MATH 465 Notes

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Assume graphs are simple!!!

Definition. *A **matching** in a graph G is a set of vertex-disjoint edges.*

*A **perfect matching** is a matching that covers all vertices in G.*

Example. Suppose we have a collection of people (V), and we know who is/is not compatible (edges). Can we pair people up into "marriages" so that each person is in exactly one pair?

Example. Suppose a company has a set of jobs open and a collection of applicants, with knowledge of what jobs an applicant is qualified for. Is it possible to fill all of the jobs with qualified applicants.

Vertices: Jobs and Appliants. Edge when applicant is qualified for a job (this is a bipartite graph). Want a matching in this bipartite graph that covers all of the "job" vertices.

Definition. *If $G = (U, V, E)$, a bipartite graph. A **matching of U into V** is a matching that covers all the vertices in U. [This defines an injection $U \to V$.]*

Definition. *Let $G = (V, E)$ be a simple graph. For $v \in V$ let $N_G(v) = \{u \in V \mid (u - v) \in E\}$.*

For $S \subseteq V$ let $N_G(S) = \bigcup_{V \in S} N_G(v)$.

Proposition 1. If $G = (U, V, E)$ is a simple bipartite graph and $S \subseteq U$ such that $|S| > |N(S)|$ then there is no matching of U into V

Proof (contrapositive). Suppose there is a matching into V, which is a collection of vertex disjoint edges $(u_1, v_1), \ldots, (u_k, v_k)$. This defines an injective function $f : U \to V$ by $u_i \mapsto v_i$. If $S \subseteq U$ then f restricts to an injective function $S \to N(S)$. By PHP, $|S| \leq |N(S)|$. $\square$

Theorem 1 (Hall's Theorem)**.** *Let $G = (U, V, E)$ be a simple bipartite graph. Then there is a matching of U into V if and only if for every $S \subseteq U$, $|S| \leq |N(S)|$.*

Proof. Let's go!

($\Rightarrow$) Done by the above proposition

($\Leftarrow$) We will use induction on $|U|$.

If $|U| = 1$ then Hall's condition says ther eis an edge incident to the single vertex in U, and this edge gives a matching of U into V.

Suppose that the statement is true for all positive integers less than $|U|$.

Let $G = (U, V, E)$ be a simple bipartite graph so that for all $S \subseteq U$, $|S| \leq |N_G(S)|$. Pick $u \in U$ and $v \in V$ connected by edge e. This must hold because $1 < |U| \leq |N_G(U)|$.

We have two cases:

- Suppose that for all nonempty proper $S \subseteq U$ $|S| < |N_G(S)|$. Consider the induced subgraph on $(U \setminus \{u\}, V \setminus \{v\})$, call it G'.

  Let $S \subseteq U \setminus \{u\}$. There are two options:

  $$N_{G'}(S) = N_G(s) \text{ or } N_G(S) \setminus \{v\}$$

  So either $|N_{G'}(S)| = |N_G(S)| \geq |S|$ or $|N_{G'}(S)| = |N_G(S)| - 1 \geq |S|$ (but this only holds in the strict inequality.

  So G' satisfies Hall's condition and by induction G' has a matching of $U \setminus \{u\}$ into $V \setminus \{v\}$. Adding e yields a matching of U into V.

- Suppose there exists a nonempty proper subset $S \subseteq U$ so that $|S| = |N_G(S)|$. Consider two induced subgraphs. G_1 on $(S, N_G(S))$ and G_2 on $(U \setminus S, V \setminus N_G(S))$.

  In G_1 let $T \subseteq S$, then:

  $$N_G(T) \subseteq N_G(S)$$

  Therefore $|N_{G_1}(T)| = |N_G(T)| \geq |T|$. By induction there is a matching M_1 of S into $N_G(S)$.

In G_2 let $T \subseteq U \setminus S$. We have:

$$\begin{aligned} N_{G_2}(T) &= N_G(T) \setminus N_G(S) \\ &= N_G(T \cup S) \setminus N_G(S) \end{aligned}$$

So we know:

$$\begin{aligned} |N_{G_2}(T)| &= |N_G(T \cup S)| - |N_G(S)| \\ &= |N_G(T \cup S)| - |S| \\ &= |T \cup S| - |S| = |T| \end{aligned}$$

By induction there is a matching M_2 of $U \setminus S$ into $V \setminus N_G(S)$. Then $M_1 \cup M_2$ is a matching of U into V.

□

Theorem 2 (Tutte's Theorem)**.** *A simple graph $G = (V, E)$ has a perfect matching if and only if for every $S \subseteq V$ there at most $|S|$-many connected components of $G - S$, the induced subgraph on $V \setminus S$ that have an odd number of vertices.*

Proof in Book, Theorem 11.20. □

Definition. *A* ***permutation matrix*** *is a square matrix with entries 0 and 1 such that every row and every column has exactly one 1. There is a clear bijection from S_n to $n \times n$ permutation matrices. A permutation σ has permutation A_σ with the (i, j)-th entry is 1 if $j = \sigma(i)$ and 0 otherwise.*

Theorem 3. *A square matrix with nonnegative integer entries whose row and columns sums all equal k is a sum of k permutation matrices.*

Proof by induction on k. Let's go!

$k = 1$ follows by definition of permutation matrix.

Assume it holds for k. Let A be an $n \times n$ matrix with nonnegative entries whose rows and columns sum to $k + 1$. It suffices to find a permutation matrix B so that $A - B$ has nonnegative integer entries. Then by induction $A - B$ would be a sum of k permutation matrices so A is a sum of $k + 1$ permutation matrices.

Trick is to construct a simple bipartite graph $G = (U, V, E)$. Where $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. And $(u_i - v_j) \in E$ if and only if the (i, j)-th entry of A is positive.

If G has a perfect matching M then define B with (i, j)-th entry where 1 if $(u_i - v_j) \in M$, and 0 otherwise. We claim that B is a permuation matrix, and $A - B$ has nonnegative integer entries.

On HW, you prove that G has a perfect matching!!! $\square$