Handout 3

- **The Cantor set.** Let us start with the interval $C = [0,1]$ and remove the middle third open interval $(\frac{1}{3}, \frac{2}{3})$. This leaves us with the set $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ formed of 2 closed subintervals. Having constructed $C_1 \supset C_2 \supset \ldots \supset C_n$ where C_n is the union of 2^n subintervals each of length $\frac{1}{3^n}$, we construct C_{n+1} as follows: To obtain C_{n+1} we remove the middle third of each of the 2^n intervals that form C_n. This leaves us with a union of 2^{n+1} intervals each of length $\frac{1}{3^{n+1}}$.

Q1) Let $C = \cap_{n=1^\infty} C_n$. Why is C non-empty? Is it compact?

Q2) Show that every point in C is a limit point. Hence C is a perfect set.

Conclusion: From the homework (HW 2), we deduce that C is uncountable, since any perfect subset of $\mathbb{R}^d$ is uncountable.

Q3) Show that C cannot contain any interval (a, b).

Conclusion: As such, C is totally disconnected (it has no non-trival connected subset) and nowhere dense (the interior of its closure is empty).

Q4) What is the total length of C_n? What would be a reasonable definition of the length of C?

- **Wish list for a measure function** Motivated by the above, it would be grand to have a measure function that tells us how big or small a subset of $\mathbb{R}^d$ is. This would be a function from the set of subsets of $\mathbb{R}^d$ into $[0, \infty]$, say $m : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$. We would like this function to satisfy the following properties:

a) If $E_1, E_2, \dots$ is a countable collection of disjoint subsets of $\mathbb{R}$, then

$$m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n).$$

b) If E is congruent to F (i.e. F can be obtained from E by applying rigid motions: translations, rotations, or a reflections) then we should have that $m(E) = m(F)$.

c) $m([0,1)^d) = 1$.

The bad news is that no such function can exist, and here's why (at least when $d = 1$). Let us define an equivalence relation between elements of $[0,1)$ as follows: We say $x \sim y$ if $x - y$ is a rational number. Let N be the subset of $[0,1]$ that contains exactly one element of each equivalence relation (the existence of this N requires invoking the axiom of choice). Now let $R = [0,1) \cap \mathbb{Q}$, and for each $r \in R$ define the set

$$N_r = \{x + r : x \in N \cap [0, 1-r]\} \cup \{x + r - 1 : x \in N \cap [1-r, 1)\}.$$

(Basically N_r is just the translate of N by r units to the right, except that we move the part that sticks out of the interval $[0,1)$ one unit to the left).

Q5) Show that $[0,1)$ is the disjoint union of N_r for $r \in R$.

Q6) Show that if a measure function satisfying a), b) and c) above exists, then $m(N) = m(N_r)$ for every $r \in R$.

Q7) Arrive at a contradiction.

Remark: One might think that possibly relaxing condition a) to cover only *finitely* many disjoint sets E_n, i.e.

$$m(\cup_{n=1}^{N} E_n) = \sum_{n=1}^{N} m(E_n).$$

would resolve the contradiction. Unfortunately, the Banach-Tarski paradox (cf. Figure 1) tells us that this is not enough to resolve this issue.

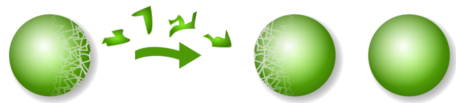

Figure 1: Banach-Tarski tells us that we can split the unit ball in $\mathbb{R}^3$ into finitely many (actually 5 is sufficient) many disjoint pieces, apply rigid motions to those pieces and then reassemble them to obtain two copies of the unit ball.

Conclusion: The problem with the above wishlist is that we insisted on being able to measure *every* subset of $\mathbb{R}^d$. We have shown that this is impossible. The solution is to be content with a measure function that is defined on some but not all subsets. Such subsets will be called measurable subsets.