-2} + 2x^3\cdot \sum_{n\geq 3} a_{n-3}x^{n-3} \\
\sum_{n\geq 3} a_n x^n &= 3x^2\cdot [A(x) - a_0] + 2x^3\cdot A(x) \\
A(x) - a_2x^2 - a_1x - a_0 &= 3x^2A(x) - 3x^2a_0 + 2x^3A(x) \\
A(x) - 3x^2A(x) - 2x^3A(x) &= a_2x^2 + a_1x + a_0 - 3x^2a_0 \\
A(x)(1 - 3x^2 - 2x^3) &= 4x^2 + x + 3 - 9x^2 \\
A(x) &= \frac{5x^2 - x - 3}{2x^3 + 3x^2 - 1} = \frac{5x^2 - x - 3}{(1+x)^2(1-2x)}
\end{aligned}$$

We look at it in terms of partial fractions:

$$\begin{aligned}
A(x) &= \frac{5x^2 - x - 3}{2x^3 + 3x^2 - 1} = \frac{5x^2 - x - 3}{(1+x)^2(1-2x)} \\
&= \frac{c_1}{1+x} + \frac{c_2}{(1+x^2)} + \frac{c_3}{(1-2x)} \\
3 + x - 5x^2 &= c_1(1-2x)(1+x) + c_2(1-2x) + c_3(x+1)^2 \\
5 &= -2c_1 + c_3 \\
1 &= -c_1 - 2c_2 + 2c_3 \\
3 &= c_1 + c_2 + c_3
\end{aligned}$$

If you did solve these you would get $c_1 = 3, c_2 = -1, c_3 = 1$.

$$\begin{aligned}
A(x) &= \frac{3}{1+x} - \frac{1}{(1+x)^2} + \frac{1}{1-2x} \\
&= \sum_{n\geq 0} 3(-x)^n - \sum_{n\geq 0}(n+1)(-x)^n + \sum_{n\geq 0}(2x)^n \\
&= \sum_{n\geq 0}[(-1)^n \cdot 3 - (-1)^n(n+1) + 2^n]x^n
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
a_n &= 3(-1)^n + (-1)^{n+1}(n+1) + 2^n \\
&= (3-1)(-1)^n - n(-1)^n + 2^n \\
&= (2-n)(-1)^n + 2^n
\end{aligned}$$

2.2 Nonhomogeneous Linear Recurrence

Example. Let $a_0 = 3, a_1 = 9$ and:

$$a_n + 4n = a_{n-1} + 2a_{n-2}$$

Set $A(x) = \sum_{n\geq 0} a_n$. Then we have that:

$$\begin{aligned}
\sum_{n\geq 2} a_n x^n &= \sum_{n\geq 2}(a_{n-1} + 2a_{n-2} - 4n)x^n \\
&= x \cdot \sum_{n\geq 2} a_{n-1}x^{n-1} + 2x^2 \cdot \sum_{n\geq 2} a_{n-2}x^{n-2} - 4 \cdot \sum_{n\geq 2} n \cdot x^n \\
&= x \cdot [A(x) - a_0] + 2x^2 A(x) - 4x \cdot \sum_{n\geq 2} nx^{n-1} \\
&= x \cdot [A(x) - a_0] + 2x^2 A(x) - 4x \cdot \sum_{n\geq 1} (n+1)x^n \\
&= x \cdot [A(x) - a_0] + 2x^2 A(x) - 4x \left[\frac{1}{(1-x)^2} - 1\right]
\end{aligned}$$

So then we get the following:

$$\begin{aligned}
A(x) - 9x - 3 &= x \cdot A(x) - 3x + 2x^2 \cdot A(x) - \frac{4x}{(1-x)^2} + 4x \\
A(x) \cdot \left[1 - x - 2x^2\right] &= 10x + 3 - \frac{4x}{(1-x)^2} \\
A(x) &= \frac{10x+3}{1-x-2x^2} - \frac{4x}{(1-x)^2(1-x-2x^2)} \\
&= \frac{c_1}{1-2x} + \frac{c_2}{1+x} + \frac{c_3}{1-x} + \frac{c_4}{(1-x)^2}
\end{aligned}$$

Then $c_1 = 0$, $c_2 = -2$, $c_3 = 3$, and $c_4 = 2$. With these you can simplify and:

$$a_n = -2(-1)^n + 3 + 2(n+1) = -2(-1)^n + 2n + 5$$

2.3 Non-Linear Recurrence

Example. $c_0 = 1$, $c_1 = 1$, and:

$$c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}$$

Call $C(x)$ the generating function, $\sum_{n\geq 0} c_n x^n$. Then:

$$(C(x))^2 = \sum_{n\geq 0} \left(\sum_{i=0}^{n} c_i c_{n-i}\right) x^n = \sum_{n\geq 0} c_{n+1} x^n$$

So then we have:

$$
\begin{aligned}
x(C(x))^2 = \sum_{n\geq 0} c_{n+1}x^{n+1} &= C(x) - c_0 \\
x(C(x))^2 - C(x) + 1 &= 0 \\
x^2(C(x))^2 - xC(x) + x &= 0 \\
x^2(C(x))^2 - xC(x) &= -x \\
x^2(C(x))^2 - x2\left(\frac{1}{2}\right)C(x) + \left(\frac{1}{2}\right)^2 &= \frac{1}{4} - x \\
\left(xC(x) - \frac{1}{2}\right)^2 &= \frac{1-4x}{4} \\
C(x) &= \frac{\sqrt{1-4x}}{2x} + \frac{1}{2x}
\end{aligned}
$$

Then we can get a formula for c_n.

OUT OF TIME ♡

MATH 465 Notes

Faye Jackson

11 February, 2020

1 The Catalan Numbers

These counts super important things, and they count a lot of things. They're super cool.

Example. Let $c_0 = 1$ and let $c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}$ for $n \geq 0$.

Consider the generating function of this thing:

$$\mathbb{C}(x) = \sum_{n \geq 0} c_n x^n$$

Now let's consider:

$$\begin{aligned}
[\mathbb{C}(x)]^2 &= \left(\sum_{n\geq 0} c_n x^n\right)\left(\sum_{n\geq 0} c_n x^n\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} c_i c_{n-i}\right) x^n \\
&= \sum_{n\geq 0} c_{n+1} x^n
\end{aligned}$$

So then we have:

$$\begin{aligned}
x\mathbb{C}(x)^2 &= \sum_{n\geq 0} c_{n+1} x^{n+1} = \mathbb{C}(x) - c_0 x^0 = \mathbb{C}(x) - 1 \\
x\mathbb{C}(x)^2 &= \mathbb{C}(x) - 1
\end{aligned}$$

We solve this by completing the square:

$$\begin{aligned}
x\mathbb{C}(x)^2 - \mathbb{C}(x) + 1 &= 0 \\
x^2\mathbb{C}(x)^2 - x\mathbb{C}(x) + x &= 0 \\
\left(x\mathbb{C}(x) - \frac{1}{2}\right)^2 - \frac{1}{4} + x &= 0 \\
\left(x\mathbb{C}(x) - \frac{1}{2}\right)^2 &= \frac{1}{4} - x \\
x\mathbb{C}(x) - \frac{1}{2} &= \pm\frac{\sqrt{1-4x}}{2} \\
&= -\frac{\sqrt{1-4x}}{2} \\
\mathbb{C}(x) &= \frac{1-\sqrt{1-4x}}{2x}
\end{aligned}$$

Note that we get the $-$ from the $\pm$ by plugging in 0 to line five. By homework we know that:

$$\mathbb{C}(x) = \sum_{n\geq 0} \binom{2n}{n} \frac{x^n}{n+1}$$

These are the Catalan Numbers $c_n = \binom{2n}{n} \cdot \frac{1}{n+1}$

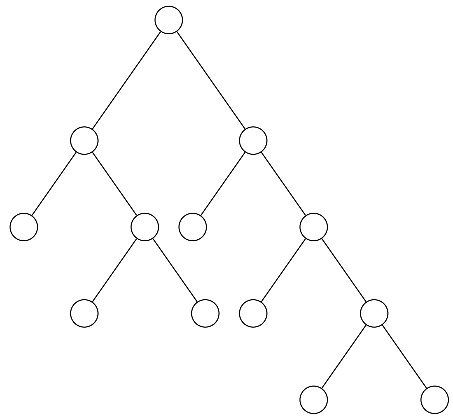
If we write this out in factorials it looks like:

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{2n+1}\binom{2n+1}{n}$$

n	c_n
0	1
1	1
2	2
3	5
4	14
5	42
6	132
7	429
8	1430
9	4862

2 Catalan Numbers: What Do They Count?

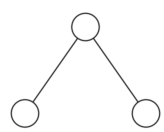
2.1 Answer #1: Rooted Binary Trees



Every node is either a branch, if it has children, and a leaf if it doesn't. Further every branch has two children. The root is always drawn at the top.

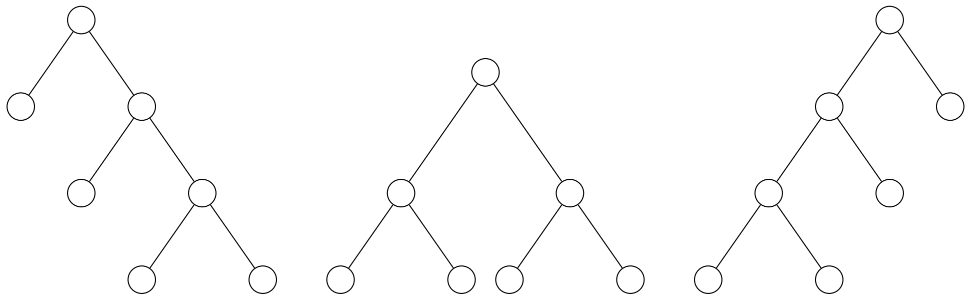
These look similar to binary choices. Let a_n be the number of binary rooted trees with n branches. Checking ti for Some values we have

Example. Note $n = 0$ has $a_0 = 1$ because it's just a root. $n = 1$, $a_1 = 1$.



$n = 2$. $a_2 = 2$.

$n = 3$, $a_3 = 5$.

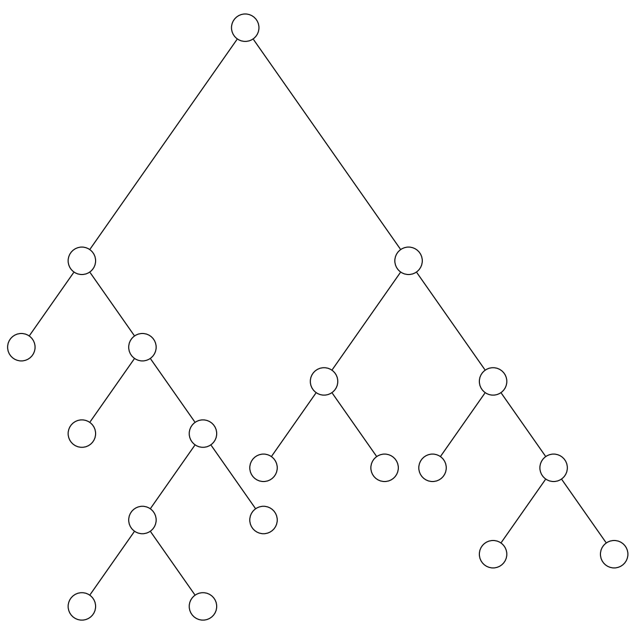


We want to show that these satisfy the linear recurrence:

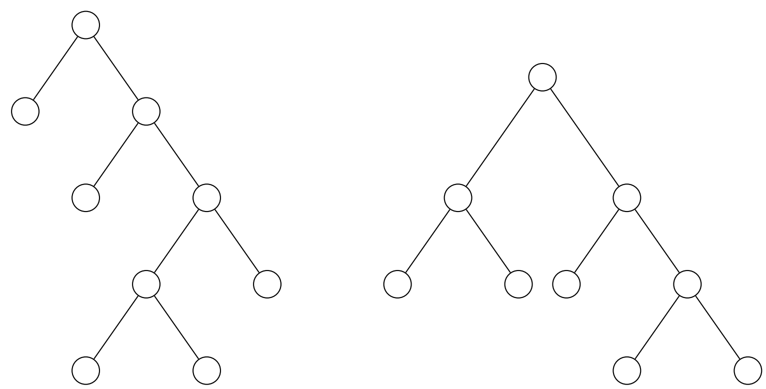
$$a_{n+1} = \sum_{i=0}^{n} a_i a_{n-i}$$

That is we want to make big trees out of two smaller trees. Well, chop the tree at the root. If we have a tree with n branches, then the left tree will have i branches and the right tree will have j branches, with $i + j = n + 1$.

Consider this as:



Splits into:



This gives a bijection between binary trees with $n+1$ branches and ordered pairs of (T, U) of binary trees with T and U together having n branches. Then we clearly have the recurrence and so:

Theorem 1. $a_n = c_n$

2.2 Answer #2: Polygon Triangulations

Theorem 2. *The number of triangulations of a convex $(n+2)$-gon is c_n.*

Let D_n be the number of such triangulations of an $(n+2)$-gon

Example. For $n = 0$ we set $D_0 = 1$ by convention.

For $n = 1$ we have $D_1 = 1$ trivially since it's just a triangle.

For $n = 2$ we can go along either diagonal of the square so $D_2 = 2$.

For $n = 3$ we actually have 5 triangulations so $D_3 = 5$.

In fact for $n = 4$ we actually have $D_4 = 14$ for a hexagon... whoo. This is hard to Latex

Proof. Consider tiling a $(n+3)$-gon, from D_{n+1}.

Well we can choose an edge, and then that edge must be in some triangle. But if we delete that edge, then the rest of the triangulation corresponds to a triangulation of an $(i+2)$-gon and a $(n-i+2)$-gon, because we count rotations differently!

Thus there's a bijection from triangulations of an $(n+3)$-gon and ordered pairs of triangulations of a $(i+2)$-gon and na $(n-i+2)$-gon for all i. The bijection in the other direction comes from gluing together two triangulations at a point and then filling in the "special" wedge that's missing with one edge.

So:

$$D_{n+1} = \sum_{n \geq 0} D_i D_{n-i}$$

And thus $c_n = D_n$ for all $n \in \mathbb{N} \cup \{0\}$ since they agree at 0. $\square$

In fact there's a bijection between binary rooted trees and triangulations. The idea is to view each triangle as rooms with "doors," and draw a node of the tree inside each triangle, where edges represent sharing a boundary, and except at the root we can "go outside" the "palace." We distinguish a root because orientation matters in these triangulations.

This bijection also tells us something about what it would mean to "rotate" binary trees since the triangulations have a natural operation of rotation.

2.3 Answer #3: Dyck Paths

Corresponds to a walk in the Cartesian Plane from $(0, 0)$, but only in two directions, one unit to the right or one unit up, ending at (n, n) and that never go strictly above $y = x$.

Let P_n denote those walks.

Example. For $n = 0$, there is only one path so $P_0 = 1$.

For $n = 1$ we have to go from $(0, 0)$ to $(1, 1)$ so $P_1 = 1$.

For $n = 2$ we have to go from $(0, 0)$ to $(2, 2)$. There are two ways to do this, "stay away from crocodiles" or "get up close", so $P_2 = 2$

For $n = 3$ we have to go from $(0, 0)$ to $(3, 3)$. It turns out there are five such paths so $P_3 = 5$.

Proof. OUT OF TIME. $\square$

MATH 465 Notes

Faye Jackson

13 February, 2020

1 Announcement

- HW6 Due Wednesday
- Today: Catalan Numbers (§8.1.2.1)
- Up Next: §5.3 Integer Partitions
- No OH tomorrow, Back on Schedule Monday

2 Review

2.1 Basics

$c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14$. And in general:

$$c_n = \frac{1}{n+1} = \binom{2n}{n}$$

For $n \in \mathbb{Z}_{\geq 0}$ then we have:

$$c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}$$

The generating function is:

$$\sum_{n \geq 0} c_n x^n = \frac{1 - \sqrt{1-4x}}{2x}$$

2.2 Finishing up Last Time

We had NE lattice paths from $(0,0)$ to (n,n) which do not cross above the diagonal $y = x$.

Proof 1. Consider $n = 3$

We want to look at:

$$c_{2+1} = c_0c_2 + c_1c_1 + c_2c_0$$

Which will better show us how this all really works (too hard to draw ☹).

We will break into cases based on the last time it touches the diagonal before the endpoint, let this point be (i, i).

Each of these paths can be constructed from a path from $(0, 0)$ to (i, i) which doesn't cross $y = x$, there are c_i many of these. Followed by a step east, a path from $(i + 1, i)$ to $(n + 1, n)$ which doesn't cross $y = x - 1$, followed by a step north. There are c_{n-i} of these.

Since there is one such path $(0, 0)$ to $(0, 0)$ and the sequence satisfies the catalan recurrence we win. □

Proof 2. We can also count these directly. Well, we will count the complement, the number of northeast paths which do cross above the diagonal.

We start with NE lattice paths from $(0, 0)$ to (n, n) which cross above the diagonal. We shift every one of these right one, giving us NE lattice paths from $(1, 0)$ to $(n+1, n)$ which touch (or cross above) the diagonal. Then we reflect the parth from $(0, 0)$ to (i, i) where i is the first place it touches the diagonal, which gets us NE lattice paths from $(0, 1)$ to $(n+, n)$ which touch the diagonal. These is equivalently any lattice path from $(0, 1)$ to $(n + 1, n)$. There are $\binom{2n}{n+1}$ of these.

These are indeed bijections. Thus there are $\binom{2n}{n+1}$ paths which do cross the diagonal. So then by previous class stuff:

$$\begin{aligned} c_n &= \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \\ &= \frac{(2n)!}{n!n!} - \frac{n(2n)!}{n(n+1)!(n-1)!} = \left(1 - \frac{n}{n+1}\right)\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n} \end{aligned}$$

□

These NE lattice paths are actually equivalent to diagonal-up $(1, 1)$ and diagonal-down $(1, -1)$ steps that never go below the x-axis

These also correspond to Ballot Sequences with n pluses and n minuses so that each initial segment has at least as many pluses as minuses.

3 New Stuff

Example. A $2 \times n$ matrix is called a <u>tableau</u> if its entries are $1, 2, 3, \ldots, 2n$ (each used exactly once) and arranged so that each entry is greater than the one above it and the one to its left. That is rows increase $L \to R$ and columns increase $T \to B$.

$n = 1$ gives:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$n = 2$ gives:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$n = 3$ gives:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

Claim. *For $n \geq 1$ there are exactly c_n tableaux*

Proof. Construct a bijection with NE lattice paths which don't cross $y = x$. Given a tableau define a lattice path from $(0, 0)$ to (n, n) whose i-th step is E if i is in the first row and N if i is in the second row. At the i-th step, if i is in the k-th column then we will have gone k steps E and at most k steps N, so it doesn't cross $y = x$.

The reason that in the k-th row we must

In fact this is a bijection. □

4 Integer Partitions

Example.

(a) How many ways are there to distribute five identical cookies to five children. These are weak compositions of 5 with 5 parts, so $\binom{9}{4} = 126$.

(b) How many ways can you put five identical cookies into identical piles:

One Pile	5	
Two Piles	1, 4	3, 2
Three Piles	3, 1, 1	2, 2, 1
Four Piles	2, 1, 1, 1	
Five Piles	1, 1, 1, 1, 1	

There's no closed formula to answer this in general.

Definition 1. *A partition of $n \in \mathbb{Z}_{\geq 0}$ is a weakly decreasing finite sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers for $\ell \in \mathbb{N}$ so that $\sum_{1 \leq i \leq \ell} \lambda_i = n$*

The λ_i are parts of λ and

$$p(n)\#\ \textit{partitions of}\ n$$

Above we found $p(5) = 7$, By convention we let $p(0) = 1$. We know $p(1) = 1$, $p(2) = 2$, ... OUT OF TIME.

MATH 465 Notes

Faye Jackson

18 February, 2020

1 Announcement

- Today: Quiz §5.3 Partitions
- HW6 Due Tomorrow
- Thursday: Quiz. Ch. 7 Inclusion / Exclusion
- Tuesday: Quiz + Review (in groups)
- HW7 Due Next Wednesday
- Next Thursday: Exam (more info TBA)
- Office Hours
  - Thursday 11:30-1:00
  - No Friday
  - Monday 4:00-5:30
  - Tuesday 11:30-1:00

2 Integer Partitions: The Main Player Today

2.1 Recalling the Definition from Last Time

Definition 1. *A partition of $n \in \mathbb{Z}_{\geq 0}$ is a weakly decreasing finite sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers such that $\lambda_1 + \cdots + \lambda_\ell = n$.*

Then let $p(n) = \#$ of partitions of n. We also call $\lambda_1, \ldots, \lambda_\ell$ is the parts of lambda and $\ell = \ell(\lambda)$ is the length of λ.

Then we have $\lambda(0) = 1, \lambda(1) = 1, \lambda(2) = 2, \lambda(3) = 3, \lambda(4) = 5, \lambda(5) = 7$.

2.2 Let's Get and Use a Generating Function

Theorem 1. *The generating function of the numbers $p(n)$ is given by:*

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Proof. The RHS is the infinite product:

$$(1 + x + x^2 + \cdots)(1 + (x^2)^1 + (x^2)^2 + \cdots)(1 + (x^3)^1 + (x^3)^2 + \cdots) \cdots$$

The coefficient of x^n is the number of ways to pick a term from each factor so that the sum of the exponents is n. We know all but finitely many such factors must be $x^0 = 1$. We want to think of think of these as follows:

$$(3, 2, 1, 1) \leftrightarrow (x^3)^1(x^2)^1(x^1)^2$$

That gives an equivalence between our picks and partitions of n. Namely the power of x^i picked from the i-th factor is the number of i's in the partition.

One can also think of this as $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ we can group together the "like terms" where there are a_i i's, then we have:

$$1a_1 + 2a_2 + 3a_3 + \cdots = n$$

□

Example. Note then that for partitions made up of 1's and 2's we'd have a generating function:

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} = (1 + x^1 + (x^1)^2 + \cdots)(1 + x^2 + (x^2)^2 + \cdots)$$

Where the coefficient of x^n gives the number of ways to pick $a, b \in \mathbb{Z}_{\geq 0}$ so that $(x^1)^a(x^2)^b$ such that $a + 2b = n$. But then this is exactly partitions made up of 1's and 2's

<u>More Generally</u>: Fix $I \subseteq \mathbb{Z}_{>0}$ and define

$$p_I(n) := \#\text{ of partitions of } n \text{ whose parts are elements of } I$$

Then:

$$\sum_{n\geq 0} p_I(n)x^n = \prod_{i\in I} \frac{1}{1-x^i}$$

Theorem 2. *The number of partitions of n into odd parts is equal to the number of partitons of n into distinct parts.*

Example. Odd parts where $n = 7$ we have:

$$(7), (3,3,1), (1,1,1,1,1,1,1), (5,1,1), (3,1,1,1,1)$$

And for distinct parts we have:

$$(7), (5,2), (4,3), (6,1), (4,2,1)$$

Proof. The generating for partitions of n into odd parts is exactly $\prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}$ (using $I = \{1,3,5,\ldots\}$).

The generating function for partitions of n into distinct parts is $\prod_{i=1}^{\infty}(1+x^i)$ because we can pick each positive integer 0 or 1 times.

It suffices to check that these two expressions are equal.

$$\prod_{k=1}^{\infty}(1+x^k) = \prod_{k=1}^{\infty}\left[(1+x^k)\cdot\frac{1-x^k}{1-x^k}\right] = \prod_{k=1}^{\infty}\frac{1-x^{2k}}{1-x^k}$$

Then observe:

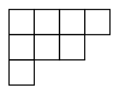
$$\prod_{k=1}^{\infty}\frac{1-x^{2k}}{1-x^k} = \frac{\cancel{1-x^2}}{1-x}\cdot\frac{1-x^4}{\cancel{1-x^2}}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{\cancel{1-x^4}}\cdots = \prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}\frac{1-x^{2i}}{1-x^{2i}} = \prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}$$

By cancelling lots of terms. $\square$

2.3 Young Diagrams

Definition 2. *The Young Diagram (Ferrer's shape) associated to a partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ *is a collection of unit squares on a rectangular grid which is made up of contiguous rows of lengths* $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ *(top-to-bottom). So that the left ends are aligned.*

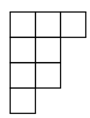
Example. Let $\lambda = (4, 3, 1)$ a partition of 8.

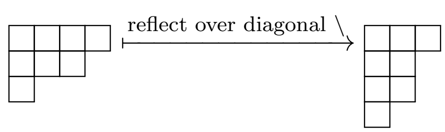


Definition 3. *If* λ *is a partition of* n *its conjugate partition is the partition of* n *whose parts are the number of boxes in the columns of* λ *in its Young Diagram.*

Notation: λ'

Example. $\lambda' = (3, 2, 2, 1)$





Observe that $(\lambda')' = \lambda$ so this defines a bijection {partitions of n} $\leftrightarrow$ {partitions of n}

This shows us that partitions with a certain number of parts is equivalent to partitions with largest part equal to that certain number of parts.

Proposition 1. The number of partitions of n with the largest part equal to at most k is equal to the number of partitions of n with exactly at most k parts

Proof. Conjugation restricts to a bijection:

$$\{\text{partitions of } n \text{ with } \lambda_1 = k\} \leftrightarrow \{\text{partitions of } n \text{ with } \ell(\lambda) = k\}$$

The Red Gives inequalities in this bijection. $\square$

Corrolary 1. *The generating function for partitions of n with at most k parts is exactly equal to the generating function for partitions of n with largest part at most k, which is exactly:*

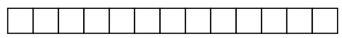
$$\prod_{i=1}^{k} \frac{1}{1-x^i}$$

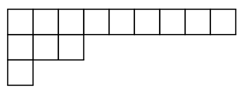
Taking $I = \{1, \ldots, k\}$.

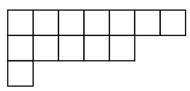
Proposition 2. The number of partitions of n into distinct odd parts is equal to the number of $\underbrace{\text{self-conjugate}}_{\lambda=\lambda'}$ partitions of n.

Example. Let $n = 13$ and consider this case.

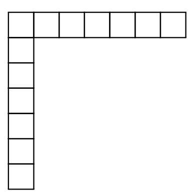
Distinct odd parts:

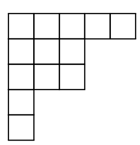
$(13) \mapsto$ 

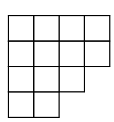
$(9, 3, 1) \mapsto$ 

$(7, 5, 1) \mapsto$ 

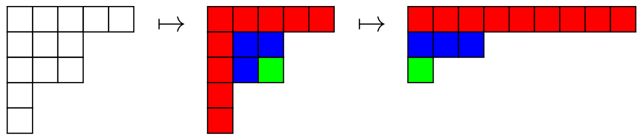
Self-conjugate:

$(7, 1, 1, 1, 1, 1, 1, 1) \mapsto$ 

$(5, 3, 3, 1, 1) \mapsto$ 

$(4, 4, 3, 2) \mapsto$ 

A bijection is given by "straightening out hooks." In one case look at:



Proof. Define a function:

$$f : \{\text{self-conjugate partitions of } n\} \to \{\text{partitions of } n \text{ with distinct odd parts}\}$$
$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \xmapsto{f} (2\lambda_1 - 1, 2\lambda_2 - 3, \ldots, 2\lambda_\ell - (2\ell - 1)\}$$

These are in fact distinct odd integers in the right prder, and this is indeed a bijection, to see that, bend hooks at the midpoint. ☐

MATH 465 Notes

Faye Jackson

20 February, 2020

1 Announcements

- Today: Quiz, Ch.7 Inclusion Exclusion
- Tuesday: Quiz, Review in Groups
- Thursday: Exam (info TBA on Canvas)
- Office Hours:
  - Monday 4-5:30
  - Tuesday 11:30-1
- HW7 Due Wednesday

2 Let's Go!

2.1 Statement and Proof

Recall. The Addition Principle. If $A_1, \ldots, A_n$ are disjoint finite sets, then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Example. $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ for all finite sets.

Example.

$$|A_1 \cup A_2 \cup A_3| = |A_1|+|A_2|+|A_3|-|A_1 \cap A_2|-|A_1 \cap A_3|-|A_2 \cap A_3|+|A_1 \cap A_2 \cap A_3|$$

Theorem 1 (The Principle of Inclusion Exclusion)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets, then:*

$$\left|\bigcup_{k=1}^{n} A_i\right| = \sum_{k=1}^{n} \sum_{(\star\star\star)} (-1)^{k-1} \left|\bigcap_{r=1}^{k} A_{i_k}\right|$$

$(\star\star\star)$ *stands for* $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

If we look at the example again

Example.

$$\begin{aligned}
& && |A_1 \cup A_2 \cup A_3| = \\
& k=1 && |A_1| + |A_2| + |A_3| \\
& k=2 && + (- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|) \\
& k=3 && + |A_1 \cap A_2 \cap A_3|
\end{aligned}$$

Proof. We show the RHS counts every element of $\bigcup_{i=1}^{n} A_i$ exactly once.

Let $x \in A_1 \cup A_2 \cup \cdots \cup A_n$. And let $S = \{i \in [n] \mid x \in A_i\}$. Then notice that $x \in A_{i_1} \cap \cdots \cap A_{i_k}$ if and only if $\{i_1, \ldots, i_k\} \subseteq S$. So the number of k-fold intersections containing x is exactly the number of of k-element subsets of S, that is $\binom{|S|}{k}$.

The contribution from x on the RHS is then:

$$\begin{aligned}
\sum_{k=1}^{n} (-1)^{k-1} \binom{|S|}{k} &= \sum_{k=1}^{|S|} (-1)^{k-1} \binom{|S|}{k} + \sum_{k=|S|+1}^{n} (-1)^{k-1} \binom{|S|}{k} \\
&= \sum_{k=1}^{|S|} (-1)^{k-1} \binom{|S|}{k} + \sum_{k=|S|+1}^{n} (-1)^{k-1} 0 \\
&= \sum_{k=1}^{|S|} (-1)^{k-1} \binom{|S|}{k} \\
&= 1 + \sum_{k=0}^{|S|} (-1)^{k-1} \binom{|S|}{k} = 1 + 0 = 1
\end{aligned}$$

Thus the right hand side counts every element fof the union $A_1 \cup \cdots \cup A_n$ exactly once and nothing else. $\square$

Example. How many n-letter words in the alphabet $\{1, 2, 3\}$ contain at least one 1, at least one 2, AND at least one 3. Assume $n \geq 3$.

Let's count the words that DON'T contain at least one 1, at least one 2, and at least one 3. Let U be the set of n-letter words in $\{1, 2, 3\}$. For $i \in \{1, 2, 3\}$ let $A_i = \{w \in U \mid w \text{ doesn't contain } i\}$. We want:

$$
\begin{aligned}
&|U| - |A_1 \cup A_2 \cup A_3| \\
&= |U| - [\ |A_1| + |A_2| + |A_3| - |A_1 \cap A_3| - |A_2 \cap A_3| - |A_1 \cap A_2| + |A_1 \cap A_2 \cap A_3|\ ] \\
&= 3^n - [2^n + 2^n + 2^n - 1 - 1 - 1 + 0] = 3^n - 3 \cdot 2^n - 3
\end{aligned}
$$

2.2 Let's Use It!

Theorem 2. *The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.*

Equivalently the number of partitions of n whose parts are not all odd is equal to the number of partitins of n whose parts are not all distinct.

Proof. Let $A_i = \{\text{partitions of } n \text{ with a part} = 2i\}$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Let $B_i = \{ \text{ partitions of } n \text{ with at least two parts} = i\}$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$

Note that:

$$
\begin{aligned}
|A_i| &= p(n - 2i) = |B_i| \\
|A_{i_1} \cap A_{i_2}| &= p(n - 2i_1 - 2i_2) = |B_{i_1} \cap B_{i_2}| \\
&\text{For } 1 \leq i_1 < \cdots < i_k < n: \\
|A_{i_1} \cap \cdots \cap A_{i_k}| &= p(n - 2i_1 - \cdots - 2i_k) = |B_{i_1} \cap \cdots \cap B_{i_k}|
\end{aligned}
$$

Thus since we know that:

$$\begin{aligned}
\#\text{ of partitions of } n \text{ with an even part} &= \left|A_1 \cup \cdots \cup A_{\lfloor \frac{n}{2} \rfloor}\right| \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le \lfloor \frac{n}{2} \rfloor} |A_{i_1} \cap \cdots \cap A_{i_k}| \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le \lfloor \frac{n}{2} \rfloor} p\left(n - \sum 2i_j\right) \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le \lfloor \frac{n}{2} \rfloor} |B_{i_1} \cap \cdots \cap B_{i_k}| \\
&= \left|B_1 \cup \cdots \cup B_{\lfloor \frac{n}{2} \rfloor}\right|
\end{aligned}$$

$$\#\text{ of partitions of } n \text{ with an even part} = \#\text{ of partitons of } n \text{ with a repeated part}$$

And so we win! $\square$

3 Stirling Numbers of the Second Kind Redux

3.1 The Thing we Missed before

Theorem 3 (Unfinished from Jan. 30)**.** *For $m, n \in \mathbb{Z}_{>0}$ we have:*

$$m^n = \sum_{k=1}^{n} S(n,k)(m)_k = \sum_{k=1}^{n} S(n,k)k!\binom{m}{k} = \sum_{k=1}^{\min\{n,m\}} S(n,k)k!\binom{m}{k}$$

The second equality holds because $S(n,k) = 0$ if $k > n$ and $\binom{m}{k} = 0$ if $k > m$.

Combinatorial Proof. The LHS counts the number of ways to color the set $[n]$ with m colors (equivalent to functions $f : [n] \to [m]$).

Alternatively, for the RHS, we can count in cases by how many colors we actually use, fix $1 \leq k \leq \min\{n, m\}$. First choose the colors which we will use, there are $\binom{m}{k}$ ways to pick k colors. Now $S(n, k)$ counts the number of partitions of our n elements into k blocks. Then we multiply by $k!$ to assign each color to each block bijectively. This gives us exactly $S(n, k)k!\binom{m}{k}$. And so this matches the right hand side:

Done! Great ☺ □

A similar argument says there are $k!S(n, k)$ surjections from $[n] \to [k]$.

Proposition 1. Let $n \geq k > 0$. Then:

$$k!S(n,k) = \sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}(k-\ell)^n = \sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n$$

Combinatorial Proof. The LHS counts surjections $[n] \to [k]$. Let's find a way to count surjections that looks like the right hand side. Let's do this using inclusion exclusion. For $i \in [k]$, let $A_i = \{f : [n] \to [k] \mid i \notin \text{range}(f)\}$. Equivalently we can think of this as the set of functions $A_i \leftrightarrow \{f : [n] \to [k] \setminus \{i\}\}$. For $1 \leq i_1 < \cdots < i_\ell \leq k$. We then have:

$$|A_{i_1} \cap \cdots A_{i_\ell}| = |\{f : [n] \to [k] \setminus \{i_1, \ldots, i_\ell\}\}| = (k-\ell)^n$$

Then by inclusion exclusion, the number of surjective functions is equal to:

$$\begin{aligned}
&(\#\text{ functions } [n] \to [k]) - |A_1 \cup \cdots \cup A_k| \\
&= k^n - \sum_{\ell=1}^{k} (-1)^{\ell-1} \sum_{1 \le i_1 < \cdots < i_\ell \le k} |A_{i_1} \cap \cdots \cap A_{i_\ell}| \\
&= k^n - \sum_{\ell=1}^{k} (-1)^{\ell-1} \sum_{1 \le i_1 < \cdots < i_\ell \le k} (k-\ell)^n \\
&= k^n - \sum_{\ell=1}^{k} (-1)^{\ell-1} \binom{k}{\ell} (k-\ell)^n \\
&= k^n + \sum_{\ell=1}^{k} (-1)^{\ell} \binom{k}{\ell} (k-\ell)^n \\
&= \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} (k-\ell)^n
\end{aligned}$$

So we are done! $\square$

MATH 465 Notes

Faye Jackson

20 February, 2020

1 Announcements

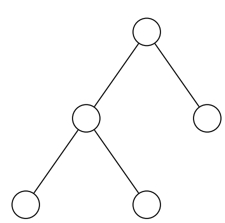
- Today: Intro to Graphs (Ch. 9) (Euler / Hamilton later)
- Next Time: Trees (Ch. 10)
- Quiz Thursday
- HW8 Due 3/18

2 Let's Go! Graphs!

2.1 Definitions and Examples

Definition 1. *A (finite, undirected) graph $G = (V, E)$ consists of a finite set V of vertices and a finite set E ofedges along with a map associating to each edge $e \in E$ an unordered pair of not necessarily distinct vertices $\{u, v\} \in V$.*

Example.



If the endpoints are equal then it is a loop.

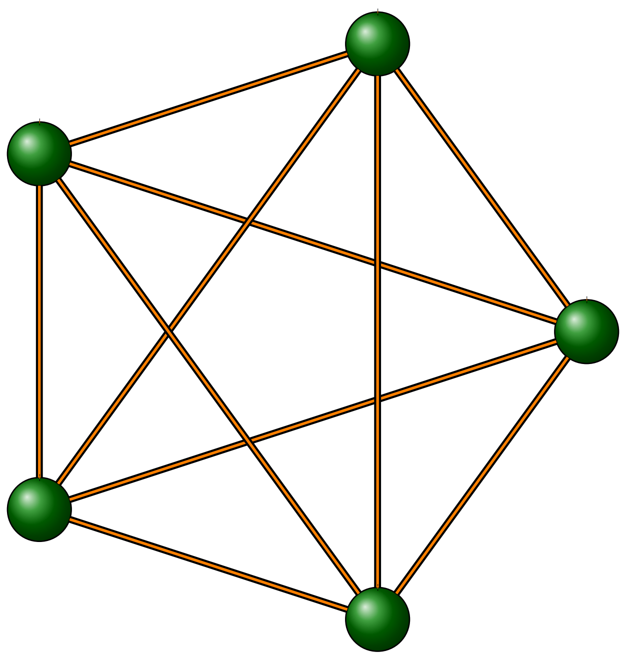
If the endpoints $\{u, v\}$ of an edge e are distinct we say u and v are adjecent and u (and v) are incident to e.

Definition 2. *A graph is simple if it has no loops or multiple edges..*

Example. Skeleta of polyhedra

Definition 3. *The complete graph on n vertices K_n is the simple graph such that any two vertices are adjacent.*

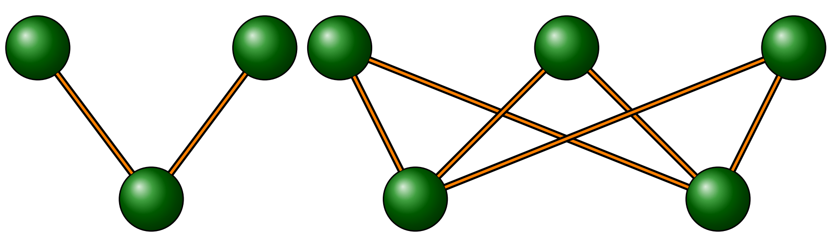
K_5

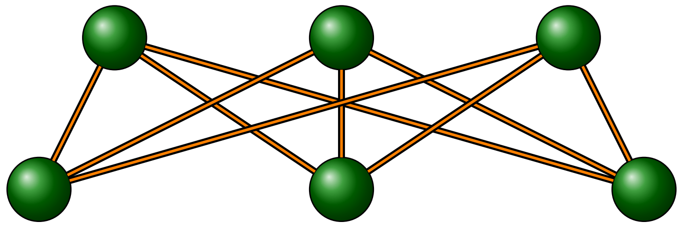


K_n has the maximum number of edges for a simple graph with n vertices, there are $\binom{n}{2}$ edges.

The complete bipartite graph $K_{m,n}$ has $m+n$ vertices partitioned into two blocks of size m and n, with mn edges connecting all vertices in different blocks.

In order, $K_{1,2}$, $K_{2,3}$, and $K_{3,3}$.



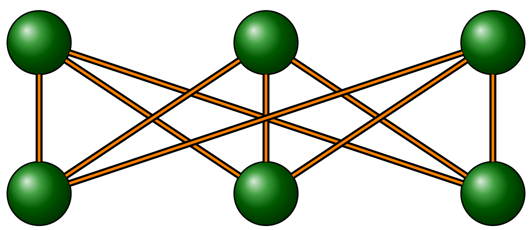


2.2 Graph Isomorphisms

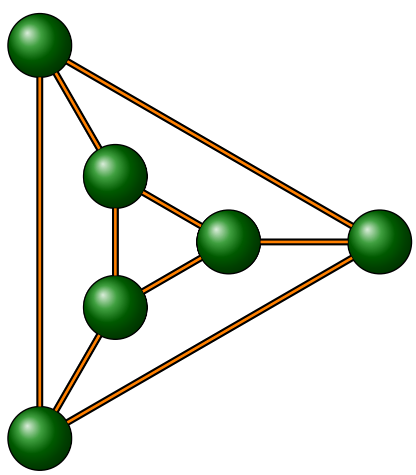
Definition 4. *Two graphs $G = (V, E)$ and $G' = (V', E')$ are* $\boxed{\textit{isomorphic}}$ *if there exist bijections $V \to V'$ and $E \to E'$ which preserves the incidence relation. That is if $v \mapsto v'$ and $e \mapsto e'$, then e is incident to v if and only if e' is incident to v'.*

Equivalently $\boxed{G \cong G'}$ *if there exists a bijection $f : V \to V'$ such that for all $u, v \in V$ the number of edges between u and v is equal to the number of edges between $f(u)$ and $f(v)$.*

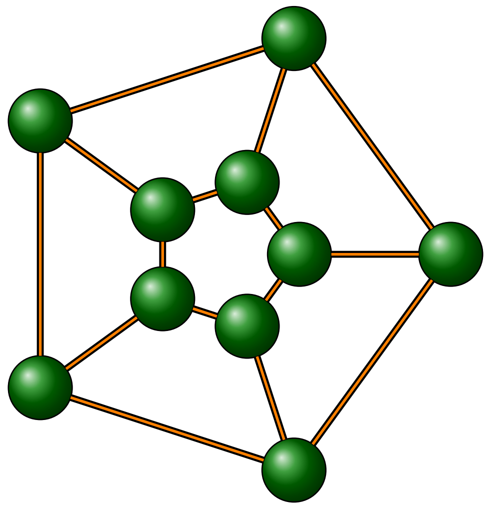
Example.



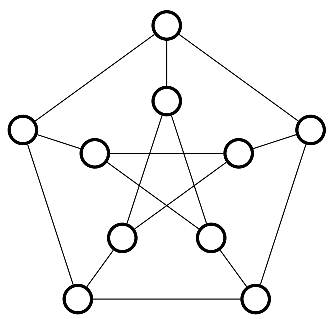
Is Not congruent to:



The reasoning is that there are a, b, c such taht $\{a-b, b-c, a-c\}$ are all edges in the second one, but not in the first.



Is not congruent to:



2.3 Properties of Graphs

Definition 5. *We say that a graph $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Note if $e \in E'$ then its endpoints are in V'.*

A subgraph G' is "induced" if whenever the endpoints of an edge $e \in E$ are in V', then $e \in E'$.

Let $G = (V, E)$. The degree of a vertex $v \in V$ is the number of "half-edges" incident to it (i.e. count loops twice)

Example.
- In K_n, every vertex has degree $n-1$
- . In $K_{m,n}$, m vertices have degree n and n vertices have degree m.

- The single edge connected to itself has degree 2.

Notation: $\deg(v) = \deg_G(v)$.

A Degree Sequence: is a non-increasing list of vertex degrees in graph

Note: If $f : V \to V'$ gives an isomorphism $G \cong G'$ then $\deg(v) = \deg(f(v))$. "isomorphism preserves degrees," but this is not enough

Proposition 1. $\sum_{v \in V} \deg(V) = 2|E|$ in a graph $G = (V, E)$.

Proof. Count the edges incident to every vertex, we've counted every edge exactly twice □

Corrolary 1. *In any graph, the number of vertices of odd degree is even.*

Proof.

$$\begin{aligned} \underbrace{2|E|}_{\text{even}} &= \sum_{v \in V} \deg(v) \\ &= \underbrace{\sum_{\substack{v \in V \\ \text{even}}} \deg(v)}_{\text{even}} + \sum_{\substack{v \in V \\ \text{odd}}} \deg(v) \\ &\implies \underbrace{\sum_{\substack{v \in V \\ \text{odd}}} \deg(v)}_{\text{even}} \end{aligned}$$

And so there is an even number of these odd numbers in the summation □

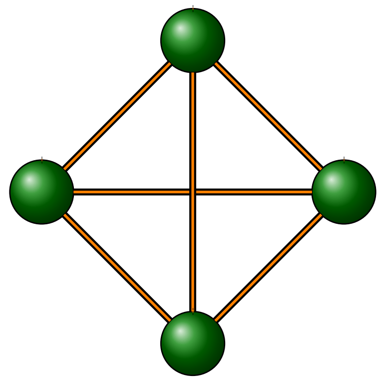
Definition 6. *A walk of length ℓ is a sequence:*

$$v_0 \underset{e_1}{\text{———}} v_1 \underset{e_2}{\text{———}} v_2 \underset{e_3}{\text{———}} \cdots \underset{e_{\ell-1}}{\text{———}} v_{\ell-1} \underset{e_\ell}{\text{———}} v_\ell$$

Of vertices and edges such that e_i has endpoints v_{i-1} and v_i for all i.

In a simple graph, it suffices to list vertices.

A closed walk returns to where it started ($v_\ell = v_1$), and a trail does not repeat any edges. A path uses each vertex at most once. Finally a cycles is a closed trail of length > 0 in which each vertex is used at most once except $v_0 = v_\ell$.



Then we have the following paths:

$$b \text{ ——— } d \text{ ——— } c \text{ ——— } b \text{ ——— } a \text{ ——— } b \qquad \text{(closed walk)}$$
$$b \text{ ——— } d \text{ ——— } c \text{ ——— } b \text{ ——— } a \qquad \text{(trail)}$$
$$b \text{ ——— } d \text{ ——— } c \text{ ——— } a \qquad \text{(path)}$$
$$b \text{ ——— } d \text{ ——— } c \text{ ——— } b \qquad \text{(3-cycle)}$$

There exists a cycle of length one if and only if there exists a loop. And there exists a cycle of length 2 if and only if there exists multiple edges. Thus, in a simple graph there exist no cycles of length < 3.

Note: If we view the edges and vertices of a cycle as a subgraph, every vertex has degree two in that subgraph, because we must enter and leave each vertex with different edges, and we cannot visit it twice. (With the exception of the start, which is easy to check has degree 2.

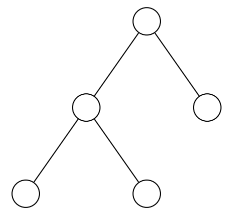
Definition 7. *A graph is connected if there exists a walk between any two vertices (note if the vertices are the same you can have a walk with no edges).*

Example. $K_n, K_{m,n}$

Non-Example. $V = [n],$, $E = \emptyset$, then for $n > 1$ this is not connected

Definition 8. *A forest is a graph with no cycles (acyclic)*

Definition 9. *A tree is a connected forest.*



MATH 465 Notes

Faye Jackson

March 17, 2020

1 Trees! The First BlueJeans Lecture

Definition 1. *A graph $G = (V, E)$ is connected if any two vertices are connected by a walk.*

Definition 2. *A connected component of a graph $G = (V, E)$ is an induced subgraph on vertices $V' \subseteq V$ such that no vertex in V' is adjacent to a vertice in $V \setminus V'$*

Definition 3. *A tree is a connected acyclic graph, and an acyclic graph is a forest*

Note:

- Trees are forests are simple because no loops means no 1-cycles and it has no multiple edges because it cannot have 2-cycles.
- A tree is a connected forest and the connected components of a forest are trees.
- A graph is connected if and only if there is a path between any two vertices.

  ($\Rightarrow$) Given a walk we can construct a path by cutting out our repeated vertices.

  ($\Leftarrow$) A path is a walk.

Definition 4. *An edge e in a connected graph $G = (V, E)$ is a bridge provided that $G' = (V, E \setminus \{e\})$ is not connected.*

Proposition 1. An edge in a connected graph is a bridge if and only if it is not contained in a cycle.

Proof. Equivalently we prove that an edge in a connected graph is not a bridge if and only if it is in a cycle.

(⇐) Suppose e is in a cycle, say:

$$b \overset{e_0}{\text{———}} v_1 \text{———} \cdots \text{———} v_\ell \overset{e_\ell}{\text{———}} a \overset{e}{\text{———}} b$$

We need to show that $G' = (V, E \setminus \{e\})$ is connected. Let $u, v \in V$. Since G is connected there is a path in G from u to v. If this path doesn't use edge e, then it is a path from u to v in G'.

Otherwise, e is an edge in this path from u to v, say:

$$u \text{———} w_1 \text{———} b \overset{e}{- - -} a \text{———} w_2 \text{———} v$$

Replace the part with e with:

$$u \text{———} w_1 \text{———} b \overset{e_0}{\text{———}} v_1 \text{———} \cdots \text{———} v_\ell \overset{e_\ell}{\text{———}} a \text{———} w_2 \text{———} v$$

Generating some walk from u to v, which we can make into a path.

(⇒) Suppose the edge e between a and b is not a bridge. So $G' = (V, E \setminus \{e\})$ is connected by definition. Thus there is a path from a to b in G', which obviously does not contain e. Adding the edge e to the end yields a cycle in G which contains e.

□

Corrolary 1. *A connected graph is a tree if and only if every edge is a bridge.*

Corrolary 2. *A graph is a tree if and only if there is a unique path between any two vertices.*

Characterization of trees via minimal number of edges

Definition 5. *A leaf is a vertex in a tree that has degree 1*

Proposition 2. A tree with at least two vertices has at least two leafs.

Proof. Consider a path of maximal length (this is possible because there are finitely many vertices and edges). Since there are at least two vertices in the tree, this path must have at least two vertices. Say:

$$v_0 \overset{e_1}{\text{———}} v_1 \overset{e_2}{\text{———}} \cdots \overset{e_\ell}{\text{———}} v_\ell$$

We claim that since the path is maximal, v_0 and v_ℓ are leaves (have degree 1). Suppose not, say v_ℓ is not a leaf. Thus there is some vertex u adjacent to v_ℓ, and $u \neq v_{\ell-1}$.

But then we can add the edge v_ℓ —— u to the end, we must get a walk because this path is maximal. Therefore we must have repeated a vertex. Therefore $u = v_i$ for some i. Then:

$$v_i \text{ —— } v_{i+1} \text{ —— } \cdots \text{ —— } v_\ell \text{ —— } u = v_i$$

Is a cycle, contradicting the fact that the graph is a tree. The proof that v_0 is a leaf is similar. Awesome! We have two leaves! □

Proposition 3. A connected graph $G = (V, E)$ must have:

$$|E| \geq |V| - 1$$

and, G is a tree if and only if equality holds $|E| = |V| - 1$.

Proof. Step 1 If G is a tree, with n vertices, then it has $n - 1$ edges. We will induct on the number of vertices, that is we will induct on n.

- When $n = 1$ then my tree must be $\cdot$, which has $n - 1 = 0$ edges, since we can't have an edge from the one vertex to itself, since this would give us a cycle.

- Let n be a positive integer and assume that every tree with n vertices has $n-1$ edges. Let G be a tree with $n+1$ vertices. We know that $n+1 \geq 2$ since $n \in \mathbb{N}$. Thus G has at least one leaf v, by the above proposition. Consider removing the leaf v from G, that is let G' be the subgraph obtained by removing v and the edge incident to v.

  G' must be a tree on n vertices, because we cannot create cycles by removing edges, and furthermore since v was a leaf we know G' is still connected. Great! Thus G' has $n - 1$ edges. Then since G has one more vertice and one more edge, we know that G has n edges. Thus the inductive step holds. Awesome!

Step 2 Show that if G is connected and not a tree (that is it contains a cycle), then $|E| > |V| - 1$. With this and Step 1 the proposition is shown! Well, let's induct! Pick an edge in G which is not a bridge, and call it e, since it is in a cycle. Then $(V, E \setminus \{e\})$ is connected. If this graph also contains a cycle, remove another non-bridge.

Continue this process, since the graph is finite there is a finite nonempty set of edged E' that we must remove. Thus $G' = (V, E \setminus E')$ is connected and contains no cycles. Hence G' is a tree, and by step 1, $|E \setminus E'| = |V| - 1$. Note that $|E'| > 0$ since E' is finite and nonempty, so:

$$|E| = \left|E'\right| + \left|E \setminus E'\right| = |E|' + |V| - 1 > |V| - 1$$

This concludes Step 2, and so we win. □

Definition 6. *A* <u>*spanning tree*</u> *in a graph $G = (V, E)$ is a subgraph of G which contains all vertices of G and is also a tree.*

Theorem 1. *A graph is connected if and only if it has a spanning tree*

Proof. Clearly if G has a spanning tree then it is connected. Furthermore, Step 2 gives us a way of constructing a spanning tree using a connected graph. □

MATH 465 Notes

Faye Jackson

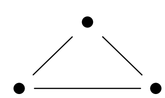
March 19, 2020

Planar Graphs

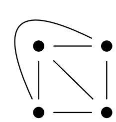
Question: Suppose 3 houses and 3 utilities, can we connect each house to each utility without wires crossing? This is essentially $K_{3,3}$
Question: Suppose 5 houses are connected by a road, can we draw a road map so that any two houses are connected by a road but no roads cross each other? This should look like K_5, and we're asking if we can draw K_5 without any of the edges crossing each other. In fact, you can't! Proof soon

Definition 1. *A graph is* ***planar*** *if it can be embedded in the plane (i.e. drawn on a planar surface) so that no two edges intersect.*

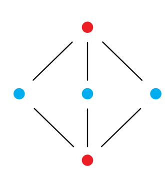
Example. K_3 is planar, with **2 Faces**:



K_4 is planar due to the following planar embedding, with **4 faces**:



$K_{2,3}$ is planar due to the planar embedding with **3 faces**:



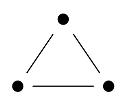
Note that trees are planar, we just start drawing and because there are no cycles we never run into problems. Note that they always have 1 face

Definition 2. *Note that the edges in a planar embedding partition the plane into regions called* ***faces****. The region outside the graph is the* ***outer, or external face*** *(as opposed to the* ***internal faces****)*

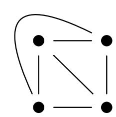
Theorem 1 (Euler's Theorem)**.** *Let $G = (V, E)$ be a connected planar graph. Let F be the set of faces obtained by drawing G in the plane without edge-crossing. Then:*

$$|V| - |E| + |F| = 2$$

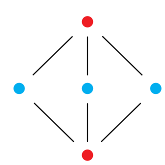
Example. K_3 satisfies the formula with $3 - 3 + 2 = 2$.



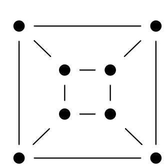
K_4 satisfies the formula with $4 - 6 + 4 = 2$



$K_{2,3}$ satisfies the formula with $5 - 6 + 3$:



A tree with 7 vertices satisfies the formula because $|E| = |V| - 1$ so $7 - 6 + 1 = 2$. Also look at the skeleton of a cube:



Then this satisfies the formula with $8 - 12 + 6 = 2$

Proof. We will prove by induction on $|E|$:

- When $|E| = 0$ we know $|V| = 1 = |F|$ and so:

$$|V| - |E| + |F| = 1 - 0 + 1 = 2$$

- Let n be a nonnegative integer and assume that every connected planar graph with n edges satisfies Euler's Formula. Let $G = (V, E)$ be a connected planar graph with $n+1$ edges. We will break this up into two cases:

  – If G is a tree then $|V| = |E| + 1$, by last time, and since trees have no cycles, $|F| = 1$. So then:

$$|V| - |E| + |F| = (n+2) - (n+1) + 1 = 2$$

  – Suppose G is not a tree. Then there is an edge which is not a bridge, call it $e \in E$. That is the subgraph $G' = (V, E \setminus \{e\})$ is connected. Note then that every subgraph of a planar graph is planar. Our inductive hypothesis implies that:

$$|V| - |E \setminus \{e\}| + \left|F'\right| = 2,$$

  Where F' is the number of faces in G'. Note that since e is not a bridge, it is in a cycle, and so it sees two different faces on each of it's sides, so removing it makes those two faces become the same face in G'. Thus $|F'| = |F| - 1$. So we have:

$$|V| - (|E| - 1) + (|F| - 1) = 2$$

  Hence:

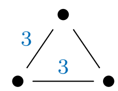
$$|V| - |E| + |F| = 2$$

□

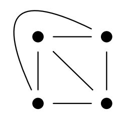
Corrolary 1. *The number of faces does not depend on the choice of planar embedding.*

Definition 3. *Given a face $f \in F$, define its degree* $\deg(f)$ *as the number of adjacent sides of edges.*

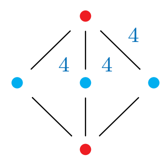
Example. K_3



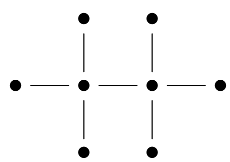
K_4



$K_{2,3}$



A tree with 7 vertices has one face with degree 12 here:



Each edge contributes 2 to the total degree so:

$$\sum_{f \in F} \deg(f) = 2\,|E| = \sum_{v \in V} \deg(v)$$

Lemma 1. *If G is a connected planar simple graph with at least three vertices, then every face has degree at least 3 and:*

$$3\,|F| \leq 2\,|E|$$

Proof. Note that the boundary of an internal face f determines a cycle whose length is $\leq \deg(f)$. Since a simple graph has no cycle of length $, 3$, every face has degree ≥ 3.

To see the outer face, note also, we need at least three vertices. Now note that:

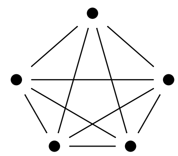
$$3\,|F| \leq \sum_{f \in F} \deg(f) = 2\,|E|$$

On homework you will prove that:

$$|E| \leq 3\,|V| - 6$$

$\square$

Proposition 1. K_5 is not planar.



Proof. For a contradiction, suppose that K_5 is planar. Since K_5 has 5 vertices and $\binom{5}{2}$ edges. A planar embedding would have:

$$|F| = 2 - |V| + |E| = 2 - 5 + 10 = 7$$

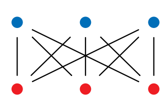
So then:

$$3\,|F| = 21 > 2\,|E| = 20$$

But since K_5 is simple, this contradicts the last lemma. $\square$

Proposition 2. $K_{3,3}$ is not planar.

Proof. Argue by contradiction, Assume $K_{3,3}$ is planar.
Note that $|V| = 6$ and $|E| = 9$:



And so by Euler's formula a planar embedding would have

$$|F| = 2 - 6 + 9 = 5$$

But we also have:

$$3\,|F| = 15 \leq 18 = 2\,|E|$$

So we cannot use the same proof as last time. The key observation is that $K_{3,3}$ has no 3-cycles. Thus each cycle has at least 4. So in fact the interior faces all have degree at least 4:

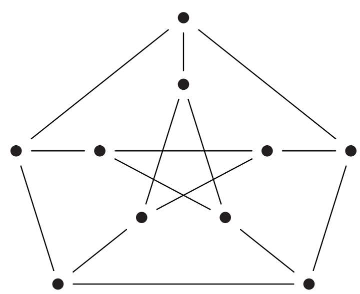
$$20 = 4\,|F| \leq \sum_{f\in F} \deg(f) = 2\,|E| = 18$$

This is a contradiction! The outer face follows because we have so many vertices, and so the boundary of the outer face will form a long cycle. $K_{3,3}$ is not planar! □

Theorem 2 (Wagner's Theorem)**.** *A graph is planar if and only if neither K_5 not $K_{3,3}$ can be obtained from G by deleting vertices deleting edges, and contracting edges (contracting an edge means we shold remove the edge and merge the endpoints)*

Proof Omitted. □

Example. Consider:



We can obtain K_5 by contracting the outside towards the inside.

MATH 465 Notes

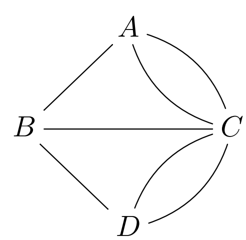
Faye Jackson

March 24, 2020

Eulerian Trails + Hamiltonian Cycles (Ch. 9)

Eulerian Trails

Example. Konigsberg Bridge Problem, how to cross all bridges between islands A, B, C, D exactly once, starting and ending at the same place. Is it possible?

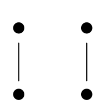


The answer is no, see theorem below.

Definition 1. *A Eulerian trail in a graph is a trail that traverses each edge exactly once. A graph is Eulerian if it has a closed Eulerian trail (AKA Eulerian circuit)*

Example. If $E = \emptyset$ then we win, pick a vertex and go nowhere:

Now what about a graph of the form:



This is not Eulerian, sice there are edges in more than one connected component. We will thus restrict out attention to connected graphs.

Theorem 1 (Euler)**.** *A connected graph has a closed Eulerian trail if and only if it has no vertices of odd degree.*

Proof. First assume that a connected graph G has a closed Eulerian trail. This means that at every time we visit vertex, we must use one edge to get there and one edge to leave (the start/end is an easy special case). Since we use every edge exactly once, this tells us that each vertex has even degree since we use two edges at each vertex.

To prove the other direction, we will proceed by strong induction on the number of edges $n = |E|$. The base case is not so bad, it is the first example above, if G has no edges, then since it is connected it consists of a single example v, and $\deg(v) = 0$. Furthermore, it has a closed Eulerian Trail (the trail v).

Now let n be a non-negative integer such that any connected graph with n or fewer edges and no vertieces of odd degree, it has a closed Eulerian trail.

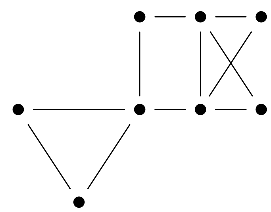
Let G be a connected graph with $n + 1$ edges and no vertex of odd degree. Because n is a nonnegative integer, we know G has at least one edge, and so G cannot be a tree, because trees with at least one edge have at least one leaf (which would have degree one).

Thus there is some cycle C in our graph, of length ℓ. Remove all the edges in the cycle C. This gives us a subgraph G' of G. Thus G' has $n + 1 - \ell$ edges, and $\ell \geq 1$, so $n + 1 - \ell \leq \ell$. We cannot directly apply the inductive hypothesis to G' since G' is not necessarily connected.

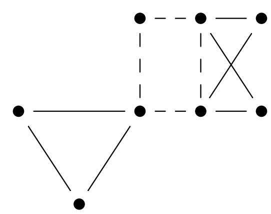
Instead we will consider each of its connected components, since each of its connected components is a connected graph with $\leq n$ edges and no vertex of odd degree. This holds because if the vertex is not in the cycle then $\deg_{G'}(v) = \deg_G(v)$, or if the vertex is in the cycle, then by removing the cycle we remove 2 edges from V, so $\deg_{G'}(v) = \deg_G(v) - 2$.

By induction, each connected component has a closed Eulerian Trail. We can sew them together using C: as we walk around the cycle, each time we encounter a vertex in a connected component of G' that we have not visited yet, we walk around the connected component using the closed Eulerian Trail in it. Then we continue.

Example.



Remove the cycle with 4 vertices in the middle to get:



It is easy to find closed Eulerian Trails in each connected component. The strategy is to walk between connected components by following the cycle, and each time you land in a connected component follow a closed Eulerian trail in it.

$\square$

Corrolary 1. *A connected graph G has an Eulerian trail if and only if G has at most 2 vertices of odd degree.*

Proof. Suppose we have an Eulerian Trail from v to w in the connected graph G. If $v = w$ then we are done, since this is a closed Eulerian trail so the previous theorem applie.

If not, add an edge between v and w. This gives us a closed Eulerian trail in a bigger graph $G' = (V, E \cup \{ v - w \})$. Therefore by the theorem G' has no vertices of odd degree. v and w are the only vertices in G allowed to have odd degree, because they are the only ones changed by adding in the edge.

Conversely, assume a connected graph G has at most 2 vertices of odd degree. If there are none, then the result follows from the theorem, because a closed Eulerian trail is in fact a Eulerian trail.

It is not possible to have one vertex of odd degree, because when we started graph theory we showed that there are always an even number of vertices of odd degree

So suppose there are 2 vertices of odd degree; say v and w. Then add the edge $v - w$ and consider the graph $G' = (V, E \cup \{ v - w \})$. By the theorem, since G' has no vertices odd degree, there is a closed Eulerian trail in G'. Removing the edge $v - w$ yields a Eulerian trail in G. $\square$

Hamiltonian Cycles

Definition 2. *A **Hamiltonian cycle (or path)** is a cycle (resp. path) that visits each vertex of a graph exactly once (unless the start and end are the same, that's allowed to be 2).*

*A graph is **Hamiltonian** if it contains a Hamiltonian cycle. There is no simple theorem that tells us when a graph is Hamiltonian like we have for closed Eulerian trails.*

Example. Let $m, n \geq 2$. Let $G_{m,n}$ be the graph on mn vertices on an $m \times n$ grid with vertical & horizontal edges: Let $m = 3$ and $n = 8$.

$$\begin{tikzcd}
\bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[d, dash] \\
\bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[d, dash] \\
\bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet
\end{tikzcd}$$

We draw a Hamiltonian Cycle as :

$$\begin{tikzcd}
\bullet \arrow[d, mapsto] & \bullet \arrow[l, mapsto] \arrow[d, dash] & \bullet \arrow[l, mapsto] \arrow[d, dash] & \bullet \arrow[l, mapsto] \arrow[d, dash] & \bullet \arrow[l, mapsto] \arrow[d, dash] & \bullet \arrow[l, mapsto] \arrow[d, dash] & \bullet \arrow[l, mapsto] \arrow[d, dash] & \bullet \arrow[l, mapsto] \\
\bullet \arrow[r, dash] \arrow[d, mapsto] & \bullet \arrow[r, mapsto] & \bullet \arrow[r, dash] \arrow[d, mapsto] & \bullet \arrow[r, mapsto] & \bullet \arrow[r, dash] \arrow[d, mapsto] & \bullet \arrow[r, mapsto] & \bullet \arrow[r, dash] \arrow[d, mapsto] & \bullet \arrow[u, mapsto] \\
\bullet \arrow[r, mapsto] & \bullet \arrow[r, dash] \arrow[u, mapsto] & \bullet \arrow[r, mapsto] & \bullet \arrow[r, dash] \arrow[u, mapsto] & \bullet \arrow[r, mapsto] & \bullet \arrow[r, dash] \arrow[u, mapsto] & \bullet \arrow[r, mapsto] & \bullet \arrow[u, mapsto]
\end{tikzcd}$$

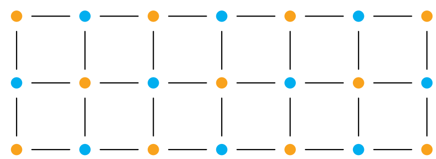
Proposition 1. $G_{m,n}$ is hamiltonian if and only if mn is even.

Proof. If nm is even, WLOG, assume n is even. Use the same technique as above. Start in the upper left corner, go all the way down the first column, up the second column to the second row, all the way down the third column, etc, up last column to the top, then left along the first row. This is a Hamiltonian cycle.

Now assume nm is odd. I.E. both n & m are odd. Color the vertices red and blue, alternating, as below, so that no two red are adjacent and no two blue are adjacent. If there was a Hamiltonian cycle, then the vertices in the cycle would have to alternate color:

$$R — B — R \cdots — R$$

But this isn't possible with an odd number of vertices. If we start at a red vertex, then the mn-th vertex would also be red and there is no edge from this vertex to the starting point. We lose!



Note that no two blue vertices are adjacent to each other and no two orange vertices are adjacent. But a Hamiltonian cycle has to alternate color. $OBOBOB \cdots O$. This can't happen because there is an odd number of vertices. $\square$

Example. The Peterson Graph is not Hamiltonian, you can use the fact that it has vertices of odd degree.

MATH 465 Notes

Faye Jackson

March 26, 2020

Vertex Coloring

<u>Question</u>: A mapmaker wants to color countries on his maps so that any two counteries which share a border have different colors. What is the fewest number of colors needed to guarantee any map can be colored this way?

$$\begin{aligned} \text{map} &\to \text{graph (simple planar)} \\ \text{countries} &\to \text{vertices} \\ \text{two countries share a border} &\to \text{two vertices adjacent} \end{aligned}$$

Definition. *A* ***proper vertex coloring*** *of a graph G is a coloring of its vertices such that any two adjacent vertices are colored differently*

G is ***k-colorable*** *if there is a proper vertex coloring which uses $\leq k$ colors.*

G is ***bipartite*** *if it is 2-colorable.*

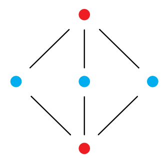
The ***chromatic number*** *$\chi(G)$ is the smallest integer k for which G is k-colorable.*

We will assume our graphs are simple here

Example. Let's get some examples.

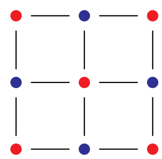
(1) The complete bipartite graph $K_{m,n}$ is in fact bipartite (as one would hope!) We color the first m vertices with one color and the n vertices another color, look

at $K_{2,3}$.



In fact $\chi(K_{m,n}) = 2$ whenever $m, n \geq 1$.

(2) Another bipartite graph we've seen is $G_{m,n}$ (grid)



In fact $\chi(G_{m,n}) = 2$ if $m > 1$ or $n > 1$

(3) If G has no edges then $\chi(G) = 1$. (i.e., G is k-colorable for every $k \in \mathbb{Z}_{>0}$).

For a simple graph, if G has at least one edge then $\chi(G) > 1$.

(4) $\chi(K_n) = n$.

Note that $\chi(K_n) \leq n$ because any graph on n vertices is n-colorable (color each vertex a different color).

And $\chi(K_n) \geq n$ because any two vertices are adjacent, and so must have a different color.

Restrict our Attention to Planar Graphs

The first thing we do is get an upper bound on the number of colors you need for planar graphs.

Theorem 1 (The Six Color Theorem)**.** *Every simple planar graph is 6-colorable, that is $\chi(G) \leq 6$ for every simple planar graph G.*

Lemma 1. *If $G = (V, E)$ is a simple planar graph then there exists a vertex $v \in V$ such that* $\deg(v) \leq 5$.

Proof. Similar to Homework 9. If not, every vertex has degree at least 6, so:

$$2|E| = \sum_{v \in V} \deg(v) \geq 6|V|$$

And from Homework 9 #5:

$$|E| \leq 3|V| - 6$$

Multiplying through by two we get:

$$2|E| \leq 6|V| - 12 < 6|V|$$

So we have a contradiction, $6|V| < 6|V|$. Oops!

$\square$

Proof of 6*-color Theorem.* We induct on $|V|$. Note that a simple graph with 1 vertex is 1-colorable, and hence 6-colorable. This is our base case.

Let $n \geq 1$ and assume every simple planar graph with n vertices is 6-colorable.

Let $G = (V, E)$ be a simple planar graph with $n + 1$ vertices. By lemma there is some $v \in V$ with $\deg(v) \leq 5$. Consider the induced subgraph on $V \setminus \{v\}$.

This is a simple planar graph with n vertices so by induction it is 6-colorable. Given a 6-coloring of this subgraph, we can extend it to a 6-coloring of G by picking a color for v.

Since $\deg(v) \leq 5$, at least one of the six colors is not used on any of v's neighbors. Choose one of these unused colors for v. $\square$

Let's prove something even better.

Theorem 2 (The Five Color Theorem)**.** *Every simple planar graph is* 5*-colorable, that is* $\chi(G) \leq 5$ *for every simple planar graph* G*.*

Proof of 5*-color Theorem.* We induct on $|V|$. Note that a simple graph with 1 vertex is 1-colorable, and hence 5-colorable. This is our base case.

Let $n \geq 1$ and assume every simple planar graph with n vertices is 5-colorable.

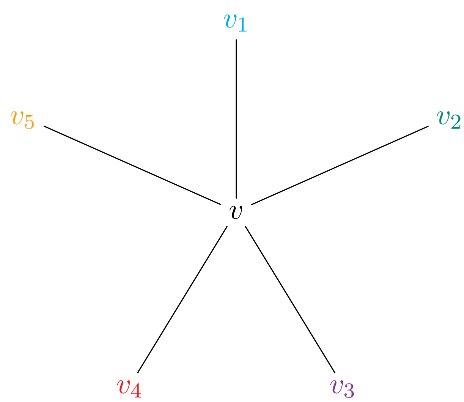
Let $G = (V, E)$ be a simple planar graph with $n + 1$ vertices. By lemma there is some $v \in V$ with $\deg(v) \leq 5$. Consider the induced subgraph on $V \setminus \{v\}$.

This is a simple planar graph with n vertices so by induction it is 5-colorable. Fix a 5-coloring of this subgraph.

If the neighbors of v use at most 4 colors, then we will win, because we can extend the 5-coloring of G with the 5-th color

If not, then there are five vertices adjacent to v, each colored by a different color.

Fix a planar embedding of G and label v's neighbors as v_1, v_2, v_3, v_4, v_5 as they appear in a clockwise order around v. Suppose v_i has color i.



Let V_{13} be the vertices colored by 1 and 3, and let G_{13} be the induced subgraph on V_{13}.

- If v_1 and v_3 lie in different connected components of G_{13} then we can swap the two colors in the entire component containing v_1. This yields a valid 5-coloring and we may color v by 1.

- If not, then there is a path from v_1 to v_3 using only vertices colored by 1 and 3. Name this path P_{13}. Let V_{24} be the vertices colored by 2 and 4 and let G_{24} be the induced subgraph on V_{24}. Follow similar motions as above

  - If v_2 and v_4 lie in different connected components of G_{24} then we can swap the two colors in the component containing v_2. But then we can color v by 2 to get a valid 5-coloring of G.
  - If not, then there is a path P_{24} from v_2 to v_4 using only vertices colored by 2 and 4. It is not possible for both of these paths to exist, P_{13} and P_{24} since they cannot contain any of the same vertices, and they must intersect. Draw the picture! Thus G could not be planar. Thus we must fall into a different case where we win.

This concludes the proof of the 5-color theorem! $\square$

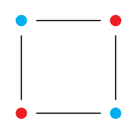
Theorem 3 (The Four Color Theorem)**.** *Every simple planar graph is* 4*-colorable. This is much much much harder. There is no known proof which does not use a computer.*

Determining the chromatic number of a graph (even a planar graph) G is very very hard, even with computers, but we do have some properties, but showing equalities in these is trickier.

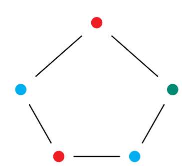
(1) If G has n vertices then $1 \leq \chi(G) \leq n$.

(2) If G' is a subgraph of G then $\chi(G') \leq \chi(G)$. (a k-coloring of G restricts to a k-coloring of G'

Example. Let C_n be an n-cycle with $n \geq 3$.



But then:



Thus:

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$$

Proposition 1. A simple graph with is bipartite if and only if every cycle has even length.

Proof. Let's go!

($\Rightarrow$) We do this by contrapositive. Suppose G contains an odd length cycle. Consider this graph as a subgraph G', then $3 = \chi(G') \leq \chi(G)$.

Thus G is not bipartite.

($\Leftarrow$) It suffices to consider connected graphs because of connected components. Assume $G = (V, E)$ is connected and has no odd length cycles. Fix two vertices $v, w \in V$.

Either every path from v to w is even or every path from v to w is odd. Otherwise we could concatenate the two paths to get an odd length cycle, with a little bit of work. (Draw some pictures).

Choose a special vertex $v \in V$. For any $w \in V$ color w RED if there is an even path from v to w and BLUE if there is an odd path from v to w.

This coloring is well-defined because of the above work. You should show that no two red vertices are adjacent and no two blue vertices are adjacent.

$\square$

MATH 465 Notes

Faye Jackson

March 31, 2020

The Chromatic Polynomial

Definition 1. *For a simple graph G and nonnegative integer k, define $p_G(k)$ the be the number of proper vertex colorings of G using $(\leq)k$ colors. We assume the k colors are fixed and swapping two colors gives different colorings.*

Example. (1) If G is a connected bipartite graph then $p_G(2) = 2$.

(2) $k < \chi(G)$ if and only if $p_G(k) = 0$, for $k \in \mathbb{Z}_{\geq 0}$.

(3) if $G = K_n$ and $k \geq n$ then:

$$p_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1) = \binom{k}{n} n!$$

(4) If G has no edges and n vertices then $p_G(k) = k^n$.

Proposition 1. If $G = (V, E)$ is a tree, then:

$$p_G(k) = k(k-1)^{|V|-1}$$

Proof. We prove by induction on $|V|$.

If $|V| = 1$ then $p_G(k) = k^1 = k$, since we can have no edges in such a tree.

Now let n be a positive integer, and assume that every tree with n vertices satisfies the above formula. Let $G = (V, E)$ be a tree with $n+1$ vertices. Let $v \in V$ be a leaf. By removing v and the edge incident to it, we obtain a tree G' with n vertices. By inductive hypothesis:

$$p_{G'}(k) = k(k-1)^{n-1}$$

Given a k-coloring of G', there are $k-1$ ways to extend this to a coloring of G by coloring v, since there is only one vertex adjacent to G. This gives us all of our colorings of G uniquely. Therefore:

$$p_G(k) = (k-1)p_{G'}(k) = (k-1)k(k-1)^{n-1} = k(k-1)^n$$

Just as desired. With this we win! □

Note: If G has connected components $G_1, \ldots, G_m$, then:

$$p_G(k) = p_{G_1}(k)p_{G_2}(k)\cdots p_{G_m}(k)$$

To color G, we just pick a coloring of each component.

Example. Suppose G is a forest with m trees on vertices $V_1, \ldots, V_m$. Then a k-coloring for G would be:

$$p_G(k) = \prod_{i=1}^{m} k(k-1)^{|V_i|-1} = k^m(k-1)^{-m+\sum_{i=1}^{m}|V_i|} = k^m(k-1)^{|V|-m}$$

Example. Let $G = C_4$ (a 4-cycle). Let u and v be vertices opposite each other. Either u and v have the same color, or u and v have different colors.

If they have the same color, the other two vertices cannot be that one color, but there are no further restrictions. So in this case we have k choices for u and v and $k-1$ choices each for the other two vertices, giving:

$$k(k-1)^2$$

If u and v have different colors, the other two vertices can't be either of these colors and we get:

$$k(k-1)(k-2)^2$$

Thus in total:

$$p_G(k) = k(k-1)^2 + k(k-1)(k-2)^2$$

Let $G = (V, E)$ be a simple graph and let $e \in E$.

Definition 2. *The deletion of* e *is the graph*

$$G - e = (V, E \setminus \{e\})$$

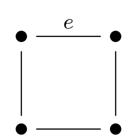
The contraction is the graph G/e *obtained from* $G - e$ *by merging the endpoints of* e.

Note: G is not necessarily simple, but has no loops, so $p_{G/e}(k)$ makes sense and:

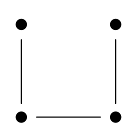
$$p_{G/e}(k) = p_{G'}(k)$$

where G' is a simple graph obtained by removing multiple edges.

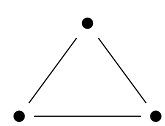
Example. Let $G = C_4$. Fix an edge e, this is G



The deletion $G - e$ is:



This is a tree, so: And the contraction G/e is:



Writing down the colorings we see that:

$$p_{G-e}(k) = k(k-1)^{4-1} = k(k-1)^3$$
$$p_{G/e}(k) = k(k-1)(k-2)$$
$$p_G(k) = k(k-1)^2 + k(k-1)(k-2)^2$$

Proposition 2. [Deletion/Contraction Formula] For any simple graph $G = (V, E)$

and edge $e \in E$ we have:

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k)$$

Proof. In each proper coloring of k-coloring of $G - e$, the (former) endpoints of e either have the same color or not.

- If they have the same color, this is a k-coloring for G/e, since their neighbors will not have that color.
- If they do not have the same color, this is a k-coloring for G since the edge e will not effect anything.

So then:

$$p_{G-e}(k) = p_{G/e}(k) + p_G(k)$$

This gives us a recursive/inductive way of computing colorings. □

Definition 3. *Viewing p_G as a function of k, we call $p_G(k)$ the chromatic polynomial*

Theorem 1. *For any simple graph $G = (V, E)$, $p_G(k)$ is a monic polynomial in k of degree $|V|$. In other words there exist constants $a_0, \ldots, a_{|V|-1}$ such that:*

$$p_G(k) = a_0 + a_1 k + \cdots + a_{|V|-1} k^{|V|-1} + k^{|V|}$$

Proof. We will induct on $|E|$ using the previous proposition.

If G has no edges and $|V|$ vertices then $p_G(k) = k^{|V|}$. This is indeed a monic polynomial whose degree is the number of vertices.

Fix n a nonnegative integer, and assume that any simple graph with $\leq n$ edges has a monic chromatic polynomial of degree its number of vertices. Let $G = (V, E)$ be a simple graph with $n + 1$ edges. Fix an edge $e \in E$.

The deletion $G - e$ is a simple graph with $|V|$ vertices and n edges. Thus $p_{G-e}(k)$ is a monic polynomial with degree $|V|$.

The contraction G/e has chromatic polynomial equal to that of a simple graph (remove any multiple edges) with $|V| - 1$ vertices and $\leq n$ edges so by induction $p_{G/e}(k)$ is a monic polynomial in K with degree $|V| - 1$.

So then by deletion/contraction:

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k)$$

is a monic polynomial in K of degree $|V|$. $\square$

Definition 4. *Let $\hat{p}_G(m)$ be the number of proper colorings of G using exactly m colors.*

Note that $\hat{p}_G(m) = 0$ unless $\chi(G) \leq m \leq |V|$.

Proposition 3. For a simple graph G:

$$p_G(k) = \sum_{m=\chi(G)}^{|V|} \hat{p}_G(m) \binom{k}{m}$$

We will not write out a proof for this, but the partition should be clear.

Example. Let $G = C_4$, then $\chi(G) = 2$, and so:

$$\begin{aligned} p_G(k) &= \hat{p}_G(2)\binom{k}{2} + \hat{p}_G(3)\binom{k}{3} + \hat{p}_G(4)\binom{k}{4} \\ &= 2\binom{k}{2} + 2 \cdot 3!\binom{k}{3} + 4!\binom{k}{4} \end{aligned}$$

$\hat{p}_G(2) = 2$, since this is a bipartite graph, $\hat{p}_G(3) = 2 \cdot 3!$, because it's two pairs can have the same color and then you distribute 3 colors among 3 things.

Other properties of $p_G(k)$:

(1) The coefficients alternate in sign, you can prove this by induction on edgse. That is we can write:

$$p_G(k) = k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} \cdots + (-1)^n a_0$$

where $a_0, \ldots, a_{n-1} \in \mathbb{Z}_{\geq 0}$.

(2) $a_{n-1} = |E|$ (HW?)

(3) $a_0, \ldots, a_{n-1}, 1$ is "unimodal"

$$a_0 \leq a_1 \leq \cdots \leq a_i \geq a_{i+1} \geq \cdots \geq a_{n-1} \geq 1$$

Furthermore it is “log-concave”

$$a_i^2 \geq a_{i-1}a_{i+1}$$

these inequalities are hard to prove. This was conjectured in 1968 and proven in 2012. It was proven by June Huh, a PhD UM-14, he proved both of them.

Remark. If G is a tree, then $p_G(k) = k(k-1)^{n-1}$, and the coefficients are binomial if you “forget” the sign. Like when $n = 3$ we get:

$$k - 2k^2 + k^3$$

When $n = 4$ we get:

$$-k + 3k^2 - 3k^2 + k^4$$

In fact this gives you Pascal’s Triangle.

If $G = K_n$, $p_G(k) = k(k-1)\cdots k(n-1+1)$, and this is the generating function for stirling #s of the first kind, so these are the coefficients. When $n = 3$:

$$2k - 3k^2 + k^3$$

When $n = 4$:

$$-6k + 11k^2 - 6k^3 + k^4$$

MATH 465 Notes

Faye Jackson

April 2, 2020

Assume graphs are simple!!!

Definition. *A* ***matching*** *in a graph G is a set of vertex-disjoint edges.*

A ***perfect matching*** *is a matching that covers all vertices in G.*

Example. Suppose we have a collection of people (V), and we know who is/is not compatible (edges). Can we pair people up into “marriages” so that each person is in exactly one pair?

Example. Suppose a company has a set of jobs open and a collection of applicants, with knowledge of what jobs an applicant is qualified for. Is it possible to fill all of the jobs with qualified applicants.

Vertices: Jobs and Appliants. Edge when applicant is qualified for a job (this is a bipartite graph). Want a matching in this bipartite graph that covers all of the “job” vertices.

Definition. *If $G = (U, V, E)$, a bipartite graph. A* ***matching of U into V*** *is a matching that covers all the vertices in U. [This defines an injection $U \to V$.]*

Definition. *Let $G = (V, E)$ be a simple graph. For $v \in V$ let $N_G(v) = \{u \in V \mid (u — v) \in E\}$.*

For $S \subseteq V$ let $N_G(S) = \bigcup_{V \in S} N_G(v)$.

Proposition 1. If $G = (U, V, E)$ is a simple bipartite graph and $S \subseteq U$ such that $|S| > |N(S)|$ then there is no matching of U into V

Proof (contrapositive). Suppose there is a matching into V, which is a collection of vertex disjoint edges $(u_1, v_1), \dots, (u_k, v_k)$. This defines an injective function $f : U \to V$ by $u_i \mapsto v_i$. If $S \subseteq U$ then f restricts to an injective function $S \to N(S)$. By PHP, $|S| \leq |N(S)|$. $\square$

Theorem 1 (Hall's Theorem)**.** *Let $G = (U, V, E)$ be a simple bipartite graph. Then there is a matching of U into V if and only if for every $S \subseteq U$, $|S| \leq |N(S)|$.*

Proof. Let's go!

($\Rightarrow$) Done by the above proposition

($\Leftarrow$) We will use induction on $|U|$.

If $|U| = 1$ then Hall's condition says ther eis an edge incident to the single vertex in U, and this edge gives a matching of U into V.

Suppose that the statement is true for all positive integers less than $|U|$.

Let $G = (U, V, E)$ be a simple bipartite graph so that for all $S \subseteq U$, $|S| \leq |N_G(S)|$. Pick $u \in U$ and $v \in V$ connected by edge e. This must hold because $1 < |U| \leq |N_G(U)|$.

We have two cases:

- Suppose that for all nonempty proper $S \subseteq U$ $|S| < |N_G(S)|$. Consider the induced subgraph on $(U \setminus \{u\}, V \setminus \{v\})$, call it G'.

  Let $S \subseteq U \setminus \{u\}$. There are two options:

  $$N_{G'}(S) = N_G(s) \text{ or } N_G(S) \setminus \{v\}$$

  So either $|N_{G'}(S)| = |N_G(S)| \geq |S|$ or $|N_{G'}(S)| = |N_G(S)| - 1 \geq |S|$ (but this only holds in the strict inequality.

  So G' satisfies Hall's condition and by induction G' has a matching of $U \setminus \{u\}$ into $V \setminus \{v\}$. Adding e yields a matching of U into V.

- Suppose there exists a nonempty proper subset $S \subseteq U$ so that $|S| = |N_G(S)|$. Consider two induced subgraphs. G_1 on $(S, N_G(S))$ and G_2 on $(U \setminus S, V \setminus N_G(S))$.

  In G_1 let $T \subseteq S$, then:

  $$N_G(T) \subseteq N_G(S)$$

  Therefore $|N_{G_1}(T)| = |N_G(T)| \geq |T|$. By induction there is a matching M_1 of S into $N_G\,(S)$.

In G_2 let $T \subseteq U \setminus S$. We have:

$$\begin{aligned} N_{G_2}(T) &= N_G(T) \setminus N_G(S) \\ &= N_G(T \cup S) \setminus N_G(S) \end{aligned}$$

So we know:

$$\begin{aligned} |N_{G_2}(T)| &= |N_G(T \cup S)| - |N_G(S)| \\ &= |N_G(T \cup S)| - |S| \\ &= |T \cup S| - |S| = |T| \end{aligned}$$

By induction there is a matching M_2 of $U \setminus S$ into $V \setminus N_G(S)$. Then $M_1 \cup M_2$ is a matching of U into V.

□

Theorem 2 (Tutte's Theorem)**.** *A simple graph $G = (V, E)$ has a perfect matching if and only if for every $S \subseteq V$ there at most $|S|$-many connected components of $G - S$, the induced subgraph on $V \setminus S$ that have an odd number of vertices.*

Proof in Book, Theorem 11.20. □

Definition. *A* ***permutation matrix*** *is a square matrix with entries 0 and 1 such that every row and every column has exactly one 1. There is a clear bijection from S_n to $n \times n$ permutation matrices. A permutation σ has permutation A_σ with the (i, j)-th entry is 1 if $j = \sigma(i)$ and 0 otherwise.*

Theorem 3. *A square matrix with nonnegative integer entries whose row and columns sums all equal k is a sum of k permutation matrices.*

Proof by induction on k. Let's go!

$k = 1$ follows by definition of permutation matrix.

Assume it holds for k. Let A be an $n \times n$ matrix with nonnegative entries whose rows and columns sum to $k + 1$. It suffices to find a permutation matrix B so that $A - B$ has nonnegative integer entries. Then by induction $A - B$ would be a sum of k permutation matrices so A is a sum of $k + 1$ permutation matrices.

Trick is to construct a simple bipartite graph $G = (U, V, E)$. Where $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. And $(u_i - v_j) \in E$ if and only if the (i, j)-th entry of A is positive.

If G has a perfect matching M then define B with (i, j)-th entry where 1 if $(u_i - v_j) \in M$, and 0 otherwise. We claim that B is a permuation matrix, and $A - B$ has nonnegative integer entries.

On HW, you prove that G has a perfect matching!!! $\square$

MATH 465 Notes

Faye Jackson

April 7, 2020

Ramsey Theory §13

Proposition 1. Suppose that the edges of K_6 have been colored two colors, then there are three vertices such that all edges connecting them are the same color [see HW1 #5]

Proof. Let v be a vertex in K_6, note $\deg(v) = 5$. By PHP three of the edges incident to v have the same color, say red. Call the other endpoints of these edges v_1, v_2, v_3. If there is a red edge connecting any two of these vertices then along with v this forms a red triangle. If not, then the three edges connecting v_1, v_2, v_3 form a blue triangle. □

Example. This is not true for K_5. Color the outside C_5 red and the inner pentagram blue.

Definition. *Let a, b be positive integers. The* ***Ramsey number R(a, b)*** *is the smallest n such that any coloring of the edges of K_n in two colors (say red and blue) must contain either a red K_a or a blue K_b.*

Example. The proposition $\iff$ $R(3,3) \leq 6$, the example coloring of K_5 shows $R(3,3) > 5$, and so $R(3,3) = 6$.

How do we know that there is a complete graph so that this is true? How do we prove that the Ramsey Number's even exist???

Convention: $R(a,1) = R(1,b) = 1$.

Example. For $n \geq 2$ what is $R(n,2)$?

Well $R(n,2) \leq n$, because K_n will either have a blue edge (a blue K_2), or it will be a red K_n. And it's greater than $n-1$, just color K_{n-1} all red. So by symmetry $R(n,2) = R(2,n) = n$, noting that $R(a,b) = R(b,a)$.

These numbers are hard to compute, for example all we know about $R(5,5)$ is that it is somewhere between 43 and 48. These are ridiculously hard, let's prove they even exist!!

Theorem 1. *For all positive integers a, b, $R(a,b)$ exists and:*

$$R(a,b) \leq \binom{a+b-2}{a-1}$$

Lemma 1. *Let $a, b \geq 2$ and assume $R(a-1,b)$ and $R(a,b-1)$ exist. Then $R(a,b)$ exists and:*

$$R(a,b) \leq R(a-1,b) + R(a,b-1)$$

Proof. Let $p = R(a-1,b)$ and $q = R(a,b-1)$. We need to show that any red/blue coloring of the edges of K_{p+q} has either a red K_a or a blue K_b.

Pick a vertex v in K_{p+q}, which has degree $p+q-1$. Among the edges incident to v there are either at least p red edges or at least q blue edges by PHP.

If we have p red edges incident to v, consider the induced subgraph on the other endpoints of those edges, it is isomorphic to K_p. Then since $p = R(a-1,b)$ we have either a red K_{a-1} or a blue K_b. If we have a blue K_b then we're done. If we have a red K_{a-1} we just add the vertex v to obtain a red K_a.

If we have q blue edges incident to v, consider the induced subgraph on the other endpoints of those edges, it is isomorphic to K_q. Then since $q = R(a,b-1)$ we have either a red K_a or a blue K_{b-1}. If we have a red K_a we are done. Then if we have a blue K_{b-1} we can add in v to obtain a blue K_b. $\square$

Theorem 2. *For all positive integers a, b, $R(a,b)$ exists and:*

$$R(a,b) \leq \binom{a+b-2}{a-1}$$

Proof by double induction. First, if $a = 1$ then $R(1,b) = 1$, and if $b = 1$ then $R(a,1) = 1$ and we get the equality:

$$R(1,b) = 1 \leq \binom{b-1}{0} \qquad\qquad R(a,1) = 1 \leq \binom{a-1}{a-1}$$

Now, let $a, b \geq 2$ and assume $R(a-1,b)$ and $R(a,b-1)$ exist. Then also assume:

$$R(a-1,b) \leq \binom{a+b-3}{a-2}$$
$$R(a,b-1) \leq \binom{a+b-3}{a-1}$$

By the lemma, $R(a,b)$ must also exist, and $R(a,b) \leq R(a-1,b) + R(a,b-1)$. But then we can write:

$$\begin{aligned} R(a,b) &\leq R(a-1,b) + R(a,b-1) \\ &\leq \binom{a+b-3}{a-2} + \binom{a+b-3}{a-1} \\ &= \binom{a+b-2}{a-1} \end{aligned}$$

Using Pascal's Recurrence. We win! $\square$

Example. What is $R(4,3)$? By lemma $R(4,3) \leq R(3,3) + R(4,2) = 6 + 4 = 10$.

And by Theorem:

$$R(4,3) \leq \binom{4+3-2}{4-1} = \binom{5}{3} = 10$$

But actually $R(4,3) \leq 9$.
Color the edges of K_9 by red and blue,

Claim. *There exists a vertex that does not have exactly* 5 *red edges incident to it.*

Otherwise the subgraph of red edges would have $\frac{9 \cdot 5}{2}$ edges, and this is impossible!

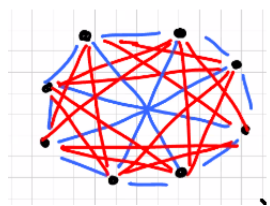
So let v be a vertex such that there are not exactly 5 red edges incident to it. There are two cases:

- If v is incident to ≤ 4 red edges, then since it has degree 8 there are at least 4 blue edges incident to it.

  Consider the edges connecting the other four endpoints. If they are all red we have a red K_4 and we're done. If not then we can form a blue triangle with v and a blue edge.

- In case two, v is incident to ≥ 6 red edges. Look at the other six endpoints of these six edges, they form a K_6, and K_6 always has a monochromatic triangle. If it's a blue triangle then we're done. If it's a red triangle then we can form a K_4 by adjoining v to the red triangle.

In fact $R(4,3) > 8$. There is a coloring of K_8 that has no blue K_3 and no red K_4. See lecture notes because drawing graphs is hard.



One can show in the book that $R(4,4) = 18$.

Let's look at something interesting

Definition. *Let $G = (V, E)$ be a simple graph. $I \subseteq V$ is* ***independent*** *if the induced subgraph on I has no edges.*

A set $Q \subseteq V$ is a ***clique*** *if the induced subgraph on Q is a complete graph.*

Definition. *$R(a, b)$ is the smallest n for which any simple graph on n vertices contains either a clique of size a or an independent set of b vertices.*

Note: Simple graphs on n vertices may be viewed as subgraphs of K_n, that means it can also be seen as a red-blue coloring of K_n. With this in mind think about why the above definition is equivalent to the one we gave before.

Theorem 3 (A Multicolor Ramsey Theorem)**.** *Let $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ then there is a smallest positive integer $N = R(n_1, \ldots, n_k)$ such that if we color the edges of K_N by k (fixed) colors, there is a K_{n_i}-subgraph whose edges are colored by i.*

MATH 465 Notes

Faye Jackson

April 9, 2020

Max Flow Min Cut (Not in Book)

Definition. *A* ***directed graph*** *is a graph in which every edge has an orientation. The edges are arrows with one endpoint the* ***tail*** *and the other the* ***head****.*
A ***directed walk*** *is a walk that always travels along arrows from tail to head.*

Example.

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \arrow[d] \\
\bullet \arrow[u] \arrow[ur] \arrow[r] & \bullet
\end{tikzcd}$$

Definition. *For a vertex v in a directed graph, we have the concepts:*

$$\begin{aligned}
\operatorname{outdeg}(v) &= \#\ \textit{arrows with } v \textit{ as tail} \\
\operatorname{indeg}(v) &= \#\ \textit{arrows with } v \textit{ as head}
\end{aligned}$$

This is neat!

Definition. *A* ***network*** *is a directed grpah $G = (V, E)$ in which:*

- *There are two special vertices* ***source*** *s and* ***sink*** *t. With the assumptions:*

  $$\boxed{\operatorname{indeg}(s) = 0 = \operatorname{outdeg}(t)}$$

- *Each edge has been assigned a nonnegative* ***capacity*** *$c(e) \in \mathbb{Z}$. With the assumptions*

Assume G is simple and has no directed cycles. This helps

Definition. *A **flow** in a network $G = (V, E)$ is a function $f : E \to \mathbb{Z}$ satisfying for all $e \in E$ and $v \in V \setminus \{s, t\}$:*

$$0 \leq f(e) \leq c(e) \qquad \text{(feasibility)}$$

$$\sum_{\bullet \xrightarrow{e} v} f(e) = \sum_{v \xrightarrow{e} \bullet} f(e) \qquad \text{(conservation)}$$

Goal Maximize the following quantity:

Definition. *The **value** of a flow in a network is:*

$$|f| = \sum_{s \xrightarrow{e} \bullet} f(e) = \sum_{\bullet \xrightarrow{e} t} f(e)$$

The equality holds by conservation.

Now let's define a new concept to help us out

Definition. *A **cut** (X, Y) is a partition of the vertices V into disjoint subsets X and Y so that $s \in X$ and $t \in Y$.*

*The **capacity** of a cut (X, Y) is:*

$$c(x, y) = \sum_{\substack{x \in X, y \in Y \\ x \xrightarrow{e} y}} c(e)$$

Lemma 1. *The value of a flow cannot exceed the capacity of a cut. That is for any flow f and cut (X, Y) in a network G, $|f| \leq c(X, Y)$.*

Proof. We'll write down:

$$
\begin{aligned}
|f| &= \sum_{s \xrightarrow{e} \bullet} f(e) = \sum_{x \in X} \left(\sum_{x \xrightarrow{e} \bullet} f(e) - \sum_{\bullet \xrightarrow{e} x} f(e) \right) && \text{(conservation)} \\
&= \sum_{\substack{x,z \in X \\ x \xrightarrow{e} z}} (f(e) - f(e)) + \sum_{\substack{x \in X, y \in Y \\ x \xrightarrow{e} y}} f(e) + \sum_{\substack{x \in X, y \in Y \\ y \xrightarrow{e} x}} (-f(e)) \\
&= \sum_{\substack{x \in X, y \in Y \\ x \xrightarrow{e} y}} f(e) - \sum_{\substack{x \in X, y \in Y \\ y \xrightarrow{e} x}} f(e) \\
&\leq \sum_{\substack{x \in X, y \in Y \\ x \xrightarrow{e} y}} c(e) - \sum_{\substack{x \in X, y \in Y \\ y \xrightarrow{e} x}} 0 = c(X,Y) && \text{(feasibility)}
\end{aligned}
$$

$\square$

Theorem 1 (Max Flow Min Cut)**.** *The maximum value of a flow is the minimum capacity of a cut, i.e.*

$$\max\{|f| \mid f \text{ is a flow in } G\} = \min\{c(X,Y) \mid (X,Y) \text{ is a cut in } G\}$$

This is great!

Idea for proof. By Lemma, it suffices to find a flow F and a cut (X,Y) so that $|F| = c(X,Y)$. By the proof of Lemma, we just ned a flow F and cut (x,y) such that for all $x \in X, y \in Y$ such that:

$$
\begin{aligned}
&\forall\ x \xrightarrow{e} y \qquad F(e) = c(e) \\
&\forall\ y \xrightarrow{e} x \qquad F(e) = 0
\end{aligned}
$$

How do we find F? Given a flow f define a directed graph G_f with vertices V and arrows:

$$
\begin{aligned}
&u \xrightarrow{e^+} v \ \text{ if } \ u \xrightarrow{e} v \ \in E \text{ and } f(e) < c(e) \\
&u \xleftarrow{e^-} v \ \text{ if } \ u \xrightarrow{e} v \ \in E \text{ and } f(e) > 0
\end{aligned}
$$

Ford-Fulkerson Algorithm:

(1) Start with a flow f in $G = (V, E)$. You can just take $f(e) = 0$ for all $e \in E$.

(2) Find a directed path P from s to t in G_f. Let:

$$\delta = \min\big(\{c(e) - f(e) \mid e^+ \in P\} \cup \{f(e) \mid e^- \in P\}\big)$$

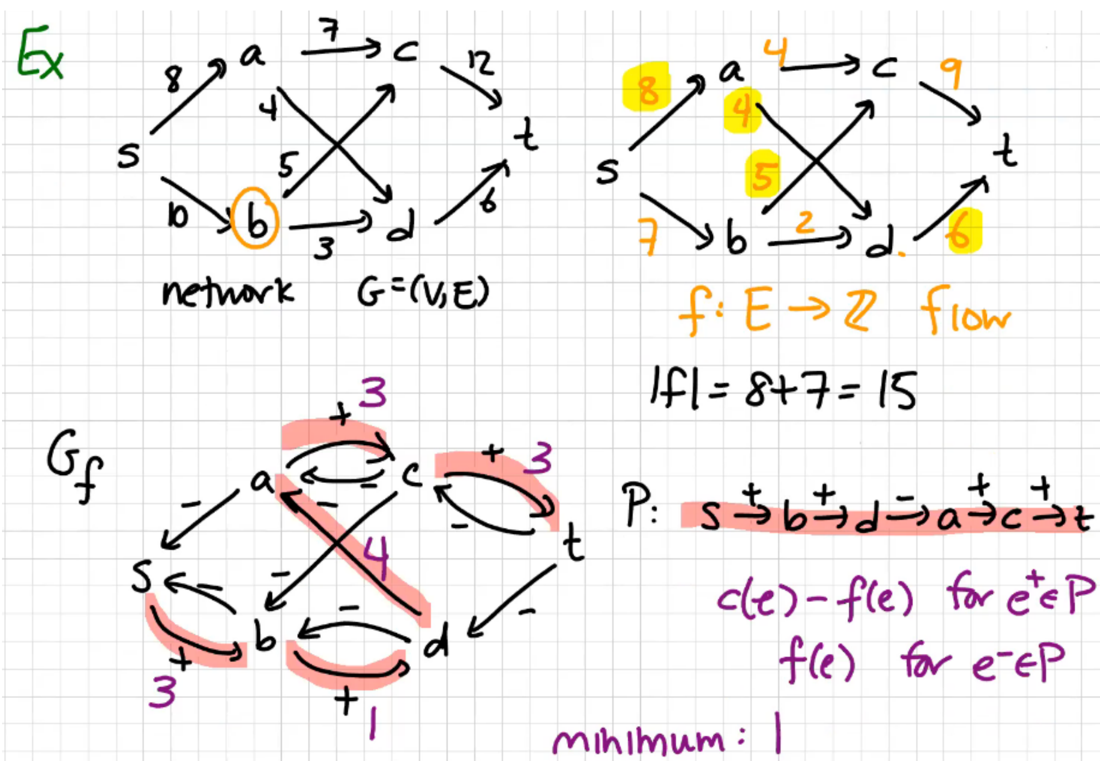
Define a flow f':

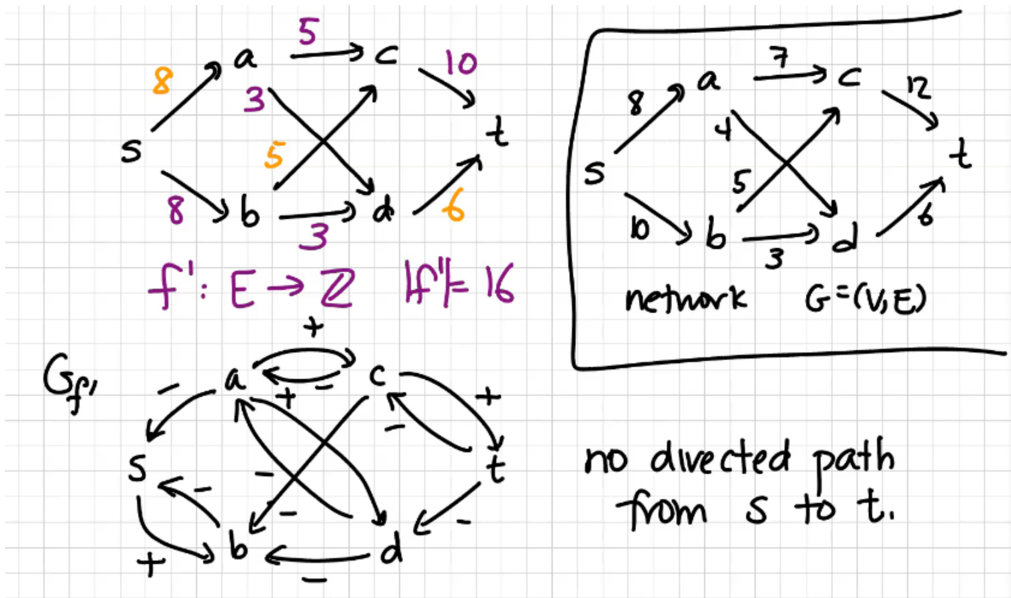
$$f'(e) = \begin{cases} f(e) + \delta & \text{if } e^+ \in P \\ f(e) - \delta & \text{if } e^- \in P \\ f(e) & \text{if } e^+, e^- \notin P \end{cases}$$

Claim. *f' is a flow with $|f'| = |f| + \delta$.*

(3) Repeat step 2, with f', until no such directed path exists.

Look at the example from the lecture notes!





Now once we terminate this process, call that flow F. Define:

$$X = \{x \mid \exists \text{ a directed path from } s \text{ in } G_F\}$$
$$Y = \{y \mid \nexists \text{ a directed path from } s \text{ in } G_F\} = V \setminus X$$

We should actually do some proof stuff! Let F be the flow obtained from the Ford Fulkerson Algorithm starting with $f = 0$, then there is no directed path from s to t in G_F. Note $s \in X$ and $t \in Y$ by the construction of F. Thus (X, Y) is a cut.

Then, for $x \xrightarrow{e} y$ for $x \in X, y \in Y$ we have $F(e) = c(e)$, because otherwise we would have some path from s to y in G_F:

$$s \longrightarrow \cdots \longrightarrow x \xrightarrow{e^+} y$$

For $y \xrightarrow{e} y$ for $x \in X, y \in Y$ we have $F(e) = 0$ because otherwise in G_F there is a directed path from s to y:

$$s \longrightarrow \cdots \longrightarrow x \xrightarrow{e^-} y$$

Then by the steps in the proof of lemma, $|F| = c(X, Y)$, and so by Lemma $\max |F| = \min c(X, Y)$. Great! $\square$

MATH 465 Notes

Faye Jackson

April 9, 2020

Meow

Definition. *A **partially ordered set (poset)** $(P, \leq)$ is a set P endowed with a binary relation which satisfies for all $x, y, z \in P$ that:*

$$x \leq x \qquad \text{(reflexive)}$$

$$x \leq y \text{ and } y \leq x \implies x = y \qquad \text{(antisymmetric)}$$

$$x \leq y \text{ and } y \leq z \implies x \leq z \qquad \text{(transitive)}$$

If $x \leq y$ and $x \neq y$ write $x < y$. And $x \geq y$ means $y \leq x$.

*For any two distinct $x, y \in P$ exactly one of the following is true: $x \leq y$ or $y \leq x$ or x and y are **incomparable***

*If any two elements of P are comparable, say that P is **linearly, or totally ordered***

Example. $(\mathbb{R}, \leq)$ is linearly ordered. Also n-letter words in (ordered) alphabet is totally ordered, see dictionaries.

Example. But subsets of $[n]$ with $\subseteq$, this is a poset but it is not totally ordered.

From here on we assume our posets are finite.

Example. Let P be a finite set of positive integers ordered by divisibility. For example $P = \{1, 2, 3\}$, and we have $1 \leq 2$ and $1 \leq 3$ but 2 and 3 are incomparable.

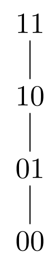
Example. Consider P to be n-tuples ordered componentwise, so that:

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \iff x_i \leq y_i \forall i$$

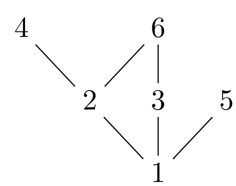
Definition. *In a poset (P, S) we say* y ***covers*** x $(x \prec y)$ *if $x < y$ and there is no $z \in P$ so that $x < z < y$. Note that $x < y$ if and only if there is $x_1, \ldots, x_k$ so that $x \prec x_1 \prec \cdots \prec x_k = y$.*

A ***Hasse Diagram*** *is a pictorial representation of $(P, \leq)$ by a graph with $V = P$ with an edge $x — y$ whenever $x \prec y$ (or $y \prec x$) drawn so taht if x, y then y is higher than x. One can also envision a directed graph.*

Example. Consider 2-letter words in alphabet $\{0, 1\}$ ordered lexicographically (with $0 < 1$). Then $00 < 01 < 10 < 11$, so the diagram is just:



Example. $P = \{1, 2, 3, 4, 5, 6\}$ ordered by divisibility.



Example. Let $G = (V, E)$ be an acyclic (no directed cycles) directed graph. Consider the set V as a poset with the order $u \leq v$ if and only if there is a directed path from u to v.

Definition. *In a poset $(P, \leq)$ we say that an element $m \in P$ is* ***maximal*** *if for all $x \in P$, $m \leq x \implies m = x$. In the Hasse diagram this corresponds to having* outdeg $= 0$.

And m is a ***maximum*** *if for all $x \in P$, $x \leq m$. We can define the corresponding ideas of* ***minimal*** *and* ***minimum*** *minimal corresponds to having* indeg $= 0$.

Note: P has maximum if and only if it has a unique maximal element.

Example. If P is a poset where no two elements are comparable, its Hasse diagram is a set of isolated vertices. Furthermore every element is both maximal and minimal. This is called an anti-chain.

Definition. *A **chain** in a poset is a subset in which any two elements are comparable, and an **antichain** is a subset where no two distinct elements are comparable A chain is called maximal provided that it is not contained in a larger chain.*

Definition. *The **Boolean lattice** B_n is subsets of $[n]$ ordered by the subset relation. Furthermore, maximal chains in B_n are in bijection with S_n.*

Definition. *The **height** of a poset is the largest size of a chain. The **width** of a poset is the largest size of an antichain.*

Example. B_3 above has width 3 and height 4.

Example. If P is totally ordered it has height $|P|$ and width 1. If P is totally unordered (i.e. an antichain), then it has height 1 and width $|P|$.

Idea: height/width roughly measure how close P is to being totally oredered or totally unordered.

Definition. *A **chain partition** of a poset P is a partition of the set P where each block is a chain, likewise we have a definition for an **antichain partition***

Theorem 1 (Dilworth)**.** *In a finite poset, the* $\overbrace{\textit{maximum size of an antichain}}^{\textit{width}}$ *is the minimum number of blocks in a chain partition*

Theorem 2 (Mirsky)**.** *In a finite poset, the* $\overbrace{\textit{maximum size of a chain}}^{\textit{height}}$ *is the minimum number of blocks in an antichain partition*

Lemma 1. *Let P be a finite poset. If $\{C_1, \ldots, C_n\}$ is a chain partition of P, then every antichain has at most n elements. Likewise if $\{A_1, \ldots, A_m\}$ is an antichain partition of P, then every chain has at most m elements.*

Proof. Let A be an antichain and $C_1, \ldots, C_n$ a chain partition of P. Every element of A is in some block C_i, but we can't have more than one element of A in the same block, since this would imply two elements were both comparable and incomparable. This defines an injective function $A \to [n]$, and so $|A| \leq n$.

The second argument is similar!!! Do it yourself as practice!!! □

MATH 465 Notes

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Posets Continued

Recall. In a poset a chain is a subset in which any two elements are comparable. An antichain is a subset in which any two elements are incomparable. A chain (or antichain) partition is a partition in which each block is a chain (or antichain)

Theorem 1 (Dilworth's Theorem)**.** *In a fintie poset, the* $\overbrace{\textit{maximum size of an antichain}}^{\textit{width}}$ *is the minimum size of a chain partition.*

Theorem 2 (Mirsky's Theorem)**.** *In a fintie poset, the* $\overbrace{\textit{maximum size of a chain}}^{\textit{height}}$ *is the minimum size of an antichain partition.*

Lemma 1. *Let P be a finite poset. If $\{C_1, \ldots, C_n\}$ is a chain partition of P and A is an antichain, then $|A| \leq n$.*

Likewise if $\{A_1, \ldots, A_m\}$ is an antichain partition of P and C is a chain, then $|C| \leq m$.

Proof of Mirsky's Theorem. Lemma shows that the maximum size of a chain is less than or equal to the minimum size of an antichain partition. Thus, letting m be the maximum size of a chain, it suffices to find an antichain partition with m blocks. Say $\{x_1, \ldots, x_m\}$ is a chain with $x_1 < \ldots < x_m$. For each $1 \leq i \leq m$ let

$$A_i = \{x \in P \mid \text{the max size of a chain with } x \text{ at the top is } i\}$$

This is a partition of P into m nonempty blocks, $x_i \in A_i$ for each i. Moreover, each A_i is an antichain: If $x < y$ and the max size of a chain with x at the top is i, then there is a chain with $i + 1$ elements and y at the top. $\square$

Proof of Dilworth's Theorem. Lemma gives that the maximum size of an antichain is less than or equal to the minimum size of a chain partition. We will use Max-Flow Min-Cut Theorem to find an antichain and a chain partition of the same size (then we're done)

Define a network G by taking two disjoint copies of P, call them (P_L, P_R) along with the sourse s and sink t as vertices. And edges:

- $x \to y$ if $x < y$
- $s \to x$ if $x \in P_L$
- $y \to t$ if $y \in P_R$.

With all capacities $c(e) = 1$. The key is to relate flows in G and chain partitions in P as follows:

Consider a flow f in G. Because all edge capacities are 1, if $x \in P_L$ with $f(s \to x) \neq 0$ there is a unique $y \in P_R$ such that x, y and $f(x \to y) \neq 0$. Consider the edges $x \to y$ with $x \in P_L$ and $y \in P_R$ such that $f(x \to y) \neq 0$. This gives a collection of pairs:

$$x_1 < y_1, \ldots, x_\ell < y_\ell$$

Where the x_i's are distinct and the y_i's are distinct. These pairs bundle into pairwise disjoint chains in P by stacking the pairs in which $x_i < y_i = x_j < y_j$. If any elements in P do not appear in these chains, add it as a singleton block to obtain a chain partition $P = C_1 \cup \cdots \cup C_n$. This chain partition constructed from a flow f satisfies:

$$|f| = \sum_{x \in P_L} f(s \to x) = \sum_{i=1}^{n} (|C|_i - 1) = |P| - n$$

Equivalently $n = |P| - |f|$. In fact every chain partition can be constructed in this way. Therefore the minimum size of a chain partition in P is equal to $|P|$ minus the maximum size of a flow in G.

Use Ford-Fulkerson algorithm to find a max flow f in G. Let (X, Y) be the corresponding minimum cut. Then let n be the maximum size of a chain partition, then:

$$|f| = |P| - n = c(X, Y)$$

By the proof of the Max-Flow Min-Cut Theorem there does not exist $x \in P_L \cap X$ and $y \in P_R \cap Y$ such that $x \to y$ is an edge in G (i.e. $x < y$). This is because there is no directed path from s to t in G_f by construction of f. Why?

Suppose $x \in P_L \cap X$ and $y \in P_R \cap Y$. Well then there is a directed path from s to x in G_f but not one from s to y. Furthermore in G_f there is no edge $x \to y$, but since $x < y$ we know there is an edge between x and y in G. Thus there must be an edge $y \to x$ in G_f. Thus $f(x \to y) = c(x \to y) = 1$. Therefore note that we must have $f(s \to x) = 1$ and so there has to be an edge $x \to s$ in G_f. Thus there exists a $y' \in P_R \cap X$ such that $y' \to x$ is in G_f. But then $f(x \to y') = 1$, contradicting conservation at x:

$$f(s \to x) = 1 < 2 = f(x \to y) + f(x \to y')$$

Define:

$$A = \{p \in P \mid p \in P_L \cap X \text{ and } p \in P_R \cap Y\}$$

We can't have two comparable elements in this by the above claim. Moreover:

$$\begin{aligned} |P| - n = c(X, Y) &= \sum_{\substack{x \to y \\ x \in X, y \in Y}} c(x \to y) \\ &= \sum_{y \in Y \cap P_L} c(s \to y) + \sum_{x \in X \cap P_R} c(x \to t) + \sum_{\substack{x \in P_L \cap X \\ y \in P_R \cap Y}} 0 \\ &= |Y \cap P_L| + |X \cap P_R| \\ &\geq |P| - |A| \end{aligned}$$

Thus $|A| \geq n$. Therefore the maximum size of an antichain is $\geq n$ which is the minimum size of a chain partition. We already have the other inequality. $\square$

MATH 465 Notes

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Graphs and Matrices

Definition. *The **adjacency matrix** of a loopless graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ is the $n \times n$ matrix $A = A(G)$ whos (i, j)-th entry is the number of edges between v_i and v_j.*

This has good properties:

(1) A is symmetric, so $A^T = A$

(2) Since G is loopless, diagonal entries are 0.

(3) Fix $i \in [n]$, then $\sum_{j=1}^{n} A_{ij} = \deg(v_i) = \sum_{j=1}^{n} A_{ji}$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} = \sum_{i=1}^{n} \deg(v_i) = 2\,|E|$$