MATH 395 Notes

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Change of Variables Theorem

Theorem. *We look at:*

$$\int_A f(g(x))\,|\det Dg(x)|\,\mathrm{d}x = \int_{g(A)} f(u)\,\mathrm{d}u$$

Intuitively we have:

$$\begin{aligned} u &= g(x) \\ \mathrm{d}u &= |\det Dg|\,\mathrm{d}x \\ x &\in A, u \in g(A) \end{aligned}$$

And so this holds whenever:

- $g : A \to g(A) = B$ *is a* C^1*-diffeomorphism*
- ***TODO***

Example. We look at Polar Coordinate Integration. Let:

$$B = \{(x, y) \in \mathbb{R}^2 \mid a^2 < x^2 + y^2 < b^2\}$$

Then there are the polar coordinates:

$$g(r, \theta) = (r\cos(\theta), r\sin(\theta))$$

Note that $B = g(A)$ where $A = \{(r, \theta) \mid a < r < b, 0 \leq \theta \leq 2\pi\}$. Then let us introduce:

$$\begin{aligned}\widetilde{A} &:= \{(r,\theta) \mid a < r < b, 0 < \theta < 2\pi\} \\ \widetilde{B} &:= g(\widetilde{A}) = B \setminus (\text{x-axis})\end{aligned}$$

And so then we have:

$$\begin{aligned}\int_{\widetilde{B}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{g(\widetilde{A})} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\widetilde{A}} f(g(r,\theta)) \cdot |\det Dg(r,\theta)|\,\mathrm{d}r\,\mathrm{d}\theta\end{aligned}$$

And we know by previous homework that:

$$\begin{aligned}Dg(r,\theta) &= \begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} \\ \det Dg(r,\theta) &= r > 0\end{aligned}$$

Since we know that Dg is locally a C^1-diffeomorphism via the inverse function theorem and it is a bijection we know that it is a C^1-diffeomorphism, which is great. Now we apply Fubini:

$$\int_{\widetilde{B}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

Now since the x-axis has Lebesgue measure zero in $\mathbb{R}^2$, we then know that:

$$\int_B f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\widetilde{B}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

We know this because for C_N a nested sequence compact Jordan measurable set contained in B and covering B we know:

$$\begin{aligned}\int_{\widetilde{B}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \lim_{N\to\infty}\int_{C_N\setminus(\text{x-axis})} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \lim_{N\to\infty}\int_{C_N} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_B f(x,y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Great!

Example. Now for Spherical coordinate integration! Suppose we have:

$$B = \{(x, y, x) \mid x > 0, y > 0, z > 0, x^2 + y^2 + z^2 < a^2\}$$

Suppose we want to evaluate $\int_B f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Suppose we take the change of coordinates:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

And we'll denote this by $g(\rho, \phi, \theta)$. We aready calculated in previous homework that $\det Dg = \rho^2 \sin\phi$, and this is greater than 0 if $\rho > 0$ and $0 < \phi < \pi$. This happens on the set:

$$A = \left\{(\rho, \phi, \theta) : 0 < \rho < a, 0 < \phi < \frac{\pi}{2}, 0 < \theta < \frac{\pi}{2}\right\}$$

And here we have $g(A) = B$. Therefore using that g is a C^1 diffeomorphism from A to B and using Fubini we have that:

$$\begin{aligned} \int_B f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{g(A)} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_A f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\rho \end{aligned}$$

Some mapping Properties of diffeomorphisms

Lemma. *Let $A \subseteq \mathbb{R}^n$ be open and let $g : A \to \mathbb{R}^n$ be a C^1 function. If $E \subseteq A$ is a set of Lebesgue measure zero, then $g(E)$ also has Lebesgue measure zero.*

Remark. This is not true if g is only assumed to be continuous. In fact, there exists a continuous $g : [0, 1] \to [0, 1]^2$ that is onto. This is called Peano's space filling curve.

Proof. Let C_N be a family of compact sets such that $A = \bigcup_{N=1}^{\infty} C_N$ and $C_N \subseteq C_{N+1}^{\circ}$.

The note that:

$$E_N := E \cap C_N \qquad\qquad E = \bigcup_{N=1}^{\infty} E_N$$

It is enough to show that each $g(E_N)$ has Lebesgue measure zero.

Fix $\varepsilon > 0$ and let $M := \sup_{C_{N+1}} \|Dg\|_{\text{op}} < \infty$, since $g \in C^1$ and C_{N+1} is compact.

Also since $C_N \subseteq C_{N+1}^{\circ}$ there exists a $\delta > 0$ such that the δ-neighborhood of C_N is a subset of C_{N+1}°.

Since E_N has Lebesgue mreasure zero we can cover E_N by countably many boxes B_j such that $\sum v(B_j) < \varepsilon$. In fact, we can assume Without Loss of Generality that all the B_j are cubes and have diameter $< \delta$ by covering them with cubes of diameter $< \delta$.

Then $g(E_N)$ is a subset of $\bigcup g(B_j)$ where B_j is a cube of diameter less than δ

Claim. $\operatorname{diam} g(B_j) \le M \operatorname{diam} B_j$.

Proof. Let $x, x' \in B_j$. By the Mean Value Theorem for some c on the line segment between x and x':

$$\begin{aligned} g(x) - g(x') &= Dg(c)(x - x') \\ \left|g(x) - g(x')\right| &\le \|Dg(c)\|_{\text{op}} \left|x - x'\right| \\ &\le M \operatorname{diam} B_j \end{aligned}$$

Great!

Therefore $g(B_j)$ is contained in a ball of radius $M \operatorname{diam} B_j$ which is then contained in a cube of $\widetilde{Q}_j$ of side length $2M \operatorname{diam} B_j$. Also:

$$\begin{aligned} \sum_j v(Q_j) &= \sum_j (2M)^n \cdot (\operatorname{diam} B_j)^n \\ &= \sum_j (2M)^n \cdot (v(B_j))^n \cdot C \end{aligned}$$

For some constant C, since the B_j are cubes, and so their diameter is proportional to their volume. But then:

$$\sum_j (Q_j) = C(2M)^n \cdot \sum_j v(B_j) < (2M)^n \cdot C \cdot \varepsilon$$

But then since $(2M)^n \cdot C$ is a constant, we can take $\varepsilon \to 0$ and we will be done. This finishes the proof.

Corrolary. *Let $g : A \to B$ be a diffeomorphism betweeen two open sets A and B. Let $K \subseteq A$ be compact. Then:*

a) $g(K^\circ) = (g(K))^\circ$ and $g(\partial K) = \partial g(K)$

b) If K is Jordan measurable, then so is $g(K)$.

These results hold if K is not compact provided that $\partial K \subseteq A$ and $\partial g(K) \subseteq B$.

Proof. Let's go!

a) This takes some work!

- Since g^{-1} is continuous, then g is open. Therefore if $B(x, \delta) \subseteq K$ then $g(B(x, \delta))$ is an open subset of $g(K)$, which implies that $g(B(x, \delta)) \subseteq (g(K))^\circ$. And so $g(K^\circ) \subseteq (g(K))^\circ$.
- Also $g(A \setminus K) \subseteq B \setminus g(K)$ since g is one-to-one. Let $y \in \partial g(K)$. Then there exists an $x \in A$ such that $y = g(x)$. We know that $x \notin K^\circ$ since then y would belong to $(g(K))^\circ$.

  We also know $x \notin A \setminus K$ since otherwise $y \in B \setminus g(K)$ which also does not intersect $\partial g(K)$ since $g(K)$ is closed. Therefore $x \in \partial K$, and so $\partial g(K) \subseteq g(\partial K)$.
- Apply the same argument to g^{-1} and $g(K)$ to obtain that:

  $$g^{-1}((g(K))^\circ) \subseteq K^\circ$$
  $$\partial K \subseteq g^{-1}(\partial g(K))$$

  And therefore:

  $$(g(K))^\circ \subseteq g(K^\circ)$$
  $$g(\partial K) \subseteq \partial g(K)$$

Combining this with the previous part gives part (a)

b) Note that if K is Jordan measurable, then ∂K has Lebesgue measure zero. Since g is C^1 we then know that $g(\partial K) = \partial g(K)$ has Lebesgue measure zero, and so $g(K)$ is Jordan measurable.

Volumes and Determinants

Theorem. *Let A be an $n \times n$ matrix and let $h : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation $h(x) = Ax$. Let S be a Jordan measurable set in $\mathbb{R}^n$ and $T := h(S)$. Then:*

$$v(T) = |\det A|\, v(S)$$

Proof. T is Jordan measurable by the above corollary. Therefore when $|\det A| \neq 0$ we have by the change of variables that:

$$\begin{aligned} v(T) = v(T^\circ) &= \int_{T^\circ} 1 \, \mathrm{d}x \\ &= \int_{h(S^\circ)} = \int_{S^\circ} |\det A| \, \mathrm{d}y \\ &= |\det A|\, v(S^\circ) = |\det A|\, v(S) \end{aligned}$$

In Case 2, when $\det A = 0$ we know that the range of h is a subspace V of $\mathbb{R}^n$ os dimension $p < n$. Since V has Lebesgue measure zero (check!), we are done, since then $T \subseteq V$ will have Lebesgue measure zero.