MATH 465 Notes

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1 Introduction

1.1 Stuff

- Quiz 4 Today
- Quiz 5 Tuesday
- HW3 due Wednesday

1.2 Today

- Finish Chapter 4
- Intro to Generating Functions (8.1)
- Next Time: Ch. 6 Permutations

1.3 Announcements

- No OH Monday
- Instead: 12-3 tomorrow or email
- Final Exam Thursday 4/30, 1:30-3:30pm
- Don't Use Words like "Obviously" or "clearly" in Homework. Justify everything. Also remember to assign pages/problems

2 Multinomial Theorem

Theorem 1 (Multinomial)**.** *For $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{N}$ we have:*

$$(x_1 + \ldots + x_m)^n = \sum_{\substack{k_i \in \mathbb{Z}_{\geq 0} \\ \sum k_i = n}} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$$

Proof. You can prove by induction on m and in the inductive step you will use the binomial theorem □

Proposition 1. For $n \in \mathbb{Z}_{\geq 0}$:

$$\sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ a+b+c=n}} \binom{n}{a,b,c} = 3^n$$

Proof. Use multinomial Theorem with $m = 3$ and $x_1 = x_2 = x_3 = 1$:

$$3^n = (1+1+1)^n = \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ a+b+c=n}} \binom{n}{a,b,c} 1^a 1^b 1^c = \text{LHS}$$

□

Combinatorial Proof. There are 3^n letter words in the alphabet $\{A, B, C\}$. We can also divide these n-letter words into categories where there are a A's, b B's, and c C's so that $a + b + c = n$. In each category there are $\binom{n}{a,b,c}$ since this is the number of n-letter words using a A's, b B's, and c C's. Thus we are done by addition principle. □

Recall. We defined:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)(n-k)(n-k-1)\ldots}{k!(n-k)!}$$
$$= \frac{n(n-1)\ldots(n-k+1)}{k!}$$

So for $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$ define:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$$

Further if $k > n$ and $n \in \mathbb{Z}_{\geq 0}$ let $\binom{n}{k} = 0$.

Look at the binomial theorem with $y = 1$:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

Well what happens if we change n to $\alpha \ldots$

Theorem 2 (Generalized Binomial Theorem)**.** *Let $\alpha \in \mathbb{C}$.*

$$(x+1)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

when it all makes sense. Holds as a "formal power series"

Proof. Omitted □

Example. Let $\alpha = -n$ for $n \in \mathbb{Z}_{>0}$. Then:

$$\begin{aligned}
\binom{-n}{k} &= \frac{-n(-n-1)(-n-2)\ldots(-n-k+1)}{k!} \\
&= \frac{(-1)^k \cdot n(n+1)(n+2)\ldots(n+k-1)}{k!} \\
&= (-1)^k \frac{(n+k-1)!}{k!(n-1)!} \\
\binom{-n}{k} &= (-1)^k \binom{n+k-1}{n-1}
\end{aligned}$$

What the Generalized Binomial Theorem then tells us $\Longrightarrow$

$$\begin{aligned}
(1-x)^{-n} &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} (-x)^k \\
(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k
\end{aligned}$$

In particular when $n = 1$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^k$$

Which you know as a geometric series.

3 The (ordinary) Generating Functions of Sequences

Let $a_0, a_1, a_2, \ldots$ be a sequence. The generating function is the formal power series:

$$a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

(x is a "symbolic variable," we will not worry about convergence)

Really this is just a tool for counting.

Example. $\frac{1}{1-x}$ is the generating function of $1, 1, 1, \ldots$

And $\frac{1}{(1-x)^n}$ is the generating function of $\binom{n-1}{n-1}, \binom{n}{n-1}, \binom{n+1}{n-1}, \ldots$. Note that these count weak compositions of k with n parts, $\binom{n+k-1}{n-1}$

For a finite sequence the generating function is a polynomial:

Example. $(1+x)^n$ is the generating function for $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$

Formal power series behave just like polynomials with addition and multiplication:

$$\sum_{k\geq 0} a_k x^k + \sum_{k\geq 0} b_k x^k = \sum_{k\geq 0} (a_k + b_k) x^k$$

$$\left(\sum_{k\geq 0} a_k x^k\right)\left(\sum_{k\geq 0} b_k x^k\right) = \sum_{k\geq 0}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k$$

These form a ring if you know what that is. Also, we will relate these to the addition and multiplication principles respectively.

Now lets do some stuff:

$$\begin{aligned}\frac{1}{(1-x)^n} &= \left(\frac{1}{1-x}\right)^n \\ &= (1+x+x^2+x^3+\ldots)(1+x+x^2+x^3+\ldots)\cdots(1+x+x^2+\ldots) \\ &= \sum_{k\geq 0} (?) x^k\end{aligned}$$

Well (?) is just how many ways to pick n nonnegative integers which add up to k. That is weak compositions of k with n parts. And so:

$$\frac{1}{(1-x)^n} = \sum_{k\geq 0} \binom{n+k-1}{n-1} x^k$$

Example. Lets look at triangular numbers. Define T_k:

$$T_k = 1 + 2 + \ldots + k$$

for $k \in \mathbb{Z}_{>0}$ and let $T_0 = 0$. We know by induction:

$$T_k = \frac{k(k+1)}{2} = \binom{k+1}{2}$$

Note that we know:

$$\frac{1}{(1-x)^3} = \sum_{k\geq 0} \binom{k+2}{2} x^k = 1 + 3x + 6x^2 + 10x^3$$

So then:

$$\sum_{k\geq 0} \binom{k+1}{2} x^k = 0 + x + 3x^2 + 6x^3 + 10x^4 + \ldots$$

Take the following:

$$\begin{aligned} \sum_{k\geq 0} T_k x^k &= \sum_{k\geq 0} \binom{k+1}{2} x^k = \sum_{k\geq 1} \binom{k+1}{2} x^k \\ &= \sum_{\ell\geq 0} \binom{\ell+2}{2} x^{\ell+1} = x \sum_{\ell\geq 0} \binom{\ell+2}{2} x^{\ell} \\ \sum_{k\geq 0} T_k x^k &= \frac{x}{(1-x)^3} \end{aligned}$$

Example. Note then by the same reasoning:

$$\frac{x^2}{(1-x)^3} = \sum_{k\geq 0} \binom{k}{2} x^k$$

Example. Find a closed formula for the generating function for $a_k = k^2$. Well:

$$2T_k - k = (k+1)k - k = k^2$$

We know the generating function of T_k. Let's find the generating function of k:

$$\frac{\mathrm{d}\left[\frac{1}{1-x}\right]}{\mathrm{d}x} = \sum_{k=0}^{\infty} k x^{k-1}$$

So then actually taking this derivative and multiplying by x:

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$$

So lets look at this as:

$$\begin{aligned}
\sum_{k\geq 0} k^2 x^k &= \sum_{k\geq 0} (2T_k - k)x^k \\
&= \frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2} \\
&= \frac{2x}{(1-x)^3} - \frac{x(1-x)}{(1-x)^3} \\
&= \frac{2x - x + x^2}{(1-x)^3} = \frac{x^2+x}{(1-x)^3}
\end{aligned}$$

We can also look at $T_k + T_{k-1} = k^2$. This is interesting:

$$\begin{aligned}
\sum_{k\geq 0} k^2 x^k &= \sum_{k\geq 0} (T_k + T_{k-1})x^k \\
&= \frac{x}{(1-x)^3} + \sum_{k\geq 0} T_{k-1}x^k \\
&= \frac{x}{(1-x)^3} + \sum_{\ell\geq 0} T_\ell x^{\ell+1} \\
&= \frac{x}{1-x^2}
\end{aligned}$$