MATH 465 Notes

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1 Stuff!

- Quiz 2 Today
- HW2 due Wednesday
- Quiz 3 Tuesday
- Today: More Counting (Finish Ch. 3 and start Ch. 4 + 5.1)
- Next Time: Binomial Theorem and Combinatorial Proofs (Ch. 4)
- No Office Hours tomorrow. Instead: 4-6pm Monday or email.

2 Bijective Proof

Proposition 1.

(a) If there is an $\boxed{\text{injection}}$ $f : M \to N$ then $|M| \leq |N|$ by the Pigeonhole Principle contrapositive

$f(a) = f(b) \implies a = b$ for all $a, b \in M$.

(b) If there is a $\boxed{\text{surjection}}$ $f : M \to N$ then $|M| \geq |N|$

for all $n \in \mathbb{N}$ there is some $m \in M$ such that $f(m) = n$

(c) If there is a $\boxed{\text{bijection}}$ $f : M \to N$ then $|M| = |N|$. Both inj. and surj.

By Pigeonhole Principle, also the converses hold (for some reason...). (c) is the heart of a bijective proof

Theorem 1. *Let $n \in \mathbb{Z}_{\geq 0}$. The number of subsets of an n-element set is 2^n.*

Proof. In order to show this we will prove there is a bijection between our subsets and the set of binary words (that is words in the alphabet $\{0, 1\}$).

Let $S = \{x_1, x_2, \ldots, x_n\}$ be our n-element set. Consider binary words. Define a function:

$$\beta : \{\text{subsets of } S\} \to \{n\text{-letter binary word}\}$$

Example. Let $n = 3$ and $S = \{1, 2, 3\}$, we want to do something like:

$$\{1,2,3\} \leftrightarrow 111 \qquad \{1,2\} \leftrightarrow 110 \qquad \{1,3\} \leftrightarrow 101 \qquad \{2\} \leftrightarrow 010 \qquad \emptyset \leftrightarrow 000$$

Fix $T \subset S$, let $\beta(T)$ be the n-letter binary word where the ith letter is 1 if $x_i \in T$ and 0 if $x_i \notin T$.

To prove that β is injective: Suppose $T_1, T_2 \subset S$ such that $\beta(T_1) = \beta(T_2)$. This means that for each i, the i-th letter of $\beta(T_1)$ is equal to the i-th letter of $\beta(T_2)$.

Thus for each i, either $x_i \in T_1$ and $x_i \in T_2$, or $x_i \notin T_1$ and $x_i \notin T_2$. That is $T_1 = T_2$.

To prove that β is surjective: Let $w = a_1 a_2 \ldots a_n$ be an n-letter binary word. Define $T = \{x_i \mid a_i = 1\}$. $T \subset S$ and $\beta(T) = w$.

Since β is a bijection, these two sets have the same number of elements, so since there are 2^n n-letter binary words, there are 2^n subsets of S. □

3 Multinomials and Binomials

Definition 1. *A multiset is a set whose elements are not necessarily distinct.*

Example. How many anagrams are there of the word "STATISTICS" There is 1A, 1C, 2I, 3T, 3S. If all the letters were distinct there'd be 10! permutations, but this overcounts by some factor.

STATISTICS
STATISTICS

This overcounting factor is 3!3!2! and so the total number is $\frac{10!}{3!3!2!}$

Theorem 2. *Let $\{A_1, A_2, \ldots, A_m\}$ be a set and consider a multiset S wich contains k_i copies of A_i, for each i. There are $\frac{(k_1+k_2+\ldots+k_m)!}{k_1!k_2!\cdots k_m!}$ permutations of the multiset S.*

Notationally if $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 0}$ and $n = k_1 + \ldots + k_m$, then:

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2!\ldots k_m!}$$

These are called "multinomial coefficients"

Corrolary 1. *The number of ways to divide an n-element set S into pairwise disjoint subsets $S_1, S_2, \ldots S_m$ of size $k_1, k_2, k_3 \ldots k_m$ is $\binom{n}{k_1, k_2, \ldots k_m}$.*

In the case of $m = 2$ we simplify to:

$$\binom{n}{k} := \binom{n}{k, n-k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \qquad \text{“binomial coefficient”}$$

Divides our set into subsets, one of size k and one of size $n - k$, this is equivalent to choosing a k-element subset of an n-element set.

Corrolary 2. *There are $\binom{n}{k}$ k-element subsets of an n-element set. There are then $\binom{n}{k}$ n-letter binary words which have k 1's (and $n - k$ 0's), [refine bijection β from earlier]*

3.1 Binomial Applications

Definition 2. *Let $n, k \in \mathbb{Z}_{>0}$. A composition of n with k parts is a positive integer solution to the equation:*

$$x_1 + x_2 + \ldots + x_k = n$$

A weak composition allows zero, so a nonzero integer solution to the above.

A solution is ordered and need not all be distinct.

Example. $n = 5$ and $k = 3$.

compositions:

$$1 + 2 + 2 = 5 \qquad\qquad 1 + 1 + 3 = 5$$

All of this form but permuted. So 6 compositions

weak compositions

$$1+2+2 \qquad 1+1+3 \qquad 0+1+4 \qquad 0+3+2 \qquad 0+0+5$$

All of this form but permuted. So $12 + 3 + 6 = 21$ weak compositions.

Another way to think of this is handing n identical cookies to k distinct kids (either so they all get one or so that they don't). This is like stars and bars:

$$\star | \star\star | \star\star$$

$$1 + 2 + 2 = 5$$

These are 7-letter words in the alphabet $\{\star, |\}$ with 5 $\star$'s and 2 $|$'s. So then:

$\{$weak composition of n with k parts$\} \leftrightarrow \{(n+k-1)$-letter binary words with n $\star$'s and $k-1$ $|$'s$\}$

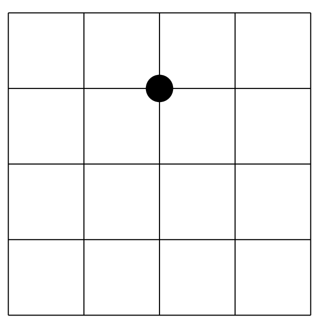
Theorem 3. *There are $\binom{n+k-1}{n}$ weak compositions of n with k parts. For compositions, first put a $\star$ into each bin, and use any weak composition of $n-k$ into k parts. So the number of compositions is $\binom{n-1}{k-1} = \binom{n-1}{(n-1)-(k-1)}$.*

A NE lattice path is awalk on the grid of points with integer coordinates that uses the steps $(1,0)$ and $(0,1)$:

$$E = ((x,y) \mapsto (x+1,y)) \qquad N = ((x,y) \mapsto (x,y+1))$$

Let $k, \ell \in \mathbb{Z}_{\geq 0}$. How many NE lattice paths are there from $(0,0)$ to (k,ℓ).

Look at lattice paths from $(0,0)$ to $(2,3)$.



Some possible paths are EENNN, NNNEE, ENENN, NENEN

It is clear that there is a bijection:

$\{$NE lattice paths from $(0,0)$ to $(k,\ell)\} \leftrightarrow \{(k+\ell)$-letter word in alphabet N, E with k E's, ℓ N's$\}$

There are $\binom{k+\ell}{k}$ NE lattice paths from $(0,0)$ to (k,ℓ).

A NE lattice path from $(0,0)$ to (k,ℓ) can be broken into exactly one of these two cases:

1. A NE lattice path from $(0,0)$ to $(k-1,\ell)$ followed by an E step, so there are $\binom{k-1+\ell}{\ell}$ or $\binom{k-1+\ell}{k-1}$
2. A NE lattice path from $(0,0)$ to $(k,\ell-1)$ followed by a N step, so there are $\binom{k-1+\ell}{k}$.

Let $n = k+\ell$, By Addition Principle:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{(Pascal's Recurrence)}$$