MATH 465 Notes

Faye Jackson

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1 Introduction

1.1 Stuff

- Quiz (5 min.) Remember to justify
- Today is Linear Recurrences
- HW4 due Wednesday

2 Linear Recurrencese (Homogeneous / Non-Homogeneous) (8.2-8.5)

2.1 Fibonacci Numbers

Example. Suppose you have an infinite supply of two types of tile: squares and dominoes (dominoes are twice as long as squares).

Question: How many ways are there to arrange these tiles into a row of some fixed length?

Answer: Let R_n be the number of ways to arrange these tiles into a row of length $n \in \mathbb{N}$. Let's look at examples

$$\begin{aligned} R_0 &= 1 & R_1 &= 1 \\ R_2 &= 2 & R_3 &= 3 \\ R_4 &= 5 & R_5 &= 8 \end{aligned}$$

In general if R_n is the number of ways to tile a row of length $n \in \mathbb{N}$,

$$R_n = R_{n-1} + R_{n-2} \qquad \text{for } n \geq 2$$

A tiling of length n is a tiling of length $n-1$ with a square added or a tiling of length $n-2$ with a domino added.

Theorem 1 (Binet's Formula)**.**

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

We let $F_0 = 0$ and $F_n = R_{n-1}$, then this satisfies $F_n = F_{n-1} + F_{n-2}$.

Note a combinatorial proof of this is quite impossible. We have to use other tools. Let's build them!

2.2 Linear Recurrence in General, a Definition

Definition 1. *A sequence of numbers $a_0, a_1, a_2, \dots$ satisfies a homogeneous linear recurrence if it satisfies:*

$$a_n + r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} = 0$$

For k fixed and $n \geq k$. The r_i are constants, and $r_k \neq 0$.

A non-homogeneous linear recurrence satisfies:

$$a_n + r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} = h(n)$$

For some constant funtion $h : \mathbb{N} \to \mathbb{N}$.

Goal: Find a Formula for a_n:

Example. $F_n - F_{n-1} - F_{n-2} = 0$ with $k = 2$, $r_1 = r_2 = -1$.

Example. Define a sequence $a_0, a_1, a_2, \dots$ by setting $a_0 = 1, a_1 = 1, a_2 = 4$, and:

$$a_n = 3a_{n-2} + 2a_{n-3} \qquad \text{for } n \geq 3$$
$$a_3 = 3a_1 + 2a_0 = 5$$
$$a_4 = 3a_2 + 2a_1 = 12 + 2 = 14$$

Write this as:

$$a_n - 0a_{n-1} - 3a_{n-2} - 2a_{n-3} = 0$$

Gives this as a homogeneous linear recurrence with $k = 3$, $r_1 = 0$, $r_2 = -3$, $r_3 = -2$.

Example. Let $a_0 = 3, a_1 = 9$, and:

$$a_n = a_{n-1} + 2a_{n-2} - 4n$$

This is a clear non-homogeneous linear recurrence with $h(n) = -4n$.

Non-Example. Let $a_0 = 1$ and $a_n = na_{n-1}$. Well this is:

$$\begin{aligned} a_0 &= 1 & a_1 &= 1 \\ a_2 &= 2 & a_3 &= 3 \cdot 2 \\ a_4 &= 4 \cdot 3 \cdot 2 & a_5 &= 5 \cdot 4 \cdot 3 \cdot 2 \end{aligned}$$

We have a guess that $a_n = n!$. We can prove it using a counting argument and induction. We know $a_0 = 1 = 0!$. Assume $a_n = n!$, we must show $a_{n+1} = (n+1)!$, but this is simply by definition $a_{n+1} = (n+1)a_n = (n+1)n! = (n+1)!$.

In general we need stronger tools to study non-linear recurrence

2.3 The Characteristic Equation Approach

Definition 2. *Given a homogeneous linear recurrence:*

$$a_n + r_1 a_{n-1} + \cdots + r_k a_{n-k} = 0$$

We write a polynomial associated to it called the characteristic polynomial:

$$q^k + r_1 q^{k-1} + \cdots + r_k = 0$$

Theorem 2. *Let k be a positive integer, and let the $r_1, \ldots, r_k \in \mathbb{R}$, let $q_1, q_2, \ldots, q_k$ be k distinct roots of the characteristic polynomial.*

Then for any $a_n + r_1 a_{n-1} + \cdots + r_k a_{n-k} = 0$. We have that the n-th term has the following form:

$$a_n = \sum_{i=1}^{k} c_i q_i^n \qquad \text{for some constants } c_i$$

Sketch of Proof. Morally this is like a <u>generalized antiderivative:</u> i.e. if we know the initial conditions $a_0, a_1, \ldots, a_{k-1}$, then we could solve for all of the c_i. Like the $+C$ in an antiderivative

(1) Consider the set V of all real-valued sequences $(a_0, a_1, a_2, \ldots)$ which satisfy the recurrence. <u>BUT</u> not necessarily the same initial conditions.

$$V = \{(a_0, a_1, \ldots) \mid a_n + r_1 a_{n-1} + \cdots r_k a_{n-k} = 0 \;\; \forall n \geq k\}$$

- Because of homogeneity note then that V is a vector space over $\mathbb{R}$.
- There's a map

$$T : V \to \mathbb{R}^k$$

$$(a_0, a_1, \ldots) \underset{\sim}{\overset{T}{\longmapsto}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix}$$

  Furthermore this map is an isomorphism, We can uniquely recover the sequence $(a_0, a_1, \ldots)$ from the first k terms using the linear recurrence.

- This tells us the dimension of our vector space V, k.

(2) If $q \neq 0$ then the geometric sequence defined by $a_n = q^n$ satisfies the linear recurrence if and only if q is a root of the characteristic polynomial.

In particular, consider k distinct roots $q_1, \ldots, q_k$ of the characteristic polynomial (which are nonzero since $r_k \neq 0$). Then the geometric sequence:

$$\begin{aligned} s_1 &= (1, q_1, q_1^2, \ldots) \\ &\vdots \\ s_k &= (1, q_k, q_k^2, \ldots) \end{aligned}$$

Are solutions of the linear recurrence. Thus for each $1 \leq i \leq k$ we have $s_i \in V$.

(3) We want to make $s_1, s_2, \ldots, s_k$ a basis for V.

Proof. We just need them to be linearly independent. Suppose they were not. Then we would have:

$$c_1 s_1 + c_2 s_2 + \cdots + c_k s_k = 0$$

with $c_i \neq 0$ for some i. We can just look at the first k terms:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix}$$

On HW1 Problem 3 we proved that this matrix is invertible, so we proved that we cannot have:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Unless $(c_1, c_2, \ldots, c_k) = (0, 0, \ldots, 0)$. □

□

Example. [Fibonacci Redux] The characteristic polynomial for the Fibonacci recurrence is $q^2 - q - 1 = 0$. It has roots:

$$q_1 = \frac{1+\sqrt{5}}{2}$$
$$q_2 = \frac{1-\sqrt{5}}{2}$$

We then know that:

$$F_n = c_1 q_1^n + c_2 q_2^n$$

For some $c_1, c_2 \in \mathbb{R}$ by $F_0 = 0 = c_1 + c_2$ and $F_1 = 1 = c_1 q_1 + c_2 q_2$. So then we must have that:

$$\begin{aligned} 1 &= c_1(q_1 - q_2) \\ c_1 &= \frac{1}{q_1 - q_2} = \frac{1}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} = \frac{1}{\sqrt{5}} \\ c_2 &= -\frac{1}{\sqrt{5}} \end{aligned}$$

Which gives that:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example. Say we have:

$$a_n + 0a_{n-1} - 3a_{n-2} - 2a_{n-3} = 0$$

Which gives the associated charateristic polynomial:

$$\begin{aligned} q^3 - 3q - 2 &= 0 \\ (q+1)^2(q-2) &= 0 \end{aligned}$$

But now we have a root with multiplicity! How do we deal with it.

Theorem 3. *Let k be a positive integer, and let the $r_1, \ldots, r_k \in \mathbb{R}$, let $q_1, q_2, \ldots, q_d$ be distinct roots of the characteristic polynomial with $d \leq k$ and multiplicities $m_1, m_2, \ldots, m_d$.*

$$q^k + r_1 q^{k-1} + \cdots = r_{k-1} q + r_k = 0$$

Then for any recurrence of the form $a_n + r_1 a_{n-1} + \cdots + r_k a_{n-k} = 0$. We have that the n-th term has the following form:

$$a_n = \sum_{i=1}^{d} \sum_{j=1}^{m_i} c_{i,j} n^{j-1} q_i^n \qquad \text{for some constants } c_{i,j}$$

This has to do with derivatives.

Proof Omitted. The idea is a root with multiplicity is a root of the derivatives. □