MATH 395 Notes

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Continuing the characterization of Jordan Measurability

Theorem. *Let S be a bounded subset of $\mathbb{R}^n$. The following are equivalent:*

1) S is Jordan measurable

2) The constant function 1 is Riemann Integrable on S

3) ∂S has Lebesgue measure zero

4) ∂S has Jordan outer measure zero.

Proof. Let's go!

$1 \implies 2)$ Suppose S is Jordan measurable. We need to show that:

$$f_S(x) = \mathbb{1}_S = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

is Riemann integrable on some box B containing S. Now let $\varepsilon > 0$ be arbitrary and pick two elementary sets $E_1 \subseteq S \subseteq E_2$ such that $m(E_2 \setminus E_1) < \varepsilon$. Without loss of generality, by dilating the component boxes of E_2 we may assume that $S \subseteq E_2^\circ$.

Choose B to be some box containing E_2. Now let P be a partition B that contains the endpoints of the intervals defining the boxes whose union is E_1 and E_2. Let $R_1, \ldots, R_m$ be some enumeration of the sub-boxes determined by this partition. Then:

$$\begin{aligned}
U(\mathbb{1}_S, P) &= \sum_{i=1}^{m} M_{R_i}(\mathbb{1}_S) v(R_i) \\
&= \sum_{R_i \cap S \neq \emptyset} M_{R_i}(\mathbb{1}_S) v(R_i) \\
&\leq \sum_{R_i \subseteq E_2} M_{R_i}(\mathbb{1}_S) v(R_i) \\
&\leq \sum_{R_i \subseteq E_2} v(R_i) = m(E_2)
\end{aligned}$$

Similarly, we can show that $L(\mathbb{1}_S, P) \geq m(E_1)$. But then:

$$U(\mathbb{1}_S, P) - L(\mathbb{1}_S, P) \leq m(E_2) - m(E_1) = m(E_2 \setminus E_1) < \varepsilon$$

Great! Therefore $\mathbb{1}_S$ is integrable and:

$$m(E_1) \leq L(\mathbb{1}_S, P) \leq \int_S 1 \, \mathrm{d}x \leq U(\mathbb{1}_S, P) \leq m(E_2)$$

and:

$$m(E_1) \leq m(S) \leq m(E_2)$$

Gives us that:

$$\left| \int_S 1 \, \mathrm{d}x - m(S) \right| < \varepsilon$$

For any $\varepsilon > 0$, and therefore:

$$m(S) = \int_S 1 \, \mathrm{d}x$$

$2 \implies 1)$ Let B be a box which contains S and take $\varepsilon > 0$ to be arbitrary. Since $\mathbb{1}_S$ is integrable on B, there exists a partition P of B such that:

$$U(\mathbb{1}_S, P) - L(\mathbb{1}_S, P) < \varepsilon$$

Let $R_1, \ldots, R_m$ be an enumeration of the sub-boxes determined by P. Now set:

$$E_1 = \bigcup_{R_i \subseteq S} R_i \subseteq S$$
$$E_2 = \bigcup_{R_i \cap S \neq \emptyset} R_i \supseteq S$$

And then we see that:

$$\begin{aligned}
U(\mathbb{1}_S, P) &= \sum_{i=1}^{m} M_{R_i}(\mathbb{1}_S) v(R_i) \\
&= \sum_{R_i \cap S \neq \emptyset} M_{R_i}(\mathbb{1}_S) v(R_i) \\
&= \sum_{R_i \cap S \neq \emptyset} v(R_i) = m(E_2) \\
L(\mathbb{1}_S, P) &= \sum_{i=1}^{m} m_{R_i}(\mathbb{1}_S) v(R_i) \\
&= \sum_{R_i \subseteq S} m_{R_i}(\mathbb{1}_S) v(R_i) \\
&= \sum_{R_i \subseteq S} v(R_i) = m(E_1)
\end{aligned}$$

Therefore!

$$m(E_2 \setminus E_1) = m(E_2) - m(E_1) = U(\mathbb{1}_S, P) - L(\mathbb{1}_S, P) < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we conclude that S is Jordan measurable.

$2 \iff 3)$ This is straightforward using our characterization of integrability and the fact that $\mathbb{1}_S$ is discontinuous exactly at the points on the boundary of S.

$3 \implies 4)$ Let $\varepsilon > 0$. Since ∂S has Lebesgue measure zero there is a collection of boxes $B_1, B_2, \ldots$ such that $\partial S \subseteq \bigcup_{j=1}^{\infty} B_j$ and $\sum v(B_j) < \frac{\varepsilon}{2}$. Dilate each B_j into a larger open box $\widetilde{B}_j$ such that $B_j \subseteq \widetilde{B}_j$ and $v(\widetilde{B}_j) < 2v(B_j)$.

Now note that the $\widetilde{B}_j$ forms an open cover of the closed and bounded set ∂S.

By compactness there is a finite sub-cover $\widetilde{B}_{j_1}, \ldots, \widetilde{B}_{j_k}$ of ∂S. But then:

$$\sum_{i=1}^{k} v(\widetilde{B}_{j_k}) \leq \sum_{j=1}^{\infty} v(\widetilde{B}_j) < 2 \sum_{j=1}^{\infty} v(B_j) < \varepsilon$$

Great! This shows that ∂S has Jordan outer measure zero.

$4 \implies 3)$ follows trivially.

Improper Integrals

Up until now in the discussion of $\int_S f$ we restricted to the case where f and S are both bounded. In this section we relax these assumptions a bit to include any open set S and any continuous function f.

Remark. The ultimate dispensing of those two restrictions on S and f comes through the theory of Lebesgue integration.

Before we proceed, we introduce some notation:

- Let $\mathcal{J}$ denote the family of Jordan measurable subsets of $\mathbb{R}^n$.
- Let $\mathcal{J}_c$ denote the collection of compact Jordan measurable sets
- For a function $f : S \to \mathbb{R}$ we define the positive part and negative part of f as:

$$f_+(x) = \max(f(x), 0) \qquad\qquad f_-(x) = \max(-f(x), 0)$$

  It is easy to veritfy that:

  - $f = f_+ - f_-$
  - $f_+, f_- \geq 0$
  - $|f| = f_+ + f_-$.
  - If f is continous then both f_+ and f_- are continuous.

Definition. *Let A be an open subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a continuous function*

- *If f is non-negative on A we define the (extended) integral of f over A as:*

$$\int_A f = \sup_{\substack{\mathscr{D}\subseteq A \\ \mathscr{D}\in\mathscr{J}_c}} \int_{\mathscr{D}} f$$

  provided that this supremum exists.

- *If f is an arbitrary continuous function on A, write $f = f_+ - f_-$, where these are the positive and negative part of f. Provided that f_+ and f_- are integrable on A in the extended sense we say f is also integrable and let:*

$$\int_A f = \int_A f_+ - \int_A f_-$$

Remark. We now have two difference definitons of $\int_A f$ when A is open and bounded and f is continuous and bounded. We shall see later that these two definitions are equivalent if both integrals exist. The extended integral might exist without having the traditional integrals existince Why?

Notice that if $B \subseteq A$ are both open then if the extended integral of f over A exists then the extended integral of f over B exists and:

$$\int_B f \leq \int_A f$$

However if $f = 1$ then $\int_B 1$ exisrts only when B is Jordan measurable, and tehre are bounded open sets that are not Jordan measurable (we'll see an example in our Friday sessions)

Convention: If A is open and f is continuous then $\int_A f$ will always denote the extended integral

Lemma. *Let $A \subseteq \mathbb{R}^n$ be open. There exists a sequence of $C_1, C-2, \ldots$ of compact Jordan measurable sets such that $A = \bigcup_{i=1}^{\infty} C_j$ and $C_j \subseteq C_{j+1}^{\circ}$. In fact, C_j can be taken to be elementary*

Proof. Define:

$$\mathscr{D}_N = \{x \in \mathbb{R}^n \mid d(x, A^c) \geq \frac{1}{N}, |x| \leq N\}$$

Thus $\mathscr{D}_N$ is bounded and closed since $x \mapsto d(x, A^c)$ and $x \mapsto |x|$ are both continuous functions. Now consider:

$$A_{N+1} = \{x \in \mathbb{R}^n \mid d(x, A^c) > \frac{1}{N+1}, |x| < N+1\}$$

And then A_{N+1} is open and:

$$\mathscr{D}_N \subseteq A_{N+1} \subseteq \mathscr{D}_{N+1}$$

This implies that:

$$\mathscr{D}_N \subseteq \mathscr{D}_{N+1}^{\circ}$$

We clearly have by the fact that A is open that:

$$A = \bigcup_{N=1}^{\infty} \mathscr{D}_N$$

The sets $\mathscr{D}_N$ may not be Jordan measurable. To fix this, note that for $x \in \mathscr{D}_N$ there exists a closed cube centered at x and contained in $\mathscr{D}_{N+1}^{\circ}$. The interior of these cubes is an open cover of $\mathscr{D}_N$ and hence by compactness there is a fintie subcover. Define C_N to be the elementary set given by the finite union of such a finite subcover of $\mathscr{D}_N$ made up of closed cubes. Thus C_N is closed and bounded, and furthermore:

$$D_N \subseteq C_N^{\circ} \subseteq C_N \subseteq \mathscr{D}_{N+1}^{\circ} \subseteq C_{N+1}^{\circ}$$

Therefore we see that C_N is compact and Jordan measurable as well as the fact that $\bigcup_{N=1}^{\infty} C_N = A$. Great! This finishes the proof.

Theorem. *Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}$ be a continuous function. Choose a sequence $C_N \in \mathscr{J}_c$ such that $A = \bigcup_{N=1}^{\infty} C_N$ and $C_N \subseteq C_{N+1}^{\circ}$ as in the above lemma. Then f is integrable over A if and only if $\int_{C_N} |f|$ is bounded by a constant which does not depend on N. In this case,*

$$\int_A f = \lim_{N \to \infty} \int_{C_N} f$$

In particular, f is integrable over A if and onl if $|f|$ is too.

We'll prove this theorem next time. In the meantime, here are some properties of the extended integral. For setup let $A \subseteq \mathbb{R}^n$ be open and let $f, g : A \to \mathbb{R}$ be continuous functions such that $\int_A f$ and $\int_A g$ exist:

a) $f + cg$ is integrable for any $c \in \mathbb{R}$ and:

$$\int_A f + cg = \int_A f + c \int_A g$$

b) If $f \leq g$ then:

$$\int_A f \leq \int_A g$$

In particular:

$$\left| \int_A f \right| \leq \int_A |f|$$

c) If A and B are both open and $A \subseteq B$ then if f is integrable over B then f is integrable over A. Furthermore if f is non-negative on B then:

$$\int_A f \leq \int_B f$$

d) If A and B are open and f is continuous on $A \cup B$, then if f is integrable on A and B then f is integrable on $A \cup B$ and $A \cap B$. Furthermore we have:

$$\int_{A \cup B} f = \int_A f + \int_B f - \int_{A \cap B} f$$