

Handout 5

The Elementary measure (Continued)

- Recall from last time that an interval I is a subset of \mathbb{R} of the form $[a, b], [a, b), (a, b], \text{ or } (a, b)$ where $a, b \in \mathbb{R}$. The length of I is defined to be $|I| := b - a$. A *box* in \mathbb{R}^d is a Cartesian product of intervals $B = I_1 \times I_2 \times \dots \times I_d$ and its *volume* is defined to be $|B| = |I_1| \dots |I_d|$. An *elementary set* is any subset of \mathbb{R}^d which is the union of a finite number of boxes.
- **Definition.** Let E be an elementary set. Last time we saw that we can write $E = B_1 \cup B_2 \cup \dots \cup B_n$ where B_1, \dots, B_n are disjoint. We define the elementary measure of E as $m(E) := |B_1| + |B_2| + \dots + |B_n|$.
- Q1) Show that $m(E)$ is well-defined in the sense that if E can be expressed in two ways as a union of disjoint boxes B_1, \dots, B_n and B'_1, \dots, B'_m , then

$$|B_1| + |B_2| + \dots + |B_n| = |B'_1| + |B'_2| + \dots + |B'_m|.$$

Hint: There's more than one approach you can take. One is to notice that for an interval I in \mathbb{R} , there holds that

$$|I| = \lim_{N \rightarrow \infty} \frac{1}{N} \# \left(I \cap \frac{1}{N} \mathbb{Z} \right).$$

(why?). And more generally for a box B ,

$$|B| = \lim_{N \rightarrow \infty} \frac{1}{N^d} \# \left(B \cap \frac{1}{N} \mathbb{Z}^d \right).$$

Here $\frac{1}{N}\mathbb{Z}^d = \{\frac{k}{N} : k \in \mathbb{Z}^d\}$. Use this to give an alternative definition of $m(E)$ for an elementary set that does rely on its decomposition into disjoint boxes.

- **Properties of Elementary measure.** Show that the following holds

Q2) Show that if E_1, \dots, E_n are disjoint elementary sets, then

$$m(E_1 \cup \dots \cup E_n) = \sum_{i=1}^n m(E_i)$$

Recall that this is called **finite additivity**.

Q3) Show that if $E \subset F$ are two elementary sets, then

$$m(E) \leq m(F).$$

This property is called **monotonicity**.

Q4) Show that if E_1, E_2, \dots, E_n is an arbitrary finite collection of elementary sets, then

$$m(E_1 \cup \dots \cup E_n) \leq m(E_1) + \dots + m(E_n).$$

This is called **finite subadditivity**.

- **Why is this unsatisfactory?** Of course, the main problem with this measure is that we can only measure relatively simple sets (namely the elementary sets). For example, we cannot measure the area of a disc. One might be tempted to generalize this measure naively as follows: For an arbitrary set $E \subset \mathbb{R}^d$, define

$$m_{\text{pixel}}(E) = \lim_{N \rightarrow \infty} \frac{1}{N^d} \# \left(E \cap \frac{1}{N} \mathbb{Z}^d \right).$$

However, this is not a particularly satisfactory definition for (at least) the following two reasons:

- Q5)** Find a subset E of \mathbb{R} for which this limit does not exist.
- Q6)** Find a subset E of \mathbb{R} such that both $m_{\text{pixel}}(E)$ and $m_{\text{pixel}}(E+x)$ exist, but $m_{\text{pixel}}(E) \neq m_{\text{pixel}}(E+x)$ for some $x \in \mathbb{R}$.