MATH 395 Notes

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1 Introduction

- Office Hours
  - Monday 8-9am
  - Wednesday 4-5pm
  - Beginning of Friday lecture
- First HW will be posted on Friday
- TAs are still not decided on
- Further info on the waitlist to come

2 Continuing Metric Spaces

2.1 Last Time

We defined metrics $d : X \times X \to [0, \infty)$ with three special properties, and we saw that this gave us a topology on X.

- Open sets, given $p \in \mathcal{O}$ we have some $\delta > 0$ so that $N_\delta(p) \subseteq \mathcal{O}$.
- Closed sets are the complements of open sets
- Limit points, p is a limit point of E if every δ-neighborhood of p intersects E in a point $q \neq p$
- Closed sets are exactly the sets where every limit point belongs to the set.

2.2 Closures!

Definition. *If X is a metric space and $E \subseteq X$ we denote by E' the set of limit points of X. The closure of E is the set $\overline{E} = E \cup E'$.*

Example. Here are some examples to look at!

- Let $E = (0,1] \subseteq \mathbb{R}$ then $E' = [0,1] = \overline{E}$
- Let $E = (0,1] \cup \{2\} \subseteq \mathbb{R}$. Then $E' = [0,1]$ and $\overline{E} = [0,1] \cup \{2\}$.

Theorem. *Let X be a metric space and $E \subseteq X$. Then:*

a) $\overline{E}$ is closed

b) $E = \overline{E}$ if and only if E is closed.

c) If $E \subseteq F$ and F is closed then $\overline{E} \subseteq F$.

Proof. Let's go!

a) Let $q \in \overline{E}^c$. Then $q \notin E' \cup E$. Thus there exists a $\delta > 0$ so that $N_\delta(q) \cap E = \emptyset$. Since $N_\delta(q)$ is open we also know that $N_\delta(q) \cap E' = \emptyset$. Therefore $N_\delta(q) \cap \overline{E} = \emptyset$ and so $\left(\overline{E}\right)^c$ is open as desired.

b) Easy exercise

c) If $E \subseteq F$ and F is closed, then $E' \subseteq F$ because any limit point of E is also a limit point of F. Therefore $\overline{E} \subseteq F$.

Theorem. *Let E be a nonempty set of real numbers which is bounded above. Then $y = \sup E$ is in $\overline{E}$. Hence $y \in E$ if E is closed.*

Proof. If $y \in E$ then we are done because $E \subseteq \overline{E}$. If $y \notin E$ then for any $\varepsilon > 0$ there exists some $x \in E$ so that:

$$y - \varepsilon < x < y$$

But this means that $x \in N_\varepsilon(y)$, and so $N_\varepsilon(y) \cap E \neq \emptyset$. This implies that $y \in E'$, and so we are done since $E' \subseteq \overline{E}$.

2.3 Compact subsets of metric spaces

Definition. *We need a couple definitions!*

- *An open cover of a set E in a metric space X is a collection $\{G_\alpha\}_{\alpha \in A}$ of open sets such that:*

$$E \subseteqq \bigcup_{\alpha \in A} G_\alpha$$

- *A subset $E \subseteq X$ is called compact provided that every open cover of E admits a finite subcover. That is we can find a finite subcollection $\{G_{\alpha_i}\}_{1 \leq i \leq n}$ of $\{G_\alpha\}_{\alpha \in A}$ such that $\{G_{\alpha_i}\}_{1 \leq i \leq n}$ covers E.*

Theorem. *Compact subsets of metric spaces are closed and bounded*

Proof of Closed. Let $K \subseteq X$ be compact and let $q \in K^c$. For each $p \in K$ there exists two subsets U_p and W_p such that $p \in U_p$, $q \in W_p$ and $U_p \cap W_p = \emptyset$. Here we use that metric spaces are Hausdorff. We can concretely take $U_p = N_\delta(p)$ and $W_p = N_\delta(q)$ with $\delta < \frac{1}{2}d(p,q)$.

Then in fact $\{U_p\}_{p \in K}$ is an open cover of K. By compactness there exists a finite subcover $U_{p_1}, \ldots U_{p_n}$ that covers K. Then let:

$$W = \bigcap_{i=1}^{n} W_{p_i}$$

Then this W is open and $W \cap U_{p_j} = \emptyset$ for all $1 \leq j \leq n$. Thus we must have $W \cap K = \emptyset$, meaning that $W \subseteq K^c$ and K^c is open.

Proof of Boundedness. Let $x \in X$ be arbitrary. The family of sets $\{N_n(x)\}_{n \in \mathbb{N}}$ is an open cover of E since $\mathbb{N}$ is unbounded. Thus by compactness E has a finite subcover, and so $E \subseteq N_k(x)$ for some $k \in \mathbb{N}$.

The main question for the rest of this section: Is the converse true? If not, what should be a workable criterion for compactness in metric spaces?

In fact it is true on $\mathbb{R}^n$ by Heine-Borel. But not the converse, particularly in infinite dimensions!

Theorem. *Closed subsets of compact sets are compact.*

Proof. Let $C \subseteq K$ be a closed subset of a compact set K and let $\{G_\alpha\}_{\alpha \in A}$ be an open cover of C. Then $\{G_\alpha\}_{\alpha \in A} \cup C^c$ is an open cover for K. Thus by compactness of K there exists $\alpha_1, \ldots, \alpha_n$ such that:

$$K \subseteq C^c \cup \bigcup_{i=1}^{n} G_{\alpha_i}$$
$$C \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

Therefore C is comapact.

Theorem (Finite intersection property)**.** *If $\{K_\alpha\}_{\alpha \in A}$ is a collection of compact sets such that the intersection of any finite subcollection of $\{K_\alpha\}_{\alpha \in A}$ is nonempty. Then, the intersection $\bigcap_{\alpha \in A} K_\alpha$ is nonempty*

Example. If $E_n = \left(0, \frac{1}{n}\right]$ then E_n has the finite intersection property since they are nested and each of them are nonempty. But $\bigcap_{n \in \mathbb{N}} E_n = \emptyset$.

Proof. Suppose that $\bigcap_{\alpha \in A} K_\alpha = \emptyset$. Then $\bigcup_{\alpha \in A} K_\alpha^c = X$, and so $\{K_\alpha^c\}_{\alpha \in A}$ is an open cover for $K_{\alpha_\star}$ where $\alpha_\star \in A$ is arbitrary. This holds because compact subsets of metric spaces are closed.

By compactness of $K_{\alpha_\star}$ there exists some $\alpha_1, \ldots, \alpha_n$ such that:

$$K_{\alpha_\star} \subseteq \bigcup_{i=1}^{n} K_{\alpha_i}^c$$

Thus the finite intersection:

$$K_{\alpha_\star} \cap \bigcap_{i=1}^{n} K_{\alpha_i} = \emptyset$$

This contradicts the finite intersection property. Oops! We win.

Theorem 1 (Compactness $\implies$ sequential compactness)**.** *Let K be a compact set and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in K. Then there exists a convergent subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ that converges to a point in K.*

Proof. Suppose that $\{x_n\}$ has no limit point in K. This means that for any $p \in K$, there exists some δ_p such that $N_{\delta_p}(p)$ contains at most one point of the sequence $\{x_n\}$. The collection $\{N_{\delta_p}(p)\}_{p \in K}$ is an open cover of K.

By compactness we have some $p_1, \dots, p_n$ such that:

$$K \subseteq \bigcup_{i=1}^{n} N_{\delta_{p_i}}(p_i)$$

But this must mean that K contains at most n points of the sequence $\{x_n\}$. This means that $\{x_n\}$ takes at most n values. Thus x_n must take one value infinitely many times, and so x_n has a convergent subsequence.

On the other hand if $\{x_n\}$ has a limit point $p \in K$, then for every $k \in \mathbb{N}$ there exists some x_{n_k} such that $d(x_{n_k}, p) < \frac{1}{k}$. Clearly $\{x_{n_k}\}$ is a convergent subsequence and so we win.

Remark. Is the converse true? Yes! But only in metric spaces.

3 Compactness in $\mathbb{R}^n$

Theorem 2 (Nested interval property on $\mathbb{R}$)**.** *Suppose that $I_n = [a_n, b_n]$ is a nested sequence of closed intervals, that is $I_n \supseteq I_{n+1}$. Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty*

Proof. We know $\{a_n\}$ is an increasing sequence thta is bounded by b_j. Let $x = \sup_{n \in \mathbb{N}} a_n$. Then $a_n \leq x$ for all n.

Also $\{b_n\}$ is decreasing so $a_n \leq b_n \leq b_m$ for all $n \geq m$. Taking the supremum in n we get $x \leq b_m$ for all m. Therefore $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$, giving us that:

$$x \in \bigcap_{i=1}^{\infty} I_n$$