MATH 395 Notes

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August 31, 2020

1 Metric Spaces

1.1 Definition

Definition. *A set X is called a metric space provided that it is equipped with a function $d : X \times X \to [0, \infty)$ such that*

1. *For all $p, q \in X$ we have $d(p, q) = 0$ if and only if $p = q$*
2. *$d(p, q) = d(q, p)$ for all $p, q \in X$.*
3. *For all $p, q, r \in X$ we have*

$$d(p, q) \leq d(p, r) + d(r, q)$$

We call d the metric on X. Formally we might write that (X, d) is a metric space, since a set X may admit many different metrics on it.

Example. Let $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$. If $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ then we define:

$$d_2(p, q) = \left[\sum_{j=1}^{n} (q_j - p_j)^2\right]^{\frac{1}{2}} = \|p - q\| = \langle q - p, q - p\rangle^{\frac{1}{2}}$$

This is commonly called the ℓ^2 metric on $\mathbb{R}^n$. The triangle inequality follows from Cauchy-Schwartz. Setting $x = p - r$ and $y = r - q$, then $x = y = p - q$ and we also have:

$$\|x+y\|^2 \leq (\|x\| + \|x\|)^2$$
$$\|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$

But since we know from Cauchy-Schwarz that $\langle x, y\rangle \leq \|x\|\|y\|$, so we win!

We can put another metric on R^n, namely the ℓ^s metric for any $1 \leq s < \infty$:

$$d_s(p,q) = \left[\sum_{j=1}^{n} |q_j - p_j|^s\right]^{\frac{1}{s}}$$

This is called the ℓ^s metric. There is also the ℓ^∞ metric denoted as:

$$d_\infty(p,q) = \max_{1\leq j\leq n} |q_j - p_j|$$

1.2 Topology on metric spaces

Definition. *A topology on a set X is some collection of subsets $\mathscr{T} \subseteq P(X)$, which we will call the open subsets of X, such that:*

- *$\emptyset$ and X are both open.*
- *Given any arbitrary family of open sets $\{U_i\}_{i\in I}$, their union $\bigcup_{i\in I} U_i$ is an open set*
- *Given any finite collection of open sets, $U_1, \ldots, U_n$, then their intersection $\bigcap_{i=1}^{n} U_i$ is open.*

Definition. *Let (X, d) be a metric space. We define a topology on X as follows:*

- *For $x_0 \in X$ and $\varepsilon > 0$ we define the ε-neighborhood of x_0 as:*

$$N_\varepsilon(x_0) := \{x \in X \mid d(x, x_0) < \varepsilon\}$$

- *A subset $U \subseteq X$ is called open provided that for every $p \in U$ there exists some $\varepsilon > 0$ so that $N_\varepsilon(p) \subseteq U$.*

Proof that this is a topology. The first property follows nearly trivially.

- Fix some arbitrary family of open sets $\{U_i\}_{i\in I}$. Fix some $p \in \bigcup_{i\in I} U_i$, then there exists some $j \in I$ so that $p \in U_j$. Since U_j is open there exists some $\varepsilon > 0$ so that:

$$N_\varepsilon(p) \subseteq U_j \subseteq \bigcup_{i\in I} U_i$$

  And so we are done ☺

- Let $p \in \bigcap_{i=1}^n U_i$ for some finite collection of open sets $U_1, \dots, U_n$. Then $p \in U_j$ for all $1 \leq j \leq n$, and so there exists an $r_j > 0$ for each j such that:

$$N_{r_j}(p) \subseteq U_j$$

  Take $r = \min(r_1, \dots, r_n$. Then for all j we have $N_r(p) \subseteq N_{r_j}(p) \subseteq U_j$. And so:

$$N_r(p) \subseteq \bigcap_{i=1}^n U_i$$

  just as desired.

Remark. This third property is not true for infinite collections! What part of the proof breaks and provide a counter-example.

With this we are done.

Exercise. *Also, as an exercise, show that for any $r > 0$ we have $N_r(p)$ is open.*

Definition. *We say a subset $C \subseteq X$ of a topological space is closed provided that its complement $X \setminus C$ is open.*

Remark. By Demorgan's laws we get three properties of closed sets:

- $\emptyset$ and X are both closed
- If $\{C_i\}_{i\in I}$ is a collection of closed sets then $\bigcap_{i\in I} C_i$ is closed
- If $C_1, \dots, C_n$ is a finite collection of closed sets then $\bigcup_{i=1}^n C_i$ is a closed set.

The proof is left as an exercise ☺

1.3 Limit Points / Accumulation Points

Definition. *A point p is called a limit point of a set E provided that every neighborhood of p contains a point $q \neq p$ such that $q \in E$.*

Example. Let $E = [0, 1) \cup \{2\}$. Then 1 is a limit point of E (note that $1 \notin E$), and also 2 is *not* a limit point of E even though $2 \in E$.

Definition. *When $p \in E$ is not a limit point of E, p is called an isolated point of E.*

Definition. *An interior point of E is a point $p \in E$ such that there exists $r > 0$ so that $N_r(p) \subseteq E$. Thus a set is open exactly when all its points are interior points. The set of all interior points of a set E is often denoted by $\mathring{E}$, this is called the interior of E.*

Example. This depends on the entire metric space

- Let $E = [0, 1) \cup \{2\}$ and $X = [0, \infty)$. Then 0 is an interior point of E (since $N_r(0) = [0, r) \subseteq E$ is r is small enough). Thus $\mathring{E} = [0, 1)$.
- Let $E = [0, 1) \cup \{2\}$ and $X = \mathbb{R}$. Then 0 is not an interior point of E, since any neighborhood of 0 will contain negative numbers, which are not contained in E.

Thus we conclude that the notion of interior (open or closed) depends on the ambient space.

Definition. *A set E is bounded provided that there eixsts a point $x \in X$ and a number $M > 0$ such that $E \subseteq N_M(x)$.*

Definition. *A set $E \subseteq X$ is dense provided that every point of X is either a limit point of E or an element in E.*

Example. Let $X = [0, 1) \cup \{\pi\}$ then $X \cap \mathbb{Q} \cup \{\pi\}$ is dense in X. Notice that $X \cap \mathbb{Q}$ is not dense in X.

Theorem. *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Exercise. *Prove this*

Corrolary. *A finite set can have no limit points*

Theorem. *A set E is closed if and only if every limit point of E is contained in E.*

Proof. Let's do it! We will use X as our ambient space.

($\Rightarrow$) Let E be closed and suppose p is a limit point of E. If $p \notin E$ then $p \in X \setminus E$, which is open, and so there exists an $r > 0$ such that $N_r(p) \subseteq X \setminus E$. Therefore $N_r(p) \cap E = \emptyset$, but this contradicts the fact that p is a limit point. Therefore $p \in E$ as desired.

($\Leftarrow$) Suppose that every limit point belongs to E and take $p \in X \setminus E$. Since p is not a limit point of E there must exist some $r > 0$ such that $N_r(p) \cap E = \emptyset$. But then $N_r(p) \subseteq X \setminus E$. Therefore $X \setminus E$ is open, and E is closed.

Awesome! We win ☺

Definition. *A set E is called perfect if E is closed and every point of E is a limit point. In other words, E consists exactly of its limit points.*

Example. $[0, 1]$ is perfect in $\mathbb{R}$, but $[0, 1] \cup \{\pi\}$ is not.

Example. Let $X = \mathbb{R}^2 = \mathbb{C}$. Consider the following sets

a) The set of all complex numbers $|z| < 1$

b) The set of all complex numbers $|z| \leq 1$

c) A finite set $F \subseteq \mathbb{C}$

d) The set of all integers $\{(n, 0) \mid n \in \mathbb{N}\}$

e) The set $z_n = \frac{1}{n}$ where $n \in \mathbb{N}$

f) The set of all complex numbers

g) The line segment (a, b) for $a, b \in \mathbb{R}$. That is the set of points $z \in \mathbb{C}$ such that $\mathrm{Im}(z) = 0$ and $a < \mathrm{Re}(z) < b$

	Closed	Open	Bounded	Perfect
a)	✗	✓	✓	✗
b)	✓	✗	✓	✓
c)	✓	✗	✓	✗
d)	✓	✗	✗	✗
e)	✗	✗	✓	✗
f)	✓	✓	✗	✓
g)	✗	✗	✓	✗