Handout 2

- **Relatively Open, closed, and compact.** We saw in class that the interval $[0,1)$ is not open in $\mathbb{R}$, but is open relative to the half-line $[0,\infty)$ (taking the usual metric on $\mathbb{R}$ and $[0,\infty)]$). Let us try to formalize and generalize this.

  Let (X,d) be a metric space and $Y \subset X$. Y is a metric space itself, by restricting the metric d to $Y \times Y$.

Q1) Let $E \subset Y$. We say that E is open relative to Y if it is open in the metric space (Y,d). Untangle what this definition means in terms of $N_\delta(p)$ neighborhood of a point $p \in E$ (i.e. restate the condition that E is open in Y in terms of the $N_\delta(p)$ neighborhoods of $p \in E$) and compare it to the condition of E being open in X.

Q2) Deduce that if there is an open subset G of X, then $G \cap Y$ is open relative to Y.

Q3) Show that E is open relative to Y if and only if there exists an open subset G of X such that $E = G \cap Y$.

Q4) Compactness on the other hand behaves better. Suppose that $K \subset Y \subset X$. Show that K is compact relative to X if and only if it is compact relative to Y.

Conclusion: We always need to specify the ambient space when we talk about open/closed sets (that's why we always say "E is an open subset of X"), but we can make statements like "K is compact (or a compact metric space)" without the need to specify the ambient space.

- **The Cantor set.** Let us start with the interval $C = [0,1]$ and remove the middle third open interval $(\frac{1}{3}, \frac{2}{3})$. This leaves us with the set $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ formed of 2 closed subintervals. Having constructed $C_1 \supset$

$C_2 \supset \ldots \supset C_n$ where C_n is the union of 2^n subintervals each of length $\frac{1}{3^n}$, we construct C_{n+1} as follows: To obtain C_{n+1} we remove the middle third of each of the 2^n intervals that form C_n. This leaves us with a union of 2^{n+1} intervals each of length $\frac{1}{3^{n+1}}$.

Q5) Let $C = \cap_{n=1^\infty} C_n$. Why is C non-empty? Is it compact?

Q6) Show that every point in C is a limit point. Hence C is a perfect set.

Conclusion: From the homework (HW 2), we deduce that C is uncountable, since any perfect subset of $\mathbb{R}^d$ is uncountable.

Q7) Show that C cannot contain any interval (a, b).

Conclusion: As such, C is totally disconnected (it has no nontrival connected subset) and nowhere dense (the interior of its closure is empty).

Q8) What is the total length of C_n? What would be a reasonable definition of the length of C?

MATH 395 Notes

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Proof of Q1. Let $N_\delta^Y(p) = \{a \in Y \mid d(p, a) < \delta\}$ denote the δ-neighborhoods in Y for $p \in Y$ and let $N_\delta(p)$ denote the neighborhood relative to X. Now the definition of an open set $E \subseteq Y$ says that for all $p \in E$ there exists a $\delta > 0$ such that $N_\delta^Y(p) \subseteq E$. Note that:

$$N_\delta^Y(p) = N_\delta(p) \cap Y$$

And so we must have that $N_\delta(p) \cap Y \subseteq E$.

If E were open in X then we would have a stronger condition, namely that the whole neighborhood $N_\delta(p) \subseteq E$.

Proof of Q2. Suppose that G is an open subset of X. Now consider some $p \in G \cap Y$. We know since $p \in G$ that there exists some $\varepsilon > 0$ so that $N_\varepsilon(p) \subseteq G$. But then we know that:

$$N_\varepsilon^Y(p) = N_\varepsilon(p) \cap Y \subseteq G \cap Y$$

By using facts from elementary set theory. This is great! We win now since this must mean that $G \cap Y$ is open as a subset of Y.

Proof of Q3. The backward direction is exactly a consequence of Q2. We work instead on the forward direction.

Suppose that E is open relative to Y. For each $p \in Y$ there exists some $\delta_p > 0$ so that:

$$N_{\delta_p}^Y(p) = N_{\delta_p}(p) \cap Y \subseteq E$$

Now consider the following union:

$$G := \bigcup_{p \in E} N_{\delta_p}(p)$$

Since each $N_{\delta_p}(p)$ is open in X we know that G must be open relative to X. We will show that $E = G \cap Y$.

($\subseteq$) Fix $p \in E$. Then we know that $p \in Y$ since E is a subset of Y, and further we know that $p \in N_{\delta_p}(p)$, and so $p \in G$.

($\supseteq$) Fix $x \in G \cap Y = Y \cap G$. Then:

$$x \in Y \cap G = Y \cap \bigcup_{p \in E} N_{\delta_p}(p) = \bigcup_{p \in E} (Y \cap N_{\delta_p}(p))$$

And thus there exists some p so that:

$$x \in N_{\delta_p}(p) \cap Y = N^Y_{\delta_p}(p) \subseteq E$$

Therefore $x \in E$ just as desired! Great.

With this we win ☺

Proof of Q4. Suppose that $K \subseteq Y \subseteq X$. Now lets go in each direction

($\Rightarrow$) Suppose that K is compact relative to X. Now fix an open cover $\{U_\alpha\}_{\alpha \in A}$ of K relative to Y. By Q3 for each $\alpha \in A$ there exists a G_α which is open in X so that $U_\alpha = G_\alpha \cap Y$. Therefore:

$$\begin{aligned} K &\subseteq \bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} (Y \cap G_\alpha) = Y \cap \bigcup_{\alpha \in A} G_\alpha \\ K &\subseteq \bigcup_{\alpha \in A} G_\alpha \end{aligned}$$

Great! Thus the $\{G_\alpha\}_{\alpha \in A}$ cover K. Since K is compact in X we know there exists a finite subcover $G_{\alpha_1}, \ldots, G_{\alpha_n}$. Then since $K \subseteq Y$ and $K \subseteq \bigcup_{i=1}^n G_{\alpha_i}$ we know:

$$K \subseteq Y \cap \bigcup_{i=1}^n G_{\alpha_i} = \bigcup_{i=1}^n (Y \cap G_{\alpha_i} = \bigcup_{i=1}^n U_{\alpha_i}$$

And therefore $U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in A}$ just as desired! Great!!

($\Leftarrow$) Suppose that K is compact relative to Y. Now fix an open cover $\{G_\alpha\}_{\alpha \in A}$ of K relative to X. By Q2 we must have that $U_\alpha := G_\alpha \cap Y$ is open in Y for each $\alpha \in A$. Note then that since $K \subseteq Y$ and $K \subseteq \bigcup_{\alpha \in A} G_\alpha$ we know:

$$K \subseteq Y \cap \bigcup_{\alpha \in A} G_\alpha = \bigcup_{\alpha \in A} (Y \cap G_\alpha) = \bigcup_{\alpha \in A} U_\alpha$$

And so $\{U_\alpha\}_{\alpha \in A}$ is an open cover of K in Y. Therefore there must exist a finite subcover for it by compactness, which we will denote by $U_{\alpha_1}, \ldots, U_{\alpha_n}$. Therefore:

$$\begin{aligned} K &\subseteq \bigcup_{i=1}^{n} U_{\alpha_i} = \bigcup_{i=1}^{n} (Y \cap G_{\alpha_i}) = Y \cap \bigcup_{i=1}^{n} G_{\alpha_i} \\ K &\subseteq \bigcup_{i=1}^{n} G_{\alpha_i} \end{aligned}$$

And so $G_{\alpha_1}, \ldots, G_{\alpha_n}$ is a finite subcover of $\{G_\alpha\}_{\alpha \in A}$ just as desired!!!

With this we win ☺

Proof of Q5. For notational convenience denote for $n \in \mathbb{N}_0$:

$$C_n = \bigcup_{i=1}^{2^n} [a_i^n, b_i^n]$$

So that inductively for $1 \leq i \leq 2^n$:

$$\begin{aligned} C_0 &= [0, 1] \\ \left[a_{2i-1}^{n+1}, b_{2i-1}^{n+1}\right] &= \left[a_i^n, \frac{2a_i^n + b_i^n}{3}\right] \\ \left[a_{2i}^{n+1}, b_{2i}^{n+1}\right] &= \left[\frac{a_i^n + 2b_i^n}{3}, b_i^n\right] \end{aligned}$$

Now lets tackle both of these questions!

- Note that $a_1^0 = 0$ will always lie at the edge of an interval because supposeing

$a_1^n = 0$ we know $a_1^{n+1} = a_{2\cdot 1-1}^{n+1} = a_1^n = 0$. Therefore since:

$$0 \in [a_1^n, b_1^n] \subseteq C_n$$

for each $n \geq 0$ we must know that $0 \in C$. A similar argument shows that $1 \in C$.

- C is compact!!! Why? Note that for every $n \geq 0$ we have that C_n is a finite union of closed intervals, so each C_n is closed. Thus, $C = \bigcap_{n=0}^{\infty} C_n$ is closed. Furthermore since $C_0 = [0, 1]$ is closed and bounded, that is compact. Therefore since $C \subseteq C_0$ is a closed subset of a compact set, C must be compact.

Perfect! We win!

Proof of Q6. Fix some point $x \in C$. Then $x \in C_n$ for all $n \geq 0$, and so for each $n \geq 0$ there exists some $1 \leq i_n \leq 2^n$ so that $x \in \left[a_{i_n}^n, b_{i_n}^n\right]$. We claim that $x_n^\ell := a_{i_n}^n$ is a sequence lying in $C \setminus \{x\}$ that converges to x or $x_n^r := a_{i_n}^n$ is a sequence lying in $C \setminus \{x\}$ that converges to x. We tackle this in steps.

- First we show that for all $n \geq 0$ and all $1 \leq i \leq 2^n$ we have a_i^n is in C. First note that $a_i^n \in [a_i^n, b_i^n] \subseteq C_n$, and thus for each $0 \leq m < n$ we must have $a_i^n \in C_n \subseteq C_m$. Inductively we will show that for $m \geq n$ if we let $j_n = i$ and $j_{m+1} = 2j_m - 1$ then:

  $$a_i^n = a_{j_m}^m \in C_m$$

  Note that it's trivial for $m = n$. Now suppose that $a_{j_m}^m = a_i^n$. Consider that:

  $$a_{j_{m+1}}^{m+1} = a_{2j_m-1}^{m+1} = a_{j_m}^m = a_i^n$$

  And so we must have that this works! Great.

- Now we show that for all $n \geq 0$ and all $1 \leq i \leq 2^n$ we have b_i^n is in C. First note that $b_i^n \in [a_i^n, b_i^n] \subseteq C_n$, and thus for each $0 \leq m < n$ we must have $b_i^n \in C_n \subseteq C_m$. Inductively we will show that for $m \geq n$ if we let $j_n = i$ and $j_{m+1} = 2j_m$ then:

  $$b_i^n = b_{j_m}^m \in C_m$$

Note that it's trivial for $m = n$. Now suppose that $b^m_{j_m} = b^n_i$. Consider that:

$$b^{m+1}_{j_{m+1}} = b^{m+1}_{2j_m} = b^m_{j_m} = b^n_i$$

And so we must have that this works! Great.

- Now we show that for each $n \geq 0$ and each $1 \leq i \leq 2^n$ the interval $[a^n_i, b^n_i]$ has length $\frac{1}{3^n}$. TODO

Proof of Q7. TODO

Proof of Q8. TODO