Handout 4

- **Wish list for a measure function** It would be grand to have a measure function that tells us how big or small a subset of $\mathbb{R}^d$ is. This would be a function from the set of subsets of $\mathbb{R}^d$ into $[0, \infty]$, say $m : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$. We would like this function to satisfy the following properties:

  a) If $E_1, E_2, \ldots$ is a countable collection of disjoint subsets of $\mathbb{R}$, then
  $$m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n).$$
  This is called **Countable Additivity.**

  b) If E is congruent to F (i.e. F can be obtained from E by applying rigid motions: translations, rotations, or a reflections) then we should have that $m(E) = m(F)$.

  c) $m([0, 1)^d) = 1$.

  The bad news is that no such function can exist, and here's why (at least when $d = 1$). Let us define an equivalence relation between elements of $[0, 1)$ as follows: We say $x \sim y$ if $x - y$ is a rational number. Let N be the subset of $[0, 1)$ that contains exactly one element of each equivalence relation (the existence of this N requires invoking the axiom of choice). Now let $R = [0, 1) \cap \mathbb{Q}$, and for each $r \in R$ define the set

  $$N_r = \{x + r : x \in N \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in N \cap [1 - r, 1)\}.$$

  (Basically N_r is just the translate of N by r units to the right, except that we move the part that sticks out of the interval $[0, 1)$ one unit to the left).

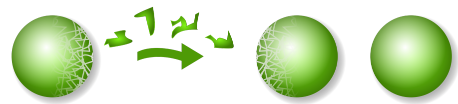

Figure 1: Banach-Tarski tells us that we can split the unit ball in $\mathbb{R}^3$ into finitely many (actually 5 is sufficient) many disjoint pieces, apply rigid motions to those pieces and then reassemble them to obtain two copies of the unit ball.

Q1) Show that $[0, 1)$ is the disjoint union of N_r for $r \in R$.

Q2) Show that if a measure function satisfying a), b) and c) above exists, then $m(N) = m(N_r)$ for every $r \in R$.

Q3) Arrive at a contradiction.

Remark: One might think that possibly relaxing condition a) to cover only *finitely* many disjoint sets E_n, i.e.

$$m(\cup_{n=1}^{N} E_n) = \sum_{n=1}^{N} m(E_n). \qquad \textbf{(Finite Additivity)}$$

would resolve the contradiction. Unfortunately, the Banach-Tarski paradox (cf. Figure 1) tells us that this is not enough to resolve this issue.

Conclusion: The problem with the above wishlist is that we insisted on being able to measure *every* subset of $\mathbb{R}^d$. We have shown that this is impossible. The solution is to be content with a measure function that is defined on some but not all subsets. Such subsets will be called measurable subsets.

The Greek method

- **Elementary measure.** An interval I is a subset of $\mathbb{R}$ of the form $[a, b], [a, b), (a, b], or (a, b)$ where $a, b \in \mathbb{R}$. The length of I is defined to be $|I| := b - a$. A *box* in $\mathbb{R}^d$ is a Cartesian product of intervals $B = I_1 \times I_2 \times \ldots I_d$ and its *volume* is defined to be $|B| = |I_1|. \ldots .|I_d|$. An *elementary set* is any subset of $\mathbb{R}^d$ which is the union of a finite number of boxes.

Q4) Show that if $E, F \subset \mathbb{R}^d$ are elementary sets, then the union $E \cup F$, the intersection $E \cap F$, the set theoretic difference $E \setminus F$, and the symmetric difference $E \Delta F = (E \setminus F) \cup (F \setminus E)$ are also elementary. Also, if $x \in \mathbb{R}^d$, then the translate $E + x := \{y + x : y \in E\}$ is also elementary.

Q5) Show that E can be expressed as the finite union of disjoint boxes. *Hint: Start with $d = 1$. Then use this result to generalize it to higher dimensions.*

- **Definition.** Let E be an elementary set. The above question allows to write $E = B_1 \cup B_2 \cup \ldots B_n$ where $B_1, \ldots, B_n$ are disjoint. We define the elementary measure of E as $m(E) := |B_1| + |B_2| + \ldots + |B_n|$.

Q6) Show that $m(E)$ is well-defined in the sense that if E can be expressed in two ways as a union of disjoint boxes $B_1, \ldots B_n$ and $B'_1, \ldots B'_m$, then

$$|B_1| + |B_2| + \ldots + |B_n| = |B'_1| + |B'_2| + \ldots + |B'_m|.$$

Hint: There's more than one approach you can take. One is to notice that for an interval I in $\mathbb{R}$, there holds that

$$|I| = \lim_{N \to \infty} \frac{1}{N} \# \left(I \cap \frac{1}{N}\mathbb{Z} \right).$$

(why?). And more generally for a box B,

$$|B| = \lim_{N \to \infty} \frac{1}{N^d} \# \left(B \cap \frac{1}{N}\mathbb{Z}^d \right).$$

Here $\frac{1}{N}\mathbb{Z}^d = \{\frac{k}{N} : k \in \mathbb{Z}^d\}$. Use this to give an alternative definition of $m(E)$ for an elementary set that does rely on its decomposition into disjoint boxes .