MATH 395 Notes

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Theorem. *Suppose $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$ is bounded. Then f is Riemann integrable if and only if the set $\mathscr{D}$ of discontinuities of f has Lebesgue measure zero.*

Proof. We've already proved the $\Longleftarrow$ direction in class. We are in the process of proving the $\Longrightarrow$ direction using the properties of osc, which we defined at a point $x \in B$ as follows:

$$\begin{aligned}
\operatorname{osc}_\delta f(x) &:= \sup_{y,y' \in B(x,\delta)\cap B} f(y) - f(y') \qquad (\delta > 0) \\
&:= \sup_{B(x,\delta)\cap B} f - \inf_{B(x,\delta)\cap B} f \\
\operatorname{osc} f(x) &:= \inf_{\delta > 0} \operatorname{osc}_\delta f(x) = \lim_{\delta \to 0} \operatorname{osc}_\delta f(x)
\end{aligned}$$

This holds because $\operatorname{osc}_\delta f(x)$ is increasing in δ.

Exercise. *Verify the properties of* osc *and* $\operatorname{osc}_\delta$*:*

a) $\operatorname{osc}_\delta f(x) = \sup_{B(x,\delta)} \cap B - \int_{B(x,\delta)} f \geq 0$

b) $\operatorname{osc}_\delta f$ *is increasing with* δ

c) f *is continuous at* $x \in B \iff \operatorname{osc} f(x) = 0$

Now we are ready to show that if f is Riemann integrable on B then $\mathscr{D}$ has Lebesgue measure zero:

$$\mathscr{D}_m := \left\{ x \in B \mid \operatorname{osc} f(x) \geq \frac{1}{m} \right\}$$

$$\mathscr{D} = \{x \in B \mid \operatorname{osc} f(x) > 0\} = \bigcup_{m=1}^{\infty} \mathscr{D}_m$$

Since $\mathscr{D}$ is a countable union of the $\mathscr{D}_m$, it suffices to show that each $\mathscr{D}_m$ has Lebesgue measure zero.

Let $\varepsilon > 0$ be arbitrary. We will cover $\mathscr{D}_m$ by countably many boxes whose total volume is less than ε. Note that since f is integrable we can find a partition P of B such that:

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2m}$$

We now write that $\mathscr{D}_m = \mathscr{D}'_m \cup \mathscr{D}''_m$ where:

$$\begin{aligned}
\mathscr{D}'_m &= \{x \in \mathscr{D}_m \mid x \in \partial R \text{ for some sub-box } R \text{ determined by } P\} \\
\mathscr{D}''_m &= \mathscr{D}_m \setminus \mathscr{D}'_m
\end{aligned}$$

Note that $\mathscr{D}'_m \subseteq \bigcup_R \partial R$ where R ranges over the finitely many sub-boxes determined by P. Therefore, since we saw last time that the boundary of any box has Lebesgue measure zero, we know $\mathscr{D}'_m$ has Lebesgue measure zero. Of course we can then cover $\mathscr{D}'_m$ by countably many boxes whose total volume is less than $\frac{\varepsilon}{2}$

It remains to cover $\mathscr{D}''_m$ by countably many boxes of total volume less than $\frac{\varepsilon}{2}$. First note that if $x \in \mathscr{D}''_m$ then:

$$\begin{aligned}
&\operatorname{osc} f(x) \geq \frac{1}{2m} \\
&x \in R^\circ \text{ for some sub-box } R \text{ determined by the partition}
\end{aligned}$$

Therefore there exists a $\delta > 0$ so that $B(x,\delta) \subseteq R$ and:

$$\begin{aligned}
\frac{1}{2m} \leq \operatorname{osc} f(x) \leq \operatorname{osc}_\delta f(x) &= \sup_{B(x,\delta)} f - \inf_{B(x,\delta)} f \\
&\leq \sup_R f - \inf_R f = M_R(f) - m_R(f)
\end{aligned}$$

We multiply by $v(R)$ and summing over all R we get:

$$\begin{aligned}
\frac{1}{2m} \sum_{\substack{R \\ R \cap \mathscr{D}''_m \neq \emptyset}} v(R) &\leq \sum_{\substack{R \\ R \cap \mathscr{D}''_m \neq \emptyset}} (M_R(f) - m_R(f)) \cdot v(R) \\
&\leq \sum_R (M_R(f) - m_R(f)) \cdot v(R) \\
&= U(f,P) - L(f,P) < \frac{\varepsilon}{2m}
\end{aligned}$$

And therefore:

$$\sum_{\substack{R \\ R \cap \mathscr{D}''_m \neq \emptyset}} < \frac{\varepsilon}{2}$$

These boxes which intersect $\mathscr{D}''_m$ provide the needed covering of $\mathscr{D}''_m$.

Remark. This theorem shows that sets of Lebesgue measure zero can be problematic for Riemann integration. In the sense that, changing a function on a set of Lebesgue measure zero can make it non-integrable. In particular consider the function:

$$\begin{gathered}
\mathbb{1}_{\mathbb{Q}} : [0,1] \to \mathbb{R} \\
\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}
\end{gathered}$$

This is only different from a constant function on a set of measure zero, namely it differs from the constant function on $\mathbb{Q} \cap [0,1]$. This indicates a kind of "incompleteness" of Riemann integration.

Corrolary. *Let B be a box in $\mathbb{R}^n$ and $f : B \to \mathbb{R}$ be Riemann integrable.*

a) If f vanishes except on a set of Lebesgue measure zero, then $\int_B f = 0$. We say that $f = 0$ almost everywhere

b) If $f \geq 0$ and $\int_B f = 0$ then f vanishes except possibly on a set of Lebesgue measure zero. That is f vanishes almost everywhere.

Remark. The corollary is not true without the assumption that f is Riemann integrable.

Proof. Let's go!

a) Let $\mathscr{D}_0$ be the set $\{x \in B \mid f(x) \neq 0\}$. By assumption, $\mathscr{D}_0$ has ℓ-measure zero. Let P be any partition of B. For any sub-box R of this partition, we have that $R \not\subseteq \mathscr{D}_0$ (since $v(R) > 0$). This implies there exists an $x \in R$ such that $f(x) = 0$, and so:

$$m_R(f) \leq 0$$
$$M_R(f) \geq 0$$

Therefore $L(f, P) \leq 0$ and $U(f, P) \geq 0$. But wait this implies that:

$$\underline{\int_B} f \leq 0 \qquad\qquad \overline{\int_B} f \geq 0$$

Since f is integrable, we then know that:

$$\int_B f = \underline{\int_B} f = \overline{\int_B} f = 0$$

And so we are done.

b) Suppose $f(x) \geq 0$ and $\int_B f = 0$. We will show that if f is continuous at some x, then $f(x) = 0$. Since the set of discontinuities of f has measure zero beause f is Riemann integrable, this shows that the set of all x where $f(x) \neq 0$ must have measure zero as well.

We will do this by contradiction. Suppose that f is continuous at some x_0 and $f(x_0) > 0$. Then there exists an $\varepsilon > 0$ and a small box R centered at x_0 such that $f(x) > \varepsilon$ for all $x \in R$.

Now consider the following function:

$$g(x) = \begin{cases} \varepsilon & \text{if } x \in \mathbb{R} \\ 0 & \text{if } x \in B \setminus R \end{cases}$$

Then g is integrable since the set of discontinuities of g has measure zero. Also $f(X) \geq g(x)$ for all $x \in B$ and so:

$$\int_B f \overset{?}{\geq} \int_B g \overset{?}{=} \varepsilon \cdot v(R) > 0$$

Hani says we should verify $\overset{?}{\geq}$ and $\overset{?}{=}$. I leave that to you ☺

Another approach is to take a partition P obtained from the endpoints of R and B. Then we get $L(f,P) \geq \varepsilon \cdot v(R_0)$. where R_0 is the sub-box of P containing x_0. But this implies that:

$$\int_B f(x)\,\mathrm{d}x = \sup_P L(f,P) \geq \varepsilon \cdot v(R_0)$$

In either case, we have an oops! Great!

Fubini's Theorem

After defining the integral, the main question remains: how to compute integrals in higher dimensions? (We know how to compute integrals in 1D using the Fundamental Theorem of Calculus and various techniques of integration)

Fubini's Theorem will allow us to compute integrals in higher dimensions by reducing them to iterated integrals in lower dimensions. This often allows us to reduce things to the one-dimensional case.

One would wish to say that if $f : Q \to \mathbb{R}$ is integrable where $Q = A \times B$ and A is a box in $\mathbb{R}^k$ and B is a box in $\mathbb{R}^\ell$. Then $x \mapsto \int_B f(x,y)\,\mathrm{d}y$ exists for every $x \in A$ and defines an integrable function over A. Furthermore:

$$\int_Q f = \int_A \left(\int_B f(x,y)\,\mathrm{d}y \right) \mathrm{d}x \tag{$\star$}$$

This requires that the function $x \mapsto \int_B f(x,y)\,\mathrm{d}y$ is defined for every x (i.e. $f(x,y)$ is integrable in y for fixed $x \in A$) and that function $x \mapsto \int_B f(x,y)\,\mathrm{d}y$ is integrable in x itself on A.

Unfortunately, such a nice property is not necessarily true for all $x \in A$. Indeed, we will see that it is true except for sets of Lebesgue measure zero. This is no problem for Lebesgue integrals (for which $\star$ holds), but since Riemann integrability can depend on sets of Lebesgue measure zero, we might lose there.

Theorem (Fubini's Theorem). *Let $Q = A \times B$ where A is a box in $\mathbb{R}^k$ and B is a box in $\mathbb{R}^\ell$. let $f(x,y) : Q \to \mathbb{R}$ be a bounded function (where $x \in A$ and $y \in B$)*

Then for each $x \in A$ consider the lower and upper integrals:

$$x \mapsto \underline{\int_B} f(x,y)\,\mathrm{d}y \qquad\qquad x \mapsto \overline{\int_B} f(x,y)\,\mathrm{d}y$$

if f is integral over Q then the above two functions are integrable over A and:

$$\int_Q f = \int_A \left(\underline{\int_B} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_A \left(\overline{\int_B} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

Of course we have lower and upper integrals here. If we get agreement of the above two functions on all of x then we would be very happy.