MATH 465 Notes

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Posets Continued

Recall. In a poset a chain is a subset in which any two elements are comparable. An antichain is a subset in which any two elements are incomparable. A chain (or antichain) partition is a partition in which each block is a chain (or antichain)

Theorem 1 (Dilworth's Theorem)**.** *In a fintie poset, the* $\overbrace{\textit{maximum size of an antichain}}^{\textit{width}}$ *is the minimum size of a chain partition.*

Theorem 2 (Mirsky's Theorem)**.** *In a fintie poset, the* $\overbrace{\textit{maximum size of a chain}}^{\textit{height}}$ *is the minimum size of an antichain partition.*

Lemma 1. *Let P be a finite poset. If $\{C_1, \ldots, C_n\}$ is a chain partition of P and A is an antichain, then $|A| \leq n$.*

Likewise if $\{A_1, \ldots, A_m\}$ is an antichain partition of P and C is a chain, then $|C| \leq m$.

Proof of Mirsky's Theorem. Lemma shows that the maximum size of a chain is less than or equal to the minimum size of an antichain partition. Thus, letting m be the maximum size of a chain, it suffices to find an antichain partition with m blocks. Say $\{x_1, \ldots, x_m\}$ is a chain with $x_1 < \ldots < x_m$. For each $1 \leq i \leq m$ let

$$A_i = \{x \in P \mid \text{the max size of a chain with } x \text{ at the top is } i\}$$

This is a partition of P into m nonempty blocks, $x_i \in A_i$ for each i. Moreover, each A_i is an antichain: If $x < y$ and the max size of a chain with x at the top is i, then there is a chain with $i + 1$ elements and y at the top. $\square$

Proof of Dilworth's Theorem. Lemma gives that the maximum size of an antichain is less than or equal to the minimum size of a chain partition. We will use Max-Flow Min-Cut Theorem to find an antichain and a chain partition of the same size (then we're done)

Define a network G by taking two disjoint copies of P, call them (P_L, P_R) along with the sourse s and sink t as vertices. And edges:

- $x \to y$ if $x < y$
- $s \to x$ if $x \in P_L$
- $y \to t$ if $y \in P_R$.

With all capacities $c(e) = 1$. The key is to relate flows in G and chain partitions in P as follows:

Consider a flow f in G. Because all edge capacities are 1, if $x \in P_L$ with $f(s \to x) \neq 0$ there is a unique $y \in P_R$ such that x, y and $f(x \to y) \neq 0$. Consider the edges $x \to y$ with $x \in P_L$ and $y \in P_R$ such that $f(x \to y) \neq 0$. This gives a collection of pairs:

$$x_1 < y_1, \ldots, x_\ell < y_\ell$$

Where the x_i's are distinct and the y_i's are distinct. These pairs bundle into pairwise disjoint chains in P by stacking the pairs in which $x_i < y_i = x_j < y_j$. If any elements in P do not appear in these chains, add it as a singleton block to obtain a chain partition $P = C_1 \cup \cdots \cup C_n$. This chain partition constructed from a flow f satisfies:

$$|f| = \sum_{x \in P_L} f(s \to x) = \sum_{i=1}^{n} (|C|_i - 1) = |P| - n$$

Equivalently $n = |P| - |f|$. In fact every chain partition can be constructed in this way. Therefore the minimum size of a chain partition in P is equal to $|P|$ minus the maximum size of a flow in G.

Use Ford-Fulkerson algorithm to find a max flow f in G. Let (X, Y) be the corresponding minimum cut. Then let n be the maximum size of a chain partition, then:

$$|f| = |P| - n = c(X, Y)$$

By the proof of the Max-Flow Min-Cut Theorem there does not exist $x \in P_L \cap X$ and $y \in P_R \cap Y$ such that $x \to y$ is an edge in G (i.e. $x < y$). This is because there is no directed path from s to t in G_f by construction of f. Why?

Suppose $x \in P_L \cap X$ and $y \in P_R \cap Y$. Well then there is a directed path from s to x in G_f but not one from s to y. Furthermore in G_f there is no edge $x \to y$, but since $x < y$ we know there is an edge between x and y in G. Thus there must be an edge $y \to x$ in G_f. Thus $f(x \to y) = c(x \to y) = 1$. Therefore note that we must have $f(s \to x) = 1$ and so there has to be an edge $x \to s$ in G_f. Thus there exists a $y' \in P_R \cap X$ such that $y' \to x$ is in G_f. But then $f(x \to y') = 1$, contradicting conservation at x:

$$f(s \to x) = 1 < 2 = f(x \to y) + f(x \to y')$$

Define:

$$A = \{p \in P \mid p \in P_L \cap X \text{ and } p \in P_R \cap Y\}$$

We can't have two comparable elements in this by the above claim. Moreover:

$$\begin{aligned}
|P| - n = c(X,Y) &= \sum_{\substack{x \to y \\ x \in X, y \in Y}} c(x \to y) \\
&= \sum_{y \in Y \cap P_L} c(s \to y) + \sum_{x \in X \cap P_R} c(x \to t) + \sum_{\substack{x \in P_L \cap X \\ y \in P_R \cap Y}} 0 \\
&= |Y \cap P_L| + |X \cap P_R| \\
&\geq |P| - |A|
\end{aligned}$$

Thus $|A| \geq n$. Therefore the maximum size of an antichain is $\geq n$ which is the minimum size of a chain partition. We already have the other inequality. $\square$