MATH 465 Notes

Faye Jackson

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The Chromatic Polynomial

Definition 1. *For a simple graph G and nonnegative integer k, define $p_G(k)$ the be the number of proper vertex colorings of G using $(\leq)k$ colors. We assume the k colors are fixed and swapping two colors gives different colorings.*

Example. (1) If G is a connected bipartite graph then $p_G(2) = 2$.

(2) $k < \chi(G)$ if and only if $p_G(k) = 0$, for $k \in \mathbb{Z}_{\geq 0}$.

(3) if $G = K_n$ and $k \geq n$ then:

$$p_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1) = \binom{k}{n} n!$$

(4) If G has no edges and n vertices then $p_G(k) = k^n$.

Proposition 1. If $G = (V, E)$ is a tree, then:

$$p_G(k) = k(k-1)^{|V|-1}$$

Proof. We prove by induction on $|V|$.

If $|V| = 1$ then $p_G(k) = k^1 = k$, since we can have no edges in such a tree.

Now let n be a positive integer, and assume that every tree with n vertices satisfies the above formula. Let $G = (V, E)$ be a tree with $n+1$ vertices. Let $v \in V$ be a leaf. By removing v and the edge incident to it, we obtain a tree G' with n vertices. By inductive hypothesis:

$$p_{G'}(k) = k(k-1)^{n-1}$$

Given a k-coloring of G', there are $k-1$ ways to extend this to a coloring of G by coloring v, since there is only one vertex adjacent to G. This gives us all of our colorings of G uniquely. Therefore:

$$p_G(k) = (k-1)p_{G'}(k) = (k-1)k(k-1)^{n-1} = k(k-1)^n$$

Just as desired. With this we win! $\square$

Note: If G has connected components $G_1, \ldots, G_m$, then:

$$p_G(k) = p_{G_1}(k)p_{G_2}(k)\cdots p_{G_m}(k)$$

To color G, we just pick a coloring of each component.

Example. Suppose G is a forest with m trees on vertices $V_1, \ldots, V_m$. Then a k-coloring for G would be:

$$p_G(k) = \prod_{i=1}^{m} k(k-1)^{|V_i|-1} = k^m(k-1)^{-m+\sum_{i=1}^{m}|V_i|} = k^m(k-1)^{|V|-m}$$

Example. Let $G = C_4$ (a 4-cycle). Let u and v be vertices opposite each other. Either u and v have the same color, or u and v have different colors.

If they have the same color, the other two vertices cannot be that one color, but there are no further restrictions. So in this case we have k choices for u and v and $k-1$ choices each for the other two vertices, giving:

$$k(k-1)^2$$

If u and v have different colors, the other two vertices can't be either of these colors and we get:

$$k(k-1)(k-2)^2$$

Thus in total:

$$p_G(k) = k(k-1)^2 + k(k-1)(k-2)^2$$

Let $G = (V, E)$ be a simple graph and let $e \in E$.

Definition 2. *The deletion of e is the graph*

$$G - e = (V, E \setminus \{e\})$$

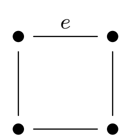
The contraction is the graph G/e obtained from $G - e$ by merging the endpoints of e.

Note: G is not necessarily simple, but has no loops, so $p_{G/e}(k)$ makes sense and:

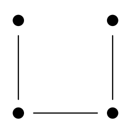
$$p_{G/e}(k) = p_{G'}(k)$$

where G' is a simple graph obtained by removing multiple edges.

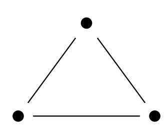
Example. Let $G = C_4$. Fix an edge e, this is G



The deletion $G - e$ is:



This is a tree, so: And the contraction G/e is:



Writing down the colorings we see that:

$$\begin{aligned} p_{G-e}(k) &= k(k-1)^{4-1} = k(k-1)^3 \\ p_{G/e}(k) &= k(k-1)(k-2) \\ p_G(k) &= k(k-1)^2 + k(k-1)(k-2)^2 \end{aligned}$$

Proposition 2. [Deletion/Contraction Formula] For any simple graph $G = (V, E)$

and edge $e \in E$ we have:

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k)$$

Proof. In each proper coloring of k-coloring of $G - e$, the (former) endpoints of e either have the same color or not.

- If they have the same color, this is a k-coloring for G/e, since their neighbors will not have that color.
- If they do not have the same color, this is a k-coloring for G since the edge e will not effect anything.

So then:

$$p_{G-e}(k) = p_{G/e}(k) + p_G(k)$$

This gives us a recursive/inductive way of computing colorings. □

Definition 3. *Viewing p_G as a function of k, we call $p_G(k)$ the chromatic polynomial*

Theorem 1. *For any simple graph $G = (V, E)$, $p_G(k)$ is a monic polynomial in k of degree $|V|$. In other words there exist constants $a_0, \ldots, a_{|V|-1}$ such that:*

$$p_G(k) = a_0 + a_1 k + \cdots + a_{|V|-1} k^{|V|-1} + k^{|V|}$$

Proof. We will induct on $|E|$ using the previous proposition.

If G has no edges and $|V|$ vertices then $p_G(k) = k^{|V|}$. This is indeed a monic polynomial whose degree is the number of vertices.

Fix n a nonnegative integer, and assume that any simple graph with $\leq n$ edges has a monic chromatic polynomial of degree its number of vertices. Let $G = (V, E)$ be a simple graph with $n + 1$ edges. Fix an edge $e \in E$.

The deletion $G - e$ is a simple graph with $|V|$ vertices and n edges. Thus $p_{G-e}(k)$ is a monic polynomial with degree $|V|$.

The contraction G/e has chromatic polynomial equal to that of a simple graph (remove any multiple edges) with $|V| - 1$ vertices and $\leq n$ edges so by induction $p_{G/e}(k)$ is a monic polynomial in K with degree $|V| - 1$.

So then by deletion/contraction:

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k)$$

is a monic polynomial in K of degree $|V|$. $\square$

Definition 4. *Let $\hat{p}_G(m)$ be the number of proper colorings of G using exactly m colors.*

Note that $\hat{p}_G(m) = 0$ unless $\chi(G) \leq m \leq |V|$.

Proposition 3. For a simple graph G:

$$p_G(k) = \sum_{m=\chi(G)}^{|V|} \hat{p}_G(m) \binom{k}{m}$$

We will not write out a proof for this, but the partition should be clear.

Example. Let $G = C_4$, then $\chi(G) = 2$, and so:

$$\begin{aligned} p_G(k) &= \hat{p}_G(2)\binom{k}{2} + \hat{p}_G(3)\binom{k}{3} + \hat{p}_G(4)\binom{k}{4} \\ &= 2\binom{k}{2} + 2 \cdot 3!\binom{k}{3} + 4!\binom{k}{4} \end{aligned}$$

$\hat{p}_G(2) = 2$, since this is a bipartite graph, $\hat{p}_G(3) = 2 \cdot 3!$, because it's two pairs can have the same color and then you distribute 3 colors among 3 things.

Other properties of $p_G(k)$:

(1) The coefficients alternate in sign, you can prove this by induction on edgse. That is we can write:

$$p_G(k) = k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} \cdots + (-1)^n a_0$$

where $a_0, \ldots, a_{n-1} \in \mathbb{Z}_{\geq 0}$.

(2) $a_{n-1} = |E|$ (HW?)

(3) $a_0, \ldots, a_{n-1}, 1$ is "unimodal"

$$a_0 \leq a_1 \leq \cdots \leq a_i \geq a_{i+1} \geq \cdots \geq a_{n-1} \geq 1$$

Furthermore it is "log-concave"

$$a_i^2 \geq a_{i-1}a_{i+1}$$

these inequalities are hard to prove. This was conjectured in 1968 and proven in 2012. It was proven by June Huh, a PhD UM-14, he proved both of them.

Remark. If G is a tree, then $p_G(k) = k(k-1)^{n-1}$, and the coefficients are binomial if you "forget" the sign. Like when $n = 3$ we get:

$$k - 2k^2 + k^3$$

When $n = 4$ we get:

$$-k + 3k^2 - 3k^2 + k^4$$

In fact this gives you Pascal's Triangle.

If $G = K_n$, $p_G(k) = k(k-1)\cdots k(n-1+1)$, and this is the generating function for stirling #s of the first kind, so these are the coefficients. When $n = 3$:

$$2k - 3k^2 + k^3$$

When $n = 4$:

$$-6k + 11k^2 - 6k^3 + k^4$$