Handout 7

Jordan measure (Continued)

- **Recall.**

  Definition 0.1 (Jordan measure)**.** Let $E \subset \mathbb{R}^d$ be a bounded set.

  – The *Jordan inner measure* $\underline{m}_J(E)$ of E is defined as

  $$\underline{m}_J(E) = \sup_{A \subset E, A \,\text{elementary}} m(A).$$

  Here $m(A)$ is the elementary measure of A.

  – The *Jordan outer measure* $\overline{m}_J(E)$ of E is defined as

  $$\overline{m}_J(E) = \inf_{A \supset E, A \,\text{elementary}} m(A).$$

  – If $\underline{m}_J(E) = \overline{m}_J(E)$, we say that E is Jordan measurable, and call the common value $m(E)$ (the Jordan measure of E).
  By convention, we do not consider unbounded sets to be Jordan measurable.

  Recall from last time that the Jordan measure extends the notion of elementary measure to more general sets. We also saw that the Jordan measure satisfies Boolean closure properties (if E, F are Jordan measurable sets, the so are $E \cup F, E \cap F, E \setminus F,$), as well as finite additivity (If $E_1, \ldots, E_k$ are disjoint are Jordan measurable, then $m(E_1 \cup \ldots \cup E_k) = m(E_1) + \ldots + m(E_k)$, and translation invariance ($m(E) = m(E + x)$ for $x \in \mathbb{R}^d$).

Q1) Show that the graph $\{(x, f(x)) : x \in B\} \subset \mathbb{R}^{d+1}$ is Jordan measurable in $\mathbb{R}^{d+1}$ and that it has Jordan measure 0. *Hint: Use that f is uniformly continuous.*

Q2) Show that the set $\{(x,t) : x \in B, 0 \leq t \leq f(x)\} \subset \mathbb{R}^{d+1}$ is Jordan measurable.

From this we conclude that some familiar sets like triangles in $\mathbb{R}^2$ and balls in $\mathbb{R}^d$ are Jordan measurable. For instance,

Q3) Show that the open and closed balls $B(x_0, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$ and $\overline{B(x,r)} = \{y \in \mathbb{R}^d : |y - x| \leq r\}$ are both Jordan measurable, and that their Jordan measure is $c_d r^d$ for some constant $c_d > 0$ that only depends on the dimension.

Q4) Establish the bound $\left(\frac{2}{\sqrt{d}}\right)^d \leq c_d \leq 2^d$.

- **Sets that are not Jordan measurable**. This shows that a lot of familiar subsets of $\mathbb{R}^d$ are Jordan measurable, however many subsets of interest aren't: a) all unbounded subsets are not Jordan measurable, and more importantly b) several interesting bounded sets are not too as the following questions show.

Q5) Let $E \subset \mathbb{R}^d$ be bounded. Show that both E and its closure $\overline{E}$ have the same Jordan outer measure.

Q6) Show that E and its interior E° have the same Jordan inner measure.

Q7) Show that E is Jordan measurable if and only if the topological boundary $\partial E = \overline{E} \setminus E^\circ$ has Jordan outer measure 0.

Q8) Show that the bullet-riddled square $[0,1] \setminus \mathbb{Q}^2$, and the set of bullets $[0,1] \cap \mathbb{Q}^2$ both have Jordan inner measure zero and Jordan outer measure one. In particular, both sets are not Jordan measurable.

Handout 7

Jordan measure and Riemann Integration

It turns out that the notion of Jordon measurability of sets is intimately related (in a way essentially equivalent) to the notion of Riemann integrability of functions. We will only display this relation in dimension 1.

- **Recall.** To define the Riemann[1] integral of a bounded function f on an interval $[a,b] \subset \mathbb{R}$, we first recall the notion of a partition $\mathcal{P}$ which is a set of points $x_0 = a < x_1 < x_2 < \ldots < x_n = b$, the norm of the partition is $\Delta\mathcal{P} = \max_{1 \le k \le n} x_k - x_{k-1}$, and we denote by $\Delta x_k = x_k - x_{k-1}$. For each such partition, we define to quantities:

$$L(f, \mathcal{P}) = \sum_{k=1}^{n} f(x_*)\Delta x_k, \quad \textit{and} \quad U(f, \mathcal{P}) = \sum_{k=1}^{n} f(x^*)\Delta x_k,$$

where $x_* = \inf_{[x_{k-1},x_k]} f$ and $x^* = \sup_{[x_{k-1},x_k]} f$.

Afterwards, we define the lower and upper Darboux integrals respectively as

$$\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(f, \mathcal{P}), \quad \textit{and} \quad \overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

where the extrema above are taken over all partitions of the interval $[a,b]$. We say that f is Riemann integrable if the above two numbers are equal. We define the common value as the Riemann (or Darboux) integral of f.

[1]Strictly speaking, we are recalling here the notion of Darboux integral, but that is equivalent to the notion of Riemann integrability that is often covered in introductory calculus classes.

Q1) Let $[a,b]$ be an interval and let $f : [a,b] \to \mathbb{R}$ be a bounded nonnegative function. Show that f is Riemann integrable if and only if the set $E := \{(x,t) : x \in [a,b] : 0 \leq t \leq f(x)\}$ is Jordan measurable in $\mathbb{R}^2$.

Q2) Let $[a,b]$ be an interval and let $f : [a,b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable if and only if the sets $E_+ := \{(x,t) : x \in [a,b] : 0 \leq t \leq f(x)\}$ and $E_- := \{(x,t) : x \in [a,b] : f(x) \leq t \leq 0\}$ are Jordan measurable in $\mathbb{R}^2$.

Remark. The above results generalize to higher dimensions. For that we will need a notion of Riemann (or Darboux) integrability on $\mathbb{R}^d$ ($d \geq 2$). We will discuss this theory in our lectures, starting next week.

MATH 395 Notes

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Worksheet 7

Proof Sketch of Q3. We talk about this by doing induction. Clearly any ball of radius r in one dimension is measurable, since this will just be a line.

Fix $d \in \mathbb{N}$ so that $B(0,r) \subseteq \mathbb{R}^d$ is measurable. We will show that $B(0,r) \subseteq \mathbb{R}^{d+1}$ is measurable. We consider the following function defined on the box $[-r,r]^d$:

$$f : [-r,r]^d \to \mathbb{R}^{d+1}$$
$$f(x) = \begin{cases} \sqrt{1 - \|x\|^2} & \text{if } \|x\| < r \\ 0 & \text{otherwise} \end{cases}$$

This will give a hemisphere of $B(0,r) \subseteq \mathbb{R}^{d+1}$, and we can glue two of these together to give the full ball. We then can take away the graphs of the functions and we will win.

Proof Sketch of Q5. Fix some bounded subset $E \subseteq \mathbb{R}^d$. We will show that E and $\overline{E}$ have the same Jordan outer measure. To do this lets show that $\overline{m}_J(E) \leq \overline{m}_J(\overline{E})$ and $\overline{m}_J(E) \geq \overline{m}_J(\overline{E})$. Let's go!

Lemma. *The closure of any elementary set A has the same elementary measure as A, and in fact $\overline{A}$ is an elementary set.*

Proof. First note that clearly $m(A) \leq m(\overline{A})$ by monotonicity. Write A as a disjoint union of a finite number of boxes $B_1, \ldots, B_n$. Now note that:

$$\overline{A} = \overline{\left(\bigcup_{k=1}^{n} B_k\right)} = \bigcup_{k=1}^{n} \overline{B}_k$$

We will justify this second equality:

- ($\subseteq$) Note that $\overline{A}$ is the smallest closed set that contains A. Now note that $\bigcup_{k=1}^n \overline{B}_k$ is closed and since $B_k \subseteq \overline{B}_k$ it contains A. Therefore $\overline{A} \subseteq \bigcup_{k=1}^n \overline{B}_k$.
- ($\supseteq$) Fix some $x \in \bigcup_{k=1}^n \overline{B}_k$. Then $x \in \overline{B}_j$ for some $1 \leq j \leq n$. Therefore since $B_j \subseteq \bigcup_{k=1}^n B_k$ that we must have $x \in \overline{B}_j \subseteq \overline{\bigcup_{k=1}^n B_k} = \overline{A}$.

Now note that B_k is a box, and so when we take its closure that is still a box, and all the intervals making up the product become closed intervals. This does not change the measure, and so $m(B_k) = m(\overline{B}_k)$. The union above demonstrates that $\overline{A}$ is elementary and by finite subadditivity:

$$m(\overline{A}) \leq \sum_{k=1}^{n} m(\overline{B}_k) = \sum_{k=1}^{n} m(B_k) = m(A)$$

And so we must have since $m(A) \leq m(\overline{A})$ that $m(A) = m(\overline{A})$.

Fix some elementary set A that contains $\overline{E}$, this must exist since E is bounded, and thus $\overline{E}$ is bounded. Then A clearly contains E. And so $\overline{m}_J(E) \leq m(E)$. This shows $\overline{m}_J(E)$ is a lower bound for the set defining $\overline{m}_J(\overline{E})$. By the definition of infimum then $\overline{m}_J(E) \leq \overline{m}_J(\overline{E})$.

Now fix some elementary set A that contains E, this must exist since E is bounded. Then $\overline{A}$ contains $\overline{E}$, and so $\overline{m}_J(\overline{E}) \leq m(\overline{A}) = m(A)$ by the lemma. But then $\overline{m}_J(\overline{E})$ is a lower bound for the set defining $\overline{m}_J(E)$. This means that $\overline{m}_J(\overline{E}) \leq \overline{m}_J(E)$

Therefore $\overline{m}_J(E) = \overline{m}_J(\overline{E})$ and we are done! Great!

Proof Sketch of Q6. This is very similar to Question 5!!! Lets show that E and its interior E° have the same Jordan inner measure! For this we a lemma:

Lemma. *The interior of any elementary set A is elementary and has the same measure as A.*

Proof. First note that $A^\circ \subseteq A$ so by monotonicity if A° is elementary then
TODO

TODO

Proof Sketch of Q7. Let's go both ways!!!

($\Rightarrow$) Suppose E is Jordan measurable. Then by Q5 and Q6:

$$m(E) = \overline{m}_J(E) = \overline{m}_J(\overline{E})$$
$$m(E) = \underline{m}_J(E) = \underline{m}_J(E^\circ)$$

Now to compute $\overline{m}_J(\partial E)$ we know that $0 \leq \overline{m}_J(\partial E)$ because for any elementary set A we know $0 \leq m(A)$. By the characterization of infima it suffices to find for every $\varepsilon > 0$ some elementary set C containing ∂E so that:

$$0 \leq m(C) \leq \varepsilon$$

Note by characterization of suprema and infima for $\overline{E}$ and E° we have an elementary set A containing $\overline{E}$ and an elementary set B contained in E° so that:

$$m(E) \leq m(A) \leq m(E) + \frac{\varepsilon}{2}$$
$$m(E) - \frac{\varepsilon}{2} \leq m(B) \leq m(E)$$

Now note that $A \setminus B$ contains ∂E since A contains $\overline{E}$ and everything we are cutting from A is in $B \subseteq E^\circ$. Now we know that $A \setminus B$ is elementary, so set $C := A \setminus B$ and we will show $m(C) \leq \varepsilon$. This is simple since:

$$\begin{aligned}
B \subseteq E^\circ \subseteq E \subseteq \overline{E} &\subseteq A \\
A &= A \setminus B \sqcup B \\
m(A) &= m(A \setminus B) + m(B) \\
m(C) &= m(A) - m(B) \\
&\leq m(E) + \frac{\varepsilon}{2} - m(E) + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

And so we are done! Great!

($\Leftarrow$) Now suppose that ∂E has outer measure 0. We must show that E is Jordan measurable. To do

By Q5 and Q6 it suffices to show that $\overline{m}_J(\overline{E}) \geq \underline{m}_J(E^\circ)$ and likewise $\overline{m}_J(\overline{E}) \leq \underline{m}_J(E^\circ)$, since these are the outer and inner measures of E respectively.

Fix some elementary set A which contains $\overline{E}$. Then since $\overline{E} \supseteq E \supseteq E^\circ$ we know A contains E°. Now fix an elementary set B so that $B \subseteq E^\circ$. Then $m(B) \leq m(A)$ by monotonicity, so by definition of supremum $\underline{m}_J(E^\circ) \leq m(A)$. Then by definition of infimum $\underline{m}_J(E^\circ) \leq \overline{m}_J(\overline{E})$.

We will prove this one by showing that for every $\varepsilon > 0$ we have:

$$\underline{m}_J(E^\circ) + \varepsilon > \overline{m}_J(\overline{E})$$

And so we get the result by taking ε to 0.

TODO

Worksheet 8