Handout 9

Jordan measure and Riemann Integration

It turns out that the notion of Jordon measurability of sets is intimately related (in a way essentially equivalent) to the notion of Riemann integrability of functions. We will only display this relation in dimension 1.

- **Recall.** To define the Riemann[1] integral of a <u>bounded</u> function f on an interval $[a, b] \subset \mathbb{R}$, we first recall the notion of a partition $\mathcal{P}$ which is a set of points $x_0 = a < x_1 < x_2 < \ldots < x_n = b$, the norm of the partition is $\Delta\mathcal{P} = \max_{1 \leq k \leq n} x_k - x_{k-1}$, and we denote by $\Delta x_k = x_k - x_{k-1}$. For each such partition, we define to quantities:

$$L(f, \mathcal{P}) = \sum_{k=1}^{n} f(x_*)\Delta x_k, \quad \textit{and} \quad U(f, \mathcal{P}) = \sum_{k=1}^{n} f(x^*)\Delta x_k,$$

where $x_* = \inf_{[x_{k-1}, x_k]} f$ and $x^* = \sup_{[x_{k-1}, x_k]} f$.

Afterwards, we define the lower and upper Darboux integrals respectively as

$$\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(f, \mathcal{P}), \quad \textit{and} \quad \overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

where the extrema above are taken over all partitions of the interval $[a, b]$. We say that f is Riemann integrable if the above two numbers are equal. We define the common value as the Riemann (or Darboux) integral of f.

[1] Strictly speaking, we are recalling here the notion of Darboux integral, but that is equivalent to the notion of Riemann integrability that is often covered in introductory calculus classes.

Q1) Let $[a, b]$ be an interval and let $f : [a, b] \to \mathbb{R}$ be a bounded nonnegative function. Show that f is Riemann integrable if and only if the set $E := \{(x, t) : x \in [a, b] : 0 \leq t \leq f(x)\}$ is Jordan measurable in $\mathbb{R}^2$.

Q2) Let $[a, b]$ be an interval and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable if and only if the sets $E_+ := \{(x, t) : x \in [a, b] : 0 \leq t \leq f(x)\}$ and $E_- := \{(x, t) : x \in [a, b] : f(x) \leq t \leq 0\}$ are Jordan measurable in $\mathbb{R}^2$.

Remark. The above results generalize to higher dimensions.

Where we are right now?

We have thus far discussed the classical theory of Jordan measure, which went as follows

(i) We define the notion of a box and its volume $|B|$ or $v(B)$,

(ii) Then we defined the notion of an elementary set and its elementary measure,

(iii) Then we defined the notion of Jordan inner and outer measure $\underline{m}_J(E)$ and $\overline{m}^J(E)$ and said that a set E is Jordan measurable if those two concepts agree.

In particular, unwinding the definition of the Jordan outer measure, we have that for any set E

$$\overline{m}_J(E) = \inf_{E \subset B_1 \cup \ldots \cup B_k} |B_1| + \ldots + |B_k|$$

where the infimum is taken over all finite coverings of E by boxes $B_1, \ldots, B_k$.

Q3) Show that a set E is Jordan measurable if and only if for every $\epsilon > 0$ there exists an elementary set U containing E such that $\overline{m}_J(U \setminus E) < \epsilon$.

The notions of Lebesgue outer measure and Lebesgue measurability are refinements of the Jordan ones as follows:

– **Lebesgue outer measure:** We modify the notion of Jordan outer measure by replacing the finite union of boxes by a countable union of boxes, i.e.

$$m^*(E) = \inf_{E \subset \cup_{j=1}^{\infty} B_j} \sum_{j=1}^{\infty} |B_j|$$

where the union above is taken over boxes $B_j \subset \mathbb{R}^d$.

Q4) Show that the Lebesgue outer measure $m^*(E)$ is zero for any countable set E. Contrast this to fact that the Jordan outer measure of the rationals in $[0, 1]$ was equal to 1.

– **Lebesgue measurability** A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing E such that $m^*(U \setminus E) \leq \epsilon$. If E is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of E.

Remarks: Note that there is no need for E to be bounded for this definition to make sense. Also, the notion of Lebesgue measurability can be seen as a (finite to countably infinite) generalization of that of Jordan measurability since it can be shown that every open set is the countable union of closed boxes.