MATH 465 Notes

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1 Stuff

- If on waitlist: please let registered students sit at desks
- Quiz 1 today (PHP)
- HW1 due tomorrow (see Canvas)
- Quiz 2 Thursdady
- Today: Basic Counting Principles
- Next Time: More Counting (Ch. 3, §5.1)

2 The Golden Rule of Counting

Golden Rule of counting: Count everything exactly once.

Things we might count:

(1) Elements of a set (Ex. How many math majors are in this room?)

(2) Choices or possibilities (Ex. How many ways could we arrange ourselves into seats)

The Additive Principle: If a finite set S can be divided into k $\boxed{\text{pairwise disjoint}}$ $S_i \cap S_j = \emptyset$ for $i \neq j$ subsets, $S_1, S_2, \ldots, S_k$, then one has:

$$|S| = |S_1| + |S_2| + \ldots + |S_k|$$

Golden: Everything in S is counted exactly once

In terms of choices, view the sets as cases or mutually exclusive events.

The Multiplicative Principle: If a finite set S can be divided into k pairwise disjoint subsets, each with n elements, then $|S| = nk$.

This has the most power when we iterate it. Think: sequences of choices

Example. Suppose we are organizing a panel discussion and we need to arrange the four panel members in a row. Two are graduate students, two are undergrads.

(a) How many possible ways could the panel members be arranged.

Well $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, choose the first person out of 4, the second person out of 3, and so on. Definite multiplicative principle.

(b) How many ways... so that grad students are next to each other.

Treat grad students as one person, but in two cases, then $2 \cdot 3! = 12$ because we're now arranging three people.

(c) How many ways... so that grad students are NOT next to each other

Well $24 - 12 = 12$... Also can consider things like this:

$$GUUG \to 4$$
$$GUGU \to 4$$
$$UGUG \to 4$$

Subtraction is like $\{(a)\} = \{(b)\} \cup \{(c)\}$, and the sets $\{(b)\}$ and $\{(c)\}$ are disjoint. So additive principle.

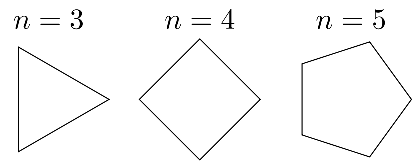
Subtraction Principle

If S is a finite set and $T \subset S$ then $|S \setminus T| = |S| - |T|$. This follows from Addition Principle easily. [Idea: Count irrelevant things]

The Division Principle If we count every possibility k times, then we may divide by k to obtain the number of distinct possibilities.

Example. Let $n \neq 3$. How many diagonals are there in a convex n-gon:

$3 \mapsto \triangle$	$4 \mapsto \square$	$5 \mapsto$ ⬠	$6 \mapsto$ ⬡
0	2	5	9



For each of the n vertices there are $n-3$ other vertices it can connect to via a diagonal. So we'd expect $n(n-3)$ diagonals, but this counts every diagonal exactly twice, so there should be $\boxed{\frac{n(n-3)}{2}}$

Example. (a) How many ways can 4 people sit in a row? $4 \cdot 3 \cdot 3 \cdot 1 = 24$

(b) How many ways can 4 people sit in the 4 seats of a car? The same! 24.

(c) How about 5 people in 5 seats. $5 \cdot 24 = 120$

(d) How do I arrange n people into n seats?

(e) A <u>permutation</u> of a finite set is a linear arrangement of its elements.

Theorem 1. *Let n be a positive integer. The number of permutations of an n-element set is $n!$ that is:*

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1 \qquad \text{n factorial}$$

As convention, we let $0! = 1$. Linear arrangement of the $\emptyset$.

Proof. When $n = 1$ we have 1 permutation of a 1-element set, and $1! = 1$.

Let n be a positive integer and assume there are $n!$ permutations of <u>any</u> n-element set.

Now consider an $(n+1)$-element set. To arrange this set in a line, we have $n+1$ choices for the first element; after it is chosen, we have $n!$ ways to arrange the remaining n elements by the Inductive Hypothesis. Then by the multiplication priciple, there are $(n+1)n! = (n+1)!$ permutations of an $(n+1)$-element set. $\square$

Example. Let $n \neq 2$. How many ways are there to arrange n distinct objects in a circle, up to notation?

There should be $(n-1)!$ ways to do so. Count all the linear arrangements, which can be glued together to give a circular arrangement, then there are n rotations which give the same circular arrangement. Thus we have counted each circle n times, so we have $\frac{n!}{n} = (n-1)!$.

Example. How many ways can 4 people sit in the 6 seats of a car?

Fill in all the seats including empty ones with "phantom people", well then there are two extra seats, and we can permute those two empty seats without changing things, so we can do this:

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2}$$

Theorem 2. *Let n, k be positive integers with $n \geq k$. The number of permutations of k-element subsets of an n-element set is:*

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Falling factorial $(n)_k$

Proof. First, permute the entire n-element set, then remove the last $n-k$ elements. There is one way to remove them, but they show up in $(n-k)!$ permutations. By the division principle we should have that the number of permutations of k-element subsets of an n-element set is $\frac{n!}{(n-k)!}$ □

Definition 1. *A word in an alphabet is any string of characters (i.e. elements of alphabet).*

Idea: A permutation is a word where letters are not repeated.

Example. Let your alphabet be $\{0, 1\}$. Three letter words are like:

000 010 111 110 011 101

We have $2 \cdot 2 \cdot 2$, 3-letter binary words.

Theorem 3. *Let n, k be positive integers. There are n^k k-lettter words in an a-letter alphabet.*

Idea: There are n ways to choose each letter, and we choose k times. It is slicker to use induction on k and the multiplication principle.

Convention: There is $n^0 = 1$ "empty word" (word with no letters).