MATH 395 Notes

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Recall. We defined directional derivatives for $f : \mathbb{R}^n \to \mathbb{R}^m$ for $u \in \mathbb{R}^n$ by:

$$D_u f(x) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(x+tu) \in \mathbb{R}^m$$

We also defined partial derivatives $\frac{\partial f}{\partial x_i} \in \mathbb{R}^m$ for $1 \leq i \leq n$ by:

$$\frac{\partial f}{\partial x_i} = D_{e_i} f(x)$$

Furthermore, if f is differentiable at x then $D_u f(x)$ exists for every u. Morevoer:

$$D_u f(x) = Df(x) \cdot u$$

The converse is not true in general!!! We will give today an example where $D_u f(x)$ exists for every n but $Df(x)$ does not

If $n \geq 1$ and $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ then we can write $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$. Then for every $u \in \mathbb{R}^n$ we have:

$$\begin{aligned} D_u f(x) &= \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} = \lim_{t \to 0} \begin{pmatrix} \frac{f_1(x+tu) - f_1(x)}{t} \\ \vdots \\ \frac{f_m(x+tu) - f_m(x)}{t} \end{pmatrix} \\ &= \begin{pmatrix} \lim_{t \to 0} \frac{f_1(x+tu) - f(x)}{t} \\ \vdots \\ \lim_{t \to 0} \frac{f_m(x+tu) - f(x)}{t} \end{pmatrix} = \begin{pmatrix} D_u f_1(x) \\ \vdots \\ D_u f_m(x) \end{pmatrix} \end{aligned}$$

That is, directional derivatives can be taken componentwise. In particular:

$$\frac{\partial f}{\partial x_i} = \begin{pmatrix} \frac{\partial f_1}{\partial x_i} \\ \vdots \\ \frac{\partial f_m}{\partial x_i} \end{pmatrix}$$

Example. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be given by:

$$F(x, y) = \begin{pmatrix} x^2 + y^2 \\ xy \\ \sin y \end{pmatrix}$$

Then computing:

$$\frac{\partial F}{\partial x}(x, y) = \begin{pmatrix} \frac{\partial}{\partial x}(x^2 + y^2) \\ \frac{\partial}{\partial x}(xy) \\ \frac{\partial}{\partial x}\sin y \end{pmatrix} = \begin{pmatrix} 2x \\ y \\ 0 \end{pmatrix}$$

$$\frac{\partial F}{\partial y}(x, y) = \begin{pmatrix} 2y \\ x \\ \cos y \end{pmatrix}$$

If $u = (1, 2)$. Then:

$$D_u F(x, y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \begin{pmatrix} (x+t)^2 + (y+2t)^2 \\ (x+t)(y+25) \\ \sin(y+2t \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ y + 2x \\ 2\cos(y) \end{pmatrix}$$

$$\begin{aligned} D_u F(x, y) &= D_{e_1 + 2e_2} F(x, y) = D_{e_1} F(x, y) + 2 D_{e_2} F(x, y) \\ &= \frac{\partial F}{\partial x}(x, y) + 2\frac{\partial F}{\partial y}(x, y) \end{aligned}$$

This suggests that F is differentiable at (x, y). But it's not a proof.

Theorem. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}^m$ where A is open and suppose $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$. Then:*

a) f is differentiable $x \in A$ if and only if each of the components of $f_1, \dots, f_n$ are differentiable at x

b) If f is differentiable at $x \in A$, then $Df(x)$ is the $(m \times n)$ matrix whose j-th *column is* $\frac{\partial f}{\partial x_j}$.

c) *Equivalently,* $Df(x)$ *is the* $(m \times n)$ *matrix whose* i*-th row is* $Df_i(x)$.

d) *Equivalently* $Df(x)$ *is the* $m \times n$ *matrix whose* (i, j)*-th entry is* $\frac{\partial f_i}{\partial x_j}(x)$.

Remark. In calculus for $f : \mathbb{R}^n \to \mathbb{R}$, $Df(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$, is often denoted $\nabla f(x)$, the gradient of f at x. Sometimes it is important to distinguish between $Df(x)$ which is a $(1 \times n)$ matrix and $\nabla f(x)$ which is an $(n \times 1)$ matrix, that is a vector.

Proof. f is differentiable at x if and only if there exists an $(m \times n)$ matrix A such that:

$$\frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} \to 0$$

as $\|h\| \to 0$. This holds if and only if each coordinate:

$$\frac{\|f_i(x+h) - f_i(x) - A_i \cdot h\|}{\|h\|} \to 0$$

As $\|h\| \to 0$, where A_i is the i-th row of A. Since the i-th coordinate of Ah is $A_i h$. But this is equivalent to saying that f_i is differentiable at x, and $Df_i(x)$ is equal to the i-th row of $Df(x)$.

The above implies parts a) and c). To obtain part b) and c) note that if f is differentiable at x then:

$$D_u f(x) = Df(x) \cdot u$$

Taking $u = e_j$ for $1 \leq j \leq n$. we get:

$$\frac{\partial f}{\partial x_j}(x) = Df(x) \cdot e_j$$

But this is exactly the j-th column of $Df(x)$. Therefore:

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Example. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ to be as before:

$$F(x,y) = \begin{pmatrix} x^2 + y^2 \\ xy \\ \sin y \end{pmatrix}$$

Then if the derivative exists we know:

$$Df(x,y) = \begin{pmatrix} \nabla(x^2 + y) \\ \nabla(xy) \\ \nabla(\sin y) \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ y & x \\ 0 & \cos y \end{pmatrix}$$

Great

Remark. Partial derivatives and even directional derivatives of a function can exist at x even if each f_n is not differentiable at x. Take $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

For $u \in \mathbb{R}^2 \setminus \{0\}$ let us compute $D_u f(0)$. Take $u = (u_1, u_2)$. Then:

$$\begin{aligned} D_u f(0) &= \lim_{t \to 0} \frac{f(0 + tu) - f(0)}{t} = \lim_{t \to 0} \frac{t^3 u_1^2 u_2}{t^5 u_1^4 + t^3 u_2^2} \\ &= \lim_{t \to 0} \frac{u_1^2 u_2}{t^2 u_1^4 + u_2^2} = \begin{cases} 0 & \text{if } u_2 = 0 \\ \frac{u_1^2}{u_2} & \text{if } u_2 \neq 0 \end{cases} \end{aligned}$$

In particular all the directional derivatives exist at $(x,y) = 0$. However, f is not differentiable at 0. There are different ways to see this

- Note that $D_u f(0)$ is not linear in u!!! This is bad, since we showed that $D_u f(0) = Df(0) \cdot u$ provided that f is differentiable, and $Df(0)$ is a linear transformation. Thus f is not differentiable.
- Note that f is not even continuous at 0. If we approach $(0,0)$ along the parabola $y = x^2$ we get that:

$$f(x, x^2) = \frac{x^4}{x^4 + x^4} = \frac{1}{2} \to 0$$

As $x \to 0$, but $f(0,0) = 0$.

The matrix whose entries are $\frac{\partial f_i}{\partial x_j}$ is called the Jacobian matrix. What we have learned up until now is:

- If f is differentiable at x then $Df(x)$ is equal to the Jacobian matrix at x.
- But the Jacobian matrix can exist without the derivative existing

Continuously differentiable functions

At this point he only criterion of differentiability at x that we can use is to go back to the definition. However, given how easy it is to compute partial derivative, it would be useful to have a criterion of differentiability based on partial derivatives.

Theorem. *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where A is open. Suppose that all partial derivatives exist $\frac{\partial f}{\partial x_j} : U \to \mathbb{R}^m$ exist in some neighborhood U of $x \in A$ and they are all continuous at x. Then f is differentiable at x.*

In particular if all partial derivatives exist and are continuous through A, then f is differentiable in A. We call such an f a continuously differentiable, or C^1, function. This implies that $Df : \mathbb{R}^n \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous as well (since each of its component functions are continuous).

Proof. Next time!