Handout 12

Where we are right now?

- **Recall:** The notion of Lebesgue outer measure of a set E:

$$m^*(E) = \inf_{E \subset \cup_{j=1}^{\infty} B_j} \sum_{j=1}^{\infty} |B_j|$$

where the union above is taken over boxes $B_j \subset \mathbb{R}^d$. A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing E such that $m^*(U \setminus E) \leq \epsilon$. If E is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of E.

We saw last time we showed that if E is an elementary set, then $m^*(E) = m(E)$ where $m(E)$ is the elementary measure of E defined before. Today, we generalize this as follows:

- We say that two boxes are *almost disjoint* if their interiors are disjoint. The above result implies that $B_1, \ldots, B_k$ are almost disjoint, then

$$m(B_1 \cup B_2 \ldots \cup B_k) = \sum_{k=1}^{k} |B_k|$$

(since the elementary measure of a box is the same as that of its interior).

Q1) Let $E = \cup_{n=1}^{\infty} B_n$ be a countable union of almost disjoint boxes B_k. Show that

$$m^*(E) = \sum_{k=1}^{\infty} |B_k|.$$

As such, $\mathbb{R}^d$ for example has infinite outer measure.

Q2) Show that any open subset E of $\mathbb{R}^d$ can be written as the countable union of almost disjoint boxes (even countable union of almost disjoint closed cubes). *Hint: Consider the so-called "dyadic cubes" of the form: Let $j \in \mathbb{Z}$*

$$[\frac{i_1}{2^j}, \frac{i_1+1}{2^j}] \times [\frac{i_2}{2^j}, \frac{i_2+1}{2^j}] \times \ldots \times [\frac{i_d}{2^j}, \frac{i_d+1}{2^j}], \qquad i_1, i_2, \ldots, i_d \in \mathbb{Z}.$$

For each $j \in \mathbb{Z}$, those dyadic cubes give a covering of $\mathbb{R}^d$ by cubes of side-length 2^{-j}. Also note that each such cube, is contained in a unique dyadic cube (called its parent) of side-length 2^{-j+1} and is the union of 2^d dyadic cubes (its children) of side-length 2^{-j-1}. Such cubes also have the nice property that given any two dyadic cubes, then they are either almost disjoint or one is contained in the other. Here, it's useful to only worth with cubes of side-length ≤ 1 (i.e. with $j \leq 0$).

Q3) Let $E \subset \mathbb{R}^d$ be an arbitrary set. Show that

$$m^*(E) = \inf_{E \subset U, U \, open} m^*(U).$$

This is called *outer regularity.*

Q4) Give an example of a set $E \subset \mathbb{R}^d$ such that the reverse statement

$$m^*(E) = \sup_{U \subset E, U \, open} m^*(U).$$

We will see that the right version of inner regularity is obtained by approximating the set E by compact sets contained in it.