MATH 395 Notes

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Theorem (Mean Value Theorem)**.** *For a differentiable function $H : \mathbb{R}^n \to \mathbb{R}$ we have that for any $x, y \in \mathbb{R}^n$ there is some c on the line segment between x and y so that:*

$$H(y) - H(x) = DH(c) \cdot (y - x)$$

Great!

Proof. Set $\phi(t) : [0, 1] \to \mathbb{R}$ as $\phi(t) = H(x + t(y - x))$. By the single-variable mean value theorem there is some $t \in (0, 1)$ so that:

$$\begin{aligned} \phi(1) - \phi(0) &= \phi'(t) \cdot (1 - 0) \\ H(y) - H(x) &= \phi'(t) \end{aligned}$$

Now by the chain rule, if we set $c := x + t(y - x)$, which is on the line segment:

$$\phi'(t) = DH(c) \cdot (y - x)$$

And so we have the statement of the mean value theorem. Of course, this is just Taylor's Theorem at degree $k = 0$.

How to estimate $R_{x_0,k}(x)$

Now for Taylor's Theorem, how do we estimate $R_{k,x_0}(x)$? This will help us to show the Taylor polynomial is a good approximation. Suppose that $f : A \to \mathbb{R}$ is sufficiently differentiable and that we can show for all $x \in A$ that $|\partial^\alpha f(x)| \leq M_{k+1}$

for $|\alpha| = k+1$. So then:

$$|(x-x_0)^\alpha| = \left|\prod_{j=1}^{n}(x_j - x_{0,j})^{\alpha_j}\right| \le \left|\prod_{j=1}^{n}|x-x_0|^{\alpha_j}\right| = |x-x_0|^{|\alpha|}$$

$$|R_{k,x_0}(x)| = \left|\sum_{|\alpha|=k+1}\frac{\partial^\alpha f(c_x)}{\alpha!}\right| \le \sum_{|\alpha|=k+1}\frac{M_{k+1}}{\alpha!}\,|x-x_0|^{k+1}$$

Worksheet Time

Handout 7

Jordan measure (Continued)

- **Recall.**

  Definition 0.1 (Jordan measure)**.** Let $E \subset \mathbb{R}^d$ be a bounded set.

  – The *Jordan inner measure* $\underline{m}_J(E)$ of E is defined as

  $$\underline{m}_J(E) = \sup_{A \subset E, A \,\text{elementary}} m(A).$$

  Here $m(A)$ is the elementary measure of A.

  – The *Jordan outer measure* $\overline{m}_J(E)$ of E is defined as

  $$\overline{m}_J(E) = \inf_{A \supset E, A \,\text{elementary}} m(A).$$

  – If $\underline{m}_J(E) = \overline{m}_J(E)$, we say that E is Jordan measurable, and call the common value $m(E)$ (the Jordan measure of E).
  By convention, we do not consider unbounded sets to be Jordan measurable.

  Recall from last time that the Jordan measure extends the notion of elementary measure to more general sets. We also saw that the Jordan measure satisfies Boolean closure properties (if E, F are Jordan measurable sets, the so are $E \cup F, E \cap F, E \setminus F,$), as well as finite additivity (If $E_1, \ldots, E_k$ are disjoint are Jordan measurable, then $m(E_1 \cup \ldots \cup E_k) = m(E_1) + \ldots + m(E_k)$, and translation invariance ($m(E) = m(E + x)$ for $x \in \mathbb{R}^d$).

Q1) Show that the graph $\{(x, f(x)) : x \in B\} \subset \mathbb{R}^{d+1}$ is Jordan measurable in $\mathbb{R}^{d+1}$ and that it has Jordan measure 0. *Hint: Use that f is uniformly continuous.*

Q2) Show that the set $\{(x,t) : x \in B, 0 \leq t \leq f(x)\} \subset \mathbb{R}^{d+1}$ is Jordan measurable.

From this we conclude that some familiar sets like triangles in $\mathbb{R}^2$ and balls in $\mathbb{R}^d$ are Jordan measurable. For instance,

Q3) Show that the open and closed balls $B(x_0, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$ and $\overline{B(x,r)} = \{y \in \mathbb{R}^d : |y - x| \leq r\}$ are both Jordan measurable, and that their Jordan measure is $c_d r^d$ for some constant $c_d > 0$ that only depends on the dimension.

Q4) Establish the bound $\left(\frac{2}{\sqrt{d}}\right)^d \leq c_d \leq 2^d$.

- **Sets that are not Jordan measurable**. This shows that a lot of familiar subsets of $\mathbb{R}^d$ are Jordan measurable, however many subsets of interest aren't: a) all unbounded subsets are not Jordan measurable, and more importantly b) several interesting bounded sets are not too as the following questions show.

Q5) Let $E \subset \mathbb{R}^d$ be bounded. Show that both E and its closure $\overline{E}$ have the same Jordan outer measure.

Q6) Show that E and its interior E° have the same Jordan inner measure.

Q7) Show that E is Jordan measurable if and only if the topological boundary $\partial E = \overline{E} \setminus E^\circ$ has Jordan outer measure 0.

Q8) Show that the bullet-riddled square $[0,1] \setminus \mathbb{Q}^2$, and the set of bullets $[0,1] \cap \mathbb{Q}^2$ both have Jordan inner measure zero and Jordan outer measure one. In particular, both sets are not Jordan measurable.