MATH 465 Notes

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Vertex Coloring

Question: A mapmaker wants to color countries on his maps so that any two counteries which share a border have different colors. What is the fewest number of colors needed to guarantee any map can be colored this way?

$$\begin{aligned} \text{map} &\to \text{graph (simple planar)} \\ \text{countries} &\to \text{vertices} \\ \text{two countries share a border} &\to \text{two vertices adjacent} \end{aligned}$$

Definition. *A* ***proper vertex coloring*** *of a graph G is a coloring of its vertices such that any two adjacent vertices are colored differently*

G is ***k-colorable*** *if there is a proper vertex coloring which uses $\leq k$ colors.*

G is ***bipartite*** *if it is 2-colorable.*

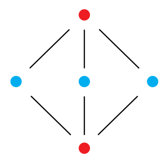
The ***chromatic number*** *$\chi(G)$ is the smallest integer k for which G is k-colorable.*

We will assume our graphs are simple here

Example. Let's get some examples.

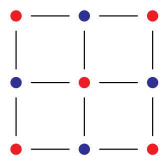
(1) The complete bipartite graph $K_{m,n}$ is in fact bipartite (as one would hope!) We color the first m vertices with one color and the n vertices another color, look

at $K_{2,3}$.



In fact $\chi(K_{m,n}) = 2$ whenever $m, n \geq 1$.

(2) Another bipartite graph we've seen is $G_{m,n}$ (grid)



In fact $\chi(G_{m,n}) = 2$ if $m > 1$ or $n > 1$

(3) If G has no edges then $\chi(G) = 1$. (i.e., G is k-colorable for every $k \in \mathbb{Z}_{>0}$).

For a simple graph, if G has at least one edge then $\chi(G) > 1$.

(4) $\chi(K_n) = n$.

Note that $\chi(K_n) \leq n$ because any graph on n vertices is n-colorable (color each vertex a different color).

And $\chi(K_n) \geq n$ because any two vertices are adjacent, and so must have a different color.

Restrict our Attention to Planar Graphs

The first thing we do is get an upper bound on the number of colors you need for planar graphs.

Theorem 1 (The Six Color Theorem)**.** *Every simple planar graph is* 6*-colorable, that is* $\chi(G) \leq 6$ *for every simple planar graph* G*.*

Lemma 1. *If* $G = (V, E)$ *is a simple planar graph then there exists a vertex* $v \in V$ *such that* $\deg(v) \leq 5$.

Proof. Similar to Homework 9. If not, every vertex has degree at least 6, so:

$$2|E| = \sum_{v \in V} \deg(v) \geq 6|V|$$

And from Homework 9 #5:

$$|E| \leq 3|V| - 6$$

Multiplying through by two we get:

$$2|E| \leq 6|V| - 12 < 6|V|$$

So we have a contradiction, $6|V| < 6|V|$. Oops!

□

Proof of 6*-color Theorem.* We induct on $|V|$. Note that a simple graph with 1 vertex is 1-colorable, and hence 6-colorable. This is our base case.

Let $n \geq 1$ and assume every simple planar graph with n vertices is 6-colorable.

Let $G = (V, E)$ be a simple planar graph with $n + 1$ vertices. By lemma there is some $v \in V$ with $\deg(v) \leq 5$. Consider the induced subgraph on $V \setminus \{v\}$.

This is a simple planar graph with n vertices so by induction it is 6-colorable. Given a 6-coloring of this subgraph, we can extend it to a 6-coloring of G by picking a color for v.

Since $\deg(v) \leq 5$, at least one of the six colors is not used on any of v's neighbors. Choose one of these unused colors for v. □

Let's prove something even better.

Theorem 2 (The Five Color Theorem)**.** *Every simple planar graph is* 5*-colorable, that is* $\chi(G) \leq 5$ *for every simple planar graph* G*.*

Proof of 5*-color Theorem.* We induct on $|V|$. Note that a simple graph with 1 vertex is 1-colorable, and hence 5-colorable. This is our base case.

Let $n \geq 1$ and assume every simple planar graph with n vertices is 5-colorable.

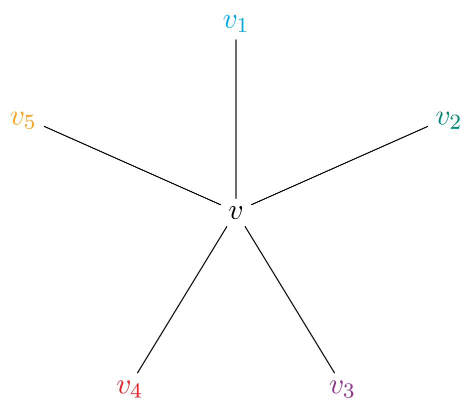
Let $G = (V, E)$ be a simple planar graph with $n + 1$ vertices. By lemma there is some $v \in V$ with $\deg(v) \leq 5$. Consider the induced subgraph on $V \setminus \{v\}$.

This is a simple planar graph with n vertices so by induction it is 5-colorable. Fix a 5-coloring of this subgraph.

If the neighbors of v use at most 4 colors, then we will win, because we can extend the 5-coloring of G with the 5-th color

If not, then there are five vertices adjacent to v, each colored by a different color.

Fix a planar embedding of G and label v's neighbors as v_1, v_2, v_3, v_4, v_5 as they appear in a clockwise order around v. Suppose v_i has color i.



Let V_{13} be the vertices colored by 1 and 3, and let G_{13} be the induced subgraph on V_{13}.

- If v_1 and v_3 lie in different connected components of G_{13} then we can swap the two colors in the entire component containing v_1. This yields a valid 5-coloring and we may color v by 1.

- If not, then there is a path from v_1 to v_3 using only vertices colored by 1 and 3. Name this path P_{13}. Let V_{24} be the vertices colored by 2 and 4 and let G_{24} be the induced subgraph on V_{24}. Follow similar motions as above

  – If v_2 and v_4 lie in different connected components of G_{24} then we can swap the two colors in the component containing v_2. But then we can color v by 2 to get a valid 5-coloring of G.

  – If not, then there is a path P_{24} from v_2 to v_4 using only vertices colored by 2 and 4. It is not possible for both of these paths to exist, P_{13} and P_{24} since they cannot contain any of the same vertices, and they must intersect. Draw the picture! Thus G could not be planar. Thus we must fall into a different case where we win.

This concludes the proof of the 5-color theorem! $\square$

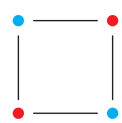
Theorem 3 (The Four Color Theorem)**.** *Every simple planar graph is* 4*-colorable. This is much much much harder. There is no known proof which does not use a computer.*

Determining the chromatic number of a graph (even a planar graph) G is very very hard, even with computers, but we do have some properties, but showing equalities in these is trickier.

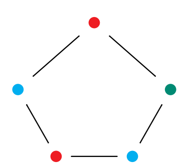
(1) If G has n vertices then $1 \leq \chi(G) \leq n$.

(2) If G' is a subgraph of G then $\chi(G') \leq \chi(G)$. (a k-coloring of G restricts to a k-coloring of G'

Example. Let C_n be an n-cycle with $n \geq 3$.



But then:



Thus:

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$$

Proposition 1. A simple graph with is bipartite if and only if every cycle has even length.

Proof. Let's go!

($\Rightarrow$) We do this by contrapositive. Suppose G contains an odd length cycle. Consider this graph as a subgraph G', then $3 = \chi(G') \leq \chi(G)$.

Thus G is not bipartite.

($\Leftarrow$) It suffices to consider connected graphs because of connected components. Assume $G = (V, E)$ is connected and has no odd length cycles. Fix two vertices $v, w \in V$.

Either every path from v to w is even or every path from v to w is odd. Otherwise we could concatenate the two paths to get an odd length cycle, with a little bit of work. (Draw some pictures).

Choose a special vertex $v \in V$. For any $w \in V$ color w RED if there is an even path from v to w and BLUE if there is an odd path from v to w.

This coloring is well-defined because of the above work. You should show that no two red vertices are adjacent and no two blue vertices are adjacent.

$\square$