MATH 395 Notes

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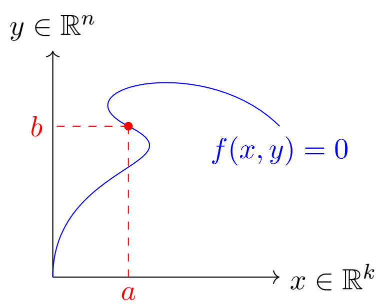
More Implicit Function Theorem

Problematic: We have $f : A \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^n$ with $f = f(x, y)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. We are interested in the level set $L = \{f(x, y) = 0\}$.



Suppose that (a, b) is on the level set, that is $f(a, b) = 0$. Now the equation $f = 0$ gives us n-equations in x

Question: Can we write the condition that $\{f(x, y) = 0\}$ near (a, b) as the graph of a function $y = g(x)$, i.e. $(x, y) \in L$ if and only if $y = g(x)$. In other words, can we solve the system of equations $f(x, y) = 0$ near (a, b) for y in terms of x? In yet other words, does the equation $\{f = 0\}$ define y implicitly in terms of x

Roughly speaking, the implicit function theorem says that the answer is yes provided that $\frac{\partial f}{\partial y} = \frac{\partial(f_1, \dots, f_n)}{\partial(y_1, \dots, y_n)}$ is non-singular.

Before stating the theorem precisely, let's state an easier result about the derivative of the implicit function:

Theorem (Implicit Differentiation). *Let $A \subseteq \mathbb{R}^{k+n}$ be open and $f : A \to \mathbb{R}^n$ be differentiable and write $f = f(x, y)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Now suppose that the equation $f(x, y) = 0$ defines y implicitly, i.e. there exists a function $g : B \to \mathbb{R}^n$*

defined on an open subset B of $\mathbb{R}^k$ such that $(x, g(x)) \in A$ and $f(x, g(x)) = 0$ for all $x \in B$.

THEN, for $x \in B$ we have:

$$\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) \cdot Dg(x) = 0$$

In particular, if $\frac{\partial f}{\partial y}(x, g(x))$ is invertible, then:

$$Dg(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \frac{\partial f}{\partial x}(x, g(x))$$

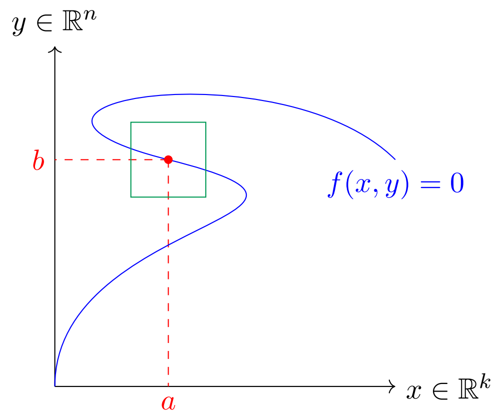
Proof. Then let $h : B \to \mathbb{R}^{k+n}$ be the function $h(x) = (x, g(x))$ then $f \circ h = 0$ by supposition. Take the derivative of this expression, and so by the chain rule:

$$\begin{aligned}
Df(h(x)) \cdot Dh(x) = 0 \\
Dh(x) &= \left(\frac{I_k}{Dg}\right)\Big|_x \\
Df &= \left(\begin{array}{c|c} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{array}\right) \\
Df(h(x))Dh(x) &= \left(\begin{array}{c|c} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{array}\right)\Big|_{h(x)} \cdot \left(\frac{I_k}{Dg}\right)\Big|_x \\
&= \frac{\partial f}{\partial x}(h(x)) + \frac{\partial f}{\partial y}(h(x))Dg(x) = 0
\end{aligned}$$

And this is what we wished to show.

The implicit function theorem tells us that the invertibility of $\frac{\partial f}{\partial y}$ is sufficient for the condition of the above theorem to hold

Theorem (Implicit Function Theorem)**.** *Let $A \subseteq \mathbb{R}^{k+n}$ be open and $f : A \to \mathbb{R}^n$ be of class C^r with $r \geq 1$. Write f in the form $f(x, y)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Suppose that $(a, b) \in A$ such that $f(a, b) = 0$.*



If $\frac{\partial f}{\partial y}(a, b)$ is non-singular, then there exists a neighborhood $B \subseteq \mathbb{R}^k$ of a and a unique continuous function $g : B \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for $x \in B$. The function g will in fact be of class C^r. In fact inside the green window, $f(x, y) = 0$ if and only if $y = g(x)$.

Remark. Of course, the variables y for which we solve for in terms of x don't have to be the last n coordinates. They can be any n of the $(n + k)$ coordinates.

Proof. Step 1 (An Auxiliary Function): Consider the auxiliary function:

$$
\begin{aligned}
F : A \subseteq \mathbb{R}^{k+n} &\to \mathbb{R}^{k+n} \\
(x, y) &\overset{F}{\mapsto} \begin{pmatrix} x \\ f(x, y) \end{pmatrix} \\
DF(x, y) &= \begin{pmatrix} DF_1 \\ DF_2 \\ \vdots \\ DF_{k+n} \end{pmatrix} = \left(\begin{array}{c} I_k \\ \hline Df \end{array} \right) \\
&= \left(\begin{array}{c|c} I_k & 0 \\ \hline \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{array} \right)
\end{aligned}
$$

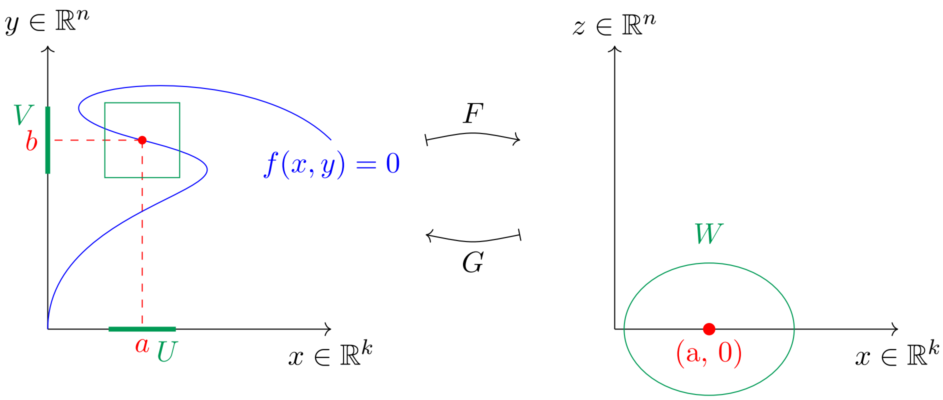
Therefore using block matrices you can check that:

$$\det DF(x,y) = \det I_k \det\left(\frac{\partial f}{\partial y}\right) = \det\left(\frac{\partial f}{\partial y}\right)$$

But we know that $\frac{\partial f}{\partial y}$ is nonsingular at (a,b) and so:

$$\det DF(a,b) = \det \frac{\partial f}{\partial y}(a,b) \neq 0$$

Thus $DF(a,b)$ is nonsingular, and so by the inverse function theorem there exists a neighborhood $U \times V$ of (a,b) such that $a \in U$ is open in $\mathbb{R}^k$ and $b \in V$ is open in $\mathbb{R}^n$ as well as a neighborhood W of $(a,0) \in \mathbb{R}^{k+n}$ such that F is a C^r-diffeomorphism from $U \times V$ onto W.



Let $G : W \to U \times V$ be the the inverse function of F. I.e. $(x,y) = G(x, f(x,y))$. for all $(x,y) \in U \times V$ and $(x,z) = F \circ G(x,z)$ for $(x,z) \in W$. This tells us that G is the identity on its first k coordinate functions. Let $h : W \to V$ be defined as $h(x,z) = (G_{k+1}, G_{k+2}, \ldots, G_{k+n})$, h is clearly C^r since G is C^r by the inverse function theorem.

Step 2 (Definition of g): Let B be a ball around a such that $B \subseteq U$ and $B \times \{0\} \subseteq W$. Now notice that $(x,y) \in B \times V$ satisfies $f(x,y) = 0$ if and only if $F(x,y) = (x,0)$ if and only if $(x,y) = G(x,0) = (x, h(x,0))$. Defining $g(x) = h(x,0)$ for $x \in B$ we have that $(x,y) \in B \times V$ satisfying $f(x,y) = 0$ if and only if $y = g(x)$ for $x \in B$. Clearly g is C^r since h is C^r.

Also note that $(a,b) = G(a,0) = (a, h(a,0))$, and so $b = h(a,0) = g(a)$ as desired.

Step 3 (Uniqueness of g): Suppose that $g' : B \to \mathbb{R}$ is another continuous function that satisfies the conclusions of the theorem. Let $S = \{x \in B \mid g(x) = g'(x)\}$. Clearly since g and g' are continuous, S is closed relative to B. Also, we must have that $a \in S$, since $b = g(a) = g(a')$. We will show that S is also open in B, which would mean that $S = B$, since B is connected. This will finish the proof. We'll leave this until next time