MATH 395 Notes

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Continue Defining the Riemann Integral

Definition. *Given a box $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$ which is closed and a function $f : B \to \mathbb{R}$ that is bounded. We defined a partition $P = (P_1, \ldots, P_n)$ of B as a tuple where each P_j is a partition of $[a_j, b_j]$. We then let $\{B_j\}_{j=1}^N$ be the set of sub-boxes determined by the partition. We then defined the lower sum and upper sum of f over a partition P:*

$$m_{B_j} := \inf_{x \in B_j} f(x)$$

$$M_{B_j} := \sup_{x \in B_j} f(x)$$

$$L(f, P) := \sum_{j=1}^{N} m_{B_j} v(B_j)$$

$$U(f, P) := \sum_{j=1}^{N} M_{B_j} v(B_j)$$

Exercise. *$U(f, P) = -L(-f, P)$.*

We then talked about refinements of a partition, saying that $Q = (Q_1, \ldots, Q_n)$ is a refinement of $P = (P_1, \ldots, P_n)$ if $P_1 \subseteq Q_1, P_2 \subseteq Q_2, \ldots$.

We defined the common refinement of $P = (P_1, \ldots, P_n)$ and $Q = (Q_1, \ldots, Q_n)$ as $P \cup Q = (P_1 \cup Q_1, \ldots, P_n \cup Q_n)$.

Lemma. *Let P be a partition of a box B and $f : B \to \mathbb{R}$ be bounded. If Q is a* *refinement of P then:*

$$L(f,P) \leq L(f,Q)$$
$$U(f,P) \geq U(f,Q)$$

Proof. We reduce first. Since $U(f,P) = -L(-f,P)$, it is enough to prove the lemma for lower sums.

Now since Q ca be obtained from P by successively adding points to the partition, we can reduce to the case where Q is obtained from $P = (P_1, \dots, P_n)$ by adding a single point. to P_k for some $1 \leq k \leq n$.

By symmetry, we assume that $k = 1$. Suppose that $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and suppose that P_1 consists of the points $a_1 = x_0 < \cdots < x_k = b$. Now Q is obtained by adding the point q that lies in the interior of (x_{p-1}, x_p) for some $1 \leq p \leq k$.

The sub-boxes determined by P are of the form $[x_{i-1}, x_i] \times S$ where S is a subbox of $[a_2, b_2] \times \cdots \times [a_n, b_n]$ determined by the partition $(P_2, \dots, P_n)$. Let us denote by $\mathscr{S}$ the set of all such subboxes.

The sub-boxes determined by Q are of the form: $[x_{i-1}, x_i] \times S$ for $1 \leq i \leq p-1$ or $p+1 \leq i \leq k$ and $S \in \mathscr{S}$ or $[x_{p-1}, q] \times S$ or $[q, x_p] \times S$ for $S \in \mathscr{S}$. Therefore:

$$\begin{aligned}
L(f,P) &= \sum_{\substack{i=1 \\ S \in \mathscr{S}}}^{k} m_{[x_{i-1},x_i] \times S}(f) \cdot v([x_{i-1}, x_i] \times S) \\
&= \sum_{\substack{i \in \{1,\dots,p\} \cup \{p+1,\dots,k\} \\ S \in \mathscr{S}}} m_{[x_{i-1},x_i] \times S}(f) \cdot v([x_{i-1}, x_i] \times S) \\
&\quad + \sum_{S \in \mathscr{S}} m_{[x_{p-1},x_p] \times S}(f) \cdot (x_p - x_{p-1}) \cdot v(S)
\end{aligned}$$

The left sum appears in the definition of $L(f,Q)$, and so we only consider the right sum. The point is that the:

$$\inf_{x \in [x_{p-1}, x_p] \times S} f(x) \leq \inf_{x \in [x_{p-1}, q] \times S} f(x), \inf_{x \in [q, x_p] \times S} f(x)$$

This implies that:

$$\begin{aligned}
m_{[x_{p-1},x_p] \times S}(f) \cdot (x_p - x_{p-1}) &= m_{[x_{p-1},x_p] \times S}(f) \cdot (q - x_{p-1}) + m_{[x_{p-1},x_p] \times S}(f) \cdot (x_p - q) \\
&\leq m_{[x_{p-1},q] \times S}(f) \cdot (q - x_{p-1}) + m_{[q,x_p] \times S}(f) \cdot (x_p - q)
\end{aligned}$$

But then:

$$\begin{aligned} L(f,Q) = &\sum_{\substack{i\in\{1,\dots,p\}\cup\{p+1,\dots,k\}\\ S\in\mathscr{S}}} m_{[x_{i-1},x_i]\times S}(f)\cdot v([x_{i-1},x_i]\times S) \\ &+ \sum_{S\in\mathscr{S}} m_{[x_{p-1},q]\times S}(f)\cdot (q-x_{p-1}) + m_{[q,x_p]\times S}(f)\cdot(x_p - q) \end{aligned}$$

And so $L(f,P)\le L(f,Q)$ because:

$$\begin{aligned} L(f,P) = &\sum_{\substack{i\in\{1,\dots,p\}\cup\{p+1,\dots,k\}\\ S\in\mathscr{S}}} m_{[x_{i-1},x_i]\times S}(f)\cdot v([x_{i-1},x_i]\times S) \\ &+ \sum_{S\in\mathscr{S}} m_{[x_{p-1},x_p]\times S}(f)\cdot (x_p-x_{p-1})\cdot v(S) \end{aligned}$$

And we know that:

$$\begin{aligned} &\sum_{S\in\mathscr{S}} m_{[x_{p-1},x_p]\times S}(f)\cdot (x_p-x_{p-1})\cdot v(S) \\ &\le \sum_{S\in\mathscr{S}} m_{[x_{p-1},q]\times S}(f)\cdot (q-x_{p-1}) + m_{[q,x_p]\times S}(f)\cdot(x_p - q) \end{aligned}$$

That was disgusting!!!

Corrolary. *If P and P' are any two partitions of B then $L(f,P)\le U(f,P')$. The proof was given last time.*

Definition (Upper integrals, lower integrals, and Riemann integrability)**.** *Let B be a box and let $f: B\to\mathbb{R}$ be a bounded function.*

a) *We define the lower and upper integral of f over B respectively as:*

$$\underline{\int_B} f(x)\,\mathrm{d}x = \sup_P L(f,P)$$
$$\overline{\int_B} f(x)\,\mathrm{d}x = \inf_P U(f,P)$$

These numbers exist because $L(f,P)$ is bounded above by $(\sup_{x\in B} f(x))\cdot v(B)$ and $U(f,P)$ is bounded below by $(\inf_{x\in B} f(x))\cdot v(B)$

b) *We say that f is Riemann integrable over B provided that the lower and upper integral agree. In this case we define the Riemann integral $\int_B f(x)\,\mathrm{d}x$ as the* *common value, aka:*

$$\int_B f(x)\,\mathrm{d}x := \underline{\int_B} f(x)\,\mathrm{d}x = \overline{\int_B} f(x)\,\mathrm{d}x$$

Remark. Strictly speaking, this is the definition of Darboux integrability. The precise definition of Riemann integrability is: A bounded function f is Riemann integrable with integral A on the box B if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if P is a partition of B with mesh $\leq \delta$, then for any choice of $x_\alpha \in B_\alpha$, where B_α are the sub-boxes determined by P:

$$\left| \sum_{B_\alpha} f(x_\alpha) v(B_\alpha) - A \right| < \varepsilon$$

We will prove these are equivaelnt on Homework 9. F

Remark. Suppose that $f : B \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a non-negative function. Then $L(f, P)$ is the total volume of a bunch of boxes under the graph of f whereas the upper sum is the total volume of a bunch of boxes that are circumscribed

Exercise. *Show that if $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$ is non-negative and bounded. Then f is Riemann integrable if and only if the region in $\mathbb{R}^{n+1}$ under the graph of f given by:*

$$R = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq x_{n+1} \leq f(x)\}$$

is Jordan measurable with $m(R) = \int_B f(x)\,\mathrm{d}x$.

Example. Let $f : [0,1]^2 \to \mathbb{R}$ be defined as:

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ and } y \text{ are rationally dependent} \\ 1 & \text{otherwise} \end{cases}$$

We call x and y rationally dependent provided that there exists $(k_1, k_2) \in \mathbb{Z}^2$ such that $(k_1, k_2) \neq 0$ and $k_1 x + k_2 y = 0$.

Now let P be any partition of $B = [0,1]^2$. For any subbox R resulting from the partition we have:

$$m_R(f) = \inf_R f = 0$$

$$M_R(f) = \sup_R f = 1$$

Since for any subbox of $[0,1]^2$ with non-empty interior, there exists $(x, y) \in R$ such that both x and y are rational numbers, and so they are rationally dependent. For the second statement, since for any sub-box of $[0,1]^2$ with non-empty interior, there exists $(x, y) \in R$ such that x is a non-zero rational and y is irrational. This implies that x, y are rationally independent.

Therefore:

$$L(f, P) = 0 \qquad\qquad U(f, P) = 1$$

For any partiton P of $[0,1]^2$. And therefore:

$$\underline{\int_B} f(x)\,\mathrm{d}x = 0 \qquad\qquad \overline{\int_B} f(x)\,\mathrm{d}x = 1$$

Therefore, f is not integrable

Theorem 1 (The Riemann Condition)**.** *Let B be a box in $\mathbb{R}^n$ and let $f : B \to \mathbb{R}$ be a bounded function. Then:*

a) We always have that $\underline{\int_B} f(x)\,\mathrm{d}x \leq \overline{\int_B} f(x)\,\mathrm{d}x$

b) f is integrable if and only if for every $\varepsilon > 0$ there exists a partition P of B for which $U(f, P) - L(f, P) < \varepsilon$.

Remark. Reminiscient of the exercise in our discussion sections that E is Jordan measurable if for any $\varepsilon > 0$ there eixsts elementary sets $A \subseteq E \subseteq B$ such that $m(B \setminus A) < \varepsilon$.

Proof. Part (a) is trivial since we saw that $L(f, P) \leq U(f, P')$ for any P and P'. Taking the sup over P and the inf over P' gives the result.

For (b), there are two directions:

($\Rightarrow$) Suppose f is integrable and $\varepsilon > 0$. Choose a partiton P_1 such that:

$$\left| L(f, P_1) - \int_B f \right| < \frac{\varepsilon}{2}$$

and another partition P_2 such that:

$$\left| U(f, P_2) - \int_B f \right| < \frac{\varepsilon}{2}$$

Then we know that $U(f, P_2) - L(f, P_1) < \varepsilon$. Take P to be the common refinement of P_1 and P_2. Then we know that:

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$$

This means that $U(f, P) - L(f, P) \leq U(f, P_2) - L(f, P_1) < \varepsilon$. Thus we win!

($\Leftarrow$) Let $\varepsilon > 0$ be arbitrary. Choose a partition P such that $U(f, P) - L(f, P) < \varepsilon$. Then:

$$\overline{\int_B} f - \underline{\int_B} f \leq U(f, P) - L(f, P) < \varepsilon$$

Since we know that:

$$\overline{\int_B} f \leq U(f, P)$$
$$\underline{\int_B} f \geq L(f, P)$$

Since $\varepsilon > 0$ is arbitrary, we can take $\varepsilon \to 0$ and so we must have that the upper and lower integrals agree. Therefore f is integrable.

With this we win! ☺

Proposition. *Let B be a box. Denote by $\mathcal{R}(B)$ the set of all Riemann integrbale functions on B. Then:*

1) $\mathcal{R}(B)$ is a vector space. That is if $f, g \in \mathcal{R}(B)$ then $f + cg \in \mathcal{R}(B)$ for all $c \in \mathbb{R}$. Furthermore, $\int_B$ is a linear function from $\mathcal{R}(B)$ to $\mathbb{R}$. That is:

$$\int_B f + cg = \int_B f + c \int_B g$$

2) Every constant function $f(x) = c$ is integrable, and in particular has integral $\int_B f = c \cdot v(B)$

3) If P is any partition of B then:

$$v(B) = \int_B 1 = \sum_Q v(Q)$$

Which is the sum taken over all sub-boxes determined by P

4) Let $B_1, \dots, B_k$ be a collection of boxes that cover B, then:

$$v(B) \leq \sum_{j=1}^{k} v(B_j)$$

Proof. Let's go!

1) We leave this as an exercise

2 & 3) For any partition P note that:

$$L(f,P) = c\sum_{Q} v(Q) = U(f,P)$$

And therefore by the Riemann condition, f is integrable. And furthermore:

$$\int_B c = c\sum_{Q} v(Q)$$

Taking P to be the trivial partition we have that $\int_B c = c \cdot v(B)$

4) Let B be a box containing $B_1, \dots, B_k$. Now let P be a partition of B that contains all the endpoints that define $B_1, \dots, B_k$ and B. By the above:

$$v(B) = \sum_{Q \subseteq B} v(Q) \leq \sum_{j=1}^{k} \sum_{Q \subseteq B_i} v(Q) = \sum_{j=1}^{k} v(B_j)$$