Handout 3

- **The Cantor set.** Let us start with the interval $C = [0,1]$ and remove the middle third open interval $(\frac{1}{3}, \frac{2}{3})$. This leaves us with the set $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ formed of 2 closed subintervals. Having constructed $C_1 \supset C_2 \supset \ldots \supset C_n$ where C_n is the union of 2^n subintervals each of length $\frac{1}{3^n}$, we construct C_{n+1} as follows: To obtain C_{n+1} we remove the middle third of each of the 2^n intervals that form C_n. This leaves us with a union of 2^{n+1} intervals each of length $\frac{1}{3^{n+1}}$.

Q1) Let $C = \cap_{n=1\infty} C_n$. Why is C non-empty? Is it compact?

Q2) Show that every point in C is a limit point. Hence C is a perfect set.

Conclusion: From the homework (HW 2), we deduce that C is uncountable, since any perfect subset of $\mathbb{R}^d$ is uncountable.

Q3) Show that C cannot contain any interval (a, b).

Conclusion: As such, C is totally disconnected (it has no non-trival connected subset) and nowhere dense (the interior of its closure is empty).

Q4) What is the total length of C_n? What would be a reasonable definition of the length of C?

- **Wish list for a measure function** Motivated by the above, it would be grand to have a measure function that tells us how big or small a subset of $\mathbb{R}^d$ is. This would be a function from the set of subsets of $\mathbb{R}^d$ into $[0, \infty]$, say $m : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$. We would like this function to satisfy the following properties:

a) If $E_1, E_2, \ldots$ is a countable collection of disjoint subsets of $\mathbb{R}$, then

$$m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n).$$

b) If E is congruent to F (i.e. F can be obtained from E by applying rigid motions: translations, rotations, or a reflections) then we should have that $m(E) = m(F)$.

c) $m([0,1)^d) = 1$.

The bad news is that no such function can exist, and here's why (at least when $d = 1$). Let us define an equivalence relation between elements of $[0,1)$ as follows: We say $x \sim y$ if $x - y$ is a rational number. Let N be the subset of $[0,1]$ that contains exactly one element of each equivalence relation (the existence of this N requires invoking the axiom of choice). Now let $R = [0,1) \cap \mathbb{Q}$, and for each $r \in R$ define the set

$$N_r = \{x + r : x \in N \cap [0, 1-r]\} \cup \{x + r - 1 : x \in N \cap [1-r, 1)\}.$$

(Basically N_r is just the translate of N by r units to the right, except that we move the part that sticks out of the interval $[0,1)$ one unit to the left).

Q5) Show that $[0,1)$ is the disjoint union of N_r for $r \in R$.

Q6) Show that if a measure function satisfying a), b) and c) above exists, then $m(N) = m(N_r)$ for every $r \in R$.

Q7) Arrive at a contradiction.

Remark: One might think that possibly relaxing condition a) to cover only *finitely* many disjoint sets E_n, i.e.

$$m(\cup_{n=1}^{N} E_n) = \sum_{n=1}^{N} m(E_n).$$

would resolve the contradiction. Unfortunately, the Banach-Tarski paradox (cf. Figure 1) tells us that this is not enough to resolve this issue.

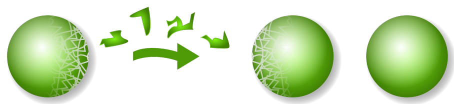

Figure 1: Banach-Tarski tells us that we can split the unit ball in $\mathbb{R}^3$ into finitely many (actually 5 is sufficient) many disjoint pieces, apply rigid motions to those pieces and then reassemble them to obtain two copies of the unit ball.

Conclusion: The problem with the above wishlist is that we insisted on being able to measure *every* subset of $\mathbb{R}^d$. We have shown that this is impossible. The solution is to be content with a measure function that is defined on some but not all subsets. Such subsets will be called measurable subsets.

MATH 395 Notes

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Proof of Q1. For notational convenience denote for $n \in \mathbb{N}_0$:

$$C_n = \bigcup_{i=1}^{2^n} [a_i^n, b_i^n]$$

So that inductively for $1 \le i \le 2^n$:

$$\begin{aligned} C_0 &= [0,1] \\ \left[a_{2i-1}^{n+1}, b_{2i-1}^{n+1}\right] &= \left[a_i^n, \frac{2a_i^n + b_i^n}{3}\right] \\ \left[a_{2i}^{n+1}, b_{2i}^{n+1}\right] &= \left[\frac{a_i^n + 2b_i^n}{3}, b_i^n\right] \end{aligned}$$

Now lets tackle both of these questions!

- Note that $a_1^0 = 0$ will always lie at the edge of an interval because supposeing $a_1^n = 0$ we know $a_1^{n+1} = a_{2\cdot 1-1}^{n+1} = a_1^n = 0$. Therefore since:

  $$0 \in [a_1^n, b_1^n] \subseteq C_n$$

  for each $n \ge 0$ we must know that $0 \in C$. A similar argument shows that $1 \in C$.

- C is compact!!! Why? Note that for every $n \ge 0$ we have that C_n is a finite union of closed intervals, so each C_n is closed. Thus, $C = \bigcap_{n=0}^{\infty} C_n$ is closed. Furthermore since $C_0 = [0,1]$ is closed and bounded, that is compact. Therefore since $C \subseteq C_0$ is a closed subset of a compact set, C must be compact.

Perfect! We win!

Proof of Q2. Fix some point $x \in C$. Then $x \in C_n$ for all $n \geq 0$, and so for each $n \geq 0$ there exists some $1 \leq i_n \leq 2^n$ so that $x \in \left[a^n_{i_n}, b^n_{i_n}\right]$. Suppose that $\varepsilon > 0$, then there is some $N \in \mathbb{N}$ so that $\frac{1}{3^N} < \varepsilon$. We claim that $a^N_{i_N}, b^N_{i_N} \in N_\varepsilon(x) \cap C$

- First we show that for all $n \geq 0$ and all $1 \leq i \leq 2^n$ we have a^n_i is in C. First note that $a^n_i \in [a^n_i, b^n_i] \subseteq C_n$, and thus for each $0 \leq m < n$ we must have $a^n_i \in C_n \subseteq C_m$. Inductively we will show that for $m \geq n$ if we let $j_n = i$ and $j_{m+1} = 2j_m - 1$ then:

  $$a^n_i = a^m_{j_m} \in C_m$$

  Note that it's trivial for $m = n$. Now suppose that $a^m_{j_m} = a^n_i$. Consider that:

  $$a^{m+1}_{j_{m+1}} = a^{m+1}_{2j_m - 1} = a^m_{j_m} = a^n_i$$

  And so we must have that this works! Great.

- Now we show that for all $n \geq 0$ and all $1 \leq i \leq 2^n$ we have b^n_i is in C. First note that $b^n_i \in [a^n_i, b^n_i] \subseteq C_n$, and thus for each $0 \leq m < n$ we must have $b^n_i \in C_n \subseteq C_m$. Inductively we will show that for $m \geq n$ if we let $j_n = i$ and $j_{m+1} = 2j_m$ then:

  $$b^n_i = b^m_{j_m} \in C_m$$

  Note that it's trivial for $m = n$. Now suppose that $b^m_{j_m} = b^n_i$. Consider that:

  $$b^{m+1}_{j_{m+1}} = b^{m+1}_{2j_m} = b^m_{j_m} = b^n_i$$

  And so we must have that this works! Great.

- Now we show that for each $n \geq 0$ and each $1 \leq i \leq 2^n$ the interval $[a^n_i, b^n_i]$ has length $\frac{1}{3^n}$. Note first that:

  $$b^0_1 - a^0_1 = 1 - 0 = 1 = \frac{1}{3^0}$$

Inductively for $1 \leq i \leq 2^n$ then we know that:

$$
\begin{aligned}
b^{n+1}_{2n-i} - a^{n+1}_{2n-i} &= \frac{2a_i^n + b_i^n}{3} - a_i^n = \frac{b_i^n - a_i^n}{3} = \frac{1}{3} \cdot \frac{1}{3^n} = \frac{1}{3^{n+1}} \\
b^{n+1}_{2n} - a^{n+1}_{2n} &= b_i^n - \frac{a_i^n + 2b_i^n}{3} = \frac{b_i^n - a_i^n}{3} = \frac{1}{3} \cdot \frac{1}{3^n} = \frac{1}{3^{n+1}}
\end{aligned}
$$

Now we're done, since in particular $a^N_{i_N}$ and $b^N_{i_N}$ are distinct, so for any ε neighborhood of x there are at least two points in $N_\varepsilon(x) \cap C$. Thus x is a limit point.

Proof of Q3. Fix $a < b$. But then if we had two points $x, y \in (a, b)$ such that $x, y \in C$ and $y > x$. Note that we then know that there exists some $N \in \mathbb{N}$ so that $\frac{1}{3^N} < \varepsilon$. This means that x and y must lie in different intervals making up C_N, since these are disjoint. But then $(a, b) \cap C$ is not an interval, since $x, y \in C \cap (a, b)$ but there is some point z between x and y so that $z \notin C$. This necessarily means so then $(a, b) \neq C \cap (a, b)$, and so $(a, b) \not\subseteq C$.

Proof of Q4. Note that the total length of C^n is:

$$\ell(C_n) = \frac{2^n}{3^n}$$

Since C_n is a union of 2^n disjoint intervals each of length 3^n. Note that for each $n \in \mathbb{N}$ we must conclude since $C \subseteq C_n$ we know:

$$\ell(C) \leq \ell(C_n) = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

Taking $n \to \infty$ we then can see that $\ell(C)$ should be zero.

Proof of Q5. Fix $r, q \in R = [0, 1) \cap \mathbb{Q}$. We will first show that if $N_r \cap N_q \neq \emptyset$ then $r = q$, so by contrapositive the $\{N_r\}_{r \in R}$ are disjoint. Fix $y \in N_r \cap N_q$. There are four cases:

- $y = x_r + r$ and $y = x_q + q$ for some $x_r, x_q \in N$. Then $x_r - x_q = q - r$ by algebra, and so since $r, q \in \mathbb{Q}$ we have that $q - r \in \mathbb{Q}$ and so $x_r \sim x_q$. By the definition of N it follows that $x_r = x_q$. Therefore $x_r + r = x_r + q$, giving that $r = q$.

- $y = x_r + r - 1$ and $y = x_q + q - 1$ for some $x_r, x_q \in N$. Then $x_r - x_q = q - r$ by algebra, and so since $r, q \in \mathbb{Q}$ we have that $q - r \in \mathbb{Q}$ and so $x_r \sim x_q$. By the definition of N it follows that $x_r = x_q$. Therefore $x_r + r - 1 = x_r + q - 1$, giving that $r = q$.

- $y = x_r + r - 1$ and $y = x_q + q$ for some $x_r, x_q \in N$. Then $x_r - x_q = q - r + 1 \in \mathbb{Q}$. Thus $x_r = x_q$. Therefore $q = r - 1$ by some quick algebra. This is clearly a contradiction! Why? Well $0 \leq r < 1$, and so $-1 \leq r - 1 < 0$, but we know $q \geq 0$!!! Oops!

- $y = x_r + r$ and $y = x_q + q - 1$ for some $x_r, x_q \in N$. Then $x_r - x_q = q - r - 1 \in \mathbb{Q}$. Thus $x_r = x_q$. Therefore $r = q - 1$ by some quick algebra. This is clearly a contradiction! Why? Well $0 \leq q < 1$, and so $-1 \leq q - 1 < 0$, but we know $r \geq 0$!!! Oops!

We want to show that:

$$[0, 1) = \bigcup_{r \in R} N_r$$

Let's go!

($\subseteq$) Fix $y \in [0, 1)$. Then by definition there is some $x \in N$ so that $y \sim x$. Note that then $y - x \in \mathbb{Q}$. Further we have $0 \leq x, y < 1$ There are two cases:

- Suppose that $y - x \geq 0$. Now set $r := y - x$. First note that since $x \geq 0$ and $y < 1$ we know $y - x < 1 - 0 = 1$. Therefore $r \in \mathbb{Q} \cap [0, 1) = R$. We claim that $y \in N_r$. In particular note that $y = x + r$. All that remains to be shown is $x \in [0, 1 - r)$. We know since $x \in N$ that $x \in [0, 1)$, so $x \geq 0$ immediately. We merely need to show that $x < 1 - y + x$. This is simple, since $y < 1$ we know $1 - y > 0$. With this we must have that $x \in [0, 1) \cap N$, and so:

$$y \in \{x' + r \mid x' \in N \cap [0, 1 - r)\} \subseteq N_r$$

  And so $y \in N_r$

- Suppose that $y - x < 0$. Set $r := y - x + 1$. Note then that $r < 1$. Since $0 \leq y$ we know $-x \leq y - x$, and then since $x < 1$ it follows that $-1 < -x \leq y - x$, and so $0 < r$. This shows since $r \in \mathbb{Q}$ that $r \in R = [0, 1) \cap \mathbb{Q}$. We claim that $y \in N_r$. Note in particular that $y = x + r - 1$ by algebra. We need merely show that $x \in [1 - r, 1)$. To do this note that $y \geq 0$ so $y \leq 0$:

$$x \geq -y + x = 1 - y + x - 1 = 1 - (y - x + 1) = 1 - r$$

And we already know $x < 1$. Therefore:

$$y \in \{x' + r - 1 \mid x' \in N \cap [1 - r, 1)\} \subseteq N_r$$

And so $y \in N_r$!

Great! Since in either case $r \in R$, we must have that $y \in \bigcup_{r \in R} N_r$. This finishes this direction!

($\supseteq$) This side follows fairly immediately. Fix $y \in \bigcup_{r \in R} N_r$. Then $y \in N_r$ for some $r \in R$. There are then two quick cases:

- We have that $y = x + r$ for some $x \in N \cap [0, 1 - r)$. Then note that since $r \geq 0$ we have:

$$\begin{aligned} &0 \leq x < 1 - r \\ &0 \leq r \leq x + r = y < 1 \end{aligned}$$

  And thus $y \in [0, 1)$
- We have that $y = x + r - 1$ for some $x \in N \cap [1 - r, 1)$. Then note that since $1 - r \leq x < 1$ that $-r \leq x - 1 < 0$. Therefore since $r < 1$ we know:

$$0 \leq x + r - 1 < r < 1$$

With this we're done!

We've finished the proof that this is a disjoint union! Wow!

Proof of Q6. Fix some $r \in R$. We wish to show that $m(N_r) = m(N)$. First note that:

$$\begin{aligned} m(N_r) &= m\left(\{x + r \mid x \in N \cap [0, 1 - r)\} \cup \{x + r - 1 \mid x \in N \cap [1 - r, 1)\right) \\ &= m(\{x + r \mid x \in N \cap [0, 1 - r)\}) + m(\{x + r - 1 \mid x \in N \cap [1 - r, 1)\}) \end{aligned}$$

This follows from axiom (a) for our measure. But then by axiom (b) note that these are translations of $N \cap [0, 1-r)$ and $N \cap [1-r, 1)$ respectively so:

$$m(N_r) = m(N \cap [0, 1-r)) + m(N \cap [1-r, 1))$$

We need to now show that:

$$[0, 1) = [0, 1-r) \cup [1-r, 1)$$

This is farily quick since we note that $r \in [0, 1)$

($\subseteq$) Fix $x \in [0, 1)$. Then if $x < 1 - r$ we have $x \in [0, 1-r)$. Otherwise we know $x \geq 1 - r$ and so $x \in [1-r, 1)$.

($\supseteq$) Fix $x \in [0, 1-r)$. Then since $r \geq 0$ we know $x < 1 - r \leq 1$. Therefore $0 \leq x < 1$, and so $x \in [0, 1)$

In the other case, fix $x \in [1-r, 1)$. Then we know since $r < 1$ that $0 < 1-r \leq x$. Therefore since $0 < x < 1$ we have $x \in [0, 1)$.

Now consider that:

$$(N \cap [0, 1-r)) \cup (N \cap [1-r, 1)) = N \cap ([0, 1-r) \cup [1-r, 1)) = N \cap [0, 1) = N$$

The last equality holds sincec N is a subset of $[0, 1)$. Therefore:

$$m(N_r) = m(N \cap [0, 1-r)) + m(N \cap [1-r, 1)) = m(N)$$

And we are done!

Proof of Q7. We wish to arrive at a contradiction. There are three quick cases:

- Suppose that $m(N) = 0$. Then since $\mathbb{Q}$ is countable we know $R = \mathbb{Q} \cap [0, 1)$ is countable, giving us by axiom (a) and (c) that:

$$1 = m([0, 1)) = m\left(\bigcup_{r \in R} N_r\right) = \sum_{r \in R} m(N_r) = \sum_{r \in R} 0 = 0$$

  This is a clear contradiction! Oops!

- Suppose that $m(N) > 0$. Note that R is countable and for $n \geq 2$ we have $0 < \frac{1}{n} < 1$ and so $\frac{1}{n} \in R$. Then using axiom (a), axiom (c), and the fact that $m(N)$ is positive we know that:

$$
\begin{aligned}
1 = m([0,1)) &= m\left(\bigcup_{r \in R} N_r\right) \\
&= \sum_{r \in R} m(N_r) \geq \sum_{n=2}^{\infty} m\left(N_{\frac{1}{n}}\right) \\
&= \sum_{n=2}^{\infty} m(N) = \infty
\end{aligned}
$$

  This is clearly true, since we know that $m(N) > 0$ doesn't go to zero, $\sum_{n=2}^{\infty} m(N)$ must diverge to infinity. This is an oops since $1 < \infty$

- Suppose that $m(N) = \infty$ Then since R is countable and $0 \in R = \mathbb{Q} \cap [0,1)$ we know that by axiom (b) and axiom (c),

$$
1 = m([0,1)) = m\left(\bigcup_{r \in R} N_r\right) = \sum_{r \in R} m(N_r) \geq m(N_0) = m(N) = \infty
$$

  This cannot be true since $1 < \infty$. Oops!

With all three of these completed, we must conclude that $m(N)$ is undefined!!! Wow! This is amazing ☺