Handout 9

Jordan measure and Riemann Integration

It turns out that the notion of Jordon measurability of sets is intimately related (in a way essentially equivalent) to the notion of Riemann integrability of functions. We will only display this relation in dimension 1.

- **Recall.** To define the Riemann[1] integral of a <u>bounded</u> function f on an interval $[a,b] \subset \mathbb{R}$, we first recall the notion of a partition $\mathcal{P}$ which is a set of points $x_0 = a < x_1 < x_2 < \ldots < x_n = b$, the norm of the partition is $\Delta\mathcal{P} = \max_{1\leq k\leq n} x_k - x_{k-1}$, and we denote by $\Delta x_k = x_k - x_{k-1}$. For each such partition, we define to quantities:

$$L(f,\mathcal{P}) = \sum_{k=1}^{n} f(x_*)\Delta x_k, \quad \textit{and} \quad U(f,\mathcal{P}) = \sum_{k=1}^{n} f(x^*)\Delta x_k,$$

  where $x_* = \inf_{[x_{k-1},x_k]} f$ and $x^* = \sup_{[x_{k-1},x_k]} f$.

  Afterwards, we define the lower and upper Darboux integrals respectively as

$$\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(f,\mathcal{P}), \quad \textit{and} \quad \overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}} U(f,\mathcal{P}).$$

  where the extrema above are taken over all partitions of the interval $[a,b]$. We say that f is Riemann integrable if the above two numbers are equal. We define the common value as the Riemann (or Darboux) integral of f.

[1] Strictly speaking, we are recalling here the notion of Darboux integral, but that is equivalent to the notion of Riemann integrability that is often covered in introductory calculus classes.

Q1) Let $[a,b]$ be an interval and let $f : [a,b] \to \mathbb{R}$ be a bounded nonnegative function. Show that f is Riemann integrable if and only if the set $E := \{(x,t) : x \in [a,b] : 0 \leq t \leq f(x)\}$ is Jordan measurable in $\mathbb{R}^2$.

Q2) Let $[a,b]$ be an interval and let $f : [a,b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable if and only if the sets $E_+ := \{(x,t) : x \in [a,b] : 0 \leq t \leq f(x)\}$ and $E_- := \{(x,t) : x \in [a,b] : f(x) \leq t \leq 0\}$ are Jordan measurable in $\mathbb{R}^2$.

Remark. The above results generalize to higher dimensions.

Where we are right now?

We have thus far discussed the classical theory of Jordan measure, which went as follows

(i) We define the notion of a box and its volume $|B|$ or $v(B)$,

(ii) Then we defined the notion of an elementary set and its elementary measure,

(iii) Then we defined the notion of Jordan inner and outer measure $\underline{m}_J(E)$ and $\overline{m}^J(E)$ and said that a set E is Jordan measurable if those two concepts agree.

In particular, unwinding the definition of the Jordan outer measure, we have that for any set E

$$\overline{m}_J(E) = \inf_{E \subset B_1 \cup \ldots \cup B_k} |B_1| + \ldots + |B_k|$$

where the infimum is taken over all finite coverings of E by boxes $B_1, \ldots, B_k$.

Q3) Show that a set E is Jordan measurable if and only if for every $\epsilon > 0$ there exists an elementary set U containing E such that $\overline{m}_J(U \setminus E) < \epsilon$.

The notions of Lebesgue outer measure and Lebesgue measurability are refinements of the Jordan ones as follows:

– **Lebesgue outer measure:** We modify the notion of Jordan outer measure by replacing the finite union of boxes by a countable union of boxes, i.e.

$$m^*(E) = \inf_{E \subset \cup_{j=1}^{\infty} B_j} \sum_{j=1}^{\infty} |B_j|$$

where the union above is taken over boxes $B_j \subset \mathbb{R}^d$.

Q4) Show that the Lebesgue outer measure $m^*(E)$ is zero for any countable set E. Contrast this to fact that the Jordan outer measure of the rationals in $[0, 1]$ was equal to 1.

– **Lebesgue measurability** A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing E such that $m^*(U \setminus E) \leq \epsilon$. If E is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of E.

Remarks: Note that there is no need for E to be bounded for this definition to make sense. Also, the notion of Lebesgue measurability can be seen as a (finite to countably infinite) generalization of that of Jordan measurability since it can be shown that every open set is the countable union of closed boxes.

Handout 10

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  (i) We define the notion of a box and its volume $|B|$ or $v(B)$,
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$$\overline{m}_J(E) = \inf_{E \subset B_1 \cup \ldots \cup B_k} |B_1| + \ldots + |B_k|$$

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Lebesgue outer measure

The notions of Lebesgue outer measure and Lebesgue measurability are refinements of the Jordan ones as follows:

- **Lebesgue outer measure:** We modify the notion of Jordan outer measure by replacing the finite union of boxes by a countable union of boxes, i.e.

$$m^*(E) = \inf_{E \subset \cup_{j=1}^{\infty} B_j} \sum_{j=1}^{\infty} |B_j|$$

where the union above is taken over boxes $B_j \subset \mathbb{R}^d$.

Q1) Show that $m^*(E) \leq \overline{m}_J(E)$ where $\bar{m}_J$ is the Jordan outer measure.

Q2) Show that in the definition above the countable cover by boxes in the definition of $m^*(E)$ can be restricted to closed boxes or open boxes.

Q3) Show that the Lebesgue outer measure $m^*(E)$ is zero for any countable set E. Contrast this to fact that the Jordan outer measure of the rationals in $[0, 1]$ was equal to 1.

- **Lebesgue measurability** A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing E such that $m^*(U \setminus E) \leq \epsilon$. If E is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of E.

  Remarks:

  (i) Note that there is no need for E to be bounded for this definition to make sense.

  (ii) The notion of Lebesgue measurability can be seen as a (finite to countably infinite) generalization of that of Jordan measurability since it can be shown that every open set is the countable union of closed boxes.

Q4) Show that $m^*(\emptyset) = 0$.

Q5) (Monotonicity) Show that if $E \subset F \subset \mathbb{R}^d$, then $m^*(E) \leq m^*(F)$.

Q6) (Countable subadditivity) If $E_1, E_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of sets, then $m^*\left(\cup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

MATH 395 Notes

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November 13, 2020

Exercise 0. *Show that a set E is Jordan measurable if and only if for every $\varepsilon > 0$ there exists an elementary set U containing E such that $\overline{m}_J(U \setminus E) < \varepsilon$.*

Proof. **TODO**

Exercise 1. *Show that $m^*(E) \leq \overline{m}_J(E)$ where $\overline{m}_J$ is the Jordan outer measure*

Proof. Fix some elementary set A which contains E and write it as the disjoint union of a finite collection of boxes $B_1, \ldots, B_n$ that cover E. Then note that:

$$m^*(E) = \inf_{E \subseteq \bigcup_{j=1}^\infty C_j} \sum_{j=1}^\infty |C_j| \leq \sum_{j=1}^n |B_j| = m(A)$$

And so taking the infimum over all elementary sets A containing E we obtain:

$$m^*(E) \leq \overline{m}_J(E)$$

Just as desired.

Exercise 2. *Show that in the definition above the countable cover by boxes in the definition of $m^*(E)$ can be restriced to closed boxes or open boxes*

Proof. We deal with closed boxes first. Consider the sets:

$$S = \left\{ \sum_{j=1}^{\infty} |B_j| \mid E \subseteq \bigcup_{j=1}^{\infty} B_j \right\}$$
$$S_c = \left\{ \sum_{j=1}^{\infty} |B_j| \mid E \subseteq \bigcup_{j=1}^{\infty} B_j, B_j \text{ closed} \right\}$$
$$S_o = \left\{ \sum_{j=1}^{\infty} |B_j| \mid E \subseteq \bigcup_{j=1}^{\infty} B_j, B_j \text{ open} \right\}$$

We know that $m^*(E) = \inf S$ and we wish to show that $\inf S = \inf S_c = \inf S_o$. Now note that of course $S_o, S_c \subseteq S$, and so $\inf S \leq \inf S_c, \inf S_o$, therefore it only remains to show that $\inf S \geq \inf S_c, \inf S_o$.

To do so, by definition of greatest lower bound, it suffices to show that $\inf S_c$ and $\inf S_o$ are both lower bounds for S. We handle each of these:

- Take some countable collection of boxes $B_1, B_2, \ldots$ such that their union contains E, giving us an element $\sum |B_j|$ of S. Then we may consider the collection of their closures $\overline{B}_1, \overline{B}_2, \ldots$. Since $B_j \subseteq \overline{B}_j$ we know that the union of all these contains E. So then $\sum \left|\overline{B}_j\right| \in S_c$. But then we are in a great spot! We know $\left|\overline{B}_j\right| = |B_j|$. So then we may write:

  $$\inf S_c \leq \sum_{j=1}^{\infty} \left|\overline{B_j}\right| \leq \sum_{j=1}^{\infty} |B_j|$$

  And so $\inf S_c$ is a lower bound for S, and so $\inf S_c \leq \inf S$ as desired.

- Take some countable collection of boxes $B_1, B_2, \ldots$ whose union contains E, giving us an element $\sum |B_j|$ of S. We will show for any $\varepsilon > 0$ that:

  $$\inf S_o \leq \varepsilon + \sum_{j=1}^{\infty} |B_j|$$

  And so taking $\varepsilon \to 0$ we see that $\inf S_o$ is a lower bound for S and so $\inf S_o \leq \inf S$ as desired.

  Fix some such $\varepsilon > 0$, and consider the open box C_j obtained from B_j by dilating B_j so that $|C_j| \leq |B_j| + \frac{\varepsilon}{2^j}$ and $B_j \subseteq C_j$. Then $\sum C_j$ lies in S_o since the union of all the C_j contains E. But then:

$$\inf S_o \leq \sum_{j=1}^{\infty} |C_j| \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} + \sum_{j=1}^{\infty} |B_j| = \varepsilon + \sum_{j=1}^{\infty} |B_j|$$

Taking $\varepsilon \to 0$ we see that $\inf S_o \leq \sum |B_j|$, and so $\inf S_o$ is a lower bound for S, giving us that $\inf S_o \leq \inf S$ as desired.

With this we are done! $m^*(E) = \inf S = \inf S_o = \inf S_c$. Great!

Exercise 3. *Show that the Lebesgue outer measure $m^*(E)$ is zero for any countable set E. Cotnrast this to the face that the Jordan outer measure of the rationals in $[0,1]$ was equal to 1*

Proof. Let E be a countable set. Then consider that:

$$E \subseteq \bigcup_{x \in E} \{x\}$$

exhibits E as a countable union of boxes, all of measure zero. Therefore:

$$0 \leq m^*(E) \leq \sum_{x \in E} |\{x\}| = 0$$

Showing us that $m^*(E) = 0$.

Let's look for another way of doing this! Write E as $x_1, x_2, \ldots$. We will allow repeats here, and if E is empty just repeat $x_n = 0$. Fix $\varepsilon >$) and then take the box whose volume is $\frac{\varepsilon}{2^j}$ around every point $x_j = (x_{j1}, \ldots, x_{jd})$. In other words:

$$B_j = \prod_{k=1}^{d} \left[x_{jk} - \frac{\sqrt[d]{\varepsilon}}{2\sqrt[d]{2^j}}, x_{jk} + \frac{\sqrt[d]{\varepsilon}}{2\sqrt[d]{2^j}} \right]$$

$$|B_j| = \prod_{k=1}^{d} \frac{\sqrt[d]{\varepsilon}}{\sqrt[d]{2^j}} = \frac{\varepsilon}{2^j}$$

$$\sum_{j=1}^{\infty} |B_j| = \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

Great! Since $E \subseteq \bigcup_{j=1}^{\infty} B_j$ this means that:

$$0 \leq m^*(E) \leq \sum_{j=1}^{\infty} |B_j| = \varepsilon$$

Now taking $\varepsilon \to 0$ we get $m^*(E) = 0$.

Exercise 4. *Show that $m^*(\emptyset) = 0$.*

Proof. Note that $\emptyset$ is a countable set, so this follows easily from Q3

Exercise 5. *Show that if $E \subseteq F \subseteq \mathbb{R}^d$ then $m^*(E) \leq m^*(F)$.*

Proof. We will show that $m^*(E)$ is a lower bound for the set defining $m^*(F)$, and so by definition of infimum we have $m^*(E) \leq m^*(F)$.

Fix some countable collection of boxes $B_1, B_2, \ldots$ containing F, then in particular they contain E since F contains E, and so by definition of infimum:

$$m^*(E) \leq \sum_{j=1}^{\infty} |B_j|$$

Taking the infimum on the right hand side we get:

$$m^*(E) \leq m^*(F)$$

Great! This is exactly what we want!

Exercise 6. *If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$ is a countable sequence of sets, then:*

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$$

Great!

Proof. Fix some $\varepsilon > 0$, we will show that:

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \varepsilon + \sum_{n=1}^{\infty} m^*(E_n)$$

and so by taking $\varepsilon \to 0$ we will obtain the result. Take $E = \bigcup_{n=1}^{\infty} E_n$ for convenience.

Consider some E_n, then by definition of infimum and the fact that $\frac{\varepsilon}{2^n} > 0$ there is some countable collection of boxes $B_{n1}, B_{n2}, \ldots$ containing E_n such that:

$$m^*(E_n) \leq \sum_{j=1}^{\infty} |B_{nj}| \leq m^*(E_n) + \frac{\varepsilon}{2^n}$$

We can then sum over all E_n to get:

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |B_{nj}| \leq \sum_{n=1}^{\infty} m^*(E_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon + \sum_{n=1}^{\infty} m^*(E_n)$$

And so now consider the countable collection of all the $\{B_{nj}\}$. This will be countable by 295, and also it will cover E, since for every $x \in E$ we know $x \in E_n$ for some n and then by construction $x \in B_{nj}$ for some j. But then by definition of infimum:

$$m^*(E) \leq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |B_{nj}| \leq \varepsilon + \sum_{n=1}^{\infty} m^*(E_n)$$

Since $\varepsilon > 0$ was chosen to be arbitrary, we can take $\varepsilon \to 0$ and we see that:

$$m^* \left(\bigcup_{n=1}^{\infty} E_n \right) = m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n)$$

Great! This is the desired result ☺.