MATH 395 Notes

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Announcements

- Final to be released on Monday December 14 in the afternoon, and due on Tuesday early morning. Say 4pm-4am
- To be submitted through gradescope

Recalling Improper Integrals

Recall. For A an open set and f continuous on A. We defined the extended $\int_A f$ as follows:

- If $f \geq 0$ then we define:

$$\int_A f = \sup_{\substack{D \in \mathcal{J}_c \\ D \subseteq A}} \int_D f$$

  Where $\mathcal{J}_c$ is the set of all compact Jordan measurable sets.

- For general f we write $f = f_+ - f_-$ and define:

$$\int_A f := \int_A f_+ - \int_A f_-$$

By convention if f is continuous and A is open then $\int_A f$ will mean the extended integral.

Problem: If A is open and bounded and f is continuous and bounded, we have two definitions for $\int_A f$. The extended integral may exist without having the ordinary integral existing. We will see today that if the ordinary integral exists then the extended integral exists and they are equal. We also proved the following

Lemma. *If $A \subseteq \mathbb{R}^n$ is open then there exists a sequence $C_1, C_2, \dots$ of elementary sets (also compact Jordan measurable) such that:*

$$C_n \subseteq C_{n+1}^{\circ}$$
$$A = \bigcup_{j=1}^{\infty} C_j$$

Theorem. *Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}$ be continuous. Choose a sequence $C_n \in \mathcal{J}_c$ as in the above lemma. Then f is integrable on A (in the extended sense) if and only if $\int_{C_n} |f|$ is bounded (uniformly in n). In this case,*

$$\int_A f = \lim_{n\to\infty} \int_{C_n} f$$

In particular, f is integrable on A if and only if $|f|$ is too.

Proof. We'll do this in cases:

- Let f be non-negative. In this case $\int_{C_n} f \, \mathrm{d}x$ is a monotonically increasing sequence of non-negative numbers, and as such it converges as $n \to \infty$ if and only if it is uniformly bounded.

  ($\Rightarrow$) Suppose that f is integrable over A. We want to show that $\int_{C_n} f$ exists and converges to $\int_A f$ as $n \to \infty$. Since f is continuous and C_n is compact, then f is bounded on C_n, and hence $\int_{C_n} f$ exists since C_n is Jordan measurable.

  Also:

  $$\int_{C_n} f \le \sup_{\substack{\mathbb{D}\subseteq A \\ D\in\mathcal{J}_c}} \int_D f = \int_A f$$

  Therefore $\int_{C_n} f$ is uniformly bounded in n. This implies that it converges, now we need to show it converges to the right thing. We must also have that:

  $$\lim_{n\to\infty} \int_{C_n} f \le \int_A f$$

Great!

($\Leftarrow$) Suppose $\lim_{n\to\infty}\int_{C_n} f$ exists. Then $\int_{C_n} f$ is uniformly bounded in n by some constant M. Now take any $D \subseteq A$ and $D \in \mathcal{J}_c$. Then we know that:

$$D \subseteq \bigcup_{n=1}^{\infty} C_j^\circ$$

By compactness of D there exists a finite subcover, and since $C_j \subseteq C_{j+1}^\circ$ there exists some $n_\heartsuit$ such that $D \subseteq C_{n_\heartsuit}^\circ$. Therefore we know that:

$$\int_D f \leq \int_{C_{n_\heartsuit}} f \leq M$$

And therefore we have a nonempty bounded set, so the supremum exists:

$$\int_A f = \sup_{\substack{D\subseteq A \\ D\in\mathcal{J}_c}} \int_D f \leq M$$

Since M can be taken to be the limit as $n \to \infty$ of $\int_{C_n} f$ then we get that:

$$\int_A f \leq \lim_{n\to\infty} \int_{C_n} f$$

Combining these two inequalities from the if and only if we win and get the equality:

$$\int_A f = \lim_{n\to\infty} \int_{C_n} f$$

Perfect!

- Let's deal with general $f : A \to \mathbb{R}$ that is continuous. f is integrable over A if and only if f_+ and f_- are integrable if and only if $\int_{C_n} f_+$ and $\int_{C_n} f_-$ are bounded sequences by case one.

  But this is if and only if $\int_{C_n} f_+ + f_-$ is a bounded sequence, since $f_+, f_- \geq 0$. But since $f_+ + f_- = |f|$ this is only when $\int_{C_n} |f|$ is a bounded sequence. Therefore applying case 1 this is if and only if $\int_A |f|$ exists.

In this case we of course have:

$$
\begin{aligned}
\int_{C_n} f_+ &\to \int_A f_+ \\
\int_{C_n} f_- &\to \int_A f_- \\
\int_{C_n} f &= \int_{C_n} f_+ - \int_{C_n} f_- \\
&\to \int_A f_+ - \int_A f_- \\
&= \int_A f
\end{aligned}
$$

So we are done!

Theorem. *Let A be a bounded open set in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded continuous function. Then:*

a) The extended integral exists

b) If the ordinary integral exists, then the two integrals are equal.

Proof.

- Let us first show that the extended integral exists. Let M be an upper bound for $|f|$ on A. If $D \in \mathcal{J}_c$ is a subset of A, then:

  $$\int_D |f| \leq M \int_D 1 = Mv(D) \leq Mv(B)$$

  Where B is any box containing A. Therefore the set defining the extended integral is bounded, and so the extended integral of $|f|$ over A exists. This of course implies that the extended integral of f over A exists by our previous theorem.

- Now suppose that the ordinary integral $\int_A f$ exists and that $f \geq 0$. Then let B be a box containing A, then:

  $$\text{(ord)} \int_A f = \int_B f_A$$

Now let $D \subseteq A$ and $D \in \mathcal{J}_c$ then we must have that:

$$\int_D f = \int_D f_A \leq \int_B f_A = (\text{ord}) \int_A f$$

Therefore taking a sup over all D we get that:

$$(\text{ext}) \int_A f \leq (\text{ord}) \int_A f$$

To show the reverse inequality, let P be any partition of B and let $R_1, \ldots, R_m$ denote the sub-boxes of this partition. Now let $D = \bigcup_{R_i \subseteq A} R_i$. Then $D \subseteq A$ and $D \in \mathcal{J}_c$. Therefore:

$$\begin{aligned} L(f_A, P) &= \sum_{i=1}^{m} m_{R_i}(f_A) v(R_i) \\ &= \sum_{R_i \subseteq A} m_{R_i}(f_A) v(R_i) \\ &\leq \sum_{R_i \subseteq A} \int_{R_i} f = \int_D f \\ &\leq (\text{ext}) \int_A f \end{aligned}$$

Take the supremum over all such P and we obtain:

$$(\text{ord}) \int_A f = \sup_P L(f_A, P) \leq (\text{ext}) \int_A f$$

These two inequalities imply that the ordinary and extended integrals agree as desired to give (b) when $f \geq 0$.

- Write $f = f_+ - f_-$ as usual. Since f is integrable over A in the ordinary sense, so are $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Therefore:

$$\begin{aligned} (\text{ord}) \int_A f &= (\text{ord}) \int_A f_+ - (\text{ord}) \int_A f_- \\ &= (\text{ext}) \int_A f_+ - (\text{ext}) \int_A f_- \\ &= (\text{ext}) \int_A f \end{aligned}$$

And this finishes the proof

Corrolary. *Let S be any bounded set and $f : S \to \mathbb{R}$ be a bounded continuous function. If f is integrable on S in the ordinary sense, then:*

$$\text{(ord)} \int_S f = \text{(ext)} \int_{S^\circ} f$$

Proof. Recall that if $\int_S f = \int_{S^\circ} f$, then apply the previous theorem.

This corollary is useful to translate results for extended integrals to ordinary integrals (like the change of variable formula in the next section).

The Change of Variables Formula

Recall. THe change of variable formula in 1D, otherwise known as u-substitution. Letting $f, g : [a, b] \to \mathbb{R}$ be functions with g C^1 and f continuous. Then letting $u = g(x)$ and $\mathrm{d}u = g'(x)\,\mathrm{d}x$ we have:

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

There's a nuance, we are using $\int_a^b f$ to denote the signed integral which is defined as:

$$\int_a^b f = \begin{cases} \int_{[a,b]} f & \text{if } a \leq b \\ -\int_{[b,a]} f & \text{if } b < a \end{cases}$$

This u-substitution holds basically due to the chain rule, since if F is an antiderivative for f then $(F \circ g)' = f(g(x)) \cdot g'(x)$

Integrating from a to b then gives u-substitution by the Fundamental Theorem of Calculus.

There is no notion of signed integrals in higher dimensions, so we first need to formulate this theorem without signed integrals. For this note that $g([a, b]) = [g(a), g(b)]$ if g is increasing, i.e. $g' \geq 0$. And also $g([a, b]) = [g(b), g(a)]$ if g is decreasing, i.e. $g' \leq 0$.

If g is increasing then we can write:

$$\int_{[a,b]} f(g(x)) \cdot g'(x) = \int_{g([a,b])} f(u)\,\mathrm{d}u$$

And if g is decreasing then we can write:

$$\int_{[a,b]} f(g(x))g'(x)\,\mathrm{d}x = -\int_{g([a,b])} f(u)\,\mathrm{d}u$$

That is:

$$\int_{[a,b]} f(g(x))(-g'(x))\,\mathrm{d}x = \int_{g([a,b])} f(u)\,\mathrm{d}u$$

In either case, we may write that if g is monotone, then:

$$\int_{[a,b]} f(g(x))\left|g'(x)\right|\mathrm{d}x = \int_{g([a,b])} f(u)\,\mathrm{d}u$$

This is the formula that generalizes easily to higher dimensions.

So we look at this genralizing this via the correspondence:

1D	higher dimension
$[a,b]$	set A
$g([a,b])$	$g(A)$
g is monotone and C^1	g is a C^1 diffeomorphism
$u = g(x)$ $\mathrm{d}u = \lvert g'(x)\rvert\,\mathrm{d}x$	$u = g(x)$ $\mathrm{d}u = \lvert\det Dg\rvert\,\mathrm{d}x$

And so we have something like:

$$\int_A f(g(x))\left|\det Dg\right|\mathrm{d}x = \int_{g(A)} f(u)\,\mathrm{d}u$$

And we use this in the same way with:

$$u = g(x)$$
$$\mathrm{d}u = \left|\det Dg\right|\mathrm{d}x$$

Definition. *Let A be open in $\mathbb{R}^n$ and let $g : A \to \mathbb{R}^n$ be a one-to-one function of class C^r such that $\det Dg(x) \neq 0$ for $x \in A$. We call such a g a change of variables* *on* A

Remark. Recall that a C^r diffeomorphism is a one-to-one and onto function such that g and g^{-1} are in C^r

The inverse function theorem tells us that $g^{-1} \in C^r$ if $g \in C^r$ and $\det Dg(x) \neq 0$.

A change of variables on A is then nothing but a C^r diffeomorphism from A to $g(A)$

Theorem (Change of Variables Theorem)**.** *Let $g : A \to B$ be a C^1-diffeomorphism of open sets in $\mathbb{R}^n$ and let $f : B \to \mathbb{R}$ be a continuous function. Then f is integrable over B if and only if $f(g(x)) \cdot |\det Dg(x)|$ is integrable over A, and:*

$$\int_A f(g(x)) \cdot |\det Dg(x)| \, \mathrm{d}x = \int_B f(u) \, \mathrm{d}u$$