MATH 395 Notes

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A Small Digression

Last time we showed that compactness in a metric space is equivalent to sequential compactness is equivalent to totally bounded and complete.

It is clear then that if the total space is complete then compactnes in that space is equivalent to closed and totally bounded

How is this useful in mathematics?

When solving an ODE or a PDE, we can often recast the problem as solving an equation of the form:

$$F(x) = 0$$

for some continuous function $F : X \to X$ and some metric space X, which will be a space of functions. Suppose we are able to find a sequence of approximate solutions to this equation, for example a sequence x_n such that:

$$F(x_n) = \varepsilon_n$$

Where we have $\|\varepsilon_n\|_X \to 0$ as $n \to \infty$. If we can then show that the sequence (x_n) belongs to a compact subset of X, then it must have a convergent subsequence. This convergent subsequence will converge to some x_0, and necessarily we will have $F(x_0) = 0$ as desired.

3 Continuous functions on metric spaces

Definition. *Let X and Y be metric spaces. We say that a function $f : X \to Y$ is continuous at $x_0 \in X$ provided that for every $\varepsilon > 0$ there exists a $\delta = \delta_{\varepsilon,x_0}$ such that whenever $d(y, x_0) < \delta$ we have $d(f(y), f(x_0)) < \varepsilon$.*

In other words, f maps $B_X(x_0, \delta)$ into $B_Y(f(x_0), \varepsilon)$. We say in particular that f is continuous when f is continuous at every point $x_0 \in X$

Proposition. *$f : X \to Y$ is continuous if and only if the inverse image of every open set $U \subseteq Y$ is open in X.*

Proof. Let's go!

- ($\Rightarrow$) Fix $x \in f^{-1}(U)$. Then since $f(x) \in U$, we know that there is an $\varepsilon > 0$ so that $B_Y(f(x), \varepsilon) \subseteq U$. By continuity there exists some $\delta > 0$ so that f maps $B_X(x, \delta)$ into $B_Y(f(x), \varepsilon$. Therefore:

$$B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \varepsilon)) \subseteq f^{-1}(U)$$

  Therefore $f^{-1}(U)$ is open.

- ($\Leftarrow$) Fix $x \in X$. Now fix $\varepsilon > 0$. Note that $B_Y(f(x), \varepsilon)$ is an open set in Y. Thus $f^{-1}(B_Y(f(x), \varepsilon))$ is open in X. Since x is in this set in particular, we know there exists a $\delta > 0$ so that:

$$\begin{aligned} B_X(x, \delta) &\subseteq f^{-1}(B_Y(f(x), \varepsilon)) \\ f(B_X(x, \delta)) &\subseteq B_Y(f(x), \varepsilon) \end{aligned}$$

  Therefore f is continuous at x. Since $x \in X$ was arbitrary, f is continuous.

Theorem. *Let X be a compact metric space and let $f : X \to Y$ be continuous, then $f(X)$ is compact*

Proof. Let $\{G_\alpha\}$ be an open cover of $f(X)$. Then $\{f^{-1}(G_\alpha)\}$ is an open cover of X. By compactness of X, there exists $\alpha_1, \dots, \alpha_n$ such that $\{f^{-1}(G_{\alpha_i})\}_{1 \le i \le n}$ is an open cover of X. But then $\{G_{\alpha_i}\}_{1 \le i \le n}$ is an open cover of $f(X)$.

Corrolary 1 (Extreme Value Theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous function. If f is compact, then f has a maximum and a minumum value.*

Proof. $f(X)$ is compact in $\mathbb{R}$. Therefore $f(X)$ is closed and bounded. Since it is bounded, $\inf f$ and $\sup f$ exist. Furthermore, since it is closed, we know that $\inf f, \sup f \in f(X)$. This shows that these are in fact a minimum and a maximum, as desired.

Definition. *Let X and Y be metric spaces. We say that $f : X \to Y$ is uniformly continuous if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that if $d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$*

Clearly uniform continuity implies continuity.

Theorem. *Let X be a compact metric space and Y be any metric space. If $f : X \to Y$ is continuous then it is in fact uniformly continuous.*

Proof. Pick some $\varepsilon > 0$. Let $\varepsilon' := \frac{\varepsilon}{2}$. Then for each $x \in X$ we know there is some $\delta_x > 0$ so that $f(B_{\delta_x}(x)) \subseteq B_{\varepsilon'}(f(x))$ by continuity. Let $\delta'_x := \frac{1}{2}\delta_x$. Now note that X is covered by these balls $\{B_{\delta'_x}(x)\}_{x \in X}$. So in particular since X is compact we have $x_1, \ldots, x_n$ and $\delta'_1, \ldots, \delta'_n > 0$ such that X is covered by $\{B_{\delta'_i}(x_i)\}_{1 \leq i \leq n}$. Note that we've notated $\delta'_i := \delta'_{x_i}$ and $\delta_i := \delta_{x_i}$ for convenience. Set:

$$\delta := \min_{1 \leq i \leq n} \delta'_i$$

Now let $x, y \in X$ so that $d(x, y) < \delta$. We know that there is some $1 \leq i \leq n$ so that $x \in B_{\delta'_i}(x_i)$. Then in particular:

$$\begin{aligned} d(x_i, y) &\overset{\triangle}{\leq} d(x_i, x) + d(x, y) < \delta'_i + \delta \\ &\leq \delta'_i + \delta'_i = \delta_i \end{aligned}$$

Therefore since $\delta'_i < \delta_i$ it is clear that $x, y \in B_{\delta_i}(x_i)$. Great! Then we must have that $f(x), f(y) \in B_{\varepsilon'}(f(x))$. Which gives:

$$d(f(x), f(y)) \overset{\triangle}{\leq} d(f(x), f(x_i)) + d(f(x_i), f(y)) < \varepsilon' + \varepsilon' = \varepsilon$$

Awesome! We win! f is uniformly continuous. See Hani's notes for an equivalent way to do this with Lemma 3' from previous lecture (it is a similar idea).

Part II

Differentiation on $\mathbb{R}^d$

1 Definition of the derviative

1.1 Recollection

Recall. For $\phi : I \to \mathbb{R}$ where I is an open subset of $\mathbb{R}$, we call ϕ differentiable at $x_0 \in I$ provided that the limit

$$\lim_{h \to 0} \frac{\phi(x_0 + h) - \phi(x_0)}{h}$$

exists. If so we call this limit $\phi'(x_0)$.

We call ϕ differentiable in I if it is differentiable at every point $x \in I$¿ If I is not open, then we say ϕ is differentiable on I if there exists an extension Φ of ϕ to some open set $J \supseteq I$ such that $\Phi = \phi$ on I and Φ is differentiable on J.

1.2 Generalization Steps

How do we generalize this? We would like to look at functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ for $n, m \in \mathbb{N}$. If $n = 1$ and $m \geq 1$ then the same definition works:

$$\phi'(x_0) = \lim_{h \to 0} \frac{\phi(x_0 + h) - \phi(x_0)}{h}$$

Exercise. *Show that $\phi = (\phi_1, \dots, \phi_m) : I \to \mathbb{R}^m$ where $I \subseteq \mathbb{R}$ where $I \subseteq \mathbb{R}$ is differentiable at x_0 if and only if ϕ_j is differentiable at x_0 ofor every $1 \leq j \leq m$ and moreover:*

$$\phi'(x_0) = (\phi_1'(x_0), \dots, \phi_m'(x_0))$$

We run into trouble when $n \geq 1$ we run into trouble because we cannot divide by a vector. Let's reinterpret the case where $n = 1$ to deal with this. Note that:

$$\lim_{h \to 0} \frac{\phi(x_0 + h) - \phi(x_0)}{h} - \phi'(x_0) = 0$$
$$\lim_{h \to 0} \frac{\phi(x_0 + h) - \phi(x_0) - \phi'(x_0) \cdot h}{h} = 0$$
$$\lim_{h \to 0} \frac{|\phi(x_0 + h) - \phi(x_0) - \phi'(x_0)h|}{|h|} = 0$$

The final definition of differntiability at x_0 makes much better since for $\phi : \mathbb{R}^n \to \mathbb{R}^m$, since $|h|$ is a nonzero real number. But we need to properly interpret $\phi'(x_0)h$.

Note that for $\phi : \mathbb{R} \to \mathbb{R}^m$, then $\phi'(x_0)$ provides the best linear approximation to $\phi(x_0 + h) - \phi(x_0)$. Namely if $\Delta_h \phi(x_0) = \phi(x_0 + h) - \phi(x_0)$ then the definition of $\phi'(x_0)$ tells us that:

$$r(h) := \Delta_h \phi(x_0) - \phi'(x_0)h$$

Satisfies $\frac{|r(h)|}{|h|} \to 0$ as $h \to 0$. Essentially, this means that $\phi'(x_0)h$ takes the increment h in x and gives us the best linear approximation to $\Delta_h \phi(x_0)$. This means that $\phi'(x_0)$ can be interepted as a linear transformation from $\mathbb{R}$ to $\mathbb{R}^m$

1.3 The Correct Generalization

Definition. *Let $E \subseteq \mathbb{R}^n$ be open and let $f : E \to \mathbb{R}^m$. We say that f is differentiable at $x \in E$ provided that there exists a linear transformation $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that:*

$$\lim_{\|h\| \to 0} \frac{\|f(x + h) - f(x) - [Df(x)](h)\|}{\|h\|} = 0$$

We can think of $Df(x_0)$ as an $m \times n$ matrix by linear alegbra. We will prove that $Df(x)$ is unique next lecture, justifying the notation.

Note that the f increment is $\Delta_h f(x) = f(x + h) - f(x)$. How good is the approximation, namely $r(h) = \Delta_h f(x) - Df(x)h$ for a fixed $x \in E$. Then:

$$\lim_{\|h\| \to 0} \frac{\|r(h)\|}{\|h\|} = 0$$