MATH 395 Notes

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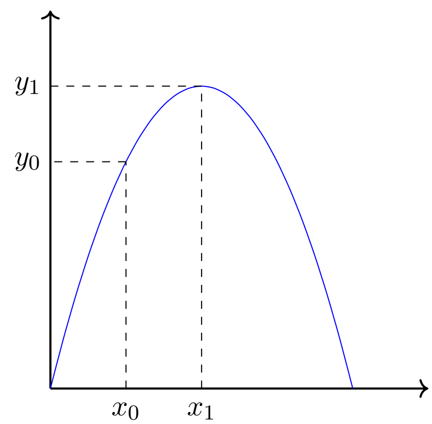
Recall. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ with A open. Let $x_0 \in A$. We say that f is locally invertible near $x_0 \in A$ provided that there eixsts $U, V \subseteq \mathbb{R}^n$ such that $x_0 \in U$, $f(x_0) \in V$, and f is bijective from U to V. Similarly we define local homeomorphism and local diffeomorphisms.

Main Question: When is a function f a local diffeomorphism? If $y = f(x)$ this means, when can we express x as a function of y.



Then clearly we can only express x as a function of y in a neighborhood of y_0 and not y_1. The reason for this difference is $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0) \neq 0$ whereas $\frac{\mathrm{d}f}{\mathrm{d}x}(x_1) = 0$.

This geometric intuition turns out to be true in any dimension if we require $Df(x_0)$ to be invertible instead of just non-zero. Of course this is equivalent to the determinant of $Df(x_0)$ being nonzero.

Recall. Last time, we showed that if f is a local diffeomorphism near x_0 and $g : U \to V$ is the inverse function with $x_0 \in U$ and $y = f(x_0) \in V$, then:

$$Dg(y_0) = [Df(x_0)]^{-1}$$

This shows the necessity of the condition $Df(x_0)$ being invertible for f to be a local diffeomorphism near x_0. The inverse function theorem (IFT) tells us that this is sufficient

Theorem (Inverse Function Theorem, IFT)**.** *Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}^n$ be of class C^r with $r \geq 1$. Suppose that $x_0 \in A$ and $Df(x_0)$ is invertible, then:*

(1) There exists an open neighborhood U of x_0 and an open neighborhood of V of $y_0 = f(x_0)$ such that f is a bijection from U to V

(2) The inverse function $g : V \to U$ is of class C^r as well, and $Dg(y) = [Df(x)]^{-1}$ when $y = f(x)$ for any $x \in U$.

Remark. Another interpretation of IFT is that it allows us to solve an equation:

$$y = f(x)$$

For x in terms of y locally around x_0 when $Df(x_0)$ is invertible. Note that if the funciton f is invertible then $f(x) = Ax$ for some $n \times n$ matrix A, then the ability to solve this equation is exactly the invertibility of A, but $A = Df(x)$ for any x. Wow! The IFT generalizes this to nonlinear functions using differentiability and we work locally.

Remark. The IFT does not guarantee the existence of a global inverse function of $f : A \to \mathbb{R}^n$, but only a local inverse, even if $Df(x)$ is invertible and continuous for all $x \in A$.

The only exception is when $n = 1$, and A is connected. In that case if $f'(x) \neq 0$ and f' is continuous then $f'(x)$ has a definite sign, and so f is either strictly increasing or decreasing. This stops being true for $n \geq 2$

Example. Here's a concrete example. Let $f : A = (1, 2) \times (-\pi, 3\pi) \to \mathbb{R}^2$ where $f(r, \theta) = (r\cos(\theta), r\sin(\theta))$. Then:

$$Df(r, \theta) = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$$

THen note that:

$$\det(Df(r, \theta)) = r \in (1, 2)$$

And so $Df(r,\theta)$ is invertible on A. However $f(r,0)=(r,0)=f(r,2\pi)$. Thus f is not globally injective, even though the IFT tells us that it is locally

Lemmas for the IFT

Lemma 1. *Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}^n$ be of class C^1. If $Df(x_0)$ is non-singular (that is invertible), then there exists an $\alpha > 0$ and a neighborhood U of x_0 such that:*

$$|f(x)-f(y)| \geq \alpha\,|x-y|$$

For any $x,y \in U$. In particular $f(x) \neq f(y)$ if $x \neq y$. Therefore f is one-to-one on U.

Proof. Let's Go! First we need the linear case:

Let $E = Df(x_0)$. If f were a linear function, that is $f(x)=Ex$, then $f(x)-f(y)=E(x-y)$. Therefore $x-y=E^{-1}(f(x)-f(y))$. This implies that:

$$|x-y| = \left|E^{-1}(f(x)-f(y))\right| \leq \|E^{-1}\| \cdot |f(x)-f(y)|$$

Where we have defined for any matrix $C : \mathbb{R}^n \to \mathbb{R}^m$ the operator norm:

$$\|C\| = \sup_{\substack{x\in\mathbb{R}^n\\ |x|=1}} |Cx|$$

Great!

Exercise. *Prove that $|Cx| \leq \|C\| \cdot |x|$ for any $x \in \mathbb{R}^n$ and that:*

$$\|C\| \leq nm \cdot \max_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} |C_{ij}|$$

This is useful for us!

Continuing we then have that:

$$|f(x)-f(y)| \geq \frac{1}{\|E^{-1}\|}\,|x-y|$$

Step 2, we need to generalize. Let $H(x) = f(x) - Ex$ where $E = Df(x_0)$. Then:

$$
\begin{aligned}
DH(x) &= Df(x) - E \\
DH(x_0) &= Df(x_0) - E = 0
\end{aligned}
$$

Since H is a C^1 function we can choose $\varepsilon > 0$ so that:

$$\|DH(x)\| \leq \frac{1}{2\|E^{-1}\|}$$

If $x \in B(x_0, \varepsilon)$. Now by the mean value theorem (that is Taylor's Theorem at order 0) we have some c between x and $y$with $x, y \in B(x_0, \varepsilon)$ so that:

$$|H(x) - H(y)| = |DH(c) \cdot (x - y)| \leq \|DH(c)\| \cdot |x - y| \leq \frac{1}{2\|E^{-1}\|} \cdot |x - y|$$

On the other hand:

$$|H(x) - H(y)| = |f(x) - f(y) - E(x - y)| \geq |E(x - y)| - |f(x) - f(y)|$$

Therefore:

$$|f(x) - f(y)| \geq |E(x - y)| - \frac{1}{2\|E^{-1}\|} |x - y|$$

But then by Step 1:

$$
\begin{aligned}
|f(x) - f(y)| &\geq |E(x - y)| - \frac{1}{2\|E^{-1}\|} |x - y| \\
&\geq \frac{1}{\|E^{-1}\|} |x - y| - \frac{1}{2\|E^{-1}\|} |x - y| = \frac{1}{2\|E^{-1}\|} |x - y|
\end{aligned}
$$

Exercise. *Suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is C^1, show that the function $x \in A \mapsto \|Df(x)\|$ is continuous. More generally we just need to know that the operator norm is continuous, that is $\mathrm{Mat}(m \times n) \to \mathbb{R}_{\geq 0}$ given by $A \mapsto \|A\|$ is continuous.*