Handout 7

Jordan measure and Riemann Integration

It turns out that the notion of Jordon measurability of sets is intimately related (in a way essentially equivalent) to the notion of Riemann integrability of functions. We will only display this relation in dimension 1.

- **Recall.** To define the Riemann[1] integral of a <u>bounded</u> function f on an interval $[a, b] \subset \mathbb{R}$, we first recall the notion of a partition $\mathcal{P}$ which is a set of points $x_0 = a < x_1 < x_2 < \ldots < x_n = b$, the norm of the partition is $\Delta\mathcal{P} = \max_{1\leq k\leq n} x_k - x_{k-1}$, and we denote by $\Delta x_k = x_k - x_{k-1}$. For each such partition, we define to quantities:

$$L(f, \mathcal{P}) = \sum_{k=1}^{n} f(x_*)\Delta x_k, \quad and \quad U(f, \mathcal{P}) = \sum_{k=1}^{n} f(x^*)\Delta x_k,$$

where $x_* = \inf_{[x_{k-1},x_k]} f$ and $x^* = \sup_{[x_{k-1},x_k]} f$.

Afterwards, we define the lower and upper Darboux integrals respectively as

$$\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(f, \mathcal{P}), \quad and \quad \overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

where the extrema above are taken over all partitions of the interval $[a, b]$. We say that f is Riemann integrable if the above two numbers are equal. We define the common value as the Riemann (or Darboux) integral of f.

[1]Strictly speaking, we are recalling here the notion of Darboux integral, but that is equivalent to the notion of Riemann integrability that is often covered in introductory calculus classes.

Q1) Let $[a,b]$ be an interval and let $f : [a,b] \to \mathbb{R}$ be a bounded nonnegative function. Show that f is Riemann integrable if and only if the set $E := \{(x,t) : x \in [a,b] : 0 \leq t \leq f(x)\}$ is Jordan measurable in $\mathbb{R}^2$.

Q2) Let $[a,b]$ be an interval and let $f : [a,b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable if and only if the sets $E_+ := \{(x,t) : x \in [a,b] : 0 \leq t \leq f(x)\}$ and $E_- := \{(x,t) : x \in [a,b] : f(x) \leq t \leq 0\}$ are Jordan measurable in $\mathbb{R}^2$.

Remark. The above results generalize to higher dimensions. For that we will need a notion of Riemann (or Darboux) integrability on $\mathbb{R}^d$ ($d \geq 2$). We will discuss this theory in our lectures, starting next week.