MATH 465 Notes

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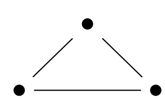
Planar Graphs

Question: Suppose 3 houses and 3 utilities, can we connect each house to each utility without wires crossing? This is essentially $K_{3,3}$

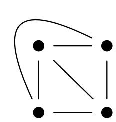
Question: Suppose 5 houses are connected by a road, can we draw a road map so that any two houses are connected by a road but no roads cross each other? This should look like K_5, and we're asking if we can draw K_5 without any of the edges crossing each other. In fact, you can't! Proof soon

Definition 1. *A graph is **planar** if it can be embedded in the plane (i.e. drawn on a planar surface) so that no two edges intersect.*

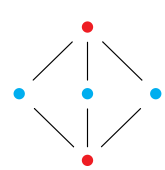
Example. K_3 is planar, with **2 Faces**:



K_4 is planar due to the following planar embedding, with **4 faces**:



$K_{2,3}$ is planar due to the planar embedding with **3 faces**:



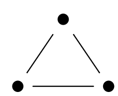
Note that trees are planar, we just start drawing and because there are no cycles we never run into problems. Note that they always have **1 face**

Definition 2. *Note that the edges in a planar embedding partition the plane into regions called **faces**. The region outside the graph is the **outer, or external face** (as opposed to the **internal faces**)*

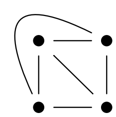
Theorem 1 (Euler's Theorem)**.** *Let $G = (V, E)$ be a connected planar graph. Let F be the set of faces obtained by drawing G in the plane without edge-crossing. Then:*

$$|V| - |E| + |F| = 2$$

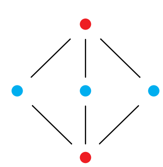
Example. K_3 satisfies the formula with $3 - 3 + 2 = 2$.



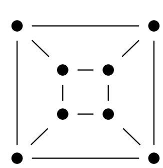
K_4 satisfies the formula with $4 - 6 + 4 = 2$



$K_{2,3}$ satisfies the formula with $5 - 6 + 3$:



A tree with 7 vertices satisfies the formula because $|E| = |V| - 1$ so $7 - 6 + 1 = 2$. Also look at the skeleton of a cube:



Then this satisfies the formula with $8 - 12 + 6 = 2$

Proof. We will prove by induction on $|E|$:

- When $|E| = 0$ we know $|V| = 1 = |F|$ and so:

$$|V| - |E| + |F| = 1 - 0 + 1 = 2$$

- Let n be a nonnegative integer and assume that every connected planar graph with n edges satisfies Euler's Formula. Let $G = (V, E)$ be a connected planar graph with $n + 1$ edges. We will break this up into two cases:

  - If G is a tree then $|V| = |E| + 1$, by last time, and since trees have no cycles, $|F| = 1$. So then:

$$|V| - |E| + |F| = (n + 2) - (n + 1) + 1 = 2$$

  - Suppose G is not a tree. Then there is an edge which is not a bridge, call it $e \in E$. That is the subgraph $G' = (V, E \setminus \{e\})$ is connected. Note then that every subgraph of a planar graph is planar. Our inductive hypothesis implies that:

$$|V| - |E \setminus \{e\}| + \left|F'\right| = 2,$$

    Where F' is the number of faces in G'. Note that since e is not a bridge, it is in a cycle, and so it sees two different faces on each of it's sides, so removing it makes those two faces become the same face in G'. Thus $|F'| = |F| - 1$. So we have:

$$|V| - (|E| - 1) + (|F| - 1) = 2$$

    Hence:

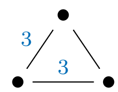
$$|V| - |E| + |F| = 2$$

□

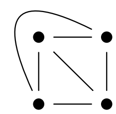
Corrolary 1. *The number of faces does not depend on the choice of planar embedding.*

Definition 3. *Given a face $f \in F$, define its degree* $\deg(f)$ *as the number of adjacent sides of edges.*

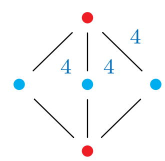
Example. K_3



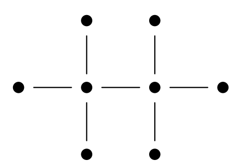
K_4



$K_{2,3}$



A tree with 7 vertices has one face with degree 12 here:



Each edge contributes 2 to the total degree so:

$$\sum_{f \in F} \deg(f) = 2\,|E| = \sum_{v \in V} \deg(v)$$

Lemma 1. *If G is a connected planar simple graph with at least three vertices, then every face has degree at least 3 and:*

$$3\,|F| \leq 2\,|E|$$

Proof. Note that the boundary of an internal face f determines a cycle whose length is $\leq \deg(f)$. Since a simple graph has no cycle of length $,3$, every face has degree ≥ 3.

To see the outer face, note also, we need at least three vertices. Now note that:

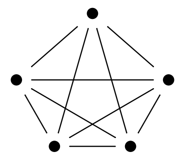
$$3\,|F| \leq \sum_{f \in F} \deg(f) = 2\,|E|$$

On homework you will prove that:

$$|E| \leq 3|V| - 6$$

$\square$

Proposition 1. K_5 is not planar.



Proof. For a contradiction, suppose that K_5 is planar. Since K_5 has 5 vertices and $\binom{5}{2}$ edges. A planar embedding would have:

$$|F| = 2 - |V| + |E| = 2 - 5 + 10 = 7$$

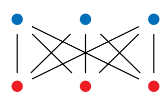
So then:

$$3|F| = 21 > 2|E| = 20$$

But since K_5 is simple, this contradicts the last lemma. $\square$

Proposition 2. $K_{3,3}$ is not planar.

Proof. Argue by contradiction, Assume $K_{3,3}$ is planar.
Note that $|V| = 6$ and $|E| = 9$:



And so by Euler's formula a planar embedding would have

$$|F| = 2 - 6 + 9 = 5$$

But we also have:

$$3|F| = 15 \leq 18 = 2|E|$$

So we cannot use the same proof as last time. The key observation is that $K_{3,3}$ has no 3-cycles. Thus each cycle has at least 4. So in fact the interior faces all have degree at least 4:

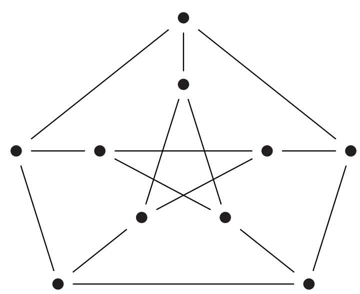
$$20 = 4\,|F| \leq \sum_{f \in F} \deg(f) = 2\,|E| = 18$$

This is a contradiction! The outer face follows because we have so many vertices, and so the boundary of the outer face will form a long cycle. $K_{3,3}$ is not planar! $\square$

Theorem 2 (Wagner's Theorem)**.** *A graph is planar if and only if neither K_5 not $K_{3,3}$ can be obtained from G by deleting vertices deleting edges, and contracting edges (contracting an edge means we shold remove the edge and merge the endpoints)*

Proof Omitted. $\square$

Example. Consider:



We can obtain K_5 by contracting the outside towards the inside.