MATH 465 Notes

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1 Introduction

1.1 Stuff

- Quiz 5 Tuesday
- Substitute

2 More Generating Functions

2.1 Some Examples

Recall.

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$$

Question: Let a_k be the number of k-card hands dealt from 2 standard 52-card decks. What is a_k.

If $k = 5$ there's not much that can happen, three possibilities, how many duplicates do you have (none, one pair, two pairs):

$$a_5 = \binom{52}{47, 5, 0} + \binom{52}{48, 3, 1} + \binom{52}{49, 1, 2} = 3,748,160$$

Generating Function:

$$\sum_{k \geq 0} a_k x^k = \left(1 + x + x^2\right)^{52}$$

This is because each card can show up 0 times, 1 time, or 2 times.

Question: How many ways can we give change for a dollar with dimes and quarters?

Either 0 quarters and 10 dimes, or 2 quarters with 5 dimes, or 4 quarters with 0 dimes. So the answer is 3.

Harder Question: Define $a_k = \#$ of integer solutions to $10d + 25q = k$ (where d is dimes and q is quarters.

Solution $(d, q) \longleftrightarrow x^{10d}x^{25q} = x^k$

$$\begin{aligned}
\sum_{k\geq 0} a_k x^k &= (1 + x^{10} + x^{20} + \ldots)(1 + x^{25} + x^{50} + \ldots) \\
&= \frac{1}{(1-x^{10})(1-x^{25})}
\end{aligned}$$

2.2 The Multiplication Principle

Proposition 1. [Multiplication Principle]
Let A, B, C be finite sets with a bijection $A \xrightarrow{f} B \times C$, with weight functions:

$$w_A : A \to \mathbb{Z}_{\geq 0} \qquad w_B : B \to \mathbb{Z}_{\geq 0} \qquad w_C : C \to \mathbb{Z}_{\geq 0}$$

Such that for $\alpha \in A \overset{f}{\mapsto} (\beta, \gamma) \in B \times C$, we have:

$$w_A(\alpha) = w_B(\beta) + w_C(\gamma)$$

Let:

$$a_k = |\{\alpha \in A \mid w_A(\alpha) = k\}| \quad b_k = |\{\beta \in B \mid w_B(\beta) = k\}| \quad c_k = |\{\gamma \in C \mid w_C(\gamma) = k\}|$$

Then:

$$\begin{aligned}
\sum_{k\geq 0} a_k x^k &= \left(\sum b_k x^k\right)\left(\sum c_k x^k\right) \\
\sum_{\alpha\in A} x^{w_A(\alpha)} &= \left(\sum_{\beta\in B} x^{w_B(\beta)}\right)\left(\sum_{\gamma\in C} x^{w_C(\gamma)}\right)
\end{aligned}$$

Example.

$$\begin{aligned} B &= \{\text{collections of dimes}\} \\ C &= \{\text{collections of quarters}\} \\ A &= \{\text{collection of both kinds of coins}\} \end{aligned}$$

Example. How many ways to change k cents into pennies, nickels, dimes, and quarters? Call this a_k. Then:

$$\sum_{k\geq 0} a_k x^k = (1 + x + x^2 + \ldots)(1 + x^5 + x^{10} + \ldots)(1 + x^{10} + x^{20} + \ldots)(1 + x^{25} + x^{50} + \ldots)$$

$$\sum_{k\geq 0} a_k x^k = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

2.3 What about Permutations

Some Notation for Convenience:

$$\begin{aligned} [n] &= \{1, 2, \ldots, n\} \\ S_n &= \text{ set of permutations of } [n] = \text{ The Symmetric Group} \end{aligned}$$

One line notation $2314 = 2, 3, 1, 4 \in S_4$

For $\sigma \in S_n$, let σ_i be the ith object

Definition 1. *An inversion of σ is a pair (i, j) with $i < j$ but $\sigma_i > \sigma_j$.*

An inversion tells you how "out of order" you are.

Goal: Count permutations by the # of inversions = $\mathrm{inv}(\sigma)$.

Example. S_3:

σ	$\mathrm{inv}(\sigma)$	$\mathrm{code}(\sigma)$
123	0	(0, 0, 0)
132	1	(0, 0, 1)
213	1	(0, 1, 0)
231	2	(0, 1, 1)
312	2	(0, 0, 2)
321	3	(0, 1, 2)

Generating function:

$$1 + 2x^1 + 2x^2 + x^3 = 1 \cdot (1+x) \cdot (1+x+x^2)$$

Definition 2. *The (inv) code of $\sigma \in S_n$ is a sequence $c = (c_1, \ldots, c_n)$ with c_k is the # of inversions (i, j) with $\sigma_i = k$.*

Note: $\mathrm{inv}(\sigma) = \sum \mathrm{code}(\sigma)$

Theorem 1. *Let a_k be the # of $\sigma \in S_n$ with $\mathrm{inv}(\sigma) = k$. THen:*

$$\begin{aligned}\sum_{k \geq 0} a_k x^k &= \sum_{\sigma \in S_n} x^{\mathrm{inv}(\sigma)} \\ &= 1(1+x)(1+x+x^2)(1+x+x^2+x^3)\cdots(1+x+\cdots+x^{n-1}) \\ &= \prod_{k=1}^{n} \frac{1-x^k}{1-x}\end{aligned}$$

Proof.

$$\mathrm{code} : S_n \to \{0\} \times \{0,1\} \times \cdots \times \{0, \ldots, n-1\} = c_n$$

This is actually a bijection! Note that $|S_n| = n! = |c_n|$, it is enough to show either code is an injection or surjection. We will show it is a surjection. By example, cosnider $c = (0, 1, 0, 2, 2, 1) \in c_n$.

We want to form a permutation $\sigma \in S_n$ with $\mathrm{code}(\sigma) = c$. We build it up piecewisely:

$$1 \mapsto 21 \mapsto 213 \mapsto 2413 \mapsto 24513 \mapsto 245163$$

We use $\mathrm{inv} : S_n \to \mathbb{Z}_{\geq 0}$ as weight function, and $\mathrm{weight}(c_i) = c_i$. With these weights is a weight preserving bijection, apply the multiplication principle and the result falls out.

TODO $\square$