

Handout 4

- **Wish list for a measure function** It would be grand to have a measure function that tells us how big or small a subset of \mathbb{R}^d is. This would be a function from the set of subsets of \mathbb{R}^d into $[0, \infty]$, say $m : \mathcal{P}(\mathbb{R}^d) \rightarrow [0, \infty]$. We would like this function to satisfy the following properties:

- a) If E_1, E_2, \dots is a countable collection of disjoint subsets of \mathbb{R} , then

$$m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n).$$

This is called **Countable Additivity**.

- b) If E is congruent to F (i.e. F can be obtained from E by applying rigid motions: translations, rotations, or a reflections) then we should have that $m(E) = m(F)$.
- c) $m([0, 1]^d) = 1$.

The bad news is that no such function can exist, and here's why (at least when $d = 1$). Let us define an equivalence relation between elements of $[0, 1)$ as follows: We say $x \sim y$ if $x - y$ is a rational number. Let N be the subset of $[0, 1)$ that contains exactly one element of each equivalence relation (the existence of this N requires invoking the axiom of choice). Now let $R = [0, 1) \cap \mathbb{Q}$, and for each $r \in R$ define the set

$$N_r = \{x + r : x \in N \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in N \cap [1 - r, 1)\}.$$

(Basically N_r is just the translate of N by r units to the right, except that we move the part that sticks out of the interval $[0, 1)$ one unit to the left).



Figure 1: Banach-Tarski tells us that we can split the unit ball in \mathbb{R}^3 into finitely many (actually 5 is sufficient) many disjoint pieces, apply rigid motions to those pieces and then reassemble them to obtain two copies of the unit ball.

- Q1)** Show that $[0, 1)$ is the disjoint union of N_r for $r \in R$.
- Q2)** Show that if a measure function satisfying a), b) and c) above exists, then $m(N) = m(N_r)$ for every $r \in R$.
- Q3)** Arrive at a contradiction.

Remark: One might think that possibly relaxing condition a) to cover only *finitely* many disjoint sets E_n , i.e.

$$m(\cup_{n=1}^N E_n) = \sum_{n=1}^N m(E_n). \quad \text{(Finite Additivity)}$$

would resolve the contradiction. Unfortunately, the Banach-Tarski paradox (cf. Figure 1) tells us that this is not enough to resolve this issue.

Conclusion: The problem with the above wishlist is that we insisted on being able to measure *every* subset of \mathbb{R}^d . We have shown that this is impossible. The solution is to be content with a measure function that is defined on some but not all subsets. Such subsets will be called measurable subsets.

The Greek method

- **Elementary measure.** An interval I is a subset of \mathbb{R} of the form $[a, b]$, $[a, b)$, $(a, b]$, or (a, b) where $a, b \in \mathbb{R}$. The length of I is defined to be $|I| := b - a$. A *box* in \mathbb{R}^d is a Cartesian product of intervals $B = I_1 \times I_2 \times \dots \times I_d$ and its *volume* is defined to be $|B| = |I_1| \cdot \dots \cdot |I_d|$. An *elementary set* is any subset of \mathbb{R}^d which is the union of a finite number of boxes.

- Q4)** Show that if $E, F \subset \mathbb{R}^d$ are elementary sets, then the union $E \cup F$, the intersection $E \cap F$, the set theoretic difference $E \setminus F$, and the symmetric difference $E \Delta F = (E \setminus F) \cup (F \setminus E)$ are also elementary. Also, if $x \in \mathbb{R}^d$, then the translate $E + x := \{y + x : y \in E\}$ is also elementary.
- Q5)** Show that E can be expressed as the finite union of disjoint boxes. *Hint: Start with $d = 1$. Then use this result to generalize it to higher dimensions.*
- **Definition.** Let E be an elementary set. The above question allows to write $E = B_1 \cup B_2 \cup \dots \cup B_n$ where B_1, \dots, B_n are disjoint. We define the elementary measure of E as $m(E) := |B_1| + |B_2| + \dots + |B_n|$.
- Q6)** Show that $m(E)$ is well-defined in the sense that if E can be expressed in two ways as a union of disjoint boxes B_1, \dots, B_n and B'_1, \dots, B'_m , then

$$|B_1| + |B_2| + \dots + |B_n| = |B'_1| + |B'_2| + \dots + |B'_m|.$$

Hint: There's more than one approach you can take. One is to notice that for an interval I in \mathbb{R} , there holds that

$$|I| = \lim_{N \rightarrow \infty} \frac{1}{N} \# \left(I \cap \frac{1}{N} \mathbb{Z} \right).$$

(why?). And more generally for a box B ,

$$|B| = \lim_{N \rightarrow \infty} \frac{1}{N^d} \# \left(B \cap \frac{1}{N} \mathbb{Z}^d \right).$$

Here $\frac{1}{N} \mathbb{Z}^d = \{ \frac{k}{N} : k \in \mathbb{Z}^d \}$. Use this to give an alternative definition of $m(E)$ for an elementary set that does rely on its decomposition into disjoint boxes.

MATH 395 Notes

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The wish list:

a) Say that E_1, E_2, E_3, \dots are disjoint subsets of \mathbb{R}^d , then:

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$$

This is Countable Additivity

b) If E is congruent to F via translations, rotations, and combinations of these. We want to have $m(E) = m(F)$.

c) We want $m([0, 1]^d) = m([0, 1]^d) = 1$.

We know from last week and Q1-Q3 that there cannot be an $m : P(\mathbb{R}^d) \rightarrow [0, \infty]$ that satisfies a), b), and c). We construct the set $N \subseteq [0, 1)$ containing exactly one element of each equivalence class for $x \sim y$ defined by $x - y \in \mathbb{Q}$. We define N_r as “essentially translates” of N by $r \in \mathbb{Q} \cap [0, 1)$. Then in fact:

$$[0, 1) = \coprod_{r \in \mathbb{Q} \cap [0, 1)} N_r$$

And this union is disjoint. Furthermore $m(N_r) = m(N)$ for each r because of congruence, so:

$$1 = m([0, 1)) = \sum_{r \in \mathbb{Q} \cap [0, 1)} m(N_r) = \sum_{n=1}^{\infty} m(N)$$

And whatever we choose for the measure of N , this produces a contradiction.

Proof of Q4. Let's go!

- First note that if $E = \bigcup_{i=1}^n A_i$ and $F = \bigcup_{k=1}^m B_k$ for some boxes A_i and B_k , then set $C_i = A_i$ if $1 \leq i \leq n$ and $C_i = B_{i-n}$ if $n < i \leq m+n$:

$$E \cup F = \bigcup_{i=1}^n A_i \cup \bigcup_{k=1}^m B_k = \bigcup_{i=1}^{m+n} C_i$$

And so we have that $E \cup F$ is an elementary set as desired.

- We wish to show that $E \cap F$ is an elementary set for elementary sets E and F , **TODO**
- We wish to show that $E \setminus F$ is an elementary set for elementary sets E and F , **TODO**
- Note now that for elementary sets E and F we know:

$$E \triangle F = (E \setminus F) \cup (F \setminus E)$$

And so by the previous bullets $E \setminus F$ and $F \setminus E$ are elementary, and so their union $E \triangle F$ is elementary

- We wish to show that the translate $E + x$ is elementary for an elementary E . **TODO**

With this we win!

Proof of Q5. induct **TODO**

Proof of Q6. **TODO**

