MATH 465 Notes

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1 Announcements

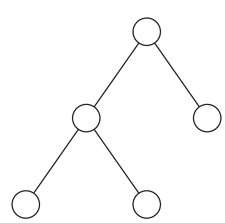
- Today: Intro to Graphs (Ch. 9) (Euler / Hamilton later)
- Next Time: Trees (Ch. 10)
- Quiz Thursday
- HW8 Due 3/18

2 Let's Go! Graphs!

2.1 Definitions and Examples

Definition 1. *A (finite, undirected) graph $G = (V, E)$ consists of a finite set V of vertices and a finite set E ofedges along with a map associating to each edge $e \in E$ an unordered pair of not necessarily distinct vertices $\{u, v\} \in V$.*

Example.



If the endpoints are equal then it is a loop.

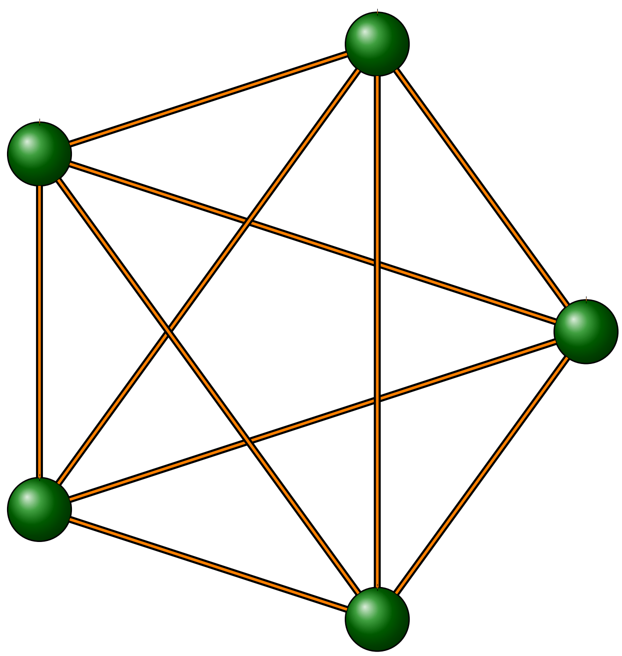
If the endpoints $\{u, v\}$ of an edge e are distinct we say u and v are adjecent and u (and v) are incident to e.

Definition 2. *A graph is simple if it has no loops or multiple edges..*

Example. Skeleta of polyhedra

Definition 3. *The complete graph on n vertices K_n is the simple graph such that any two vertices are adjacent.*

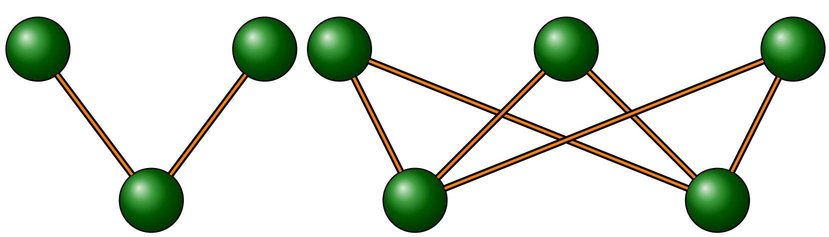
K_5

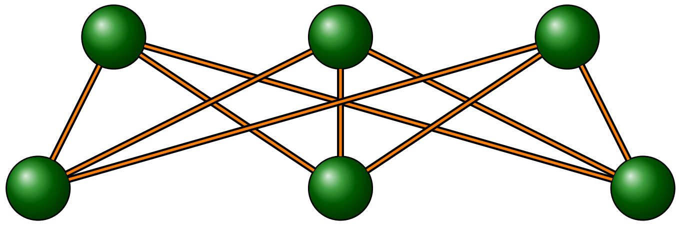


K_n has the maximum number of edges for a simple graph with n vertices, there are $\binom{n}{2}$ edges.

The complete bipartite graph $K_{m,n}$ has $m+n$ vertices partitioned into two blocks of size m and n, with mn edges connecting all vertices in different blocks.

In order, $K_{1,2}$, $K_{2,3}$, and $K_{3,3}$.



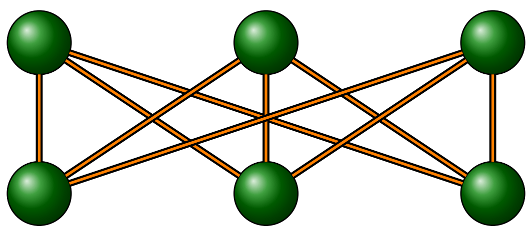


2.2 Graph Isomorphisms

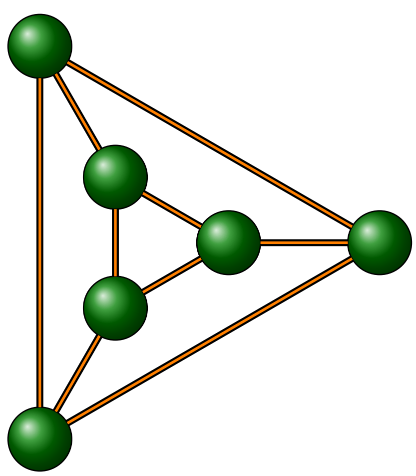
Definition 4. *Two graphs $G = (V, E)$ and $G' = (V', E')$ are $\boxed{\text{isomorphic}}$ if there exist bijections $V \to V'$ and $E \to E'$ which preserves the incidence relation. That is if $v \mapsto v'$ and $e \mapsto e'$, then e is incident to v if and only if e' is incident to v'.*

Equivalently $\boxed{G \cong G'}$ if there exists a bijection $f : V \to V'$ such that for all $u, v \in V$ the number of edges between u and v is equal to the number of edges between $f(u)$ and $f(v)$.

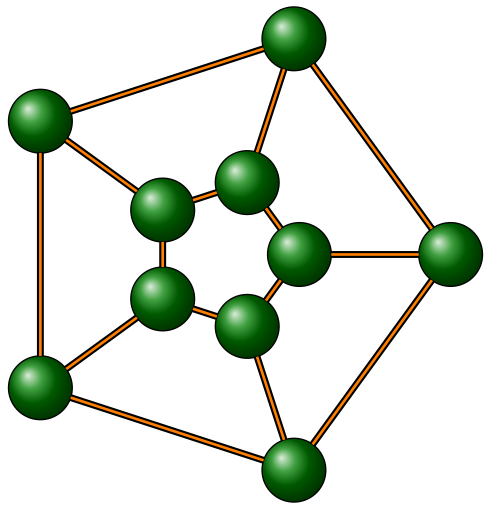
Example.



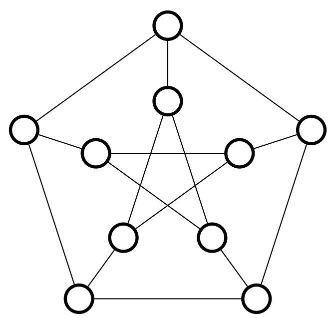
Is Not congruent to:



The reasoning is that there are a, b, c such taht $\{a-b, b-c, a-c\}$ are all edges in the second one, but not in the first.



Is not congruent to:



2.3 Properties of Graphs

Definition 5. *We say that a graph $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Note if $e \in E'$ then its endpoints are in V'.*

A subgraph G' is "induced" if whenever the endpoints of an edge $e \in E$ are in V', then $e \in E'$.

Let $G = (V, E)$. The degree of a vertex $v \in V$ is the number of "half-edges" incident to it (i.e. count loops twice)

Example.

- In K_n, every vertex has degree $n - 1$
- . In $K_{m,n}$, m vertices have degree n and n vertices have degree m.

- The single edge connected to itself has degree 2.

Notation: $\deg(v) = \deg_G(v)$.

A Degree Sequence: is a non-increasing list of vertex degrees in graph

Note: If $f : V \to V'$ gives an isomorphism $G \cong G'$ then $\deg(v) = \deg(f(v))$. "isomorphism preserves degrees," but this is not enough

Proposition 1. $\sum_{v\in V} \deg(V) = 2\,|E|$ in a graph $G = (V, E)$.

Proof. Count the edges incident to every vertex, we've counted every edge exactly twice $\square$

Corrolary 1. *In any graph, the number of vertices of odd degree is even.*

Proof.

$$\begin{aligned}
\underbrace{2\,|E|}_{\text{even}} &= \sum_{v\in V} \deg(v) \\
&= \underbrace{\sum_{\substack{v\in V\\ \text{even}}} \deg(v)}_{\text{even}} + \sum_{\substack{v\in V\\ \text{odd}}} \deg(v) \\
&\implies \underbrace{\sum_{\substack{v\in V\\ \text{odd}}} \deg(v)}_{\text{even}}
\end{aligned}$$

And so there is an even number of these odd numbers in the summation $\square$

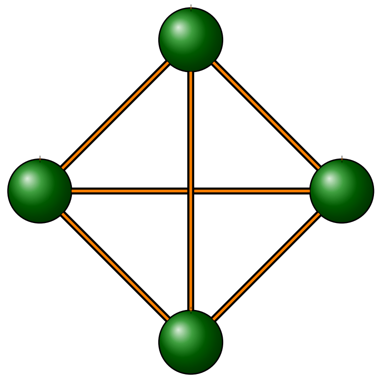
Definition 6. *A walk of length ℓ is a sequence:*

$$v_0 \underset{e_1}{\longrule} v_1 \underset{e_2}{\longrule} v_2 \underset{e_3}{\longrule} \cdots \underset{e_{\ell-1}}{\longrule} v_{\ell-1} \underset{e_\ell}{\longrule} v_\ell$$

Of vertices and edges such that e_i has endpoints v_{i-1} and v_i for all i.

In a simple graph, it suffices to list vertices.

A closed walk returns to where it started ($v_\ell = v_1$), and a trail does not repeat any edges. A path uses each vertex at most once. Finally a cycles is a closed trail of length > 0 in which each vertex is used at most once except $v_0 = v_\ell$.



Then we have the following paths:

$$b \text{ —— } d \text{ —— } c \text{ —— } b \text{ —— } a \text{ —— } b \qquad \text{(closed walk)}$$

$$b \text{ —— } d \text{ —— } c \text{ —— } b \text{ —— } a \qquad \text{(trail)}$$

$$b \text{ —— } d \text{ —— } c \text{ —— } a \qquad \text{(path)}$$

$$b \text{ —— } d \text{ —— } c \text{ —— } b \qquad \text{(3-cycle)}$$

There exists a cycle of length one if and only if there exists a loop. And there exists a cycle of length 2 if and only if there exists multiple edges. Thus, in a simple graph there exist no cycles of length < 3.

Note: If we view the edges and vertices of a cycle as a subgraph, every vertex has degree two in that subgraph, because we must enter and leave each vertex with different edges, and we cannot visit it twice. (With the exception of the start, which is easy to check has degree 2.

Definition 7. *A graph is <u>connected</u> if there exists a walk between any two vertices (note if the vertices are the same you can have a walk with no edges).*

Example. $K_n, K_{m,n}$

Non-Example. $V = [n],, E = \emptyset$, then for $n > 1$ this is not connected

Definition 8. *A <u>forest</u> is a graph with no cycles (acyclic)*

Definition 9. *A <u>tree</u> is a connected forest.*