Handout 10

Where we are right now?

- We have thus far discussed the classical theory of Jordan measure, which went as follows
  (i) We define the notion of a box and its volume $|B|$ or $v(B)$,
  (ii) Then we defined the notion of an elementary set and its elementary measure,
  (iii) Then we defined the notion of Jordan inner and outer measure $\underline{m}_J(E)$ and $\overline{m}^J(E)$ and said that a set E is Jordan measurable if those two concepts agree.

  In particular, unwinding the definition of the Jordan outer measure, we have that for any set E
  $$\overline{m}_J(E) = \inf_{E \subset B_1 \cup \ldots \cup B_k} |B_1| + \ldots + |B_k|$$
  where the infimum is taken over all finite coverings of E by boxes $B_1, \ldots, B_k$.

Q0) Show that a set E is Jordan measurable if and only if for every $\epsilon > 0$ there exists an elementary set U containing E such that $\overline{m}_J(U \setminus E) < \epsilon$.

Lebesgue outer measure

The notions of Lebesgue outer measure and Lebesgue measurability are refinements of the Jordan ones as follows:

- **Lebesgue outer measure:** We modify the notion of Jordan outer measure by replacing the finite union of boxes by a countable union of boxes, i.e.
  $$m^*(E) = \inf_{E \subset \cup_{j=1}^{\infty} B_j} \sum_{j=1}^{\infty} |B_j|$$

where the union above is taken over boxes $B_j \subset \mathbb{R}^d$.

Q1) Show that $m^*(E) \le \overline{m}_J(E)$ where $\bar{m}_J$ is the Jordan outer measure.

Q2) Show that in the definition above the countable cover by boxes in the definition of $m^*(E)$ can be restricted to closed boxes or open boxes.

Q3) Show that the Lebesgue outer measure $m^*(E)$ is zero for any countable set E. Contrast this to fact that the Jordan outer measure of the rationals in $[0, 1]$ was equal to 1.

- **Lebesgue measurability** A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing E such that $m^*(U \setminus E) \le \epsilon$. If E is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of E.

  Remarks:

  (i) Note that there is no need for E to be bounded for this definition to make sense.

  (ii) The notion of Lebesgue measurability can be seen as a (finite to countably infinite) generalization of that of Jordan measurability since it can be shown that every open set is the countable union of closed boxes.

Q4) Show that $m^*(\emptyset) = 0$.

Q5) (Monotonicity) Show that if $E \subset F \subset \mathbb{R}^d$, then $m^*(E) \le m^*(F)$.

Q6) (Countable subadditivity) If $E_1, E_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of sets, then $m^*\left(\cup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m^*(E_n)$.