

MATH 465 Notes

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Meow

Definition. A *partially ordered set (poset)* (P, \leq) is a set P endowed with a binary relation which satisfies for all $x, y, z \in P$ that:

$$x \leq x \quad (\text{reflexive})$$

$$x \leq y \text{ and } y \leq x \implies x = y \quad (\text{antisymmetric})$$

$$x \leq y \text{ and } y \leq z \implies x \leq z \quad (\text{transitive})$$

If $x \leq y$ and $x \neq y$ write $x < y$. And $x \geq y$ means $y \leq x$.

For any two distinct $x, y \in P$ exactly one of the following is true: $x \leq y$ or $y \leq x$ or x and y are *incomparable*

If any two elements of P are comparable, say that P is *linearly, or totally ordered*

Example. (\mathbb{R}, \leq) is linearly ordered. Also n -letter words in (ordered) alphabet is totally ordered, see dictionaries.

Example. But subsets of $[n]$ with \subseteq , this is a poset but it is not totally ordered.

From here on we assume our posets are finite.

Example. Let P be a finite set of positive integers ordered by divisibility. For example $P = \{1, 2, 3\}$, and we have $1 \leq 2$ and $1 \leq 3$ but 2 and 3 are incomparable.

Example. Consider P to be n -tuples ordered componentwise, so that:

$$(x_1, \dots, x_n) \leq (y_1, \dots, y_n) \iff x_i \leq y_i \forall i$$

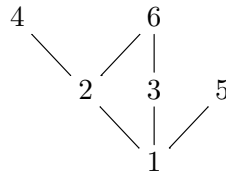
Definition. In a poset (P, \leq) we say y **covers** x ($x \prec y$) if $x < y$ and there is no $z \in P$ so that $x < z < y$. Note that $x < y$ if and only if there is x_1, \dots, x_k so that $x \prec x_1 \prec \dots \prec x_k = y$.

A **Hasse Diagram** is a pictorial representation of (P, \leq) by a graph with $V = P$ with an edge $x \text{ --- } y$ whenever $x \prec y$ (or $y \prec x$) drawn so that if x, y then y is higher than x . One can also envision a directed graph.

Example. Consider 2-letter words in alphabet $\{0, 1\}$ ordered lexicographically (with $0 < 1$). Then $00 < 01 < 10 < 11$, so the diagram is just:



Example. $P = \{1, 2, 3, 4, 5, 6\}$ ordered by divisibility.



Example. Let $G = (V, E)$ be an acyclic (no directed cycles) directed graph. Consider the set V as a poset with the order $u \leq v$ if and only if there is a directed path from u to v .

Definition. In a poset (P, \leq) we say that an element $m \in P$ is **maximal** if for all $x \in P$, $m \leq x \implies m = x$. In the Hasse diagram this corresponds to having $\text{outdeg} = 0$.

And m is a **maximum** if for all $x \in P$, $x \leq m$. We can define the corresponding ideas of **minimal** and **minimum** minimal corresponds to having $\text{indeg} = 0$.

Note: P has maximum if and only if it has a unique maximal element.

Example. If P is a poset where no two elements are comparable, its Hasse diagram is a set of isolated vertices. Furthermore every element is both maximal and minimal. This is called an anti-chain.

Definition. A *chain* in a poset is a subset in which any two elements are comparable, and an *antichain* is a subset where no two distinct elements are comparable. A chain is called maximal provided that it is not contained in a larger chain.

Definition. The *Boolean lattice* B_n is subsets of $[n]$ ordered by the subset relation. Furthermore, maximal chains in B_n are in bijection with S_n .

Definition. The *height* of a poset is the largest size of a chain. The *width* of a poset is the largest size of an antichain.

Example. B_3 above has width 3 and height 4.

Example. If P is totally ordered it has height $|P|$ and width 1. If P is totally unordered (i.e. an antichain), then it has height 1 and width $|P|$.

Idea: height/width roughly measure how close P is to being totally ordered or totally unordered.

Definition. A *chain partition* of a poset P is a partition of the set P where each block is a chain, likewise we have a definition for an *antichain partition*

Theorem 1 (Dilworth). In a finite poset, the $\overbrace{\text{maximum size of an antichain}}^{\text{width}}$ is the minimum number of blocks in a chain partition

Theorem 2 (Mirsky). In a finite poset, the $\overbrace{\text{maximum size of a chain}}^{\text{height}}$ is the minimum number of blocks in an antichain partition

Lemma 1. Let P be a finite poset. If $\{C_1, \dots, C_n\}$ is a chain partition of P , then every antichain has at most n elements. Likewise if $\{A_1, \dots, A_m\}$ is an antichain partition of P , then every chain has at most m elements.

Proof. Let A be an antichain and C_1, \dots, C_n a chain partition of P . Every element of A is in some block C_i , but we can't have more than one element of A in the same block, since this would imply two elements were both comparable and incomparable. This defines an injective function $A \rightarrow [n]$, and so $|A| \leq n$.

The second argument is similar!!! Do it yourself as practice!!!

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