MATH 395 Notes

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3 Compactness on $\mathbb{R}^d$

Last time we proved the nested interval property on $\mathbb{R}$, namely

Theorem (Nested Interval Property). *Let $I_n = [a_n, b_n]$ be a sequence of closed and bounded intervals that is nested, aka $I_n \supseteq I_{n+1}$. Then we have that:*

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Note that we need "closedness." Take $I_n = \left(0, \frac{1}{n}\right]$. In fact what we really need is compactness.

Definition. *A closed box in $\mathbb{R}^d$ is a set of the form:*

$$\prod_{j=1}^{d} [a_j, b_j]$$

Corrolary (The nested box property of $\mathbb{R}^d$). *Let B_n be a sequence of closed and nested boxes. Then:*

$$\bigcap_{n=1}^{\infty} \neq \emptyset$$

Great!

Proof. Let:

$$B_n = \prod_{j=1}^{d} \left[a_j^{(n)}, b_j^{(n)}\right]$$

$B_n \subseteq B_{n+1}$ implies for every $1 \le j \le d$ that the intervals $I_j^{(n)} = \left[a_j^{(n)}, b_j^{(n)}\right]$ are nested. By the previous theorem, for every $1 \le j \le d$ there exists some:

$$x_j \in \bigcap_{n=1}^{\infty} I_j^{(n)}$$

Therefore:

$$x = (x_1, \dots, x_d) \in \bigcap_{n=1}^{\infty} B_n$$

And so we win!

Definition. *Define in a metric space for any subset E of a metric space X the diameter when the following supremum exists:*

$$\operatorname{diam} E = \sup_{x,y \in E} d(x,y)$$

Great!

Exercise. *Show that for any box $B = \prod_{j=1}^{d} [a_j, b_j]$ that:*

$$\operatorname{diam} B = \left[\sum_{j=1}^{d} (b_j - a_j)^2\right]^{\frac{1}{2}}$$

Where we use the standard Euclidean metric on $\mathbb{R}^d$

Proof. We will do this with induction on d

- Suppose $d = 1$. We wish to prove that $\operatorname{diam}[a,b] = |b-a| = b-a$. Note that $b-a$ is in the set we are taking a supremum over, and so we merely need to show it is an upper bound. Fix $x, y \in [a,b]$. Without loss of generality take $y \ge x$. Then note that:

$$b - a = (b-y) + (y-x) + (x-a) \ge y - x$$

And so we win

- Suppose that the result holds for $d \in \mathbb{N}$. We must show it holds for $d+1$. Note then that $a = (a_1, \dots, a_{d+1})$ and $b = (b_1, \dots, b_{d+1})$ are in B, and so:

  $$d(a,b) = \left[\sum_{j=1}^{d+1} (b_j - a_j)^2\right]^{\frac{1}{2}}$$

  Is in the set we are taking a supremum over. We need only show that it is a maximum. Fix $x = (x_1, \dots, x_{d+1})$ and $y = (y_1, \dots, y_{d+1})$ in the box B and without loss of generality assume $y_{d+1} \geq x_{d+1}$.

  Define $x' = (x_1, \dots, x_d)$ and $y' = (y_1, \dots, y_d)$. Then we have:

  $$d(x', y') \leq \delta := \operatorname{diam} \prod_{j=1}^{d} [a_j, b_j] = \left[\sum_{j=1}^{d} (b_j - a_j)^2\right]^{\frac{1}{2}}$$

  Now note that:

  $$\begin{aligned} d(x,y) &= \sqrt{(d(x',y'))^2 + (y_{d+1} - x_{d+1})^2} \\ &\leq \sqrt{\delta^2 + (b_{d+1} - a_{d+1}^2} \\ &= \left(\left[\sum_{j=1}^{d} (b_j - a_j)^2\right] + (b_{d+1} - a_{d+1})^2\right)^{\frac{1}{2}} \\ &= \left[\sum_{j=1}^{d+1} (b_j - a_j)^2\right]^{\frac{1}{2}} = d(a,b) \end{aligned}$$

  But this is exactly what we want ☺

Awesome!

Theorem. *Every closed box in $\mathbb{R}^d$ is compact.*

Proof. Let $B = \prod_{j=1}^d [a_j, b_j]$ be any closed box. Set:

$$\delta_0 := \operatorname{diam} B = \left[\sum_{j=1}^{d} (b_j - a_j)^2 \right]^{\frac{1}{2}}$$

Suppose for the sake of contradiction that $\{G_\alpha\}_{\alpha \in A}$ is an open cover of B that has no finite subcover

Split B into 2^d subboxes of equal size. That is let $c_j = \frac{a_j + b_j}{2}$. Then the subboxes are $\prod_{j=1}^d I_j$ where $I_j \in \{[a_j, c_j], [c_j, b_j]\}$.

Since B cannot be covered by any finite collection of the $\{G_\alpha\}_{\alpha \in A}$, there must exist a subbox, B_1 such that B_1 cannot be covered by any finite subcollection of the $\{G_\alpha\}_{\alpha \in A}$. Note also that $\operatorname{diam} B_1 = \frac{\operatorname{diam} B}{2}$. Set $\delta_1 = \operatorname{diam} B_1$.

Continue inductively, having constructed $B \supseteq B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n$ such that $\operatorname{diam} B_n = \delta_n = \frac{\operatorname{diam} B}{2^n}$ and B_n cannot be covered by any finite collection of the $\{G_\alpha\}_{\alpha \in A}$. We construct B_{n+1} by splitting B_n into 2^d subboxes of equal size as in the previous paragraph and noting that one of those subboxes cannot be covered by any finite collection of the $\{G_\alpha\}_{\alpha \in A}$. Let B_{n+1} be this subbox of B_n. Also note:

$$\operatorname{diam} B_{n+1} = \frac{\operatorname{diam} B_n}{2} = \frac{\operatorname{diam} B}{2^{n+1}}$$

This is a sequence of closed nested boxes. Applying the nested box property we know that $\bigcap_{n=1}^\infty B - n \neq \emptyset$.

Claim. *$\bigcap_{n=1}^\infty B_n$ is a singleton x.*

Proof. Suppose $x, y \in \bigcap_{n=1}^\infty$. Then $x, y \in B_n$ for every n, and therefore $d(x, y) \leq \operatorname{diam} B_n = \frac{\operatorname{diam} B}{2^n}$. Letting n go to infinity we get $d(x, y) = 0$ and so $x = y$.

Now $x \in B$ implies there exists an $\alpha_x \in A$ so that $x \in G_{\alpha_x}$. But then this implies that there is an $r > 0$ so that $N_r(x) \subseteq G_{\alpha_x}$.

For n large enough we know $B_n \subseteq N_r(x)$. In fact if $\delta_n < r$ then $B_n \subseteq N_r(x)$. Thus since $\delta_n \to 0$ we know $\delta_n < r$ eventually. But then obviously B_n is covered by a finite collection of the $\{G_\alpha\}_{\alpha \in A}$. Oops! The box B must then be compact.

Theorem (Heine-Borel)**.** *A subset K of $\mathbb{R}^d$ is compact if and only if it is closed and bounded.*

Proof. Let's go!

- ($\Rightarrow$) We already showed this direction in general metric spaces.
- ($\Leftarrow$) If K is bounded then K is contained in some large closed box B which is compact. Therefore K is a closed subset of a compact set. This implies that K is compact (we showed this last time in Hausdorff spaces).

4 Compactness in Metric Spaces

It turns out that being closed and bounded is not sufficient to guarantee compactness in infinite-dimensional metric spaces.

Example. Let $\ell^\infty(\mathbb{N})$ denote the set of bounded sequences $(a_n)_{n\in\mathbb{N}}$. The metric on $\ell^\infty(\mathbb{N})$ is defined as:

$$d((a_n),(b_n)) = \sup_{n\in\mathbb{N}} |a_n - b_n|$$

Consider the set $B = \{(a_n) \in \ell^\infty(\mathbb{N}) \mid \sup_{n\in\mathbb{N}} |a_n| \le 1\}$.

Exercise. *This set is closed and bounded (check ✓).*

Proof. To note that it's bounded consider that:

$$d((a_n),0) = \sup_{n\in\mathbb{N}} |a_n| \le 1$$

So this is trivial. Now consider a sequence of sequences $\left(a_n^{(j)}\right)_{j\in\mathbb{N}}$ which are all in B which converges to some $(a_n)_{n\in\mathbb{N}}$. We will show 1 is an upper bound for the set $\{|a_n|\}_{n\in\mathbb{N}}$, and so:

$$\sup_{n\in\mathbb{N}} |a_n| \le 1$$

Fix $n \in \mathbb{N}$. Now fix $\varepsilon > 0$. We know there is some large $j \in \mathbb{N}$ so that:

$$d\left(\left(a_n^{(j)}\right), (a_n)\right) = \sup_{n \in \mathbb{N}} \left|a_n - a_n^{(j)}\right| < \varepsilon$$

Now note that:

$$\begin{aligned} |a_n| &\overset{\triangle}{\leq} \left|a_n^{(j)}\right| + \left|a_n - a_n^{(j)}\right| \\ &< 1 + \varepsilon \end{aligned}$$

And so since this holds for all $\varepsilon > 0$ we must have $|a_n| \leq 1$ as desired.

Claim. *This set B is not compact!*

Proof. Consider the sequence of sequences:

$$\left(a_n^{(k)}\right) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Therefore:

$$d\left(\left(a_n^{(k)}\right), \left(a_n^{(k')}\right)\right) = 1$$

Thus this sequence of sequences $\left(a_n^{(k)}\right)$ can have no convergent subsequence. And thus B is not sequentially compact, and so B is not compact.

How do we fix this? It turns out we need to strengthen our conditions

- Replace closed by Cauchy Complete
- Replace bounded by total boundedness

Definition. *A subset E of a metric space X is totally bounded if for every $\varepsilon > 0$ there is a finite cover of E by balls of radius $\varepsilon > 0$.*

Exercise. *Show that:*

- *On $\mathbb{R}^d$ we have boundedness if and only if total boundedness*
  - *Totally bounded implies bounded on every metric space*
  - *For bounded implies totally bounded. Since any box B of the form $[-N, N]^d$ can be split into finitely many subboxes of diameter less than ε, and each sub-box is contained in a ball of radius ε.*
- *On $\mathbb{R}^d$ we have closed if and only if Cauchy complete. Of course Cauchy complete implies closed, and for the other direction we just use Cauchy completeness of $\mathbb{R}^d$.*
- *On $\ell^\infty(\mathbb{N})$ we have that total boundedness is stronger than boundedness. In fact:*

  Exercise. *Show that the set B in the above is bounded but not totally bounded. Use the exact same sequence as in the example and use pigeonhole principle.*

  Proof. We've already proved it is bounded. Let $\varepsilon = \frac{1}{2}$ and suppose for the sake of contradiction that we have a finite cover by balls of radius ε. Call these balls $B_1, \ldots, B_N$. Without loss of generality assume we have $\left(a_n^{(k)}\right) \in B_j$ for each $1 \leq k \leq N$ where we have:

  $$\left(a_n^{(k)}\right) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

  Now consider the sequence $\left(a_n^{(m)}\right)$ where we set $m := N + 1$. We know there is some k so that $\left(a_n^{(m)}\right) \in B_k$. But then letting $\left(x_n^{(k)}\right)$ be the center of the ball B_k we have that:

  $$\begin{aligned} 1 = d\left(\left(a_n^{(m)}\right), \left(a_n^{(k)}\right)\right) &\overset{\triangle}{\leq} d\left(\left(a_n^{(m)}\right), \left(x_n^{(k)}\right)\right) + d\left(\left(x_n^{(k)}\right), \left(a_n^{(k)}\right)\right) \\ &< \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

  Oops! We win ☺

Theorem. *Let X be a metric space and $E \subseteq X$. The following are equivalent:*

1) E is compact

2) E is sequentially compact

3) E is complete and totally bounded.

Remark. If X is a complete metric space then 3) above can be replaced by closed and totally bounded.

Lemma. *Completeness of $E \subseteq X$ implies E is closed.*

Proof. Let E be complete and $x_n \in E$ such that $x_n \to x \in X$. Since (x_n) converges it must be Cauchy, and so since E is compelte we know (x_n) converges to some point in E. But limits are unique in metric spaces so $x \in E$, so E is closed!!!