MATH 465 Notes

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1 Announcement

- HW6 Due Wednesday
- Today: Catalan Numbers (§8.1.2.1)
- Up Next: §5.3 Integer Partitions
- No OH tomorrow, Back on Schedule Monday

2 Review

2.1 Basics

$c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14$. And in general:

$$c_n = \frac{1}{n+1} = \binom{2n}{n}$$

For $n \in \mathbb{Z}_{\geq 0}$ then we have:

$$c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}$$

The generating function is:

$$\sum_{n \geq 0} c_n x^n = \frac{1 - \sqrt{1-4x}}{2x}$$

2.2 Finishing up Last Time

We had NE lattice paths from $(0,0)$ to (n,n) which do not cross above the diagonal $y = x$.

Proof 1. Consider $n = 3$

We want to look at:

$$c_{2+1} = c_0c_2 + c_1c_1 + c_2c_0$$

Which will better show us how this all really works (too hard to draw ☹).

We will break into cases based on the last time it touches the diagonal before the endpoint, let this point be (i, i).

Each of these paths can be constructed from a path from $(0, 0)$ to (i, i) which doesn't cross $y = x$, there are c_i many of these. Followed by a step east, a path from $(i + 1, i)$ to $(n + 1, n)$ which doesn't cross $y = x - 1$, followed by a step north. There are c_{n-i} of these.

Since there is one such path $(0, 0)$ to $(0, 0)$ and the sequence satisfies the catalan recurrence we win. □

Proof 2. We can also count these directly. Well, we will count the complement, the number of northeast paths which do cross above the diagonal.

We start with NE lattice paths from $(0, 0)$ to (n, n) which cross above the diagonal. We shift every one of these right one, giving us NE lattice paths from $(1, 0)$ to $(n+1, n)$ which touch (or cross above) the diagonal. Then we reflect the parth from $(0, 0)$ to (i, i) where i is the first place it touches the diagonal, which gets us NE lattice paths from $(0, 1)$ to $(n+, n)$ which touch the diagonal. These is equivalently any lattice path from $(0, 1)$ to $(n + 1, n)$. There are $\binom{2n}{n+1}$ of these.

These are indeed bijections. Thus there are $\binom{2n}{n+1}$ paths which do cross the diagonal. So then by previous class stuff:

$$\begin{aligned} c_n &= \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \\ &= \frac{(2n)!}{n!n!} - \frac{n(2n)!}{n(n+1)!(n-1)!} = \left(1 - \frac{n}{n+1}\right)\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n} \end{aligned}$$

□

These NE lattice paths are actually equivalent to diagonal-up $(1, 1)$ and diagonal-down $(1, -1)$ steps that never go below the x-axis

These also correspond to Ballot Sequences with n pluses and n minuses so that each initial segment has at least as many pluses as minuses.

3 New Stuff

Example. A $2 \times n$ matrix is called a <u>tableau</u> if its entries are $1, 2, 3, \ldots, 2n$ (each used exactly once) and arranged so that each entry is greater than the one above it and the one to its left. That is rows increase $L \to R$ and columns increase $T \to B$.

$n = 1$ gives:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$n = 2$ gives:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$n = 3$ gives:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

Claim. *For $n \geq 1$ there are exactly c_n tableaux*

Proof. Construct a bijection with NE lattice paths which don't cross $y = x$. Given a tableau define a lattice path from $(0, 0)$ to (n, n) whose i-th step is E if i is in the first row and N if i is in the second row. At the i-th step, if i is in the k-th column then we will have gone k steps E and at most k steps N, so it doesn't cross $y = x$.

The reason that in the k-th row we must

In fact this is a bijection. □

4 Integer Partitions

Example.

(a) How many ways are there to distribute five identical cookies to five children. These are weak compositions of 5 with 5 parts, so $\binom{9}{4} = 126$.

(b) How many ways can you put five identical cookies into identical piles:

One Pile	5	
Two Piles	$1, 4$	$3, 2$
Three Piles	$3, 1, 1$	$2, 2, 1$
Four Piles	$2, 1, 1, 1$	
Five Piles	$1, 1, 1, 1, 1$	

There's no closed formula to answer this in general.

Definition 1. *A partition of $n \in \mathbb{Z}_{\geq 0}$ is a weakly decreasing finite sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers for $\ell \in \mathbb{N}$ so that $\sum_{1 \leq i \leq \ell} \lambda_i = n$*

The λ_i are parts of λ and

$$p(n)\# \textit{ partitions of } n$$

Above we found $p(5) = 7$, By convention we let $p(0) = 1$. We know $p(1) = 1$, $p(2) = 2$, $\ldots$ OUT OF TIME.