**Notes on**
**MATH 396**
**(Analysis on Manifolds)**
**Syllabus**

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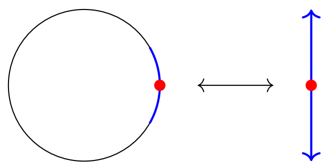
Contents

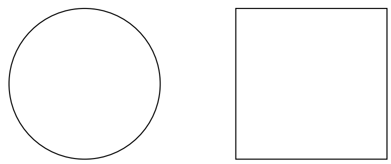
**Part A. Introduction to Manifolds**

In layman's terms, a manifold is a topological space that looks locally like Euclidean space $\mathbb{R}^n$. The number $n$ is called the dimension of the manifold, and it must be constant across the manifold, spaces like the disjoint union of a sphere and a circle are not manifolds.

Here is what we mean by locally homeomorphic, we can unbend and strech the blue section into the right hand side, which is a copy of $\mathbb{R}$.



But to a topologist, a small section of a corner is like $\mathbb{R}^n$. However, as an analyst, these is not diffeomorphic. To a topologist the square is the same as a circle, but not to an analyst:



If we allow for such objects with corners, we obtain topological manifolds, that is spaces that are locally homeomorphic to $\mathbb{R}^n$.

However, we cannot do calculus on such manifolds. For this, we will need to introduce an additional structure called a smooth structure to obtain a smooth manifold.

If we further would like to measure distance on such manifolds, another structure is needed. This is given by a Riemannian metric, making the manifold into a Riemannian manifold

# I. Topological Manifolds

## I.1. The Basic Definition

**Definition I.1.1**

Suppose $M$ is a topological space. We say $M$ is a topological manifold of dimension $n$ (or an $n$-manifold) if it has the following properties:

(1) $M$ is Hausdorff (i.e. if $p \neq q \in M$ then there exists disjoint neighborhoods $U$ and $V$ of $p$ and $q$ respectively)

(2) $M$ is second countable (i.e. there exists a countable basis of the topology)

A basis is a collection of open sets $\{\mathcal{O}_i\}$ such that if $U$ is open in $M$ and $x \in U$ then there exists some $\mathcal{O}_\alpha$ so that $x \in \mathcal{O}_\alpha \subseteq U$.

(3) $M$ is locally Euclidean of dimension $n$. I.e. every point $p \in M$ has a neighborhood $U \subseteq M$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

More explicitly, for each $p \in M$ there exists a homeomorphism $\phi_p : U_p \to V_p$ for some open subset $U_p$ of $M$ and some open subset $V_p$ of $\mathbb{R}^n$. The particular choice of homeomorphism is not part of the data of the manifold, but it also doesn't really matter if it were for topological manifolds. This will come up however for smooth manifolds.

**Remark I.1.1**

Condition (3) is the main condition, (1) and (2) are added to avoid pathological behaviors. Some books do not adopt both of them

(3) means that for each $p \in M$ there exists an open set $V$

**Definition I.1.2**

A coordinate chart on a space $M$ is a pair $(U, \phi)$ where $U \subseteq M$ is open and $\phi : U \to \widehat{U} \subseteq \mathbb{R}^n$ is a homeomorphism.

$U$ is often called the domain of the chart, and $\phi$ is called a local coordinate map since it gives coordinates to every point $q \in U$ via $\phi(q) = (x^1(q), \dots, x^n(q))$

Further, if $\widehat{U}$ is a ball in $\mathbb{R}^n$, then $(U, \phi)$ is called a coordinate ball.

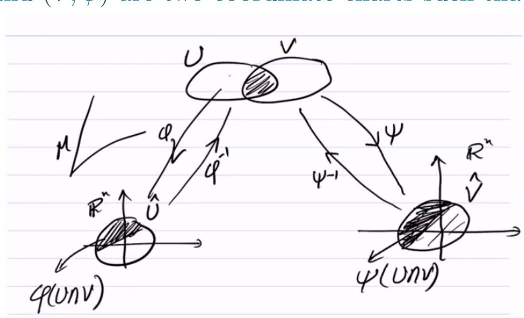
A collection of coordinate charts $(U_\alpha, \phi_\alpha)$ such that $\{U_\alpha\}$ covers $M$ is called an atlas, and such an atlas makes $M$ into a topological manifold.

**Remark I.1.2**

By definition, for a manifold $M$ every point $p \in M$ belongs to the domain of a coordinate chart.

**Definition I.1.3**

Question Suppose $(U, \phi)$ and $(V, \psi)$ are two coordinate charts such that $U$ and $V$ intersect.




This gives a map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$. This map is a homeomorphism and we call this map a transition map between the two coordinate systems on $U \cap V$.

**Example I.1.1** (Graphs of continuous functions)

Let $U \subseteq \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^k$ be a continuous function. Then the graph

$$\Gamma_F = \{(x, y) \in U \times \mathbb{R}^k \mid y = F(x)\} \subseteq \mathbb{R}^n \times \mathbb{R}^k$$

is a topological manifold with the subspace topology. Clearly $\Gamma$ is both Hausdorff and second countable because it is a subspace of $\mathbb{R}^n$ which is both Hausdorff and second countable.

Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the projection $\pi_1(x, y) = x$ and let $\phi : \Gamma \to U$ be the restriction of $\pi_1$. Then $(\Gamma_F, \phi)$ is a coordinate chart, since the inverse given by $\phi^{-1}(x) = (x, F(x))$ is also continuous.

Therefore $\Gamma_F$ is a topological manifold and $(\Gamma, \phi)$ is a coordinate chart and an atlas.

**Example I.1.2** (Spheres)

Let $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ denote the unit sphere:

$$\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$$

with the subspace topology.

$\mathbb{S}^n$ is in fact a topological manifold. Again, it is Hausdorff and second countable as in the previous example. Define the open sets:

$$U_i^+ = \{(x_1, \dots, x_{n+1}) \in \mathbb{S}^n \mid x_i > 0\}$$
$$U_i^- = \{(x_1, \dots, x_{n+1}) \in \mathbb{S}^n \mid x_i < 0\}$$

Then simply note that $U_i^\pm$ is the graph of the function:

$$x_i = \pm\sqrt{1 - (x_1^2 + \cdots \widehat{x}_i^2 + x_{n+1})^2}$$

Here we use the notation:

$$(x_1, x_2, \dots, \widehat{x}_i, \dots, x_{n+1}) = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \in \mathbb{R}^n$$

Hence, each $U_i^{\pm}$ is locally Euclidean, and the coordinate maps $\phi_i^{\pm} : U_i^{\pm} \to B(0,1)$ given by $\phi_i^{\pm} : (x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1})$ give us coordinate charts $(U_i^{\pm}, \phi_i^{\pm})$

Therefore $\mathbb{S}^n$ is a topological manifold. Great ☺

## I.2. Back to Manifolds

Last time, we defined topological manifolds in definition I.1.1. To remind us:

**Definition I.2.1**

Suppose $M$ is a topological space. We say $M$ is a topological manifold of dimension $n$ (or an $n$-manifold) if it has the following properties:

(1) $M$ is Hausdorff (i.e. if $p \neq q \in M$ then there exists disjoint neighborhoods $U$ and $V$ of $p$ and $q$ respectively)

(2) $M$ is second countable (i.e. there exists a countable basis of the topology)

A basis is a collection of open sets $\{\mathcal{O}_i\}$ such that if $U$ is open in $M$ and $x \in U$ then there exists some $\mathcal{O}_\alpha$ so that $x \in \mathcal{O}_\alpha \subseteq U$.

(3) $M$ is locally Euclidean of dimension $n$. I.e. every point $p \in M$ has a neighborhood $U \subseteq M$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

More explicitly, for each $p \in M$ there exists a homeomorphism $\phi_p : U_p \to V_p$ for some open subset $U_p$ of $M$ and some open subset $V_p$ of $\mathbb{R}^n$. The particular choice of homeomorphism is not part of the data of the manifold, but it also doesn't really matter if it were for topological manifolds. This will come up however for smooth manifolds.

**Definition I.2.2**

We call $(U, \phi)$ a coordinate chart where $U$ is open in $M$ and $\phi : U \to \widehat{U}$ is a homeomorphism onto an open subset of $\mathbb{R}^n$.

A collection of coordinate charts which cover $M$ is called an atlas

One last example of topological manifolds!

**Proposition I.2.1**

Suppose that $M_1, M_2, \ldots, M_k$ are topological manifolds of dimension $n_1, n_2, \ldots, n_k$. Then $M_1 \times M_2 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + n_2 \cdots + n_k$.

*Proof.* Hausdorffness and second countability are not too hard to show.

If $(U_1, \phi_1), \ldots, (U_k, \phi_k)$ are coordinate charts on $M_1, \ldots, M_k$ then $(U_1 \times \cdots \times U_k; \phi_1 \times \cdots \phi_k)$ is a coordinate chart on $M_1 \times \cdots M_k$.

**Example I.2.1**

In particular, the torus $\mathbb{T}^k = \underbrace{S^1 \times \cdots \times S^1}_{k \text{ times}}$ is a topological manifold.

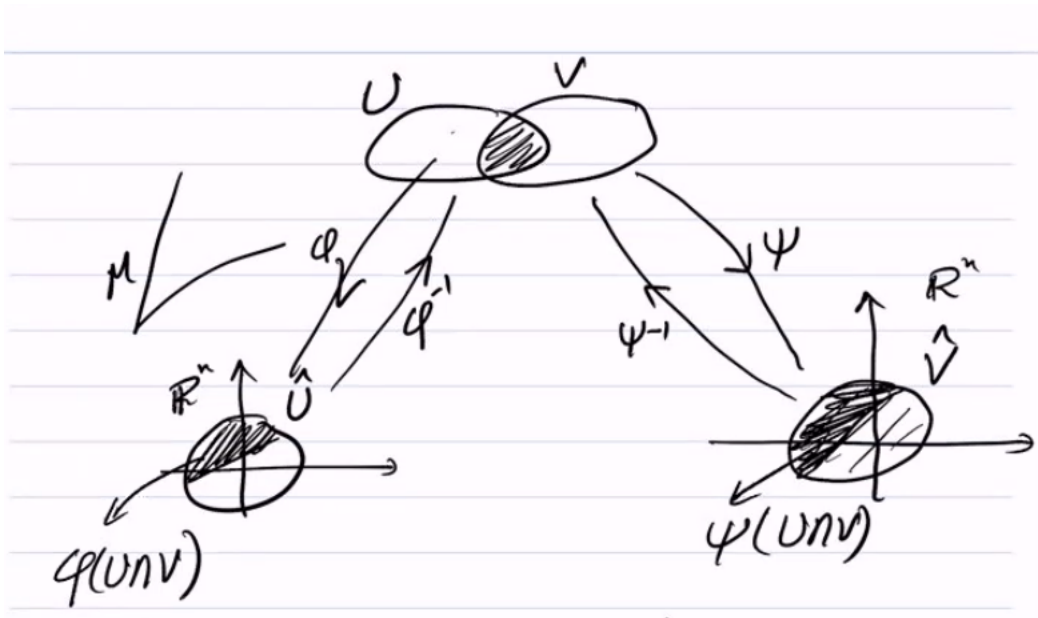
# II. Smooth Manifolds

## II.1. Motivation

One cannot make sense of derivatives on topological manifolds. To make sense of derivatives and to be able to do calculus on manifolds, we need an extra structure. This structure will be called a "smooth structure"

To motivate the definition below: suppose we try to define a smooth (or differentiable) function on a topological manifold $M$, say $f : M \to \mathbb{R}$. Naturally, we shall require that if $\phi : U \to \widehat{U}$ is a coordinate chart, then $f \circ \phi^{-1} : \widehat{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ is smooth.

While this seems to be the only plausible definition, it is not well-defined on topological manifolds. Why? Well suppose you have another coordinate chart $(V, \psi)$ such that $U$ and $V$ intersect.




Now note that $\phi$ and $\psi$ are defined on $U \cap V$. For $x \in \psi(U \cap V)$ we have:

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$$

Now $\phi \circ \psi^{-1} : \psi(U \cap V) \subseteq \mathbb{R}^n \to \phi(U \cap V) \subseteq \mathbb{R}^n$. But wait! If $f \circ \phi^{-1}$ is required to be smooth, then the only way to guarantee that $f \circ \psi^{-1}$ is smooth would be to guarantee that $\phi \circ \psi^{-1}$ is a smooth map itself. Otherwise, we get two different notions of smoothness with respect to $(U, \phi)$ and $(V, \psi)$.

This condition is not necessarily true on topological manifolds, which only guarantee that $\phi \circ \psi^{-1}$ is a homeomorphism.

Conclusion: In order to define smooth functions on a topological manifold, we need the following:

## II.2. **Definitions**

**Definition II.2.1**

If $(U, \phi)$ and $(V, \psi)$ are two coordinate charts and $U \cap V$ is nonempty, then we require that the map $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a diffeomorphism, noting that this is a map between open sets in $\mathbb{R}^n$. Note that a diffeomorphism is a smooth map with smooth inverse.

This condition is called smooth compatibility

**Definition II.2.2**

A smooth manifold is a topological manifold equipped with a particular atlas whose coordinate charts are smoothly compatible. Such an atlas is called a smooth atlas. In other words, a topological space $M$ is called a smooth $n$-manifold provided that:

1) It is Hausdorff
2) It is second countable
3) It is locally Euclidean, i.e. for each $p \in M$ there exists an open neighborhood $U \subseteq M$ of $p$ and $\phi : U \to \widehat{U} \subseteq \mathbb{R}^n$ a homeomorphism onto an open neighborhood $\widehat{U}$. $(U, \phi)$ is called a coordinate chart
4) We have a particular atlas on $M$ such that if $(U, \phi)$ and $(V, \psi)$ are two charts in the atlas such that $U \cap V \neq \emptyset$ then $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is smooth. It is enough to just check smoothness because it will be a diffeomorphism by switching the role of $(U, \phi)$ and $(V, \psi)$.

   We say $(U, \phi)$ and $(V, \psi)$ are smoothly compatible. The atlas on $M$ is then called a smooth atlas

**Remark II.2.1**

A $C^k$-manifold is defined exactly as above but replacing the word smooth by $C^k$. That is we require the transition maps in the atlas to be $C^k$ and we call such an atlas a $C^k$-atlas.

**Definition II.2.3**

Let $M$ be a smooth (resp. $C^k$) manifold. A smooth (resp. $C^k$) function $f : M \to R$ is one that satisfies the condition $f \circ \phi^{-1} : \widehat{U} \to \mathbb{R}$ is smooth (resp. $C^k$) for any coordinate chart $(U, \phi)$ on $M$ such that $\widehat{U} = \phi(U) \subseteq \mathbb{R}^n$.

Thanks to smooth compatibility, this definition makes sense.

**Remark II.2.2**

The above definition of a smooth manifold says that a Hausdorff, second countable topological space is a smooth manifold if and only if it admits a smooth atlas.

However, two smooth atlases might give the same notion of what a smooth function on the manifold is, and in such cases we would like not to distinguish between between the resulting smooth manifold.

**Example II.2.1**

$(\mathbb{R}^d, \mathrm{Id})$ is a smooth atlas on $\mathbb{R}^d$. Similarly, $\{B(x, 1), \mathrm{Id}\}_{x \in \mathbb{R}^d}$. However they yield an equivalent notion of smooth functions. We would like to consider these two smooth atlases to be the same

**Definition II.2.4**

We say that two smooth atlases $\mathcal{A}_1 = (U_\alpha, \phi_\alpha)$ and $\mathcal{A}_2 = (V_\beta, \psi_\beta)$ are equivalent if their union atlas $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas

**Exercise II.2.2**

This is equivalent to saying that $(U_\alpha, \phi_\alpha)$ and $(V_\beta, \psi_\beta)$ are smoothly compatible for each $\alpha$ and $\beta$. Furthermore, this is an equivalence relation on smooth atlases.

**Definition II.2.5**

This gives equivalence classes of smooth atlases on the same topological manifold $M$. Each such equivalence class contains a unique maximal smooth atlas $\mathcal{A}_{\max}$ (maximal = one that is not contained in any strictly larger atlas). This means that if $(U, \phi)$ is a chart that is smoothly compatible with every element of $\mathcal{A}_{\max}$ then $(U, \phi) \in \mathcal{A}_{\max}$¿ Such an atlas is called complete

As such, strictly speaking one should define a smooth manifold as a pair $(M, \mathcal{A})$ where $M$ is a topological manifold and $\mathcal{A}$ is a maximal smooth atlas on $M$. $\mathcal{A}$ is called a “smooth structure” on $M$.

**Example II.2.3**

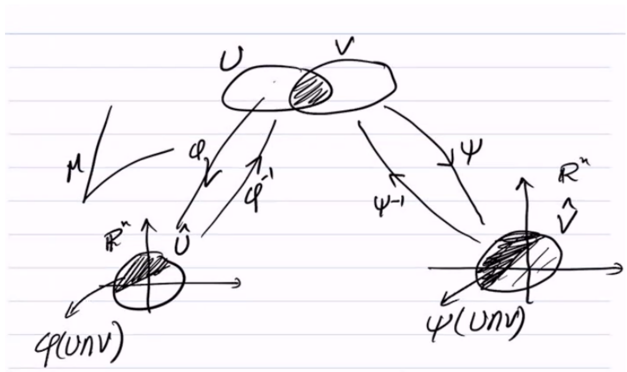
Another smooth structure on $\mathbb{R}$. Consider the single chart $(\mathbb{R}, \psi)$ where $\psi(x) = x^3$. This gives a smooth atlas on $\mathbb{R}$, and hence a smooth structure that is different than the standard structure given by the chart $(\mathbb{R}, \mathrm{Id})$.

Indeed $\phi \circ \psi^{-1}(y) = y^{1/3}$ is not smooth with $\phi = \mathrm{Id}$, and so these smooth structures are different.

**Review**

A smooth manifold $M$ is a topological manifold equipped with a smooth atlas, that is a collection of smoothly compatible charts $(U_\alpha, \varphi_\alpha)$ that covers $M$.

$(U_\alpha, \varphi_\alpha)$ and $(U_\beta, \varphi_\beta)$ are smoothly compatible if:




The transition map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism. That is it's smooth and it has a smooth inverse. This covered automatically if it's smooth when we're talking about an atlas, because we can just swap the role of $\alpha$ and $\beta$.

A topological manifold $M$ may be equipped with different atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ which give different smooth manifolds $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$.

We defined an equivalence relation on atlases. Namely $\mathcal{A}_1 \sim \mathcal{A}_2$ whenever $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas. This is equivalent to saying that if $(U, \varphi) \in \mathcal{A}_1$ and $(V, \psi) \in \mathcal{A}_2$ then $(U, \varphi)$ and $(V, \psi)$ are smoothly compatible.

For the purposes of this class we identify $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ when $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent. Since every equivalence class of a smooth element has a unique maximal element $\mathcal{A}_{\max}$, we can describe any smooth manifold as a pair $(M, \mathcal{A}_{\max})$ where $M$ is a topological manifold and $\mathcal{A}_{\max}$ is a maximal smooth atlas. We call such a maximal smooth atlas a smooth structure.

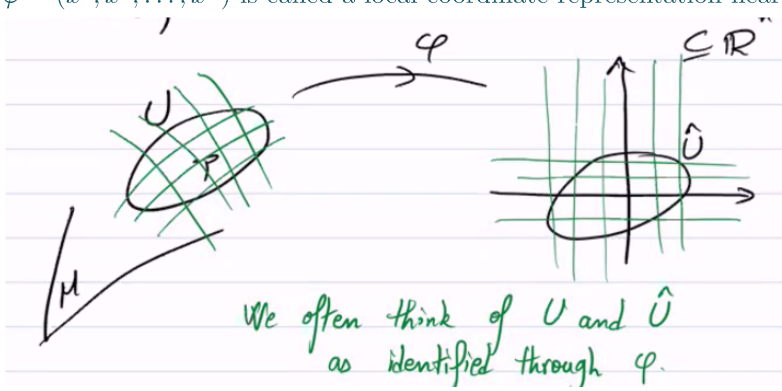
**Exercise II.2.4** (Homework)

Show that every smooth atlas of $M$ determines a unique maximal smooth atlas

**Definition II.2.6** (Notational Convenience)

$(U, \varphi)$ is called a smooth coordinate chart. $U$ is called a smooth coordinate domain (a smooth coordinate ball if $\widehat{U} \in \mathbb{R}^n$ is a ball)

If $p \in U$, then $\varphi = (x^1, x^2, \ldots, x^n)$ is called a local coordinate representation near $p$.



## II.3. **Examples of Smooth Manifolds**

**Example II.3.1** (Trivial Examples)

Let's find some very easy examples first:

(1) Zero dimensional topological manifolds are countable discrete spaces, since $\mathbb{R}^0$ is a singleton, and so each point $p$ has a neighborhood $U$ which is homeomorphic to $\mathbb{R}^0$. $U$ must be equal to $\{p\}$. All charts on $M$ are trivially smoothly compatible, so countable discrete spaces are also smooth manifolds of dimension zero.

(2) Euclidean space $\mathbb{R}^n$ is a smooth manifold via the coordinate chart $(\mathbb{R}^n, \mathrm{Id})$. This single chart gives us a smooth atlas on $\mathbb{R}^n$ since any single chart is trivially compatible with itself. Therefore $\mathbb{R}^n$ is a smooth manifold.

**Remark II.3.1**

If a topological manifold can be covered by a single coordinate chart, then the smooth compatibility condition is trivial, so it is automatically a smooth manifold.

**Example II.3.2** (Less trivial examples)

Let's find a bit more interesting examples!

(3) Finite dimensional vector spaces $V$ over $\mathbb{R}$ are smooth manifolds. Let $V$ be an $n$-dimensional vector space. We saw in 395 that all norms on $V$ determine the same topology. We just let $E_1, \ldots, E_n$ be a basis of $V$ and let $(V, \psi)$ be the coordinate chart defined by $\psi : V \to \mathbb{R}^n$:

$$\psi^{-1}(x^1, \ldots, x^n) = \sum_{j=1}^n x^i E_j =: x^j E_j$$

Where we have adopted the Einstein notation, which says that repeated indices that appear once above and once below are summed from 1 to the dimension $n$. Note that this will be a homeomorphism since $\psi$ and $\psi^{-1}$ are linear, and all linear maps between normed vector spaces are continuous.

This gives a smooth atlas and hence a smooth structure on $V$. This structure is independent of the choice of the basis. To see this let $\widetilde{E}_1, \ldots, \widetilde{E}_n$ be any other basis and denote by $\varphi : V \to \mathbb{R}^n$ given by $\varphi^{-1}(x^1, \ldots, x^n) = \sum_{j=1}^n x^i \widetilde{E}_j$. It is enough to check that $(V, \psi)$ and $(V, \varphi)$ are smoothly compatible, but note that these are linear maps, so any composition $\varphi \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ or $\psi \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ will be linear, and all linear maps are smooth.

One could check this by hand by writing the following, adopting Einstein notation:

$$E_i = \sum_{j=1}^n A_i^j \widetilde{E}_j = A_i^j \widetilde{E}_j$$

$$\begin{aligned} \varphi \circ \psi^{-1}(x^1, \ldots, x^n) &= \varphi(x^j E_j) = \varphi(x^j A_j^k \widetilde{E}_k) \\ &= \varphi\left(\sum_{k=1}^n (x^j A_j^k) \widetilde{E}_k\right) \\ &= (x^j A_j^1, x^j A_j^2, \ldots, x^j A_j^n) \end{aligned}$$

(4) Recall that we showed the sphere $\mathbb{S}^n$ is a topological manifold with charts $(U_i^\pm, \varphi_i^\pm)$ with:

$$\begin{aligned} U_i^+ &= \{(x_1, \ldots, x_n) \in \mathbb{S}^n \mid x_i > 0\} \\ U_i^- &= \{(x_1, \ldots, x_n) \in \mathbb{S}^n \mid x_i < 0\} \\ \varphi_i^\pm &: U_i^\pm \to B(0, 10 \subseteq \mathbb{R}^n \end{aligned}$$

$$\varphi_i^\pm = \pi_i\big|_{U_i^\pm}$$

Where we've taken $\pi_i : \mathbb{R}^{n+1} \to \mathbb{R}^n$ as the orthogonal projection in the direction of $e_i$. Clearly $U_i^+ \cap U_i^- = \emptyset$ for all $i$. We only check on $U_i^+ \cap U_j^+$ since the other cases are similar:

$$\varphi_i^+ \circ (\varphi_j^+)^{-1}(y^1, \dots, y^n) = \varphi_i^+(y^1, \dots, \underbrace{\sqrt{1-|y|^2}}_{j\text{-th entry}}, \dots, y^n)$$
$$= (y^1, \dots, \widehat{y}_i, \dots, \sqrt{1-|y|^2}, \dots, y^n)$$

Clearly this is smooth on $B(0,1)$. You can check it similarly for $\varphi_i^- \circ (\varphi_j^-)^{-1}$ and $\varphi_i^+ \circ (\varphi_j^-)^{-1}$.

Therefore $(U_i, \varphi_i)$ is a smooth atlas which defines the standard smooth structure on $\mathbb{S}^n$.

**Remark II.3.2**

In Einstein notation, we will also denote basis vectors with lower indices (like $E_1, \dots, E_n$) and the components of a vector with upper indices $x^1, \dots, x^n$, so that a vector is writtena s $x^i E_i$.

**Example II.3.3**

Let's get some more examples!

(5) Let $M$ be a smooth $n$-manifold and let $U \subseteq M$ be open. Now suppose that $\mathcal{A}$ is a smooth atlas on $M$. THen define an atlas as follows:

$$\mathcal{A}_U = \{(U \cap V, \varphi\big|_{U\cap V}) \mid (V, \varphi) \in \mathcal{A}\}$$

This gives a smooth atlas on $U$ since restrictions of smooth maps are smooth. This gives $U$ a smooth structure, with which $U$ is called an open submanifold of $M$.

(6) We can have matrix manifolds! Let $M(m \times n, \mathbb{R})$ denote the vector space of $(m \times n)$ matrix with real entries over $\mathbb{R}$. This gives it a standard (vector space smooth structure. For convenience we also write $M(n, \mathbb{R}) := M(n \times n, \mathbb{R})$.

Furthermore any $M(m \times n, \mathbb{C})$ has complex $(m \times n)$ matrices which is a vector space over $\mathbb{R}$ of dimension $2mn$, so it is a $2mn$ dimensional manifold.

(7) The set of all invertible $(n \times n)$ matrices $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $M(n, \mathbb{R})$ and hence by the above two examples it is a smooth manifold. Note that this is an open subset because the determinant is continuous on $M(n, \mathbb{R})$.

(8) Smooth product manifolds. If $M_1, \dots, M_k$ are smooth manifolds of $n_1, \dots, n_k$, we saw that $M_1 \times \cdots \times M_k$ is a topological manifolds with charts given by $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \varphi_k)$.

We then check compatibility given compatibility for the charts from$M_1, \dots, M_n$. This is because:

$$\varphi \times \cdots \times \varphi_k(p_1, \dots, p_k) = (\varphi(p_1), \dots, \varphi(p_k))$$
$$(\psi_1 \times \cdots \times \psi_k) \circ (\varphi \times \cdots \times \varphi_k)^{-1} = (\psi_1 \circ \varphi_1^{-1}) \times \cdots \times (\psi_k \circ \varphi_k^{-1})$$

And this is smooth in each coordinate.

(9) By the previous example the $n$-dimensional torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a smooth $n$-manifold.

**Back to Lecture!**

We will continue our discussion of smooth manifolds. We recall the three conditions we need from these in definition II.2.2:

(1) Hausdorff second countable topological spaces
(2) Locally Euclidean
(3) Equipped with an atlas such that all transition maps are smooth (diffeomorphisms)

These are $C^\infty$-manifolds. We cna also talk about $C^r$-manifolds ($r \geq 1$) replace the smoothness condition for transition maps by a $C^r$-condition.

Last time, we saw many exaples of smooth manifolds. This time we want to give one more example:

## II.4. Submanifolds of $\mathbb{R}^m$

**Definition II.4.1**

Let $M \subseteq \mathbb{R}^m$. Suppose that for each $p \in M$ there exists an open set $U \subseteq M$ (in the subspace topology) and a map $\varphi : U \to \mathbb{R}^n$ such that:

a) $\varphi$ is a homeomorphism from $U$ onto $\widehat{U} = \phi(U)$.
b) $\varphi^{-1} : \widehat{U} \to \mathbb{R}^m$ is of class $C^r$ and $D\varphi^{-1}$ has rank $n$
c) If $(V, \psi)$ is another chart such that $U \cap V \neq \emptyset$, then:

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a $C^r$-diffeomorphism

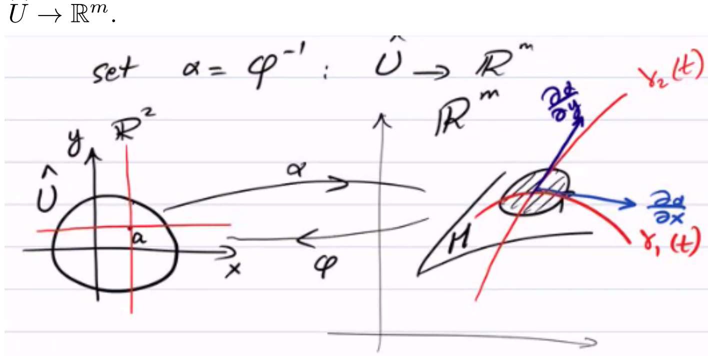
Then $M$ is called a $C^r$-submanifold of $\mathbb{R}^m$. If $r = \infty$ then $M$ is called a smooth submanifold of $\mathbb{R}^m$

**Remark II.4.1**

We have a few nice properties:

1) Clearly by a) and c) any smooth (or $C^r$) submanifold of $\mathbb{R}^m$ is a smooth (or $C^r$ manifold)
2) In fact, condition c) above is redundant. You can use a) and b) to prove c). Likewise you can use a) and c) to prove b). We'll discuss that when we talk about manifolds with boundary in the next section. We'll discuss that when we talk about manifolds with boundary in the next section.
3) Consider the case when $n = 2$. The condition that $\varphi^{-1} : \widehat{U} \to \mathbb{R}^m$ satisfies $D\varphi^{-1}$ has rank 2 means the following:

Set $\alpha = \varphi^{-1} : \widehat{U} \to \mathbb{R}^m$.




$D\alpha$ having rank 2 means that $\frac{\partial \alpha}{\partial x}$ and $\frac{\partial \alpha}{\partial y}$ are independent vectors. Recall that $\frac{\partial \alpha}{\partial x}$ is tangent to the curve:

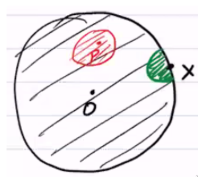
$$\gamma_1(t) = \alpha(a + te_1)$$

And $\frac{\partial \alpha}{\partial y}$ is tangent to the curve $\gamma_2(t) = \alpha(a + te_2)$. Then $\frac{\partial \alpha}{\partial x}$ and $\frac{\partial \alpha}{\partial y}$ span a 2-dimensional "tangent plane" to $M$ at the point $p = \alpha(a)$.

# III. Manifolds with boundary

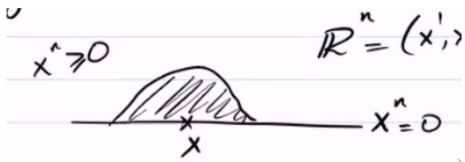
## III.1. Definitions

We saw that an open subset of a manifold $M$ (like $\mathbb{R}^n$) is also a manifold. What about closed subsets? For example, look at the closed unit ball $\overline{B(0,1)}$:




Then if we pick $x$ such that $|x| < 1$ then locally we look like and open subset of $\mathbb{R}^n$. However for $|x| = 1$, then locally near $x$, $M$ does not look like an open neighborhood of $\mathbb{R}^n$

In fact, what this looks like (standing at $x$) is an open subset of "half-space" $(x^1, x^2, \dots, x^n)$ with $x^n \geq 0$:




This motivates a few definitions:

**Definition III.1.1**

We define the closed $n$-dimensional upper half space

$$\mathbb{H}^n \subseteq \mathbb{R}^n = \{(x^1, \dots, x^n) \mid x^n \geq 0\}$$
$$\operatorname{Int} \mathbb{H}^n = \{(x^1, \dots, x^n) \mid x^n > 0\}$$
$$\partial \mathbb{H}^n = \{(x^1, \dots, x^n) \mid x^n = 0\}$$

Great!

**Definition III.1.2**

A topological manifold with boundary is a Hausdorff, second countable topological space such that each point $p \in M$ has a neighborhood $U$ that is homeomorphic to a (relatively) open subset of $\mathbb{H}^n$.

That is, there exists a homeomorphism $\varphi : U \to \widehat{U}$ where $\widehat{U} \subseteq \mathbb{H}^n$ is an open subset in the subspace topology on $\mathbb{H}^n$

$(U, \varphi)$ is called a coordinate chart. If $\widehat{U} \subseteq \operatorname{Int} \mathbb{H}^n$, this is called an interior chart, and otherwise it is a boundary chart.

Recall: $\widehat{U}$ is relatively open in $\mathbb{H}^n$ if and only if $\widehat{U} = V \cap \mathbb{H}^n$ where $V$ is open in $\mathbb{R}^n$. Two cases:

- We can have an interior chart $\widehat{U} \cap \partial \mathbb{H}^n = \emptyset$ then $\widehat{U}$ is open in $\mathbb{R}^n$
- Otherwise we have a boundary chart when $\widehat{U} \cap \partial \mathbb{H}^n \neq \emptyset$.

Warning: There can be points which have an interior and a boundary chart around them.

To define smooth structures on such manifolds with boundary we need to recall what it means for a function on an open subset $\widehat{U}$ of $\mathbb{H}^n$ to be smooth. This is clear when $\widehat{U}$ is open in $\mathbb{R}^n$ (i.e. $\widehat{U} \cap \partial H = \emptyset$). What about if $\widehat{U} \cap \partial \mathbb{H}^n \neq \emptyset$.

**Definition III.1.3**

We say that a function $f : \widehat{U} \to \mathbb{R}^m$ is smooth (or $C^r$) on a relatively open set $\widehat{U} \subseteq \mathbb{H}^n$ provided that there exists an open set $V \subseteq \mathbb{R}^n$ that contains $\widehat{U}$ and a function $\widetilde{f} : V \to \mathbb{R}^m$ that is smooth (or $C^r$) which extends $f$. Aka:

$$f = \widetilde{f}\Big|_{\widehat{U}}$$

In other words, $f$ admits a smooth (or $C^r$) extension

**Exercise III.1.1**

If $f$ is smooth on $\widehat{U}$ if and only if $f$ is smooth on $\operatorname{Int}\widehat{U}$ and $f$ and its derivatives are continuous on $\widehat{U}$. Where we take interior with respec to $\mathbb{R}^n$.

**Definition III.1.4** (Smooth manifold with boundary)

A smooth manifold with boundary is a topological manifold with boundary equipped with a smooth atlas, i.e. a collection of charts $(U_\alpha, \varphi_\alpha)$ that cover $M$ and such that the transition map:

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi(U_\alpha \cap U_\beta)$$

are smooth.

**Remark III.1.1**

As before, we identify manifolds with boundary that have equivalent atlases. Thus, strictly speaking, a smooth manifold with boundary is a topological manifold with boundary equipped with a maximal smooth atlas called a "smooth structure"

We recall that $(U_\alpha, \varphi_\alpha)$, $(V_\beta, \psi_\beta)$ are equivalent if and only their union is also an atlas, which holds if and only if every $(U_\alpha, \varphi_\alpha)$ and $(V_\beta, \psi_\beta)$ are smoothly compactible.

**Remark III.1.2**

Every smooth submanifold (definition II.2.2) is a smooth manifold with boundary. Ultimately, this is because $\mathbb{R}^n$ is diffeomorphic to $\operatorname{Int}\mathbb{H}^n$ via the map:

$$(x^1, x^2, \ldots, x^n) \mapsto (x^1, \ldots, x^{n-1}, e^{x^n})$$

and hence any chart $(U, \varphi)$ with $\varphi(U) \subseteq \mathbb{R}^n$ can be replaced by a chart $(U, \widetilde{\varphi})$ such that $\widetilde{\varphi(U)} \subseteq \operatorname{Int}\mathbb{H}^n$. This will form a smooth atlas for a manifold with boundary if we start with a smooth atlas for a manifold.

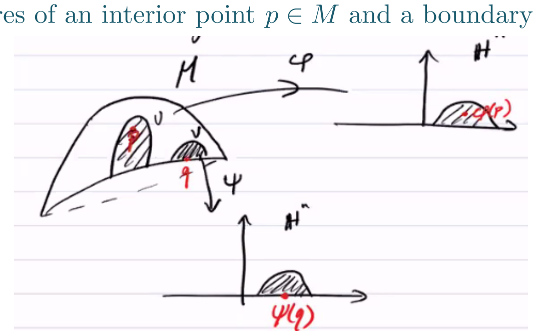
In other words, manifolds from the previous section are nothing but manifolds with boundary all of whose charts are interior charts.

**Definition III.1.5**

A point $p \in M$ is called a boundary point of $M$ if its image under some smooth chart $(U, \phi)$ is in $\partial\mathbb{H}^n$. We then call $\partial M$ the set of boundary points.

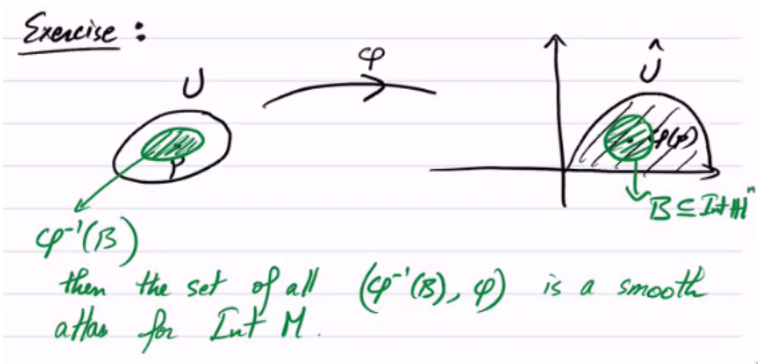
On the other hand a point $p \in M$ is called an interior point of $M$ if its image under some smooth chart $(U, \phi)$ is in $\operatorname{Int}\mathbb{H}^n$. We then call $\operatorname{Int} M$ the set of interior points.

Here are some nice pictures of an interior point $p \in M$ and a boundary point $q \in M$:




**Exercise III.1.2**

Let $M$ be an $n$-manifold with boundary. Then $\operatorname{Int} M$ is a $n$-manifold without boundary. Here is a picture to help with the proof:




**Exercise III.1.3**

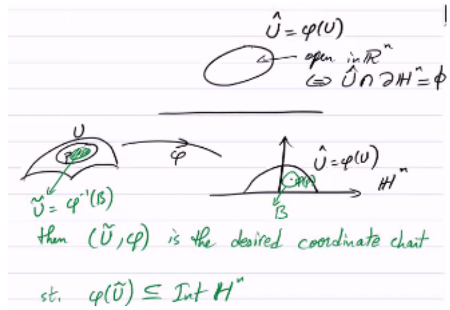
Show that the above definition is well-defined. That is, if you have an interior point (or boundary point) with respect to some chart $(U, \phi)$ then it is an interior point (or boundary point) with respect to any other chart $(V, \psi)$. We will also do this next time!

Last time we defined manifolds with boundary, and we can recall this at definition III.1.4
We also defined interior points, which we recall now at definition III.1.5.
In summary, we say:

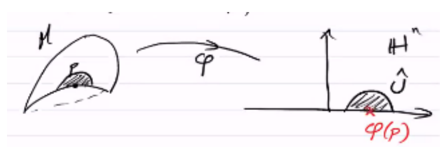
**Definition III.1.6**

$p \in M$ is an interior point if there exists a chart $(U, \varphi)$ such that $\varphi(p) \in \operatorname{Int} \mathbb{H}^n$. Equivalently, there exists a chart $(U, \varphi)$ such that $\varphi(U) \subseteq \operatorname{Int} \mathbb{H}^n$ or in other words $\varphi(U)$ is open in $\mathbb{R}^n$.

We include pictures:




A point $p \in M$ is called a boundary point (we say $p \in \partial M$) if there exists a coordinate chart $(U, \varphi)$ such that $\varphi(p) \in \partial \mathbb{H}^n$:




**Remark III.1.3**

These two definitions are mutually exclusive, in the sense that we cannot have two coordinate charts $(U, \varphi)$ and $(V, \psi)$ such that $p \in U \cap V$ and $\varphi(p) \in \operatorname{Int} \mathbb{H}^n$ and $\varphi(p) \in \partial \mathbb{H}^n$.

*Proof.* Why? As we discussed above, we may assume that $\varphi(U) \subseteq \operatorname{Int} \mathbb{H}^n$ by shrinking $U$ as needed by the above. Then look at $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$. We see that $\varphi(U \cap V) \subseteq \varphi(U) \subseteq \operatorname{Int} \mathbb{H}^n$ will be open in $\mathbb{R}^n$ since $\varphi(U \cap V)$ is also open in $\mathbb{H}^n$.

Since $\psi \circ \varphi^{-1}$ is a $C^r$ diffeomorphism on an open subset $\mathcal{O} = \varphi(U \cap V)$ of $\mathbb{R}^n$ and $D(\psi, \varphi^{-1})$ is non-singular. By the inverse function theorem we can conclude that $(\psi \circ \varphi^{-1})(\mathcal{O})$ is open in $\mathbb{R}^n$. This in fact shows that $\psi(U \cap V)$ is open in $\mathbb{R}^n$, so $\psi(U \cap V)$ does not intersect the boundary of $\mathbb{H}^n$. This leads to a contradiction if $\psi(p)$ lies on this boundary.

The conclusion is that a manifold $M$ with boundary is the disjoint union of its interior and its boundary

**Remark III.1.4**

$\operatorname{Int} M$ and $\partial M$ defined here might not be the same as the topological interior or boundary of $M$ in the case when $M$ is a subset of another topological space.

**Example III.1.4**

Take $M = \overline{B(0,1)} \subseteq \mathbb{R}^n$. In Homework 3 we will show that this is a manifold with boundary. In this case we have $\partial M = \mathbb{S}^{n-1}$

But if we regard $M$ as a topological space by itself, then the topological boundary of $M$ is the empty set.

In other words, smooth manifolds have a more intrinsic notion of boundary than the one we get from topology

**Theorem III.1.1**

Let $M$ be a smooth $n$-manifold with boundary. Then $\operatorname{Int} M$ is a smooth $n$-manifold without boundary. Similarly, if $\partial M \neq \emptyset$ then $\partial M$ is a smooth $(n-1)$-manifold without boundary.

*Proof.* The statement about Int $M$ we already proved by the picture above where we showed that if $p$ is an interior point then we can find some chart $(U, \varphi)$ around $p$ so that $\varphi(U)$ is an open subset of $\mathbb{R}^n$.

To show that $\partial M$ is a smooth $(n-1)$ manifold we first note that it inherits the Hausdorff and second countable properties from $M$. It remains to exhibit a smooth atlas.

Let $(U_\alpha, \varphi_\alpha)$ be a smooth atlas of $M$. Consider the charts $U_\alpha$ such that $U_\alpha \cap \partial M \neq \emptyset$ (equivalently $\widehat{U}_\alpha \cap \partial \mathbb{H}^n \neq \emptyset$.)

Let $V_\alpha = U_\alpha \cap \partial M$. Obviously $\{V_\alpha\}$ covers $\partial M$. We note that $\varphi : U_\alpha \to \widehat{U} \subseteq \mathbb{H}^n$, and $\varphi_\alpha$ maps $V_\alpha$ to the set $\widehat{U} \cap \partial \mathbb{H}^n = \widehat{U}_\alpha \cap \{x^n = 0\}$. Therefore, we can write $\varphi_\alpha\big|_{V_\alpha} = (\psi_\alpha, 0)$.

**Claim**

The collection of charts $(V_\alpha, \psi_\alpha)$ is a smooth atlas for $\partial M$.

From the above we see that $\psi_\alpha : V_\alpha \to \widehat{V}_\alpha = \psi_\alpha(V_\alpha)$, and $\psi_\alpha(V_\alpha) = \widehat{U}_\alpha \cap \{x^n = 0\}$, and so since $\widehat{U}_\alpha$ is open in $\mathbb{H}^n$ we know that $\widehat{V}_\alpha$ will be an open subset of $\mathbb{R}^{n-1}$

$\psi_\alpha$ is continuous, since it is the restriction of a continuous function. Also note that $\psi_\alpha^{-1}$ is the restriction of $\varphi_\alpha^{-1}$ to $(\widehat{V}_\alpha, 0)$, and so it is also continuous, hence $\psi_\alpha$ is a homeomorphism.

It remains to show that for any $\alpha, \beta$ such that $V_\alpha \cap V_\beta \neq \emptyset$ that the map $\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(V_\alpha \cap V_\beta) \to \psi_\alpha(V_\alpha \cap V_\beta)$ is a smooth diffeomorphism. We note that:

$$\psi_\beta^{-1} = \varphi_\beta^{-1}\Big|_{(\widehat{V}_\beta, 0)} \varphi_\alpha \circ \psi_\beta^{-1} \qquad \subseteq \widehat{U}_\alpha \cap \{x^n = 0\} = (\widehat{V}_\alpha, 0)$$

$$(\psi_\alpha \circ \psi_\beta^{-1}, 0) = \varphi_\alpha \circ \psi_\beta^{-1} = \varphi_\alpha \circ \varphi_\beta^{-1}\Big|_{(\widehat{V}_\beta, 0)}$$

Since $\psi_\beta^{-1}(x)$ always lies in $\partial M$. This shows that $\psi_\alpha \circ \psi_\beta^{-1}$ is the restriction of a smooth function, and therefore it is smooth itself. The same holds for $\psi_\beta \circ \psi_\alpha^{-1}$, and so $\psi_\alpha \circ \psi_\beta^{-1}$ is a smooth diffeomorphism as well.

## III.2. Submanifolds of $\mathbb{R}^d$ with boundary

**Definition III.2.1**

Let $M \subseteq \mathbb{R}^d$. We call $M$ a $C^r$-submanifold of $\mathbb{R}^d$ with boundary of dimension $n$ provided that that for each $p \in M$ there exists an open set $U \subseteq M$ and a map $\varphi : U \to \mathbb{H}^n$ so that:

a) $\varphi$ is a homeomorphism from $U$ onto $\widehat{U} = \varphi(U)$ which is an open subset of $\mathbb{H}^n$
b) $\varphi^{-1} : \widehat{U} \to \mathbb{R}^d$ is of class $C^r$ and $D\varphi^{-1}$ has rank $n$
c) If $(V, \psi)$ is another chart such that $U \cap V \neq \emptyset$ then:

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is a $C^r$ (or smooth) diffeomorphism.

Great! Of course if we take $r = \infty$ we get a smooth submanifold of $\mathbb{R}^d$ with boundary

**Remark III.2.1**

By part c) any $C^r$-submanifold of $\mathbb{R}^d$ is a $C^r$-manifold with boundary

The following propositions tells us that condition c) is actually redundant. In fact a), b) $\implies$ c) and also a), c) $\implies$ b)

**Proposition III.2.1**

Suppose $M \subseteq \mathbb{R}^d$ satisfies conditions a) and b) in the above definition. Then condition c) is automatically satisfied.

*Proof.* We have to show that if $(V, \psi)$ and $(U, \varphi)$ are two coordinate patches satisfying $V \cap U \neq \emptyset$, then the map:

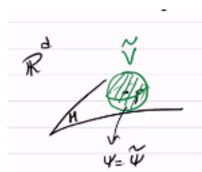
$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

Is a $C^r$-diffeomorphism. Since both $\psi$ and $\varphi$ are homeomorphisms we know that $\psi \circ \varphi^{-1}$ is a homeomorphism. It is then sufficient to show that $\psi \circ \varphi^{-1}$ is of class $C^r$, since the same argument will apply to $\varphi \circ \psi^{-1}$, which is the inverse function, showing that $\psi \circ \varphi^{-1}$ is a $C^r$-diffeomorphism.

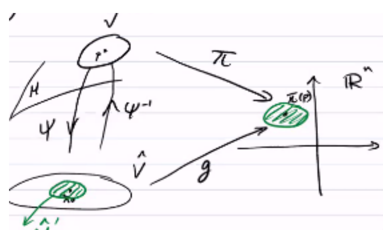
We will show this by working locally for every point lying in $\varphi(U \cap V)$ there is an open neighborhood of that point on which $\psi \circ \varphi^{-1}$ is a $C^r$ function.

We know that $\varphi^{-1} : \widehat{U} \to \mathbb{R}^d$ is of class $C^r$. It is sufficient to show that for each $p \in V$ there exists an open neighborhood $\widetilde{V}_p \subseteq \mathbb{R}^d$ and a $C^r$ function $\widetilde{\psi} : \widetilde{V}_p \to \mathbb{R}^n$ such that $\psi$ is the restriction of $\widetilde{\psi}$ to the set $V \cap \widetilde{V}_p$. This would imply that $\psi \circ \varphi^{-1} = \widetilde{\psi} \circ \varphi^{-1}$, which will be a composition of two $C^r$ functions, so it must have been $C^r$. Here we have $\psi \circ \varphi^{-1} : \phi(U \cap V \cap \widetilde{V}_p) \to \psi(U \cap V \cap \widetilde{V}_p)$. In this case, we know $\phi(U \cap V \cap \widetilde{V}_p)$ will be a neighborhood of $p$ if $p \in U \cap V$.

Here is a picture:




To prove that, we argue as follows. Let $x_0 = \psi(p)$. Since $D\psi^{-1}(x_0)$ has rank $n$, then $D\psi^{-1}(x_0)$ has $n$ linearly independent rows. Let us say, for the sake of concreteness, that these are the first $n$ rows. Let $\pi : \mathbb{R}^d \to \mathbb{R}^n$ be the projection on the first $n$ coordinates. Consider then $g = \pi \circ \psi^{-1} : \widehat{V} \to \mathbb{R}^n$ is a $C^r$-function such that $Dg(x_0)$ is invertible, since it has the first $n$ linearly independent rows of $D\psi^{-1}(x_0)$. Here's a nice picture again:




By the inverse function theorem, there exists an open neighborhood $\widehat{V}'$ of $x_0$ and a neighborhood $S$ of $\pi(p)$ so that $g$ is a $C^r$ diffeomorphism from $\widehat{V}' \to S$. This means that $g^{-1} : S \to \widehat{V}'$ exists and is $C^r$ and also $\psi = g^{-1} \circ \pi$ on $V \cap \pi^{-1}(S)$. But $g^{-1} \circ \pi$ is defined as a smooth function on all of $\pi^{-1}(S)$, which is open in $\mathbb{R}^d$. Now taking $\widetilde{V}_p = \pi^{-1}(S)$ and $\widetilde{\psi} = g^{-1} \circ \pi$ gives the needed claim.

**Back to Lecture!**

We recall the definition of submanifolds of $\mathbb{R}^d$ given at Definition III.2.1. For convenience we restate this definition:

**Definition III.2.2**

Let $M \subseteq \mathbb{R}^d$. We call $M$ a $C^r$-submanifold of $\mathbb{R}^d$ with boundary of dimension $n$ provided that that for each $p \in M$ there exists an open set $U \subseteq M$ and a map $\varphi : U \to \mathbb{H}^n$ so that:

a) $\varphi$ is a homeomorphism from $U$ onto $\widehat{U} = \varphi(U)$ which is an open subset of $\mathbb{H}^n$
b) $\varphi^{-1} : \widehat{U} \to \mathbb{R}^d$ is of class $C^r$ and $D\varphi^{-1}$ has rank $n$
c) If $(V, \psi)$ is another chart such that $U \cap V \neq \emptyset$ then:

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is a $C^r$ (or smooth) diffeomorphism.

Great! Of course if we take $r = \infty$ we get a smooth submanifold of $\mathbb{R}^d$ with boundary. We also have that condition c) is redundant in the presence of a) and b).

**Theorem III.2.2**

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}$ be of class $C^r$. Let $M$ be the set of points $\{x \mid f(x) = 0\}$ and $N$ be the set of points $\{x \mid f(x) \geq 0\}$.

Now suppose that $M$ is non-empty and $Df(x) \neq 0$ at each point $x \in M$. Then $N$ is an $n$-dimensional $C^r$-submanifold of $\mathbb{R}^n$ with boundary and $\partial N = M$. In particular $M$ is an $(n-1)$ dimensional $C^r$-submanifold of $\mathbb{R}^n$.

**Example III.2.1**

Let $f(x) = 1 - |x|^2 : \mathbb{R}^n \to \mathbb{R}$. Then $N = \{f(x) \geq 0\}$ is nothing but the ball $\overline{B(0,1)}$ and $M = \partial N = \{x \mid f(x) = 0\} = \{x \mid |x| = 1\} = \mathbb{S}^{n-1}$. This gives another manifestation of the smooth structure of $\mathbb{S}^{n-1}$ as an $(n-1)$-submanifold of $\mathbb{R}^n$.

*Proof.* Suppose $p \in N$ is a point such that $f(p) > 0$. Then let $U = \{x \mid f(x) > 0\}$, which is open by continuity of $f$. So we may consider the chart $(U, \mathrm{Id})$, which clearly satisfies conditions a) and b), so we are done.

Now we handle the other case and suppose that $f(p) = 0$. Since $Df(p) \neq 0$ there exists a coordinate $x^n$ such that $\frac{\partial f}{\partial x^n}(p) \neq 0$. We then consider the function $F : \mathcal{O} \to \mathbb{R}^n$ given by:

$$F(x^1, \ldots, x^n) = (x^1, \ldots, x^{n-1}, f(x))$$

Then we have that:

$$DF(p) = \begin{pmatrix} I_{n-1} & 0 \\ \frac{\partial f}{\partial(x^1, \ldots, x^{n-1})} & \frac{\partial f}{\partial x^n} \end{pmatrix}$$

This is nonsingular since $\det DF(p) = \frac{\partial f}{\partial x^n}(p) \neq 0$¿ Hence by the inverse function theorem there is an open set $p \in A \subseteq \mathbb{R}^n$ and another open set $B \subseteq \mathbb{R}^n$ such that $F : A \to B$ is a $C^r$-diffeomorphism. Notice that $A \cap = A \cap \{x \mid f(x) \geq 0\}$ is relatively open in $N$ and contains $p$. Also with $\varphi := F\big|_{A \cap N}$ is a coordinate map from $A \cap N$ onto $B \cap \mathbb{H}^n$.

Why? If $q \in A \cap N$ then $F(q) \in B$ and $f(q) \geq 0$, so $F(q) \in B \cap \{x \mid x^n \geq 0\} = B \cap \mathbb{H}^n$. Thus $\varphi$ maps $A \cap N$ into $B \cap \mathbb{H}^n$. $\varphi$ is bijective since $F$ is so. It also satisfies condition b) since $\varphi^{-1} = F^{-1}\big|_{B \cap \mathbb{H}^n}$ is of class $C^r$ since $F$ is a $C^r$-diffeomorphism on $A$, and it satifies $D\varphi^{-1}$ has rank $n$(since $D\varphi^{-1} = DF^{-1}$ by the above applicatin of the IFT).

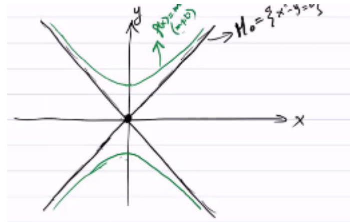
Therefore, $N$ is an $n$-dimensional $C^r$-submanifold of $\mathbb{R}^n$ and $\partial N = \{f(x) = 0\} = M$.

**Remark III.2.2**

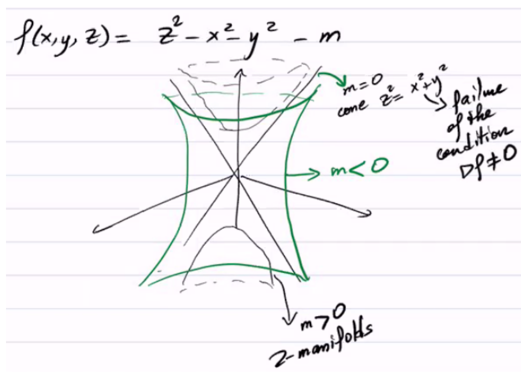
We are often interested in the set $\{x \in \mathbb{R}^n \mid f(x) = 0\}$ where $f : \mathbb{R}^n \to \mathbb{R}$.. The theorem tells you that this is an $(n-1)$-dimensional $C^r$-submanifold provided that$Df(p) \neq 0$ for every $p \in M$.

**Example III.2.2**

Let $f(x, y) = x^2 - y^2 - m$. The set $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = m\}$ is a 1-dimensional submanifold of $\mathbb{R}^2$ provided that $Df(x, y) = 2(x - y) \neq 0$. I.e., provided that $(x, y) = (0, 0) \notin M$.




You can also do $f(x, y, z) = z^2 - x^2 - y^2 - m$ and you get:




What about the intersection of such hyperboloids with spheres? Do I get a one-dimensional manifold out of it? The intersection is the zero set of two functions $f = 0$ and $g(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$.

Equivalently, the question becomes the following: Suppose $F : \mathbb{R}^n \to \mathbb{R}^k$ is a $C^r$ function. When does the set $\{x \in \mathbb{R}^n \mid F(x) = 0\}$ define an $(n - k)$-dimensional $C^r$-submanifold? For the answer to that, see HW4.

**Non-Example III.2.3**

We have $F(x, y) = x^3$. Then $DF(x, y) = (3x^2, 0) = 0$ at $x = 0$, but $\{(x, y) \mid F(x, y) = 0\}$ is just the $y$-axis, which is a manifold. I.e., the converse of the theorem does not hold.

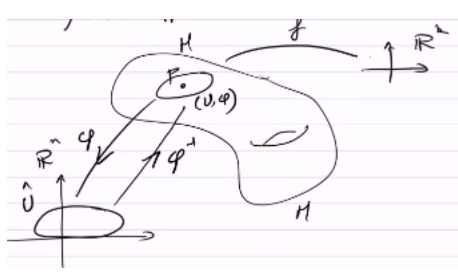
# IV. Smooth Maps

Some say that we define smooth manifolds in order to study smooth maps. This turns out to be the case

## IV.1. Definitions and Whitney's Embedding Theorem

**Definition IV.1.1**

Let $M$ be a smooth $n$-manifold. A function $f : M \to \mathbb{R}^k$ is said to be smooth if for every $p \in M$ there existsa chart $(V, \psi)$ for $M$ whose domain contains $p$ and such that $f \circ \varphi^{-1}$ is smooth on the open subset $\widehat{U} = \varphi(U)$ of $\mathbb{R}^n$ or $\mathbb{H}^n$.




Similarly one defines a $C^r$-map from a $C^r$-manifold into $\mathbb{R}^k$.

**Remark IV.1.1**

Some nice things:

1) The map $\widehat{f} := f \circ \varphi^{-1} : \widehat{U} \to \mathbb{R}^k$ is called the coordinate representation of $f$. I.e., $f$ is smooth if and only if its coordinate representation is smooth in some chart around each point.
2) If $f : M \to \mathbb{R}^k$ is smooth, then $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$ is smooth for every chart $(V, \psi)$ on $M$. I.e., smoothness is independent of the choice of coordinate chart.

   *Proof.* We prove this locally, since smoothness is a local property. Let $x_0 \in \widehat{V} = \psi(V)$ and $p_0 = \psi^{-1}(x_0)$. By the definition of smoothness there exists a coordinate chart $(U, \varphi)$ around $p_0$ such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth.

   Note that $p_0 \in U \cap V$, implies that $x_0 \in \psi(U \cap V)$. For $x \in \psi(U \cap V)$ we have:

   $$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1})$$

   And so this is smooth via compatibility and composition of smooth functions. Then $f \circ \psi^{-1}$ is smooth on $\psi(U \cap V)$ which contains $x_0$. Since smoothness is a local property we then also have $f \circ \psi^{-1}$ is smooth on $\psi(V)$.
3) As a corollary, every coordinate chart $(U, \varphi)$ where $\varphi : U \to \widehat{U} \subseteq \mathbb{R}^n$ is a smooth map on the open submanifold $U$ of $M$.

**Example IV.1.1**

Let $f(x, y) = x^2 + y^2$ on $\mathbb{R}^2$. Using polar coordinates on the open set $U = \{(x, y) \mid x > 0\}$, $f$ has the coordinate representation $f(r, \theta) = r^2$.

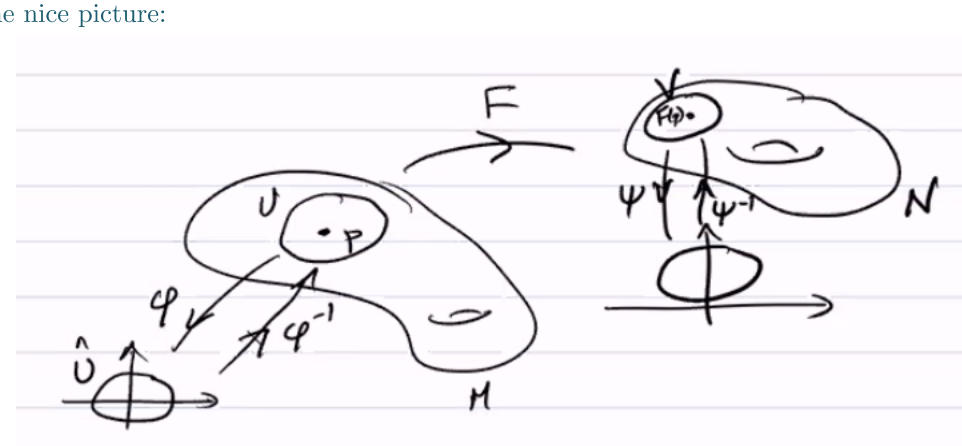
(Often we don't distinguish $f$ and $\widehat{f}$ and just say that $f(r, \theta) = r^2$ in polar coordinates on $U$)

**Definition IV.1.2**

Let $M$ and $N$ be smooth manifolds and let $F : M \to N$ be a map. We say $F$ is a smooth map if for every $p \in M$, there exists a coordinate chart $(U, \varphi)$ around $p$ and a coordinate chart $(V, \psi)$ on $N$ with $f(p) \in V$ such that:

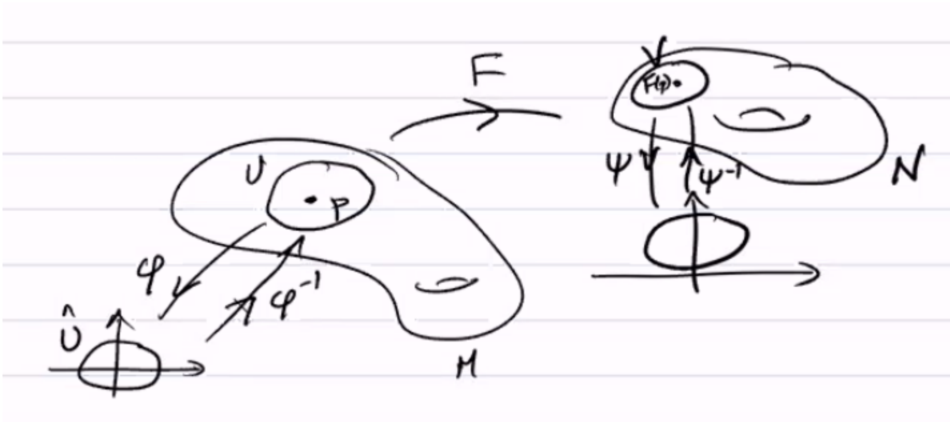
- $f(U) \subseteq V$
- $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

Here's the nice picture:



### Continue Discussion of Smooth Maps between Manifolds

Here's a picture for our discussion:




We now recall Definition IV.1.2 of a smooth map.

Recall: A function $F : M \to N$ is said to be smooth provided that for every $p \in M$ we can find a coordinate chart $(U, \varphi)$ around $p$ and another coordinate chart $(V, \psi)$ on $N$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1} : \widehat{U} \to \widehat{V}$ is smooth for every choice of charts around $p$ and $F(p)$.

**Exercise IV.1.2**

Show that $F : M \to N$ is smooth if and only if for each $p \in M$ there exists a neighborhood $U$ such that $F\big|_U$ is smooth.

**Proposition IV.1.1**

Every smooth map is continuous.

*Proof.* Let $(U, \varphi)$ and $(V, \psi)$ be as in the definition. Then consider that

$$F\big|_U = \psi^{-1} \circ \underbrace{(\psi \circ F \circ \varphi^{-1})}_{\text{smooth}} \circ \varphi$$

And so since $F\big|_U$ is continuous since it is a composition of continuous functions. Since continuity is a local property this means that $F$ is continuous.

**Exercise IV.1.3**

Let $M, N,$ and $P$ be smooth manifolds and $F : M \to N$ and $G : N \to P$ be smooth maps. Then $G \circ F : M \to P$ is also smooth.

**Example IV.1.4**

Consider the inclusion map $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$. This is a smooth map because its coordinate representation w.r.t. the charts $(U_i^{\pm}, \varphi^{\pm})$ looks like:

$$i \circ (\varphi_i^{\pm})^{-1}(u_1, \dots, u_n) = (u_1, \dots, u_{i-1}, \pm\sqrt{1 - |u|^2}, u_i, \dots, u_n)$$

This is clearly a smooth map from $B(0, 1)$ into $\mathbb{R}^{n+1}$.

**Example IV.1.5**

The quotient map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is also smooth. Again, using the same charts as in HW1 for $\mathbb{RP}^n$ we have:

$$\varphi_i \circ \pi(x^1, \dots, x^{n+1}) = \varphi_i([x^1, \dots, x^{n+1}]) = \frac{(x^1, \dots, \widehat{x}_i, \dots, x^{n+1})}{x^i}$$

This is obviously a smooth map from $\mathbb{R}^{n+1} \setminus \{x^i = 0\}$ to $\mathbb{R}^n$ so we're done.

Therefore $\pi$ is a smooth map

**Example IV.1.6**

Let $M$ be a submanifold of $\mathbb{R}^d$. Consider the inclusion map $\iota : M \to \mathbb{R}^d$. Then $\iota$ is smooth. Take a point $p \in M$ and a coordinate patch $(U, \varphi)$. Then $\iota \circ \varphi^{-1} = \varphi^{-1}$. By our assumptions of a submanifold, this is a smooth function $\widehat{U} \to \mathbb{R}^d$, by part b) of the submanifold definition.

**Example IV.1.7**

The map $\widetilde{\pi} : \mathbb{S}^n \to \mathbb{RP}^n$ given by $\widetilde{\pi} = \pi\big|_{\mathbb{S}^n}$ is also smooth. This holds because it can be expressed as the composition of two smooth maps since $\widetilde{\pi} = \pi \circ \iota$ where $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the inclusion map.

**Definition IV.1.3**

A diffeomorphism between two manifolds $M$ and $N$ is a smooth bijection whose inverse is also smooth. We say that $M$ and $N$ are diffeomorphic provided that such a map exists.

**Exercise IV.1.8**

Show that if $M$ is diffeomorphic to $N$, then $\dim M = \dim N$.

**Example IV.1.9**

Let $F : B(0,1) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be given by $F(x) = \frac{x}{1-|x|^2}$. Then $F$ is a diffeomorphism.

Clearly, $F$ is smooth. To show that $F$ is a bijection we show that the equation$F(x) = y$ has a unique solution for each $y$. This is clear for $y = 0$, so let $y \neq 0$. Write $y = \rho\omega_1$ where $\rho = |y| > 0$ and $\omega_1 = \frac{y}{|y|}$, and likewise let $x = r\omega_2$ where $r = |x| > 0$ and $\omega_2 = \frac{x}{|x|}$. Then:

$$F(x) = \frac{r}{1-r^2} \cdot \omega_1 = \rho \cdot \omega_2$$

Therefore $\omega_1 = \omega_2$, and $\frac{r}{1-r^2} = \rho$. Then note that the function $r \mapsto \frac{r}{1-r^2}$ is one-to-one and onto from $(0,1)$ to $(0,\infty)$. Therefore $F$ is a bijection.

It remains to show that $F^{-1}$ is smooth via the inverse function theorem. We can see that $DF(x)$ is nonsingular for every $x \in B(0,1)$.

Differential topology deals with properties of manifolds that are invariant under diffeomorphisms. (i.e. if $M$ is diffeomorphic to $N$ and $M$ has property $P$, then $N$ also has property $P$).

**Theorem IV.1.2** (Whitney's Embedding Theorem)

Whitney's embeding theorem tells us that for any abstract $n$-manifold $M$, there exists a submanifold $\widetilde{M}$ of $\mathbb{R}^d$ (with $d \leq 2n$) such that $M$ is diffeomorphic to $\widetilde{M}$.

*Proof.* Take 591.

We will mostly be restricting ourselves to studying submanifolds of $\mathbb{R}^d$ for the remainder of this course. Whitney's embedding theorem tells us that there is no loss of generality from a theoretical point of view.

**Part** B. **Differential Forms**

# V. **Multilinear Algebra**

## V.1. **Multilinear Forms**

We develop the algebraic framework of differential forms first.

**Definition V.1.1**

Let $V$ be a vector space over $\mathbb{R}$ and denote by $V^k = \underbrace{V \times \cdots \times V}_{k\text{-times}}$, and let $W$ be a vector space.

(i) A function $f : V^k \to W$ is said to be linear in the $i$-th variable provided that, given fixed vectors $v_j$ for $j \neq i$, the function:

$$T : V \to WT(v) \qquad = f(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_n)$$

is a linear transformation.

(ii) The function $f$ is called multilinear provided that it is linaer in every variable. In the case where $W = \mathbb{R}$ such functions are called $k$-tensors or tensors of order $k$

**Remark V.1.1**

We have a few interesting thigns to note:

1) A tensor of order 1 is also called a linear functional
2) The space of $k$-tensors will be denoted by $\mathscr{L}^k(V)$. Note that $\mathscr{L}^1(V) = V^*$, the dual space of $V$. More generally the space of $k$-multilinear functions from $V^k$ to $W$ is denoted $\mathscr{L}^k(V, W)$.

**Theorem V.1.1**

We have a few nice things:

a) $\mathscr{L}^k(V, W)$ is always vector space over $\mathbb{R}$ if we define:

$$(f + g)(v_1, \ldots, v_k) = f(v_1, \ldots, v_k) + g(v_1, \ldots, v_k)$$
$$(cf)(v_1, \ldots, v_k) = cf(v_1, \ldots, v_k)$$

b) let $B = \{a_1, \ldots, a_n\}$ be a basis of $V$. If $f, g : V^k \to W$ are two $k$-multilinear maps such that:

$$f(a_{i_1}, \ldots, a_{i_k}) = g(a_{i_1}, \ldots, a_{i_k})$$

for every $k$-tuple $(i_1, \ldots, i_k)$ of integers from the set $\{1, \ldots, n\}$, then we have $f = g$.

c) Let $I = (i_1, \ldots, i_k)$ be a fixed tuple of integers from the set $\{1, \ldots, n\}$. Then there exists a unique $k$-tensor $\Phi^I$ such that:

$$\Phi^I(a_{i_1}, \ldots, a_{i_k}) = 1$$
$$\Phi^I(a_{j_1}, \ldots, a_{j_k}) = 0$$

Whenever $(j_1, \ldots, j_k) \neq (i_1, \ldots, i_k)$. The set of all such $\Phi^I$ where $I$ ranges over such $k$-tuples is a basis of $\mathscr{L}^k(V)$. These are called elementary tensors relative to the basis $B$. In particular $\mathscr{L}^k(V)$ has dimension $n^k$.

We can use a similar technique to construct a basis for $\mathscr{L}^k(V, W)$.

**Exercise V.1.1**

Prove parts a) and b). We will give the proof of part c) in class.

Last time, we defined $k$-tensors on a vector space $V$. A $k$-tensor is a function $f : V^k \to \mathbb{R}$ that is multilinear (i.e. it is linear in each of the components when we fix all other components). See Definition V.1.1. We denoted the spae of $k$-tensors by $\mathscr{L}^k(V)$, and so $\mathscr{L}^1(V)$ is just the dual space $V^*$. We stated Theorem V.1.1 last time, and left parts a) and b) as exercises:

**Theorem V.1.2**

We have a few nice things:

a) $\mathscr{L}^k(V, W)$ is always vector space over $\mathbb{R}$ if we define:

$$(f + g)(v_1, \ldots, v_k) = f(v_1, \ldots, v_k) + g(v_1, \ldots, v_k)$$
$$(cf)(v_1, \ldots, v_k) = cf(v_1, \ldots, v_k)$$

b) let $B = \{a_1, \ldots, a_n\}$ be a basis of $V$. If $f, g : V^k \to W$ are two $k$-multilinear maps such that:

$$f(a_{i_1}, \ldots, a_{i_k}) = g(a_{i_1}, \ldots, a_{i_k})$$

for every $k$-tuple $(i_1, \ldots, i_k)$ of integers from the set $\{1, \ldots, n\}$, then we have $f = g$.

c) Let $I = (i_1, \ldots, i_k)$ be a fixed tuple of integers from the set $\{1, \ldots, n\}$. Then there exists a unique $k$-tensor $\Phi^I$ such that:

$$\Phi^I(a_{i_1}, \ldots, a_{i_k}) = 1$$
$$\Phi^I(a_{j_1}, \ldots, a_{j_k}) = 0$$

Whenever $(j_1, \ldots, j_k) \neq (i_1, \ldots, i_k)$. The set of all such $\Phi^I$ where $I$ ranges over such $k$-tuples is a basis of $\mathscr{L}^k(V)$. These are called elementary tensors relative to the basis $B$. In particular $\mathscr{L}^k(V)$ has dimension $n^k$.

We can use a similar technique to construct a basis for $\mathscr{L}^k(V, W)$.

*Proof of part c).* Let $\phi^1, \ldots, \phi^n$ be the dual basis of $V^*$ defined by:

$$\phi^i(v) = \phi^i\left(\sum_{j=1}^{n} \alpha_j a_j\right) = \alpha_i$$

In other words, we have that:

$$\phi^i(a_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

For $k \geq 2$ if $I = (i_1, \ldots, i_k)$, then we set:

$$\Phi^I(v_1, \ldots, v_k) = \phi^{i_1}(v_1) \cdots \phi^{i_k}(v_k)$$

Clearly, this is a $k$-tensor that satisfies the conditions of the theorem.

To show that these $\Phi^I$ form a basis of $\mathscr{L}^k(V)$, let $g \in \mathscr{L}^k(V)$ and for any $k$-tuple $I = (i_1, \ldots, i_k)$ from the set $\{1, \ldots, n\}$, let:

$$d_I = g(a_{i_1}, \ldots, a_{i_k})$$

Then we define:

$$\widetilde{g} = \sum_I d_I \Phi^I$$

Then $g$ and $\widetilde{g}$ satisfy the conditions of part b), and thus $g = \widetilde{g}$. Thus the set $\{\Phi^I\}_I$ is a basis.

Now we just note that the number of $k$-tuples of $\{1, \ldots, n\}$ is $n \times n \times \cdots \times n = n^k$.

**Example V.1.2**

Let $V = \mathbb{R}^n$ and $e_1, \ldots, e_n$ be the standard basis, and let $\phi^1, \ldots, \phi^n$ be the dual basis. Then given a vector $v = v^i e_i$ (Einstein notation). We have $\phi^j(v) = \phi^j(v^i e_i) = v^i \phi^j(e_i) = v^i \delta^j_i = v^j$.

Given $I = (i_1, \ldots, i_k)$ the elementary tensor $\Phi^I$ satisfies:

$$\Phi^I(v_1, \ldots, v_k) = \phi^{i_1}(v_1) \cdots \phi^{i_k}(v_k)$$

And so if $v_\ell = x^i_\ell e_i$ then:

$$\Phi^I(v_1, \dots, v_k) = x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$$

This is a monomial of degree $k$ in the components of $v_1, \dots, v_k$. Any general $k$-tensor is a linear combination of such monomials. For example, a general 2-tensor looks like:

$$g(v, w) = \sum_{i,j=1}^{n} d_{ij} x^i y^j$$

Where $v = \sum x^i e_i$ and $w = \sum y^j e_j$.

## V.2. The Tensor Product

**Definition V.2.1**

Let $f$ be a $k$-tensor and $g$ be an $\ell$-tensor on $V$. We define $f \otimes g$ as the $(k+\ell)$-tensor defined by:

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k) g(v_{k+1}, \dots, v_{k+\ell})$$

**Exercise V.2.1**

Check that this is a tensor, and it defines a multilinear map $\mathscr{L}^k(V) \times \mathscr{L}^\ell(V) \to \mathscr{L}^{k+\ell}(V)$.

**Theorem V.2.1**

Let $f, g, h$ be tensors on $V$. Then the following holds:

(1) $f \otimes (g \otimes h) = (f \otimes g) \otimes h$
(2) $(cf) \otimes g = c(f \otimes g) = f \otimes (cg)$
(3) If $f$ and $g$ have the same order then $(f+g) \otimes h = f \otimes h + g \otimes h$ and $h \otimes (f+g) = h \otimes f + h \otimes g$.
(4) Given a basis $a_1, \dots, a_n$ of $V$, then the elementary $k$-tensors $\Phi^I$ satisfy:

$$\Phi^I = \phi^{i_1} \otimes \phi^{i_2} \otimes \cdots \otimes \phi^{i_k}$$

With this, any $k$-tensor is a linear combination of tensor products of 1-tensors.

**Exercise V.2.2**

Proof is left as an exercise!

The action of a linear transformation

**Definition V.2.2**

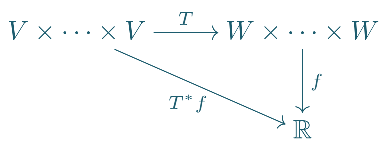
Suppose we have a linear map $T : V \to W$. $T$ allows us to pull-back $k$-tensors on $W$ into $k$-tensors on $V$ by composition. One can define a dual transformation (called pullback operation) $T^* : \mathscr{L}^k(W) \to \mathscr{L}^k(V)$ by defining for all $f \in \mathscr{L}^k(W)$:

$$T^* f = f \circ T$$

More explicitly, given $(v_1, \dots, v_k \in V^k$ we have:

$$T^* f(v_1, \dots, v_k) = f(Tv_1, \dots, Tv_k)$$

In yet other words, we have the following commutative diagram:




Great!

**Proposition V.2.2**

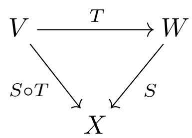
Let $T : V \to W$ be a linear transformation. Let $T^* : \mathscr{L}^k(W) \to \mathscr{L}^k(V)$ be the pullback operator, then:

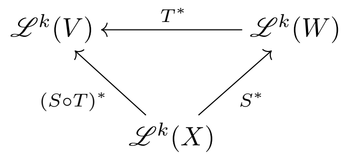
1) $T^*$ is linear

2) $T^*(f \otimes g) = (T^* f) \otimes (T^* g)$
3) If $S : W \to X$ is a linear transformation then:

$$(S \circ T)^* f = T^* S^* f$$

That is we have:




And these diagrams both commute as desired

**Exercise V.2.3**
Prove this

## V.3. **Alternating Tensors**

**Definition V.3.1**
Let $k \geq 2$. A permutation of the set of integers $\{1, \ldots, k\}$ is a one-to-one and onto mapping from $\{1, \ldots, k\}$ to itself. We denote the set of all permutations by $S_k$.
If $\sigma$ and $\tau$ are elements of $S_k$, so are $\sigma \circ \tau$ and $\sigma^{-1}$. This makes $S_k$ into a group, called the symmetric group (or the permutation group). There are $k!$ elements in this group.

**Definition V.3.2**
Given $1 \leq i \leq k$, let $e_i$ be the element of $S_k$ defined by setting $e_i(j)$ if $j \neq i, i+1$, $e_i(i) = i+1$, and $e_i(i+1) = i$. We call $e_i$ an elementary permutation (it permutes $i$ and $i+1$).

**Lemma V.3.1**
Any $\sigma \in S_k$ is the composite of alternating permutations.

**Exercise V.3.1** (On Homework)
Prove this

**Definition V.3.3**
Let $\sigma \in S_k$. Consider the set of all pairs of integers $(i, j)$ from $\{1, \ldots, k\}$ such that $i < j$ but $\sigma(i) > \sigma(j)$. The pair $(i, j)$ is called an inversion of $\sigma$. Let $p$ be the number of such couples. Then, the sign of $\sigma$ is defined as $\operatorname{sgn} \sigma = (-1)^p$.
If $p$ is odd, we say that $\sigma$ is an odd permutation, and if $p$ is even, we sa that $\sigma$ is an even permutation

**Proposition V.3.2**
Let $\sigma, \tau \in S_k$. Then:
(a) If $\sigma$ is the composite of $m$ elementary permutations, then:

$$\operatorname{sgn} \sigma = (-1)^m$$

(b) $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn} \sigma \cdot \operatorname{sgn} \tau$
(c) $\operatorname{sgn} \sigma^{-1} = \operatorname{sgn} \sigma$
(d) If $i \neq j$, and $\sigma$ is the permutation that only exchanges $i$ and $j$, leaving all other integers in $\{1, \ldots, k\}$ fixed, then $\operatorname{sgn} \sigma = -1$.

**Exercise V.3.2** (On Homework)
Prove this.

**Definition V.3.4**

Let $f$ be an arbitrary $k$-tensor on $V$. If $\sigma$ is a permutation of $\{1, \ldots, k\}$ we define:

$$f^\sigma(v_1, \ldots, v_k) = f(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)})$$

Because $f$ is linear in all of its arguments, so is $f^\sigma$, so $f^\sigma$ is a $k$-tensor as well.

A $k$-tensor $f$ is called symmetric provided that $f^\sigma = f$ for all $\sigma \in S_k$. It is said to be alternating provided that $f^\sigma = \operatorname{sgn}\sigma \cdot f$ for all $\sigma \in S_k$. Equivalently $f^e = -f$ for any non-trivial elementary permutation $e \in S_k$.

**Definition V.3.5**

Let $V$ be a vector space. We denote the set of alternating tensors by $\mathcal{A}^k(V)$. It is easy to check that this is a subspace of $\mathscr{L}^k(V)$.

Since the condition that a 1-tensor be alternating is vacuous, we adopt that convention that $\mathcal{A}^1(V) = \mathscr{L}^1(V) = V^*$.

**Example V.3.3**

We have the following nice examples:

1) Elementary tensors $\Phi^I$ in general are not alternating. For example, if $I = (1, 2)$ then $\Phi^I(v, w) = \phi^(v)\phi^2(w) = v^1 w^2$. Whereas $\Phi^{(2,1)}(v, w) = v^2 w^1$. But of course $\Phi^{(2,1)} \neq -\Phi^{(1,2)}$.

   However, note that we can define:

$$\Phi^{\text{alt}} = \Phi^{(1,2)} - \Phi^{(2,1)} = v^1 w^2 - v^2 w^1 = \det \begin{pmatrix} v^1 & v^2 \\ w^1 & w^2 \end{pmatrix}$$

   Which is alternating. Also note that $\phi^{(2,1)} = \left(\phi^{(1,2)}\right)^\sigma$ where $\sigma$ is the swapping permutation of $(1, 2)$. This kind of construction will be generalized next time.

Last time we were talking about alternating tensors. We've already seen an interesting example:

**Example V.3.4**

Let $(a_1, \dots, a_n)$ be a basis of $V$ and set $v_i = \sum_{j=1}^n v_i^j a_j = v_i^j a_j$.
Then define:

$$\phi(v_1, \dots, v_n) = \det(v_i^j)$$

Then $\phi$ is an alternating $n$-tensor by the properties of the determinant.

**Lemma V.3.3**

Let $f \in \mathscr{L}^k(V)$ be any $k$-tensor and let $\sigma, \tau \in S_k$. Then:

(a) The transformation $f \mapsto f^\sigma$ is a linear transformation on $\mathscr{L}^k(V)$ and $(f^\sigma)^\tau = f^{\tau \circ \sigma}$.
(b) If $f$ is alternating and if $v_p = v_q$ for som $p \neq q$ then $f(v_1, \dots, v_k) = 0$.

*Proof.* Part a) is an exercise. For part b) the idea is to let $\sigma$ be the permutation that switches $p$ and $q$. Then:

$$f(v_1, \dots, v_k) = f^\sigma(v_1, \dots, v_k) = -f(v_1, \dots, v_k)$$

And so $f(v_1, \dots, v_k) = 0$

**Example V.3.5**

Let us consider $\mathcal{A}^k(V)$ with $k > n$. Take $f \in \mathcal{A}^k(V)$, then we saw last time that $f$ is completely determined by its values on a basis $\{a_1, \dots, a_n\}$ of $V$. Computing $f(a_{i_1}, \dots, a_{i_k})$ we see that one of the $a_i$ must be repeated since $k > n$ by the pigeonhole principle. By the lemma we then have that:

$$f(a_{i_1}, \dots, a_{i_k}) = 0$$

Therefore the only alternating $k$-tensor for $k > n$ is the trivial tensor.

**Lemma V.3.4**

Let $a_1, \dots, a_n$ be a basis of $V$. If $f$ and $g$ are two alternating $k$-tensors that satisfy:

$$f(a_{i_1}, \dots, a_{i_k}) = g(a_{i_1}, \dots, a_{i_k})$$

For any ascending $k$-tuple $I = (i_1, \dots, i_k)$ from the set $\{1, \dots, n\}$. Then:

$$f = g$$

**Remark V.3.1**

Compare this to the analogous lemma for $\mathscr{L}^k(V)$.

*Proof.* Let $J = (j_1, \dots, j_k)$ be a $k$-tuple. By the analogous lema for $k$-tensors, $f = g$ if and only if:

$$f(a_{j_1}, \dots, a_{j_k}) = g(a_{j_1}, \dots, a_{j_k})$$

Therefore it is enough to show that this holds. If two of the indices in $J$ are the same, then we are done by the previous lemma, since both sides of this equation will be zero.

If no two indices of $J$ are the same, let $I$ be the ascending rearrangement of $J$ by some permutation $\sigma$ such that $I = (j_{\sigma(1)}, \dots, j_{\sigma(k)})$..

Then we have that:

$$\begin{aligned} f(a_{i_1}, \dots, a_{i_k}) &= f(a_{j_{\sigma(1)}}, \dots, a_{j_{\sigma(k)}}) \\ &= f^\sigma(a_{j_1}, \dots, a_{j_k}) \\ &= (\operatorname{sgn} \sigma) \cdot f(a_{j_1}, \dots, a_{j_k}) \end{aligned}$$

Similarly we have:

$$\begin{aligned} g(a_{i_1}, \dots, a_{i_k}) &= g(a_{j_{\sigma(1)}}, \dots, a_{j_{\sigma(k)}}) \\ &= g^\sigma(a_{j_1}, \dots, a_{j_k}) \\ &= (\operatorname{sgn} \sigma) \cdot g(a_{j_1}, \dots, a_{j_k}) \end{aligned}$$

By the assumption we have that $f(a_{i_1}, \dots, a_{i_k}) = g(a_{i_1}, \dots, a_{i_k})$. Therefore:

$$f(a_{j_1}, \dots, a_{j_k}) = g(a_{j_1}, \dots, a_{j_k})$$

And so we are done!

**Theorem V.3.5** (Basis for $\mathcal{A}^k(V)$)
Let $V$ be a vector space with basis $a_1, \dots, a_n$. Let $I = (i_1 \dots, i_k)$ be tan ascending $k$-tuple from the set $\{1, \dots, n\}$. Then there is a unique alternating $k$-tensor $\Psi^I$ on $V$ such that for every ascending $k$-tuple $J = (j_1, \dots, j - K0$ from $\{1, \dots, n\}$ such that:

$$\Psi^I(a_{j_1}, \dots, a_{j_k}) = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$$

In fact, we take a "anti-symmetrization" of $\Phi^I$ from before:

$$\Psi^I = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \cdot \left(\Phi^I\right)^\sigma$$

The tensors $\Psi^I$ form a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these tensors, since htere are $\binom{n}{k}$ ascending $k$-tuples. These are called the elementary alternating tensors corresponding to the basis $\{a_1, \dots, a_n\}$.
Recall: We have the following:

$$\Phi^I = \Phi^{i_1} \otimes \cdots \otimes \Phi^{i_k}$$

$$\Phi^I(a_{\ell_1}, \dots, a_{\ell_k}) = \begin{cases} 1 & \text{if } I = (\ell_1, \dots, \ell_k) \\ 0 & \text{if } I \neq (\ell_1, \dots, \ell_k) \end{cases}$$

*Proof.* Uniqueness of the $\Psi^I$ follows from the previous lemma. Take $\Psi^I$ as given in the theorem:

$$\Psi^I = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \cdot \left(\Phi^I\right)^\sigma$$

We first show that this is alternating:

$$\begin{aligned} \left(\Psi^I\right)^\tau &= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \cdot \left(\left(\Phi^I\right)^\sigma\right)^\tau \\ &= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \cdot \left(\Phi^I\right)^{\tau \circ \sigma} \\ &= \operatorname{sgn} \tau \cdot \sum_{\sigma \in S_n} (\operatorname{sgn} \tau)(\operatorname{sgn} \sigma) \cdot \left(\Phi^I\right)^{\tau \circ \sigma} \\ &= \operatorname{sgn} \tau \cdot \sum_{\sigma \in S_n} (\operatorname{sgn} \tau \circ \sigma) \cdot \left(\Phi^I\right)^{\tau \circ \sigma} \\ &= \operatorname{sgn} \tau \cdot \sum_{\mu \in S_n} (\operatorname{sgn} \mu) \cdot \left(\Phi^I\right)^{\mu} = (\operatorname{sgn} \tau) \cdot \Psi^I \end{aligned}$$

And therefore $\Psi^I \in \mathcal{A}^k(V)$.
Next let $J = (j_1, \dots, j_k)$ be another ascending $k$-tuple. Look at:

$$\Psi^I(a_{j_1}, \dots, a_{j_k}) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)(\Phi^I)^\sigma(a_{j_1}, \dots, a_{j_k}) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)(\Phi^I)(a_{j_{\sigma(1)}}, \dots, a_{j_{\sigma(k)}})$$

And then this is 0 unless $(j_{\sigma(1)}, \dots, j_{\sigma(k)}) = I$. Since $I$ and $J$ are both ascending this only happens when $\sigma$ is the identity and $I = J$:

$$\Psi^I(a_{j_1}, \dots, a_{j_k}) = \begin{cases} (\operatorname{sgn} \mathrm{Id})\Phi^I(a_{i_1}, \dots, a_{i_K}) & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$$
$$= \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$$

To show that this is a basis let $f \in \mathcal{A}^k(V)$ and let $d_I = f(a_{i_1}, \dots, a_{i_k})$ for every ascending $k$-tuple $I = (i_1, \dots, i_k)$. Then we consider:

$$\widetilde{f} = \sum_I d_I \Psi^I$$

Then $f$ and $\widetilde{f}$ satisfy the conditions of the previous lemma, which you should check, and then $f = \widetilde{f}$.

The uniqueness of this representation also follows from the previous lemma. If we have that:

$$\sum_I d_I \Psi^I = 0$$

Then necessarily for any ascending $J = (j_1, \dots, j_k)$ we have:

$$\sum_I d_I \Psi^I(a_{j_1}, \dots, a_{j_k}) = d_J \cdot 1 = 0$$

So $d_J = 0$. This happens because all the terms where $I \neq J$ are zero, and the term when $I = J$ is $d_J$.

**Remark V.3.2**

$\mathcal{A}^1(V)$ has dimension $n$ and $\mathcal{A}^n(V)$ has dimension 1. In particular, any alternating $n$-tensor is a multiple of the determinant tensor discussed in a previous example.

Just like general $k$-tensors, alternating tensors can be pulled back by linear transformations as follows.

**Theorem V.3.6**

Let $T : V \to W$ be a linear transformation. If $f$ is an alternating tensor on $W$, then $T^*f$ is an alternating tensor on $V$.

*Proof left as an Exercise.* Recall $T^*f(v_1, \dots, v_k) = f(Tv_1, \dots, Tv_k)$.

### V.3.1. The space $\mathcal{A}^k(\mathbb{R}^n)$

By the aboe, $\mathcal{A}^n(\mathbb{R}^n)$ has dimension 1, and we already saw that the tensor defined by:

$$\Psi(x_1, \dots, x_n) = \det[x_1 \mid \cdots \mid x_n]$$

is an alternating $n$-tensor thanks to the properties of the determinant function. If $I = (1, \dots, n)$ then $\Psi^I = c\Psi$ for some constant $c \in \mathbb{R}$. In fact $\Psi^I = \Psi$. We see this because:

$$1 = \Psi^I(e_1, \dots, e_n) = c\Psi(e_1, \dots, e_n) = c \det I_n = c$$

This has a generalization for $k < n$

**Theorem V.3.7**

Let $\Psi^I$ be an elementary alternating tensor on $\mathbb{R}^n$ corresponding to the standard basis, where $I = (i_1, \dots, i_k)$ is an ascending $k$-tuple from $\{1, \dots, n\}$.

Then given $x_1, \dots, x_k \in \mathbb{R}^n$, let $X$ be the $(n \times k)$ matrix $X = [x_1 \mid \cdots \mid x_k]$. Then we have:

$$\Psi^I(x_1, \dots, x_k) = \det X^I$$

Where $X^I$ is the $(k \times k)$ matrix whose successive rows are the rows $i_1, \dots, i_k$ of $X$.

This holds for $1 \leq k \leq n$. Recall that if $k > n$ then $\mathcal{A}^k(\mathbb{R}^n) = 0$.

> **Remark V.3.3**
> Any $k$-form on $\mathbb{R}^n$ is a linear combination of $\det X^I$ for different $(k \times k)$ submatrices of the $(n \times k)$ matrix $X$.

*Proof.* Compute $\Psi^I(x_1, \ldots, x_k)$. Then:

$$\begin{aligned}\Psi^I(x_1, \ldots, x_k) &= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \cdot (\Phi^I)^\sigma (x_1, \ldots, x_k) \\ &= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \cdot \Phi^I (x_{\sigma(1)}, \ldots, x_{\sigma(K)}) \\ &= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \cdot x^{i_1}_{\sigma(1)} \cdots x^{i_n}_{\sigma(n)} = \det X^I\end{aligned}$$

Great! This is exactly what we wanted.

## V.4. **The Wedge Product**

The wedge product for alternating tensors is the analogue of the tensor product for general tensors. Recall that if $f \in \mathscr{L}^k(V)$ and $g \in \mathscr{L}^m(V)$, then $f \otimes g \in \mathscr{L}^{k+m}(V)$ is defined by:

$$[f \otimes g](v_1, \dots, v_{k+m}) = f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+m})$$

Clearly if $f \in \mathcal{A}^k(V)$ and $g \in \mathcal{A}^m(V)$. Then $f \otimes g$ might not be alternating. (Take the case $k = m = 1$, then $(f \otimes g)(v_1, v_1) = f(v_1)g(v_1)$ may not be zero, which must hold for alternating tensors).

**Lemma V.4.1**

Let $V$ be a vector space. There exists a linear transformation $A : \mathscr{L}^k(V) \to \mathcal{A}^k(V)$ (called the ănti-symmetrization map) given by:

$$Af = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) f^\sigma$$

For every $f \in \mathscr{L}^k(V)$. Moreover $A$ is onto and satisfies $Af = k!f$ when $f \in \mathcal{A}^k(V)$.

**Remark V.4.1**

For all practical purposes, $A$ can be considered a projection onto $\mathcal{A}^k(V)$, and in fact $\frac{1}{k!} \cdot A$ should be such a projection.

*Proof.* Clearly $A$ is linear because $(f+g)^\sigma = f^\sigma + g^\sigma$. To show that $Af$ is actually alternating, notice that:

$$\begin{aligned}(Af)^\tau &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \cdot (f^\sigma)^\tau = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \cdot f^{\tau \circ \sigma} \\ &= (\operatorname{sgn} \tau) \cdot \sum_{\sigma \in S_k} (\operatorname{sgn} \tau) \cdot (\operatorname{sgn} \sigma) \cdot f^{\tau \circ \sigma} = (\operatorname{sgn} \tau) \cdot \sum_{\sigma \in S_k} \operatorname{sgn}(\tau \circ \sigma) \cdot f^{\tau \circ \sigma} \\ &= (\operatorname{sgn} \tau) \cdot \sum_{\mu \in S_k} (\operatorname{sgn} \mu) \cdot f^\mu = (\operatorname{sgn} \tau) \cdot Af\end{aligned}$$

Finally, if $f \in \mathcal{A}^k(V)$ then:

$$Af = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \cdot f^\sigma = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)^2 f = f \sum_{\sigma \in S_k} 1 = k!f$$

Awesome!

**Definition V.4.1**

Let $V$ be a vector space and let $f \in \mathcal{A}^k(V)$ and $g \in \mathcal{A}^\ell(V)$. We define $f \wedge g \in \mathcal{A}^{k+\ell}(V)$ to be given by:

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g)$$

This alternating $k + \ell$ tensor is called the wedge product of $f$ and $g$:

**Remark V.4.2**

The factor $\frac{1}{k!\ell!}$ is introduced to retain some nice properties of the wedge product (namely associativity). Sometimes it is defined with $\frac{1}{(k+\ell)!}$ instead, but it doesn't really matter that much.

**Theorem V.4.2** (Properties of the wedge product)

Let $V$ be a vector space. The wedge product satisfies the following properties:

(1) Associativity, $(f \wedge g) \wedge k) = f \wedge (g \wedge k)$.
(2) Homogeneity, $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$
(3) Distributivity, if $f$ and $g$ have the same order, then we have:

$$(f + g) \wedge h = f \wedge h + g \wedge h$$

$$h \wedge (f+g) = h \wedge f + h \wedge g$$

(4) Anti-commutativity, if $f$ and $g$ have orders $k$ and $\ell$ respectively, then:

$$f \wedge g = (-1)^{k\ell} \cdot g \wedge f$$

In particular, if $f$ has odd order, then $f \wedge f = 0$.

(5) Given a basis $a_1, \dots, a_n$ of $V$, let $\phi^i$ be the dual basis of $V^*$ (equivalently, $\phi^i$ is the basis of $\mathscr{L}^1(V) = \mathcal{A}^1(V)$). Then let $\Psi^I$ denote the elementary alternating tensor where $I = (i_1, \dots, i_k)$ is an ascending $k$-tuple from $\{1, \dots, n\}$. Then we have:

$$\Psi^I = \phi^{i_1} \wedge \phi^{i_2} \wedge \cdots \wedge \phi^{i_k}$$

(6) If $T : V \to W$ is a linear transformation, then $T^*(f \wedge g) = (T^* f) \wedge T^*(T^* g)$.

(7) The product $\wedge$ is the unique operation satisfying properties (1)-(5).

*Proof.* The proof of (1) will be outlined in HW5. (2) and (3) follow from the corresponding results for tensor product and the linearity of $A$, for example:

$$\begin{aligned}(f+g) \wedge h &= \frac{1}{k!\ell!} A((f+g) \otimes h) = \frac{1}{k!\ell!} A(f \otimes h + g \otimes h) \\ &= \frac{1}{k!\ell!} A(f \otimes h) + \frac{1}{k!\ell!} A(g \otimes h) = f \wedge h + g \wedge h\end{aligned}$$

(4), that is anti-commutativity, will follow by showing that $A(f \otimes g) = (-1)^{k\ell} A(g \otimes f)$ for any $f \mathscr{L}^k(V)$ and $g \in \mathscr{L}^\ell(V)$. To see this, consider $\pi \in S_{k+\ell}$ given by $(\pi(1), \dots, \pi(k+\ell) = (k+1, \dots, k+\ell, 1, \dots, k)$. Then by counting the inversions we see that $\operatorname{sgn} \pi = (-1)^{k\ell}$ (check!).

But then we see that:

$$(g \otimes f)^\pi (v_1, \dots, v_{k+\ell}) = (g \otimes f)(v_{k+1}, \dots, v_{k+\ell}, v_1, \dots, v_k) = (f \otimes g)(v_1, \dots, v_{k+\ell})$$

Great! This is the key, that $(g \otimes f)^\pi = (f \otimes g)$. We write:

$$\begin{aligned}A(f \otimes g) &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(f \otimes g)^\sigma = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)((g \otimes f)^\pi)^\sigma \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(g \otimes f)^{\sigma \circ \pi} = (\operatorname{sgn} \pi) \cdot \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \cdot (\operatorname{sgn} \pi) \cdot (g \otimes f)^{\sigma \circ \pi} \\ &= (-1)^{k\ell} \cdot \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma \circ \pi) \cdot (g \otimes f)^{\sigma \circ \pi} = (-1)^{k\ell} A(g \otimes f)\end{aligned}$$

Perfect!

Now for (5) recall that the elementary alternating tensors $\Psi^I$ are defined for $I = (i_1, \dots, i_k)$ as:

$$\Phi^I = \phi^{i_1} \otimes \cdots \otimes \phi^{i_k}$$

$$\Psi^I = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(\Phi^I)^\sigma = A(\Phi^I)$$

This will follow once we show that if $f_1, \dots, f_k$ are 1-tensors, then:

$$A(f_1 \otimes \cdots \otimes f_k) = f_1 \wedge \cdots \wedge f_k \tag{$\star$}$$

From Homework 5, we know that if $g \in \mathcal{A}^\ell(V)$ and $f \in \mathscr{L}^k(V)$ then:

$$A(f \otimes g) = \frac{1}{\ell!} \cdot (Af) \wedge g \tag{$\star\star$}$$

To prove ($\star$) we induct on $k$. For $k = 1$ this is trivially true. In general, assuming that:

$$A(f_1 \otimes \cdots \otimes f_k) = f_1 \wedge \cdots \wedge f_k$$

Then we know by associativity that since $f_{k+1}$ will be an alternating tensor we can use ($\star\star$) to get:

$$A(f_1 \otimes \cdots \otimes f_k \otimes f_{k+1}) = A(f_1 \otimes \cdots f_k) \wedge f_k = f_1 \wedge \cdots f_k \wedge f_{k+1}$$

Therefore ($\star$) is true by induction, and the result follows.

For propert (6) we consider that:

$$\begin{aligned}T^*(f\wedge g) &= \frac{1}{k!\ell!}T^*(A(f\otimes g))\\ &= \frac{1}{k!\ell!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\cdot T^*((f\otimes g)^\sigma)\\ &= \frac{1}{k!\ell!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\cdot [T^*((f\otimes g))]^\sigma\end{aligned}$$

Why? Well $T^*$ is linear and also:

$$\begin{aligned}T^*(F^\sigma)(v_1,\dots,v_m) &= F^\sigma(Tv_1,\dots,Tv_m) = F(Tv_{\sigma(1)},\dots,Tv_{\sigma(m)})\\ &= T^*F(v_{\sigma(1)},\dots,v_{\sigma(m)}) = (T^*F)^\sigma(v_1,\dots,v_m)\end{aligned}$$

Then:

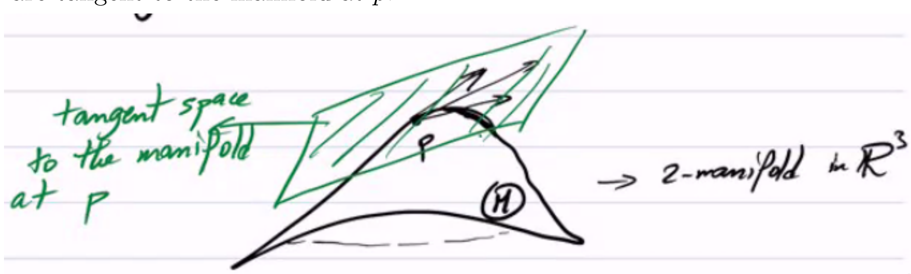
$$\frac{1}{k!\ell!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\cdot [T^*((f\otimes g))]^\sigma = \frac{1}{k!\ell!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\cdot [T^*f\otimes T^*g]^\sigma = \frac{1}{k!\ell!}A(T^*f\otimes T^*g) = T^*f\wedge T^*g$$

Awesome! (7) will be part of Homework 6

# VI. Tangent Vectors and the Tangent Space

## VI.1. Definitions

Here's the picture which we will try to formalize. We want to pick a point $p\in M$ and describe the set of manifolds that are tangent to the manifold at $p$:




Let us start with the easy case when $M=\mathbb{R}^n$.

**Definition VI.1.1**

Given $x\in\mathbb{R}^n$, we define a tangent vector to $\mathbb{R}^n$ at $x$ to be the pair $(x;\vec{v})$ (sometimes denoted $\vec{v}_x$ where $\vec{v}\in\mathbb{R}^n$. That is we attach a vector $\vec{v}\in\mathbb{R}^n$ to a point $x\in\mathbb{R}^n$.

The set of all tangent vectors to $\mathbb{R}^n$ at $x$ is denoted $T_x\mathbb{R}^n$, and is called the tangent space to $\mathbb{R}^n$ at $x$. This is a vector space with the operations:

$$\begin{aligned}(x;\vec{v})+(x;\vec{w}) &= (x;\vec{v}+\vec{w})\\ c(x;\vec{v}) &= (x;c\vec{v})\end{aligned}$$

Notice that of course $T_x\mathbb{R}^n\cong\mathbb{R}^n$ as vector spaces.

**Remark VI.1.1**

Let $\vec{v}_x\in T_x\mathbb{R}^n$ be a tangent vector at $x$. Then there exists a smooth curve $\gamma:(-\varepsilon,\varepsilon)\to\mathbb{R}^n$ such that $\gamma(0)=x$ and $\gamma'(0)=\vec{v}_x$. Namely, the curve $\gamma(t)=x+tv$ is such a curve. In other words, the tangent space to $\mathbb{R}^n$ at $x$ is exactly the set of tangents $\gamma'(0)$ to curves $\gamma(t)$ with $\gamma(0)=x$.

Last time, we defined tangent vectors and the tangent space to $\mathbb{R}^n$ (Definition VI.1.1). For each $p \in \mathbb{R}^n$, we defined the tangent vectors as the pair $(p, \vec{v})$ where $\vec{v} \in \mathbb{R}^n$, we sometimes write this as $\vec{v}_p$. We had the following notes:

- Then $T_p\mathbb{R}^n$ is the tangent space to $\mathbb{R}^n$ at $p$, and it is isomorphic to $\mathbb{R}^n$.
- We also saw that $T_p\mathbb{R}^n$ is exactly the set of velocity vectors $\gamma'(0)$ of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ satisfying $\gamma(0) = p$.

**Definition VI.1.2**

Let $M$ be an $n$-dimensional smooth manifold and let $p \in M$. Suppose that $(U, \varphi)$ is a coordinate chart near $p$ where $\varphi : U \to \widehat{U} \subseteq \mathbb{R}^n$ with $\varphi(p) = x \in \mathbb{R}^n$. A tangent vector to $M$ at $p$ consists of a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to U$ such that $\gamma(0) = p$. We consider two such curves $\gamma, \delta$ to be equivalent as tangent vectors whenever $(\varphi \circ \gamma)'(0) = (\varphi \circ \delta)'(0)$, and we denote the equivalence class of such a curve as $[\gamma]$, this is a tangent vector. The set of such tangent vectors is called the tangent space of $M$ at $p$ and is denoted $T_pM$. We define a vector space structure on $T_pM$ by defining:

$$c[\gamma] = [t \mapsto \gamma(ct)]$$
$$[\gamma] + [\delta] = [t \mapsto \varphi^{-1}(\varphi(\gamma(t)) + \varphi(\delta(t)))]$$

We of course must restrict the domains of these maps to be smaller neighborhoods $(-\varepsilon, \varepsilon)$ for this to work out, but that's fine.

Really, this just translates the vector space structure of $\mathbb{R}^n$ to $T_pM$. In a miracle, $T_pM$ does not depend on the chart $U$ chosen. It turns out that $T_pM \cong \mathbb{R}^n$, and this is given by the linear isomorphism $[\gamma] \to (\varphi \circ \gamma)'(0)$.

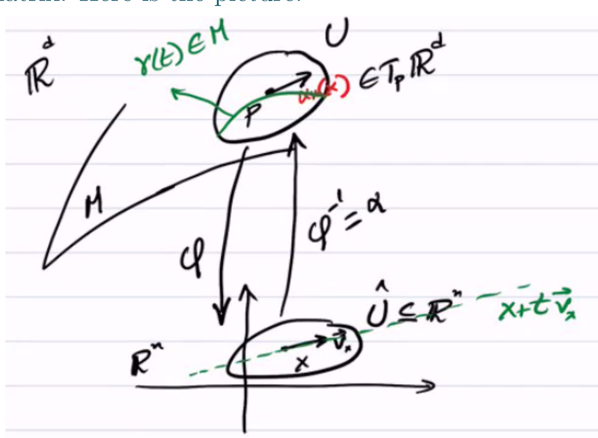
We now give a simpler definition for submanifolds, which is what we will be dealing with in this course

**Definition VI.1.3**

Let $M$ be an $n$-dimensional smooth submanifold of $\mathbb{R}^d$ and let $p \in M$. Suppose that $(U, \varphi)$ is a coordinate chart near $p$ where $\varphi : U \to \widehat{U} \subseteq \mathbb{R}^n$ with $\varphi(p) = x \in \mathbb{R}^n$. Let $\alpha = \varphi^{-1} : \widehat{U} \to \mathbb{R}^d$. We define the pushforward of $\alpha$ as $\alpha_* : T_x\mathbb{R}^n \to T_p\mathbb{R}^d$ as:

$$\alpha_*(\vec{v}_x) = D\alpha(x) \cdot \vec{v}_x$$

Since $D\alpha(x)$ is a $d \times n$ matrix. Here is the picture:



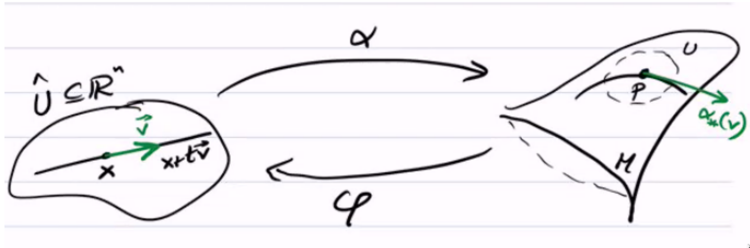
**Remark VI.1.2**

Notice that by the chain rule $\alpha_*(\vec{v}_x)$ is the velocity or tangent vector $\gamma'(0)$ of the curve:

$$\gamma(t) = \alpha(x + t\vec{v}_x) : (-\varepsilon, \varepsilon) \to M$$

which satisfies $\gamma(0) = p$.

In another nice picture:

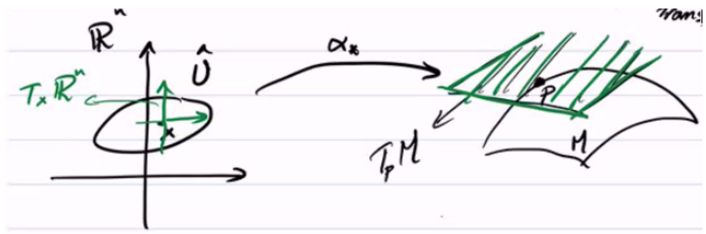



**Definition VI.1.4**

With the same notation as in the above definition. The following set $T_pM$ is the tangent space to $M$ at $p$:

$$T_pM := \operatorname{im} \alpha_* = \{\alpha_*(v_x) \mid v_x \in T_x\mathbb{R}^n\}$$

Where $\alpha$ is the inverse to some chart around $p$ as in the previous definition. A tangent vector to $M$ at $p$ is just a member of this tangent space. $T_pM$ is a subspace of $T_p\mathbb{R}^d$, since it is the image of a linear transformation:




This definition does not depend on the coordinate chart ☺

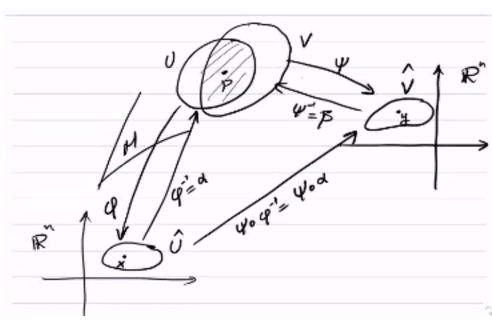
In what follows, we will show the following for the submanifold definition:

(A) $T_pM$ is independent of the choice of coordinate chart $(U, \varphi)$ (and hence $\alpha$)
(B) $T_pM$ is an $n$-dimensional subspace of $T_p\mathbb{R}^d$
(C) With $\alpha$ as above, $\alpha_* : T_x\mathbb{R}^n \to T_pM$ is a vector space isomorphism.

In fact, (B) and (C) are not that hard to show, since $T_pM$ is the range of $\alpha_*$ which is a linear transformation $T_x\mathbb{R}^n \to T_p\mathbb{R}^d$ given by matrix multiplication by $D\alpha(x)$. But this matrix has rank $n$, which means that its image is $n$-dimensional. Furthermore, $\alpha_* : T_x\mathbb{R}^m \to T_pM$ is an isomorphism since it's onto and the dimension of $T_x\mathbb{R}^n$ and $T_pM$ agree (they are both $n$-dimensional).

There is more than one way to show (A). One is direct, and the other will follow from the characterization of $T_pM$ using curves in $M$ passing through $p$.

*Direct proof of (A).* Let $(U, \varphi)$ be the coordinate chart used in the definition with $\alpha = \varphi^{-1}$. Let $(V, \psi)$ be another coordinate patch neat $p \in M$ such that $\psi(p) = y$, and denote $\beta = \psi^{-1}$.




We know that $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a $C^r$-diffeomorphism, which means that $D(\psi \circ \alpha)$ is an invertible matrix. Now note that $\alpha = \psi^{-1} \circ \psi \circ \alpha = \beta \circ \psi \circ \alpha$. Therefore by the chain rule:

$$D\alpha(x) = D\beta(y) \circ D(\psi \circ \alpha)(x) \tag{$\star$}$$

This gives that $\Im D\alpha(x) = \Im D\beta(y)$. In other words $\alpha_*(T_x\mathbb{R}^n) = \beta_*(T_y\mathbb{R}^n)$. This is because we have:

$$\begin{array}{ccc} & T_p\mathbb{R}^d & \\ {\scriptstyle \alpha_*}\nearrow & & \nwarrow{\scriptstyle \beta_*} \\ T_x\mathbb{R}^n & \xrightarrow[(\psi\circ\varphi^{-1})_*]{} & T_y\mathbb{R}^n \end{array}$$

Thus $T_pM$ is well-defined. In more concrete terms:

$$\alpha_*(T_x\mathbb{R}^n) = \beta_*([\psi \circ \varphi^{-1}]_*(T_x\mathbb{R}^n) = \beta_*(T_y\mathbb{R}^n)$$

Great!

**Remark VI.1.3**

Notice that from ($\star$) we have the following. If $z_p \in T_pM$ is such that $z_p = \alpha_*(v_x) = \beta_*(w_y)$. The question is then what is $w_y$ in terms of $v_x$? Well, we see via ($\star$) that:

$$z_p = D\alpha(x) \cdot v_x = D\beta(y) \cdot (D(\psi \circ \alpha)(x) \cdot v_x) = D\beta(y) \cdot w_y$$

Therefore since $D\beta(y)$ is injective from previous work, we see that:

$$w_y = D(\psi \circ \alpha)(x) \cdot v_x$$

In other words, the coordinate representation of $z_p$ in the $\psi$ coordinates ($w_y$) is given by applying the derivative of the transition map from $\varphi$ coordinates to $\psi$ coordinates ($D(\psi \circ \varphi^{-1})$) to the coordinate representation of $z_p$ in the $\varphi$ coordinates ($v_x$).

**Theorem VI.1.1** (Characterization of $T_pM$ using curves through $p$)

This theorem links our concrete definition for submanifolds to the definition for abstract manifolds. Let $M$ be an $n$-dimensional submanifold of $\mathbb{R}^d$. Suppose that $p \in M \setminus \partial M$, and $z_p \in T_p\mathbb{R}^d$.

Then $z_p \in T_pM$ if and only if there exists a smooth curve $\gamma(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}^d$ such that $\operatorname{im}\gamma \subseteq M$ (aka $\gamma$ lies in $M$), $\gamma(0) = p$, and $\gamma'(0) = z_p$.

In other words, $T_pM$ is the set of velocity vectors to the curves in $M$ passing through $p$.

*Proof.* If $z_p \in T_pM$, then by definition $z_p = D\alpha(x) \cdot v_x$ for some $v_x \in T_x\mathbb{R}^n$ (using the same notation from earlier). We may then take $\gamma$ to be the curve:

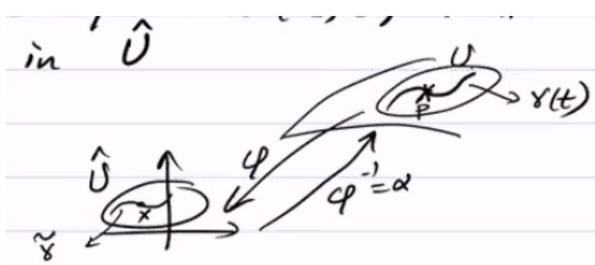
$$\begin{aligned} \gamma : (-\varepsilon, \varepsilon) &\to U \subseteq M \\ t &\mapsto \alpha(x + tv_x) \end{aligned}$$

noting that $x + tv_x$ belongs to $\widehat{U}$ is $\varepsilon > 0$ is small enough. Then by the chain rule:

$$\gamma'(t) = D\alpha(x + tv_x) \cdot v_x \implies \gamma'(0) = D\alpha(x) \cdot v_x = z_p$$

This takes care of one direction of the proof.

Conversely, suppose that $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^d$ is smooth and $\operatorname{im}\gamma \in M$, $\gamma(0) = p$, and $\gamma'(0) = z_p \in T_p\mathbb{R}^d$. We wish to show that $z_p \in T_pM$. Let $\widetilde{\gamma}(t) = \varphi \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$. This is a curve in $\widehat{U}$.



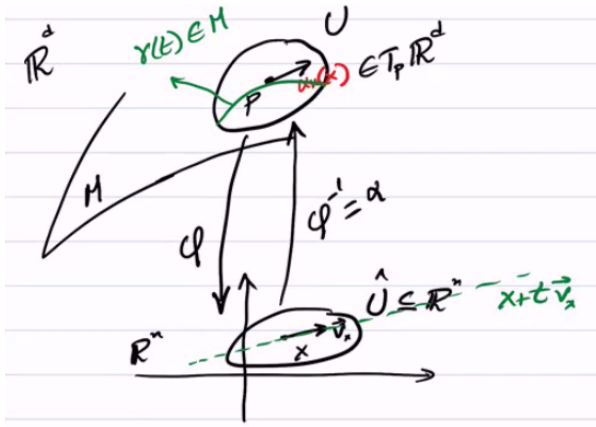
Then $\gamma = \varphi^{-1} \circ \widetilde{\gamma} = \alpha \circ \widetilde{\gamma}$. Hence, provided that $\widetilde{\gamma}$ can be shown to the smooth, we have by the chain rule that:

$$\gamma'(0) = D\alpha(x) \cdot \widetilde{\gamma}'(0)$$

So $z_p = \alpha_*(\widetilde{\gamma}'(0)) \in T_pM$. We now prove that $\widetilde{\gamma}$ is smooth (we cannot use composition of smooth maps, because $\varphi : U \to \widehat{U}$ is not a map from an open set of $\mathbb{R}^d$)

To show that $\widetilde{\gamma}$ is a smooth curve, recall that we showed in Lecture 5 that $\varphi : U \to \mathbb{R}^n$ extends to a smooth map $\Phi : U_\heartsuit \to \mathbb{R}^n$ where $U_\heartsuit$ is an open subset of $\mathbb{R}^d$ containing $U$ which agrees with $\varphi$ on $M$. But then $\widetilde{\gamma} = \Phi \circ \gamma$ is smooth being the composite of two smooth maps.

The case when $p \in \partial M$ is similar and is left to the homework.




Last time we defined for an inverse chart $\alpha : \widehat{U} \to U$ for a submanifold $M$ a push-forward map $\alpha_* : T_x\mathbb{R}^n \to T_p\mathbb{R}^d$, and then $T_pM$ was the image of this map:

$$\alpha_*(v_x) = D\alpha(x) \cdot v_x$$

This image $T_pM = \operatorname{im}\alpha_*$ was called the tangent space to $M$ at $p$, and this was an $n$-dimensional subspace isomorphic to $T_x\mathbb{R}^n$ via $\alpha_*$.

We then showed that $z_p \in T_pM$ if and only if there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^d$ such that $\gamma(t) \in M$, $\gamma(0) = p$, $\gamma'(0) = z_p$.

**Remark VI.1.4**

Suppose that $M = \mathbb{R}^d$, and let $f : \widehat{U} \subseteq \mathbb{R}^n \to \mathbb{R}^d$ then:

$$f_*(v_x) = Df(x) \cdot v_x$$

Then the right interpretation of $Df(x)$ is as a linear transformation $T_x\mathbb{R}^n \to T_p\mathbb{R}^d$ where $p = f(x)$.

## VI.2. **Vector Fields and the tangent bundle**

We saw that if $M = \mathbb{R}^n$ then $T_pM \cong \mathbb{R}^n$ for every $p \in M$.

**Definition VI.2.1**

We define the tangent bundle of $\mathbb{R}^n$, denoted $T\mathbb{R}^n$ as $\bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$ and give it the topology of $\mathbb{R}^n \times \mathbb{R}^n$. This means that $(p, v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ corresponds to the point $p \in \mathbb{R}^n$ along with the tangent vector $v$ at $p$.

**Definition VI.2.2** (Tangent Bundle)

Let $M$ be a smooth $n$-dimensional submanifold of $\mathbb{R}^d$. The tangent bundle $TM$ is defined as the smooth submanifold of $T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ given by:

$$TM = \{(p, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid p \in M, v \in T_pM\}$$

This submanifold is $2n$ dimensional, and its submanifold structure defined as follows:

Suppose that $(U_j, \varphi_j)$ is a coordinate atlas of $M$, and denote $\alpha_j := \varphi_j^{-1} : \widehat{U}_j \to \mathbb{R}^d$ as usual.

Define $\Lambda_j : \widehat{U}_j \times \mathbb{R}^n \to T\mathbb{R}^d$ as follows:

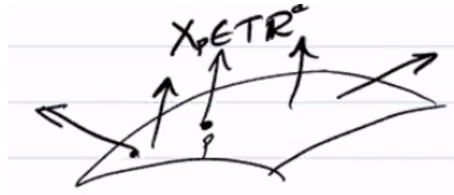
$$\Lambda_j(x, v) = (\alpha_j(x), D\alpha_j(x) \cdot v) = (\alpha_j(x), (\alpha_j)_*(v_x))$$

This gives an atlas for $TM$. This will be homework. Thus $TM$ is a $2n$-dimensional submanifold of $T\mathbb{R}^d$.

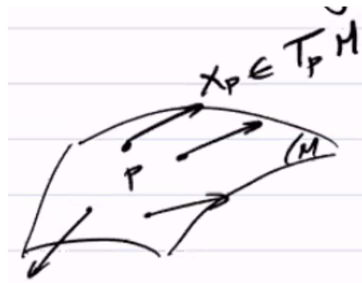
**Definition VI.2.3** (Tangent vector fields)

Let $M$ be a smooth $n$-submanifold of $\mathbb{R}^d$. Then:

(a) A $C^k$ vector field $X$ on $M$ is a $C^k$ map from $M \to T\mathbb{R}^d$ such that $X(p) \in T_p\mathbb{R}^d$ for every $p \in M$.

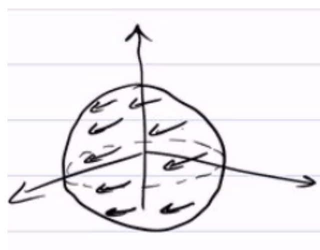



(b) A $C^k$ tangent vector field $X$ on $M$ is a $C^k$ map from $M \to TM$ such that $X(p) \in T_pM$ for every $p \in M$.




**Example VI.2.1**

The wind velocity at each point $p$ on the surface of the earth is a vector field.



**Exercise VI.2.2**

Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be any smooth function. Then $F$ can be regarded as a vector field as follows. Define:

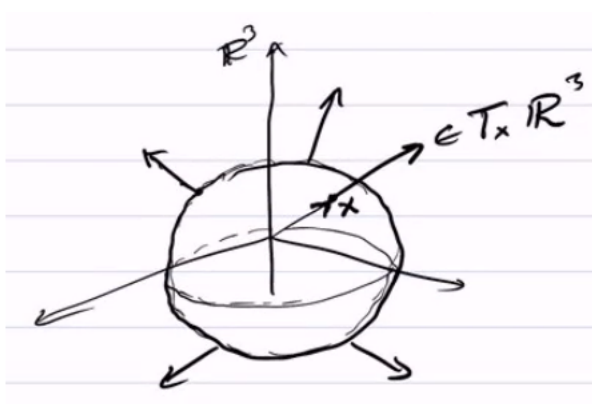
$$\overline{F} : x \mapsto (x, F(x)) \in T_x\mathbb{R}^d$$

**Example VI.2.3**

Let $F : \mathbb{S}^2 \to \mathbb{R}^3$ be given by $F(x) = \vec{x}$ for every $x \in \mathbb{S}^2$. Then $F$ gives a vector field on $\mathbb{S}^2$ via the identification $\overline{F}$:

$$\overline{F} : x \mapsto (x, x) \in T_x\mathbb{R}^3$$

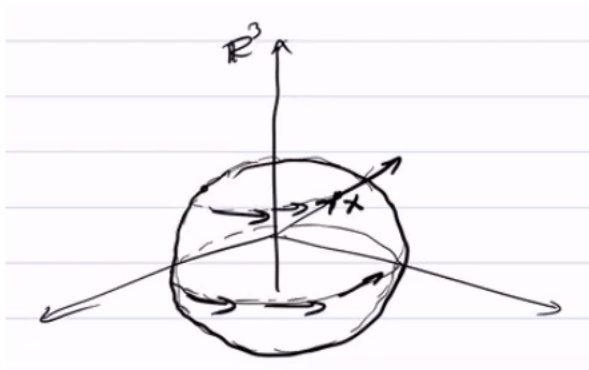
Pictorially we have a vector field:



This is called the normal field to $\mathbb{S}^2$.

**Example VI.2.4**

Let $G : \mathbb{S}^2 \to \mathbb{R}^3$ be defined by $G(x_1, x_2, x_3) = (-x_2, x_1, 0)$. Then one can easily see that $(x, G(x)) \in T_x\mathbb{S}^2$. This gives a smooth tangent vector field on $\mathbb{S}^2$.




## VI.3. The Pushforward Map

**Lemma VI.3.1**

Let $F : M \to N$ be some map. $F$ is a smooth map if and only if $F$ admits local extensions near every point $p \in M$ as a smooth map from an open set $\mathcal{O} \subseteq \mathbb{R}^{d_1}$ into $\mathbb{R}^{d_2}$.

More precisely, for every $p \in M$, there exists a neighborhood $\mathcal{O}$ of $p$ such that $\mathcal{O}$ is open in $\mathbb{R}^{d_1}$ and there exists a smooth map $\overline{F} : \mathcal{O} \to \mathbb{R}^{d_2}$ such that:

$$\overline{F}\big|_{\mathcal{O}\cap M} = F$$

*Proof.* Let's go!

($\Leftarrow$) Trivial–from Hani.

($\Rightarrow$) Fix some $p \in M$, and by the definition of smoothness pick charts $(U, \varphi)$ around $p$ and $(V, \psi)$ around $F(p)$ such that $\psi \circ F \circ \varphi^{-1}$ is smooth and $F(U) \subseteq V$.

Then by lecture 5, there exists an open $\mathcal{O} \subseteq \mathbb{R}^{d_1}$ and a function $\Phi : \mathcal{O} \to \widehat{U}$ such that $\Phi$ is smooth and $\Phi\big|_{\mathcal{O}\cap M} = \varphi$.

Then for each $q \in U$, $q = \varphi^{-1} \circ \Phi(q)$, so $F(q) = F \circ \varphi^{-1} \circ \Phi(q)$. Then let $\overline{F} : \mathcal{O} \to \mathbb{R}^d$ be given by $\overline{F} = F \circ \varphi^{-1} \circ \Phi$. Then $\overline{F}\big|_{\mathcal{O}\cap M} = M$.

What remains is to show that $\overline{F}$ is smooth. Note that since $\overline{F}(x) \in V$, then:

$$\psi^{-1} \circ \psi \circ \overline{F} = \overline{F}$$

Therefore:

$$\overline{F} = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \Phi$$

These are all smooth functions defined open subsets of some $\mathbb{R}^\ell$, and so $\overline{F}$ is smooth.

**Definition VI.3.1** (General Pushforward)

Let $M$ be an $n$-dimensional submanifold of $\mathbb{R}^{d_1}$ and $N$ be a $k$-dimensional submanifold of $\mathbb{R}^{d_2}$. Then let $F : M \to N$ be some smooth map. We have the following picture in charts:

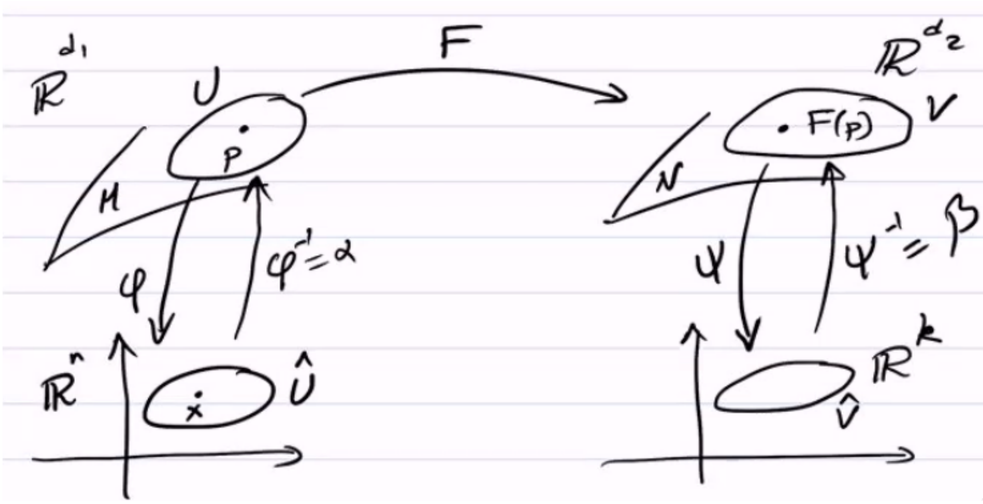
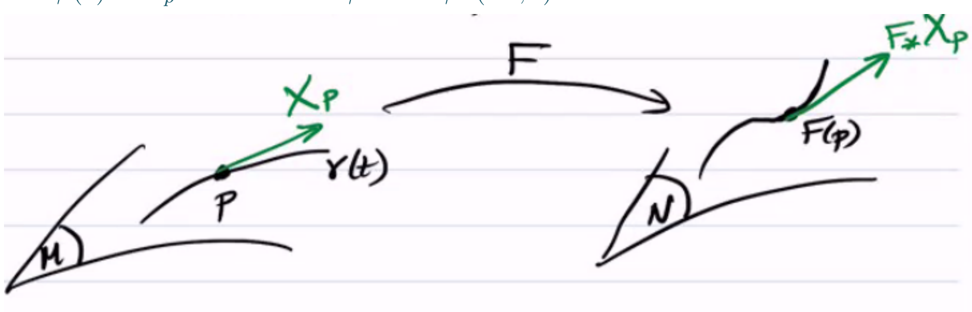



FIGURE 1. The charts for the pushforward map

Recall that a function $F : M \to N$ is smooth if $\psi \circ F \circ \varphi^{-1} : \widehat{U} \to \widehat{V}$ is smooth for some coordinate charts $(U, \varphi)$ and $(V, \psi)$ on $M$ and $N$ respectively such that $p \in U$, $F(U) \subseteq V$.

Let $X_p$ be some tangent vector in $T_pM$. Then there exists a curve $\gamma(t) : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. Then define $\widetilde{\gamma} = F \circ \gamma : (-\varepsilon, \varepsilon) \to N$.




Note that $\widetilde{\gamma}$ is also smooth (since $\widetilde{\gamma} = \overline{F} \circ \gamma$ where $\overline{F}$ is the lcoal smooth extension of $F$ given by Lemma VI.3.1).

Then $\widetilde{\gamma}'(0) \in T_{F(p)}N$. This vector is called $F_*X_p$. The push forward of $X_p$ by $F$.

**Proposition VI.3.2**

Let $M, N$ be as above, and let $F : M \to N$ be a smooth map. For $p \in M$, the push-foward map:

$$F_* : T_pM \to T_{F(p)}N$$

defined above, is a well-defined linear transformation. Moreover, let $(U, \varphi)$ be local coordinates near $p$ with $\alpha = \varphi^{-1} : \widehat{U} \to U$, $(V, \psi)$ local coordinates near $F(p)$ with $\beta = \psi^{-1} : \widehat{V} \to V$.

If $X_p \in T_pM$ is given by $\alpha_*(v_x)$ where $v_x \in T_x\widehat{U}$ and if $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ is the coordinate representation of $F$, then:

$$w_y := D\widetilde{F}(x) \cdot v_x \qquad (y = \psi(F(p)))$$
$$F_*(X_p) = \beta_*(w_y)$$

Recall Figure 1 for a nice picture. In a commutative diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{F_*} & T_{F(p)}N \\ \alpha_* \uparrow \downarrow (\alpha_*)^{-1} & & (\beta_*)^{-1} \downarrow \uparrow \beta_* \\ T_x\mathbb{R}^n & \xrightarrow[\widetilde{F}_*]{} & T_y\mathbb{R}^n \end{array}$$

Basically, all that the proposition is saying is that this diagram commutes.

*Proof.* Linearity of $F_*$ follows from the coordinate representation above. Let $X_p^{(1)}, X_p^{(2)} \in T_pM$, then $X_p^{(j)} = \alpha_*(v_x^{(j)})$ where $v_x^{(j)} \in T_x\mathbb{R}^n$ for $j = 1, 2$.

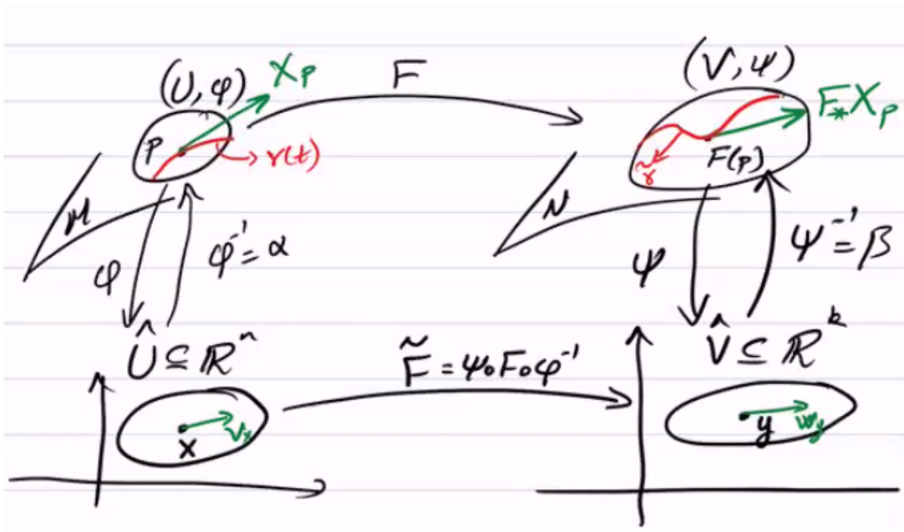
Then:

$$\begin{aligned} X_p^{(1)} + cX_p^{(2)} &= \alpha_*(v_x^{(1)} + cv_x^{(2)}) \\ F_*(X_p^{(1)} + cX_p^{(2)}) &= \beta_*(D\widetilde{F}(x) \cdot (v_x^{(1)} + cv_x^{(2)})) \\ &= \beta_* D\widetilde{F}(x) \cdot v_x^{(1)} + c\beta_* D\widetilde{F}(x) \cdot v_x^{(2)} \\ &= F_*(X_p^{(1)}) + cF_*(X_p^{(1)}) \end{aligned}$$

Similarly, one can use this coordinate representation to check that $F_*(X_p)$ is well-defined, which we'll leave as an exericse.

It then suffices to prove the coordinate representation.

Last time we defined the pushforward map (Definition VI.3.1). This definition gave us for $F : M \to N$ a smooth function between submanifolds a map $F_*TM \to TN$, or at points $F_* : T_pM \to T_{F(p)}N$. In pictures this looked like




In particular, given $X_p \in T_pM$, we know that there is a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. Then we let $\widetilde{\gamma} = F \circ \gamma$ and then $\widetilde{\gamma}'(0)$ is a tangent vector to $N$ at $F(p)$. So $F_*(X_p) = \widetilde{\gamma}'(0)$.

**Properties of the pushforward map**:

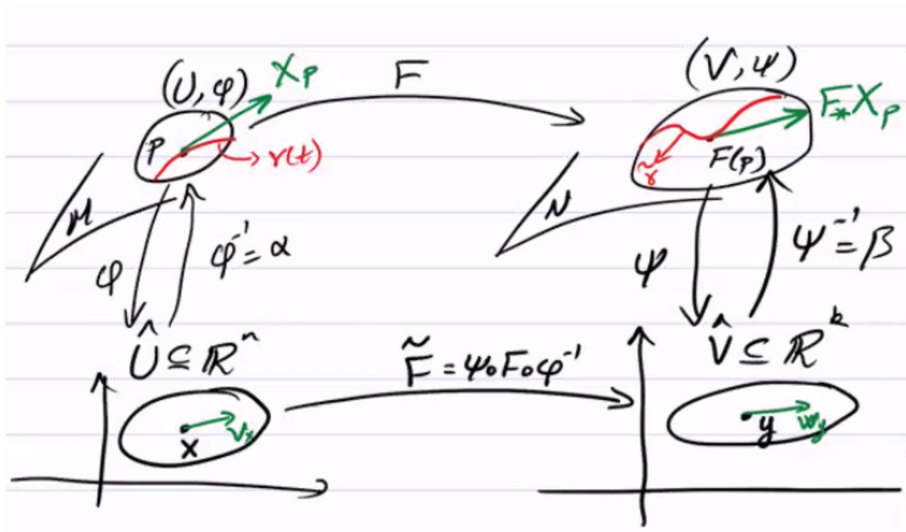
a) It is well-defined (i.e. $F_*X_p$ is independent of the choice of $\gamma$)
b) $F_*$ is a linear transformation from $T_pM$ into $T_{F(p)}N$.
c) The coordinate representation of $F_*$ with the charts and notation given above we have a commutative diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{F_*} & T_{F(p)}N \\ \alpha_* \uparrow \downarrow (\alpha_*)^{-1} & & (\beta_*)^{-1} \downarrow \uparrow \beta_* \\ T_x\mathbb{R}^n & \xrightarrow[\widetilde{F}_*]{} & T_y\mathbb{R}^n \end{array}$$

At points this just looks like:

$$\begin{array}{ccc} X_p & \xmapsto{F_*} & F_*(X_p) \\ \alpha_* \uparrow \downarrow (\alpha_*)^{-1} & & (\beta_*)^{-1} \downarrow \uparrow \beta_* \\ v_x & \xmapsto[\widetilde{F}_*]{} & w_y \end{array}$$

In pictures this is:




And so if $X_p = \alpha_*(v_x)$ then $F_*(X_p) = \beta_*(w_y)$ where $w_y = D\widetilde{F}_*(v_x) = D\widetilde{F}(x) \cdot v_x$.

**Exercise VI.3.1**

Question: What if $F$ has a smooth extension $\overline{F} : \overline{U} \to N \subseteq \mathbb{R}^{d_2}$ where $\overline{U}$ is an open subset of $\mathbb{R}^{d_1}$ (where $\dim M = d_1$, $\dim N = d_2$) such that $\overline{F}(p) = F(p)$ for all $p \in M$.

In that case, the natural thing happens $F_*(X_p) = D\overline{F}(p) \cdot X_p$.

**Remark VI.3.1**

fF $U \subseteq \mathbb{R}^n$ is open and $V \subseteq \mathbb{R}^k$ is open, then $F : U \to V$, then $F_* = DF$. More precisely, $F_*(X_p) = DF(p) \cdot X_p$. That is it's a linear transformation between the tangent spaces $T_pU$ to $T_{F(p)}V$.

$F_*$ can also be regarded as a map from $TM \to TN$ as follows:

$$(p, X_p) \mapsto (F(p), F_*(X_p))$$

*Proof of Coordinate Representation of $F_*$.* Let $X_p \in T_pM$, then $X_p = \alpha_*(v_x)$ where $v_x \in T_x\mathbb{R}^n$. Take $\gamma(t) = \alpha(x + tv)$, which satisfies $\gamma(0) = p$ and $\gamma'(0) = X_p = D\alpha(x) \cdot v$.

Then we compute:

$$\begin{aligned}
\widetilde{\gamma}(t) &= (F \circ \gamma)(t) = (F \circ \alpha)(x + tv)| \qquad = (F \circ \varphi^{-1})(x + tv) \\
&= (\psi^{-1} \circ \psi \circ F \circ \varphi^{-1})(x + tv) \\
&= (\psi^{-1} \circ \widetilde{F})(x + tv) \\
&= (\beta \circ \widetilde{F})(x + tv)
\end{aligned}$$

These are smooth functions on open subsets of euclidean space and so by the chain rule:

$$\begin{aligned}
\widetilde{\gamma}'(0) &\qquad = D\beta(y) \cdot D\widetilde{F}(x) \cdot v_x \\
F_*X_p &= \beta_*(D\widetilde{F}(x) \cdot v_x)
\end{aligned}$$

which is exactly what we wanted to show.

Special Important case: $N = \mathbb{R}$

Let $f : M \to \mathbb{R}$ be smooth. Then $f_* : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$ is a linear transformation from $T_pM \to \mathbb{R}$. In fact:

$$f_*(X_p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \gamma(t)$$

where $\gamma$ is any curve such that $\gamma(0) = p$ and $\gamma'(0) = X_p$.

This linear transformation is also denoted by $\mathrm{d}f_p$ and one can regard it as an element of the dual space $T_p^*M := (T_pM)^*$, which is often called the cotangent space at $p$ (see Definition VII.1.2).

## VI.4. Basis for the tangent space

Let $M$ be an $n$-dimensional submanifold of $\mathbb{R}^d$ and let $(U, \varphi)$ be a coordinate chart near $p \in M$ and set $\alpha = \varphi^{-1} : \widehat{U} \to U$ and $x = \varphi(p)$. Then we know that $T_pM = \alpha_*(T_x\mathbb{R}^n) = D\alpha(T_x\mathbb{R}^n)$. Then since $D\alpha$ has rank $n$, the vectors $D\alpha(e_1), D\alpha(e_2), \ldots, D\alpha(e_n)$ are $n$ linearly independent vectors in $T_pM$, forming a basis for this space.

Notation: Suppose $\varphi(p) = (x^1(p), \ldots, x^n(p))$. Then $\alpha_*(e_j)$ is often denoted by $\frac{\partial}{\partial x^j}(p)$. That is:

$$\frac{\partial}{\partial x^j}(p) = D\alpha(\varphi(p)) \cdot (e_j) = \frac{\partial \alpha}{\partial x^j}(\varphi(p))$$

Each $\frac{\partial}{\partial x^j}$ is a tangent vector field to $M$ defined for $p \in U$.

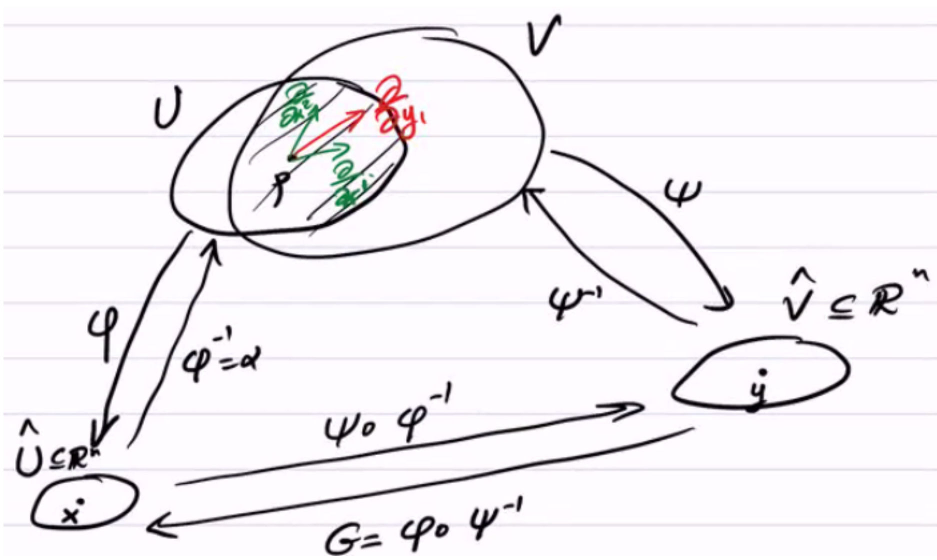
**Lemma VI.4.1**

Suppose that $(U, \varphi)$ and $(V, \psi)$ are two coordinate charts on $M$ and let $p \in U \cap V$. Denote $\alpha = \varphi^{-1}$ and $\beta = \psi^{-1}$.

Let $\varphi = (x^1, \ldots, x^n)$ and $\psi = (y^1, \ldots, y^n)$. Then the vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^j}$ for $1 \le i \le n$ and $1 \le j \le n$ coexist on the set $U \cap V$. If we denote $G = \varphi \circ \psi^{-1}$, then we have that:

$$\frac{\partial}{\partial y^i}(p) = (\alpha_*)_x (G_*)_y (e_i) = \sum_{j=1}^{n} \frac{\partial G^j}{\partial y^i} \frac{\partial}{\partial x^j}(p)$$

Where $G^j(y) = (x^j \circ \psi^{-1})(y)$. In a picture:




Abusing notation, we can write $\frac{\partial G^j}{\partial y^i} = \frac{\partial x^j}{\partial y^i}$. And then by einstein summation notaton:

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$$

*Proof.* For $p \in U \cap V$, we have that $\beta(y) = \psi^{-1}(y) = \varphi^{-1} \circ \varphi \circ \psi^{-1}(y) = \alpha \circ G(y)$. By the chain rule then:

$$\beta_*(v_y) = D\beta(y) \cdot v_y = D\alpha(x) DG(y) v_y = (\alpha_*)_x \cdot (G_*)_y (v_y)$$

Setting $v_y = (e_i)_y$ we see that:

$$\frac{\partial}{\partial y^i} = (\alpha_*)_x((G_*)_y(e_i)) = D\alpha(x) \cdot DG(y) \cdot e_i$$

$$DG(y) = \frac{\partial G}{\partial y^i} = \sum_{j=1}^{n} \frac{\partial G^j}{\partial y^i} (e_j)_x$$

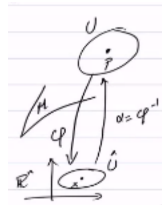
$$\frac{\partial}{\partial y^i} = \sum_{j=1}^{n} \frac{\partial G^j}{\partial y^i} \alpha_*(e_j) = \sum_{j=1}^{n} \frac{\partial G^j}{\partial y^i} \frac{\partial}{\partial x^j}$$

**Proposition VI.4.2**

Let $X$ be a vector field on $M$ (i.e. $X$ is a map from $M$ to $TM$ such that $X_p \in T_pM$).

$X$ is smooth if and only if for every coordinate chart $(U, \varphi)$, there exist smooth functions $f^1, \dots, f^n : U \to \mathbb{R}$ such that:

$$X_p = \sum_{i=1}^{n} f^i(p) \frac{\partial}{\partial x^i}(p)$$



Basis for $T_pM$. If $x = \varphi(p)$ then $\{e_1, \ldots, e_n\}$ is a basis for $T_x\mathbb{R}^n = T_x\widehat{U}$.

THen a basis of $T_pM$ is given by $D\alpha(x) \cdot e_1, \ldots, D\alpha(x) \cdot e_n$ (since $D\alpha(x)$ has rank $n$).

Notation: If we denote $\varphi(p) = (x^1(p), \ldots, x^n(p))$ then $D\alpha(\varphi(p)) \cdot e_j$ is often denoted by $\frac{\partial}{\partial x^j}(p)$. In effect, each $\frac{\partial}{\partial x^j}$ gives us a locally defined vector field on $U$.

Caution: Those $n$ vector fields are only defined on $U$ and they depend on the choice of coordinates on $U$. In fact, last time we showed that if $(V, \psi)$ is another coordinate system near $p$ and we denote $\psi = (y^1, \ldots, y^n)$, then we have on $U \cap V$ that:

$$\frac{\partial}{\partial y^i} = \sum_{j=1}^{n} \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$$

Here we what we really mean by $\frac{\partial x^j}{\partial y^i}$ is the $i$-th partial derivative of of of $y \mapsto x^j \circ \psi^{-1}(y)$. Essentially we're conflating $x^j$ with $x^j \circ \psi^{-1}$ in an abuse of notation.

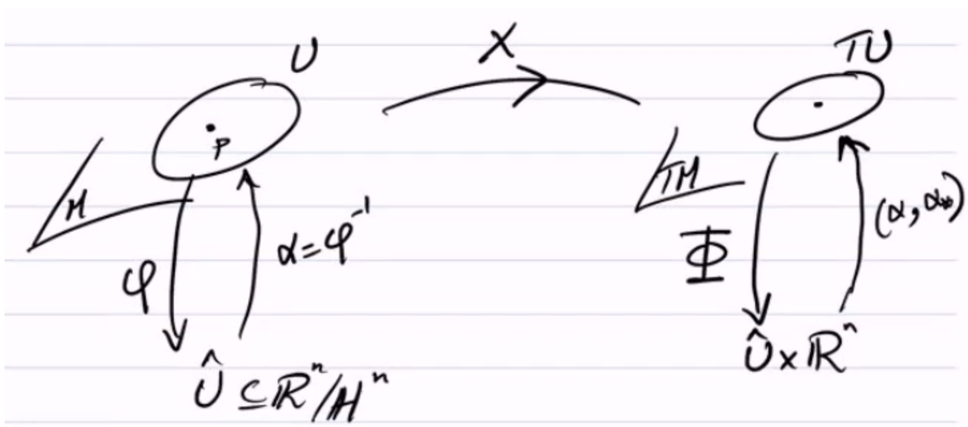
**Proposition VI.4.3**

Let $X$ be a vector field on $M$ (i.e. $X$ is a map from $M$ to $TM$ such that $X_p \in T_pM$).

$X$ is smooth if and only if for every coordinate system $(U, \varphi)$ there exist smooth functions $f^1, \ldots, f^n : U \to \mathbb{R}$ such that:

$$X_p = \sum_{i=1}^{n} f^i(p) \cdot \frac{\partial}{\partial x^i}(p)$$

*Proof.* Suppose that $X : M \to TM$ is a vector field. Let $(U, \varphi)$ be a coordinate chart on $M$, and take the corresponding coordinate chart on $TM$. In pictures:



Since $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ is a basis of $T_pM$. For all $p \in U$, there exist $f^1(p), \ldots, f^n(p)$ such that:

$$X_p = \sum_{i=1}^{n} f^i(p) \frac{\partial}{\partial x^i}(p)$$

We have that $X$ is smooth if and only if $\Phi \circ X \circ \varphi^{-1} : \widehat{U} \to \widehat{U} \times \mathbb{R}^n$ is smooth. This holds if and only if:

$$\begin{aligned} &\Phi \circ \left( \sum_{i=1}^{n} f^i(p) \frac{\partial}{\partial x^i}(p) \right) \circ \varphi^{-1} \text{ is smooth} \\ \iff\ &\Phi \circ \left[ \sum_{i=1}^{n} f^i(\varphi^{-1}(x)) \frac{\partial}{\partial x^i}(\varphi^{-1}(x)) \right] \text{ is smooth} \\ A \iff\ &\left( x, \sum_{i=1}^{n} f^i(\varphi^{-1}(x)) e_i \right) \text{ is smooth} \\ \iff\ &f^i(\varphi^{-1}(x)) \text{ is smooth } \forall\, 1 \le i \le n \\ \iff\ &f^i(p) \text{ is smooth } \forall\, 1 \le i \le n \end{aligned}$$

Perfect!

# VII. Differential Forms

## VII.1. Definitions and Operations on $k$-forms

**From now on, saying $M$ is an $n$-manifold means that $M$ is an $n$-dimensional submanifold of $\mathbb{R}^d$ for some $d$**

Given any vector space $V$, we define and manipulate tensors on this vector space, $k$-tensors $\mathscr{L}^k(V)$ (the space of $k$-tensors). In particular, we are concerned with the alternating $k$-tensors $\mathcal{A}^k(V)$.

In differential topology, the vector space $V$ is taken to be the tangent space $T_pM$ to a manifold at a point $p \in M$.

**Definition VII.1.1**

Let $M$ be a smooth manifold with or without boundary

(a) A $k$-tensor field $h$ on $M$ is a function which assigns to each $p \in M$ a $k$-tensor $h(p)$ on the tangent space $T_pM$; i.e. $h(p) \in \mathscr{L}^k(T_pM)$.

(b) A $k$-form is an alternating $k$-tensor field; i.e. it is a function $\omega$ that assigns to each $p \in M$, $\omega(p) \in \mathcal{A}^k(T_pM)$.

**Operations on $k$-forms**

a) Two $k$-forms $\omega_1$ and $\omega_2$ may be added to create a new $k$-form, and we can also take scalar multiples of $\omega$ by $c \in \mathbb{R}$:

$$(\omega_1 + \omega_2)(p) = \omega_1(p) + \omega_2(p)$$
$$(c\omega)(p) = c\omega(p)$$

b) (Wedge product). If $\omega$ is a $k$-form and $\theta$ is an $\ell$-form on $M$¡ then the $(k+\ell)$-form $\omega \wedge \theta$ is given by:

$$(\omega \wedge \theta)(p) = \omega(p) \wedge \theta(p)$$

We recall the anti-commutativity of the wedge product, which says that $\omega \wedge \theta = (-1)^{k\ell}\theta \wedge \omega$.

Convention: Smooth functions $f : M \to \mathbb{R}$ are identified with 0-forms.

### VII.1.1. Understanding 1-forms

Let's try to understand 1-forms. Let $\omega$ be a 1-form on $M$. Then $\omega(p) \in \mathcal{A}^1(T_pM) = \mathscr{L}^1(T_pM)$. I.e. $\omega(p)$ is a linear transformation $T_pM \to \mathbb{R}$, that is $\omega(p) \in (T_pM)^*$.

**Definition VII.1.2**

For a manifold $M$ and a point $p \in M$, we call $(T_pM)^*$ the cotangent space at $p$

Suppose we are given a vector field $X$ on $M$ (i.e. $X(p) \in T_pM$ for every $p \in M$) and a 1-form $\omega$. Then the function $p \mapsto (\omega(p))(X(p)) \in \mathbb{R}$ is denoted by $\omega(X)$.

The main example of 1-forms comes from taking the derivative (or pushforward map) of a smooth function $\phi : M \to \mathbb{R}$. (i.e. derivatives of 0-forms).

**Definition VII.1.3**

Let $\phi : M \to \mathbb{R}$ be a 0-form (aka a smooth function). We defined previously the push-forward map $\phi_* : T_pM \to T_{\phi(p)}\mathbb{R} = \mathbb{R}$ via:

$$\phi_*(v) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t))\Big|_{t=0}$$

where $\gamma(t)$ is any smooth curve into $M$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. We showed that this was a linear transformation from $T_pM \to T_{\phi(p)}\mathbb{R} = \mathbb{R}$. Therefore $\phi_*$ gives us a 1-form!!! Great!

This linear transformation from $T_pM \to \mathbb{R}$ is also denoted by $\mathrm{d}\phi(p)$ and it generalizes the notion of the derivative to real-valued functions on manifolds. We call $\mathrm{d}\phi$ the 1-form $\phi_*$.

Remark: We were given a 0-form $\phi$ on $M$ and we defined out of it a 1-form $\mathrm{d}\phi$ by taking a derivative of $\phi$. In the next section, we will define a generalization of this operation which will taken in a $k$-form and give out a $(k+1)$-form.

### VII.1.2. **Understanding $k$-forms on open subsets of $\mathbb{R}^n$**

Let us consider an open subset $U$ of $\mathbb{R}^n$. The coordinate functions $x^1, \dots, x^n$ are smooth functions from $U \to \mathbb{R}$. I.e. they are 0-forms.

At each point $p \in \mathbb{R}^n$ we have a basis $e_1, e_2, \dots, e_n$ of $T_p\mathbb{R}^n$. Now $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ are 1-forms on $U$. We want to understand these forms more precisely. Given a vector $v \in T_pU$, then:

$$\mathrm{d}x^j(p)(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x^j(p+tv) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(p^j + tv^j) = v^j$$

But wait! this means that $\mathrm{d}x^j(p)$ is the standard basis of $(T_pU)^*$ (i.e. the dual basis of $e_1, e_2, \dots, e_n$). More generally from our discussion on linear algebra, this means that there is a basis on $\mathcal{A}^k(T_pU)$ given by taking increasing sequences $I = (i_1, \dots, i_k)$ from $\{1, \dots, n\}$ and then considering the basis vectors

$$\mathrm{d}x^I := \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

Awesome! This will allow us to fully understand these $k$-forms on $U$.

Let $f : M \to N$ be a smooth map. We defined before the pushforward map $f_* : T_pM \to T_{f(p)}N$. We define this map by taking a tangent vector $v_p \in T_pM$ and finding a smooth curve $\gamma$ representing $v_p$, aka so that $\gamma$ lies in $M$, $\gamma(0) = p$, and $\gamma'(0) = v_p$. We then say that $f_*(v_p) = (f \circ \gamma)'(0) \in T_{f(p)}N$, noting that $f \circ \gamma$ is a smooth curve lying in $N$ and $(f \circ \gamma)(0) = f(\gamma(0)) = f(p)$. We showed that this is an unambiguous definition that generalizes the notion of a derivative.

Sometimes we write this linear transformation as $Df(p) = f_* : T_pM \to T_{f(p)}N$ (some books use $\mathrm{d}f(p)$, etc.)

Last time, we defined the notion of a $k$-form $\omega$ as a function which assigns to each $p \in M$ an alternating $k$-tensor $\omega(p)$ on the tangent space $T_pM$. We also said 0-forms are just functions $f : M \to \mathbb{R}$. We also understood 1-forms $\omega$, so that $\omega(p) \in \mathcal{A}^1(T_pM) = \mathscr{L}^1(T_pM)$, aka $\omega(p)$ is a linear map from $T_pM$ into $\mathbb{R}$. That is $\omega(p) \in (T_pM)^*$, which is the cotangent space.

The most important example of 1-forms are derivatives of 0-forms. Let $\phi : M \to \mathbb{R}$ be a smooth function (i.e. a 0-form). Then $D\phi(p) : T_pM \to T_{\phi(p)}\mathbb{R} \cong \mathbb{R}$ is a linear transformation. Therefore $D\phi$ (often written $\mathrm{d}\phi$) is a 1-form on $M$. Given $\vec{v} \in T_pM$ we have that:

$$\mathrm{d}\phi_p(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi(\gamma(t))$$

Where $\gamma(t)$ is any smooth curve in $M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$

Check: On $\mathbb{R}^d$ we have $\mathrm{d}\phi_p(v) = D\phi(p) \cdot v$, that is the directional derivative of $\phi$ at $p$ in the direction of $v$.

Also last time we investigated $k$-forms on open subsets of $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$ be open and let $x^1, \dots, x^n$ denote the standard coordinate functions. These are smooth, so $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ are 1-forms on $U$. We saw that $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ is actually the dual basis to the basis $e_1, \dots, e_n$ of $T_pU \cong \mathbb{R}^n$. Therefore a basis for $\mathcal{A}^k(T_pM)$ is given by taking wedge products:

$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

Where $I = (i_1, \dots, i_K)$ is an ascending index set from $\{1, \dots, n\}$.

**Proposition VII.1.1**

Every $k$-form on an open subset $U \subseteq \mathbb{R}^n$ can be uniquely expressed as a linear combination as below:

$$\omega = \sum_I f_I \, \mathrm{d}x^I$$

Where the sum is taken over all ascending index sets $I = (i_1, \dots, i_k)$ and $f_I$ is a function $U \to \mathbb{R}$, and $\mathrm{d}x^I$ is given as above.

**Definition VII.1.4**

We say that a $k$-form on an open subset $U$ of $\mathbb{R}^n$ is smooth provided that every $f_I$ in the expansion $\omega = \sum_I f_I \, \mathrm{d}x^I$ is smooth.

**Example VII.1.1**

Let $\phi : M \to \mathbb{R}$. We just defined the 1-form $\mathrm{d}\phi$. This means:

$$\mathrm{d}\phi = \sum_{i=1}^n f_i \, \mathrm{d}x^i$$

What is $f_i$? Well:

$$\begin{aligned}
\mathrm{d}\phi_p(v^1, \dots, v^n) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi(p + tv) \\
&= D\phi(p) \cdot v \\
&= \sum_{i=1}^n \frac{\partial \phi}{\partial x^i}(p) \cdot v^i \\
&= \sum_{i=1}^n \frac{\partial \phi}{\partial x^i}(p) \cdot \mathrm{d}x^i(v)
\end{aligned}$$

Therefore:

$$\mathrm{d}\phi_p = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x^i}(p)\,\mathrm{d}x^i$$
$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x^i}\,\mathrm{d}x^i$$

This is an important formula!

## VII.2. **Pullback on $k$-forms**

Two things to recall:

(1) If $T : V \to W$ is a linear transformation between vector spaces and if $\omega$ is a $k$-tensor on $W$ then $T^*\omega$ is a $k$-tensor on $V$ defined by:

$$T^*\omega(v_1, \dots, v_k) = \omega(Tv_1, \dots, Tv_k)$$

(2) Given $f : M \to N$ smooth, we have a natural linear transformation $Df(p) : T_pM \to T_{f(p)}N$

Combining these two points, we arrive at the following definition:

**Definition VII.2.1**

Let $f : M \to N$ be a smooth map between two smooth manifolds. Denote by $Df(p)$ the linear transformation from $T_pM$ into $T_{f(p)}N$ given by the pushforward map.

Given a $k$-form $\omega$ on $N$, we define the pullback of $\omega$ by $f$, denoted $f^*\omega$, to be the $k$-form defined by the formula:

$$f^*\omega(p) = [Df(p)]^*\omega(f(p))$$
$$f^*\omega(p)(v_1, \dots, v_k) = \omega(f(p))(Df(p) \cdot v_1, \dots, Df(p) \cdot v_k)$$

**Remark VII.2.1**

We pushforward tangent vectors, but we pullback cotangent vectors and more generally $k$-forms using smooth functions $f : M \to N$.

**Proposition VII.2.1**

Suppose $f : M \to N$ and $h : N \to K$ are smooth. Let $\omega_1, \omega_2$ be $k$-forms on $N$ and $\theta$ be an $\ell$-form on $N$.

(1) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$
(2) $f^*(\omega_1 \wedge \theta) = f^*\omega \wedge f^*\theta$
(3) $(f \circ h)^*\omega = h^* f^*\omega$

*Proof.* The first piece is an exercise. For the second part, write $y = f(p)$ and see:

$$\begin{aligned}
f^*(\omega \wedge \theta)_p(v_1, \dots, v_{k+\ell}) &= (\omega \wedge \theta)_y(f_* v_1, \dots, f_* v_{k+\ell}) \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn}\sigma)(\omega_y \otimes \theta_y)(Df(p) \cdot v_{\sigma(1)}, \dots, Df(p) \cdot v_{\sigma(k+\ell)}) \\
&= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)[Df(p)]^*\omega_y \otimes [Df(p)]^*\theta_y(v_{\sigma(1)}, \dots, v_{\sigma(k+\ell)}) \\
&= [Df(p)]^*\omega_y \wedge [Df(p)]^*\theta_y(v_1, \dots, v_{k+\ell}) \\
&= f^*\omega_p \wedge f^*\theta_p(v_1, \dots, v_{k+\ell})
\end{aligned}$$

Therefore $f^*(\omega \wedge \theta) = f^*\omega \wedge f^*\theta$.

(3) is also an exercise, using the following theorem

**Theorem VII.2.2** (Chain rule on manifolds)
et $f : M \to N$ and $g : N \to \mathcal{O}$ be smooth. Then $g \circ f : M \to \mathcal{O}$ is smooth, and furthermore:

$$(g \circ f)_* = g_* \circ f_*$$

Or equivaently:

$$D(g \circ f)(p) = Dg(f(p)) \circ Df(p)$$

*Proof.* Homework!

**Remark VII.2.2**
For (3) in the proposition, what we use is that for linear maps $T : V \to W$, $S : W \to X$, then:

$$(S \circ T)^* = T^* \circ S^*$$

We use this and the theorem above to give the proof of (3).

### VII.2.1. Pullback operation on $\mathbb{R}^n$ in coordinates

Let $V \subseteq \mathbb{R}^\ell$ and $U \subseteq \mathbb{R}^n$ be open, and let $f : V \to U$ be smooth. The question is what is $f^*\omega$ in coordinates. Let $x^1, \dots, x^n$ be the standard coordinates on $U$ and $y^1, \dots, y^\ell$ the standard coordinates on $V$.

Write the $k$-from $\omega$ as $\omega = \sum_I a_I \, \mathrm{d}x^I$ as before where $I = (i_1, \dots, i_k)$ ranges over all ascending index sets and $a_I : U \to \mathbb{R}$ is a function (aka 0-form) and $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$.

Then we see that:

$$\begin{aligned} f^*\omega &= f^*\left(\sum_I a_I \, \mathrm{d}x^I\right) \\ &= \sum_I f^*(a_I \, \mathrm{d}x^I) \\ &= \sum_I f^*(a_I) f^*(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) \\ &= \sum_I f^*(a_I) f^*(\mathrm{d}x^{i_1}) \wedge \cdots \wedge f^*(\mathrm{d}x^{i_k}) \end{aligned}$$

Where we've used the fact that since $a_I$ is a 0-form that $a_I \, \mathrm{d}x^I = a_I \wedge \mathrm{d}x^I$. Now the question is what is $f^*(a_I)$ and $f^*(\mathrm{d}x^j)$? Well then $f^*(a_I)$ is the pullback of the 0-form $a_I$ which is $f^*(a_I) = a_I \circ f$.

Let $M, N$ be manifolds and $f : M \to N$ be a smooth map. Given a $k$-form $\omega$ on $N$, we can define the pullback of $\omega$ by $f$ as:

$$f^*\omega_p(v_1, \ldots, v_k) = \omega_{f(p)}(f_*v_1, \ldots, f_*v_k) = \omega_{f(p)}(Df(p) \cdot v_1, \ldots, Df(p) \cdot v_k)$$

We have some nice properties of pullback of $k$-forms given in Proposition VII.2.1. We copy them here:

(1) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$
(2) $f^*(\omega_1 \wedge \theta) = f^*\omega \wedge f^*\theta$
(3) $(f \circ h)^*\omega = h^* f^*\omega$

Also for 0-forms on $N$, aka functions of the form $N \to \mathbb{R}$, we define $f^*$ via the simple formula $f^*\phi = \phi \circ f$, which pulls back $\phi$ to a function $M \to \mathbb{R}$.

Last time we were trying to understand pullback on open subsets of euclidean space. Namely if we have open sets $V \subseteq \mathbb{R}^\ell$ and $U \subseteq \mathbb{R}^m$, a smooth map $f : V \to U$, and a $k$-form $\omega$ on $U$. Then:

$$\omega = \sum_I a_I \, \mathrm{d}x^I$$

Where $I = (i_1, \ldots, i_k)$ is an ascending index set from $\{1, \ldots, n\}$, $a_I : U \to \mathbb{R}$, and:

$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

Since $\mathrm{d}x^I(p)$ is a basis for $\mathcal{A}^k(T_pU)$. Our questions is what is $f^*\omega$ in the standard coordinates $y^1, \ldots, y^\ell$ on $V$.

By the given properties of the pullback discussed above, we may compute that:

$$\begin{aligned}
\omega &= \sum_I a_I \, \mathrm{d}x^I \\
f^*\omega &= f^*\left(\sum_I a_I \, \mathrm{d}x^I\right) = \sum_I f^*(a_I \, \mathrm{d}x^I) \\
&= \sum_I f^*(a_I \, \mathrm{d}x^I) = \sum_I f^*(a_I) f^*(\mathrm{d}x^I) \\
&= \sum_I (a_I \circ f) f^*(\mathrm{d}x^I)
\end{aligned}$$

So now we reach the natural question (where we stopped last time), what is $f^*(\mathrm{d}x^I)$. Well we know:

$$f^*(\mathrm{d}x^I) = (f^* \, \mathrm{d}x^{i_1}) \wedge \cdots \wedge (f^* \, \mathrm{d}x^{i_k})$$

Wel what is $f^* \, \mathrm{d}x^i$? Let $z \in V$ and $v_z \in T_zV$. Then we write:

$$\begin{aligned}
f^* \, \mathrm{d}x^i_z(v_z) &= \mathrm{d}x^i_{f(z)}(Df(z) \cdot v_z) \\
&= \mathrm{d}x^i_{f(z)}\left(\sum_{j=1}^{\ell} v_z^j \frac{\partial f}{\partial y^i}\right) = \sum_{j=1}^{\ell} \mathrm{d}x^i_{f(z)}\left(v_z^j \frac{\partial f}{\partial y^i}\right) \\
&= \sum_{j=1}^{\ell} \frac{\partial f^i}{\partial y^j} v_z^j = \sum_{j=1}^{\ell} \frac{\partial f^i}{\partial y^j} \, \mathrm{d}y^j(v_z) \\
f^* \, \mathrm{d}x^i &= \sum_{j=1}^{\ell} \frac{\partial f^i}{\partial y^j} \, \mathrm{d}y^j
\end{aligned}$$

Another way to write this is:

$$f^* \, \mathrm{d}x^i = \mathrm{d}f^* x^i = \mathrm{d}(x^i \circ f) = \mathrm{d}f^i$$

This identity here is general. We'll prove this on the current homework that $f^*(\mathrm{d}\phi) = \mathrm{d}(f^*\phi) = \mathrm{d}(\phi \circ f)$. Then of course we have:

$$\begin{aligned} \mathrm{d}f^I &= \mathrm{d}f^{i_1} \wedge \cdots \wedge \mathrm{d}f^{i_k} \\ f^*\omega &= f^*\left(\sum_I a_I \,\mathrm{d}x^I\right) = \sum_I f^*(a_I \,\mathrm{d}x^I) \\ &= \sum_I f^*(a_I \,\mathrm{d}x^I) = \sum_I f^*(a_I) f^*(\mathrm{d}x^I) \\ &= \sum_I (a_I \circ f) f^*(\mathrm{d}x^I) \\ &= \sum_I (a_I \circ f) \,\mathrm{d}f^I \end{aligned}$$

**Corollary VII.2.3**
Suppose that $\omega = \sum_I a_I \,\mathrm{d}x^I$ is a smooth $k$-form on an open subset $U \subseteq \mathbb{R}^n$ (i.e. the functions $a_I : U \to \mathbb{R}$ are smooth). Then let $f : V \to U$ be a smooth map where $V \subseteq \mathbb{R}^\ell$ is open. Then $f^*\omega$ is smooth as well.

*Proof.* We write

$$\begin{aligned} f^*\omega &= \sum_I (a_I \circ f) \,\mathrm{d}f^I \\ \mathrm{d}f^i &= \sum_{j=1}^{\ell} \frac{\partial f^i}{\partial y^j} \,\mathrm{d}y^j \\ f^*\omega &= \sum_{I=(i_1,\ldots,i_k)} (a_I \circ f) \bigwedge_{m=1}^{k} \left( \sum_{j=1}^{\ell} \frac{\partial f^{i_m}}{\partial y^j} \,\mathrm{d}y^j \right) \end{aligned}$$

Then since $a_I \circ f$ is always smooth and $\frac{\partial f^{i_m}}{\partial y^i}$ is always smooth, we know that this will break down into a linear combination of smooth functions for the coefficients. Thus $f^*\omega$ is a smooth $k$-form on $V$.

Special important case
Suppose we have open sets $U, V \subseteq \mathbb{R}^n$ and a smooth map $f : V \to U$. Then let $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Then we have that:

$$f^*\omega_y = \mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^n$$

So we recall that:

$$\begin{aligned} \mathrm{d}f^i &= \sum_{j=1}^{n} \frac{\partial f^i}{\partial y^j} \,\mathrm{d}y^j \\ \mathrm{d}f^i_y(e_j) &= \frac{\partial f^i}{\partial y^j}(y) \end{aligned}$$

And this allows us to compute:

$$\begin{aligned} f^*\omega_y(e_1,\dots,e_n) &= (\mathrm{d}f^1\wedge\cdots\wedge \mathrm{d}f^n)(e_1,\dots,e_n) \\ &= \sum_{\sigma\in S_n}(\operatorname{sgn}\sigma)\,\mathrm{d}f^1\otimes\cdots\otimes(e_{\sigma(1)},\dots,e_{\sigma(n)}) = \sum_{\sigma\in S_n}(\operatorname{sgn}\sigma)\frac{\partial f^1}{\partial y^{\sigma(1)}}\cdots\frac{\partial f^n}{\partial y^{\sigma(n)}} \\ &= \det\begin{pmatrix}\frac{\partial f^1}{\partial y^1} & \cdots & \frac{\partial f^1}{\partial y^n}\\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial y^1} & \cdots & \frac{\partial f^n}{\partial y^n}\end{pmatrix} = \det[Df(y)]^T = \det Df(y)\end{aligned}$$

But then $f^*\omega_y$ is an alternating multilinear $n$-form on $T_yV\cong\mathbb{R}^n$. So then:

$$f^*\omega_y = c\,\mathrm{d}y^1\wedge\cdots\wedge dy^n$$

To determine $c$, we compute:

$$\det Df(y) = f^*\omega_y(e_1,\dots,e_n) = c\,\mathrm{d}y^1\wedge\cdots\wedge dy^n(e_1,\dots,e_n) = c\det I = c$$

therefore, we get the following formula:

$$f^*\omega_y = (\det Df(y))\,\mathrm{d}y^1\wedge\cdots\wedge\mathrm{d}y^n$$

**Theorem VII.2.4**

Let $f : V\to U$ be a smooth map for open subsets $V, U\subseteq\mathbb{R}^n$. Let $x^1,\dots,x^n$ be the standard coordinates on $U$ and $y^1,\dots,y^n$ the standard coordinates on $V$. Then we have that for $y\in V$:

$$f^*(\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n)_y = (\det Df(y))\,\mathrm{d}y^1\wedge\cdots\wedge\mathrm{d}y^n$$

The above calculation gives the proof

**Remark VII.2.3**

In other words, a change of coordinates from $y\in V$ to $f(y)\in U$ gives a multiplicative factor of $\det Df(y) = \frac{\partial(f_1,\dots,f_n)}{\partial y^1\cdots\partial y^n}$. This is the same multiplicative factor (up to signs) that appears in the change of coordinates theorem for integration last semester.

## VII.3. Smooth Forms on Manifolds

**Definition VII.3.1**

Let $\omega$ be a $k$-form on an $n$-dimensional manifold $M$. We say that $\omega$ is a smooth $k$-form provided that for every coordinate chart $(U,\varphi)$ on $M$, where $\varphi : U\to\widehat{U}\subseteq\mathbb{R}^n$ and $\alpha := \varphi^{-1} : \widehat{U}\to U\subseteq\mathbb{R}^d$, we have that $\alpha^*\omega$ is a smooth form on $\widehat{U}\subseteq\mathbb{R}^n$.

Smooth forms are also called differential forms

**Remark VII.3.1**

Of course, to check that a $k$-form $\omega$ on $M$ is smooth, it is enough to show that for some atlas $(U_\gamma,\varphi_\gamma)$ of $M$, there holds that $\alpha_\gamma^*\omega$ is smooth for every $\gamma$ where $\alpha_\gamma = \varphi_\gamma^{-1}$.

Here is an equivalent way to phrase the definition of smoothness for a $k$-form:

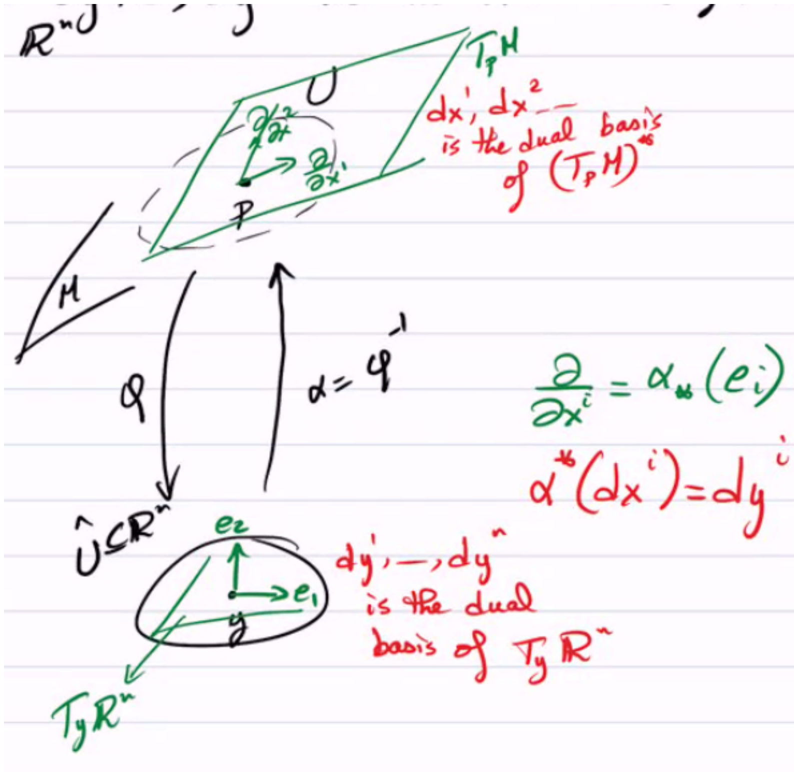
**Proposition VII.3.1**

Let $(U,\varphi)$ be given as $\varphi = (x^1,\dots,x^n)$, were each $x^i : U\to\mathbb{R}$ is a smooth function. Then we have $n$ 1-forms $\mathrm{d}x^1,\dots,\mathrm{d}x^n$ defined on $U$. Recall that the vector fields $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}$ give a basis for $T_pM$ for every $p\in U$.

a) There holds $\mathrm{d}x_p^j\left(\frac{\partial}{\partial x^i}(p)\right) = \delta_i^j = \begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$

In other words $\mathrm{d}x_p^1,\dots,\mathrm{d}x_p^n$ is the basis of $(T_pM)^*$ dual to $\frac{\partial}{\partial x^1}(p),\dots,\frac{\partial}{\partial x^n}(p)$.

b) With $\alpha = \varphi^{-1}$ we have that $\alpha^*(\mathrm{d}x^i) = \mathrm{d}y^i$ where $\mathrm{d}y^1,\dots,\mathrm{d}y^n$ are the standard 1-forms on $\mathbb{R}^n$.




*Proof.* Let $\alpha = \varphi^{-1}$ and recall that $\frac{\partial}{\partial x^i}(p) = \alpha_*(e_i) = D\alpha(y) \cdot e_i$, where $y = \varphi(p)$. Using that $\varphi \circ \alpha = \text{Id}$ we have that:

$$\begin{aligned} \varphi \circ \alpha(y) &= y \\ x^i \circ \alpha(y) &= y^i \end{aligned}$$

Now then we have that:

$$\mathrm{d}x^j(p)\left(\frac{\partial}{\partial x^i}(p)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} x^j(\gamma(t))$$

Where $\gamma$ is any curve in $M$ such that $\gamma(0) = p$ and $\gamma'(0) = \frac{\partial}{\partial x^i}$. One such curve is $\gamma(t) = \alpha(y + te_i)$. Then we have:

$$\begin{aligned} \mathrm{d}x^j(p)\left(\frac{\partial}{\partial x^i}(p)\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} x^j(\alpha(y + te_i)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (y^j + \delta_i^j) = \delta_i^j \end{aligned}$$

This gives part (a). Now for part (b) we use Homework 8 to write that:

$$\alpha_*(\mathrm{d}x^i) = \mathrm{d}(\alpha^* x^i) = \mathrm{d}(x^i \circ \alpha) = \mathrm{d}y^i$$

This finishes the proof ☺.

Next time, we will formulate an equivalent definition of the smoothness of a $k$-form $\omega$ in terms of the forms $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ defined above.

**Proposition VII.3.2**
Let $\omega$ be a $k$-form defined on $M$. $\omega$ is smooth if and only if for every coordinate patch $(U, \varphi)$ where $\varphi = (x^1, \dots, x^n)$ the coefficients functions $f_I : U \to \mathbb{R}$ in the below expression are smooth:

$$\omega = \sum_I f_I \, \mathrm{d}x^I$$

*Proof.* $\omega$ is smooth if and only if $\alpha^*$ is smooth on $\widehat{U} \subseteq \mathbb{R}^n$ where $\alpha = \varphi^{-1} : \widehat{U} \to U$ for every chart $(U, \varphi)$.

Using $\mathrm{d}y^1, \dots, \mathrm{d}y^n$ to denote the standard 1-forms on $\mathbb{R}^n$, we have:

$$\alpha^* \omega = \sum_I g_I \, \mathrm{d}y^I$$

for some $g_I : \widehat{U} \to \mathbb{R}$. Thus $\omega$ is smooth if and only if each $g_I$ are smooth

**Claim**
$f_I(p) = g_I \circ \varphi(p)$

The claim gives the the result since $f_I$ is smooth on $U$ if and only if $f_I \circ \varphi^{-1}$ is smooth on $\widehat{U}$ if and only if $g_I$ is smooth if and only if $\omega$ is smooth.

To prove this claim, we note that:

$$\begin{aligned}
\alpha^* \omega = \alpha^* \left( \sum_I f_I \, \mathrm{d}x^I \right) &= \sum_I \alpha^*(f_I \, \mathrm{d}x^I) \\
&= \sum_I \alpha^*(f_I) \alpha^*(\mathrm{d}x^I) = \sum_I (f_I \circ \alpha) \alpha^*(\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}) \\
&= \sum_I (f_I \circ \alpha) \alpha^*(\mathrm{d}x^{i_1}) \wedge \dots \wedge \alpha^*(\mathrm{d}x^{i_k}) = \sum_I (f_I \circ \alpha) \, \mathrm{d}y^{i_1} \wedge \dots \wedge \mathrm{d}y^{i_k} \\
&= \sum_I (f_I \circ \alpha) \, \mathrm{d}y^I
\end{aligned}$$

And thus $g_I = f_I \circ \alpha = f_I \circ \varphi^{-1}$. With this $f_I = g_I \circ \varphi$ as desired.

# VIII. The Exterior Derivative

We already saw that if $\phi : M \to \mathbb{R}$ is a smooth 0-form (i.e., a smooth function), then we obtain a 1-form $\mathrm{d}\phi$ defined as:

$$\mathrm{d}\phi_p(v) = \phi_*(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\phi \circ \gamma)(t)$$

Where $\gamma(t)$ is a smooth curve in $M$ such that $\gamma(0) = 0$ and $\gamma'(0) = v$. Intuitively this is "the directional derivative of $\phi$ at $p$ in the direction of $v$."

In this section, we generalize this to $k$-forms. We start on Euclidean Space

## VIII.1. Exterior Derivative on Euclidean Spaces

**Definition VIII.1.1**
Let $U \subseteq \mathbb{R}^n$ be open and suppose that $\omega$ is a differential $k$-form on $U$. Then:

$$\omega = \sum_I a_I \, \mathrm{d}x^I$$

where $a_I : U \to \mathbb{R}$ are smooth. We define the exterior derivative f $\omega$ to be the following $(k+1)$-form:

$$\mathrm{d}\omega = \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I$$

**Example VIII.1.1**

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a 0-form. Then in last section we established that:

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \,\mathrm{d}x^j$$

**Example VIII.1.2**

Consider the 1-form on $\mathbb{R}^3$ given by $\omega = f_1 \,\mathrm{d}x^1 + f_2 \,\mathrm{d}x^2 + f_3 \,\mathrm{d}x^3$. Then:

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}f_1 \wedge \mathrm{d}x^1 + \mathrm{d}f_2 \wedge \mathrm{d}x^2 + \mathrm{d}f_2 \wedge \mathrm{d}x^2 \\
\mathrm{d}f_1 \wedge \mathrm{d}x^1 &= \left( \frac{\partial f_1}{\partial x^1} \,\mathrm{d}x^1 + \frac{\partial f_1}{\partial x^2} \,\mathrm{d}x^2 + \frac{\partial f_1}{\partial x^3} \,\mathrm{d}x^3 \right) \wedge \mathrm{d}x^1 \\
&= \frac{\partial f_1}{\partial x^1} \,\mathrm{d}x^1 \wedge \mathrm{d}x^1 + \frac{\partial f_1}{\partial x^2} \,\mathrm{d}x^2 \wedge \mathrm{d}x^1 + \frac{\partial f_1}{\partial x^3} \,\mathrm{d}x^3 \wedge \mathrm{d}x^1 \\
&= -\frac{\partial f_1}{\partial x^2} \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \frac{\partial f_1}{\partial x^3} \,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \\
\mathrm{d}f_2 \wedge \mathrm{d}x^2 &= \frac{\partial f_2}{\partial x^1} \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \frac{\partial f_2}{\partial x^3} \,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
\mathrm{d}f_3 \wedge \mathrm{d}x^3 &= \frac{\partial f_3}{\partial x^1} \,\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \frac{\partial f_3}{\partial x^2} \,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
\mathrm{d}\omega &= \left( \frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&\quad + \left( \frac{\partial f_1}{\partial x^3} - \frac{\partial f_3}{\partial x^1} \right) \mathrm{d}x^3 \wedge \mathrm{d}x^1 \\
&\quad + \left( \frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= g_1 \,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + g_2 \,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + g_3 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2
\end{aligned}$$

If we stare at this for a while, we realize it is *eerily* similar to the curl of $(f_1, f_2, f_3)$.

$$g = (g_1, g_2, g_3) = \operatorname{curl}(f_1, f_2, f_3)$$

As we see:

$$\begin{aligned}
\operatorname{curl}(f_1, f_2, f_3) &= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{j} \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ f_1 & f_2 & f_3 \end{vmatrix} \\
&= \left( \frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) \widehat{i} - \left( \frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3} \right) \widehat{j} + \left( \frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) \widehat{k}
\end{aligned}$$

**Example VIII.1.3**

On $\mathbb{R}^3$ any 2-form can be written as:

$$\omega = f_1 \,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + f_2 \,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + f_3 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

Then in fact we will have that:

$$\begin{aligned}
\mathrm{d}\omega &= \frac{\partial f_1}{\partial x^1} \cdot \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \frac{\partial f_2}{\partial x^2} \cdot \mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \frac{\partial f_3}{\partial x^3} \cdot \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \left( \frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3} \right) \cdot \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3
\end{aligned}$$

**Theorem VIII.1.1**

Let $U \subseteq \mathbb{R}^n$ be open. The exterior differentiation operator d defined on smooth forms $\omega$ on $U$ satisfies the following properties

a) Linearity: $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$
b) Multiplication Law: If $\omega$ is a $k$-form and $\theta$ any $\ell$-form:
$$\mathrm{d}(\omega \wedge \theta) = \mathrm{d}\omega \wedge \theta + (-1)^k \omega \wedge \mathrm{d}\theta$$
c) The cocycle condition. $\mathrm{d}(\mathrm{d}\omega) = 0$

Furthermore, this is the only operator on $k$-forms that exhibits these properties and agrees with the previous definition of $\mathrm{d}f$ for smooth functions $f : U \to \mathbb{R}$

*Proof.* a) is left as an exercise.

For b). Write $\omega = \sum_{I_1} a_{I_1}\, \mathrm{d}x^{I_1}$ and $\theta = \sum_{I_2} b_{I_2}\, \mathrm{d}x^{I_2}$. Then we have that:
$$\omega \wedge \theta = \sum_{I_1, I_2} a_{I_1} b_{I_2}\, \mathrm{d}x^{I_1} \wedge \mathrm{d}x^{I_2}$$
$$\mathrm{d}(\omega \wedge \theta) = \sum_{I_1, I_2} \mathrm{d}(a_{I_1} b_{I_2}) \wedge \mathrm{d}x^{I_1} \wedge \mathrm{d}x^{I_2}$$
Now we use that for smooth functions $f, g : U \to \mathbb{R}$ that by the product rule:
$$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$$
Then we get that:
$$\begin{aligned}
\mathrm{d}(\omega \wedge \theta) &= \sum_{I_1, I_2} (b_{I_2}\, \mathrm{d}a_{I_1} + a_{I_1}\, \mathrm{d}b_{I_2}) \wedge \mathrm{d}x^{I_2} \\
&= \sum_{I_1, I_2} b_{I_2}\, \mathrm{d}a_{I_1} \wedge \mathrm{d}x^{I_1} \wedge \mathrm{d}x^{I_2} + \sum_{I_1, I_2} a_{I_1}\, \mathrm{d}b_{I_2} \wedge \mathrm{d}x^{I_1} \wedge \mathrm{d}x^{I_2} \\
&= \sum_{I_2} b_{I_2} \left( \sum_{I_1} \mathrm{d}a_{I_1} \wedge \mathrm{d}x^{I_1} \right) \wedge \mathrm{d}x^{I_2} + \sum_{I_2} a_{I_1} \left[ (-1)^k\, \mathrm{d}x^{I_1} \wedge \mathrm{d}b_{I_2} \right] \wedge \mathrm{d}x^{I_2} \\
&= \sum_{I_2} b_{I_2}\, \mathrm{d}\omega \wedge \mathrm{d}x^{I_2} + (-1)^k \sum_{I_1} a_{I_1}\, \mathrm{d}x^{I_1} \wedge \left( \sum_{I_2} \mathrm{d}b_{I_2} \wedge \mathrm{d}x^{I_2} \right) \\
&= \mathrm{d}\omega \wedge \left( \sum_{I_2} b_{I_2} \wedge \mathrm{d}x^{I_2} \right) + (-1)^k \left( \sum_{I_1} a_{I_1}\, \mathrm{d}x^{I_1} \right) \wedge \mathrm{d}\theta \\
&= \mathrm{d}\omega \wedge \theta + (-1)^k \omega \wedge \mathrm{d}\theta
\end{aligned}$$
c) is left as a homework. It's a similar proof to part (b).

To show uniqueness, suppose that $D$ is another operator satisfying a), b), and c) so that $Df = \mathrm{d}f$ for smooth functions $f : U \to \mathbb{R}$. We observe that by b) and c):
$$\begin{aligned}
D(\mathrm{d}x^I) &= D(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) \\
&= \sum \pm\, \mathrm{d}x^{i_1} \wedge \cdots \wedge (D\, \mathrm{d}x^{i_j}) \wedge \cdots \wedge \mathrm{d}x^{i_k} \\
D(\mathrm{d}x^k) &= D(Dx^k) = 0 \\
D(\mathrm{d}x^I) &= 0
\end{aligned}$$
Now if $\omega = \sum_I a_I\, \mathrm{d}x^I$ is any $k$-form, then we compute by a) and b) that:
$$\begin{aligned}
D\omega &= \sum_I D(a_I\, \mathrm{d}x^I) \\
&= \sum_I D(a_I) \wedge \mathrm{d}x^I + (-1)^k a_I \wedge D(\mathrm{d}x^I) \\
&= \sum_I D(a_I) \wedge \mathrm{d}x^I = \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I = \mathrm{d}\omega
\end{aligned}$$

Last time, we defined the exterior derivative on Euclidean space (Definition VIII.1.1).

**Corollary VIII.1.2**
Suppose that $g : V \to U$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ (or $\mathbb{H}^n$). Then for every $k$-form $\omega$ on $U$, we have

$$\mathrm{d}(g^*\omega) = g^*(\mathrm{d}\omega)$$

**Remark VIII.1.1**
We will show later that this is actually true for any smooth $g$.

*Proof.* Let $\omega$ be a $k$-form on $U$, then:

$$(g^{-1})^* g^* \omega = \omega$$

Since by HW8: $f^* g^* \omega = (g \circ f)^* \omega$.

To show the corollary, it is enough to show that $(g^*)^{-1}\,\mathrm{d}(g^*\omega) = \mathrm{d}\omega$.

Let $D = (g^{-1})^*\,\mathrm{d}g^*$. Then $D$ is an operator on differential $k$-forms that satisfies the defining properties of the exterior derivative (see Theorem VIII.1.1). Furthermore, for any smooth function $f : U \to \mathbb{R}$ we have:

$$\begin{aligned} Df &= (g^{-1})^*\,\mathrm{d}(g^* f) = (g^{-1})^*\,\mathrm{d}(f \circ g) \\ &= \mathrm{d}(g^{-1})^* f \circ g) = \mathrm{d}(f \circ g \circ g^{-1}) = \mathrm{d}f \end{aligned}$$

Therefore $D = \mathrm{d}$ by the uniqueness property of the exterior derivative.

## VIII.2. The Exterior Derivative on Manifolds

The relation $\mathrm{d}g^* = g^* \circ \mathrm{d}$ that we just proved on Euclidean space is exactly what we need to extend the definition of d to smooth manifolds.

**Definition VIII.2.1**
Suppose that $\omega$ is a smooth $k$-form on a manifold $M$ (with or without boundary). We define $\mathrm{d}\omega$ locally as follows.

Let $(U, \varphi)$ be a coordinate chart on $M$, and let $\alpha = \varphi^{-1}$. Define:

$$\mathrm{d}\omega = \varphi^*\,\mathrm{d}\alpha^*\omega$$

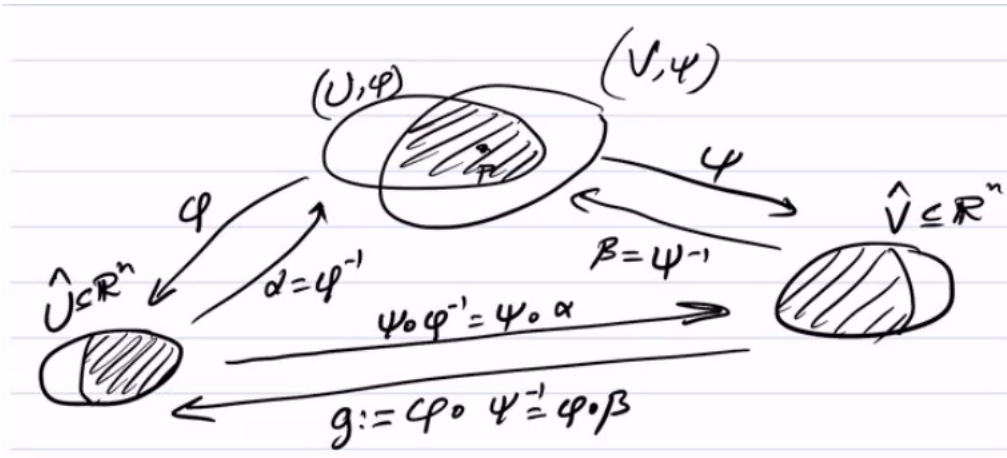
**Remark VIII.2.1**
We need to check some things:

1) This is a definition using coordinates, so we need to prove that it is well-defined (see the next proposition)
2) $\varphi_*$ is just the inverse of $\alpha_* : T_x\mathbb{R}^n \to T_pM$ by the chain rule on manifolds. This implies that $\varphi^*$ is the inverse of $\alpha^* : \mathcal{A}^k(T_pM) \to \mathcal{A}^k(T_x\mathbb{R}^n)$.

**Proposition VIII.2.1**
The above definition makes sense. If $(V, \psi)$ is another coordinate chart such that $\beta = \psi^{-1}$ and $V \cap U \neq \emptyset$, then on $U \cap V$ we have that:

$$\phi^*\,\mathrm{d}\alpha^*\omega = \psi^*\,\mathrm{d}\beta^*\omega$$

*Proof.* Here is the picture:




Let $g := \varphi \circ \psi^{-1}$, then this is a diffeomorphism on open subsets of $\mathbb{R}^n$, we have by the previous corollary that:

$$\begin{aligned} g^* \,\mathrm{d}(\alpha^*\omega) &= \mathrm{d}(g^*\alpha^*\omega) \\ &= \mathrm{d}((\alpha \circ g)^*\omega) \\ &= \mathrm{d}(\beta^*\omega) \end{aligned}$$

Therefore we have that:

$$\begin{aligned} (\psi^{-1})^*\varphi^* \,\mathrm{d}(\alpha^*\omega) &= \mathrm{d}(\beta^*\omega) \\ \varphi^* \,\mathrm{d}(\alpha^*\omega) &= \psi^* \,\mathrm{d}(\beta^*\omega) \end{aligned}$$

since $(\psi^*)^{-1} = (\psi^{-1})^*$. This is what we wanted to show!

**Theorem VIII.2.2**

The exterior derivative of $k$-forms on manifolds enjoys the following properties

1) $\mathrm{d}(\omega_1 + c\omega_2) = \mathrm{d}\omega_1 + c\,\mathrm{d}\omega_2$ for $c \in \mathbb{R}$.
2) $\mathrm{d}(\omega \wedge \theta) = \mathrm{d}\omega \wedge \theta + (-1)^k \omega \wedge \mathrm{d}\theta$ whenever $\omega$ is a $k$-form and $\theta$ is an $\ell$-form
3) $\mathrm{d}(\mathrm{d}\omega) = 0$.
4) If $f$ is a smooth function, then $\mathrm{d}f$ agrees with the previous definition.

Such an operator is in fact unique

*Proof.* 1) is direct. 2) follows because:

$$\begin{aligned} \alpha^*(\omega \wedge \theta) &= (\alpha^*\omega) \wedge (\alpha^*\theta) \\ \mathrm{d}\alpha^*(\omega \wedge \theta) &= \mathrm{d}(\alpha^*\omega) \wedge \alpha^*\theta + (-1)^k \alpha^*\omega \wedge \mathrm{d}(\alpha^*\theta) \\ \varphi^* \,\mathrm{d}\alpha^*(\omega \wedge \theta) &= \varphi^* \left(\mathrm{d}(\alpha^*\omega) \wedge \alpha^*\theta + (-1)^k \alpha^*\omega \wedge \mathrm{d}(\alpha^*\theta)\right) \\ &= \varphi^* \,\mathrm{d}(\alpha^*\omega) \wedge \varphi^*\alpha^*\theta + (-1)^k \varphi^*\alpha^*\omega \wedge \varphi^* \,\mathrm{d}(\alpha^*\theta) \\ &= \mathrm{d}\omega \wedge \theta + (-1)^k \omega \wedge \mathrm{d}\theta \end{aligned}$$

3) is similar, and 4) works because:

$$\begin{aligned} \varphi^* \,\mathrm{d}(\alpha^* f) &= \varphi^* \,\mathrm{d}(f \circ \alpha) \\ &= \mathrm{d}(f \circ \alpha \circ \varphi) = \mathrm{d}f \end{aligned}$$

**Remark VIII.2.2**

An equivalent way to define $\mathrm{d}\omega$ is as follows. Let $\varphi = (x^1, \dots, x^n)$ be a local coordinate system, then we can write $\omega$ as:

$$\omega = \sum_I f_I \, \mathrm{d}x^I$$

Where $I$ is an ascending $k$-tuple and $f_I : U \to |R$ are smooth. Then by the above theorem (or from the definition) we have that:

$$\begin{aligned} \mathrm{d}\omega &= \sum_I \mathrm{d}f_I \wedge \mathrm{d}x^I + f_I \, \mathrm{d}(\mathrm{d}x^I) \\ &= \sum_I \mathrm{d}f_I \wedge \mathrm{d}x^I \end{aligned}$$

This is really just another way to write the original definition.

**Theorem VIII.2.3**

Let $g : N \to M$ be any smooth map between manifolds with boundary. Then for every form $\omega$ on $M$,

$$\mathrm{d}(g^*\omega) = g^*(\mathrm{d}\omega) \qquad (\star\star)$$

*Proof.* We go in steps.

Step 1: $(\star\star)$ holds if $\omega$ is a 0-form by Homework 8.

Step 2: $(\star\star)$ holds if $\omega = \mathrm{d}f$ for some smooth function $f$. In this case, $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}f = 0)$, and so the RHS of $(\star\star)$ is zero. The LHS of $(\star\star)$ is given by:

$$\mathrm{d}g^*(\mathrm{d}f) = \mathrm{d}(\mathrm{d}g^* f) = 0$$

Step 3: Let $\omega$ be any $k$-form on $M$, then $\omega$ can be represented in a local coordinate system $(U, \varphi)$ where $\varphi = (x^1, \ldots, x^n)$ as follows:

$$\begin{aligned} \omega &= \sum_I f_I \, \mathrm{d}x^I \\ g^*\omega &= \sum_I (g^* f_I)(g^* \, \mathrm{d}x^{i_1} \wedge \cdots g^* \, \mathrm{d}x^{i_k}) \\ \mathrm{d}(g^*\omega) &= \sum_I \mathrm{d}(g^* f_I) \wedge (g^* \, \mathrm{d}x^{i_1} \wedge \cdots g^* \, \mathrm{d}x^{i_k}) \\ &\quad + \sum_I \pm (g^* f_I)(g^* \, \mathrm{d}x^{i_1}) \wedge \cdots \mathrm{d}(g^* \, \mathrm{d}x^{\ell}) \wedge \cdots \wedge g^* \, \mathrm{d}x^{i_k} \\ &= \sum_I \mathrm{d}(g^* f_I) \wedge (g^* \, \mathrm{d}x^{i_1} \wedge \cdots g^* \, \mathrm{d}x^{i_k}) \\ &= \sum_I g^*(\mathrm{d}f_I) \wedge (g^* \, \mathrm{d}x^{i_1} \wedge \cdots g^* \, \mathrm{d}x^{i_k}) \\ &= g^* \left( \sum_I \mathrm{d}f_I \wedge \mathrm{d}x^I \right) \\ &= g^* \, \mathrm{d}\omega \end{aligned}$$

**Part C. Integration of Forms on Smooth Manifolds**

## IX. Orientable Manifolds

Recall from 395 that we called a basis $\{a_1, \ldots, a_n\}$ right-handed (or positive) if $\det(a_1, \ldots, a_n) > 0$. More generally, for any $n$-dimensional vector space $V$, we defined an equivalence relation on bases of $V$ as follows: Given two basis $B_1 = (v_1, \ldots, v_n)$ and $B_2 = (w_1, \ldots, w_n)$, we say that $B_1 \sim B_2$ (or $B_1, B_2$ have the same orientation) if the change of coordinate matrix ${}_{B_2}[\mathrm{Id}_V]_{B_1}$ defined by the matrix $A_i^j$ where:

$$w_i = \sum_{i=1}^{n} A_i^j v_j$$

satisfies $\det(A_i^j) > 0$. Clearly, there are only two equivalence classes. An orientation of $V$ is a choice of one of those two equivalence classes We then can say that a basis is positive or right-handed if it belongs to the chosen equivalence class.

**Example IX.0.1**

On $\mathbb{R}^n$ we chose the equivalence class containing the standard basis $(e_1, \ldots, e_n)$ to give the standard orientation defined above.

**Definition IX.0.1**

Let $g : A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^n$. We say $g$ is orientation preserving if $\det Dg > 0$ on $A$ and orientation reversing if $\det Dg < 0$.

**Remark IX.0.1**

Some relevant notes:

a) If $A$ is connected, then either $\det Dg > 0$ for all $x \in A$ or $\det Dg < 0$ for all $x \in A$, since $A_+ = \{x \in A \mid \det Dg > 0\}$ and $A_- = \{x \in A \mid \det Dg < 0\}$ are two disjoint open subsets covering $A$ (since $\det Dg \neq 0$ everywhere).

b) Recall that $Dg(x)$ is interpreted as the pushforward map $Dg(x) = g_* : T_x\mathbb{R}^n \to T_{g(x)}\mathbb{R}^n$. Then $g$ is orientation preserving if and only if for every $x \in A$, $g_*$ is orientation preserving if and only if for every $x \in A$ we have $g_*(e_1), \ldots, g_*(e_n)$ is a positively oriented basis.

**Definition IX.0.2** (Orientable Manifolds)

Let $M$ be a smooth $n$-manifold (possibly abstract).

a) Given two coordinate charts $(U, \varphi)$ and $(V, \psi)$ on $M$, we say that the two charts overlap positively if $U \cap V \neq \emptyset$ and the transition map $\varphi \circ \psi^{-1}$ is orientation preserving. i.e. $\det D(\varphi \circ \psi) > 0$.

b) If $M$ can be covered by a collection of coordinate charts, each pair of which overlap positively or don't overlap at all, then $M$ is said to be orientable. If this is not possible, $M$ is non-orientable

c) Suppose $M$ is orientable and choose a collection of coordinate charts covering $M$ that overlap positively (or don't overlap t all).

Let us adjoin to this collection all other smooth coordinate charts on $M$ that overlap these patches positively. It is easy to check that this expanded collection also overlaps itself positively.

This expanded collection defines an orientation on $M$. A manifold $M$ together with an orientaiton is called an oriented manifold

**Remark IX.0.2**

In short an orientation of $M$ is a choice of atlas like the one in b).

**Example IX.0.2**

We saw that any vector space $V$ is an $n$-manifold. The two notions of orientation here are the same. Note that $V$ is orientable, since it can be covered by one coordinate parameterization

$$(x^1, \ldots, x^n) \in \mathbb{R}^n \mapsto \sum_{j=1}^n x^j v_j$$

For any choice of basis $v_1, \ldots, v_n$ of $V$.

Given such a basis the orientation of $V$ defined in part c) of the above definition includes all coordinate charts

$$(y^1, \ldots, y^n) \in \mathbb{R}^n \mapsto \sum_{i=1}^n y^i w_i$$

Where $w_1, \ldots, w_n$ is a basis with the same orientation as $v_1, \ldots, v_n$ according to the equivalence relation defined last class between bases on $V$.

Thus the two notions of orientation agree.

Suppose we are given an atlas $\{(U_\gamma, \varphi_\gamma)\}$ of $M$ that defines the orientation. THen we can give an orientation to every $T_pM$ as follows. A basis $v_1, \ldots, v_n$ of $T_pM$ is positively oriented if it has the same orientation as $\frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^n}(p)$ for some coordinate chart $(U, \varphi)$ around $p$ with $\varphi = (x^1, \ldots, x^n)$.

To show that this orientation of $T_pM$ is well-defined, we need to show that if $(V, \psi)$ is another chart containing $p$ with $\psi = (y^1, \ldots, y^n)$ then $\frac{\partial}{\partial y^1}(p), \ldots, \frac{\partial}{\partial y^n}(p)$ has the same orientation as $\frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^n}(p)$. But this follows since $U$ and $V$ overlap positively. This works from homework, since the change of coordinates matrix between these two bases is given by $D(\varphi \circ \psi^{-1})$, and so its determinant is greater than 0.

We see this another way, since if $\alpha = \varphi^{-1}$, $\beta = \psi^{-1}$, $x = \varphi(p)$, and $y = \psi(p)$ this is equivalent to having

$$\psi_* \alpha_*(x) e_1, \ldots, \psi_* \alpha_*(x) e_n$$

In $T_y\mathbb{R}^n$ positively oriented. But this is just $(\psi \circ \alpha)_* = (\varphi \circ \psi^{-1})_*$, and tehrefore this basis is positively oriented becuase $\det(\varphi \circ \psi^{-1})_*(p) = \det D(\varphi \circ \psi^{-1})(p) > 0$.

**Here we have implicitly used that if $T : V \to W$ is a linaer map between two oriented vector spaces. Then if $T$ is orientation preserving then $T^{-1}$ is also orientation preserving**

**Theorem IX.0.1**
a) An orientation of a manifold $M$ gives a smooth choice of orientation for $T_pM$ for eahc $p \in M$. By smooth choice we mean that for each $p \in M$ there exists a coordinate chart $(U, \varphi)$ such that with $\alpha = \varphi^{-1}$ we have $\alpha_*(x)$ maps the basis $e_1, \ldots, e_n$ of $T_x\mathbb{R}^n$ into a positively oriented basis of $T_{\alpha(x)}M$ for every $x \in \widehat{U} = \varphi(U)$.
b) The converse holds as well, $M$ is orientable if and only if there exists a smooth choice of orientation for each $T_pM$.

*Proof.* a), which is the forward direction of b), is what we just proved.

We just need to show the backward implication in b). Suppose we have a smooth choice of orientation for each $T_pM$. THen this means that for each $p \in M$ there exists a coordinate chart $(U, \varphi)$ such that if $\alpha = \varphi^{-1}$ and $x \in \widehat{U} = \varphi(U)$, then $\alpha_*(x)e_1, \ldots, \alpha_*(x)e_n$ is positively oriented in $T_{\alpha(x)}M$.

Let $\{(U_\gamma, \varphi_\gamma)\}$. denote the collection of all such charts. We show that this gives an orientation on $M$, aka that this is a positively overlapping atlas. I.e. we must show that $\det D(\varphi_{\gamma'} \circ \varphi_\gamma^{-1}) > 0$ for all $\gamma, \gamma'$. But this follows from noticing that the two bases:

$$(\alpha_\gamma)_* e_1, \ldots, (\alpha_\gamma)_* e_n \qquad\qquad (\alpha_{\gamma'})_* e_1, \ldots, (\alpha_\gamma)_* e_n$$

are both positively oriented bases of $T_pM$. This means exactly that the change of coordinates matrix between them has positive determiant, but by homework this change of coordinates matrix is exactly $D(\varphi_{\gamma'} \circ \varphi_\gamma^{-1})$. Thus we have the desired property.

Last time we defined orientable in two different ways Definition IX.0.2 and Theorem IX.0.1

**Definition IX.0.3**

Let $f : M \to N$ be a diffeomorphism between two oriented manifolds. We say $f$ is orientation preserving if $f_* : T_pM \to T_{f(p)}N$ is orientation preserving (i.e. it maps a positively oriented basis into a positively oriented bases)

**Definition IX.0.4**

For 0-manifolds, which are discrete collections of points, an orientation is just a choice of $+1$ or $-1$ at each point. This implies that all zero dimensional manifolds are orientatable.

**Theorem IX.0.2**

We need a theorem to verify some intuition about oriented manifolds

a) Let $M$ be an oriented manifold. Then $M$ admits a reverse orientation that we denote by $-M$ that assigns the opposite orientation for every tangent space $T_pM$
b) If $M$ is a conneted orientable manifold then $M$ admits exactly two orientations

*Proof.* We prove a) and leave b) as homework. This is clear for zero-dimensional manifolds. For $n$-dimensional manifolds, given an orientation provided by an atlas $(U_\gamma, \varphi_\gamma)$ consider the atlas $(U_\gamma, \widetilde{\varphi}_\gamma\}$ where:

$$\widetilde{\varphi}_\gamma = A \circ \varphi_\gamma$$
$$A : \mathbb{H}^n \to \mathbb{H}^n$$
$$A(x^1, \dots, x^n) = (-x^1, x^2, \dots, x^n)$$

So then we have $\widetilde{\varphi}_\gamma^{-1} = \varphi_\gamma^{-1} \circ A^{-1} = \varphi_\gamma \circ A$.

This reverses the orientation of $T_pM$ from that given by $\{\alpha_{|}ast(x)e_1, \dots, \alpha_*(x)e_n\}$ where $\alpha = \varphi_\gamma^{-1}$ to that given by $\{-\alpha_*(x)e_1, \dots, \alpha_*(x)e_n\}$ which has the opposite orientation.

It is easy to check that $(U_\gamma, \widetilde{\varphi}_\gamma)$ is still positively overlapping since:

$$\begin{aligned}\widetilde{\varphi}_\gamma^{-1} \circ \widetilde{\varphi}_\gamma &= \widetilde{\varphi}_\gamma^{-1} \circ A^{-1} \circ A \circ \varphi_\gamma \\ &= \varphi_\gamma^{-1} \circ \varphi_\gamma\end{aligned}$$

Caveat: When $n = 1$, $A$ does not map $\mathbb{H}^1$ into $\mathbb{H}^1$ but rather into $\mathbb{L}^1 = \{x \in \mathbb{R} \mid x \leq 0\}$. To solve this caveat, we allow for coordinate charts of 1-dimensional manifolds to map into $\mathbb{H}^1$ or $\mathbb{L}^1$. THis does not change the class of smooth manifolds with boundary in 1D.

## IX.1. Oriented manifolds in $\mathbb{R}^d$ of dimensions 1, $d-1$, and $d$

### IX.1.1. Manifolds of dimension 1

**Definition IX.1.1**

Let $M$ be an oriented 1-dimensional manifold in $\mathbb{R}^d$. We define the unit tangent vector field $T$ on $M$ as follows: Given $p \in M$, choose a coordinate chart $(U, \varphi)$ containing $p$ in the orientation of $M$ and let $|alpha : \widehat{U} \to \mathbb{R}^d$ by $\alpha = \varphi^{-1}$.

Then we define $T(p) = \frac{D\alpha(t_0)\cdot 1}{|D\alpha(t_0)\cdot 1|} = \frac{\alpha'(t_0)}{|\alpha'(t_0)|}$ where $\alpha(t_0) = p$.

**Exercise IX.1.1**

$T$ is well-defined and smooth

$T$ is called the unit tangent vector field corresponding to the orientaiton of $M$. This allows us to think of $M$ as a directed curve.

**Remark IX.1.1**

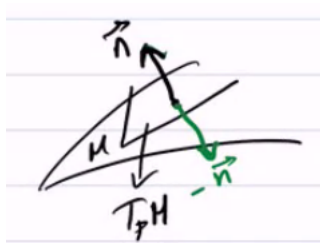
Note that if $M$ has boundary like below


We have to allow for one of the parameterizations of the two boundary points above to be from $\mathbb{L}^1$ instead of $\mathbb{H}^1$. In fact if $\alpha : \widehat{U} \to M$ is a paramaterization such that $\widehat{U} \subseteq \mathbb{H}^1$ and $\alpha(0) = p$, then the unit tangent vector at $p$ has to point into $M$.

The same argument would apply at $q$ as well, but this would give you disagreeing orientations to be from $\mathbb{L}^1$ instead of $\mathbb{H}^1$.

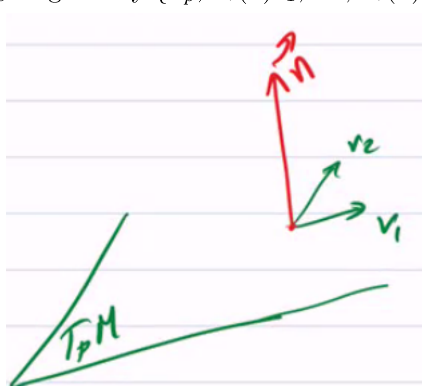
IX.1.2. $(d-1)$ **manfiolds of** $\mathbb{R}^d$

Let $M$ be a $(d-1)$-manifold in $\mathbb{R}^d$. If $p \in M$, let $\vec{n}_p$ be a unit normal vector to $T_pM$ in $T_p\mathbb{R}^d$ (since $T_pM$ has codimension 1 in $T_p\mathbb{R}^d$, there are only two choices).




Then $\vec{n}_p$ is uniquely determine up to sign.

Given an orientation of $M$, choose a coordinate chart $(U, \varphi)$ in the orientation such that $\alpha = \varphi^{-1}$ and $x = \varphi(p)$, and $\{\alpha_*(x)e_1, \dots, \alpha_*(x)e_{d-1}\}$ is a positively oriented basis of $T_pM$. Then we specify the sign of $\vec{n}_p$ by requiring that the basis of $T_p\mathbb{R}^d$ given by $\{\vec{n}_p, \alpha_*(x)e_1, \dots, \alpha_*(x)e_{d-1}\}$ is a positively oriented basis.




Since $\alpha_*(x)e_j = \frac{\partial \alpha}{\partial x^j}$, this is equivalent to asking that the matrix $\begin{bmatrix} \vec{n} & D\alpha(x) \end{bmatrix}$ given below has positive determinant.

**Exercise IX.1.2** (Homework)

This vector field $\vec{n}$ is well-defined and smooth. It is called the unit normal vector field to $M$.

Conversely, given a smooth (or continuous) unit normal vector field $\vec{n}$ to a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$, this gives an orientation of $M$ as follows.

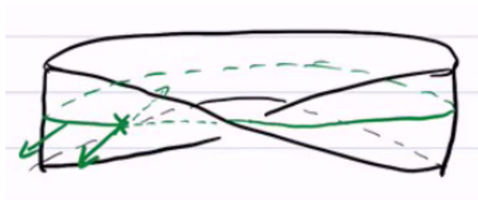
If $\{v_1, \dots, v_{d-1}\}$ is a basis for $T_pM$, we declare this basis for be positively oriented if $\{n, v_1, \dots, v_{d-1}\}$ is a positively oriented basis of $T_p\mathbb{R}^d = \mathbb{R}^d$.

**Remark IX.1.2**

If $M$ is given by a level set $L_a = \{x \in \mathbb{R}^d \mid f(x) = a\}$ of a function $f : \mathbb{R}^d \to \mathbb{R}$ and if $\nabla f \neq 0$ for all $x \in M$, then $\vec{n} = \frac{\nabla f}{|\nabla f|}$ is a smooth normal vector field to $M$, which implies that $M$ is orientable.

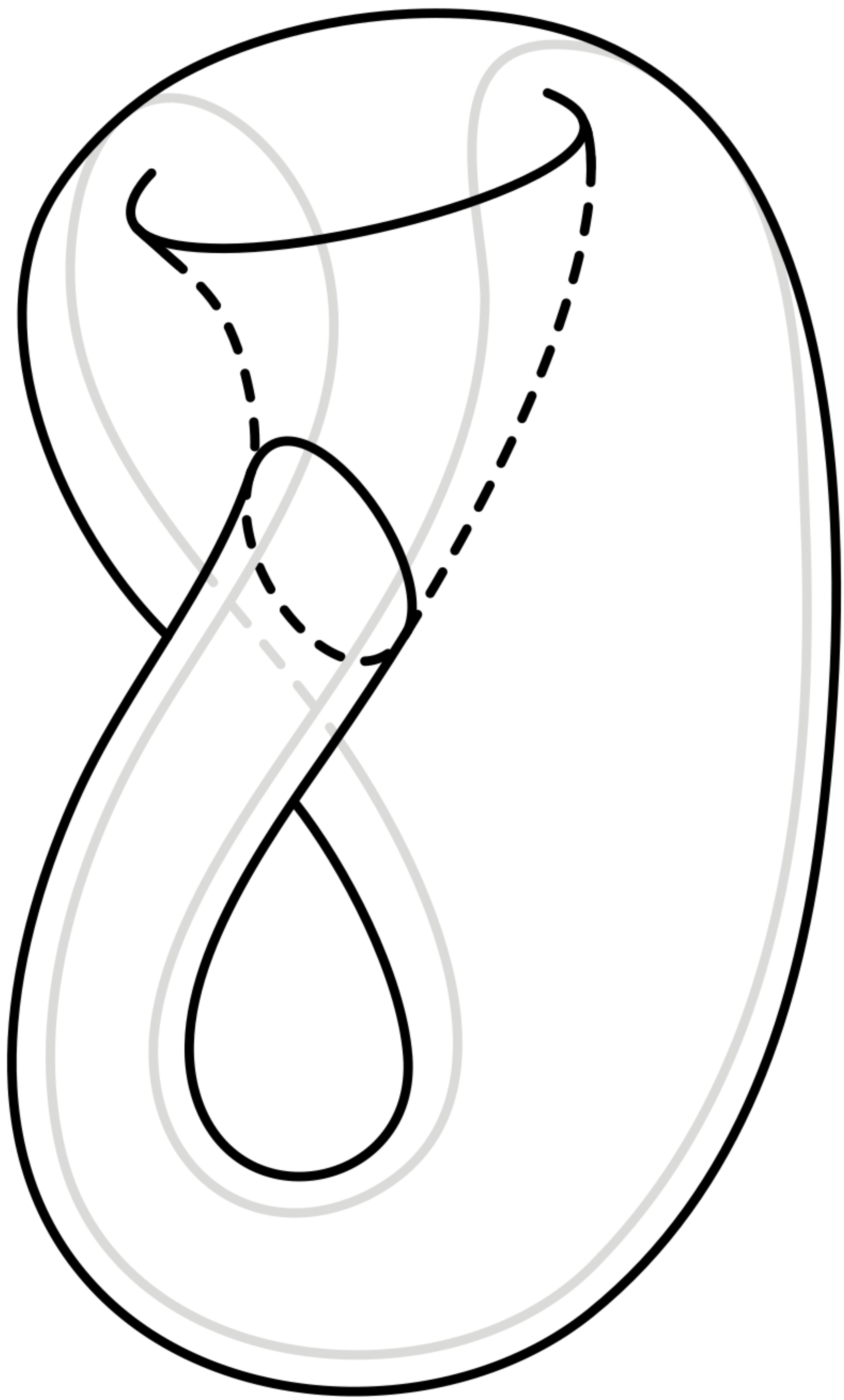
**Example IX.1.3** (Not all manifolds are orientable)

Consider the 2-manifold in $\mathbb{R}^3$ depicted below, which is called the Mobius band:



We cannot have a continuous normal vector field to $M$, because as you travel along the curve continuously, you eventually come around to the same point, but on the oppsoite side, meaning you will orient the normal vector with two different signs at that point. Intuitively, the mobius band has no "inside" or "outside," and a choice of orientation is a choice of inside and outside.

Another example of a non-orientable manifold is the Klein bottle (it contains a copy of the mobius band).

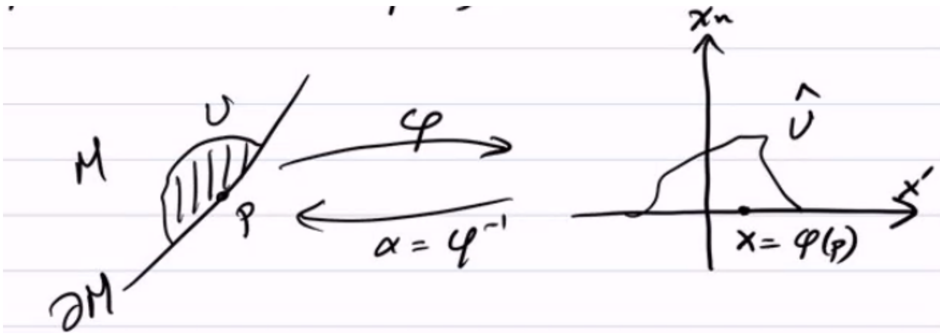


### IX.1.3. $d$-dimensional submanifolds in $\mathbb{R}^d$

## IX.2. Boundary Orientation

**Theorem IX.2.1**
Let $n \geq 1$. If $M$ is an orientable $n$-manifolds iwth non-empty boundary then $\partial M$ is orientable.

*Proof.* If $n = 1$, then $\partial M$ is zero dimensional, so it is always orientable. For the remainder of this proof let $n > 1$, let $p \in \partial M$, and let $(U, \varphi)$ be a coordinate chart near $p$ on $M$ such that $\varphi(U) = \widehat{U} \subseteq \mathbb{H}^n$.




We saw previously this gives a coordinate chart for $\partial M$ given by $(U_\heartsuit, \varphi_\heartsuit)$ where $U_\heartsuit = U \cap \partial M$ and $\varphi_\heartsuit = \varphi\big|_{U_\heartsuit}$. Then $\widehat{U}_\heartsuit = \varphi(U_\heartsuit) = \widehat{U} \cap \{x^n = 0\}$, which we can regard as an open subset of $\mathbb{R}^{n-1}$.

Let $\alpha_\heartsuit = \varphi_\heartsuit^{-1} = \alpha\big|_{U_\heartsuit \times \{0\}}$. Such coordinate charts cover $\partial M$.

Given an orientation of $M$ given by a positively overlapping atlas will restrict to a positively overlapping atlas on $\partial M$.

We only need to show that if $(U, \varphi)$ and $(V, \psi)$ are two positively overlapping charts on $M$, then $(U_\heartsuit, \varphi_\heartsuit)$ and $(V_\heartsuit, \psi_\heartsuit)$ are positively overlapping on $\partial M$.

We continue the proof of Theorem IX.2.1

*Proof.* We just needed to show that if $(U, \varphi)$ and $(V, \psi)$ are two positively overlapping charts on $M$, then the restricted charts $(U_\heartsuit, \varphi_\heartsuit)$ and $(V_\heartsuit, \psi_\heartsuit)$ are positively overlapping on $\partial M$.

That is, we need to show that $\det D(\psi_\heartsuit \circ \varphi_\heartsuit^{-1})(x_\heartsuit) > 0$ for all $x_\heartsuit \in \varphi(U_\heartsuit \cap V_\heartsuit)$. But note that:

$$(\psi_\heartsuit \circ \varphi_\heartsuit^{-1})(x) = \pi(\psi(\varphi^{-1}(x, 0)))$$

where $\pi$ is the projection from $\mathbb{H}^n$ onto $\mathbb{R}^{n-1}$ which gives the first $n-1$ coordinates. Therefore $D(\psi_\heartsuit \circ \varphi_\heartsuit^{-1})(x_\heartsuit)$ is nothing but the $(n-1) \times (n-1)$ submatrix of $D(\psi \circ \varphi^{-1})(x_\heartsuit, 0)$ obtained by removing the last row and the last column In other words if $g = \psi \circ \varphi^{-1}$):

$$Dg(x_\heartsuit, 0) = \begin{pmatrix} D(\psi_\heartsuit \circ \varphi_\heartsuit^{-1}(x_\heartsuit) & 0 \\ \frac{\partial g}{\partial(x^1, \ldots, x^{n-1})} & \frac{\partial g_n}{\partial x^n} \end{pmatrix}$$

Why do we have the upper right being zero? Well note that for any $x \in \varphi(U_\heartsuit \cap V_\heartsuit)$ we have $\varphi^{-1}(x, 0) \in \partial M$ so $g(x, 0) = \psi(\varphi^{-1}(x, 0)) = 0$. Therefore $\frac{\partial g_n}{\partial x^i}(x_\heartsuit, 0) = 0$ for $1 \leq i \leq n-1$.

Also $g_n(x, t) \geq 0$ for all $(x, t) \in \varphi(U \cap V)$. This means that as $g$ increases in the direction of $x^n$ at $(x_\heartsuit, 0)$, and so $\frac{\partial g_n}{\partial x^n}(x_\heartsuit, 0) \geq 0$.

Therefore:

$$0 < \det Dg(x_\heartsuit, 0) = \det D(\psi_\heartsuit \circ \varphi_\heartsuit^{-1})(x_\heartsuit) \cdot \frac{\partial g_n}{\partial x^n}(x_\heartsuit, 0)$$

And so since the right piece is greater than or equal to zero, we must have that:

$$\det D(\psi_\heartsuit \circ \varphi_\heartsuit^{-1})(x_\heartsuit) > 0$$

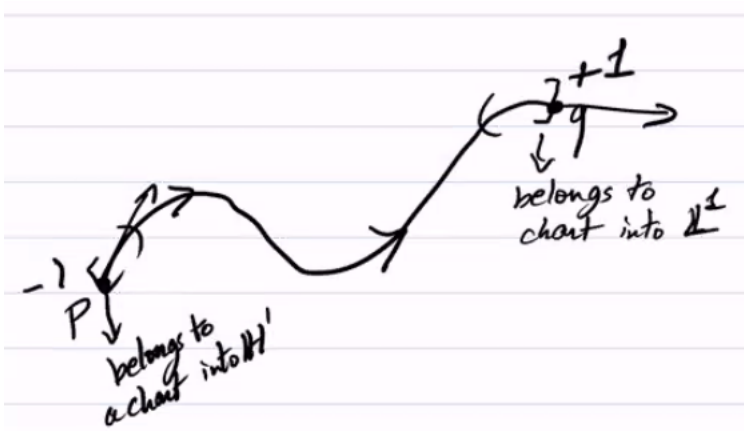
This completes the proof

**Definition IX.2.1** ($\partial$ Orientation)

Let $M$ be an orientable manifold with nonempty $\partial$ and dimension ¿ 1. Given an orientation of $M$, the corresponding induced boundary orientation on $\partial M$ is defined as follows:

a) If $n$ is even, it is exactly the orientation obtained by restricting coordinate charts in the orientation on $M$ to $\partial M$ as in the proof of Theorem IX.2.1
b) If $n$ is odd, we take the opposite orientation.

If $\dim M = 1$, the boundary orientation is defined as follows. Well $\partial M$ consists of discrete points because it is a zero-dimensional manifold. We give a point $p \in \partial M$ the orientation $+1$ if $p$ belongs to a coordinate chart $(U, \varphi)$ such that $\varphi(U) \subseteq \mathbb{L}^1$ and orientation $-1$ if it belongs to a chart $(U, \varphi)$ with $\varphi(U) \subseteq \mathbb{H}^1$. Consider the picture below:



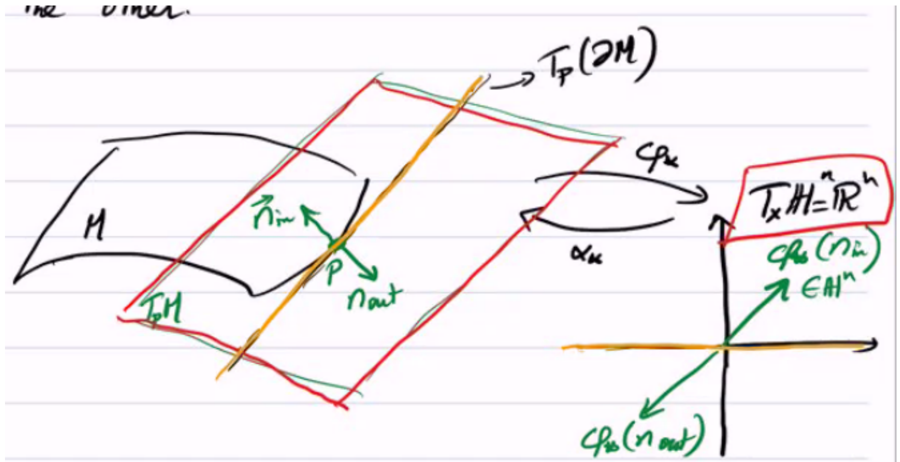

So then it's kinda like + on the endpoint and $-1$ on the starting point.

**Remark IX.2.1**

This choice of induced orientation is so that Stoke's Theorem can be stated properly. On Homework 11, you will show that this choice of orientaiton for 1D manifolds is well-defined.

We have an alternative and equivalent definition of $\partial$ orientation. Let $M$ be an oriented $n$-manifold. At every point $p \in \partial M$ we will give an orientation of $T_p\partial M$.

a) $T_p(\partial M)$ has codimension 1 inside $T_pM$. Therefore, there are two unit vectors in $T_pM$ that are orthogonal to $T_p(\partial M)$ (using the inner product on the ambient space $\mathbb{R}^d$). One is the negative of the other.




More precisely, let $(U, \varphi)$ be a chart near $p$ and $\alpha := \varphi^{-1} : \widehat{U} \subseteq \mathbb{H}^n \to U$ such that $x = \varphi(p)$. Then $\alpha_*(x) = D\alpha(x)$ is an isomorphism of $T_x\mathbb{H}^n$ and $T_pM$, whose inverse is $\varphi_*(p) = [\alpha_*(x)]^{-1}$.

b) The <u>inward unit normal</u> is the normal vector whose image under $\varphi_*(p)$ belongs to $\mathbb{H}^n$ and the <u>outward unit normal</u> is the one whose image belongs to $-\mathbb{H}^n$.

This distinction between the unit normals is independent of the choice of $(U, \varphi)$ in the same orientation of $M$ (Homework 11).

c) We orient $T_p(\partial M)$ by declaring that a basis $\vec{v}_1, \ldots, \vec{v}_{n-1}$ in $T_p\partial M$ if:

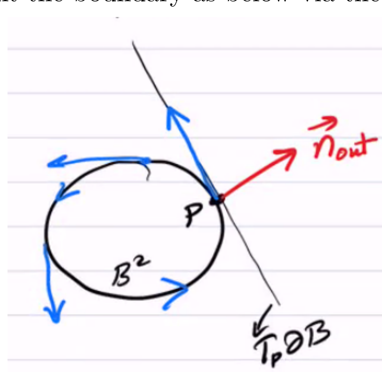
$$\{\vec{n}_{\text{out}}, \vec{v}_1, \ldots, \vec{v}_{n-1}\}$$

is positively oriented in $T_pM$.

We will check int he homework that this is a smooth choice of orientations on the tangent spaces, and it is the same orientation as the one we defined using restrictions of coordinate charts.

**Example IX.2.1**

Consider the closed unit ball $B^2$ in $\mathbb{R}^2$. It inherits from $\mathbb{R}^2$ the standard orientation at each $p$ since $T_pB^2 = T_p\mathbb{R}^2 = \mathbb{R}^2$. Then we orient the boundary as below via the right-hand rule.



$\vec{v}$ is positively oriented in $T_p\partial B$ if and only if $\{\vec{n}_{\text{out}}, \vec{v}\}$ is positively orineted in $\mathbb{R}^2 = T_pB^2$, which means that $\vec{v}$ must be oriented counterclockwise.

# X. Integration of $n$-forms

## X.1. Integrating $n$-forms on $\mathbb{R}^n$

Recall the change of variables theorem for integration. Let $g : V \to U$ be a diffeomorphism of open subsets of $\mathbb{H}^n$ and suppose $a : U \to \mathbb{R}$ is integrable on $U$, then:

$$\int_U a(y)\,\mathrm{d}y = \int_V a(g(x))\,|\det Dg(x)|\,\mathrm{d}x = \int_V g^*a\,|\det Dg(x)|\,\mathrm{d}x \tag{$*$}$$

In particular,

$$\int_U a(y)\,\mathrm{d}y \neq \int_V g^*a(x)\,\mathrm{d}x$$

Equality here is needed to define $\int_U a$ in a coordinate-independent way (and hence to define the integral of a function on a manifold). Here $g$ is understood as a transition map between two coordinate charts. We say integration of functions on $\mathbb{R}^n$ is not coordinate invariant.

However, looking at $(*)$, we recall that if $\omega$ is an $n$-form on $U$ given by $\omega = a(y) \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$ then $g^*\omega = g^*ag^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n)$, and so from before:

$$\begin{aligned} g^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n)(x) &= \det Dg(x) \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ g^*\omega(x) &= a(g(x)) \cdot \det Dg(x) \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= a(g(x)) \cdot |\det Dg(x)| \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \end{aligned}$$

provided that $g$ is orientation preserving. This motivates the following definition:

**Definition X.1.1**

Suppose that $\omega$ is an $n$-form on an open subset $U$ of $\mathbb{H}^n$. Then $\omega(y) = a(y)\,\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$ for some $a : U \to \mathbb{R}$. Suppose that $a : U \to \mathbb{R}$ is integrable on $U$, then we say that $\omega$ is integrable on $U$ and define:

$$\int_U \omega = \int_U a(y)\,\mathrm{d}y$$

The above computation gives that:

**Proposition X.1.1** (Coordinate Invariance of $\int \omega$)

Suppose $\omega$ is an integrable $n$-form on an open set $U \subseteq \mathbb{H}^n$, and let $g : V \to U$ be an orientation-preserving diffeomorphism. Then $g^*\omega$ is integrable on $V$, and:

$$\int_U \omega = \int_V g^*\omega$$

*Proof.* The key point is that for $\omega = a\,\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$ as in the definition of integration, we have:

$$g^*\omega(x) = a(g(x))\,|\det Dg(x)|\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

So then the result follows from the change of variables formula:

$$\int_U \omega = \int_U a(y)\,\mathrm{d}y = \int_V a(g(x))\,|\det Dg(x)|\,\mathrm{d}x = \int_V g^*\omega$$

**Remark X.1.1**

The key point was that $g^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n) = \det Dg(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ which came from the anti-commutativity $\mathrm{d}x^1 \wedge \mathrm{d}x^2 = -\,\mathrm{d}x^2 \wedge \mathrm{d}x^1$.

The whole apparatus of alternating forms on manifolds exists so that this coordinate invariance holds.

This coordinate invariance of $\int_U \omega$ allows us to define integration on manifolds

Last time, we define $\int_U \omega$ when $U$ is an open subset of $\mathbb{H}^n$ and $\omega$ is an $n$-form on $U$. How? Write $\omega = a(y)\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^n$. Just define:

$$\int_U \omega := \int_U a(y)\,\mathrm{d}y$$

provided that $a$ is integrable.

**Remark X.1.2**

There is no need for $\omega$ to be a smooth $n$-form in order to define this integral. Continuity is sufficient, and not even necessary.

We showed that this definition was invariant under orientation-preserving changes of coordinates, see Proposition X.1.1. This is exactly the property that will allow the extension of this integration theory to manifolds.

## X.2. Intgrating $n$-forms on Oriented $n$-Manifolds

**Definition X.2.1**

Let $M$ be a smooth oriented manifold with boundary. Let $n = \dim M$, and $\omega$ be an $n$-form on $M$ (continuity of $\omega$ is sufficient, but not necessary).

The support of $\omega$ (denoted $\operatorname{supp}\omega$) is defined to be the closure of the set $\{p \in M \mid \omega_p \neq 0\}$. We shall assume that $\operatorname{supp}\omega$ is compact. The definition of $\int_M \omega$ is done in two steps:

(Step 1) Suppose first that $\operatorname{supp}\omega$ is contained in some chart $(U, \varphi)$ which is in the orientation on $M$. Then let $\alpha := \varphi^{-1} : \widehat{U} \to U$. Then $\widehat{U} \subseteq \mathbb{H}^n$ is open, and defin

$$\int_M \omega := \int_{\widehat{U}} \alpha^* \omega$$

Note that $\alpha^*\omega$ is a compactly supported continuous $n$-form on $\widehat{U}$, so it is integrable.

**Exercise X.2.1**

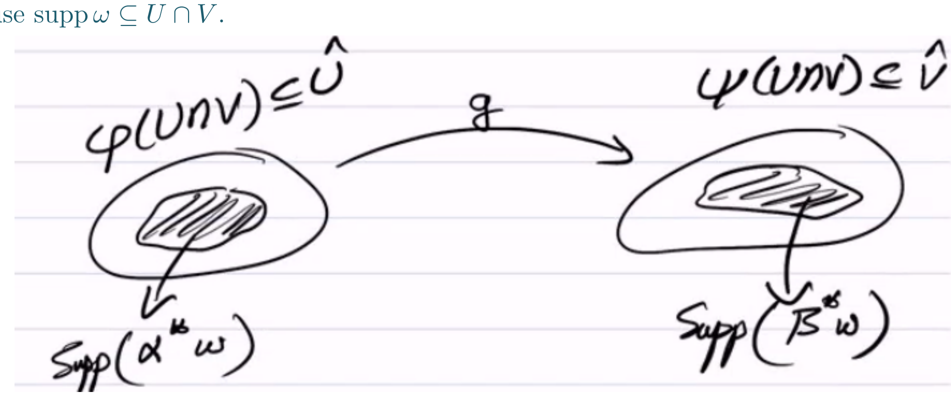
Suppose $\varphi = (x^1, \ldots, x^n)$, then $\omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Then:

$$\int_M \omega = \int_{\widehat{U}} f(\alpha(x))\,\mathrm{d}x$$

This definition makes sense, once we show that it does not depend on the chart chosen. Let $(V, \psi)$ be another coordinate chart on $M$ in the orientation such that $\operatorname{supp}\omega \subseteq V$, and let $\beta := \psi^{-1} : \widehat{V} \to V$. That is we must show:

$$\int_{\widehat{V}} \beta^*\omega = \int_{\widehat{U}} \alpha^*\omega$$

For this, we let $g := \psi \circ \alpha^{-1} = \psi \circ \alpha$, then $g : \varphi(U \cap V) \to \psi(U \cap V)$ is an orientation preserving diffeomorphism. Note that $\operatorname{supp}(\alpha^*\omega) \subseteq \varphi(U \cap V)$ and $\operatorname{supp}(\beta^*\omega) \subseteq \psi(U \cap V)$ because $\operatorname{supp}\omega \subseteq U \cap V$.



Now note that $\alpha^*\omega = (\beta \circ g)^*\omega = g^*(\beta^*\omega)$ on $\varphi(U \cap V)$. Therefore by invariance under change of coordinates (Proposition X.1.1) we have:

$$\int_{\widehat{U}} \alpha^*\omega = \int_{\varphi(U\cap V)} \alpha^*\omega = \int_{\varphi(U\cap V)} g^*(\beta^*\omega)) = \int_{\psi(U\cap V)} \beta^*\omega = \int_{\widehat{V}} \beta^*\omega$$

Thus $\int_M \omega$ is well-defined when $\operatorname{supp}\omega$ is a subset of a coordinate chart in the orientation.

(Step 2) Now we integrate arbitrary compactly supported continuous $n$-forms. We simply use a partition of unity to break up $\omega$ into finitely many pieces $\omega_i$ such that each $\omega = \sum_i \omega_i$ and $\operatorname{supp}\omega_i$ is always a subset of some coordinate chart in the orientation.

More precisely, using the partition of unity theorem (see IBL), we first cover $\operatorname{tsupp}\omega$ by finitely many coordinate charts $(U_k, \varphi_k)$ in the orientation and obtain a smooth partition of unity subordinate to this covering, namely smooth functions $\rho_i : M \to \mathbb{R}$ such that:

- $\sum_i \rho_i = 1$ on $\operatorname{supp}\omega$
- $\operatorname{supp}\rho_i \subseteq U_k$ for some $k$.

Therefore, we have that $\omega = \sum_i \rho_i\omega$, and $\operatorname{supp}\rho_i\omega \subseteq U_k$ for some $k$. We can then define

$$\int_M \omega = \sum_i \int_M \rho_i\omega$$

Where the latter integral was defined in the previous step. For this definition to make sense, we need to show that this does not depend on charts or the partition of unity chosen.

Support first that $\operatorname{supp}\omega \subseteq U$, where $(U, \varphi)$ si a coordinate chart in the orientation. Step 1 gives us one definition of the integral, and step 2 gives us another definition. We need to show these definitions are the same. Writing $\omega = \sum_i \rho_i\omega$, then we have that for $\alpha := \varphi^{-1}$:

$$\begin{aligned}\alpha^*\omega &= \sum_i \alpha^*(\rho_i\omega)\\ \int_{\widehat{U}} \alpha^*\omega &= \sum_i \int_{\widehat{U}} \alpha^*(\rho_i\omega)\\ \int_M \omega &= \sum_i \int_M \rho_i\omega\end{aligned}$$

Great! The two definitions agree as desired.

Now suppose that $(V_\ell, \psi_\ell)$ are another finite collection of charts in the orientation covering $\operatorname{supp}\omega$ and we have a partition of unity $\rho'_j$ here so that $\sum_j \rho'_j = 1$ on $\operatorname{supp}\omega$ and $\operatorname{supp}\rho'_j \subseteq V_\ell$ for soem $\ell$.

Let $\omega_i = \rho_i\omega$. Then $\omega_i = \sum_j \rho'_j\omega_i$ and then:

$$\begin{aligned}\int_M \omega_i &= \sum_j \int_M \rho'_j\omega_i\\ &= \sum_j \int_M \rho'_j\rho_i\omega\\ \sum_i \int_M \omega_i &= \sum_i \sum_j \int_M \rho'_j\rho_i\omega\\ &= \sum_j \sum_i \int_M \rho_i\rho'_j\omega\\ &= \sum_j \int_M \rho'_j\omega\end{aligned}$$

Perfect! This is exactly what we wanted to show ☺

Amazing! This gives us a definition of integration on manifolds.

**Remark X.2.1**

It is trivial from the definition to check that the integral $\int_M$ satisfies the linearity properties. That is for a scalar $c \in \mathbb{R}$, and two compactly supported continuous $n$-forms $\omega_1, \omega_2$ we have:

$$\int_M (c\omega_1 + \omega_2) = c \int_M \omega_1 + \int_M \omega_2$$

**Theorem X.2.1**

Let $g : N \to M$ be an orientation-preserving diffeomorphism between oriented $n$-manifolds, and let $\omega$ be a compactly supported continuous $n$-form on $M$. Then we have that:

$$\int_M \omega = \int_N g^* \omega$$

*Proof.* Homework (See final problem set).

**Exercise X.2.2**

Let $\omega = f_1 \, dx^1 + f_2 \, dx^2 + f_3 \, dx^3$ be a 1-form on $\mathbb{R}^3$. Suppose that $\gamma : [0,1] \to \mathbb{R}^3$ be a diffeomorphism of the unit interval and a smooth curve $C = \gamma([0,1])$ which is a 1-manifold with boundary. $C$ is naturally oriented by $\gamma$. In fact, $\gamma$ gives us an atlas for $C$. Take $(U_1, \varphi_1)$ and $U_2, \varphi_2$ such that:

$$U_1 = C \setminus \{\gamma(1)\} \qquad U_2 = C \setminus \{\gamma(0)\}$$
$$\varphi_1^{-1} = \gamma\big|_{[0,1)} \qquad \varphi_2^{-1} = \gamma(- + 1)\big|_{(-1,0]}$$

This gives the $\partial$ orientation for $\partial C$ as $\gamma(1)$ has $+1$ orientation and $\gamma(0)$ has $-1$ orientation.

Thenn we have by the above theorem that:

$$\int_C \omega = \int_{[0,1]} \gamma^* \omega$$
$$\gamma^*\omega = \gamma^* \sum_{j=1}^{3} f_j \, dx^j = \sum_{j=1}^{3} \gamma^*(f_j \, dx^j)$$
$$= \sum_{j=1}^{3} (f_j \circ \gamma) \, d(\gamma^* x^j)$$
$$= \sum_{j=1}^{3} (f_j \circ \gamma) \frac{d\gamma^j}{dt} \, dt$$

Let $\vec{F} = (f_1, f_2, f_3)$. Then $\gamma^*(\omega) = \vec{F}(\gamma(t))\vec{\gamma}'(t) \, dt$. That is:

$$\int_C \omega = \int_{[0,1]} \gamma^* \omega = \int_0^1 \vec{F}(\gamma(t)) \cdot \vec{\gamma}'(t) \, dt$$

This is sometimes called the line integral of $\vec{F}$ over $C$.

## X.3. Stokes' Theorem

Stokes' theorem is the generalization of the fundamental theorem of calculus to manifolds. To see this, we start by reinterpreting the Fundamental Theorem of Calculus using our new langauge:

$$\int_a^b f'(t) \, dt = f(b) - f(a)$$

Consider the 1-manifold $M = [a,b]$ with its natural orientation from $a$ to $b$. Then $\partial M = \{a, b\}$ has an induced orientation $+1$ for $b$ and $-1$ for $a$. Let $f(t)$ be a 0-form on $M$, then $df = f'(t) \, dt$ is a 1-form on $M$, which we can integrate. Then we have:

$$\int_M \mathrm{d}f = \int_{[a,b]} f'(t)\,\mathrm{d}t = \int_a^b f'(t)\,\mathrm{d}t$$
$$\int_{\partial M} f = f(b) - f(a)$$

Thus the Fundamental Theorem of Calculus is equivalent to the statement that:

$$\int_M \mathrm{d}f = \int_{\partial M} f$$

This is true for any 0-form on the 1-manifold $M = [a,b]$ given. We picked the induced orientation of $\partial M$ exactly so that this would hold.

**Theorem X.3.1** (The Generalized Stokes' Theorem)

Let $M$ be an oriented $n$-manifold and let $\omega$ be a smooth compactly supported $(n-1)$-form on $M$. Then! We have something amazing:

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$

Here, $\partial M$ is given its induced boundary orientation, and if $\partial M = \emptyset$ then $\int_{\partial M} \omega = 0$ by convention.

**Remark X.3.1**

It will be clear from the proof that one actually only needs that $\omega$ is a $C^1$-form, i.e. its coefficients in any smooth coordinate system $\varphi = (x^1, \ldots, x^n)$ are $\omega = \sum_I f_I \,\mathrm{d}x^I$ where $f_I \in C^1(U)$.

We wish to prove Theorem X.3.1!!! Let's go!

*Proof of Stoke's Theorem.* Since both sides of Stokes' theorem are linear in $\omega$, we may assume without loss of generality (using a partition of unity) that $\operatorname{supp}\omega \subseteq U$ where $(U, \varphi)$ is a coordinate chart in the orientation of $M$. As usual, we set $\widehat{U} := \varphi(U) \subseteq \mathbb{H}^n$ open and $\alpha : \widehat{U} \to U = \varphi^{-1}$.

There are two cases:

(Case 1) Suppose that $\widehat{U} \cap \partial\mathbb{H}^n = \emptyset$, i.e. $\widehat{U}$ is open in $\mathbb{R}^n$. Then of course $\omega$ is zero on $\partial M$ so:

$$\int_{\partial M} \omega = 0$$

So then we want to show that:

$$\int_M \mathrm{d}\omega = 0$$

That is, we want to show the following:

$$\int_M \mathrm{d}\omega = \int_{\widehat{U}} \alpha^*(\mathrm{d}\omega) = \int_{\widehat{U}} \mathrm{d}(\alpha^*\omega) = \int_{\widehat{U}} \mathrm{d}\nu$$

Where we have set $\nu = \alpha^*\omega$. So then we have to show that if $\nu$ is an $(n-1)$-form on the open set $\widehat{U}$ such that $\operatorname{supp}\nu$ is a compact set contained in $\widehat{U}$, then:

$$\int_{\widehat{U}} \mathrm{d}\nu = 0$$

We may write $\nu$ in the following way since it is an $(n-1)$-form on $\widehat{U} \in \mathbb{R}^n$:

$$\nu = \sum_I g_I \,\mathrm{d}x^I = \sum i = 1^n (-1)^{i-1} f_i \,\mathrm{d}x^1 \wedge \cdots \widehat{\mathrm{d}x}^i \wedge \cdots \wedge \mathrm{d}x^n$$

Where $\widehat{\mathrm{d}x^i}$ means that the term $\mathrm{d}x^i$ is omitted from the product, and the signs $(-1)^{i-1}$ are just there to make the proof cleaner. Also note that the $f_i \in C^1(\widehat{U})$ and $\operatorname{supp} f_I$ is a compact subset of $\widehat{U}$.

Then we have the following:

$$\begin{aligned}
\mathrm{d}\nu &= \sum_{i=1}^n (-1)^{i-1} \,\mathrm{d}f_i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^n (-1)^{i-1} \left( \sum_{j=1}^n \frac{\partial f_i}{\partial x^j} \,\mathrm{d}x^j \right) \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^n (-1)^{i-1} \frac{\partial f_i}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^n \frac{\partial f_i}{\partial x^i} \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n
\end{aligned}$$

We then write that:

$$\int_{\widehat{U}} = \int_{\widehat{U}} \left( \sum_{i=1}^n \frac{\partial f_i}{\partial x^i} \right) \mathrm{d}x^1 \dots \mathrm{d}x^n$$

Extending $f_i$ to all of $\mathbb{R}^n$ to be 0 outside of $\widehat{U}$ by using that $\operatorname{supp} f_i$ is a compact subset of $\widehat{U}$ and using bump functions, we can then take a box $B = \prod_{j=1}^n [a_j, b_j]$ so that $\widehat{U} \subseteq B$, and no point in $\widehat{U}$

has coordinates $a_j, b_j$, just by taking $a_j, b_j$ sufficiently large. Then we necessarily have that:

$$\begin{aligned}\int_{\widehat{U}} \mathrm{d}\nu &= \int_B \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x^i}\right) \mathrm{d}x^1 \dots \mathrm{d}x^n \\ &= \sum_{i=1}^n \int_B \frac{\partial f_i}{\partial x^i} \mathrm{d}x^1 \dots \mathrm{d}x^n \\ &= \sum_{i=1}^n \int_{B_i'} \left(\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x^i} \mathrm{d}x^i\right) \mathrm{d}x_i'\end{aligned}$$

where $x_i' = (x^1, \dots, x^i, \dots, x^n)$, and $B_i' = [a_1, b_1] \times \cdots \times \widehat{[a_i, b_i]} \times \cdots \times [a_n, b_n]$. By the fundamental theorem of calculus, we see that:

$$\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x^i}(x^i, x_i') \,\mathrm{d}x^i = f_i(b_i, x_i') - f_i(a_i, x_i') = 0 - 0 = 0$$

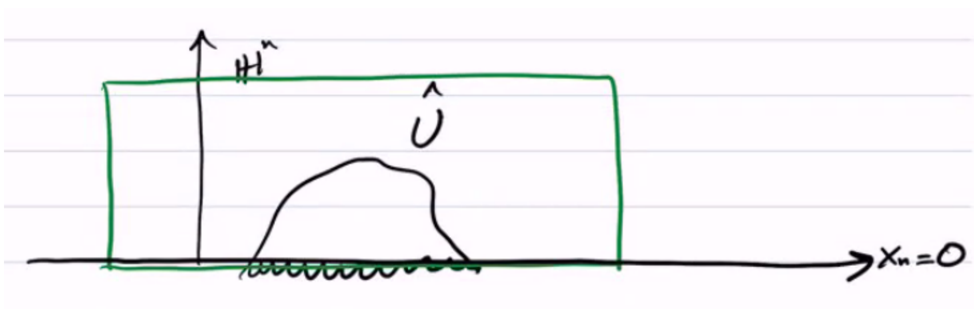
since $(b_i, x_i'), (a_i, x_i') \notin \widehat{U} \supseteq \operatorname{supp} f_i$ for all $x_i' \in \mathbb{R}^{n-1}$ by how we chose $B$.

Thus we have that:

$$\int_{\widehat{U}} \mathrm{d}\nu = 0$$

just as needed. Perfect!

(Cbse 2) Now suppose that $U \cap \partial M \neq 0$. We argue similarly to the above, but we take $B$ to be the box $B = \prod_{j=1}^n [a_j, b_j]$ with $a_n = 0$. This is given in the below picture, where $B$ is in green:




And we can also choose things so that:

$$\operatorname{supp} f_i \subseteq \widehat{U} \subseteq \prod_{j=1}^{n-1} [a_j - 1, b_j - 1] \times [0, b_n - 1]$$

As before, we extend $f_i$ to all of $B$ by setting it to be zero outside of $\widehat{U}$. We then apply Fubini and the Fundamental Theorem of Calculus to see that:

$$\begin{aligned}\int_{\widehat{U}} \mathrm{d}\nu &= \sum_{i=1}^n \int_B \frac{\partial f_i}{\partial x^j} \mathrm{d}x^1 \dots \mathrm{d}x^n \\ &= \sum_{i=1}^n \int_{B_i'} \left(\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x^i} \mathrm{d}x^i\right) \mathrm{d}x_i'\end{aligned}$$

Where that with $x_i' = (x^1, \dots, \widehat{x^i}, \dots, x^n)$ and $B_i' = [a_1, b_1] \times \cdots \times \widehat{[a_i, b_i]} \times \cdots \times [a_n, b_n]$. Then we see for $1 \leq i \leq n-1$ that

$$\int_{B_i'} \int_{a_i}^{b_i} \frac{\partial f_i}{\partial x^i}(x^i, x_i') \,\mathrm{d}x^i = f_i(b_i, x_i') - f(a_i, x_i') = 0 - 0 = 0$$

Just as before. Then for $i = n$ we just have:

$$\int_{\widehat{U}} \mathrm{d}\nu = \int_{B_n'} \left( \int_0^{b_n} \frac{\partial f_n}{\partial x^n} \,\mathrm{d}x^n \right) \mathrm{d}x_n'$$
$$= \int_{B_n'} [f_n(x_n', b_n) - f_n(x_n', 0)] \,\mathrm{d}x_n'$$
$$= -\int_{B_n'} f_n(x_n', 0) \,\mathrm{d}x_n'$$

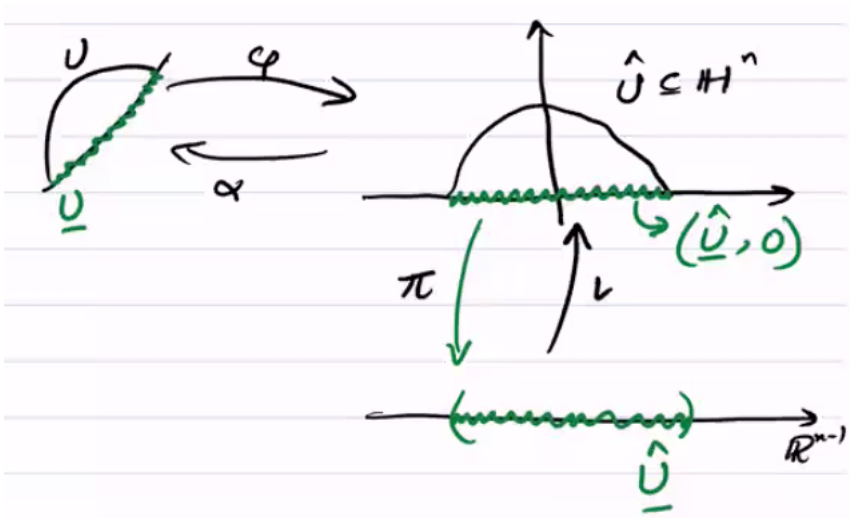
where $B - n' = B_n \cap \{x^n = 0\}$. Great! This is exactly what we need. We then have our identity:

$$\int_M \mathrm{d}\omega = \int_{\widehat{U}} \mathrm{d}\nu = -\int_{B_n'} f_n(x_n', 0) \,\mathrm{d}x_n' \qquad \text{(Identity 1)}$$

We now see what is $\int_{\partial M} \omega$.

Since we're given $\partial M$ the induced boundary orientation in which the restricted chart $(\underline{U}, \underline{\varphi})$ is positively oriented if and only if $n$ is even where $\underline{U} = U \cap \partial M$, $\underline{\varphi} = \varphi\big|_{\underline{U}}$ and $\underline{\varphi} : \underline{U} \to \widehat{\underline{U}} \subseteq \mathbb{R}^{n-1}$ is open. Then $\underline{\alpha} = \underline{\varphi}^{-1} = \alpha\big|_{(\widehat{\underline{U}}, 0)}$

These charts can be pictured as:




Where $\pi(x^1, \ldots, x^n) = (x^1, \ldots, x^{n-1})$ and $\iota(x^1, \ldots, x^{n-1}) = (x^1, \ldots, x^{n-1}, 0)$. So $\underline{\alpha} = \alpha \circ \iota$. We then may write:

$$\int_{\partial M} \omega = (-1)^n \int_{\widehat{\underline{U}}} \underline{\alpha}^* \omega$$

Where the $(-1)^n$ term comes out because $(\underline{U}, \underline{\varphi})$ has positive orientation if and only if $n$ is even. Now recall that:

$$\alpha^* \omega = \nu = \sum_{i=1}^n (-1)^{i-1} f_i \,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$$

We then have because $\underline{\alpha} = \alpha \circ \iota$ that $\underline{\alpha}^* = \iota^* \alpha^*$, so:

$$\underline{\alpha}^* \omega = \iota^* \alpha^* \omega = \iota^* \nu$$

Therefore, we may write:

$$\underline{\alpha}^* \omega = \iota^* \nu = \sum_{i=1}^n (-1)^{n-1} \iota^* f_i (\iota^* \,\mathrm{d}x^1) \wedge \cdots \wedge \widehat{\iota^* \,\mathrm{d}x^i} \wedge \cdots (\iota^* \,\mathrm{d}x^n)$$

Now note that $\iota^* f_i = f_i \circ \iota$, that is the restriction of $f_i$ to $(\widehat{\underline{U}}, 0)$. But then, we see that:

$$\iota^* \mathrm{d}x^j = \mathrm{d}(\iota^* x^j) = \mathrm{d}(x^j \circ \iota) = \mathrm{d}x^j \qquad (j \neq n)$$
$$\iota^* \mathrm{d}x^n = \mathrm{d}(\iota^* x^n) = \mathrm{d}(x^n \circ \iota) = 0$$

Therefore, we see that:

$$\begin{aligned}\underline{\alpha}^* \omega = \iota^* \nu &= \sum_{i=1}^n (-1)^{i-1} \iota^* f_i(x', 0)\, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots 0 \\ &= (-1)^{n-1} f_n(x', 0)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}\end{aligned}$$

Because all terms in the sum with $i \neq n$ vanish. Therefore:

$$\int_{\partial M} \omega = \int_{\widehat{U}} \omega = (-1)^n \int_{\widehat{\underline{U}}} (-1)^{n-1} f_n(x', 0)\, \mathrm{d}x^1 \ldots \mathrm{d}x^{n-1} = -\int_{B'} f_n(x'_n, 0)\, \mathrm{d}x'_n$$

Using (Identity 1), we get that;

$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega$$

This proves the result in this case.

With this we're done! Perfect!

**Corollary X.3.2** (Green's Formula in $\mathbb{R}^2$)
Let $W$ be a compact 2-dimensional submanifold of $\mathbb{R}^2$ and denote by $C = \partial W$.
Then we have that:

$$\int_C (f\, \mathrm{d}x + g\, \mathrm{d}y) = \int_W \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y$$

for any $f, g \in C^1(W)$.

**Remark X.3.2**
As we saw in the last section,

$$\int_C f\, \mathrm{d}x + g\, \mathrm{d}y = \int_C \vec{F} \cdot \mathrm{d}\vec{s} = \int_a^b \vec{F}(\gamma(s)) \cdot \gamma'(s)\, \mathrm{d}s$$

Where $\vec{F} = (f, g)$ and $\gamma : [a, b] \to \mathbb{R}^2$ is a curve parameterizing $C$, that is the line integral of $\vec{F}$ along $C$.
The RHS of Green's formula is the integral:

$$\int_W \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y = \int_W \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y$$

since $W$ is a 2-manifold in $\mathbb{R}^2$, so you can use itself as a coordinate chart.
Also, the quantity $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$ is called $\operatorname{curl} \vec{F} = \nabla \times \vec{F}$, which is a scalar if $\vec{F} : \mathbb{R} \to \mathbb{R}^2$. Therefore Green's Formula has the form:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{s} = \int_W (\operatorname{curl} \vec{F})\, \mathrm{d}x\, \mathrm{d}y$$

*Proof.* Let $\omega = f\, \mathrm{d}x + g\, \mathrm{d}y$. Then:

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\ &= \left( \frac{\partial f}{\partial x} \mathrm{d}x + \frac{\partial f}{\partial y} \mathrm{d}y \right) \wedge \mathrm{d}x + \left( \frac{\partial g}{\partial x} \mathrm{d}x + \frac{\partial g}{\partial y} \mathrm{d}y \right) \wedge \mathrm{d}y \\ &= \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

Then Stokes' theorem on $W$ gives the result.

**Part** D. **Riemannian Metrics and the Volume Form**

All objects we have thus introduced (forms, integrals on $n$-forms, Stokes' theorem) are purely topological or differential objects.

In particular, they do not distinguish between a manifold $M$ and a diffeomorphic copy of $M$ (like the open unit ball and all of $\mathbb{R}^n$).

Now, we introduce a geometric structure to our differentiable manifolds, namely a Riemannian metric.

**Definition X.3.1** (Riemannian Metrics and Manifolds)

Let $M$ be a differentiable manifold. A Riemannian metric $g$ on $M$ is a smooth, symmetric, 2-tensor that is positive definite at each point $p \in M$. The pair $(M, g)$ is called a Riemannian manifold.

Let's unwind this definition a bit.

Recall a symmetric 2-tensor. For each point $p \in M$, $g_p$ is a 2-tensor on $T_pM$, that is $g_p(-,-) : T_pM \times T_pM \to \mathbb{R}$ is bilinear, and $g_p(v, w) = g_p(w, v)$.

Positive-definite means $g_p(v, v) > 0$ for every $v \neq 0$ lying in $T_pM$.

For smoothness of a 2-tensor: Let $(U, \varphi)$ be a coordinate chart such that $\varphi = (x^1, \ldots, x^n)$. Then for each $p \in U$, we have $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ is a basis for $(T_pM)^*$ and $\mathrm{d}x^i \otimes \mathrm{d}x^j$ is a basis for $\mathscr{L}^2(T_pM)$ (2-tensors on $T_pM$). Therefore we can write that:

$$g_p = \sum_{1 \leq i,j \leq m} g_{ij}(p) \, \mathrm{d}x^i \otimes \mathrm{d}x^j$$

So the claim that $g$ is smooth is that the functions $g_{ij} : M \to \mathbb{R}$ are smooth. $g$ being symmetric is equivalent to saying $g_{ij} = g_{ji}$.

**Exercise X.3.1**

Show that $g$ is smooth if and only if for any two smooth vector fields $X, Y$ on $M$, there holds that $g(X, Y) : M \to \mathbb{R}$ is smooth, where this function is given by $p \mapsto g_p(X_p, Y_p)$.

Recall from Definition X.3.1 that a metric $g$ on $M$ is a smooth symmetric 2-tensor that is positive definite at each point $p \in M$. We call $(M, g)$ a Riemannian manifold.

If $(U, \varphi)$ is a coordinate chart, then since $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ is a basis for $(T_pM)^*$¿ Then for $p \in U$ we have for $p \in U$:

$$g(p) = \sum_{i,j} g_{ij}(p)\,\mathrm{d}x^i \otimes \mathrm{d}x^j$$

With $g_{ij} : U \to \mathbb{R}$ satisfying $g_{ij} = g_{ji}$. And in fact:

$$g_{ij}(p) = g(p) \cdot \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_g$$

If $M$ is a $k$-submanifold of $\mathbb{R}^d$, then it inherits the Euclidean metric from $\mathbb{R}^d$. We call this the induced Euclidean metric on $M$. That is for $v, w \in T_pM \subseteq T_p\mathbb{R}^d = \mathbb{R}^d$:

$$g_p(v, w) = \langle v, w \rangle_{\mathbb{R}^d}$$

## X.4. **The Volume Form**

We know from last Friday's discussion that if $(M, g)$ is an oriented Riemannian manifold, then there exists a smooth orthonormal frame (ONF) $\{E_1, \ldots, E_n\}$ in a neighborhood of each point $p \in M$.

By replacing $E_1$ by $-E_1$ if needed, we may assume that this orthonormal frame is positively oriented.

**Theorem X.4.1** (The volume form on oriented Riemannian manifolds)

Let $(M, g)$ be an oriented Riemannian $n$-manifold. Then:

a) There is a unique smooth $n$-form $\Omega$ on $M$ such that:

$$\Omega(E_1, \ldots, E_n) = 1$$

for any positively oriented orthonormal frame $\{E_1, \ldots, E_n\}$ on $M$. $\Omega$ is denoted by $\mathrm{d}V_g$ or $\mathrm{d}V_M$ (or $\mathrm{d}A$ if $n = 2$ or $\mathrm{d}s$ if $n = 1$).

Caution: This notation does not mean that $\mathrm{d}V_M$ is exact (that is $V_M$ is just notation, not an $(n-1)$-form), in fact it is not

b) Let $(U, \varphi)$ be a coordinate chart in the orientation of $M$, and suppose that $\varphi = (x^1, \ldots, x^n)$. Then:

$$\mathrm{d}V_M = \sqrt{|g|}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

where:

$$|g| = \det(g_{ij}) > 0$$

Where $g_{ij}$ are the components of $g$ in these coordinates $(x^1, \ldots, x^n)$.

This unique smooth $n$-form is called the volume form on $(M, g)$.

*Proof.* First we show uniqueness. Suppose that such an $n$-form $\Omega$ exists. Then let $E_1, \ldots, E_n$ be a smooth positively oriented orthonormal frame on some open set $U \subseteq M$. Let $\varepsilon^1(p), \ldots, \varepsilon^n(p)$ be the basis of $(T_pM)^*$ dual to $E^1(p), \ldots, E^n(p)$. The 1-forms $\varepsilon^1, \ldots, \varepsilon^n$ are called the dual coframe to $E_1, \ldots, E_n$. We showed on Friday that in coordinates we have:

$$\varepsilon^i = g^{ij} E_j$$

However, we won't use this for our proof.

Then we know that:

$$\Omega(p) = f(p) \cdot \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$$

for some smooth function $f : U \to \mathbb{R}$. This holds because $\dim \mathscr{A}^n(T_pM) = 1$. But then:

$$1 = \Omega(E_1, \ldots, E_n) = f(p) \cdot (\varepsilon^1 \wedge \cdots \wedge \varepsilon^n)(E_1, \ldots, E_n) = f(p)$$

Therefore $\Omega = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$. This implies that $\Omega$ is unique because it has this very precise form.

Now for existence. Since for every $p \in M$, there exists an open neighborhood $U$ of $p$ and a positively oriented orthonormal frame $\{E_1, \dots, E_n\}$ on $U$, we define $\Omega(p) = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ where $\varepsilon^1, \dots, \varepsilon^n$ is the dual coframe to $E_1, \dots, E_n$.

To show that this definition makes sense, we need to show that it does not depend on the positively oriented orthonormal frame chosen. Let $\widetilde{E}_1, \dots, \widetilde{E}_n$ be another positively oriented orthonormal frame near $p$ on a neighborhood $V$ of $p$. Let $\widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_n$ be its dual coframe. Then we must show that:

$$\varepsilon^1 \wedge \cdots \wedge \varepsilon^n = \widetilde{\varepsilon}^1 \wedge \cdots \wedge \widetilde{\varepsilon}^n$$

To see this, write $\widetilde{E}_i(q) = A_i^j(q) E_j(q)$ for all $q \in U \cap V$ for some matrix $A_i^j$ of smooth $A_i^j : U \cap V \to \mathbb{R}$.

The fact that both frames are orthonormal means that $A_i^j(q)$ is an orthogonal matrix (linear algebra). Therefore $\det A_i^j(q) = \pm 1$. Since both frames are positively oriented , then $\det A_i^j(q) > 0$, so $\det A_i^j(q) = 1$ for all $q \in U \cap V$

Then we may write:

$$\Omega = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n \qquad\qquad \widetilde{\Omega} = \widetilde{\varepsilon}^1 \wedge \cdots \wedge \widetilde{\varepsilon}^n$$

Then we may write that:

$$\begin{aligned}
\Omega(\widetilde{E}_1, \dots, \widetilde{E}_n) &= (\varepsilon^1 \wedge \cdots \wedge \varepsilon^n)(\widetilde{E}_1, \dots, \widetilde{E}_n) \\
&= \det[\varepsilon^j(\widetilde{E}_i)] = \det(A_i^j) = 1 \\
&= \widetilde{\varepsilon}^1 \wedge \cdots \wedge \widetilde{\varepsilon}^n(\widetilde{E}_1, \dots, \widetilde{E}_n) \\
&= \widetilde{\Omega}(\widetilde{E}_1, \dots, \widetilde{E}_n)
\end{aligned}$$

Therefore $\Omega$ and $\widetilde{\Omega}$ agree on a basis, and so $\Omega = \widetilde{\Omega}$.

Now for the proof of part (b). Let $(U, \varphi)$ be some coordinate chart with $\varphi = (x^1, \dots, x^n)$. Then in those coordinates we have $\mathrm{d}V_M$ can be given as:

$$\mathrm{d}V_M(p) = f(p) \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

for some smooth function $f : U \to \mathbb{R}$. To compute $f(p)$ we let $E_1, \dots, E_n$ be a positively oriented orthonormal frame defined on $U$ and let $\varepsilon^1, \dots, \varepsilon^n$ be its dual coframe. Then we may write that:

$$\frac{\partial}{\partial x^j} = \sum B_i^j E_j$$

Then applying the above equality to $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ we obtain that:

$$\begin{aligned}
f(p) &= \mathrm{d}V_M(p)\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = (\varepsilon^1 \wedge \cdots \wedge \varepsilon^n)\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) \\
&= \det\left[\varepsilon^j \cdot \frac{\partial}{\partial x^i}\right] = \det[B_i^j]
\end{aligned}$$

On the other hand, we may write with Einstein Summation notation:

$$\begin{aligned}
g_{ij} := g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= g(B_i^k E_k, B_j^\ell E_\ell) \\
&= B_i^k B_j^\ell g(E_k, E_\ell) = B_i^k B_j^\ell \delta_{k\ell} = \sum_{k=1}^n B_i^k B - j^k = (B^T B)_{ij}
\end{aligned}$$

Therefore we have that:

$$\begin{aligned}
\det(g_{ij}) &= \det(B^T B) = (\det B)^2 > 0 \\
|g| &= \det(g_{ij}) > 0 \\
\det B &= \pm\sqrt{|g|}
\end{aligned}$$

But then both $E_1, \dots, E_n$ and $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ and positively oriented, so $\det B > 0$. Thus $\det B = \sqrt{|g|}$. Great! This then tells us that:

$$\mathrm{d}V_M(p) = f(p) \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = (\det B) \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \sqrt{|g|} \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

This is exactly what we wanted to show.

**Example X.4.1**
Let $M$ be a submanifold of $\mathbb{R}^d$ and let $g$ be the induced Euclidean metric on $M$.
Let $(U, \varphi)$ be a coordinate chart on$M$ and suppose $\varphi = (\varphi^1, \dots, \varphi^n)$ and $\alpha = \varphi^{-1}$ and set. Then $g_{ij}$ is just:

$$g_{ij} = g\left(\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}\right) = \frac{\partial \alpha}{\partial x^i} \cdot \frac{\partial \alpha}{\partial x^j} = \alpha_*(e_i) \cdot \alpha_*(e_j)$$
$$g_{ij} = (D\alpha^T(x) \cdot D\alpha(x))_{ij}$$

Why? Well suppose that $A$ has columns $v_1, \dots, v_n$. Then $A^T$ has rows $v_1^T, \dots, v_n^T$. But then:

$$(A^T A)_{ij} = v_i^T v_j = v_i \cdot v_j$$

This means that the volume form $\mathrm{d}V_M$ on $U$ is given by:

$$\begin{aligned}\mathrm{d}V_M(p) &= \sqrt{|g|} \cdot \mathrm{d}\varphi^1 \wedge \cdots \wedge \mathrm{d}\varphi^n \\ &= \sqrt{\det\left[D\alpha^T(x) \cdot D\alpha(x)\right]} \cdot \cdot \mathrm{d}\varphi^1 \wedge \cdots \wedge \mathrm{d}\varphi^n\end{aligned}$$

where $x = \varphi(p)$

**Definition X.4.1**
Let $(M, g)$ be an oriented Riemanian manifold, and let $f : M \to \mathbb{R}$ be continuous and compactly supported. Then the integral of $f$ over $M$ is defined as $\int_M f \, \mathrm{d}V_M$.
Also, the volume of $M$ is defined as $V(M) = \int_M 1 \, \mathrm{d}V_M$ when $M$ is compact

**Exercise X.4.2**
Check that $V(M) > 0$.

**Exercise X.4.3**
If we reverse the orientation of $M$, then $\int_M f \, \mathrm{d}V$ does not change

**Example X.4.4**
Let $C$ be a 1-dimensional submanifold of $\mathbb{R}^d$ and let $\gamma : (a, b) \to C$ be a parameterization of $C$ (i.e. $C = \gamma(a, b)$. We denote $\mathrm{d}V_C = \mathrm{d}s$. Then:

$$\mathrm{d}s = \sqrt{\gamma'^T(t)\gamma'(t)} \, \mathrm{d}t = \sqrt{|\gamma'(t)|^2} \, \mathrm{d}t = |\gamma'(t)| \, \mathrm{d}t$$

If $f : C \to \mathbb{R}$ is continuous and compactly supported, then:

$$\int_C f \, \mathrm{d}s = \int_a^b f(\gamma(t)) \, |\gamma'(t)| \, \mathrm{d}t$$

the length of $C$ is then just:

$$\int_C \mathrm{d}s = \int_a^b |\gamma'(t)| \, \mathrm{d}t$$

**Example X.4.5**
Let $S \subseteq \mathbb{R}^d$ be a surface (i.e. a 2-dimensional submanifold) and let $\alpha : U \to \mathbb{R}^d$ be its parameterization, i.e. $S = \alpha(U)$. Denote $\mathrm{d}V_S = \mathrm{d}A$ and let $\mathrm{d}u^1 \wedge \mathrm{d}u^2$ be the coordinate form given by $\alpha$ (i.e. if $\varphi = \alpha^{-1}$ then $\mathrm{d}u^i = \mathrm{d}\varphi^i$ as before).
Then we have that:

$$\mathrm{d}A = \sqrt{\det D\alpha^T(u) D\alpha(u)} \, \mathrm{d}u^1 \wedge \mathrm{d}u^2$$
$$D\alpha^T(u) D\alpha(u) = \begin{pmatrix} \frac{\partial \alpha}{\partial u^1} \cdot \frac{\partial \alpha}{\partial u^1} & \frac{\partial \alpha}{\partial u^1} \cdot \frac{\partial \alpha}{\partial u^2} \\ \frac{\partial \alpha}{\partial u^2} \cdot \frac{\partial \alpha}{\partial u^1} & \frac{\partial \alpha}{\partial u^2} \cdot \frac{\partial \alpha}{\partial u^2} \end{pmatrix}$$

We may then write:

$$\det D\alpha^T(u)D\alpha(u) = \left|\frac{\partial\alpha}{\partial u^1}\right|^2 \left|\frac{\partial\alpha}{\partial u^2}\right|^2 - \left(\frac{\partial\alpha}{\partial u^1}\cdot\frac{\partial\alpha}{\partial u^2}\right)^2$$
$$|g| = \left|\frac{\partial\alpha}{\partial u^1}\right|^2 \left|\frac{\partial\alpha}{\partial u^2}\right|^2 - \left(\frac{\partial\alpha}{\partial u^1}\cdot\frac{\partial\alpha}{\partial u^2}\right)^2$$
$$\mathrm{d}A = \sqrt{|g|}\,\mathrm{d}u^1 \wedge \mathrm{d}u^2$$

If $d = 3$, then recall that:

$$\left|\frac{\partial\alpha}{\partial u^1}\right|^2 \left|\frac{\partial\alpha}{\partial u^2}\right|^2 - \left(\frac{\partial\alpha}{\partial u^1}\cdot\frac{\partial\alpha}{\partial u^2}\right)^2 = \left|\frac{\partial\alpha}{\partial u^1}\right|^2 \left|\frac{\partial\alpha}{\partial u^2}\right|^2 (1-\cos^2\theta) = \left|\frac{\partial\alpha}{\partial u^1}\times\frac{\partial\alpha}{\partial u^2}\right|$$

In this case, we see that:

$$\int_S f\,\mathrm{d}A = \int_U f(\alpha(u))\cdot\left|\frac{\partial\alpha}{\partial u^1}\times\frac{\partial\alpha}{\partial u^2}\right|\cdot\mathrm{d}u^1\,\mathrm{d}u^2$$

**Remark X.4.1**

Last time, we saw that if $M$ is an orientable Riemannian manifold, then we could define a unique volume form $\mathrm{d}V_M$ and then we used that for integrate functions on $M$ as:

$$\int_M f \,\mathrm{d}V_M$$

In coordinates, if $\operatorname{supp} f \subseteq (U, \varphi)$ then if $\widehat{U} = \varphi(U)$ and $\alpha = \varphi^{-1}$ we have:

$$\int_M f \,\mathrm{d}V_M = \int_{\widehat{U}} f(\alpha(x))\sqrt{|g|(x)}\,\mathrm{d}x \tag{$\star$}$$

where $|g|(x) = \det D\alpha^T(x)D\alpha(x)$ when $(M, g)$ is the induced Euclidean metric.

I the geeral case, we use a partition of unity to ue the above formula on each coordinate chart.

$M$ does not need to be orientable to define this integral using $(\star)$. To see this, we can check that $(\star)$ is indeendent of the choice of coordinates.

We often use the notation $\int_M f\,|\mathrm{d}V_M|$ for the left hand side of $(\star)$. $|\mathrm{d}V_M|$ is called a Riemannian density.

# XI. Theorems of vector calculus

Here we shall prove the divergence theorem and the classical Stokes' theorem.

**Definition XI.0.1**

Given an alternating $k$-form $\omega$ on a vector space $V$ ($\omega \in \mathscr{A}^k(V)$) and a vector $x \in V$, we define $i_X\omega$ (called the interior multiplication) as the $(k-1)$-form:

$$i_x(\omega)(v_1, \ldots, v_{k-1}) = \omega(x, v_1, \ldots, v_{k-1})$$

**Exercise XI.0.1**

Show that $i_x(\omega) \in \mathscr{A}^{k-1}(V)$

We will use this notation when $\omega = \mathrm{d}V_{\mathbb{R}^n}$ is the volume form on $\mathbb{R}^n$. Then $i_x\,\mathrm{d}V_{\mathbb{R}^n}$ is an $(n-1)$-form for any $x \in T_p\mathbb{R}^n$. It is given by:

$$\begin{aligned}
i_X\,\mathrm{d}V_{\mathbb{R}^n}(v_1, \ldots, v_{k-1}) &= \mathrm{d}V_{\mathbb{R}^n}(X, v_1, \ldots, v_{k-1}) \\
&= \sum_{i=1}^n X^i\,\mathrm{d}V_{\mathbb{R}^n}(e_i, v_1, \ldots, v_{n-1}) \\
&= \sum_{i=1}^n X^i \det(e_i, v_1, \ldots, v_{n-1}) \\
&= \sum_{i=1}^n X^i(-1)^{i+1}\det(\widehat{v}_1, \ldots, \widehat{v}_{n-1})
\end{aligned}$$

where $\widehat{v}_j \in \mathbb{R}^{n-1}$ in the above formula is obtained from $v_j$ by dropping the $i$-th coordinate. Then we know:

$$i_X\,\mathrm{d}V_{\mathbb{R}^n}(v_1, \ldots, v_{n-1}) = \sum_{i=1}^n (-1)^{i+1}X^i\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n(v_1, \ldots, v_{n-1})$$

$$i_X\,\mathrm{d}V_{\mathbb{R}^n} = \sum_{i=1}^n (-1)^{i+1}X^i\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$$

**Proposition XI.0.1**

Let $(M, g)$ be an oriented $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ (aka a hypersurface), and let $g$ be the induced Euclidean metric on $M$. Let $\vec{N}$ be the smooth unit normal vector to $M$ corresponding to its orientation.

(recall that this means that $\{v_1, \ldots, v_{k-1}\}$ is positively orinted in$T_pM$ if and only if $\{\vec{N}, v_1, \ldots, v_{n-1}\}$ is positively oriented in $T_p\mathbb{R}^n$)

Then in fact:

$$\mathrm{d}V_M = i_N \mathrm{d}V_{\mathbb{R}^n}\big|_M$$

*Proof.* Let $E_1, \ldots, E_{n-1}$ be any positively oriented orthonormal frame on $U \subseteq M$ open. Then we know:

$$i_N\, \mathrm{d}V_{\mathbb{R}^n}(E_1, \ldots, E_{n-1} = \mathrm{d}V_{\mathbb{R}^n}(N, e_1, \ldots, E_{n-1}) = 1$$

And this holds for every such orthonormal frame. However this is just the definition of the volume form by uniqueness. Therefore $\mathrm{d}V_M = i_N \mathrm{d}V_{\mathbb{R}^n}\big|_M$.

**Proposition XI.0.2**

With the same notation as above, let $X$ be any vector field along $M$ (not necessarily tangent to $M$), then we have:

$$i_X \mathrm{d}V_{\mathbb{R}^n}\big|_M = \langle X, N\rangle\, \mathrm{d}V_M$$

*Proof.* Write $X = X^T + X^\perp$ where $X^\perp = \langle X, N\rangle N \in (T_pM)^\perp$ and $X^T = X - X^\perp \in T_pM$.

Then we have:

$$\begin{aligned} i_X\, \mathrm{d}V_{\mathbb{R}^n} &= i_{X^\perp}\, \mathrm{d}V_{\mathbb{R}^n} + i_{X^T}\, \mathrm{d}V_{\mathbb{R}^n} \\ &= i_{\langle X,N\rangle N}\, \mathrm{d}V_{\mathbb{R}^n} + i_{X^T}\, \mathrm{d}V_{\mathbb{R}^n} \end{aligned}$$

Therefore we may write:

$$i_X \mathrm{d}V_{\mathbb{R}^n}\big|_M = \langle X, N\rangle i_N \mathrm{d}V_{\mathbb{R}^n}\big|_M + i_{X^T} \mathrm{d}V_{\mathbb{R}^n}\big|_M$$

But then we just need to notice that $i_{X^T} \mathrm{d}V_{\mathbb{R}^n}\big|_M = 0$. Why? Wel if $v_1, \ldots, v_{n-1} \in T_pM$ then:

$$i_{X^T} \mathrm{d}V_{\mathbb{R}^n}\big|_M(v_1, \ldots, v_{n-1}) = \mathrm{d}V_{\mathbb{R}^n}(X^T, v_1, \ldots, v_{n-1}) = \det(X^T, v_1, \ldots, v_{n-1}) = 0$$

Because $T_pM$ is an $(n-1)$-dimensional subspace, so the vectors given above cannot be linearly independent. Therefore as claimed above:

$$i_X \mathrm{d}V_{\mathbb{R}^n}\big|_M = \langle X, N\rangle\, \mathrm{d}V_M$$

**Theorem XI.0.3** (The Divergence Theorem)

Let $M$ be an $n$-dimensional submanifold of $\mathbb{R}^n$ with its induced Euclidean metric and orientation. Let $\vec{N}$ be the outward unit normal vector to $\partial M$.

Suppose that $F$is a smooth ($C^1$ is enough) vector field on $M$. Then $F = F(^1, \ldots, F^n)$ where $F^i : M \to \mathbb{R}$.

Then we have that:

$$\int_{\partial M} \langle F, N\rangle\, \mathrm{d}V_{\partial M} = \int_M (\operatorname{div} F)\, \mathrm{d}V_{\mathbb{R}^n}$$

WHere $\operatorname{div} F = \sum_{i=1}^n \frac{\partial F^i}{\partial x^i}$

**Remark XI.0.1**

The integral $\int_{\partial M} \langle F, N\rangle\, \mathrm{d}V_{\partial M}$ is called the flow of $F$ through $\partial M$.

For a small ball $B$, then $\int_{\partial B} \langle F, N\rangle\, \mathrm{d}V_{\partial B}$ is how much $F$ is pointing / flowing into $B$. If the ball has radius $\varepsilon$ then the right hand side is approximately $\operatorname{div} F(p)\varepsilon^3$ for $p$ the center of the ball.

If $F$ is the velocity of a liquid, then $\operatorname{div} F = 0$, which means the liquid is incompressible which holds if and only if the amount of fluid that enters a closed region is equal to the amount that exits that region.

*Proof.* Let $\omega = i_F \, \mathrm{d}V_{\mathbb{R}^n}$. Then $\omega = \sum_{i=1}^n (-1)^{i+1} F^i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$.

So then we have that:

$$\begin{aligned}
\mathrm{d}\omega &= \sum_{i=1}^n (-1)^{i+1} \, \mathrm{d}F^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^n (-1)^{i+1} \left( \sum_{k=1}^n \frac{\partial F^i}{\partial x^k} \, \mathrm{d}x^k \right) \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^n (-1)^{i+1} \frac{\partial F^i}{\partial x^i} \, \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^n (-1)^{i+1} (-1)^{i-1} \frac{\partial F^i}{\partial x^i} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^n \\
&= \left( \sum_{i=1}^n \frac{\partial F^i}{\partial x^i} \right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= (\operatorname{div} F) \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n
\end{aligned}$$

Then we proceed from Stokes' theorem. We know that:

$$\begin{aligned}
\int_M \mathrm{d}\omega &= \int_M (\operatorname{div} F) \, \mathrm{d}V_{\mathbb{R}^n} \\
\int_{\partial M} \omega &= \int_{\partial M} (i_F \, \mathrm{d}V_{\mathbb{R}^n})\big|_{\partial M} \\
&= \int_{\partial M} \langle F, N \rangle \, \mathrm{d}V_{\partial M}
\end{aligned}$$

Recalling that the boundary orientation of $\partial M$ is determined by the outward normal vector $N$. Great! We then know:

$$\int_{\partial M} \langle F, N \rangle \, \mathrm{d}V_{\partial M} = \int_{\partial M} \omega = \int_M \mathrm{d}\omega = \int_M (\operatorname{div} F) \, \mathrm{d}V_{\mathbb{R}^n}$$

This finishes the proof

**Theorem XI.0.4** (The classical Stokes' theorem)

Let $M$ be an oriented 2-dimensional submanifold of $\mathbb{R}^3$ with its induced Euclidean metric.

Let $N$ be the unit normal vector field to $M$ corresponding to its orientation. Let $F = (F^1, F^2, F^3)$ be a $C^1$-vector field defined in an open set containing $M$ . Then:

$$\int_{\partial M} F \cdot \mathrm{d}s = \int_M \langle \operatorname{curl} F, N \rangle \, \mathrm{d}A$$

where $\mathrm{d}A = \mathrm{d}V_M$ and $\mathrm{d}s = \mathrm{d}V_{\partial M}$

**Remark XI.0.2**

This is the generalization of Green's formula which deals with the case where $M$ is a subset of the $(x, y)$ plane. In that case, $\operatorname{curl} F = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ and hence $\langle \operatorname{curl} F, N \rangle = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ because $N = (0, 0, 1)$.

*Proof.* Let $\omega = F^1 \, \mathrm{d}x^1 + F^2 \, \mathrm{d}x^2 + F^3 \, \mathrm{d}x^3$.

We have seen before that:

$$\int_{\partial M} F \cdot \mathrm{d}s = \int_{\partial M} \omega$$

We also saw that:

$$\mathrm{d}\omega = G_1 \,\mathrm{d}x^2 \wedge \mathrm{d}x^3 - G_2 \,\mathrm{d}x^1 \wedge \mathrm{d}x^3 + G_3 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 = \sum_{i=1}^{3} G_i (-1)^{i-1} \,\mathrm{d}x^1 \wedge \widehat{\mathrm{d}x^i} \wedge \mathrm{d}x^3$$

where $G = (G_1, G_2, G_3) = \operatorname{curl} F$. Where of course:

$$\operatorname{curl} F = \left( \partial F^2 \partial y - \frac{\partial F^2}{\partial z}, \frac{\partial F^1}{\partial z} - \frac{\partial F^3}{\partial x}, \frac{\partial F^2}{\partial x} - \frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y} \right)$$

Hence $\mathrm{d}\omega = i_G \,\mathrm{d}V_{\mathbb{R}^3}$ and

$$\mathrm{d}\omega\big|_M = i_G \mathrm{d}V_{\mathbb{R}^3}\big|_M = \langle G, N \rangle \,\mathrm{d}V_M = \langle \operatorname{curl} F, N \rangle \,\mathrm{d}V_M$$

But we also know that:

$$(\mathrm{d}\omega)\big|_= \mathrm{d}(\omega\big|_M)$$

Why? Well if $j : M \to \mathbb{R}^3$ is the inclusion then:

$$\omega\big|_M = j^*\omega \implies \mathrm{d}(\omega\big|_M) = \mathrm{d}j^*\omega = j^* \,\mathrm{d}\omega = (\mathrm{d}\omega)\big|_M$$

By Stokes Theorem applied to $\omega\big|_M$ we have:

$$\int_{\partial M} \omega\big|_M = \int_M \mathrm{d}(\omega\big|_M) = \int_M (\mathrm{d}\omega)\big|_M$$

$$\int_{\partial M} F \cdot \mathrm{d}s = \int_M \langle \operatorname{curl} F, N \rangle \,\mathrm{d}A$$

This completes the proof

**Appendix**

## Appendix A. IBL: Measure Theory and Lebesgue Integration

This is the IBL section of the course. These problems are done in groups, and not all of them are completed here. These problems are primarily in measure theory and the construction of the Lebesgue Integral

## Handout 1

### Where did we learn in 395?

The notion of Lebesgue outer measure of a set $E$:

$$m^*(E) = \inf_{E \subset \cup_{j=1}^{\infty} B_j} \sum_{j=1}^{\infty} |B_j|$$

where the union above is taken over boxes $B_j \subset \mathbb{R}^d$. A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing $E$ such that $m^*(U \setminus E) \leq \epsilon$. If $E$ is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of $E$.

We have proven the following facts:

(i) Properties of the outer measure

- $m^*(\emptyset) = 0$.
- (Monotonicity) If $E \subset F \subset \mathbb{R}^d$, then $m^*(E) \leq m^*(F)$.
- (Countable subadditivity) If $E_1, E_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of sets, then $m^*\left(\cup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

(ii) If $\text{dist}(E, F) > 0$, then $m^*(E \cup F) = m^*(E) + m^*(F)$.

(iii) If $E$ is an elementary set, then $m^*(E) = m(E)$ where $m(E)$ is the elementary measure of $E$ defined before. More generally,

(iv) Let $E = \cup_{n=1}^{\infty} B_n$ be a countable union of almost disjoint boxes $B_k$ (this means that their interiors are disjoint) then

$$m^*(E) = \sum_{k=1}^{\infty} |B_k|.$$

As such, $\mathbb{R}^d$ for example has infinite outer measure.

(v) Let $E \subset \mathbb{R}^d$ be an arbitrary set. There holds

$$m^*(E) = \inf_{E \subset U, U\, open} m^*(U).$$

This is called *outer regularity*.

---

Show the following (Warning: some of those questions are trivial one-liners).

**Q1)** Every open set is Lebesgue measurable.

**Q2** If $E_1, E_2, E_3, \ldots \subset \mathbb{R}^d$ are a sequence of Lebesgue measurable sets, then the union $\cup_{n=1}^{\infty} E_n$ is Lebesgue measurable

**Q3)** Every closed set is Lebesgue measurable. *Hint: Reduce to the compact case. Then, use that any open set is the countable union of almost disjoint closed cubes, as well as some of the properties reviewed above.*

**Q4)** Every set of Lebesgue outer-measure 0 is measurable (such sets are called null sets).

**Q5)** the empty set $\emptyset$ is Lebesgue measurable.

**Q6)** If $E \subset \mathbb{R}^d$ is Lebesgue measurable, the so is its complement $\mathbb{R}^d \setminus E$.

**Q7)** If $E_1, E_2, E_3, \ldots \subset \mathbb{R}^d$ are a sequence of Lebesgue measurable sets, then the intersection $\cap_{n=1}^{\infty} E_n$ is Lebesgue measurable.

**Q8)** A set $E$ is measurable iff and only for every $\epsilon > 0$ one can find an open set $U$ such that $m^*(E \Delta U) \leq \epsilon$ (in other words $E$ differs from an open set by a set of outer measure $\epsilon$.)

**Q9)** A set $E$ is measurable iff and only for every $\epsilon > 0$ one can find a closed set $F$ such that $m^*(E \Delta F) \leq \epsilon$ (in other words $E$ differs from a closed set by a set of outer measure $\epsilon$.)

**Q10)** If $E_1, E_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of disjoint Lebesgue measurable sets, then $m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$.

## I.1. **IBL Week 1**

**Problem I-3**

Every closed set is Lebesgue measurable. *Hint: Reduce to the compact case. Then, use that any open set is the countable union of almost disjoint closed cubes, as well as some of the properties*

*Solution.* First we handle the case of a compact set $C$. Fix some $\varepsilon > 0$ and let $U$ be an open set so that $C \subseteq U$ and $m^*(U) - m^*(C) < \frac{\varepsilon}{2}$ by outer regularity. Now we express the open set $U \setminus C$ as the countable union of almost disjoint closed cubes $B_1, B_2, \ldots$. We wish to show that the following by property (iv):

$$m^*(U \setminus C) = \sum_{k=1}^{\infty} |B_k| \leq \frac{\varepsilon}{2} < \varepsilon$$

To do so it suffices to show that any finite sum $\sum_{k=1}^{N} |B_k| \leq \frac{\varepsilon}{2}$.

To do this, note that $B_k \subseteq U \setminus C$ so $B_k \cap C = \emptyset$. We wish to show that $\operatorname{dist}(B_k, C) > 0$ in order to use property (ii). We prove a small lemma

**Lemma I.1.1**

For any two disjoint compact sets $K_1$ and $K_2$ we know that $\operatorname{dist}(K_1, K_2) > 0$

*Proof.* Suppose that $\operatorname{dist}(K_1, K_2) = 0$. Then construct sequences $x_n \in K_1$ and $y_n \in K_2$ such that $d(x_n, y_n) < \frac{1}{n}$ by definition of the distance.

Now by compactness is a convergent subsequence $x_{n_k}$ of $x_n$ and $y_{n_{k_\ell}}$ of $y_{n_k}$. These converge to some $x \in K_1$ and some $y \in K_2$ respectively. We show that $x = y$ and so $K_1$ and $K_2$ are not disjoint. To do this fix some $\varepsilon > 0$ and let $\ell \in \mathbb{N}$ be so that $\frac{1}{\ell} < \frac{\varepsilon}{3}$, $d(x_{n_{k_\ell}}, x) < \frac{\varepsilon}{3}$, and $d(y_{n_{k_\ell}}, y) < \frac{\varepsilon}{3}$. Then note that

$$\begin{aligned} d(x,y) &\overset{\triangle}{\leq} d(x, x_{n_{k_\ell}}) + d(x_{n_{k_\ell}}, y_{n_{k_\ell}}) + d(y_{n_{k_\ell}}, y) \\ &< \frac{\varepsilon}{3} + \frac{1}{n_{k_\ell}} + \frac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + \frac{1}{\ell} + \frac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

And so $x = y$ by taking $\varepsilon \to 0$. This shows that $x \in K_1 \cap K_2$ so $K_1$ and $K_2$ are not disjoint. With this we've shown the contrapositive of the lemma.

Therefore since $\bigcup_{k=1}^{N} B_k$ is compact since each $B_k$ is compact and $C$ is compact, we know that:

$$\operatorname{dist}\left(\bigcup_{k=1}^{N} B_k, C\right) > 0$$

Now by property (ii) and monotonicity we know that:

$$\begin{aligned} m^*(C) + \sum_{k=1}^{N} |B_k| &= m^*\left(C \cup \bigcup_{k=1}^{N} B_k\right) \leq m^*(U) \\ \sum_{k=1}^{N} |B_k| &\leq m^*(U) - m^*(C) < \frac{\varepsilon}{2} \end{aligned}$$

And so passing to the infinite case:

$$m^*(U \setminus C) = \sum_{k=1}^{\infty} |B_k| \leq \frac{\varepsilon}{2} < \varepsilon$$

This shows that $C$ is Lebesgue measurable just as desired.

Now consider any closed set $C$. Now write $\mathbb{R}^n$ as a countable union of almost disjoint closed boxes $B_1, B_2, \ldots$ by just taking points on the integer lattice as corners for these boxes. Then we know that:

$$\begin{aligned} C = C \cap \mathbb{R}^n &= C \cap \left( \bigcup_{k=1}^{\infty} B_k \right) \\ &= \bigcup_{k=1}^{\infty} C \cap B_k \end{aligned}$$

Now since $C \cap B_k$ is a closed subset of the compact set $B_k$ we know that $C \cap B_k$ is compact, and so by the previous case it is Lebesgue measurable. But then by Question 2 we know that $C$ is Lebesgue measurable just as desired! Great!

## Handout 1

### Where did we learn in 395?

The notion of Lebesgue outer measure of a set $E$:

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where the union above is taken over boxes $B_j \subset \mathbb{R}^d$. A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing $E$ such that $m^*(U \setminus E) \leq \epsilon$. If $E$ is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of $E$.

We have proven the following facts:

(i) Properties of the outer measure

- $m^*(\emptyset) = 0$.
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(ii) If $\text{dist}(E, F) > 0$, then $m^*(E \cup F) = m^*(E) + m^*(F)$.

(iii) If $E$ is an elementary set, then $m^*(E) = m(E)$ where $m(E)$ is the elementary measure of $E$ defined before. More generally,

(iv) Let $E = \cup_{n=1}^{\infty} B_n$ be a countable union of almost disjoint boxes $B_k$ (this means that their interiors are disjoint) then

$$m^*(E) = \sum_{k=1}^{\infty} |B_k|.$$

As such, $\mathbb{R}^d$ for example has infinite outer measure.

(v) Let $E \subset \mathbb{R}^d$ be an arbitrary set. There holds

$$m^*(E) = \inf_{E \subset U, U \, open} m^*(U).$$

This is called *outer regularity.*

---

Show the following (Warning: some of those questions are trivial one-liners).

**Q1)** Every open set is Lebesgue measurable.

**Q2** If $E_1, E_2, E_3, \ldots \subset \mathbb{R}^d$ are a sequence of Lebesgue measurable sets, then the union $\cup_{n=1}^{\infty} E_n$ is Lebesgue measurable

**Q3)** Every closed set is Lebesgue measurable. *Hint: Reduce to the compact case. Then, use that any open set is the countable union of almost disjoint closed cubes, as well as some of the properties reviewed above.*

**Q4)** Every set of Lebesgue outer-measure 0 is measurable (such sets are called null sets).

**Q5)** the empty set $\emptyset$ is Lebesgue measurable.

**Q6)** If $E \subset \mathbb{R}^d$ is Lebesgue measurable, the so is its complement $\mathbb{R}^d \setminus E$.

**Q7)** If $E_1, E_2, E_3, \ldots \subset \mathbb{R}^d$ are a sequence of Lebesgue measurable sets, then the intersection $\cap_{n=1}^{\infty} E_n$ is Lebesgue measurable.

**Q8)** A set $E$ is measurable iff and only for every $\epsilon > 0$ one can find an open set $U$ such that $m^*(E \Delta U) \leq \epsilon$ (in other words $E$ differs from an open set by a set of outer measure $\epsilon$.)

**Q9)** A set $E$ is measurable iff and only for every $\epsilon > 0$ one can find a closed set $F$ such that $m^*(E \Delta F) \leq \epsilon$ (in other words $E$ differs from a closed set by a set of outer measure $\epsilon$.)

**Q10)** If $E_1, E_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of disjoint Lebesgue measurable sets, then $m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$.

## I.2. **IBL Week 2**

> **Problem I-6**
> If $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, then so is its complement $\mathbb{R}^d \setminus E$

*Solution.* Fix some sequence $U_1, U_2, \ldots$ of open sets so that each $U_n$ contains $E$ and:

$$m^*(U_n \setminus E) \leq \frac{1}{n}$$

Now name the following set:

$$A = \bigcup_{n \geq 1} \mathbb{R}^d \setminus U_n$$

Here we know that each $\mathbb{R}^d \setminus U_n$ is closed, so by Questions 2 and 3 $A$ is measurable. We claim that $A \subseteq E^c$, and we will show that $E^c \setminus A$ is Lebesgue measurable. With this we will see that:

$$E^c = (E^c \setminus A) \cup A$$

So then $E^c$ will be measurable. We'll quickly show that $A \subseteq E^c$. Fix some $a \in A$, then $a \in \mathbb{R}^d \setminus U_n$ for some $n$, and then $a \notin U_n$, and $U_n$ contains $E$, so $a \notin E$, so $a \in E^c$. Great!

To show that $E^c \setminus A$ is Lebesgue measurable we will use Question 4, showing that it has measure zero:

$$E^c \setminus A \subseteq E^c \setminus U_n^c = (\mathbb{R}^d \setminus E) \cap U_n = U_n \setminus E$$

So applying monotonicity we have for all $n$ that:

$$m^*(E^c \setminus A) \leq m^*(U_n \setminus E) \leq \frac{1}{n}$$

And so $E^c \setminus A$ has measure zero. Thus $E^c \setminus A$ is Lebesgue measurable and so $E^c = (E^c \setminus A) \cup A$ is Lebesgue measurable as desired!

> **Problem I-7**
> If $E_1, E_2, E_3, \ldots$ are a sequence of Lebesgue measurable sets then the intersection $\bigcap_{n=1}^{\infty} E_n$ is Lebesgue measurable

*Solution.* This is simple, note by by Problem 2 and Problem 6 since $E_1^c, E_2^c, \ldots$ are all Lebesgue measurable that $\bigcup_{n=1}^{\infty} E_n^c$ is Lebesgue measurable. Then we see that:

$$\left(\bigcup_{n=1}^{\infty} E_n^c\right) = \bigcap_{n=1}^{\infty} (E_n^c)^c = \bigcap_{n=1}^{\infty} E_n$$

Must be Lebesgue measurable by Question 6.

> **Problem I-8**
> A set $E$ is measurable if and only if for every $\varepsilon > 0$ one can find an open set $U$ such that $m^*(E \triangle U) \leq \varepsilon$ (in other words $E$ differs from an open set by a set of outer measure $\varepsilon$).

*Solution.* Let's do this in each direction!

($\Rightarrow$) Suppose that $E$ is measurable and fix $\varepsilon > 0$. Then there is some open set $U$ containing $E$ so that $m^*(U \setminus E) \leq \varepsilon$. Since $U$ contains $E$ we know that $E \setminus U = \emptyset$, and so:

$$m^*(U \setminus E) = m^*(E \triangle U) \leq \varepsilon$$

And so we are done!

($\Leftarrow$) Fix some $\varepsilon > 0$, and pick an open $U$ such that:

$$m^*(E \triangle U) \leq \frac{\varepsilon}{53}$$

in particular we will have by monotonicity that:

$$m^*(E \setminus U) \leq \frac{\varepsilon}{53} \qquad\qquad m^*(U \setminus E) \leq \frac{\varepsilon}{53}$$

Now by outer regularity find some open set $\mathcal{O}$ containing $E \setminus U$ so that:

$$m^*(\mathcal{O}) \leq m^*(E \setminus U) + \frac{\varepsilon}{53} \leq \frac{2\varepsilon}{53}$$

Now take $V = U \cup \mathcal{O}$. $V$ is open and we claim that $V$ contains $E$ since $\mathcal{O}$ contains $E \setminus U$ and $U$ will necessarily contain the rest of $E$. Then we compute by monotonicity and subadditivity that:

$$\begin{aligned} m^*(V \setminus E) &\leq m^*(U \setminus E) + m^*(\mathcal{O} \setminus E) \\ &\leq \frac{\varepsilon}{53} + m^*(\mathcal{O}) \\ &\leq \frac{2\varepsilon}{53} + m^*(E \setminus U) \leq \frac{3\varepsilon}{53} < \varepsilon \end{aligned}$$

And so since we can repeat this construction for any $\varepsilon > 0$ we see that $E$ is Lebesgue measurable.

Great!

**Problem I-9**

A set $E$ is measurable if and only if for every $\varepsilon > 0$ one can find a closed set $F$ such that $m^*(E \triangle F) \leq \varepsilon$ (in other words $E$ differs from a closed set of outer measure $\varepsilon$).

*Solution.* We apply Question 6 and Question 8. Note that $E$ is measurable if and only if $E^c$ is measurable, and so $E$ is measurable if and only if for every $\varepsilon > 0$ one can find an open set $U$ such that $m^*(E^c \triangle U) \leq \varepsilon$. Now consider that if we set $F = U^c$ then:

$$\begin{aligned} E^c \triangle U &= E^c \setminus U \cup U \setminus E^c = (E^c \cap U^c) \cup (U \cap E) \\ &= (F \setminus E) \cup (F^c \cap E) = (F \setminus E) \cup (E \setminus F) = E \triangle F \end{aligned}$$

Now note that since $U$ is open if and only if $F$ is closed and we see that:

$$m^*(E^c \triangle U) = m^*(E \triangle F)$$

It is clear that we can find an open set $U$ such that $m^*(E^c \triangle U) \leq \varepsilon$ if and only if we can find a closed set $F$ such that $m^*(E \triangle F) \leq \varepsilon$.

**Problem I-10**

If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$ is a countable sequence of disjoint Lebesgue measurable sets, then $m\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty m(E_n)$.

*Solution.* We have the inequality in one way by countable subadditivity.

**Claim**

If a set $E$ is measurable then for every $\varepsilon > 0$ we can find some closed set $F$ contained in $E$ so that $m^*(E \setminus F) \leq \varepsilon$.

*Proof.* Fix $\varepsilon > 0$, and note that $E^c$ is measurable by Question 6. Then there is some open set $U$ containing $E^c$ so that $m^*(U \setminus E^c) \leq \varepsilon$. Then take $F = U^c$. We then $F$ is closed and contains $E$, and also:

$$m^*(E \setminus F) = m^*(U \cap E) = m^*(U \setminus E^c) \leq \varepsilon$$

And so we're done

**Claim**

If all $E_j$ are compact then the statement is true. TODO

*Proof.* TODO

**Claim**

If all $E_j$ were bounded then the statement is true by approximating with the first claim, reducing to the compact case. TODO

*Proof.* TODO

Now we finally handle the general case.

## Handout 2

## Integration of Simple Functions

**Definition 0.1** (Simple function)**.** A (complex-valued) simple function $f : \mathbb{R}^d \to \mathbb{C}$ is a finite linear combination

$$f = c_1 1_{E_1} + \ldots + c_k 1_{E_k}$$

of indicator functions $1_{E_i}$ of Lebesgue measurable sets $E_i \subset \mathbb{R}^d$. Here, $i = 1, \ldots, k$ where $k \geq 1$ is a natural number and $c_1, c_2, \ldots, c_k \in \mathbb{C}$ are complex numbers. (Recall that the indicator function of set is the function that is equal to 1 on this set and zero otherwise). Clearly, this is a complex vector space.

An unsigned simple function $f : \mathbb{R}^d \to [0, \infty]$ is defined similarly but the $c_i$ take their values in $[0, \infty]$ rather than $\mathbb{C}$.

**Definition 0.2** (Integral of an unsigned simple function)**.** If $f = c_1 1_{E_1} + \ldots + c_k 1_{E_k}$ is an unsigned simple function, the integral $\text{Simp}\int_{\mathbb{R}^d} f(x)dx$ is defined by the formula

$$\text{Simp} \int_{\mathbb{R}^d} f(x)dx := c_1 m(E_1) + \ldots + c_k m(E_k). \tag{1}$$

Thus $\text{Simp}\int_{\mathbb{R}^d} f(x)dx$ takes values in $[0, \infty]$. (Here we adopt the convention that $0 \cdot \infty = 0$ in doing computations with the extended non-negative real numbers $[0, \infty]$).

A simple function has different representations as a linear combination of indicator functions of measurable sets, so for the above definition to make sense, we need to show that if

$$f = c_1 1_{E_1} + \ldots + c_k 1_{E_k} = c_1' 1_{E_1'} + \ldots + c_{k'}' 1_{E_{k'}'}$$

then we get the same answer when applying the formula (1), i.e. that

$$c_1 m(E_1) + \ldots + c_k m(E_k) = c_1' m(E_1') + \ldots + c_{k'}' m(E_k').$$

**Q1)** Show this! *Partition* $\mathbb{R}^d$ *into at most* $2^{k+k'}$ *disjoint sets formed by taking intersections of the* $k+k'$ *sets* $E_k$ *and* $E'_{k'}$*and their complements. Then write what needs to be proved in terms of those disjoint sets.*

In the following questions, let $f, g : \mathbb{R}^d \to [0, \infty]$ be simple unsigned function.

**Q2)** (Unsigned Linearity) Show that

$$\text{Simp} \int_{\mathbb{R}^d} f(x) + g(x) dx = \text{Simp} \int_{\mathbb{R}^d} f(x) dx + \text{Simp} \int_{\mathbb{R}^d} g(x) dx$$

and for any $c \in [0, \infty]$

$$\text{Simp} \int_{\mathbb{R}^d} cf(x) dx = c \, \text{Simp} \int_{\mathbb{R}^d} f(x) dx.$$

**Q3)** (Finiteness) Show that $\text{Simp}\int_{\mathbb{R}^d} f(x)dx < \infty$ if and only if $f$ is finite almost every where, and its support (defined here as the set $\{f(x) \neq 0\}$) has finite measure.

*Notation:* A property $P(x)$ of a point $x \in \mathbb{R}^d$ is said to hold *almost everywhere* in $\mathbb{R}^d$ (or for almost every point $x \in \mathbb{R}^d$) if the set of $x \in \mathbb{R}^d$ for which $P(x)$ fails has Lebesgue measure 0. For example, two functions $f$ and $g$ agree almost everywhere if one has that $f(x) = g(x)$ for almost every $x \in \mathbb{R}^d$.

**Q4)** We have $\text{Simp}\int_{\mathbb{R}^d} f(x)dx = 0$ if and only if $f$ is 0 almost everywhere. In particular, $f$ and $g$ agree almost everywhere then $\text{Simp}\int_{\mathbb{R}^d} f(x)dx =\text{Simp}\int_{\mathbb{R}^d} g(x)dx$.

**Q6)** If $f(x) \leq g(x)$ for almost every $x \in \mathbb{R}^d$, then $\text{Simp}\int_{\mathbb{R}^d} f(x)dx \leq\text{Simp}\int_{\mathbb{R}^d} g(x)dx$.

---

**Definition 0.3** (Absolutely convergent simple integral)**.** A complex-valued simple function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be absolutely integrable if

$$\text{Simp} \int_{\mathbb{R}^d} |f(x)| dx < \infty.$$

If $f$ is absolutely integrable, the integral $\text{Simp}\int_{\mathbb{R}^d} f(x)dx$ is defined for real signed $f$ by the formula

$$\text{Simp}\int_{\mathbb{R}^d} f(x)dx := \text{Simp}\int_{\mathbb{R}^d} f_+(x)dx - \text{Simp}\int_{\mathbb{R}^d} f_-(x)dx$$

where $f_+ = \max(f(x), 0)$ is the positive part of $f$, and $f_- = \max(-f(x), 0)$ is the negative part of $f$. Note that since $f_+, f_- \leq |f|$, they have a finite integral. Finally, for complex-valued $f$, we define

$$\text{Simp}\int_{\mathbb{R}^d} f(x)dx := \text{Simp}\int_{\mathbb{R}^d} \text{Re}\, f(x)dx + i\,\text{Simp}\int_{\mathbb{R}^d} \text{Im}\, f(x)dx$$

**Q7)** Show the linearity property of this integral as in **Q2)** (but with $c \in \mathbb{C}$). *Hint: Start with establishing it for real-valued simple functions.*

## I.3. **IBL Week 3**

**Problem I-1**

Show this! *Partition $\mathbb{R}^d$ into at most $2^{k+m}$ disjoint sets formed by taking intersections of the $k+m$ sets $E_i$ and $E'_j$ and their complements. Then write what needs to be proved in terms of those disjoint sets*

*Solution.* Start by writing $F_j := E'_j$ for convenience. Now write $E_j^0 = E_j$, $E_j^1 = E_j^c$, $F_j^0 = F_j$, and $F_j^1 = F_j^c$. Now for $\varepsilon_E \in \{0,1\}^k$ and $\varepsilon_F \in \{0,1\}^m$ we take the following set:

$$A(\varepsilon_E, \varepsilon_F) = \bigcap_{i=1}^{k} E_i^{\varepsilon_E^i} \cap \bigcap_{j=1}^{m} F_j^{\varepsilon_F^j}$$

These are disjoint since if we have $(\varepsilon_E, \varepsilon_F) \neq (\delta_E, \delta_F)$ then either $\varepsilon_E^i \neq \delta_E^i$ or $\varepsilon_F^j \neq \varepsilon_F^j$, and so we intersected $A(\varepsilon_E, \varepsilon_F)$ with $A(\delta_E, \delta_F)$ we would end up with something like $E_i \cap E_i^c$ or $F_j \cap F_j^c$, which must be empty.

Furthermore they partition $\mathbb{R}^d$ since for any $x \in \mathbb{R}^d$ we may set $\varepsilon_E^i$ to be 0 if $x \in E_i$ and 1 if $x \in E_i^c$, and likewise $\varepsilon_F^j$ to be 0 if $x \in F_i$ and 1 if $x \in F_i^c$. We can always do this, and by definition we will then have $x \in A(\varepsilon_E, \varepsilon_F)$.

Now we note that if $\mathcal{E}_i \subseteq \{0,1\}^k$ is the subset where the $i$-th coordinate is 0 we see that:

$$\begin{aligned} E_i = E_i \cap \mathbb{R}^d &= E_i \cap \left( \bigcup_{\substack{\varepsilon_E \in \{0,1\}^k \\ \varepsilon_F \in \{0,1\}^m}} A(\varepsilon_E, \varepsilon_F) \right) \\ &= \bigcup_{\substack{\varepsilon_E \in \{0,1\}^k \\ \varepsilon_F \in \{0,1\}^m}} E_i \cap A(\varepsilon_E, \varepsilon_F) = \bigcup_{\substack{\varepsilon_E \in \mathcal{E}_i \\ \varepsilon_F \in \{0,1\}^m}} A(\varepsilon_E, \varepsilon_F) \end{aligned}$$

Because either we intersect with a subset of $E_i$ or we intersect with $E_i$ and $E_i^c$. Likewise for $\mathcal{F}_j \subseteq \{0,1\}^m$ is the subset where the $j$-th coordinate is 0 we see that:

$$F_j = \bigcup_{\substack{\varepsilon_E \in \{0,1\}^k \\ \varepsilon_F \in \mathcal{F}_j}} A(\varepsilon_E, \varepsilon_F)$$

With this established we now write the sum, noting that since these are unions and intersections of Lebesgue measurable sets they are Lebesgue measurable:

$$\sum_{i=1}^{k} c_i m(E_i) = \sum_{i=1}^{k} c_i \sum_{\substack{\varepsilon_E \in \mathcal{E}_i \\ \varepsilon_F \in \{0,1\}^m}} m(A(\varepsilon_E, \varepsilon_F))$$

$$\sum_{j=1}^{m} c'_j m(F_j) = \sum_{j=1}^{m} c'_j \sum_{\substack{\varepsilon_E \in \{0,1\}^k \\ \varepsilon_F \in \mathcal{F}_j}} m(A(\varepsilon_E, \varepsilon_F))$$

## Handout 3

## Integration of Simple Functions

**Definition 0.1** (Simple function)**.** A (complex-valued) simple function $f : \mathbb{R}^d \to \mathbb{C}$ is a finite linear combination

$$f = c_1 1_{E_1} + \ldots + c_k 1_{E_k}$$

of indicator functions $1_{E_i}$ of Lebesgue measurable sets $E_i \subset \mathbb{R}^d$. Here, $i = 1, \ldots, k$ where $k \geq 1$ is a natural number and $c_1, c_2, \ldots, c_k \in \mathbb{C}$ are complex numbers. (Recall that the indicator function of set is the function that is equal to 1 on this set and zero otherwise). Clearly, this is a complex vector space.

An unsigned simple function $f : \mathbb{R}^d \to [0, \infty]$ is defined similarly but the $c_i$ take their values in $[0, \infty]$ rather than $\mathbb{C}$.

**Definition 0.2** (Integral of an unsigned simple function)**.** If $f = c_1 1_{E_1} + \ldots + c_k 1_{E_k}$ is an unsigned simple function, the integral $\text{Simp}\int_{\mathbb{R}^d} f(x)dx$ is defined by the formula

$$\text{Simp} \int_{\mathbb{R}^d} f(x)dx := c_1 m(E_1) + \ldots + c_k m(E_k). \tag{1}$$

Thus $\text{Simp}\int_{\mathbb{R}^d} f(x)dx$ takes values in $[0, \infty]$. (Here we adopt the convention that $0 \cdot \infty = 0$ in doing computations with the extended non-negative real numbers $[0, \infty]$).

Last time, we showed that this is integral is well-defined, and explored some properties of this integral (like linearity, monotonicity, when it is finite, when it is zero). Then we used it to define

**Definition 0.3** (Absolutely convergent simple integral)**.** A complex-valued simple function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be absolutely integrable if

$$\text{Simp} \int_{\mathbb{R}^d} |f(x)|dx < \infty.$$

If $f$ is absolutely integrable, the integral $\text{Simp} \int_{\mathbb{R}^d} f(x)dx$ is defined for real signed $f$ by the formula

$$\text{Simp} \int_{\mathbb{R}^d} f(x)dx := \text{Simp} \int_{\mathbb{R}^d} f_+(x)dx - \text{Simp} \int_{\mathbb{R}^d} f_-(x)dx$$

where $f_+ = \max(f(x), 0)$ is the positive part of $f$, and $f_- = \max(-f(x), 0)$ is the negative part of $f$. Note that since $f_+, f_- \leq |f|$, they have a finite integral. Finally, for complex-valued $f$, we define

$$\text{Simp} \int_{\mathbb{R}^d} f(x)dx := \text{Simp} \int_{\mathbb{R}^d} \operatorname{Re} f(x)dx + i\, \text{Simp} \int_{\mathbb{R}^d} \operatorname{Im} f(x)dx$$

Last time, we showed that this definition of integral is linear.

**Q1)** If $f$ and $g$ are two absolutely integrable simple functions that agree almost everywhere, show that their integral is the same.

**Definition 0.4** (Unsigned measurable functions)**.** An unsigned function $f : \mathbb{R}^d \to [0, \infty]$ is said to be Lebesgue measurable if it is the pointwise limit of unsigned simple functions, i.e. if there exists a sequence $f_1, f_2, f_3, \ldots : \mathbb{R}^d \to [0, \infty]$ of unsigned simple functions such that $f_n(x) \to f(x)$ for every $x \in \mathbb{R}^d$.

**Q2)** Show that the following are equivalent for an unsigned function $f : \mathbb{R}^d \to [0, \infty]$

(a) $f$ is Lebesgue measurable.

(b) $f$ is the supremum $f(x) = \sup_n f_n(x)$ of an increasing sequence $0 \leq f_1 \leq f_2 \leq \ldots$ of unsigned simple functions $f_n$, each of which are bounded with finite measure support.

(c) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) > \lambda\}$ is Lebesgue measurable.

(d) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) \geq \lambda\}$ is Lebesgue measurable.

(e) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) < \lambda\}$ is Lebesgue measurable.

(f) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) < \lambda\}$ is Lebesgue measurable.

(g) For every interval $I \subset [0, \infty)$, the set $f^{-1}(I) := \{x \in \mathbb{R}^d : f(x) \in I\}$ is Lebesgue measurable.

(h) For every (relatively) open subset $U$ of $[0, \infty)$, the set $f^{-1(U)}$ is Lebesgue measurable.

(i) For every (relatively) closed subset $K$ of $[0, \infty)$, the set $f^{-1(K)}$ is Lebesgue measurable.

*Hints: The following are not so hard to prove b) $\Rightarrow$ a), and (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g) $\Leftrightarrow$ (h) $\Leftrightarrow$ (i). You can start with those. Then one is left with proving that a) $\Rightarrow$ c) and (c) $-$ (i) $\Rightarrow$ b). To prove that a) $\Rightarrow$ c), use the identity*

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \inf_{N>0} \sup_{n\geq N} f_n(x).$$

*Finally to obtain (c)$-$(i) $\Rightarrow$ b), assume $f$ obeys (c)$-$(i), and define $f_n(x)$ to be the largest integer multiple of $2^{-n}$ that is smaller than* $\min(f(x), n)$ *when $|x| < n$ and 0 otherwise. Verify that this is an increasing sequence of simple functions that satisfy the conditions of (b).*

## I.4. **IBL Week 4**

**Problem I-2**

Show that the following are equivalent for an unsigned function $f : \mathbb{R}^d \to [0, \infty]$.

(a) $f$ is Lebesgue measurable.
(b) $f$ is the supremum $f(x) = \sup_n f_n(x)$ of an increasing sequence $0 \leq f_1 \leq f_2 \leq \cdots$ of unsigned simple functions $f_n$, each of which are bounded with finite measure support
(c) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d \mid f(x) > \lambda\}$ is Lebesgue measurable
(d) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d \mid f(x) \geq \lambda\}$ is Lebesgue measurable
(e) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d \mid f(x) < \lambda\}$ is Lebesgue measurable
(f) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d \mid f(x) \leq \lambda\}$ is Lebesuge measurable
(g) For every interval $I \subseteq [0, \infty)$ the set $f^{-1}(I) := \{x \in \mathbb{R}^d \mid f(x) \in I\}$ is Lebesgue measurable
(h) For every (relatively) open subset $U$ of $[0, \infty)$, the set $f^{-1}(U)$ is Lebesgue measurable
(i) For every (relatively) closed subset $K$ of $[0, \infty)$, the set $f^{-1}(K)$ is Lebesgue measurable

*Solution.* Let's prove these duderinos:

b) $\implies$ a)

## Handout 4

## Measurable functions (Continued)

**Definition 0.1** (Unsigned measurable functions)**.** An unsigned function $f : \mathbb{R}^d \to [0, \infty]$ is said to be Lebesgue measurable if it is the pointwise limit of unsigned simple functions, i.e. if there exists a sequence $f_1, f_2, f_3, \ldots : \mathbb{R}^d \to [0, \infty]$ of unsigned simple functions such that $f_n(x) \to f(x)$ for every $x \in \mathbb{R}^d$.

Last time, we showed that the following are equivalent definitions of measurability for unsigned functions $f : \mathbb{R}^d \to [0, \infty]$

(i) $f$ is the supremum $f(x) = \sup_n f_n(x)$ of an increasing sequence $0 \le f_1 \le f_2 \le \ldots$ of unsigned simple functions $f_n$, each of which are bounded with finite measure support.

(ii) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) > \lambda\}$ is Lebesgue measurable.

(iii) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) \ge \lambda\}$ is Lebesgue measurable.

(iv) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) < \lambda\}$ is Lebesgue measurable.

(v) For each $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) \le \lambda\}$ is Lebesgue measurable.

(vi) For every interval $I \subset [0, \infty)$, the set $f^{-1}(I) := \{x \in \mathbb{R}^d : f(x) \in I\}$ is Lebesgue measurable.

(vii) For every (relatively) open subset $U$ of $[0, \infty)$, the set $f^{-1}(U)$ is Lebesgue measurable.

(viii) For every (relatively) closed subset $K$ of $[0, \infty)$, the set $f^{-1}(K)$ is Lebesgue measurable.

**Q1)** Show that every continuous function $f : \mathbb{R}^d \to [0, \infty)$ is measurable.

**Q2)** Show that the supremum, infimum, limit superior, or limit inferior of sequences of unsigned measurable functions is measurable.

**Q3)** Show that an unsigned function that is equal almost everywhere to an unsigned measurable function, is itself measurable. *Remark.* This means that one can define the concept of measurability for an unsigned function that is only defined almost everywhere on $\mathbb{R}^d$, rather than everywhere on $\mathbb{R}^d$, by extending that function arbitrarily (say setting it to be 0) on the null set where it is currently undefined.

**Q4)** Show that if a sequence $f_n$ of unsigned measurable functions converges pointwise almost everywhere to an unsigned limit $f$, then $f$ is also measurable.

**Q5)** If $f : \mathbb{R}^d \to [0, +\infty)$ is measurable and $\phi : [0, \infty) \to [0, \infty)$ is continuous, show that $\phi \circ f : \mathbb{R}^d \to [0, \infty)$ is measurable.

**Q6)** If $f, g : \mathbb{R}^d \to [0, +\infty]$ are measurable, show that $f + g$ and $fg$ are measurable too.

---

We can now define the concept of measurability for complex-valued functions. As discussed in the above remark, it is convenient to allow for such function to be only defined *almost everywhere*, rather than *everywhere*, to allow for the possibility that the function becomes singular or otherwise undefined on a set of measure zero.

**Definition 0.2** (Complex measurability)**.** An almost everywhere defined complex-valued function $f : \mathbb{R}^d \to \mathbb{C}$ is Lebesgue measurable, or measurable for short, if it is the pointwise almost everywhere limit of complex-valued simple functions.

As before, there are several equivalent definitions:

**Q7)** Let $f : \mathbb{R}^d \to \mathbb{C}$ be an almost everywhere defined complex-valued function. The the following are equivalent:

(a) $f$ is measurable.

(b) The positive and negative parts of $\text{Re} f$ and $\text{Im} f$ are unsigned measurable functions.

(c) $f^{-1}(U)$ is measurable for every open set $U \subset \mathbb{C}$.

(d) $f^{-1}(K)$ is Lebesgue measurable for every closed set $K \subset \mathbb{C}$.

*Remark.* Part (ii) above (or even the definition) shows that $f$ is measurable iff its real and imaginary parts are measurable, and that a real-valued function is measurable if and only if its positive and negative parts are measurable.

## I.5. **IBL Week 5**

**Problem I-1**
Show that every continuous function $f : \mathbb{R}^d \to [0,\infty)$ is measurable.

*Solution.* Note that if $U$ is an open subset of $[0,\infty)$, then by definition of continuity $f^{-1}(U)$ is an open set, and we know that open sets of $\mathbb{R}^d$ are Lebesgue measurable from previous work. Thus by definition (vii) we know $f$ is measurable.

**Problem I-2**
Show that the supremum, infimum, limit superior, or limit inferior of sequences of unsigned measurable functions is measurable.

*Solution.* We will first show that the supremum / infimum of a sequence of functions is Lebesgue measurable. Fix some sequence of functions $(f_n)$, and let $f = \sup f_n$ and $g = \inf f_n$. We will use the definitions (iii) and (v) for this. Namely we write that for any $\lambda \in [0,\infty]$:

$$\{x \in \mathbb{R}^d \mid f(x) \le \lambda\} = \bigcap_{n\in\mathbb{N}} \{x \in \mathbb{R}^d \mid f_n(x) \le \lambda\}$$
$$\{x \in \mathbb{R}^d \mid g(x) \ge \lambda\} = \bigcap_{n\in\mathbb{N}} \{x \in \mathbb{R}^d \mid f_n(x) \ge \lambda\}$$

These holds since by definition of supremum $f(x) \le \lambda$ if and only if $\lambda$ is an upper bound of $f_n(x)$, that is $f_n(x) \le \lambda$ for every $n \in \mathbb{N}$. Similarly by definition of infimum $g(x) \ge \lambda$ if and only if $\lambda$ is a lower bound of $f_n(x)$, that is $f_n(x) \ge \lambda$ for all $n \in \mathbb{N}$. Great! Then since each $f_n$ is measurable we have written the above sets as countable intersections of measurable sets, and so these are measurable as well. Thus by definitions (iii) and (v) we have that $f$ and $g$ are measurable.

Now note that we have the following definitions of limit superior and limit inferior from 295/296:

$$\limsup f_n = \inf_N \sup_{n\ge N} f_n$$
$$\liminf f_n = \sup_N \inf_{n\ge N} f_n$$

Now notice that $\sup_{n\ge N} f_n$ ad $\inf_{n\ge N} f_n$ are infimums and supremums of the sequence $f_N, f_{N+1}, \ldots$, and we know that these must be measurable by the previous work we've done. But then $\limsup f_n$ is an infimum of measurable functions and $\liminf f_n$ is a supremum of measurable functions, and so again by previous work these are measurable, and so we are done.

**Problem I-3**
Show that an unsigned function that is equal almost everywhere to an unsigned measurable function, is itself measurable.

*Solution.* Let $f, g$ be usigned functions that agree almost everywhere, and let $g$ be measurable. In particular, suppose that $A$ is the set on which $f$ and $g$ agree, so that $m(A^c) = 0$.

We invoke definition (ii), and consider that since $f$ and $g$ agree on $A$:

$$\begin{aligned}\{x \in \mathbb{R}^d \mid g(x) > \lambda\} &= (\{x \in \mathbb{R}^d \mid g(x) > \lambda\} \cap A) \cup (\{x \in \mathbb{R}^d \mid g(x) > \lambda\} \cap A^c) \\ &= (\{x \in \mathbb{R}^d \mid f(x) > \lambda\} \cap A) \cup (\{x \in \mathbb{R}^d \mid g(x) > \lambda\} \cap A^c)\end{aligned}$$

Now the left hand part of this union is measurable since $f$ is measurable and $A$ is the complement of a measurable set. Likewise we know by monotonicity that:

$$m^*(\{x \in \mathbb{R}^d \mid g(x) > \lambda\} \cap A^c) \le m(A^c) = 0$$

And so since any outer measure zero set is Lebesgue measurable we know that this is measurable as well. Since both parts of our union is measurable, $\{x \in \mathbb{R}^d \mid g(x) > \lambda\}$ is Lebesgue measurable, and so $g$ is a measurable function. Awesome!

**Problem I-4**

Show that if a sequence $f_n$ of unsigned measurable functions converges pointwise almost everywhere to an unsigned limit $f$, then $f$ is also measurable.

*Solution.* This is not too hard. Note that since a limit supremum always exists we can take $g = \limsup f_n$, and by Question 2 this will be a measurable function. Now note that $g$ and $f$ agree on the set $A$ where $f_n$ converges to $f$, since when a limit exists it is equal to a limit supremum. Great! Then since $A^c$ has measure zero by the setup, we know that $f$ and $g$ agree almost everywhere, and so by Question 3 we know that $f$ is measurable. Awesome!

**Problem I-5**

If $f : \mathbb{R}^d \to [0, \infty)$ is measurable and $\phi : [0, \infty) \to [0, \infty)$ is continuous, show that $\phi \circ f : \mathbb{R}^d \to [0, \infty)$ is measurable

*Solution.* We use definition (vii), and this makes this very easy. Fix some open set $U$ of $[0, \infty)$. Then by definition of continuous we know $\phi^{-1}(U)$ is open, and so by definition (vii) we know $f^{-1}(\phi^{-1}(U))$ is measurable. But this is great, because this is exactly $(\phi \circ f)^{-1}(U)$ by set theory / 295. And therefore, by definition (vii) we know $\phi \circ f$ is a measurable function.

**Problem I-6**

If $f, g : \mathbb{R}^d \to [0, \infty]$ are measurable, show that $f + g$ and $fg$ are measurable too.

*Solution.* This is fairly simple. By definition (i), fix some sequences $f_1 \leq f_2 \leq \cdots$ and $0 \leq g_1 \leq g_2 \leq \cdots$ of unsigned simple functions $(f_n), (g_n)$ which are all bounded with finite measure support such that $f = \sup_n f_n$ and $g = \sup_n g_n$. Then by 295/296 work with supremums since these are unsigned we know that:

$$f + g = \sup_n f_n + \sup_n g_n = \sup_n (f_n + g_n)$$
$$fg = \sup_n f_n \cdot \sup_n g_n = \sup_n (f_n g_n)$$

It now suffices to show that $f_n + g_n$ and $f_n g_n$ are also going to be measurable functions by Question 2. In particular, we will show that these are unsigned simple functions

TODO

# Handout 5

## Unsigned Lebesgue Integrals

After defining the notion of measurability, both for unsigned functions taking values in $[0, \infty]$ and complex-valued functions taking values in $\mathbb{C}$, we are now ready to start defining the Lebesgue integral of such functions. As usual, we start with unsigned functions.

**Definition 0.1** (Lower unsigned Lebesgue integral)**.** Let $f : \mathbb{R}^d \to [0, \infty]$ be an unsigned function (not necessarily measurable). We define the *lower unsigned Lebesgue integral* $\underline{\int_{\mathbb{R}^d}} f(x)dx$ to be the quantity

$$\underline{\int_{\mathbb{R}^d}} f(x)dx : \sup_{0 \leq g \leq f; g \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} g(x)dx$$

where $g$ ranges over all unsigned simple functions $g : \mathbb{R}^d \to [0, \infty]$ that are pointwise bounded by $f$. One can also define the *upper Lebesgue integral* as

$$\overline{\int_{\mathbb{R}^d}} f(x)dx : \inf_{f \leq h; h \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} h(x)dx,$$

but we will use this integral very rarely.

In what follows, we establish some properties of the lower and upper integrals. Let $f, g : \mathbb{R}^d \to [0, \infty]$ be unsigned functions (not necessarily measurable)

**Q1)** If $f$ is simple, then $\underline{\int_{\mathbb{R}^d}} f(x)dx = \overline{\int_{\mathbb{R}^d}} f(x)dx = \text{Simp} \int_{\mathbb{R}^d} f(x)dx$.

**Q2)** If $f \leq g$ pointwise almost everywhere, then we have that $\underline{\int_{\mathbb{R}^d}} f(x)dx \leq \underline{\int_{\mathbb{R}^d}} g(x)dx$ and $\overline{\int_{\mathbb{R}^d}} f(x)dx \leq \overline{\int_{\mathbb{R}^d}} g(x)dx$.

**Q3)** If $c \in [0, \infty)$, then $\underline{\int_{\mathbb{R}^d}} cf(x)dx = c\underline{\int_{\mathbb{R}^d}} f(x)dx$.

**Q4)** If $f, g$ agree almost everywhere, then $\underline{\int_{\mathbb{R}^d}} f(x)dx = \underline{\int_{\mathbb{R}^d}} g(x)dx$ and $\overline{\int_{\mathbb{R}^d}} f(x)dx = \overline{\int_{\mathbb{R}^d}} g(x)dx$.

**Q5)** (Superadditivity of lower integral) $\underline{\int_{\mathbb{R}^d}} f(x) + g(x)dx \geq \underline{\int_{\mathbb{R}^d}} f(x)dx + \underline{\int_{\mathbb{R}^d}} g(x)dx$.

**Q6)** (Subadditivity of upper integral) $\overline{\int_{\mathbb{R}^d}} f(x) + g(x)dx \leq \overline{\int_{\mathbb{R}^d}} f(x)dx + \overline{\int_{\mathbb{R}^d}} g(x)dx$.

**Q7)** For any measurable set $E$, one has $\underline{\int_{\mathbb{R}^d}} f(x)dx = \underline{\int_{\mathbb{R}^d}} f(x)1_E(x)dx + \underline{\int_{\mathbb{R}^d}} f(x)1_{E^c}(x)dx$.

**Q8)** (Horizontal Truncation) As $n \to \infty$, $\underline{\int_{\mathbb{R}^d}} \min(f(x), n)dx$ converges to $\underline{\int_{\mathbb{R}^d}} f(x)dx$.

**Q9)** (Vertical Truncation) As $n \to \infty$, $\underline{\int_{\mathbb{R}^d}} f(x)1_{|x| \leq n}dx$ converges to $\underline{\int_{\mathbb{R}^d}} f(x)dx$. *Hint: Recall that one has that for any measurable set $E$, $m(E \cap \overline{B(0,n)}) \to m(E)$ as $n \to \infty$.*

**Q10)** If $f + g$ is a simple function that is bounded with finite measure support (i.e. it is absolutely integrable), then we have that $\text{Simp} \int_{\mathbb{R}^d} f(x) + g(x)dx = \underline{\int_{\mathbb{R}^d}} f(x)dx + \overline{\int_{\mathbb{R}^d}} g(x)dx$.

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**Definition 0.2.** If $f : \mathbb{R}^d \to [0, \infty]$ is measurable, we define the unsigned Lebesgue integral $\int_{\mathbb{R}^d} f(x)dx$ to equal the lower unsigned integral $\underline{\int_{\mathbb{R}^d}} f(x)dx$. For unmeasurable functions, we leave the integral undefined.

**Q11)** Let $f$ be an unsigned measurable function that is bounded, and vanishing outside a set of finite measure. Then, the lower and upper integrals agree. *Hint: Start by showing that a unsigned measurable function is bounded if and only if it is the uniform limit of bounded simple functions.*

## I.6. **IBL Week 6**

**Problem I-7**

For any measurable set $E$, one has $\underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x = \underline{\int_{\mathbb{R}^d}} f(x)1_E(x)\,\mathrm{d}x + \underline{\int_{\mathbb{R}^d}} f(x)1_{E^c}(x)\,\mathrm{d}x$

*Solution.* We get that $\underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x \geq \underline{\int_{\mathbb{R}^d}} f(x)1_E(x)\,\mathrm{d}x + \underline{\int_{\mathbb{R}^d}} f(x)1_{E^c}(x)\,\mathrm{d}x$ directly from Problem 5 by the fact that $f = f1_E + f1_{E^c}$.

For the other direction, we can use by the fact that this lower integral is a least upper bound of some set, so we can show that the right hand side is an upper bound for that same set. To do this, fix $g \leq f$ to be simple, and then $g1_E \leq f1_E$ and $g1_{E^c} \leq f1_{E^c}$ and $g1_E, g1_{E^c}$ are both simple. Of course $g = g1_E + g1_{E^c}$, and so by linearity of the simple integral

$$\mathrm{Simp}\int_{\mathbb{R}^d} g(x)\,\mathrm{d}x = \mathrm{Simp}\int_{\mathbb{R}^d} g(x)1_E(x)\,\mathrm{d}x + \mathrm{Simp}\int_{\mathbb{R}^d} g(x)1_{E^c}(x)\,\mathrm{d}x \leq \underline{\int_{\mathbb{R}^d}} f(x)1_E(x)\,\mathrm{d}x + \underline{\int_{\mathbb{R}^d}} f(x)1_{E^c}(x)\,\mathrm{d}x$$

$$\underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x \leq \underline{\int_{\mathbb{R}^d}} f(x)1_E(x)\,\mathrm{d}x + \underline{\int_{\mathbb{R}^d}} f(x)1_{E^c}(x)\,\mathrm{d}x$$

Great!!!

# Handout 5

## Unsigned Lebesgue Integrals

After defining the notion of measurability, both for unsigned functions taking values in $[0, \infty]$ and complex-valued functions taking values in $\mathbb{C}$, we are now ready to start defining the Lebesgue integral of such functions. As usual, we start with unsigned functions.

**Definition 0.1** (Lower unsigned Lebesgue integral)**.** Let $f : \mathbb{R}^d \to [0, \infty]$ be an unsigned function (not necessarily measurable). We define the *lower unsigned Lebesgue integral* $\underline{\int_{\mathbb{R}^d}} f(x)dx$ to be the quantity

$$\underline{\int_{\mathbb{R}^d}} f(x)dx : \sup_{0 \leq g \leq f; g \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} g(x)dx$$

where $g$ ranges over all unsigned simple functions $g : \mathbb{R}^d \to [0, \infty]$ that are pointwise bounded by $f$. One can also define the *upper Lebesgue integral* as

$$\overline{\int_{\mathbb{R}^d}} f(x)dx : \inf_{f \leq h; h \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} h(x)dx,$$

but we will use this integral very rarely.

Last time, we established some properties of the lower and upper integrals. Let $f, g : \mathbb{R}^d \to [0, \infty]$ be unsigned functions (not necessarily measurable)

(i) If $f$ is simple, then $\underline{\int_{\mathbb{R}^d}} f(x)dx = \overline{\int_{\mathbb{R}^d}} f(x)dx = \text{Simp} \int_{\mathbb{R}^d} f(x)dx$.

(ii) If $f \leq g$ pointwise almost everywhere, then we have that $\underline{\int_{\mathbb{R}^d}} f(x)dx \leq \underline{\int_{\mathbb{R}^d}} g(x)dx$ and $\overline{\int_{\mathbb{R}^d}} f(x)dx \leq \overline{\int_{\mathbb{R}^d}} g(x)dx$.

(iii) If $c \in [0, \infty)$, then $\underline{\int_{\mathbb{R}^d}} cf(x)dx = c \underline{\int_{\mathbb{R}^d}} f(x)dx$.

(iv) If $f, g$ agree almost everywhere, then $\underline{\int_{\mathbb{R}^d}} f(x)dx = \underline{\int_{\mathbb{R}^d}} g(x)dx$ and $\overline{\int_{\mathbb{R}^d}} f(x)dx = \overline{\int_{\mathbb{R}^d}} g(x)dx$.

(v) (Superadditivity of lower integral) $\underline{\int_{\mathbb{R}^d}} f(x)+g(x)dx \geq \underline{\int_{\mathbb{R}^d}} f(x)dx + \underline{\int_{\mathbb{R}^d}} g(x)dx$.

(vi) (Subadditivity of upper integral) $\overline{\int_{\mathbb{R}^d}} f(x)+g(x)dx \leq \overline{\int_{\mathbb{R}^d}} f(x)dx + \overline{\int_{\mathbb{R}^d}} g(x)dx$.

(vii) For any measurable set $E$, one has $\underline{\int_{\mathbb{R}^d}} f(x)dx = \underline{\int_{\mathbb{R}^d}} f(x)1_E(x)dx + \underline{\int_{\mathbb{R}^d}} f(x)1_{E^c}(x)dx$.

(viii) (Horizontal Truncation) As $n \to \infty$, $\underline{\int_{\mathbb{R}^d}} \min(f(x), n)dx$ converges to $\underline{\int_{\mathbb{R}^d}} f(x)dx$.

(ix) (Vertical Truncation) As $n \to \infty$, $\underline{\int_{\mathbb{R}^d}} f(x)1_{|x|\leq n}dx$ converges to $\underline{\int_{\mathbb{R}^d}} f(x)dx$.

(x) If $f + g$ is a simple function that is bounded with finite measure support (i.e. it is absolutely integrable), then we have that $\text{Simp}\int_{\mathbb{R}^d} f(x) + g(x)dx = \underline{\int_{\mathbb{R}^d}} f(x)dx + \overline{\int_{\mathbb{R}^d}} g(x)dx$.

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**Definition 0.2.** If $f : \mathbb{R}^d \to [0, \infty]$ is measurable, we define the unsigned Lebesgue integral $\int_{\mathbb{R}^d} f(x)dx$ to equal the lower unsigned integral $\underline{\int_{\mathbb{R}^d}} f(x)dx$. For unmeasurable functions, we leave the integral undefined.

**Q1)** Show that an unsigned measurable function is bounded if and only if it is the uniform limit of bounded simple functions.

**Q2)** Let $f$ be an unsigned measurable function that is bounded, and vanishing outside a set of finite measure. Then, the lower and upper integrals agree.

**Q3)** (Finite Additivity of the Lebesgue Integral) Let $f, g : \mathbb{R}^d \to [0, \infty]$ be measurable. Then $\int_{\mathbb{R}^d} f(x) + g(x)dx = \int_{\mathbb{R}^d} f(x)dx + \int_{\mathbb{R}^d} g(x)dx$.

*Hint: Use* **Q2)** *Remark.* One of the major theorems on Lebesgue integrals is that this finite additivity can be improved to countable additivity. This is known as the *monotone convergence theorem*, which we will prove later.

**Q4)** (Translation Invariance) Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Show that $\int_{\mathbb{R}^d} f(x + v)dx = \int_{\mathbb{R}^d} f(x)dx$ for any $v \in \mathbb{R}^d$.

**Q5)** (Linear change of variables) Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable, and let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transformation. Show that $\int_{\mathbb{R}^d} f(T^{-1}x)dx = |\det T| \int_{\mathbb{R}^d} f(x)dx$.

*Hint: You will need to show that* $m(T(E)) = |\det T| m(E)$ *for any measurable function. Start with the case when* $\det T = 0$*, and then deal with the invertible case.*

**Q6)** (Compatibility with the Riemann Integral) Let $f : [a, b] \to [0, \infty)$ be Riemann integrable. If we extend $f$ to $\mathbb{R}$ by declaring $f$ to equation 0 outside of $[a, b]$, show that $\int_{\mathbb{R}} f(x)dx = \int_a^b f(x)dx$.

## I.7. **IBL Week 7**

> **Problem I-1**
> Show that an unsigned measurable function is bounded if and only if it is the uniform limit of bounded simple functions.

*Solution.* First we show the converse, since this is simpler, in fact the uniform limit of bounded functions is always bounded. Let $f_1, f_2, \ldots : \mathbb{R}^d \to [0, \infty]$ be a sequence of bounded functions converging uniformly to a function $f : \mathbb{R}^d \to [0, \infty]$. By uniform convergence, we may take some large enough $n \in \mathbb{N}$ so that for every $x \in \mathbb{R}^d$ we have:

$$|f(x) - f_n(x)| \leq 1$$

Then since $f_n$ is bounded, we may let $M \in \mathbb{R}$ be its bound. Then:

$$|f(x)| \overset{\triangle}{\leq} |f(x) - f_n(x)| + |f_n(x)| \leq M + 1$$

Great! This means that $M + 1$ is an upper bound for $f$, so $f$ is bounded.

Awesome. Now we must prove the converse. Let $f : \mathbb{R}^d \to [0, \infty]$ be some bounded measurable function, and say it is bounded by $M \in \mathbb{N}$. First consider a set $D_n(x)$ for any $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$:

$$D_n(x) = \{k \cdot 2^{-n} \mid k \in \mathbb{N}_0, k \cdot 2^{-n} \leq f(x)\}$$

Then $D_n(x)$ is a finite set, since we know that $k \cdot 2^{-n}$ goes to infinity as $k \to \infty$, and so for some $K \in \mathbb{N}$ we have $K \cdot 2^{-n} > M \geq f(x)$, and for any $k \geq K$ we know $k \cdot 2^{-n} \geq K \cdot 2^{-n} > f(x)$, so that $k\dot{2}^{-n} \notin D_n(x)$. Great! We also know that $D_n(x)$ is nonempty since $0 \cdot 2^{-n} = 0 \leq f(x)$. Therefore, $0 \in D_n(x)$. With this in mind we may make the following definition:

$$f_n(x) = \max D_n(x)$$

First note that $f_n(x)$ is bounded above by $f(x)$, which is bounded by $M$, namely because $f_n(x) \in D_n(x)$ and for any $y \in D_n(x)$ we have $y \leq f(x) \leq M$ by definition. Thus $f_n \leq f \leq M$. Also $f_n(x) \geq 0$ because $f_n(x) = \max D_n(x) = k \cdot 2^{-n} \geq 0$ for some $k \in \mathbb{N}_0$. This means that the $f_n$ are unsigned.

Furthermore, the sequence $f_1, f_2, \ldots$ increases (this isn't relevant for this problem, but will be in Problem 2). Fix some $n \in \mathbb{N}$. Then we claim first that $D_n(x) \subseteq D_{n+1}(x)$. Why? Well fix some $a = k \cdot 2^{-n} \in D_n(x)$. Then we have that $a = 2k \cdot 2^{-(n+1)}$, and $2k$ is integer, furthermore $a \leq f(x)$. Therefore $a \in D_{n+1}(x)$. With this in mind we know that $\max D_n(x) \leq \max D_{n+1}(x)$. Great! Thus $f_n(x) \leq f_{n+1}(x)$.

Now we verify that the sequence $f_1, f_2, \ldots$ converges uniformly to $f$. To do this, fix some $\varepsilon > 0$. Since $\frac{1}{2^n} \to 0$ as $n \to \infty$, there is some $N \in \mathbb{N}$ so that if $n \geq N$ then $0 < \frac{1}{2^n} < \varepsilon$. We claim that $|f(x) - f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}^d$. Why? Well let $f_n(x) = k \cdot 2^{-n} = \max D_n(x)$ for some $k \in \mathbb{N}_0$ and suppose that $|f(x) - f_n(x)| \geq \varepsilon > \frac{1}{2^n}$. Then since $f(x) \geq f_n(x)$ this gives that $f(x) - f_n(x) > \frac{1}{2^n}$, and so $f(x) > f_n(x) + \frac{1}{2^n}$. This is bad! With this we see that $f_n(x) < f_n(x) + \frac{1}{2^n} < f(x)$, and because $f_n(x) + \frac{1}{2^n} = (k+1) \cdot 2^{-n}$ this means that $\max D_n(x)$ is strictly less than some member of $D_n(x)$. This is nonsense, and thus $|f(x) - f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}^d$.

We now finally just need to verify that each $f_n$ is a simple function. Define the sets $E_k^n$ for $k \in \mathbb{N}_0$ by the following:

$$E_k^n = \{x \in \mathbb{R}^d \mid k \cdot 2^{-n} \leq f(x) < (k+1)2^{-n}\}$$

This set is exactly $f^{-1}([k \cdot 2^{-n}, (k+1)2^{-n}))$ and so it is measurable since $f$ is measurable. Therefore $E_k^n$ is measurable for all such $k$. Awesome! Now let $K$ be the largest integer so that $K \cdot 2^{-n} \leq M$, this must exist since the sequence $k \mapsto k \cdot 2^{-n}$ is monotonically increasing to $\infty$, and $0 \cdot 2^{-n} \leq M$.

To show $f_n$ is simple we now claim that:

$$f_n = \sum_{k=0}^{K} (k \cdot 2^{-n}) \cdot \mathbb{1}_{E_k^n}$$

Fix $x \in \mathbb{R}^d$. We need to show that:

$$\max D_n(x) = \sum_{k=0}^{K} (k \cdot 2^{-n}) \cdot \mathbb{1}_{E_k^n}$$

First note that each $E_k^n$ is disjoint, and that they cover $\mathbb{R}^d$. Why? Well first to show they're disjoint note that if $x \in E_k^n$ and $x \in E_j^n$, then $k \cdot 2^{-n} \leq f(x)$ and $f(x) < (j+1) \cdot 2^{-n}$. Thus $k \leq 2^n f(x) < j + 1$, and so $k + 1 \leq j + 1$, so $k \leq j$. We can carry out this argument in reverse as well, since $k$ and $j$ were arbitrary, and so we get $k \geq j$, so that $k = j$. Great! Now these cover $\mathbb{R}^d$ because for such $x$ we know that $f(x) \leq M$ and so:

$$[0, \infty] = \bigcup_{k=0}^{\infty} [k \cdot 2^{-n}, (k+1)2^{-n})$$
$$\mathbb{R}^d = f^{-1}([0, \infty]) = f^{-1}\left( \bigcup_{k=0}^{\infty} [k \cdot 2^{-n}, (k+1)2^{-n}) \right)$$
$$\mathbb{R}^d = \bigcup_{k=0}^{\infty} f^{-1}([k \cdot 2^{-n}, (k+1)2^{-n})$$

But for $k > K$ we know that $f^{-1}([k \cdot 2^{-n}, (k+1)2^{-n})$ is empty, because $f(x) \leq M < (K+1) \cdot 2^{-n} \leq k \cdot 2^{-n}$. Thus:

$$\mathbb{R}^d = \bigcup_{k=0}^{K} f^{-1}([k \cdot 2^{-n}, (k+1)2^{-n}) = \bigcup_{k=0}^{K} E_k^n$$

Great! Therefore we know that for any $x \in \mathbb{R}^d$ there is a unique $0 \leq j \leq K$ so that $x \in E_j^n$. With this in mind we then have that:

$$\sum_{k=0}^{K} (k \cdot 2^{-n}) \cdot \mathbb{1}_{E_k^n} = j \cdot 2^{-n}$$

It now suffices to show that $\max D_n(x) = j \cdot 2^{-n}$. Note first that $j \cdot 2^{-n} \in D_n(x)$. Why? Well consider that that $j \cdot 2^{-n} \leq f(x)$ because $x \in E_k^n = f^{-1}([j \cdot 2^{-n}, (j+1) \cdot 2^{-n}))$. With this $j \cdot 2^{-n} \in D_n(x)$. Now fix $k > j$. Then $k \geq j + 1$ and so $k \cdot 2^{-n} \geq (j+1) \cdot 2^{-n} > f(x)$. Therefore $k \cdot 2^{-n} \notin D_n(x)$. With this established, we know that:

$$f_n(x) = \max D_n(x) = j \cdot 2^{-n} = \sum_{k=0}^{K} (k \cdot 2^{-n}) \cdot \mathbb{1}_{E_k^n}$$

This shows that $f_n$ is a simple function! Awesome! This verifies that a bounded unsigned measurable function is a uniform limit of bounded simple functions. Combined with the previous proof of the other direction, we have the desired statement of the problem ☺.

**Problem I-2**

Let $f$ be an unsigned measurable function that is bounded, and vanishing outside a set of finite measure. Then, the lower and upper integrals agree.

*Solution.* Let $f$ be such a unsigned measurable function which is bounded and vanishing outside a set of finite measure. Let $A$ be set so that $f$ vanishes on $A^c$ and $m(A) < \infty$. In the previous problem, we constructed a sequence of unsigned bounded simple functions $f_1 \leq f_2 \leq f_3, \ldots$ which converged to $f$ uniformly. Furthermore, we had that each $f_n \leq f$. We claim the following equalities, which provide the result:

$$\underline{\int_{\mathbb{R}^d}} f(x) \, \mathrm{d}x = \sup_n \ \operatorname{Simp} \int_{\mathbb{R}^d} f_n(x) \, \mathrm{d}x = \overline{\int_{\mathbb{R}^d}} f(x) \, \mathrm{d}x$$

First note that we get both of the following by monotonicity of our integrals (noting that $f_n \leq f$) and how supremums work from 295 (letting $n \in \mathbb{N}$ be arbitrary):

$$\begin{aligned}
\text{Simp} \int_{\mathbb{R}^d} f_n(x)\, \mathrm{d}x &= \underline{\int_{\mathbb{R}^d}} f_n(x)\, \mathrm{d}x \leq \underline{\int_{\mathbb{R}^d}} f(x)\, \mathrm{d}x \\
\text{Simp} \int_{\mathbb{R}^d} f_n(x)\, \mathrm{d}x &= \overline{\int_{\mathbb{R}^d}} f_n(x)\, \mathrm{d}x \leq \overline{\int_{\mathbb{R}^d}} f(x)\, \mathrm{d}x \\
\sup_n \ \text{Simp} \int_{\mathbb{R}^d} f_n(x)\, \mathrm{d}x &\leq \underline{\int_{\mathbb{R}^d}} f(x)\, \mathrm{d}x \\
\sup_n \ \text{Simp} \int_{\mathbb{R}^d} f_n(x)\, \mathrm{d}x &\leq \overline{\int_{\mathbb{R}^d}} f(x)\, \mathrm{d}x
\end{aligned}$$

Therefore we just need to show the nontrivial sides of these inequalities, the $\geq$ side. To show these, we show that for all $\varepsilon > 0$ we have:

$$\begin{aligned}
\sup_n \ \text{Simp} \int_{\mathbb{R}^d} f_n(x)\, \mathrm{d}x + \varepsilon &\geq \underline{\int_{\mathbb{R}^d}} f(x)\, \mathrm{d}x \\
\sup_n \ \text{Simp} \int_{\mathbb{R}^d} f_n(x)\, \mathrm{d}x + \varepsilon &\geq \overline{\int_{\mathbb{R}^d}} f(x)\, \mathrm{d}x
\end{aligned}$$

By uniform convergence in the previous problem, there is some $N \in \mathbb{N}$ so that for $n \geq N$ we have that for all $x \in \mathbb{R}^d$, $f(x) - f_n(x) = |f(x) - f_n(x)| < \frac{\varepsilon}{m(A)+1}$, noting that $m(A) + 1 \geq 1 > 0$. Now we see that $\mathbb{1}_A(x) \geq 0$ for all $x \in \mathbb{R}^d$, and also for $x \in A$ and $y \in A^c$ we have:

$$\begin{aligned}
f(x) \cdot \mathbb{1}_A(x) &= f(x) & f_n(x) \cdot \mathbb{1}_A(x) &= f_n(x) \\
f(y) \cdot \mathbb{1}_A(y) &= 0 & f_n(y) \cdot \mathbb{1}_A(y) &= 0
\end{aligned}$$

And for such $y$ we know $0 = f_n(y) \leq f(y) = 0$ so $f_n(y) = 0$ as well. With this in mind. We now get inequalities as follows, using linearity of the simple integral and the fact that $f_n \mathbb{1}_A + \frac{\varepsilon}{m(A)+1} \mathbb{1}_A$ is simple (being a product / sum of simple functions):

$$\begin{aligned}
f \mathbb{1}_A &\leq f_N \mathbb{1}_A + \frac{\varepsilon}{m(A)+1} \cdot \mathbb{1}_A \\
\underline{\int_{\mathbb{R}^d}} f(x) \mathbb{1}_A(x)\, \mathrm{d}x &\leq \underline{\int_{\mathbb{R}^d}} f_N(x) \mathbb{1}_A(x) + \frac{\varepsilon}{m(A)+1} \cdot \mathbb{1}_A(x)\, \mathrm{d}x \\
&= \text{Simp} \int_{\mathbb{R}^d} f_N(x) \mathbb{1}_A(x) + \frac{\varepsilon}{m(A)+1} \cdot \mathbb{1}_A(x)\, \mathrm{d}x \\
&= \text{Simp} \int_{\mathbb{R}^d} f_N(x) \mathbb{1}_A(x)\, \mathrm{d}x + \text{Simp} \int_{\mathbb{R}^d} \frac{\varepsilon}{m(A)+1} \cdot \mathbb{1}_A(x)\, \mathrm{d}x \\
&= \text{Simp} \int_{\mathbb{R}^d} f_N(x)\, \mathrm{d}x + \frac{\varepsilon \cdot m(A)}{m(A)+1} \\
&\leq \sup_n \ \text{Simp} \int_{\mathbb{R}^d} f_n(x)\, \mathrm{d}x + \varepsilon \\
\underline{\int_{\mathbb{R}^d}} f(x)\, \mathrm{d}x &\leq \sup_n \ \text{Simp} \int_{\mathbb{R}^d} f_n(x)\, \mathrm{d}x + \varepsilon
\end{aligned}$$

And likewise:

$$\begin{aligned}
f\mathbb{1}_A &\leq f_N\mathbb{1}_A + \frac{\varepsilon}{m(A)+1}\cdot \mathbb{1}_A \\
\overline{\int_{\mathbb{R}^d}} f(x)\mathbb{1}_A(x)\,\mathrm{d}x &\leq \overline{\int_{\mathbb{R}^d}} f_N(x)\mathbb{1}_A(x) + \frac{\varepsilon}{m(A)+1}\cdot \mathbb{1}_A(x)\,\mathrm{d}x \\
&= \mathrm{Simp}\int_{\mathbb{R}^d} f_N(x)\mathbb{1}_A(x) + \frac{\varepsilon}{m(A)+1}\cdot \mathbb{1}_A(x)\,\mathrm{d}x \\
&= \mathrm{Simp}\int_{\mathbb{R}^d} f_N(x)\mathbb{1}_A(x)\,\mathrm{d}x + \mathrm{Simp}\int_{\mathbb{R}^d} \frac{\varepsilon}{m(A)+1}\cdot \mathbb{1}_A(x)\,\mathrm{d}x \\
&= \mathrm{Simp}\int_{\mathbb{R}^d} f_N(x)\,\mathrm{d}x + \frac{\varepsilon\cdot m(A)}{m(A)+1} \\
&\leq \sup_n \ \mathrm{Simp}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x + \varepsilon \\
\overline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x &\leq \sup_n \ \mathrm{Simp}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x + \varepsilon
\end{aligned}$$

This is exactly what we wanted to show. With this by taking $\varepsilon \to 0$ we get that:

$$\begin{aligned}
\underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x &\leq \sup_n \ \mathrm{Simp}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \\
\overline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x &\leq \sup_n \ \mathrm{Simp}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x
\end{aligned}$$

And thus with the other inequalities:

$$\begin{aligned}
\underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x &= \sup_n \ \mathrm{Simp}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \\
\overline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x &= \sup_n \ \mathrm{Simp}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x
\end{aligned}$$

This finishes the problem!

**Problem I-3** (Finite Additivity of the Lebesgue Integral)
Let $f, g : \mathbb{R}^d \to [0,\infty]$ be measurable. Then:

$$\int_{\mathbb{R}^d} f(x) + g(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} g(x)\,\mathrm{d}x$$

*Hint: Use Problem 2. Remark:* One of the major theorems on Lebesgue integrals is that this finite additivity can be improved to countable additivity. This is known as the *monotone convergence theorem*, which we will prove later

*Solution.* First we note that the result holds for measurable functions $f, g : \mathbb{R}^d \to [0,\infty]$ which are bounded and have finite measure support. Why? Well we know that for such functions by Problem 2 that their upper and lower integrals agree. Then by superadditivity/sub-additivity of the lower/upper integrals respectively we know that:

$$\begin{aligned}
\underline{\int_{\mathbb{R}^d}} f(x) + g(x)\,\mathrm{d}x &\geq \underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x + \underline{\int_{\mathbb{R}^d}} g(x)\,\mathrm{d}x \\
\overline{\int_{\mathbb{R}^d}} f(x) + g(x)\,\mathrm{d}x &\leq \overline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x + \overline{\int_{\mathbb{R}^d}} g(x)\,\mathrm{d}x
\end{aligned}$$

This gives the desired result in this case.

We now extend our result to bounded functions. Let $f, g : \mathbb{R}^d \to [0,\infty]$ be bounded measurable functions. Now note that for any $n \in \mathbb{N}$ we have that $f\mathbb{1}_{\overline{B(0,n)}}$ and $g\mathbb{1}_{\overline{B(0,n))}}$ are unsigned bounded measurable functions with finite measure support. Why? Well $f\mathbb{1}_{\overline{B(0,n)}} \leq f$ since $0 \leq \mathbb{1}_{\overline{B(0,n)}} \leq 1$, and $f$ is bounded. Furthermore

$\mathbb{1}_{\overline{B(0,n)}}$ is a simple function, so it is measurable, and the product of measurable functions is measurable. Finally it has finite measure support because it is supported on at most $\overline{B(0,n)}$ and $\overline{B(0,n)}$ has finite measure. Great! Then we apply vertical truncation along with the previous case to see that:

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)+g(x)\,\mathrm{d}x &= \lim_{n\to\infty}\int_{\mathbb{R}^d}(f(x)+g(x))\mathbb{1}_{|x|\le n}\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d} f(x)\mathbb{1}_{|x|\le n}+g(x)\mathbb{1}_{|x|\le n}\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d} f(x)\mathbb{1}_{|x|\le n}\,\mathrm{d}x+\lim_{n\to\infty}\int_{\mathbb{R}^d} g(x)\mathbb{1}_{|x|\le n}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x+\int_{\mathbb{R}^d} g(x)\,\mathrm{d}x\end{aligned}$$

Finally we extend the result to general unsigned measurable functions. Let $f,g:\mathbb{R}^d\to[0,\infty]$ be such unsigned measurable functions. Note that for any $a\in[0,\infty)$ that $\min(f,a)$ and $\min(g,a)$ are also unsigned measurable functions by previous worksheets, and that they are bounded because $\min(f(x),a),\min(g(x),a)\le a$. Great! Now we note that for $n\in\mathbb{N}$ and $x\in\mathbb{R}^d$:

$$\begin{aligned}&\min(f(x)+g(x),n)\le f(x)+g(x)\\ &\min(f(x)+g(x),n)\le n\le f(x)+n,n+g(x),2n\\ &\min(f(x)+g(x),n)\le \min(f(x),n)+\min(g(x),n)\end{aligned}$$

Therefore we have by monotonicity, horizontal truncation, as well as the case for bounded functions that:

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)+g(x)\,\mathrm{d}x &= \lim_{n\to\infty}\int_{\mathbb{R}^d}\min(f(x)+g(x),n)\,\mathrm{d}x\\ &\le \lim_{n\to\infty}\int_{\mathbb{R}^d}\min(f(x),n)+\min(g(x),n)\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d}\min(f(x),n)\,\mathrm{d}x+\lim_{n\to\infty}\int_{\mathbb{R}^d}\min(g(x),n)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x+\int_{\mathbb{R}^d} g(x)\,\mathrm{d}x\end{aligned}$$

The other inequality follows directly from superadditivity of the lower integral and the fact that these are defined as lower integrals, so we have both of the inequalities:

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)+g(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x+\int_{\mathbb{R}^d} g(x)\,\mathrm{d}x\\ \int_{\mathbb{R}^d} f(x)+g(x)\,\mathrm{d}x \ge \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x+\int_{\mathbb{R}^d} g(x)\,\mathrm{d}x\end{aligned}$$

Perfect! This means that the two quantities are equal, and so we've finished the problem ☺.

**Problem I-4** (Translation Invariance)
Let $f:\mathbb{R}^d\to[0,\infty]$ be measurable. Show that $\int_{\mathbb{R}^d} f(x+v)\,\mathrm{d}x=\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$ for any $v\in\mathbb{R}^d$

*Solution.* First we show the statement for simple functions. This is fairly simple. Fix a simple function $g:\mathbb{R}^d\to[0,\infty]$ and $v\in\mathbb{R}^d$. Now write $g$ as below for coefficients $c_1,\dots,c_k\in[0,\infty]$ and measurable sets $E_1,\dots,E_k\subseteq\mathbb{R}^d$:

$$g=\sum_{i=1}^k c_i\mathbb{1}_{E_i}$$

We claim that $x \mapsto g(x+v)$ is a simple function. Why? Well consider that:

$$g(x+v) = \sum_{i=1}^{k} c_i \cdot \mathbb{1}_{E_i}(x+v) = \sum_{i=1}^{k} c_i \cdot \mathbb{1}_{E_i - v}(x)$$

Why? Well if $x + v \in E_i$, then clearly $x \in E_i - v$ since $x = (x+v) - v$. Then if $x \in E_i - v$ then $x = y - v$ for some $y \in E_i$, so $y = x + v \in E_i$¿ Thus $\mathbb{1}_{E_i}(x+v) = \mathbb{1}_{E_i - v}(x)$. Great! But then by translation invariance of the Lebesgue measure from previous worksheets:

$$\text{Simp} \int_{\mathbb{R}^d} g(x)\,\mathrm{d}x = \sum_{i=1}^{k} c_i \cdot m(E_i) = \sum_{i=1}^{k} c_i \cdot m(E_i - v) = \text{Simp} \int_{\mathbb{R}^d} g(x+v)\,\mathrm{d}x$$

Great! Since the simple integral agrees with the Lebesgue integral for simple functions this gives the result in this case.

Now we prove the desired result. Let $v \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to [0, \infty]$ be any function. Now fix some simple function $g$ so that $0 \le g \le f$. Then of course for all $x \in \mathbb{R}^d$ we have $0 \le g(x+v) \le f(x+v)$. Therefore by the above case and definitions:

$$\text{Simp} \int_{\mathbb{R}^d} g(x)\,\mathrm{d}x = \text{Simp} \int_{\mathbb{R}^d} g(x+v)\,\mathrm{d}x \le \underline{\int_{\mathbb{R}^d}} f(x+v)\,\mathrm{d}x$$

By the definition of supremum, we then have that:

$$\underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x \le \underline{\int_{\mathbb{R}^d}} f(x+v)\,\mathrm{d}x$$

Perfect! This actually will give the result for all functions. Why? Well replace $f$ with $x \mapsto f(x+v)$, and then replace $v$ with $-v$. This gives that:

$$\underline{\int_{\mathbb{R}^d}} f(x+v)\,\mathrm{d}x \le \underline{\int_{\mathbb{R}^d}} f((x+v)-v)\,\mathrm{d}x = \underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x$$

And therefore combining the two inequalities we have that:

$$\underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x = \underline{\int_{\mathbb{R}^d}} f(x+v)\,\mathrm{d}x$$

Since the Lebesgue integral is defined to be the lower integral for measurable functions, this means that we now just need to verify that if $f : \mathbb{R}^d \to [0, \infty]$ is measurable then $x \mapsto f(x+v)$ (which we'll call $f_v : \mathbb{R}^d \to [0, \infty]$) is measurable.

Why does this hold? Well fix any $\lambda \in [0, \infty]$. We then consider that:

$$\{x \in \mathbb{R}^d \mid f_v(x) \ge \lambda\} = \{x \in \mathbb{R}^d \mid f(x+v) \ge \lambda\} = \{y \in \mathbb{R}^d \mid f(y) \ge \lambda\} - v$$

The last equality is the only one that is nontrivial. It holds since if $x \in \mathbb{R}^d$ such that $f(x+v) \ge \lambda$ then $x + v \in \{y \in \mathbb{R}^d \mid f(y) \ge \lambda\}$, and so $x = (x+v) - v \in \{y \in \mathbb{R}^d \mid f(y) \ge \lambda\} - v$. For the other direction if we have $y \in \mathbb{R}^d$ with $f(y) \ge \lambda$ then $x = y - v \in \mathbb{R}^d$ has the property that $f(x+v) = f(y) \ge \lambda$. Great! Well the right hand side is a translate of a measurable set since $f$ is measurable, and so $x \mapsto f_v(x) = f(x+v)$ is measurable as well. With this in mind, we may use the above equality and definitions to write the result:

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x+v)\,\mathrm{d}x$$

Perfect!

## Handout 6/7

### Unsigned Lebesgue Integrals

After defining the notion of measurability, both for unsigned functions taking values in $[0, \infty]$ and complex-valued functions taking values in $\mathbb{C}$, we are now ready to start defining the Lebesgue integral of such functions. As usual, we start with unsigned functions.

**Definition 0.1** (Lower unsigned Lebesgue integral)**.** Let $f : \mathbb{R}^d \to [0, \infty]$ be an unsigned function (not necessarily measurable). We define the *lower unsigned Lebesgue integral* $\underline{\int_{\mathbb{R}^d}} f(x)dx$ to be the quantity

$$\underline{\int_{\mathbb{R}^d}} f(x)dx : \sup_{0 \leq g \leq f; g \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} g(x)dx$$

where $g$ ranges over all unsigned simple functions $g : \mathbb{R}^d \to [0, \infty]$ that are pointwise bounded by $f$. One can also define the *upper Lebesgue integral* as

$$\overline{\int_{\mathbb{R}^d}} f(x)dx : \inf_{f \leq h; h \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} h(x)dx,$$

but we will use this integral very rarely.

Last time, we established some properties of the lower and upper integrals. Let $f, g : \mathbb{R}^d \to [0, \infty]$ be unsigned functions (not necessarily measurable)

(i) If $f$ is simple, then $\underline{\int_{\mathbb{R}^d}} f(x)dx = \overline{\int_{\mathbb{R}^d}} f(x)dx = \text{Simp} \int_{\mathbb{R}^d} f(x)dx$.

(ii) If $f \leq g$ pointwise almost everywhere, then we have that $\underline{\int_{\mathbb{R}^d}} f(x)dx \leq \underline{\int_{\mathbb{R}^d}} g(x)dx$ and $\overline{\int_{\mathbb{R}^d}} f(x)dx \leq \overline{\int_{\mathbb{R}^d}} g(x)dx$.

(iii) If $c \in [0, \infty)$, then $\underline{\int_{\mathbb{R}^d}} cf(x)dx = c\underline{\int_{\mathbb{R}^d}} f(x)dx$.

(iv) If $f, g$ agree almost everywhere, then $\underline{\int_{\mathbb{R}^d}} f(x)dx = \underline{\int_{\mathbb{R}^d}} g(x)dx$ and $\overline{\int_{\mathbb{R}^d}} f(x)dx = \overline{\int_{\mathbb{R}^d}} g(x)dx$.

(v) (Superadditivity of lower integral) $\underline{\int_{\mathbb{R}^d}} f(x)+g(x)dx \geq \underline{\int_{\mathbb{R}^d}} f(x)dx + \underline{\int_{\mathbb{R}^d}} g(x)dx$.

(vi) (Subadditivity of upper integral) $\overline{\int_{\mathbb{R}^d}} f(x)+g(x)dx \leq \overline{\int_{\mathbb{R}^d}} f(x)dx + \overline{\int_{\mathbb{R}^d}} g(x)dx$.

(vii) For any measurable set $E$, one has $\underline{\int_{\mathbb{R}^d}} f(x)dx = \underline{\int_{\mathbb{R}^d}} f(x)1_E(x)dx + \underline{\int_{\mathbb{R}^d}} f(x)1_{E^c}(x)dx$.

(viii) (Horizontal Truncation) As $n \to \infty$, $\underline{\int_{\mathbb{R}^d}} \min(f(x), n)dx$ converges to $\underline{\int_{\mathbb{R}^d}} f(x)dx$.

(ix) (Vertical Truncation) As $n \to \infty$, $\underline{\int_{\mathbb{R}^d}} f(x)1_{|x|\leq n}dx$ converges to $\underline{\int_{\mathbb{R}^d}} f(x)dx$.

(x) If $f + g$ is a simple function that is bounded with finite measure support (i.e. it is absolutely integrable), then we have that $\text{Simp}\int_{\mathbb{R}^d} f(x) + g(x)dx = \underline{\int_{\mathbb{R}^d}} f(x)dx + \overline{\int_{\mathbb{R}^d}} g(x)dx$.

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**Definition 0.2.** If $f : \mathbb{R}^d \to [0, \infty]$ is measurable, we define the unsigned Lebesgue integral $\int_{\mathbb{R}^d} f(x)dx$ to equal the lower unsigned integral $\underline{\int_{\mathbb{R}^d}} f(x)dx$. For unmeasurable functions, we leave the integral undefined.

**Q1)** Show that an unsigned measurable function is bounded if and only if it is the uniform limit of bounded simple functions.

**Q2)** Let $f$ be an unsigned measurable function that is bounded, and vanishing outside a set of finite measure. Then, the lower and upper integrals agree.

**Q3)** (Finite Additivity of the Lebesgue Integral) Let $f, g : \mathbb{R}^d \to [0, \infty]$ be measurable. Then $\int_{\mathbb{R}^d} f(x) + g(x)dx = \int_{\mathbb{R}^d} f(x)dx + \int_{\mathbb{R}^d} g(x)dx$.

*Hint: Use* **Q2)** *Remark.* One of the major theorems on Lebesgue integrals is that this finite additivity can be improved to countable additivity. This is known as the *monotone convergence theorem*, which we will prove later.

**Q4)** (Translation Invariance) Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Show that $\int_{\mathbb{R}^d} f(x + v)dx = \int_{\mathbb{R}^d} f(x)dx$ for any $v \in \mathbb{R}^d$.

**Q5)** (Linear change of variables) Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable, and let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transformation. Show that $\int_{\mathbb{R}^d} f(T^{-1}x)dx = |\det T| \int_{\mathbb{R}^d} f(x)dx$.

*Hint: You will need to show that $m(T(E)) = |\det T| m(E)$ for any measurable set. It might be helpful to recall that every invertible linear transformation is the composite of elementary transformations. You can also use previous results we proved similar to this for Jordan measure in 395.*

**Q6)** (Compatibility with the Riemann Integral) Let $f : [a, b] \to [0, \infty)$ be Riemann integrable. If we extend $f$ to $\mathbb{R}$ by declaring $f$ to equation 0 outside of $[a, b]$, show that $\int_{\mathbb{R}} f(x)dx = \int_a^b f(x)dx$.

**Q7)** (Markov property) Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable. Then for any $0 < \lambda < \infty$, we have

$$m(\{x \in \mathbb{R}^d : f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x)dx.$$

*Hint: Look at the indicator function of the* $\{x \in \mathbb{R}^d : f(x) > \lambda\}$.

**Q8)** Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable. Show that if $\int_{\mathbb{R}^d} f(x)dx < \infty$, then $f$ is finite almost everywhere. Give a counterexample to show that the opposite is not true.

**Q9)** Show that if $\int_{\mathbb{R}^d} f(x)dx = 0$ if and only if $f$ is zero almost everywhere.

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**Definition 0.3** (Absolute integrability)**.** An almost everywhere defined measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be absolutely integrable if the unsigned integral

$$\|f\|_{L^1(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f(x)|dx$$

is finite. We refer to this quantity as the $L^1(\mathbb{R}^d)$ norm of $f$, and use $L^1(\mathbb{R}^d)$ to denote the space of absolutely integrable functions. If $f$ is real-valued and absolutely integrable, we define $\int_{\mathbb{R}^d} f(x)dx$ by the formula

$$\int_{\mathbb{R}^d} f(x)dx := \int_{\mathbb{R}^d} f_+(x)dx - \int_{\mathbb{R}^d} f_-(x)dx,$$

where $f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$ are the magnitudes of the positive and negative components of $f$ (note that the two unsigned integrals on the right-hand side are finite, as $f_+, f_-$ are pointwise dominated by $|f|$. If $f$ is complex-valued and absolutely integrable, we define the Lebesgue integral $\int_{\mathbb{R}^d}$ by the formula

$$\int_{\mathbb{R}^d} f(x)dx := \int_{\mathbb{R}^d} \operatorname{Re} f(x)dx + \int_{\mathbb{R}^d} \operatorname{Im} f(x)dx.$$

**Q10)** Show that this integral is a linear operation, i.e. it satisfies that if $f, g \in L^1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} f(x)+g(x)dx = \int_{\mathbb{R}^d} f(x)dx + \int_{\mathbb{R}^d} g(x)dx, \qquad \int_{\mathbb{R}^d} cf(x)dx = c\int_{\mathbb{R}^d} f(x)dx.$$

## I.8. **IBL Week 8**

> **Problem I-1** (Markov property)
> Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Then for any $0 < \lambda < \infty$ we have:
> $$m(\{x \in \mathbb{R}^d \mid f(x) > \lambda\}) \le \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x) \, \mathrm{d}x$$
> *Hint: Look at the indicator function of* $\{x \in \mathbb{R}^d \mid f(x) > \lambda\}$.

*Solution.* Take any measurable function $f : \mathbb{R}^d \to [0, \infty]$ and any $\lambda$ with $0 < \lambda < \infty$. We will prove a slightly stronger result, namely that:

$$m(\{x \in \mathbb{R}^d \mid f(x) \ge \lambda\}) \le \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x) \, \mathrm{d}x$$

This is stronger because for any $x \in \mathbb{R}^d$ if $f(x) > \lambda$ then $f(x) \ge \lambda$, and so applying monotonicity along with the fact that both of these are measurable sets by measurability of $f$, we know that:

$$m(\{x \in \mathbb{R}^d \mid f(x) > \lambda\} m(\{x \in \mathbb{R}^d \mid f(x) \ge \lambda\}) \le \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x) \, \mathrm{d}x$$

Great!

Now call $A = \{x \in \mathbb{R}^d \mid f(x) \ge \lambda\}$. Since $0 < \lambda < \infty$ we know $0 < \frac{1}{\lambda} < \infty$. By the unsigned linearity of the lower integral combined with definitions, we wish to show that:

$$m(A) = m(\{x \in \mathbb{R}^d \mid f(x) \ge \lambda\}) \le \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x) \, \mathrm{d}x = \underline{\int_{\mathbb{R}^d}} \frac{1}{\lambda} \cdot f(x) \, \mathrm{d}x$$

But this follows by definition of the lower integral. Why? Well we know that if $x \in A$ and $y \notin A$ that:

$$\mathbb{1}_A(x) = 1 = \frac{\lambda}{\lambda} \le \frac{f(x)}{\lambda}$$
$$\mathbb{1}_A(y) = 0 \le \frac{f(y)}{\lambda}$$

Great! Since $A$ is measurable by measurability of $f$, this is a simple function. Thus we write by definition of the lower integral:

$$m(A) = \mathrm{Simp} \int_{\mathbb{R}^d} \mathbb{1}_A(x) \, \mathrm{d}x \le \underline{\int_{\mathbb{R}^d}} \frac{1}{\lambda} \cdot f(x) \, \mathrm{d}x$$

Great! This proves the desired result by tracing back through the above equalities ☺.

> **Problem I-2**
> Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Show that if $\int_{\mathbb{R}^d} f(x) \, \mathrm{d}x < \infty$ then $f$ is finite almost everywhere. Give a counterexample to show that the opposite is not true.

*Solution.* We prove the contrapositive. Fix a measurable function $f : \mathbb{R}^d \to [0, \infty]$ and let $A$ be the set $\{x \in \mathbb{R}^d \mid f(x) \ge \infty\} = \{x \in \mathbb{R}^d \mid f(x) = \infty\}$. Now suppose that $m^*(A) \ne 0$, that is $f$ is not finite almost everywhere. Since $f$ is measurable, we know that $A$ is measurable, and so we have $m(A) \ge 0$ and $m(A) \ne 0$. Therefore $m(A) > 0$. Consider now the unsigned simple function $\infty \cdot \mathbb{1}_A$. We show that $\infty \cdot \mathbb{1}_A \le f$.

Fix some $x \in A$ and some $y \notin A$. Then we compute that:

$$\infty \cdot \mathbb{1}_A(x) = \infty = f(x)$$
$$\infty \cdot \mathbb{1}_A(y) = 0 \le f(y)$$

Awesome! With this in mind we see by the definition of the unsigned lebesgue integral as the lower unsigned integral that:

$$\infty = \infty \cdot m(A) = \mathrm{Simp} \int_{\mathbb{R}^d} \infty \cdot \mathbb{1}_A(x) \, \mathrm{d}x \le \int_{\mathbb{R}^d} f(x) \, \mathrm{d}x$$

Where the first equality follows from the fact that $m(A) > 0$. This proves the desired result! Perfect!

Now we give the desired counterexample. Consider the indicator function $\mathbb{1}_{\mathbb{R}^d}$. Then we know that $\mathbb{1}_{\mathbb{R}^d}$ is finite everywhere because it only takes on the values 0 and 1. However:

$$\int_{\mathbb{R}^d} \mathbb{1}_{\mathbb{R}^d}(x)\,\mathrm{d}x = m(\mathbb{R}^d) = \infty$$

And therefore it is not true that if the function is finite almost everywhere that its integral is finite.

> **Problem I-3**
> Show that $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = 0$ if and only if $f$ is zero almost everywhere for a measurable function $f : \mathbb{R}^d \to [0, \infty]$.

*Solution.* Let's go! We show both directions:

($\Rightarrow$) For this we'll do the contrpositive. Let $f$ be a measurable function which is not zero almost everywhere. That is, let $A = \{x \in \mathbb{R}^d \mid f(x) \neq 0\} = \{x \in \mathbb{R}^d \mid f(x) > 0\}$. Then suppose that $m^*(A) \neq 0$. In this case, since $f$ is measurable we know that $A$ is measurable, and so $m(A) \geq 0$, and $m(A) > 0$. Great!

Now let $A_n = \{x \in \mathbb{R}^d \mid f(x) > 1/n\}$. Each of these are measurable since $f$ is a measurable function. Then we claim the following set equality, which gives the below equalities by the properties of the Lebesgue measure:

$$A = \bigcup_{n=1}^{\infty} A_n$$
$$0 < m(A) \leq \sum_{n=1}^{\infty} m(A_n)$$

The set equality holds since if $x \in \mathbb{R}^d$ and there is an $n \in \mathbb{N}$ so that $x \in A_n$, aka $f(x) > 1/n$, then since $1/n > 0$ we know $f(x) > 0$ and $x \in A$. For the other direction if $x \in A$ then $f(x) > 0$, then by the Archimedean principle there is an $n \in \mathbb{N}$ so that $f(x) > 1/n$, so $f(x) \in A_n$.

Now by the above sum, since each $m(A_n) \geq 0$ this implies that there is some $n \in \mathbb{N}$ so that $m(A_n) > 0$, because otherwise each would be equal to 0 and so $m(A)$ would be zero. Great! By the Markov Property in Problem 7 we then have that:

$$m(A_n) \leq \frac{1}{\frac{1}{n}} \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$$
$$0 < \frac{1}{n} \cdot m(A_n) \leq \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$$

Therefore $\int_{\mathbb{R}^d} f(x) \neq 0$, proving the contrapositive just as desired! Woot!

($\Leftarrow$) Now suppose that $f(x) = 0$ almost everywhere and $f : \mathbb{R}^d \to [0, \infty]$ is any function. We actually don't need the measurability hypothesis for this direction, only for the fact that the integral is defined, which it is as the lower unsigned integral. In this case, we know that $\underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x = \sup F$ where $F$ is defined as the set below:

$$F = \left\{ \mathrm{Simp} \int_{\mathbb{R}^d} g(x)\,\mathrm{d}x \mid 0 \leq g \leq f, g \text{ simple} \right\}$$

We show that $F = \{0\}$, and so $\sup F = 0$. This implies the result for measurable functions because then:

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \underline{\int_{\mathbb{R}^d}} f(x)\,\mathrm{d}x = \sup F = 0$$

Great! To do this, let $0 \leq g \leq f$ where $g$ is a simple function. Now we show that $g$ is 0 almost everywhere. Why? Well let $A$ be the set on which $f$ is not zero, and let $B$ be the set on which $g$ is not zero. By definition of almost everywhere, $m(A) = 0$. Now suppose that $x \in B$, $g(x) > 0$, so $f(x) \geq g(x) > 0$, so $x \in A$. Therefore $B \subseteq A$, giving us that $m^*(B) \leq m(A) = 0$, and so $m^*(B) = 0$, showing by previous work that $m(B) = 0$.

Therefore $g$ is zero almost everywhere. By work with the simple integral, we showed that changing a simple function on a set of measure zero does not change its simple integral. Therefore:

$$\operatorname{Simp} \int_{\mathbb{R}^d} g(x) \, \mathrm{d}x = \operatorname{Simp} \int_{\mathbb{R}^d} 0 \, \mathrm{d}x = 0$$

Great! This shows that $F = \{0\}$ as desired, and so tracing back we have the result via the equalities:

$$\int_{\mathbb{R}^d} f(x) \, \mathrm{d}x = \underline{\int_{\mathbb{R}^d}} f(x) \, \mathrm{d}x = \sup F = 0$$

Now let's tie the bow on this problem.

With this we're done! Perfect ☺.

**Problem I-4**

Show that this integral is a linear operation, i.e. it satisfies that if $f, g \in L^1(\mathbb{R}^d)$ and $c \in \mathbb{C}$ then:

$$\int_{\mathbb{R}^d} f(x) + g(x) \, \mathrm{d}x = \int_{\mathbb{R}^d} f(x) \, \mathrm{d}x + \int_{\mathbb{R}^d} g(x) \, \mathrm{d}x$$

$$\int_{\mathbb{R}^d} cf(x) \, \mathrm{d}x = c \int_{\mathbb{R}^d} f(x) \, \mathrm{d}x$$

*Solution.* Let $f, g \in L^1(\mathbb{R}^d)$ and $c \in \mathbb{C}$. First we claim that $f + g \in L^1(\mathbb{R}^d)$ and $cf \in L^1(\mathbb{R}^d)$, so this is even well defined. This is clear since by the properties of the absolute value for any $x \in \mathbb{R}^d$ we have $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ and $|cf(x)| = |c| \, |f(x)|$. We now may write by monotonicity and the unsigned lienearity of the unsigned integral that:

$$\begin{aligned}
\|f + g\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |(f+g)(x)| \, \mathrm{d}x = \int_{\mathbb{R}^d} |f(x) + g(x)| \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} |f(x)| + |g(x)| \, \mathrm{d}x = \int_{\mathbb{R}^d} |f(x)| \, \mathrm{d}x + \int_{\mathbb{R}^d} |g(x)| \, \mathrm{d}x \\
&= \|f\|_{L^1(\mathbb{R}^d)} + \|g\|_{L^1(\mathbb{R}^d)} < \infty \\
\|cf\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |cf(x)| \, \mathrm{d}x = \int_{\mathbb{R}^d} |c| \, |f(x)| \, \mathrm{d}x \\
&= |c| \int_{\mathbb{R}^d} |f(x)| \, \mathrm{d}x = |c| \cdot \|f\|_{L^1(\mathbb{R}^d)} < \infty
\end{aligned}$$

Great! Thus $f + g, cf \in L^1(\mathbb{R}^d)$ as desired. We now tackle showing the linearity of this integral in pieces.

- TODO
- TODO
- TODO

Great! With these all put together we have the full result!

# Handout 7

## Absolute Integrability

**Q1)** (Markov property) Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable. Then for any $0 < \lambda < \infty$, we have

$$m(\{x \in \mathbb{R}^d : f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x)dx.$$

*Hint: Look at the indicator function of the* $\{x \in \mathbb{R}^d : f(x) > \lambda\}$.

**Q2)** Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable. Show that if $\int_{\mathbb{R}^d} f(x)dx < \infty$, then $f$ is finite almost everywhere. Give a counterexample to show that the opposite is not true.

**Q3)** Show that if $\int_{\mathbb{R}^d} f(x)dx = 0$ if and only if $f$ is zero almost everywhere.

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**Definition 0.1** (Absolute integrability)**.** An almost everywhere defined measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be absolutely integrable if the unsigned integral

$$\|f\|_{L^1(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f(x)|dx$$

is finite. We refer to this quantity as the $L^1(\mathbb{R}^d)$ norm of $f$, and use $L^1(\mathbb{R}^d)$ to denote the space of absolutely integrable functions. If $f$ is real-valued and absolutely integrable, we define $\int_{\mathbb{R}^d} f(x)dx$ by the formula

$$\int_{\mathbb{R}^d} f(x)dx := \int_{\mathbb{R}^d} f_+(x)dx - \int_{\mathbb{R}^d} f_-(x)dx,$$

where $f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$ are the magnitudes of the positive and negative components of $f$ (note that the two unsigned integrals on the right-hand side are finite, as $f_+, f_-$ are pointwise dominated by $|f|$. If $f$ is complex-valued and absolutely integrable, we define the Lebesgue integral $\int_{\mathbb{R}^d}$ by the formula

$$\int_{\mathbb{R}^d} f(x)dx := \int_{\mathbb{R}^d} \operatorname{Re} f(x)dx + i \int_{\mathbb{R}^d} \operatorname{Im} f(x)dx.$$

**Q4)** Show that this integral is a linear operation, i.e. it satisfies that if $f, g \in L^1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} f(x)+g(x)dx = \int_{\mathbb{R}^d} f(x)dx+\int_{\mathbb{R}^d} g(x)dx, \quad \int_{\mathbb{R}^d} cf(x)dx = c\int_{\mathbb{R}^d} f(x)dx.$$

**Q5)** Show that $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$, and $\|cf\|_{L^1} = |c|\|f\|_{L^1}$. (This makes $L^1$ a *seminorm* on the space of absolutely integrable functions. It is not a norm because of the following (fixable) small caveat.

**Q6)** Show that $\|f\|_{L^1} = 0$ if and only if $f$ is zero almost everywhere.

**Q7)** (The triangle inequality) Let $f \in L^1(\mathbb{R}^d \to \mathbb{C})$. Show that

$$\left|\int_{\mathbb{R}^d} f(x)\,dx\right| \leq \int_{\mathbb{R}^d} |f(x)|\,dx.$$

*Hint: This is easy when $f$ is real-valued, but one has to be a bit more careful with the argument when $f$ is complex-valued.*

## I.9. **IBL Week 9**

# Handout 8

## Convergence Theorems-Part I

The main power of Lebesgue integration comes from the (much) stronger convergence theorems that hold in its context. Let $f_1, f_2, \ldots : \mathbb{R}^d \to [0, \infty]$ be a sequence of measurable unsigned functions. Suppose that as $n \to \infty$, $f_n(x)$ converges pointwise to a measurable limit $f$. A basic question in analysis is to determine the conditions under which the pointwise convergence implies the convergence of the integral,

$$\int_{\mathbb{R}^d} f_n(x)dx \overset{?}{\to} \int_{\mathbb{R}^d} f(x)dx. \qquad (**)$$

or in other words can one interchange the order of the limits

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n(x)dx?$$

**Q1)** Let $E \subset \mathbb{R}^d$ be a measurable set of finite measure. Suppose that $f_n : E \to [0, \infty]$ is a sequence of unsigned measurable functions that converge *uniformly* to $f$. Show that $\int_E f_n(x)$ converges to $\int_E f(x)dx$.

*Remark.* : This statement is actually true for absolutely integrable functions as well. It even holds for Riemann integrals! The condition of uniform convergence such an overkill; we shall see that much less is needed for Lebesgue integrals.

**Q2)** Give an example of a sequence of functions $f_n$ that violates $(**)$ such that all $f_n$ are bounded and are supported on a set of measure $\leq 1$.

**Q3)** Give an example of a sequence of functions $f_n$ that violates $(**)$ such that all $f_n$ are bounded and are supported on a set of measure $\geq n$.

**Q4)** Give an example of a sequence of functions $f_n$ that violates $(**)$ such that all $f_n$ are supported on a set of $E_n$ such that $m(E_n) \to 0$.

*Remark.* : The example in **Q2), Q3), Q4)** correspond to the sequence $(f_n)$ doing the following three things respectively: *escaping to horizontal infinity, escaping to width infinity, escaping to vertical infinity.* A deep principle of analysis (whose formulation is well-beyond the scope of this course) states that these are the only three avenues for (**) to fail. The *monotone convergence theorem below* is one manifestation of this principle.

---

**Theorem 0.1** (The Monotone Convergence Theorem)**.** Let $0 \leq f_1 \leq f_2 \leq \ldots$ be a monotone non-decreasing sequence of unsigned measurable function on $\mathbb{R}^d$. Then we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} f(x)dx.$$

**Q5)** Show the theorem when $f_n$ are indicator functions, i.e.: Let $E_1 \subset E_2 \subset \ldots \subset \mathbb{R}^d$ be a countable nested sequence of measurable sets. Show that $m(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$. *Hint: Use countable additivity of Lebesgue measure.*

**Q6)** Show that $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx \leq \int_{\mathbb{R}^d} \lim_{n\to\infty} f(x)dx$.

As such, it remains to show that $\int_{\mathbb{R}^d} f(x)dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx$, where $f := \lim_{n\to\infty} f_n(x)$.

**Q7)** Why is it enough to show that

$$\int_{\mathbb{R}^d} g(x)dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx$$

for any simple function $g$ that is bounded above by $f$ and such that $g$ is finite everywhere.

**Q8)** Suppose that $g$ is a function as above, and write $g = \sum_{i=1}^{k} c_i \mathbf{1}_{A_i}$, where $0 \le c_i < \infty$ and $A_i$ are disjoint. Let $0 < \varepsilon < 1$ be arbitrary, and let

$$A_{i,n} := \{x \in A_i : f_n(x) > (1-\varepsilon)c_i\}.$$

Show that $lim_{n\to\infty} m(A_{i,n}) = m(A_i)$.

**Q9)** Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx \ge (1-\varepsilon)\sum_{i=1}^{k} c_i m(A_i) = (1-\varepsilon)\int_{\mathbb{R}^d} g(x)dx.$$

This concludes the proof by letting $\varepsilon \to 0$.

## I.10. **IBL Week 10**

# Handout 9

## Partition of Unity

**Theorem 0.1.** Let $X$ be an arbitrary subset of $\mathbb{R}^d$. For each covering of $X$ by (relatively) open subsets $\{U_\alpha\}$, there exists a sequence of smooth[1] functions $\theta_i$ on $X$, called a *partition of unity* subordinate to the open cover $\{U_\alpha\}$ with the following properties:

(i) $0 \leq \theta_i \leq 1$ for all $x \in X$ and all $i$.

(ii) Each $x \in X$ has a neighborhood on which all but finitely many functions $\theta_i$ are identically zero.

(iii) Each function $\theta_i$ is identically zero except on some closed set contained in one of a the $U_\alpha$.

(iv) For each $x \in X$,
$$\sum_i \theta_i(x) = 1$$
(Note that according to (ii), this sum is always finite).

Below, we present the proof of this result through a series of questions.

**Q1)** Write each $U_\alpha = X \cap W_\alpha$ where $W_\alpha$ is open in $\mathbb{R}^d$. Set $W = \cup_\alpha W_\alpha$. Show that there exists a *nested* sequence of compact sets $K_j$ such that
$$\cup_{j=1}^\infty K_j = W.$$

**Q2)** For each $x \in K_2$, one can find a ball centered at $x$ and whose closure is contained in one of the $W_\alpha$. Cover $K_2$ by $r$ such balls $B_1^{(2)}, \ldots, B_r^{(2)}$ (why is this possible?). Find $r$ smooth functions

[1] Recall that a function on $X$ is smooth if it admits a smooth extension to an open subset containing $X$.

$\eta_1, \ldots, \eta_r$ such that $\eta_k$ is 1 on $B_k^{(2)}$ and zero outside another ball contained in one of the $W_\alpha$.

*Hint: Given any two nested balls, the existence of a smooth $\eta$ that is 1 on the smaller ball and zero outside the outer one was part of our Midterm.*

**Q3)** Repeat the above step with $K_j$ replaced by $K_j \setminus Int\, K_{j-1}$ and $W$ replaced by $W \setminus K_{j-2}$, to obtain for each $j$, a finite collection of functions $\eta_i$ (that we add to the previous collection at step $j$); each such function is to be equal 1 on a ball $B_i^{(j)}$ and zero outside a closed ball contained in both $W \setminus K_{j-2}$ and in one of the $W_\alpha$. The union of the $B_i^{(j)}$ covers $K_j \setminus Int\, K_{j-1}$.

**Q4)** Show that $\sum_i \eta_i$ is finite in a neighborhood of every point of $W$, and at least one term of the sum is nonzero at any point of $W$.

**Q5)** Find the function $\theta_i$ and finish the proof.

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**Back to Lebesgue theory and convergence theorems**

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**Theorem 0.2** (The Monotone Convergence Theorem)**.** Let $0 \leq f_1 \leq f_2 \leq \ldots$ be a monotone non-decreasing sequence of unsigned measurable function on $\mathbb{R}^d$. Then we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} f(x)dx.$$

**Q7)** Let $f_1, f_2, \ldots : \mathbb{R}^d \to [0,\infty]$ be a sequence of unsigned measurable functions. Then one has

$$\int_{\mathbb{R}^d} \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} f_n(x)dx.$$

**Q8)** (Borel-Cantelli) Let $E_1, E_2, \ldots$ be a sequence of measurable sets such that

$$\sum_{n=1}^{\infty} m(B_n) < \infty$$

Show that the set of points contained in infinitely many of the $E_n$ has measure zero, in other words the measure of the set

$$\cap_{n=1}^{\infty} \cup_{k \geq n} E_k$$

is zero.

## I.11. **IBL Week 11**

# Handout 10

## More convergence theorems

**Theorem 0.1** (Fatou's lemma)**.** Let $0 \leq f_1, f_2, \ldots$ be a sequence of unsigned measurable function on $\mathbb{R}^d$. Then we have

$$\int_{\mathbb{R}^d} \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx.$$

Informally speaking, Fatou's lemma tells us that when taking the pointwise limit of unsigned functions $f_n$, the mass $\int_X f_n dx$ can be destroyed in the limit (as was the case of the three avenues of escape to $\infty$ we discussed in Worksheet 8) but it cannot be created in the limit.

**Q1)** Prove Fatou's lemma. Recall first that $\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf_{n\geq N} a_n$. Let $F_N(x) = \inf_{n\geq N} f_n(x)$. Show that $F_N(x)$ is monotone, and apply the monotone convergence theorem to $F_N(x)$.

---

The third major convergence theorem for Lebesgue integrals is the *dominated convergence theorem.*

**Theorem 0.2** (Dominated Convergence Theorem)**.** Let $f_1, f_2, \ldots : \mathbb{R}^d \to \mathbb{C}$ be a sequence of measurable functions that converge pointwise almost everywhere to a measurable limit $f : \mathbb{R}^d \to \mathbb{C}$. Suppose that there is an unsigned absolutely integrable function $G : \mathbb{R}^d \to [0, +\infty]$ such that $|f_n(x)| \leq G(x)$ for almost every $x \in \mathbb{R}^d$ and every $n$. Then we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \, dx = \int_{\mathbb{R}^d} f \, dx.$$

**Q3)** Why can we reduce to the case when $f_n$ converges to $f$ and $f_n \leq G$ everywhere and not almost everywhere.

**Q4)** Reduce to the case when $f_n$ are real-valued. Hence $-G(x) \leq f_n(x) \leq G(x)$ pointwise everywhere.

**Q5)** Apply Fatou's lemma to the unsigned functions $f_n + G$ to conclude that

$$\int_{\mathbb{R}^d} f(x)\, dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n\, dx.$$

**Q6)** Apply Fatou now to the unsigned function $G - f_n$ to finish the proof of the theorem.

**Q7)** Under the hypothesis of the dominated convergence theorem, establish the stronger bound $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$.

## I.12. **IBL Week 11**

**Problem I-1**

Prove Fatou's lemma. Recall first that $\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \int_{n\geq N} a_n$. Let $F_n(X) = \inf_{n\geq N} f_n(x)$. Show that $F_N(x)$ is monotone, and apply the monotone convergence theorem to $F_N(x)$

*Solution.* Note that $F_{N+1}(x) = \int_{n\geq N+1} f_n(x) \geq \inf_{n\geq N} f_n(x) = F_N(x)$ because $F_N(x)$ is a lower bound for the set defining $F_{N+1}(x)$, from 295 work.

Furthermore, since $f_1, f_2, \ldots \geq 0$ we know that $F_1, F_2, \ldots \geq 0$. Therefore we can write the following by the monotonic convergence theorem:

$$\int_{\mathbb{R}^d} \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \lim_{n\to\infty} F_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} F_n(x)\,\mathrm{d}x = \liminf_{n\to\infty} \int_{\mathbb{R}^d} F_n(x) \leq \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x$$

Because each $F_n \leq f_n$.

**Problem I-2**

Why in the proof of the Dominated Convergence Theorem can we reduce to the case when $f_n$ converges to $f$ and $|f_n| \leq G$ everywhere and not almost everywhere.

*Solution.* Let $D_f$ be the set where $f_n$ does not converge to $f$ and $D_G$ to be the set where $|f_n(x)|$ is not less than or equal to $G(x)$. Define $f_n'$ to be $f_n$ on $(D_f \cup D_G)^c$ and 0 on $D_f \cup D_G$, likewise defined $f$ to be $f$ on $(D_f \cup D_G)^c$ and 0 on $D_f \cup D_G$. Since integrals only care about sets of non-zero measure and $D_f \cup D_g$ has measure zero, we know that when either of these integrals exist:

$$\int_{\mathbb{R}^d} f_n'(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x$$
$$\int_{\mathbb{R}^d} f'(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$$

Then the limits also agree. Furthemore $f_n'$ converges to $f$ everywhere and $|f_n(x)| \leq G(x)$ everywhere. We then have the reduction.

Also note that under this reduction, the integrals always exist. Why? Well if $f_n'$ converges to $f'$, then $|f_n'|$ converges to $|f'|$. With this we have that because $|f_n'(x)| \leq G(x)$ for all $n$ and $x \in \mathbb{R}^d$ that $|f'(x)| \leq G(x)$ for all $x \in \mathbb{R}^d$. Therefore we have that, since $|f|$ agrees with $|f'|$ almost everywhere and $|f_n'|$ agrees with $|f_n|$ almost everywhere:

$$\int_{\mathbb{R}^d} |f_n|(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |f_n'(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}^d} G(x)\,\mathrm{d}x < \infty$$
$$\int_{\mathbb{R}^d} |f(x)|\,\mathrm{d}x = \int_{\mathbb{R}^d} |f'(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}^d} G(x)\,\mathrm{d}x < \infty$$

Perfect! This shows all the integrals are defined and the problem is well-posed.

**Problem I-3**

Reduce to the case when $f_n$ are real-valued. Hence $-G(x) \leq f_n(x) \leq G(x)$ pointwise everywhere.

*Solution.* Let $f_n(x) = u_n(x) + iv_n(x)$ and $f(x) = u(x) + iv(x)$ where $u_n, v_n, u, v : \mathbb{R}^d \to \mathbb{R}$. We know that $u_n$ and $v_n$ converge to $u$ and $v$ since $f_n$ converges to $f$. Likewise, we know that whenever $|f_n(x)| \leq G(x)$ that $|u_n(x)|, |v_n(x)| \leq |f_n(x)| \leq G(x)$, and so $u_n$ and $v_n$ are bounded by $G$.

We then apply the result to $u_n, v_n$:

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x &= \lim_{n\to\infty}\int_{\mathbb{R}^d} u_n(x)\,\mathrm{d}x + i\cdot\lim_{n\to\infty}\int_{\mathbb{R}^d} v_n(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x + i\cdot\int_{\mathbb{R}^d} v(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x\end{aligned}$$

Perfect! This demonstrates that the result reduces to the real-valued case.

**Problem I-4**

Apply Fatou's lemma to the unsigned functions $f_n + G$ to conclude that:

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x$$

*Solution.* Note that since $-G(x) \le f_n(x) \le G(x)$ everwhere, we know that $f_n(x) + G(x) \ge 0$ everywhere. Then these are unsigned measurable functions we know by Fatou's Lemma that:

$$\int_{\mathbb{R}^d} f(x) + G(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \liminf_{n\to\infty} f_n(x) + G(x)\,\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^d} f_n(x) + G(x)\,\mathrm{d}x$$

Now because $G(x)$ is absolutely integrable, its integral is finite, so we can cancel on both sides as follows, using that $G(x)$ is constant in $n$:

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} G(x)\,\mathrm{d}x &\le \liminf_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} G(x)\,\mathrm{d}x \\ \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x &\le \liminf_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x\end{aligned}$$

This is the result we wanted ☺

**Problem I-5**

Apply Fatou now to the unsigned function $G - f_n$ to finish the proof of the theorem.

*Solution.* Now note that $G(x) - f_n(x) \ge 0$ everywhere since $G(x) \ge f_n(x)$ everywhere. Therefore $G_n - f_n$ is an unsigned measurable function. We then apply Fatou's Lemma to write:

$$\int_{\mathbb{R}^d} G(x) - f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \liminf_{n\to\infty} G(x) - f_n(x)\,\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^d} G(x) - f_n(x)\,\mathrm{d}x$$

Again because $G(x)$ is absolutely integrable, its integral is finite, so we can cancel on both sides as follows, using that $G(x)$ is constant in $n$:

$$\begin{aligned}\int_{\mathbb{R}^d} G(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x &\le \int_{\mathbb{R}^d} G(x)\,\mathrm{d}x + \liminf_{n\to\infty} -\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \\ -\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x &\le \liminf_{n\to\infty} -\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \\ -\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x &\le -\limsup_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \\ \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x &\ge \limsup_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x\end{aligned}$$

Perfect! Then from the previous problem:

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \le \limsup_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$$

Therefore the lim inf and the lim sup agree, and so the limit exists and we have:

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$$

Just as desired. Tracing through the previous questions we see that this special case of real-valued functions where everything is pointwise everywhere generalizes to the full theorem.

**Problem I-6**

Under the hypothesis of the dominated convergence theorem, establish the stronger bound that $\|f_n - f\|_{\mathscr{L}^1} \to 0$ as $n \to \infty$

*Solution.* Note that since $f_n$ converges to $f$ almost everywhere that $|f_n - f|$ converges to 0 almost everywhere. Furthermore, $|f_n - f|$ is bounded above almost everywhere by the absolutely integrable function $G + |f|$, since $|f|$ was previously established to be absolutely integrable and $G$ is absolutely integrable, the fact that their sum is absolutely integrable follows from previous work. This holds from the triangle inequality, since for almost every $x \in \mathbb{R}^d$ we have:

$$|f_n(x) - f(x)| \le |f_n(x)| + |f(x)| \le G(x) + |f(x)|$$

Perfect! Then we apply the dominated convergence theorem to $|f_n - f|$ to see that:

$$\lim_{n\to\infty}\|f_n - f\|_{\mathscr{L}^1} = \lim_{n\to\infty}\int_{\mathbb{R}^d} |f_n(x) - f(x)|\,\mathrm{d}x = \int_{\mathbb{R}^d} 0\,\mathrm{d}x = 0$$

And therefore the result holds, namely that $\|f_n - f\|_{\mathscr{L}^1} \to 0$ as $n \to \infty$. Awesome ☺

# Handout 11

## Riemannian Manifolds

***Introduction and examples**

Recall that we defined, in class, a Riemannian metric to be a symmtric 2-tensor that is positive definite at every point $p \in M$ ($M$ is a differentiable manifold). This 2-tensor gives us an inner product on each $T_pM$ as follows: Given any two vectors $v, w \in T_pM$, we set

$$\langle v, w\rangle_g := g_p(v, w), \qquad g_p = g(p).$$

**Q1)** Check that this is indeed makes $T_pM$ an inner product space.

Once this is set, one can define a bunch of geometric constructions on a Riemannian manifold $(M, g)$, such as:

- The length or norm of tangent vector $X \in T_pM$ is defined to be $|X|_g = \sqrt{\langle X, X\rangle_g} = \sqrt{g_p(X, X)}$.
- The angle between two nonzero tangent vectors $X, Y \in T_pM$ is the unique $\theta \in [0, \pi]$ such that $\cos\theta = \frac{\langle X,Y\rangle_g}{|X|_g|Y|_g}$.
- Two tangent vectors are said to be orthogonal if $\langle X, Y\rangle_g = 0$.
- If $\gamma : [a, b] \to M$ is a continuous piece-wise smooth curve (this means that there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that $\gamma|_{[x_i,x_{i+1}]}$ is smooth), the length of $\gamma$ is defined by

  $$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt.$$

  Since $|\gamma'(t)|_g$ is continuous at all but finitely many points, the integral is well-defined. It is not hard to check that this length is independent of the parametrization. We won't do that today though.

- If $M$ is connected, then given two points $p, q \in M$, one can define the distance between $p$ and $q$ as

$$d_g(p,q) = \inf_{\gamma} L_g(\gamma),$$

where the infimum is taken over all piecewise-smooth curves connecting $p$ to $q$. Again, one can show that this is indeed a metric on $M$ that makes $M$ into a metric space (where the metric topology coincides with the manifold topology of $M$). Also, we won't check that today.

The simplest example of a Riemannian manifold is Euclidean space $\mathbb{R}^n$ with the metric $\bar{g}$ defined by

$$\bar{g}(p) = \delta_{ij}\, dx^i \otimes dx^j, \qquad \delta_{ii} = 1, \delta_{ij} = 0 \text{ if } i \neq j$$

where we are using the Einstein summation notation here. This means for any two vectors $\vec{v}, \vec{w} \in T_p\mathbb{R}^n$, then $\bar{g}(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w} = \sum_{i=1}^n v_i w_i$.

**Q2)** Let $M$ be a differentiable submanifold of $\mathbb{R}^d$. Then $M$ inherits from $\mathbb{R}^d$ its Riemannian metric as follows: For any $p \in M$, and any vectors $\vec{v}, \vec{w} \in T_pM \subset T_p\mathbb{R}^d$, we set $g(p)(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w}$. Show that this is indeed a Riemannian metric on $M$. This is called the induced Riemannian metric. In particular, since any (abstract) differentiable manifold can be regarded as a submanifold of $\mathbb{R}^d$, one can put a metric on any differentiable manifold.

***Coordinate representation**

Given a coordinate chart $(U, \varphi)$ where $\varphi = (x^1, \ldots, x^n)$, we saw in class that a metric $g$ on a manifold $M$ has the following coordinate representation

$$g(p) = g_{ij}(p)\, dx^i \otimes dx^j$$

where $g_{ij} : U \to \mathbb{R}$ are smooth functions (actually this is how defined smoothness of $g$!).

**Q3)** Now suppose that $(V, \psi)$ is another coordinate chart such that $U \cap V \neq \emptyset$, and write $\psi = (y^1, \ldots, y^n)$, then $g$ can also be expressed as $g(p) = \widetilde{g}_{ij}\, dy^i \otimes dy^j$. What is the relation between the two matrices $g_{ij}$ and $\widetilde{g}_{ij}$? *Hint: Recall that* $dy^i = \sum_{k=1}^{n} \frac{\partial y^i \circ \varphi^{-1}(x)}{\partial x^k} dx^k$

***Orthonormal Frames**

Let $(M, g)$ be a Riemannian $n-$manifold, and let $U \subset M$ be open. We define an *orthonormal frame* for $M$ on $U$ to be collection of $n-$smooth vector fields $E_1, E_2, \ldots, E_n$ on $U$ such that for each $p \in U$, $\{E_1(p), \ldots, E_n(p)\}$ forms an orthonormal basis for $T_pM$.

**Q4)** Check that the coordinate frame $(\partial/\partial x^i)$ is a global orthonormal frame on $\mathbb{R}^n$. *Hint: Recall that* $\partial/\partial x^i(p) = \vec{e}_i$.

*Remark.* The fact that the orthonormal frame is also a coordinate frame is a very special property that generally cannot be achieved. It is a reflection of the zero curvature on $\mathbb{R}^d$. So, in general, we don't expect orthonormal frames to be coordinate frames.

**Q5)** (Existence of Orthonormal Frames) Show that for each $p \in M$, there is a smooth orthonormal frame on a neighborhood of $p$. *Hint: Gram-Schmidt to coordinate frame. Why is the resulting frame smooth?*

***The tangent-cotangent isomorphism**

**Q6)** Let $V$ be an inner product vector space. The inner product gives an isomorphism between $V$ and its dual space $V^*$ as follows: Let $\omega \in V^*$, show that there exists a unique $v \in V$ such that $\omega(\cdot) = \langle \cdot, v \rangle$. Check that the map $L : V \to V^*$ given by $Lv = \langle \cdot, v \rangle$ is a linear isomorphism.

**Q7)** We apply the above question to $T_pM$ of some Riemannian manifold $(M, g)$. Let $(U, \varphi)$ be coordinate chart near $p$ and let $\varphi = (x^1, \ldots, x^n)$. Let $X \in T_pM$ be written as $X = X^i \frac{\partial}{\partial x^i}$. Check that $L(X)$ defined in the above problem (using the inner product $\langle \cdot, \cdot \rangle_g$)

is given by

$$L(X) = X_i dx^i, \quad \text{where} \quad X_i := g_{ij} X^j.$$

Therefore, $g_{ij}$ is the matrix of the transformation $L$. For this reason, applying the operator $L$ is sometimes called *lowering an index.*

**Q8)** (Raising indices) As a result, the matrix of the transformation $L^{-1} : V^* \to V$ is given by the inverse of the matrix $(g_{ij})$. Why is $g_{ij}$ invertible? The inverse matrix is usually denoted $g^{ij}$ so that the following holds using Einstein's summation notation

$$g^{ij} g_{jk} = g_{kj} g^{ji} = \delta^i_k.$$

Thus given a covector $\omega \in (T_pM)^*$ with $\omega = \omega_i dx^i$, then the vector $X = L^{-1}\omega$ is given by

$$X = \omega^i \frac{\partial}{\partial x^i}, \qquad \omega^i = g^{ij}\omega_j.$$

We say that applying $L^{-1}$ amounts to raising an index.

**Q9)** The most important manifestation of this tangent-cotangent isomorphisms happens when we apply $L^{-1}$ to the 1-form $df(p)$ when $f : M \to \mathbb{R}$ is a smooth function. We call the resulting vector the gradient of $f$ at $p$ or $\nabla f(p)$. Show that

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, \qquad \frac{\partial f}{\partial x^i}(p) := \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(x), x = \varphi(p).$$

Hence, $\nabla f$ is a smooth vector field on $M$.

## I.13. IBL Week 12

**Problem I-1**

Check that $\langle v, w\rangle_g := g_p(v, w)$ makes $T_pM$ into an inner product space

*Solution.* This follows directly, $g$ being a symmetric 2-tensor means that $g_p$ is symmetric and bilinear, and positive definiteness gives us the remaining properties of an inner-product.

**Problem I-2**

Let $M$ be a differentiable submanifold of $\mathbb{R}^d$. Then $M$ inherits from $\mathbb{R}^d$ its Riemannian metric as follows: For any $p \in M$, and any vectors $v, w : T_pM \subseteq T_p\mathbb{R}^d$, we set $g_p(v, w) = v \cdot w$. Show that this is indeed a Riemannian metric on $M$. This is called the induced Riemannian metric. In particular, since any (abstract) differentiable manifold can be regarded as a submanifold of $\mathbb{R}^d$, one can put a metric on any differentiable manifold.

*Solution.* $g$ as defined above is just a pullback under the smooth map $\iota : M \to \mathbb{R}^d$ of the smooth 2-tensor $(v, w) \mapsto v \cdot w$ on $\mathbb{R}^d$, which is smooth.

**Problem I-3**

Now suppose that $(V, \psi)$ is another coordinate chart such $U \cap V \neq \emptyset$ and write $\psi = (y^1, \ldots, y^n)$, then $g$ can also be expressed as $g_p = \widetilde{g}_{ij}\, \mathrm{d}y^i \otimes \mathrm{d}y^j$. What is the relation between the two matrices $g_{ij}$ and $\widetilde{g}_{ij}$.

*Hint: Recall that* $\mathrm{d}y^i = \sum_{k=1}^n \frac{\partial[y^i \circ \varphi^{-1}]}{\partial x^k}\, \mathrm{d}x^k$

*Solution.* We may write that:

$$\begin{aligned} g_p &= \sum_{1\le i,j\le n} \widetilde{g}_{ij}\, \mathrm{d}y^i \otimes \mathrm{d}y^j \\ &= \sum_{1\le i,j\le n} \widetilde{g}_{ij} \cdot \left(\sum_{k=1}^n \frac{\partial[y^i \circ \varphi^{-1}]}{\partial x^k}\, \mathrm{d}x^k\right) \otimes \left(\sum_{r=1}^n \frac{\partial[y^j \circ \varphi^{-1}]}{\partial x^r}\, \mathrm{d}x^r\right) \\ &= \sum_{1\le k,r\le n} \left(\sum_{1\le i,j\le n} \frac{\partial[y^i \circ \varphi^{-1}]}{\partial x^k} \cdot \frac{\partial[y^i \circ \varphi^{-1}]}{\partial x^r} \cdot \widetilde{g}_{ij}\right) \mathrm{d}x^k \otimes \mathrm{d}x^r \end{aligned}$$

So then because of the uniqueness of these expressions, we see that:

$$g_{kr} = \sum_{1\le i,j\le n} \frac{\partial[y^i \circ \varphi^{-1}]}{\partial x^k} \cdot \widetilde{g}_{ij} \cdot \frac{\partial[y^j \circ \varphi^{-1}]}{\partial x^r}$$

Then define:

$$A_{ik} = \frac{\partial[y^i \circ \varphi^{-1}]}{\partial x^k} \qquad\qquad A = D(\psi \circ \varphi^{-1})$$

Then we have with $G_{kr} = g_{kr}$ a matrix representation of $g$ at $p$ and likewise $\widetilde{G}_{ij} = g_{ij}$ that:

$$\begin{aligned} g_{kr} &= \sum_{1\le i,j\le n} A_{ik} \cdot \widetilde{g}_{ij} \cdot A_{jr} \\ G &= A^T \cdot \widetilde{G} \cdot A \end{aligned}$$

Great!

**Problem I-4**

Check that the coordinate frame $\frac{\partial}{\partial x^i}$ is a global orthonormal frame on $\mathbb{R}^n$. *Hint: Recall that* $\frac{\partial}{\partial x^i}(p) = e_i$.

*Remark.* The fact that the orthonormal frame is also a coordinate frame is a very special property that generally cannot be achieved. It is a reflection of the zero curvature on $\mathbb{R}^d$. So, in general, we don't expect orthonormal frames to be coordinate frames.

*Solution.* First note that by class since $(\mathbb{R}^n, \mathrm{Id})$ is a coordinate chart, $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ is a smooth global frame. Then this is orthonormal because at each $p$ we have this is just $e_1, \ldots, e_n$, which is orthonormal with respect to the standard inner product.

**Problem I-5** (Existence of Orthonormal Frames)

Show that for each $p \in M$, there is a smooth orthonormal frame on a neighborhood of $p$. *Hint: Gram-Schmidt to coordinate frame. Why is the resulting frame smooth?*

*Solution.* Let $(U, \varphi)$ be a coordinate chart around $p$ with $\varphi = (x^1, \ldots, x^n)$. Then consider the smooth frame on $U$ given by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. We may apply Gram-Schmidt to this to get vector fields $E_1, \ldots, E_n$ which is an orthonormal frame. Furthermore, these are smooth because in Gram-Schmidt we only use the inner product of vector fields, sums of vector fields, and scalar multiples of vector fields, which are all smooth operations because the metric $g$ is smooth.

**Problem I-6**

Let $V$ be an inner product vector space. The inner product gives an isomorphism between $V$ and its dual space $V^*$ as follows: Let $\omega \in V^*$, show that there exists a unique $v \in V$ such that $\omega(\cdot) = \langle \cdot, v\rangle$. Check that the map $L : V \to V^*$ given by $Lv = \langle \cdot, v\rangle$ is a linear isomorphism

*Solution.* First note that $L$ is well-defined and a linear since for $c \in \mathbb{R}$, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in V$ we have:

$$\begin{aligned} [L(v)](cw_1 + w_2) &= \langle cw_1 + w_2, v\rangle = c\langle w_1, v\rangle + \langle w_2, v\rangle = c[L(v)](w_1) + [L(v)](w_2) \\ [L(cv_1 + v_2)](w) &= \langle w, cv_1 + v_2\rangle = c\langle w, v_1\rangle + \langle w, v_2\rangle \end{aligned}$$

Great! Then $L$ is injective because if $v \in \ker L$ then $L(v) = 0$, so $\langle v, v\rangle = 0$, and then $v = 0$. Therefore since $V$ and $V^*$ have the same dimension, $L$ is a linear isomorphism, which proves the claim.

**Problem I-7**

We apply the above question to $T_pM$ of some Riemannian manifold $(M, g)$. Let $(U, \varphi)$ be a coordinate chart near $p$ and let $\varphi = (x^1, \ldots, x^n)$. Let $X \in T_pM$ be written as $X = X^i \frac{\partial}{\partial x^i}$. Check that $L(X)$ defined in the above problem (using the inner product $\langle \cdot, \cdot\rangle_g$) is given by:

$$L(X) = X_i \, \mathrm{d}x^i \text{ where } X_i := g_{ij} X^j$$

Therefore, $g_{ij}$ is the matrix of the transformation $L$. For this reason, applying the operator $L$ is sometimes called *lowering an index*

*Solution.* We apply both $L(X)$ and $X_i \, \mathrm{d}x^i$ to an element of the basis $\frac{\partial}{\partial x^k}$. If they agree then we're done since linear maps are determined by where they send the basis. We then see that:

$$\begin{aligned} L(X) \cdot \frac{\partial}{\partial x^k} &= \left\langle X, \frac{\partial}{\partial x^k} \right\rangle_g \\ &= g_{ij} \, \mathrm{d}x^i(X) \cdot \mathrm{d}x^j \left(\frac{\partial}{\partial x^k}\right) \\ &= g_{ij} X^i \cdot \delta^j_k = g_{ik} X^i = X_k \\ (X_i \, \mathrm{d}x^i) \cdot \frac{\partial}{\partial x^k} &= X_i \, \mathrm{d}x^i \left(\frac{\partial}{\partial x^k}\right) \\ &= X_i \delta^i_k = X_k \end{aligned}$$

As demonstrated, we get the same thing in either case, and so:

$$L(X) = X_i \, \mathrm{d}x^i$$

Perfect!