Handout 10

Where we are right now?

- **Lebesgue outer measure:** We modify the notion of Jordan outer measure by replacing the finite union of boxes by a countable union of boxes, i.e.
$$m^*(E) = \inf_{E \subset \cup_{j=1}^{\infty} B_j} \sum_{j=1}^{\infty} |B_j|$$
where the union above is taken over boxes $B_j \subset \mathbb{R}^d$. We saw last time that this is smaller than the Jordan outer measure and that the boxes above can be taken to be open or closed. We also saw that any countable set has zero Lebesgue outer measure.

- **Lebesgue measurability** A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing E such that $m^*(U \setminus E) \leq \epsilon$. If E is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of E.

  We saw last time some properties of this definition:

  - Show that $m^*(\emptyset) = 0$.
  - (Monotonicity) Show that if $E \subset F \subset \mathbb{R}^d$, then $m^*(E) \leq m^*(F)$.
  - (Countable subadditivity) If $E_1, E_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of sets, then $m^*\left(\cup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

  A natural question is whether one has that an additivity property for the outer measure: namely that if E, F are disjoint sets then $m^*(E \cup F) = m^*(E) + m^*(F)$? While this turns out to be correct for some sets E and F (to be called Lebesgue-measurable sets), we already saw at the start of our discussion of measures that this cannot hold for general sets (cf. the Banach-Tarski paradox). The enemy here is that we might have the two sets E and F too intertwined or entangled together which can cause the additivity property to fail.

Q1) Show that if $\operatorname{dist}(E, F) > 0$, then $m^*(E \cup F) = m^*(E) + m^*(F)$.

Q2) Show that if E is an elementary set, then $m^*(E) = m(E)$ where $m(E)$ is the elementary measure of E defined before.

Q3) Conclude that if E is any bounded set, then $\underline{m}(E) \leq m^*(E) \leq \overline{m}(E)$ where $\underline{m}(E)$ and $\overline{m}(E)$ are the inner and outer Jordan measures of E.

Q4) Construct a bounded open subset U of $\mathbb{R}$ that is not Jordan measurable. *Hint: Start with an enumeration of the rationals in* $[0, 1]$ *and create an open set whose Lebesgue outer-measure is arbitrarily small but the Jordan outer measure is* ≥ 1.