MATH 395 Notes

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Last time, we proved that:

Theorem. *Let A be an $n \times n$ matrix and $h : \mathbb{R}^n \to \mathbb{R}^n$ be given by $h(x) = A \cdot x$. If S is Jordan measurable then $h(S)$ is Jordan measurable and:*

$$\operatorname{vol}(h(S)) = |\det A| \cdot \operatorname{vol}(S)$$

Corrolary. *Let $a_1, \ldots, a_n$ be n linearly independent vectors of $\mathbb{R}^n$. Let $A = [a_1, \ldots, a_n]$ be the $n \times n$ matrix whose columns are $a_1, \ldots, a_n$ and let P be the parallelopiped given by:*

$$P = \left\{ \sum c_i a_i \mid 0 \leq c_i \leq 1 \right\}$$

Then $v(P) = |\det A|$

Proof. Let $h(x) = Ax$, then h takes the unit cube in $\mathbb{R}^n$ to P. Therefore:

$$\operatorname{vol}(P) = \operatorname{vol}(h(S)) = |\det A| \cdot \operatorname{vol}([0,1]^n) = |\det A|$$

Orientations

Definition. *Let $\beta = (a_1, \ldots, a_n)$ be a basis of $\mathbb{R}^n$. We call this basis right-handed if $\det(a_1, \ldots, a_n) > 0$ and left-handed if $\det(a_1, \ldots, a_n) < 0$.*

On a general vector space V. Let $\beta = (v_1, \ldots, v_n)$ and $\beta' = (w_1, \ldots, w_n)$ be two bases of V. Let $w_j = a_{j1} v_1 + \cdots + a_{jn} v_n$. Then the matrix $A = (a_{jk})$ is invertible *since:*

$$A = {}_{\beta'}[\mathrm{Id}]_{\beta}$$

is a change of basis matrix. We say that β *and* β' *have the same orientation if* $\det A > 0$ *and opposite orientation if* $\det A < 0$.

Remark. The choice of notation is motivated by the 2D and 3D cases in which we have the right-hand rule

Exercise. *Show that:*

1) *This gives an equivalence relation on the set of bases of* V *with two equivalence classes.*

2) *Another way to define this equivalence relation is as follows. Pick* $T : \mathbb{R}^n \to V$ *a linear isomorphism. Any basis* β *of* V *can be writte as* $\{Ta_1, \ldots, Ta_n\}$ *where* $(a_1, \ldots, a_n)$ *is a basis of* $\mathbb{R}^n$. *So given two bases* $\beta = \{Ta_1, \ldots, Ta_n\}$ *and* $\beta' = \{Tb_1, \ldots, Tb_n\}$.

   β *and* β' *have the same orientation if and only if* $(a_1, \ldots, a_n$ *and* $(b_1, \ldots, b_n)$ *have the same orientation in* $\mathbb{R}^n$.

Theorem. *Let* C *be a non-singular* $n \times n$ *matrix and let* $h : \mathbb{R}^n \to \mathbb{R}^n$ *be given by* $h(x) = Cx$. *Let* $(a_1, \ldots, a_n)$ *be a basis in* $\mathbb{R}^n$. *Then the two bases* $(a_1, \ldots, a_n)$ *and* $(h(a_1), \ldots, h(a_n))$ *have the same orientation if and only if* $\det C > 0$.

Proof. Let $b_j = h(a_j)$. Then $C[a_1, \ldots, a_n] = [b_1, \ldots, b_n]$. But then:

$$\det C \cdot \det(a_1, \ldots, a_n) = \det(b_1, \ldots, b_n)$$

And so $\det C > 0$ if and only if $\det(a_1, \ldots, a_n)$ and $\det(b_1, \ldots, b_n)$ have the same sign, which is exactly when they have the same orientation.

Isometries of $\mathbb{R}^n$

Definition. *Let* $h : X \to Y$ *be a map between metric spaces* (X, d_X) *and* (Y, d_Y). *We say that* h *is an isometry provided that:*

$$d_Y(h(x_1), h(x_2)) = d_X(x_1, x_2) \qquad (x_1, x_2 \in X)$$

Remark. Isometries are always one-to-one, but they might not be onto. For example $h : \mathbb{R} \to \mathbb{R}^2$ where $h(x) = (x, 0)$.

Here we will discuss isometries from $\mathbb{R}^n \to \mathbb{R}^n$ with the same Euclidean metrix

Example. Lets grab some examples!

1) Consider $h : \mathbb{R}^n \to \mathbb{R}^n$ where $h(x) = x - a$ for a constant $a \in \mathbb{R}^n$, since:

$$h(x) - h(y) = x - a - y + a = x - y \implies \|h(x) - h(y)\| = \|x - y\|$$

2) Let $h : \mathbb{R}^n \to \mathbb{R}^n$ where $h(x) = Ax$ and A is an orthogonal matrix. Then h is an isometry:

   Recall. A is orthogonal means $A^T A = AA^T = \mathrm{Id}$. In other words:

$$\langle Ax, Ay \rangle = \langle A^T Ax, y \rangle = \langle x, y \rangle$$

   That is A preserves inner products

   But then we know that:

$$\begin{aligned} \|Ax - Ay\|^2 &= \langle Ax - Ay, Ax - Ay \rangle \\ &= \langle A(x - y), A(x - y) \rangle \\ &= \langle x - y, x - y \rangle = \|x - y\|^2 \end{aligned}$$

   And therefore h is an isometry.

The interesting fact is that these are the only two examples of isometries on $\mathbb{R}^n$

Theorem. *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a map such that $h(0) = 0$. Then:*

a) h is an isometry if and only if h preserves inner products

b) h is an isometry if and only if $h = Ax$ where A is an orthogonal matrix.

Proof. Let's go!

a) Consider that:

$$\begin{aligned} \|h(x) - h(y)\|^2 &= \langle h(x) - h(y), h(x) - h(y) \rangle \\ &= \langle h(x), h(x) \rangle - 2\langle h(x), h(y) \rangle + \langle h(y), h(y) \rangle \end{aligned}$$

And:

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle \\ &= \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle\end{aligned}$$

Now we can do this. Therefore if h preserves inner products we must have $\|h(x)-h(y)\| = \|x-y\|$, and so h is an isometry.

On the other hand, if h is an isometry and $h(0)=0$ then:

$$\langle h(x), h(x)\rangle = |h(x)|^2 = |h(x)-h(0)|^2 = |x-0|^2 = \langle x,x\rangle$$

We also know for every $x, y \in \mathbb{R}^n$ that $|h(x)-h(y)|^2 = |x-y|^2$ and so using the above two equations again we see that:

$$2\langle h(x), h(y)\rangle = 2\langle x,y\rangle \implies \langle h(x), h(y)\rangle = \langle x,y\rangle$$

b) The backwards implication was discussed in the previous direction. For the forward direction consider $\{h(e_1), h(e_2), \ldots, h(e_n)\}$ where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Since h preserves inner products $\{h(e_1), \ldots, h(e_n)\}$ is an orthonormal set, which implies that it is an orthonormal basis.

Therefore for any $x \in \mathbb{R}^n$ we can express:

$$h(x) = \sum_{j=1}^{n} \alpha_j(x) h(e_j)$$

And then we know that:

$$\begin{aligned}\langle h(x), h(e_k)\rangle &= \left\langle \sum_{j=1}^{n} \alpha_j(x) h(e_j), h(e_k)\right\rangle \\ &= \sum_{j=1}^{n} \alpha_j(x) \cdot \langle h(e_j), h(e_k)\rangle \\ &= \sum_{j=1}^{n} \alpha_j(x) \cdot \langle e_j, e_k\rangle \\ &= \alpha_k(x)\end{aligned}$$

But then we have that:

$$\alpha_k(x) = \langle h(x), h(e_k) \rangle = \langle x, e_k \rangle = x_k$$

And therefore:

$$h(x) = \sum_{j=1}^{n} x_j h(e_j) = Ax$$

where $A = [h(e_1), \ldots, h(e_n)]$. Since this is an orthonormal basis, A is orthogonal and so we are done.

Corrolary. *Let $h : \mathbb{R}^n \to \mathbb{R}^n$. Then:*

1) h is an isometry if and only if it is an orthogonal transformation followed by a translation. I.e. $h(x) = Ax + p$ where A is an orthogonal matrix and $p \in \mathbb{R}^n$.

2) If h is an isometry, then h preserves volumes as well. That is if S is Jordan measurable, then $h(S)$ is Jordan measurable and:

$$v(h(S)) = v(S)$$

Proof. This is pretty cool!

1) Let $\widetilde{h}(x) = h(x) - h(0)$. Then h is an isometry if and only if $\widetilde{h}$ is an isometry with $\widetilde{h}(0) = 0$, and this holds by the previous theorem if and only if $\widetilde{h}(x) = Ax$ for A some orthogonal matrix.

   Then by rearrangement h is an isometry if and only if:

$$h(x) = \widetilde{h}(x) + h(0) = Ax + h(0)$$

   For some orthogonal matrix A.

2) We know that $A \cdot S$ is Jordan measurable with volume $|\det A| \cdot v(S) = v(S)$ since $|\det A| = 1$ when A is orthogonal. Of course $A \cdot S + p$ has the same measure as $A \cdot S$, and so $h(S) = A \cdot S + p$, and therefore $v(h(S)) = v(S)$ as desired!!!

Great!