MATH 465 Notes

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Today: Pigeonhole Principle (Ch. 1) Next Time: Basic Counting Principles (Ch. 3)

Theorem 1 (The Pigeonhole Principle, PHP)**.** *If we have two positive integers with $m > n$, and if we have m pigeons to put into n pigeonholes, then at least one pigeonhole contains more than one pigeon.*

In other words, if you place m objects into n boxes, then at least one box has more than one object in it.

In rigourous terms. Let M and N be finite sets with $|M| > |N|$. Then any function $f : M \to N$ cannot be injective.

Example. Suppose that we select 11 different integers from the set $\{1, 2, 3 \ldots 20\}$. Prove that there will always be two among the selected integers, whose difference is two.

Proof. We construct ten boxes of the form:

$$\begin{array}{ccccc} \{1,3\} & \{5,7\} & \{9,11\} & \{13,15\} & \{17,19\} \\ \{2,4\} & \{6,8\} & \{10,12\} & \{14,16\} & \{18,20\} \end{array}$$

Place the 11 integers into the 10 boxes above. By the Pigeonhole Principle there is a box with two selected integers in it. By construction, these integers have a difference of two. □

Theorem 2 (The Generalized Pigeonhole Principle, GPHP)**.** *Let m, n, r be positive integers with $m > nr$. If we place m objects into n boxes, then at least one box has more than r objects.*

The Pigeonhole Principle is in the case where $r = 1$

Proof. Done by contradiction. We will assume the contrary, that is assume that we can place m objects into n boxes such that every box contains at most r objects.

Then the total number of objects (m) is at most $r+r+r\ldots+r$, n times, that is nr. And so, $m \leq nr$. This contradicts the assumption that $m > nr$, and so we are done. □

Example. The Michigan stadium can hold up to 107,601 spectators. Prove that, during a sell-out crowd, there is a group of 294 spectators with the same birthday.

Proof. There are 366 days on which someone could have a birthday. These are our boxes. Place the spectators into each of these boxes. We know that $107,601 > 107238 = 366 * 293$. Therefore, by the The Generalized Pigeonhole Principle, there is a box with more than 293 people in it. That is, there is a group of at least 294 people with the same birthday ☺ □

Example. What is the minimum number of spectators necessary to ensure 100

$99 \cdot 366 + 1 = 36235$. The proof is a quick generalization.

Example. Let k be a positive integer which is not divisible by 2 or 5. Prove that k divides a number of the form $99\ldots9 = 10^N - 1$, for some N.

Proof. Consider the remainders of:

$$10^1 - 1, 10^2 - 1, \ldots 10^{k+1} - 1 \qquad \text{(Objects)}$$

after division by k. There are k possible remainders $(0, 1\ldots, k-1)$ [Boxes], and there are $k+1$ numbers.

Therefore there are two such $1 \leq m \leq n \leq k+1$ such that $10^n - 1$ and $10^m - 1$ have the same remainder upon dividing by k by the Pigeonhole Principle.

Then k divides $10^n - 10^m = 10^m(10^{n-m} - 1)$. Since k is not divisible by 2 or 5, $10^m = 2^m 5^m$ and k share no prime factors. Therefore k divides $10^{n-m} - 1$. $\square$

Theorem 3 (Erdős-Szekeres)**.** *Let n, m be positive integers. Any sequence of $nm+1$ distinct real numbers contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$.*

Example. Let $n = 2$ and $m = 3$,

$\underline{5}$, $\underline{6}$, $3, 4, 1, 2,$ $\underline{7}$

This is 7 numbers. The underlined numbers are an increasing subsequence of length $2+1$.

$\underline{5}$, 6, $\underline{3}$, 4,$\underline{1}$, 2, $\underline{0}$

This is 7 numbers. The underlined numbers are a decreasing subsequence of length $3+1$

$5, 6, 3, 4, 1, 2$

This is six numbers with no increasing or decreasing subsequence of the correct length

Proof. Let $a_1, a_2, \ldots a_{nm+1}$ be our sequence of distinct real numbers. For each $1 \leq k \leq nm+1$ define t_k to be the length of the longest increasing subsequence starting at a_k.

If there is a k with $t_k > n$, then we have found an increasing subsequence of the right length and we are done.

So suppose not, that is assume $t_k \leq n$ for all k. So $t_k \in \{1, 2, \ldots n\}$. By the The Generalized Pigeonhole Principle, there are $nm+1 > nm$ of these numbers, so there is a value attained by at least $m+1$ of these numbers. With the following:

$$\{t_1, t_2, \ldots, t_{nm+1}\} \qquad \text{(Objects)}$$
$$\{1, 2, \ldots, n\} \qquad \text{(Boxes)}$$

That is we have a subsequence $a_{k_1}, a_{k_2}, \ldots, a_{k_{m+1}}$ such that $t_{k_i} = t_{k_j}$ for every $1 \leq i, j \leq m+1$. We want to show that:

$$a_{k_1} > a_{k_2} > \ldots > a_{k_{m+1}}$$

This follows from the claim given since these numbers are distinct.

Claim. *If* $1 \leq k < \ell \leq nm+1$ *and* $t_k = t_\ell$*, then* $a_k > a_\ell$*.*

To prove the claim, suppose otherwise, that is assume $a_k < a_\ell$ [not $\leq$ because these are distinct].

Then we could add a_k to the beginning of the longest increasing subsequence starting at a_ℓ, and we would have a subsequence of length $t_\ell + 1$ starting at a_k. This is impossible, because the longest subsequence starting at a_k is of length $t_k = t_\ell$. Oops ☺. ☐

Example. 51 of the 100 squares on a 10×10 checkerboard are marked. Prove that there exists three marked squares which form three corners of a 2×2 square.

Proof. So begin by tiling the checkerboard with squares that are 2×2, place each of the marked squares in one of these boxes. Then since $51 > 2 * 25$, where 25 is the number of boxes. Then there is a box with more than 2 marked squares in it. There are then at least three marked tiles and so we are done! ☺. ☐