MATH 465 Notes

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1 Today

1.1 Stuff

- Substitute Anna (Bibby is out)
- Quiz (Ch. 3 and 5.1)
- HW2 due Wednesday

1.2 Topics

- Binomial Theorem
- Combinatorial Proofs

2 Lets look at Binomial Coefficients again

2.1 4.2 Pascal's Triangle

Recall. The binomial coefficient is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (n, k \in \mathbb{Z}_{\geq 0})$$

We'll add the convention that $\binom{n}{k} = 0$ if $k < 0$ or $k > n$

Proposition 1. Let $n, k \in \mathbb{Z}_{>0}$. Then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

We've already shown this one way, now lets do a combinatorial proof, i.e. we will show that they count the same set.

Combinatorial Proof. Let S be an n-element set and fix $x \in S$. THe Left Hand Side counts the number of k-element subsets of S.

The Right Hand Side: By the addition principle, the number of k-element subsets of S is the same as the number of k-element subsets which don't contain x plus the number of k-element subsets which do contain x. Lets count these:

- How many k-element subsets of S contain x. Well we are choosing from $n-1$ things ($S \setminus \{x\}$) and we need $k-1$ of them, that is $\binom{n-1}{k-1}$
- How many k-element subsets of S don't contain x. Well we are choosing from $n-1$ things ($S \setminus \{x\}$) and we need k of them, that is $\binom{n-1}{k}$.

Thus we are done. $\square$

The triangle looks like:

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \end{array} \qquad \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Some patterns:

- Symmetry gives us $\binom{n}{k} = \binom{n}{n-k}$ (we've already proved this)
- Sum of the n-th row is 2^n
- Alternating sums of rows are always 0

Proposition 2. For $n, k \in \mathbb{Z}_{\geq 0}$ we have the symmetry $\binom{n}{k} = \binom{n}{n-k}$ and the sum of each row is:

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

Proof. We observed symmetry on Thursday. We will use a combinatorial proof again for the second claim.

We know that 2^n counts subsets of an n-element set S. We will break this down:

$$2^n = |\{T \subset S\}| = \sum_{i=0}^{n} |\{T \subset S \mid |S| = i\}| = \sum_{i=0}^{n} \binom{n}{i}$$

$\square$

Proposition 3. If $n > 0$ then:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Equivalently:

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$$

Bijective Proof. Let S be an n-element set. Notice that a subset of S has either an even or odd cardinality, but not both.

Fix $x > 0$ and for an $T \in S$ define a piecewise function:

$$f(T) = \begin{cases} T \cup \{x\} & \text{if } x \notin T \\ T \setminus \{x\} & \text{if } x \in T \end{cases}$$

Note that T is even if and only if $f(T)$ is odd:

$$f : \{T \subset S \mid |T| \text{ is even}\} \to \{T \subset S \mid |T| \text{ is odd}\}$$
$$f : \{T \subset S \mid |T| \text{ is odd}\} \to \{T \subset S \mid |T| \text{ is even}\}$$

Omitting the restrictions. It is enough to show that $f(f(T)) = T$. Time for two quick cases:

- Suppose $x \in T$. Then $f(f(T)) = f(T \setminus \{x\}) = (T \setminus \{x\}) \cup \{x\} = T$ This works since $x \in T$.

- Suppose $x \notin T$. Compute $f(f(T))$:

$$f(f(T)) = f(T \cup \{x\}) = (T \cup \{x\}) \setminus \{x\} = T$$

This works since $x \notin T$.

The result follows easily since f is now a bijection. $\square$

Theorem 1 (The Binomial Theorem)**.** *Let n be a non-negative integer and x, y be "variables" (or complex numbers... or more). Then:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example.

$$\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^1 &= xx + xy + yx + yy \\
&= x^2 + 2xy + y^2 \\
(x+y)^3 &= xxx + xxy + xyx + yxx + xyy + yxy + yyx + yyy \\
&= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}$$

Idea: If we expand $(x+y)^n$, each term in this expansion corresponds to an n-letter word in the alphabet $\{x, y\}$. The coefficient of $x^k y^{n-k}$ is the number of n-letter words with k x's and $n-k$ y's. This is counted by $\binom{n}{k}$. This is a combinatorial proof.

Proof by Pascal's Recurrence and Induction. In the base case, $n = 0$, the formula works since:

$$(x+y)^0 = 1 = \binom{0}{0} x^0 y^0$$

Let $n \in \mathbb{Z}_{\geq 0}$. Assume that:

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
(x+y)^{n+1} &= (x+y)\left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right) \\
(x+y)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k+1}
\end{aligned}$$

Lets now do some clever reindexing:

$$
\begin{aligned}
(x+y)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k+1} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} x^{k} y^{n-k+1} + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k+1} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^{k} y^{n-k+1} + \sum_{k=1}^{n} \binom{n}{k} x^{k+1} y^{n-k+1} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} - \binom{n}{k} \right) x^{k} y^{n-k+1} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{k} y^{(n+1)-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{k} y^{(n+1)-k}
\end{aligned}
$$

Thus the binomial theorem holds for $n+1$ and so by induction we're done. $\square$

Set $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Set $x = -1$ and $y = 1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Proposition 4.

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m}{\ell - k} = \binom{n+m}{\ell}$$

Proof 1. Note that $(x+1)^n (x+1)^m = (x+1)^{n+m}$, and then look at the coefficient of x^ℓ.

We know then that:

$$\left(\sum_{k=0}^{n} \binom{n}{k} x^k \right) \left(\sum_{i=0}^{n} \binom{m}{i} x^i \right) = \sum_{\ell=0}^{m+n} \binom{n+m}{\ell} x^\ell$$

Look for ways that $k + i = \ell$, and so $i = \ell - k$. $\square$

Combinatorial Proof. Take disjoint sets S, T such that $|S| = n$ and $|T| = m$.

Count the ℓ-element subsets of the set $S \cup T$.

- The Right Hand Side surely counts this from thursday, $\binom{n+m}{\ell}$
- The Left Hand Side breaks into cases, let $0 \leq k \leq n$. Let A_k be the set of ways to select a k-element subset of S and an $\ell - k$-element subset of T.

  These are separate events so by the multiplication principle so $|A_k| = \binom{n}{k}\binom{n}{\ell-k}$

  Every ℓ-element subset of $S \cup T$ will be in exactly one of these, and so the total number of ℓ-element subsets of $S \cup T$ by the addition principle is:

$$\sum_{k=0}^{n} |A_k| = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{\ell - k} = \binom{n+m}{\ell}$$

□

Proposition 5.

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Proof. Set $\ell = m = n$, by the previous proposition:

$$\sum_{k=0}^{k} \binom{n}{k}\binom{n}{n-k} = \binom{n+n}{n}$$

By symmetry:

$$\sum_{k=0}^{k} \binom{n}{k}^2 = \binom{2n}{n}$$

□

Proposition 6. $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

Proof 1. Use the binomial theorem:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Take the derivative with respect to x.

$$n(x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

Then set $x = 1$

$$n2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k$$

$\square$

Combo Proof. Count ways to choose a subset of an n-element set with a distinguished element (Pick a committee and its chair).

On the Left Hand Side we will split into cases based on the size of the subset: There are $\binom{n}{k}$ ways to choose the committee, and then k choices for the chairperson. So there are $k\binom{n}{k}$ to choose a k-element subset with a distinguished element.

On the Right Hand Side we will pick our chairperson first. There are n ways to pick our chairperson, then we must choose from an $n-1$-element set to fill out the rest of our committee. There are 2^{n-1} ways to fill out the committee. So this gives us $n2^{n-1}$.

And so we are done. $\square$