Handout 5

The Elementary measure (Continued)

- Recall from last time that an interval I is a subset of $\mathbb{R}$ of the form $[a,b],[a,b),(a,b],or(a,b)$ where $a,b\in\mathbb{R}$. The length of I is defined to be $|I|:=b-a$. A *box* in $\mathbb{R}^d$ is a Cartesian product of intervals $B=I_1\times I_2\times\ldots I_d$ and its *volume* is defined to be $|B|=|I_1|.\ldots.|I_d|$. An *elementary set* is any subset of $\mathbb{R}^d$ which is the union of a finite number of boxes.

- **Definition.** Let E be an elementary set. Last time we saw that we can write $E=B_1\cup B_2\cup\ldots B_n$ where $B_1,\ldots,B_n$ are disjoint. We define the elementary measure of E as $m(E):=|B_1|+|B_2|+\ldots+|B_n|$.

Q1) Show that $m(E)$ is well-defined in the sense that if E can be expressed in two ways as a union of disjoint boxes $B_1,\ldots B_n$ and $B_1',\ldots B_m'$, then

$$|B_1|+|B_2|+\ldots+|B_n|=|B_1'|+|B_2'|+\ldots+|B_m'|.$$

Hint: There's more than one approach you can take. One is to notice that for an interval I in $\mathbb{R}$, there holds that

$$|I|=\lim_{N\to\infty}\frac{1}{N}\#\left(I\cap\frac{1}{N}\mathbb{Z}\right).$$

(why?). And more generally for a box B,

$$|B|=\lim_{N\to\infty}\frac{1}{N^d}\#\left(B\cap\frac{1}{N}\mathbb{Z}^d\right).$$

Here $\frac{1}{N}\mathbb{Z}^d = \{\frac{k}{N} : k \in \mathbb{Z}^d\}$. Use this to give an alternative definition of $m(E)$ for an elementary set that does rely on its decomposition into disjoint boxes.

- **Properties of Elementary measure.** Show that the following holds

Q2) Show that if $E_1, \ldots, E_n$ are disjoint elementary sets, then

$$m(E_1 \cup \ldots \cup E_n) = \sum_{i=1}^{n} m(E_i)$$

Recall that this is called **finite additivity**.

Q3) Show that if $E \subset F$ are two elementary sets, then

$$m(E) \leq m(F).$$

This property is called **monotonicity**.

Q4) Show that if $E_1, E_2, \ldots, E_n$ is an arbitrary finite collection of elementary sets, then

$$m(E_1 \cup \ldots E_n) \leq m(E_1) + \ldots + m(E_n).$$

This is called **finite subadditivity**.

- **Why is this unsatisfactory?** Of course, the main problem with this measure is that we can only measure relatively simple sets (namely the elementary sets). For example, we cannot measure the area of a disc. One might be tempted to generalize this measure naively as follows: For an arbitrary set $E \subset \mathbb{R}^d$, define

$$m_{\text{pixel}}(E) = \lim_{N\to\infty} \frac{1}{N^d} \# \left(E \cap \frac{1}{N}\mathbb{Z}^d \right).$$

However, this is not a particurlary satisfactory definition for (at least) the following two reasons:

Q5) Find a subset E of $\mathbb{R}$ for which this limit does not exist.

Q6) Find a subset E of $\mathbb{R}$ such that both $m_{\text{pixel}}(E)$ and $m_{\text{pixel}}(E+x)$ exist, but $m_{\text{pixel}}(E) \neq m_{\text{pixel}}(E+x)$ for some $x \in \mathbb{R}$.

MATH 395 Notes

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Proof of Q1.

Lemma. *For any interval I in $\mathbb{R}$ we have that:*

$$|I| = \lim_{N\to\infty} \frac{1}{N} \cdot \# \left(I \cap \frac{1}{N}\mathbb{Z} \right)$$

Proof. Consider that the following sets are in bijection:

$$\begin{aligned} f : I \cap \frac{1}{N}\mathbb{Z} &\to NI \cap \mathbb{Z} \\ x &\mapsto N \cdot I \end{aligned}$$

This maps its domain into the codomain by definition, since $N \cdot I = \{N \cdot x \mid x \in I\}$ and $\frac{1}{N}\mathbb{Z} = \left\{\frac{1}{N} \cdot m \mid m \in \mathbb{Z}\right\}$. We also know since $N > 0$ that this is an injection from linear algebra. We also know surjectivity as well by quick definition from the sets. Now say I has endpoints $a \leq b$, then NI has endpoints aN and bN.

Now note that the cardinality $\#(NI \cap \mathbb{Z})$ is between $bN - aN - 5$ and $bN - aN + 5$. So then note that:

$$\begin{aligned} bN - aN - 5 \leq \#(NI \cap \mathbb{Z}) &\leq bN - aN + 5 \\ b - a - \frac{5}{N} \leq \frac{1}{N}\#\left(I \cap \frac{1}{N}\mathbb{Z}\right) &\leq b - a + \frac{5}{N} \\ b - a \leq \lim_{N\to\infty} \frac{1}{N}\#\left(I \cap \frac{1}{N}\mathbb{Z}\right) &\leq b - a \end{aligned}$$

By squeeze theorem! We win! This limit is equal to $|I| = b - a$.

Lemma. *For any box $B \subseteq \mathbb{R}^d$, we have:*

$$|B| = \lim_{N\to\infty} \frac{1}{N^d} \# \left(B \cap \frac{1}{N}\mathbb{Z}^d \right)$$

Proof. First write $B = \prod_{k=1}^d I_k$ for intervals I_k and note that:

$$B \cap \frac{1}{N}\mathbb{Z}^d = \left(\prod_{k=1}^d I_k \right) \cap \prod_{k=1}^d \frac{1}{\mathbb{N}} \cdot \mathbb{Z} = \prod_{k=1}^d \left(I_k \cap \frac{1}{\mathbb{N}} \cdot \mathbb{Z} \right)$$

$$\# \left(B \cap \frac{1}{N}\mathbb{Z}^d \right) = \# \left(\prod_{k=1}^d \left(I_k \cap \frac{1}{\mathbb{N}} \cdot \mathbb{Z} \right) \right) = \prod_{k=1}^d \# \left(I_k \cap \frac{1}{\mathbb{N}} \cdot \mathbb{Z} \right)$$

So now we write that:

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N^d} \# \left(B \cap \frac{1}{N}\mathbb{Z}^d \right) &= \lim_{N\to\infty} \frac{1}{N^d} \prod_{k=1}^d \# \left(I_k \cap \frac{1}{\mathbb{N}} \cdot \mathbb{Z} \right) \\ &= \prod_{k=1}^d \lim_{N\to\infty} \frac{1}{N} \# \left(I_k \frac{1}{N}\mathbb{Z} \right) \\ &= \prod_{k=1}^d |I_k| = |B| \end{aligned}$$

And therefore the lemma is proved!

We prove one final lemma, and then the result will fall out!

Lemma. *Suppose that we have two disjoint sets $X, Y \subseteq \mathbb{R}^d$ and the limits:*

$$\lim_{N\to\infty} \frac{1}{N^d} \# \left(X \cap \frac{1}{N}\mathbb{Z}^d \right) \qquad \lim_{N\to\infty} \frac{1}{N^d} \# \left(Y \cap \frac{1}{N}\mathbb{Z}^d \right)$$

both exist, then:

$$\lim_{N\to\infty} \frac{1}{N^d} \# \left((X \cup Y) \cap \frac{1}{N}\mathbb{Z}^d \right) = \lim_{N\to\infty} \frac{1}{N^d} \# \left(X \cap \frac{1}{N}\mathbb{Z}^d \right) + \lim_{N\to\infty} \frac{1}{N^d} \# \left(Y \cap \frac{1}{N}\mathbb{Z}^d \right)$$

Proof. This is fairly simple to prove. Note that:

$$(X \cup Y) \cap \frac{1}{N}\mathbb{Z}^d = \left(X \cap \frac{1}{N}\mathbb{Z}^d\right) \cup \left(Y \cap \frac{1}{N}\mathbb{Z}^d\right)$$

And since these are disjoint:

$$\begin{aligned} \#\left((X \cup Y) \cap \frac{1}{N}\mathbb{Z}^d\right) &= \#\left(\left(X \cap \frac{1}{N}\mathbb{Z}^d\right) \cup \left(Y \cap \frac{1}{N}\mathbb{Z}^d\right)\right) \\ &= \#\left(X \cap \frac{1}{N}\mathbb{Z}^d\right) + \#\left(Y \cap \frac{1}{N}\mathbb{Z}^d\right) \end{aligned}$$

We then know that we can take the limit as $N \to \infty$ on either side by real analysis and we must get the same limit as desired in the lemma

Now fix an elementary set $E \subseteq \mathbb{R}^d$ and let it be the union of disjoint boxes $B_1, \ldots, B_n$. By applying the lemmas multiple times:

$$\begin{aligned} m(E) = \sum_{k=1}^{n} |B_k| &= \sum_{k=1}^{n} \lim_{N\to\infty} \frac{1}{N^d} \# \left(B_k \cap \frac{1}{N}\mathbb{Z}^d\right) \\ &= \lim_{N\to\infty} \frac{1}{N^d} \# \left(\left(\bigcup_{k=1}^{n} B_k\right) \cap \frac{1}{N}\mathbb{Z}^d\right) \\ &= \lim_{N\to\infty} \frac{1}{N^d} \left(E \cap \frac{1}{N}\mathbb{Z}^d\right) \end{aligned}$$

Now note that the limit does not depend on the choice of disjoint boxes $B_1, \ldots, B_n$, so if we choose another choice of disjoint boxes $B_1', \ldots, B_m'$ that union to E then we know:

$$\sum_{k=1}^{n} |B_k| = \lim_{N\to\infty} \frac{1}{N^d} \left(E \cap \frac{1}{N}\mathbb{Z}^d\right) = \sum_{k=1}^{m} |B_k'|$$

And so the measure of E is well-defined.