# MATH 395 Notes

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## Announcements

- Midterm is Wednesday (class time)
  - Cameras should be on
  - Be ready 5 minutes earlier
  - Exam from 1pm $\to$ 2:20pm
  - From 2:30 $\to$ 2:30 pm upload your answers to gradescope
- No class on Friday October 30th because Hani has to work with the NSF

## Concluding Remarks on the Inverse Function Theorem

The IFT says that if $y = f(x)$ for $x \in \mathbb{R}^n$ satisfies $Df(x_0)$ being non-singular, then there exists an inverse function near $x_0$. In other words, this means that specifying $(y_1, \ldots, y_n)$ completely determines $(x_1, \ldots, x_n)$ at least locally around $x_0$ for $y_0 = f(x_0)$.

This means that we can use $(y_1, \ldots, y_n)$ as a coordinate system around $x_0$ instead of $(x_1, \ldots, x_n)$.

**Example.** Let $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ be given by:

$$f(\rho, \phi, \theta) = \rho(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

This is the spherical coordinate system, note that:

$$Df = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}$$

$$\begin{aligned} \det Df(\rho,\phi,\theta) &= -\rho\sin\phi\sin\theta(-\rho\sin\theta(\sin^2\phi+\cos^2\phi)) \\ &\quad + \rho\sin\phi\cos\theta(-\rho\cos\theta(\sin^2\phi+\cos^2\phi)) \\ &= \rho^2\sin\phi(\sin^2\theta+\cos^2\theta) = \rho^2\sin\phi \end{aligned}$$

Now note that $\det Df \neq 0$ whenever $\rho \neq 0$ and $\sin\phi \neq 0$. That is for any $(\rho_0, \phi_0, \theta_0)$ such that $\phi_0 \neq 0$ and $\phi_0 \neq 0$ there is a neighborhood $U$ of $(\rho_0, \phi_0, \theta_0)$ on which $f$ is a diffeomorphism. In particular, we can use $(f_1, f_2, f_3)$ as coordinates on $U$.

In this example, the inverse function can be computed using:

$$\rho = \sqrt{f_1^2 + f_2^2 + f_3^2} \qquad \phi = \arccos\left(\frac{f_3}{\rho}\right) \qquad \theta = \arctan\left(\frac{f_2}{f_1}\right)$$

Around a point for which $\rho \neq 0$ and $\sin\phi \neq 0$, this holds whenever $f_1^2 + f_2^2 \neq 0$.

# The Implicit Function Theorem

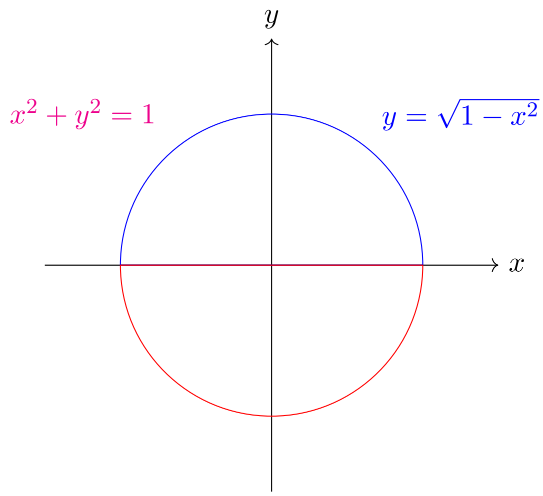
## Geometric Motivations

**Definition.** *A level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is of the form $\{x \in \mathbb{R}^n \mid f(x) = C\}$ for some constant $C \in \mathbb{R}$*

Consider the function:

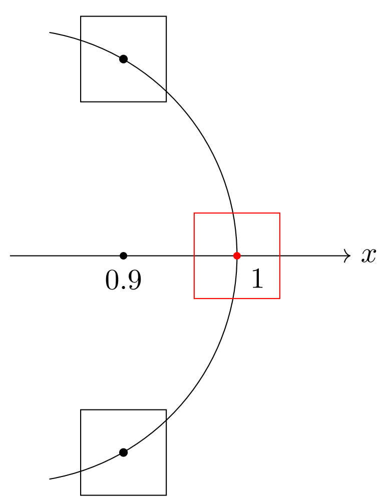
$$\begin{aligned} &f : \mathbb{R}^2 \to \mathbb{R} \\ &(x, y) \overset{f}{\mapsto} x^2 + y^2 - 1 \end{aligned}$$

We know that the equation $f(x, y) = 0$, a level set of $f$, is the unit circle.




But the upper part of the unit circle is also defined by the function $y = \sqrt{1 - x^2}$. In other words, when does the equation $f(x, y) = 0$ define the graph of a function $y = g(x)$. In this case, we say that $f(x, y) = 0$ defines $y$ implicitly in terms of $x$.

For $(a, b)$ on the unit circle, we can write the equation $f(x, y) = 0$ as $y = g(x)$ in a small neighborhood of $(a, b)$ so long as $(a, b) \neq (1, 0)$ and $(a, b) \neq (-1, 0)$ by the vertical line test




Clearly any red box will violate the vertical line test, and so we can't do this trick near $(1, 0)$.

These are exactly the points where $\frac{\partial f}{\partial y} = 0$. In the context of the implicit function theorem, we are given a function $f(x, y)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$ and

$f : A \subseteq \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$. When can the level set $\{f = C\}$ locally be described as the graph of a function $y = g(x)$.

### Calculus Motivation (Implicit Differentiation)

Suppose that the equation $f(x, y) = 0$ defines $y$ as a function of $x$ (the main assumption). What is $\frac{\mathrm{d}y}{\mathrm{d}x}$. Well:

$$\begin{aligned} f(x, y(x)) &= 0 \\ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \end{aligned}$$

Indeed the condition $\frac{\partial f}{\partial y} \neq 0$ is again needed to compute $\frac{\mathrm{d}y}{\mathrm{d}x}$. We will see that the Implicit Function Theorem Tells us this is a sufficient condition to be able to express $y$ as a function of $x$.

### Dimension Counting

We would like to find and prove the right generalization of those conditions so that the equation $f(x, y) = 0$ with $x \in \mathbb{R}^k, y \in \mathbb{R}^n$ and $f(x, y) \in \mathbb{R}^p$ can be solved uniquely in terms of $x$ in a sufficiently small neighborhood of $(a, b)$ on the level set.

Let us study the linear problem, i.e. when $f(x, y) = L(x, y)$ and $L$ is a linaer function from $\mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^p$, that is $L$ is a $p \times (n + k)$ matrix. Write $L$ as:

$$L = \left( \begin{array}{c|c} A & B \end{array} \right)$$

Where $A$ is $p \times k$ and $B$ is $p \times n$. Then $L(x, y) = Ax + By$, and so $L(x, y) = 0$ if and only if $Ax + By = 0$. Therefore $y$ is uniquely solvable in terms of $x$ when $By = Ax$ is uniquely solvable, which happens exactly when $B$ is an invertible matrix. Therefore we must have that $p = n$.

Notice that the matrix $B$ has its columns as $\frac{\partial L(x,y)}{\partial y_j}$ for $1 \leq j \leq n$. This motivates the following:

**Definition.** *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable and let $f_1, \ldots, f_m$ be the components of $f$. We denote:*

- *First:*

$$Df = \frac{\partial(f_1, \dots, f_m)}{\partial(x_1, \dots, x_n)} = \frac{\partial f}{\partial \vec{x}}$$

  *The matrix whose columns are* $\frac{\partial f}{\partial x_j} \in \mathbb{R}^m$ *for* $1 \leq j \leq n$*.*

- *Now suppose that* $(x_1, \dots, x_n) = (y, z)$ *for* $y \in \mathbb{R}^k$ *and* $y \in \mathbb{R}^{n-k}$*. We denote then:*

$$\frac{\partial f}{\partial \vec{y}} = \frac{\partial(f_1, \dots, f_n)}{\partial(y_1, \dots, y_k)} = \left(\frac{\partial f}{\partial y_j}\right)_{1 \leq j \leq k}$$
$$\frac{\partial f}{\partial \vec{z}} = \frac{\partial(f_1, \dots, f_n)}{\partial(z_1, \dots, z_{n-k})} = \left(\frac{\partial f}{\partial z_j}\right)_{1 \leq j \leq n-k}$$

The Implicit function theorem states (roughly) that given $f : A \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^n$ where $f(v) = f(x, y)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$, then the level set $\{f(x, y) = 0\}$ defines $y$ as a function of $x$ in a neighborhood of any point $(a, b)$ on the level set <u>if</u> $\frac{\partial f}{\partial \vec{y}}$ is non-singular.