## Taylor's Theorem on $\mathbb{R}^d$

**Lemma** (The multinomial lemma)**.** *Let $x = (x_1, \ldots, x_n)$. We would like to look at:*

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha$$

*With:*

$$|\alpha| = \alpha_1 + \cdots \alpha_n$$
$$\alpha! = \alpha_1! \cdots \alpha_n!$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

*This generalizes the binomial theorem.*

*Proof.* The proof proceeds by induction on $n$. The binomial theorem gives the case $n = 2$. Suppose that the multinomial theorem holds up to $n - 1$. We want to show it holds for $n$, where $n \geq 3$. So then we write:

$$\begin{aligned}(x_1 + x_2 + \cdots + x_n)^k = (x_1 + (x_2 + \cdots + x_n))^k &= \sum_{a+b=k} \frac{k!}{a!b!} x_1^a (x_2 + \cdots + x_n)^b \\ &= \sum_{a+b=k} \frac{k!}{a!b!} x_1^a \sum_{|\beta|=b} \frac{b!}{\beta!} (x_2, \ldots, x_n)^\beta \\ &= \sum_{a+b=k} \sum_{\substack{|\beta|=b \\ \beta\in\mathbb{N}_0^{n-1}}} \frac{k!}{a!\beta!} x_1^a x_2^{\beta_1} \cdots x_n^{\beta_{n-1}}\end{aligned}$$

Now set $\alpha = (a, \beta)$. Then:

$$\begin{aligned}(x_1 + x_2 + \cdots + x_n)^k &= \sum_{a+b=k} \sum_{\substack{|\beta|=b \\ \beta\in\mathbb{N}_0^{n-1}}} \frac{k!}{a!\beta!} x_1^a x_2^{\beta_1} \cdots x_n^{\beta_{n-1}} \\ &= \sum_{\substack{|\alpha|=k \\ \alpha\in\mathbb{N}_0^{n}}} \frac{k!}{\alpha!} x^\alpha\end{aligned}$$

Therefore the result follows by induction. Great!!!

**Lemma** (Higher order product rule)**.** *For any* $\alpha \in \mathbb{N}_0^n$ *and* $f, g : \mathbb{R}^n \to \mathbb{C}$ *we have:*

$$\partial^\alpha (fg) = \sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma\in\mathbb{N}_0^n}} \frac{\alpha!}{\beta!\gamma!} \partial^\beta f \partial^\gamma g$$

*Whenever* $f$ *and* $g$ *are differentiable up to order* $|\alpha|$*. This generalizes Leibniz Rule.*

**Recall.** We take as notation:

$$\partial_j^a f = \partial_{x_j}^a f = \frac{\partial^a f}{\partial x_j^a}$$

For convenience

*Proof.* Again the proof is by induction on $n$. For $n = 1$, let $\alpha = k \in \mathbb{N}_0$, we want to show that:

$$\partial^k (fg) = \sum_{p+q=k} \frac{k!}{p!q!} \partial^p f \partial^q g = \sum_{p=0}^{k} \frac{k!}{p!(k-p)!} \partial^p f \partial^{k-p} g$$

This is part of your homework. Press F to pay respects. THerefore the result is true when $n = 1$. Now assume the result is true for $n - 1$, we will show it holds for $n$.

Take $f, g : \mathbb{R}^n \to \mathbb{C}$ and take $\alpha \in \mathbb{N}_0^n$. Write $\alpha = (a, \theta)$ where $a \in \mathbb{N}_0$, $\theta \in \mathbb{N}_0^{n-1}$, and $x = (x_1, x')$ where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$. Then:

$$\begin{aligned}
\partial_x^\alpha (fg) = \partial_{x_1}^a \partial_{x'}^\theta (fg) &= \partial_{x_1}^\alpha \left[ \sum_{\substack{\mu+\nu=\theta \\ \mu,\nu\in\mathbb{N}_0^{n-1}}} \frac{\theta!}{\mu!\nu!} \partial_{x'}^\mu f \partial_{x'}^\nu g \right] \\
&= \sum_{\substack{\mu+\nu=\theta \\ \mu,\nu\in\mathbb{N}_0^{n-1}}} \frac{\theta!}{\mu!\nu!} \partial_{x_1}^\alpha \left[ \partial_{x'}^\mu f \partial_{x'}^\nu g \right] \\
&= \sum_{\substack{\mu+\nu=\theta \\ \mu,\nu\in\mathbb{N}_0^{n-1}}} \frac{\theta!}{\mu!\nu!} \sum_{m+k=a} \frac{a!}{m!k!} \partial_{x_1}^m \partial_{x'}^\mu f \partial_{x_1}^k \partial_{x'}^\nu
\end{aligned}$$

So then we may write:

$$\begin{aligned}
\partial_x^\alpha(fg) &= \sum_{\substack{\mu+\nu=\theta \\ \mu,\nu\in\mathbb{N}_0^{n-1}}} \sum_{m+k=a} \frac{a!\theta!}{(\mu!m!)(\nu!k!)} \partial_{x_1}^m \partial_{x'}^\mu f \partial_{x_1}^k \partial_{x'}^\nu \\
&= \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial^\beta f \partial^\gamma g
\end{aligned}$$

The result now follows by induction. Great! Here we take:

**Recall.** We recall Taylor's Theorem for single-variable functions. Suppose $f : [a,b] \to \mathbb{R}$ is $C^k([a,b])$ and $\partial^k f : (a,b) \to \mathbb{R}$ is differentiable. Then for any $a \leq x \leq b$ then:

$$\begin{aligned}
f(x) &= R_{a,k}(x) + \sum_{j=0}^{k} \frac{(x-a)^j \cdot f^{(j)}(a)}{j!} \\
R_{a,k}(x) &= \frac{(x-a)^{k+1}}{(k+1)!} f^{(k+1)}(c)
\end{aligned}$$

For some $a \leq c \leq x$.

We will study the generalization of this theorem for functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

**Recall.** If $f = (f_1, \ldots, f_m)$ and $\alpha$ is a multi-index then:

$$\partial^\alpha f = \begin{pmatrix} \partial^\alpha f_1 \\ \cdots \\ \partial^\alpha f_m \end{pmatrix}$$

Thus we only need to consider the case $m = 1$

**Definition.** *We call a subset $G \subseteq \mathbb{R}^n$ convex provided that for every $x, y \in G$ and every $t \in [0,1]$ we have $tx + (1-t)y \in G$.*

The Plan: We would like to derive the Taylor Expansion of $f$ at some point $a$ of its domain (which should be open and convex). At order $k$ this should give us a polynomial in $x_1, \ldots, x_n$ of degree $\leq k$ that approximates the function near $a$.

## The General Statement and Proof

**Theorem** (Taylor's Theorem)**.** *Let $G \subseteq \mathbb{R}^n$ be an open convex set. Suppose that $f : G \to \mathbb{C}$ is of class $C^{k+1}$. If $a \in G$, then for any $x \in G$ we have:*

$$f(x) = R_{a,k}(x) + \sum_{\substack{|\alpha| \leq k \\ \alpha \in \mathbb{N}_0^n}} \frac{1}{\alpha!}(x-a)^\alpha \partial^\alpha f(a)$$

*where we have:*

$$R_{a,k}(x) = \sum_{\substack{|\alpha| = k+1 \\ \alpha \in \mathbb{N}_0^n}} \frac{1}{\alpha!}(x-a)^\alpha \partial^\alpha f(c)$$

*For some $c \in G$ on the line segment connecting $a$ and $x$, that is $c = ta + (1-t)x$ for some $t \in [0,1]$.*

**Recall.** Recall the following formula

$$D_u f(x+tu) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f(x+tu+su) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f(x+(t+s)u) = \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=t} f(x+ru)$$
$$D_u f(x+tu) = \frac{\mathrm{d}}{\mathrm{d}t} f(x+tu)$$

Which is nice

*Proof.* To avoid confusion, let us denote $x$ by $x_0$. We will deduce this result from the single-variable case. To do so we will look at the restriction of $f$ along the line segment connecting $a$ and $x_0$, by convexity this line segment belongs to $G$. Set:

$$\begin{aligned} \phi : [0,1] &\to \mathbb{C} \\ t &\overset{\phi}{\mapsto} f(ta + (1-t)x_0) \end{aligned}$$

Notice that $\phi(0) = f(a)$ and $\phi(1) = f(x_0)$, furthermore note that $\phi \in C^{k+1}([0,1])$ since $f \in C^{k+1}(G)$. By Taylor's Formula in one dimension at $t = 0$ we know:

$$\phi(1) = R_{0,k}(1) + \sum_{p=0}^{k} \frac{\phi^{(p)}(0) \cdot 1^p}{p!}$$
$$R_{0,k} = \frac{\phi^{k+1}(c)}{(k+1)!} \cdot 1^{k+1}$$

What is $\phi^{(p)}(0)$? For $p = 0$ we know $\phi^{(0)}(0) = \phi(0) = f(a)$. For $p = 1$ we have

$$\begin{aligned}\phi^{(1)}(t) = \phi'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f(a + t(x_0 - a)) \\ &= Df(a + t(x_0 - a)) \cdot (x_0 - a) = D_u f(a + tu)\end{aligned}$$

Where $u = x_0 - a$. But then this is equal to:

$$\phi'(t) = \left(u_1 \frac{\partial}{\partial x_1} + \cdots + u_n \frac{\partial}{\partial x_n}\right) f(a + tu)$$

So then we know that:

$$\phi'(0) = \left(u_1 \frac{\partial}{\partial x_1} + \cdots + u_n \frac{\partial}{\partial x_n}\right) f(a)$$

Now for $p = 2$:

$$\begin{aligned}\phi''(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right) f(a + tu) \\ &= \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right) D_u f(a + tu) = \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right)^2 f(a + tu)\end{aligned}$$

Think of these as operators on functions that we're manipulating and consider:

$$\frac{\mathrm{d}}{\mathrm{d}t} u_1 \frac{\partial f}{\partial x_1}(a + tu) = u_1 D_u \left(\frac{\partial f}{\partial x_1}\right)(x + tu)$$

And so in general we want to think about:

$$\begin{aligned}\phi^{(p)}(t) &= \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right)^p f(a + tu) \\ \phi^{(p)}(0) &= \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right)^p f(a)\end{aligned}$$