MATH 465 Notes

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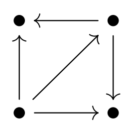
Max Flow Min Cut (Not in Book)

Definition. *A* ***directed graph*** *is a graph in which every edge has an orientation. The edges are arrows with one endpoint the* ***tail*** *and the other the* ***head****.*
A ***directed walk*** *is a walk that always travels along arrows from tail to head.*

Example.



Definition. *For a vertex v in a directed graph, we have the concepts:*

$$\begin{aligned} \operatorname{outdeg}(v) &= \#\ \textit{arrows with } v \textit{ as tail} \\ \operatorname{indeg}(v) &= \#\ \textit{arrows with } v \textit{ as head} \end{aligned}$$

This is neat!

Definition. *A* ***network*** *is a directed grpah $G = (V, E)$ in which:*

- *There are two special vertices* ***source*** *s and* ***sink*** *t. With the assumptions:*

$$\boxed{\operatorname{indeg}(s) = 0 = \operatorname{outdeg}(t)}$$

- *Each edge has been assigned a nonnegative* ***capacity*** *$c(e) \in \mathbb{Z}$. With the assumptions*

Assume G is simple and has no directed cycles. This helps

Definition. *A* ***flow*** *in a network $G = (V, E)$ is a function $f : E \to \mathbb{Z}$ satisfying for all $e \in E$ and $v \in V \setminus \{s, t\}$:*

$$0 \le f(e) \le c(e) \tag{feasibility}$$

$$\sum_{\bullet \xrightarrow{e} v} f(e) = \sum_{v \xrightarrow{e} \bullet} f(e) \tag{conservation}$$

<u>Goal</u> Maximize the following quantity:

Definition. *The* ***value*** *of a flow in a network is:*

$$|f| = \sum_{s \xrightarrow{e} \bullet} f(e) = \sum_{\bullet \xrightarrow{e} t} f(e)$$

The equality holds by conservation.

Now let's define a new concept to help us out

Definition. *A* ***cut*** *(X, Y) is a partition of the vertices V into disjoint subsets X and Y so that $s \in X$ and $t \in Y$.*

The ***capacity*** *of a cut (X, Y) is:*

$$c(x, y) = \sum_{\substack{x \in X, y \in Y \\ x \xrightarrow{e} y}} c(e)$$

Lemma 1. *The value of a flow cannot exceed the capacity of a cut. That is for any flow f and cut (X, Y) in a network G, $|f| \le c(X, Y)$.*

Proof. We'll write down:

$$
\begin{aligned}
|f| &= \sum_{s \xrightarrow{e} \bullet} f(e) = \sum_{x \in X} \left(\sum_{x \xrightarrow{e} \bullet} f(e) - \sum_{\bullet \xrightarrow{e} x} f(e) \right) && \text{(conservation)} \\
&= \sum_{\substack{x,z \in X \\ x \xrightarrow{e} z}} (f(e) - f(e)) + \sum_{\substack{x \in X, y \in Y \\ x \xrightarrow{e} y}} f(e) + \sum_{\substack{x \in X, y \in Y \\ y \xrightarrow{e} x}} (-f(e)) \\
&= \sum_{\substack{x \in X, y \in Y \\ x \xrightarrow{e} y}} f(e) - \sum_{\substack{x \in X, y \in Y \\ y \xrightarrow{e} x}} f(e) \\
&\leq \sum_{\substack{x \in X, y \in Y \\ x \xrightarrow{e} y}} c(e) - \sum_{\substack{x \in X, y \in Y \\ y \xrightarrow{e} x}} 0 = c(X,Y) && \text{(feasibility)}
\end{aligned}
$$

$\square$

Theorem 1 (Max Flow Min Cut). *The maximum value of a flow is the minimum capacity of a cut, i.e.*

$$\max\{|f| \mid f \text{ is a flow in } G\} = \min\{c(X,Y) \mid (X,Y) \text{ is a cut in } G\}$$

This is great!

Idea for proof. By Lemma, it suffices to find a flow F and a cut (X,Y) so that $|F| = c(X,Y)$. By the proof of Lemma, we just ned a flow F and cut (x,y) such that for all $x \in X, y \in Y$ such that:

$$
\begin{aligned}
&\forall\ x \xrightarrow{e} y \qquad F(e) = c(e) \\
&\forall\ y \xrightarrow{e} x \qquad F(e) = 0
\end{aligned}
$$

How do we find F? Given a flow f define a directed graph G_f with vertices V and arrows:

$$
\begin{aligned}
&u \xrightarrow{e^+} v \ \text{ if } \ u \xrightarrow{e} v \ \in E \text{ and } f(e) < c(e) \\
&u \xleftarrow{e^-} v \ \text{ if } \ u \xrightarrow{e} v \ \in E \text{ and } f(e) > 0
\end{aligned}
$$

Ford-Fulkerson Algorithm:

(1) Start with a flow f in $G = (V, E)$. You can just take $f(e) = 0$ for all $e \in E$.

(2) Find a directed path P from s to t in G_f. Let:

$$\delta = \min\left(\{c(e) - f(e) \mid e^+ \in P\} \cup \{f(e) \mid e^- \in P\}\right)$$

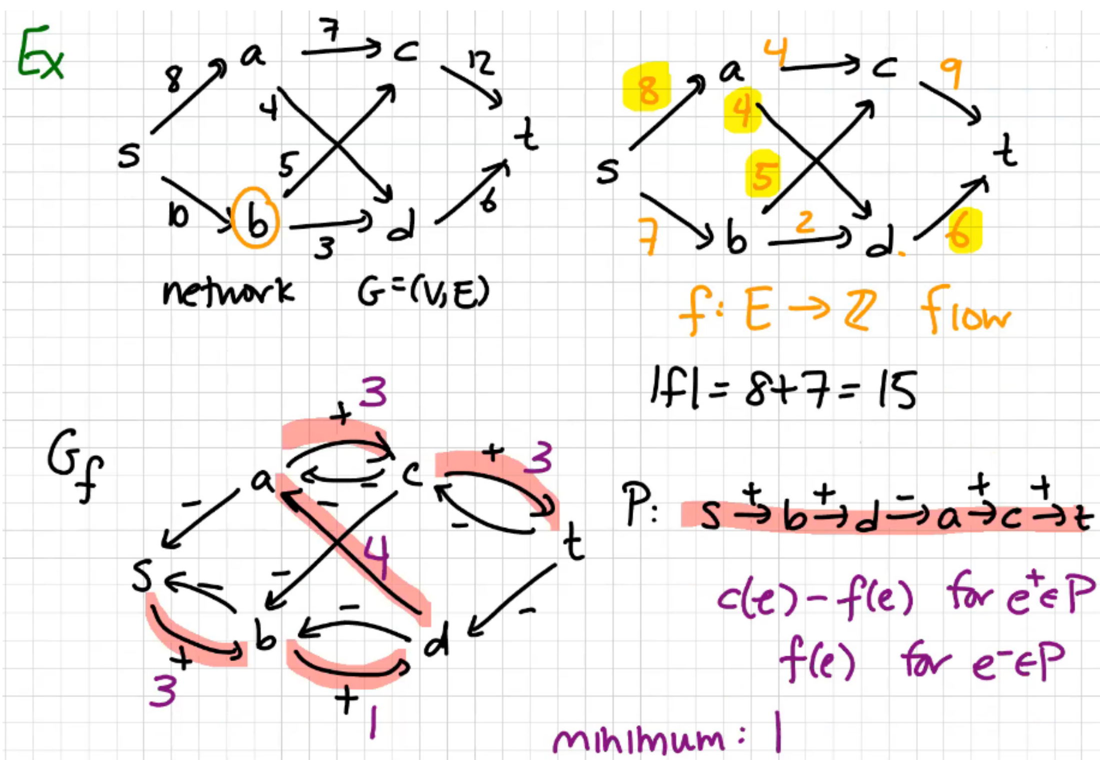
Define a flow f':

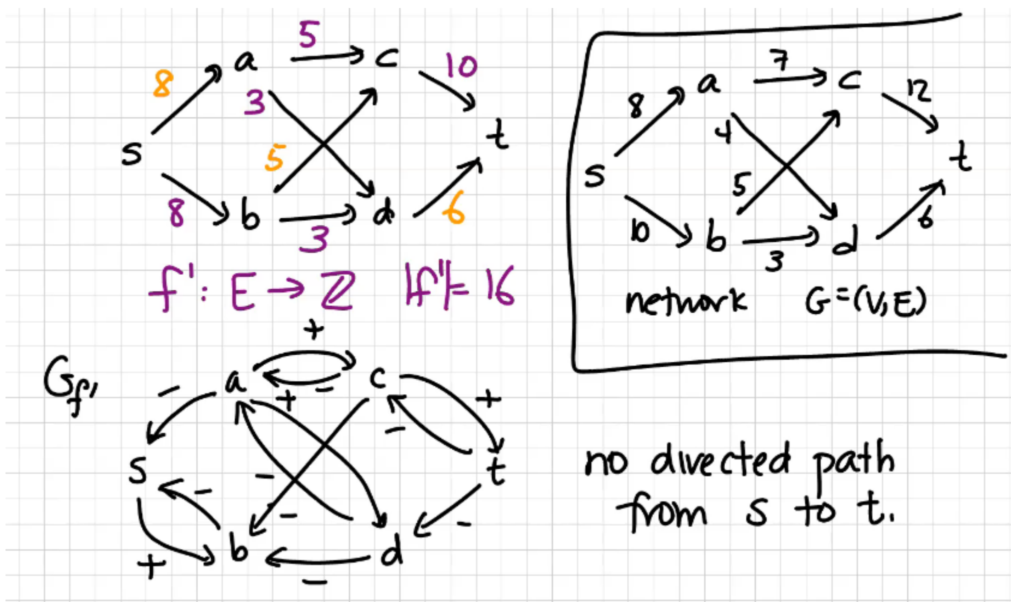
$$f'(e) = \begin{cases} f(e) + \delta & \text{if } e^+ \in P \\ f(e) - \delta & \text{if } e^- \in P \\ f(e) & \text{if } e^+, e^- \notin P \end{cases}$$

Claim. *f' is a flow with $|f'| = |f| + \delta$.*

(3) Repeat step 2, with f', until no such directed path exists.

Look at the example from the lecture notes!





Now once we terminate this process, call that flow F. Define:

$$X = \{x \mid \exists \text{ a directed path from } s \text{ in } G_F\}$$
$$Y = \{y \mid \nexists \text{ a directed path from } s \text{ in } G_F\} = V \setminus X$$

We should actually do some proof stuff! Let F be the flow obtained from the Ford Fulkerson Algorithm starting with $f = 0$, then there is no directed path from s to t in G_F. Note $s \in X$ and $t \in Y$ by the construction of F. Thus (X, Y) is a cut.

Then, for $x \xrightarrow{e} y$ for $x \in X, y \in Y$ we have $F(e) = c(e)$, because otherwise we would have some path from s to y in G_F:

$$s \longrightarrow \cdots \longrightarrow x \xrightarrow{e^+} y$$

For $y \xrightarrow{e} y$ for $x \in X, y \in Y$ we have $F(e) = 0$ because otherwise in G_F there is a directed path from s to y:

$$s \longrightarrow \cdots \longrightarrow x \xrightarrow{e^-} y$$

Then by the steps in the proof of lemma, $|F| = c(X, Y)$, and so by Lemma $\max |F| = \min c(X, Y)$. Great! $\square$