Handout 2

- **Relatively Open, closed, and compact.** We saw in class that the interval $[0, 1)$ is not open in $\mathbb{R}$, but is open relative to the half-line $[0, \infty)$ (taking the usual metric on $\mathbb{R}$ and $[0, \infty)]$). Let us try to formalize and generalize this.

  Let (X, d) be a metric space and $Y \subset X$. Y is a metric space itself, by restricting the metric d to $Y \times Y$.

Q1) Let $E \subset Y$. We say that E is open relative to Y if it is open in the metric space (Y, d). Untangle what this definition means in terms of $N_\delta(p)$ neighborhood of a point $p \in E$ (i.e. restate the condition that E is open in Y in terms of the $N_\delta(p)$ neighborhoods of $p \in E$) and compare it to the condition of E being open in X.

Q2) Deduce that if there is an open subset G of X, then $G \cap Y$ is open relative to Y.

Q3) Show that E is open relative to Y if and only if there exists an open subset G of X such that $E = G \cap Y$.

Q4) Compactness on the other hand behaves better. Suppose that $K \subset Y \subset X$. Show that K is compact relative to X if and only if it is compact relative to Y.

Conclusion: We always need to specify the ambient space when we talk about open/closed sets (that's why we always say "E is an open subset of X"), but we can make statements like "K is compact (or a compact metric space)" without the need to specify the ambient space.

- **The Cantor set.** Let us start with the interval $C = [0, 1]$ and remove the middle third open interval $(\frac{1}{3}, \frac{2}{3})$. This leaves us with the set $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ formed of 2 closed subintervals. Having constructed $C_1 \supset$

$C_2 \supset \ldots \supset C_n$ where C_n is the union of 2^n subintervals each of length $\frac{1}{3^n}$, we construct C_{n+1} as follows: To obtain C_{n+1} we remove the middle third of each of the 2^n intervals that form C_n. This leaves us with a union of 2^{n+1} intervals each of length $\frac{1}{3^{n+1}}$.

Q5) Let $C = \cap_{n=1} \infty C_n$. Why is C non-empty? Is it compact?

Q6) Show that every point in C is a limit point. Hence C is a perfect set.

Conclusion: From the homework (HW 2), we deduce that C is uncountable, since any perfect subset of $\mathbb{R}^d$ is uncountable.

Q7) Show that C cannot contain any interval (a, b).

Conclusion: As such, C is totally disconnected (it has no nontrivial connected subset) and nowhere dense (the interior of its closure is empty).

Q8) What is the total length of C_n? What would be a reasonable definition of the length of C?