MATH 465 Notes

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1 Announcements

- Today: Quiz, Ch.7 Inclusion Exclusion
- Tuesday: Quiz, Review in Groups
- Thursday: Exam (info TBA on Canvas)
- Office Hours:
  - Monday 4-5:30
  - Tuesday 11:30-1
- HW7 Due Wednesday

2 Let's Go!

2.1 Statement and Proof

Recall. The Addition Principle. If $A_1, \ldots, A_n$ are disjoint finite sets, then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Example. $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ for all finite sets.

Example.

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Theorem 1 (The Principle of Inclusion Exclusion)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets, then:*

$$\left|\bigcup_{k=1}^{n} A_i\right| = \sum_{k=1}^{n} \sum_{(\star\star\star)} (-1)^{k-1} \left|\bigcap_{r=1}^{k} A_{i_k}\right|$$

$(\star\star\star)$ *stands for* $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

If we look at the example again

Example.

$$\begin{aligned}
& && |A_1 \cup A_2 \cup A_3| = \\
k &= 1 && |A_1| + |A_2| + |A_3| \\
k &= 2 && + (-|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|) \\
k &= 3 && + |A_1 \cap A_2 \cap A_3|
\end{aligned}$$

Proof. We show the RHS counts every element of $\bigcup_{i=1}^{n} A_i$ exactly once.

Let $x \in A_1 \cup A_2 \cup \cdots \cup A_n$. And let $S = \{i \in [n] \mid x \in A_i\}$. Then notice that $x \in A_{i_1} \cap \cdots \cap A_{i_k}$ if and only if $\{i_1, \ldots, i_k\} \subseteq S$. So the number of k-fold intersections containing x is exactly the number of of k-element subsets of S, that is $\binom{|S|}{k}$.

The contribution from x on the RHS is then:

$$\begin{aligned}
\sum_{k=1}^{n} (-1)^{k-1} \binom{|S|}{k} &= \sum_{k=1}^{|S|} (-1)^{k-1} \binom{|S|}{k} + \sum_{k=|S|+1}^{n} (-1)^{k-1} \binom{|S|}{k} \\
&= \sum_{k=1}^{|S|} (-1)^{k-1} \binom{|S|}{k} + \sum_{k=|S|+1}^{n} (-1)^{k-1} 0 \\
&= \sum_{k=1}^{|S|} (-1)^{k-1} \binom{|S|}{k} \\
&= 1 + \sum_{k=0}^{|S|} (-1)^{k-1} \binom{|S|}{k} = 1 + 0 = 1
\end{aligned}$$

Thus the right hand side counts every element fof the union $A_1 \cup \cdots \cup A_n$ exactly once and nothing else. $\square$

Example. How many n-letter words in the alphabet $\{1, 2, 3\}$ contain at least one 1, at least one 2, AND at least one 3. Assume $n \geq 3$.

Let's count the words that DON'T contain at least one 1, at least one 2, and at least one 3. Let U be the set of n-letter words in $\{1, 2, 3\}$. For $i \in \{1, 2, 3\}$ let $A_i = \{w \in U \mid w \text{ doesn't contain } i\}$. We want:

$$|U| - |A_1 \cup A_2 \cup A_3|$$
$$= |U| - [\ |A_1| + |A_2| + |A_3| - |A_1 \cap A_3| - |A_2 \cap A_3| - |A_1 \cap A_2| + |A_1 \cap A_2 \cap A_3|\ ]$$
$$= 3^n - [2^n + 2^n + 2^n - 1 - 1 - 1 + 0] = 3^n - 3 \cdot 2^n - 3$$

2.2 Let's Use It!

Theorem 2. *The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.*

Equivalently the number of partitions of n whose parts are not all odd is equal to the number of partitins of n whose parts are not all distinct.

Proof. Let $A_i = \{\text{partitions of } n \text{ with a part} = 2i\}$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Let $B_i = \{\text{ partitions of } n \text{ with at least two parts} = i\}$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

Note that:

$$|A_i| = p(n - 2i) = |B_i|$$
$$|A_{i_1} \cap A_{i_2}| = p(n - 2i_1 - 2i_2) = |B_{i_1} \cap B_{i_2}|$$
$$\text{For } 1 \leq i_1 < \cdots < i_k < n :$$
$$|A_{i_1} \cap \cdots \cap A_{i_k}| = p(n - 2i_1 - \cdots - 2i_k) = |B_{i_1} \cap \cdots \cap B_{i_k}|$$

Thus since we know that:

$$
\begin{aligned}
\#\text{ of partitions of } n \text{ with an even part} &= \left|A_1 \cup \cdots \cup A_{\lfloor \frac{n}{2} \rfloor}\right| \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le \lfloor \frac{n}{2} \rfloor} |A_{i_1} \cap \cdots \cap A_{i_k}| \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le \lfloor \frac{n}{2} \rfloor} p\left(n - \sum 2i_j\right) \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le \lfloor \frac{n}{2} \rfloor} |B_{i_1} \cap \cdots \cap B_{i_k}| \\
&= \left|B_1 \cup \cdots \cup B_{\lfloor \frac{n}{2} \rfloor}\right|
\end{aligned}
$$

$$\#\text{ of partitions of } n \text{ with an even part} = \#\text{ of partitons of } n \text{ with a repeated part}$$

And so we win! $\square$

3 Stirling Numbers of the Second Kind Redux

3.1 The Thing we Missed before

Theorem 3 (Unfinished from Jan. 30)**.** *For $m, n \in \mathbb{Z}_{>0}$ we have:*

$$m^n = \sum_{k=1}^{n} S(n,k)(m)_k = \sum_{k=1}^{n} S(n,k)k!\binom{m}{k} = \sum_{k=1}^{\min\{n,m\}} S(n,k)k!\binom{m}{k}$$

The second equality holds because $S(n,k) = 0$ if $k > n$ and $\binom{m}{k} = 0$ if $k > m$.

Combinatorial Proof. The LHS counts the number of ways to color the set $[n]$ with m colors (equivalent to functions $f : [n] \to [m]$).

Alternatively, for the RHS, we can count in cases by how many colors we actually use, fix $1 \leq k \leq \min\{n, m\}$. First choose the colors which we will use, there are $\binom{m}{k}$ ways to pick k colors. Now $S(n, k)$ counts the number of partitions of our n elements into k blocks. Then we multiply by $k!$ to assign each color to each block bijectively. This gives us exactly $S(n,k)k!\binom{m}{k}$. And so this matches the right hand side:

Done! Great ☺ □

A similar argument says there are $k!S(n, k)$ surjections from $[n] \to [k]$.

Proposition 1. Let $n \geq k > 0$. Then:

$$k!S(n,k) = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (k-\ell)^n = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n$$

Combinatorial Proof. The LHS counts surjections $[n] \to [k]$. Let's find a way to count surjections that looks like the right hand side. Let's do this using inclusion exclusion. For $i \in [k]$, let $A_i = \{f : [n] \to [k] \mid i \notin \text{range}(f)\}$. Equivalently we can think of this as the set of functions $A_i \leftrightarrow \{f : [n] \to [k] \setminus \{i\}\}$. For $1 \leq i_1 < \cdots < i_\ell \leq k$. We then have:

$$|A_{i_1} \cap \cdots A_{i_\ell}| = |\{f : [n] \to [k] \setminus \{i_1, \ldots, i_\ell\}\}| = (k-\ell)^n$$

Then by inclusion exclusion, the number of surjective functions is equal to:

$$
\begin{aligned}
&(\#\text{ functions } [n] \to [k]) - |A_1 \cup \cdots \cup A_k| \\
&= k^n - \sum_{\ell=1}^{k} (-1)^{\ell-1} \sum_{1 \le i_1 < \cdots < i_\ell \le k} |A_{i_1} \cap \cdots \cap A_{i_\ell}| \\
&= k^n - \sum_{\ell=1}^{k} (-1)^{\ell-1} \sum_{1 \le i_1 < \cdots < i_\ell \le k} (k-\ell)^n \\
&= k^n - \sum_{\ell=1}^{k} (-1)^{\ell-1} \binom{k}{\ell} (k-\ell)^n \\
&= k^n + \sum_{\ell=1}^{k} (-1)^{\ell} \binom{k}{\ell} (k-\ell)^n \\
&= \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} (k-\ell)^n
\end{aligned}
$$

So we are done! □