Handout 1

- **What is a topology on a set X?** Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X that are called *open sets* satisfying the following three conditions:

C1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

C2) Given a collection $O_\alpha \in \mathcal{T}$ of index sets, then $\cup_\alpha O_\alpha \in \mathcal{T}$ as well; We say that $\mathcal{T}$ is closed under unions,

C3) Given a *finite* collection of open set $O_1, \ldots, O_n$, then $\cap_1^n O_n \in \mathcal{T}$; We say that $\mathcal{T}$ is closed under finite intersections.

- A topology can be equivalently defined by specifying the collection of *closed sets* which satisfy the same conditions as above except that we switch unions $\cup$ with intersections $\cap$ in conditions C2) and C3). The couple $(X, \mathcal{T})$ is called a topological space, or sometimes we just say X is a topological space if we're only playing with one agreed upon topology

- A space X can have more than one topology defined on it. A topology $\mathcal{T}_1$ is said to be finer or stronger than $\mathcal{T}_2$ if $\mathcal{T}_2 \subset \mathcal{T}_1$ (we say $\mathcal{T}_2$ is courser or weaker). Notice that the trivial topology $\{\emptyset, X\}$ is the weakest topology on X.

- One way to describe a topology on a set X is to define precisely all open sets. This is what we did for metric spaces. Occasionally, we want to define the smallest topology that designates a particular collection $\mathcal{B}$ of subsets of X as open. This is done as follows:

Q1) Let $\overline{\mathcal{B}}$ be the collection of subsets of X that contains the empty set, X, as well as all sets obtained as finite intersections of elements of $\mathcal{B}$. Show that the collection $\mathcal{T}$ obtained by taking unions of elements of $\overline{\mathcal{B}}$ is a topology on X.

Q2) Show that any other topology on X that contains $\mathcal{B}$ as open sets, contains $\mathcal{T}$. We call $\mathcal{T}$ the topology generated by $\mathcal{B}$. It is the coarsest topology containing $\mathcal{B}$.

- (Product Topology) One example where this construction is useful is to define a topology on the product of topological spaces. Suppose $(X_\alpha, \mathbb{T}_\alpha)$ are topological spaces for $\alpha \in A$ (where A is an index set that could be infinite). We would like to define a "natural" topology on $\prod_\alpha X_\alpha$. One reasonable requirement is that the *cylindrical sets* are open (cylindrical sets are those of the form $\prod_\alpha U_\alpha$ where all the U_α are open in X_α and all but one of them is equal to X_α. The topology generated by this collection is called the product or Tychonoff topology.

Q3) Consider the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as defined above. Why is this the same as the standard topology on $\mathbb{R}^2$ defined in class.

- We saw in class that the interval $[0, 1)$ is not open in $\mathbb{R}$, but is open relative to the half-line $[0, \infty)$ (taking the usual metric on $\mathbb{R}$ and $[0, \infty)$]).
Let us try to formalize and generalize this.

  Let (X, d) be a metric space and $Y \subset X$. Y is a metric space itself, by restricting the metric d to $Y \times Y$.

Q4) Let $E \subset Y$. We say that E is open relative to Y if it is open in the metric space (Y, d). Untangle what this definition means in terms of $N_\delta(p)$ neighborhood of a point $p \in E$. Deduce that if there is an open subset G of X, then $G \cap Y$ is open relative to Y.

Q5) Show that E is open relative to Y if and only if there exists an open subset G of X such that $E = G \cap Y$.

Q6) Compactness on the other hand behaves better. Suppose that $K \subset Y \subset X$. Then K is compact relative to X if and only if it is compact relative to Y.

Remark: As such, we always need to specify the ambient space when we talk about open/closed sets (that's why we always say "E is an open subset of X"), but we can make statements like "K is compact (or a compact metric space)" without the need to specify the ambient space.