Handout 1

- **What is a topology on a set X?** Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X that are called *open sets* satisfying the following three conditions:

C1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

C2) Given a collection $O_\alpha \in \mathcal{T}$ of index sets, then $\cup_\alpha O_\alpha \in \mathcal{T}$ as well; We say that $\mathcal{T}$ is closed under unions,

C3) Given a *finite* collection of open set $O_1, \ldots, O_n$, then $\cap_1^n O_n \in \mathcal{T}$; We say that $\mathcal{T}$ is closed under finite intersections.

- A topology can be equivalently defined by specifying the collection of *closed sets* which satisfy the same conditions as above except that we switch unions $\cup$ with intersections $\cap$ in conditions C2) and C3). The couple $(X, \mathcal{T})$ is called a topological space, or sometimes we just say X is a topological space if we're only playing with one agreed upon topology

- A space X can have more than one topology defined on it. A topology $\mathcal{T}_1$ is said to be finer or stronger than $\mathcal{T}_2$ if $\mathcal{T}_2 \subset \mathcal{T}_1$ (we say $\mathcal{T}_2$ is courser or weaker). Notice that the trivial topology $\{\emptyset, X\}$ is the weakest topology on X.

- One way to describe a topology on a set X is to define precisely all open sets. This is what we did for metric spaces. Occasionally, we want to define the smallest topology that designates a particular collection $\mathcal{B}$ of subsets of X as open. This is done as follows:

Q1) Let $\overline{\mathcal{B}}$ be the collection of subsets of X that contains the empty set, X, as well as all sets obtained as finite intersections of elements of $\mathcal{B}$. Show that the collection $\mathcal{T}$ obtained by taking unions of elements of $\overline{\mathcal{B}}$ is a topology on X.

Q2) Show that any other topology on X that contains $\mathcal{B}$ as open sets, contains $\mathcal{T}$. We call $\mathcal{T}$ the topology generated by $\mathcal{B}$. It is the coarsest topology containing $\mathcal{B}$.

- (Product Topology) One example where this construction is useful is to define a topology on the product of topological spaces. Suppose $(X_\alpha, \mathbb{T}_\alpha)$ are topological spaces for $\alpha \in A$ (where A is an index set that could be infinite). We would like to define a "natural" topology on $\prod_\alpha X_\alpha$. One reasonable requirement is that the *cylindrical sets* are open (cylindrical sets are those of the form $\prod_\alpha U_\alpha$ where all the U_α are open in X_α and all but one of them is equal to X_α. The topology generated by this collection is called the product or Tychonoff topology.

Q3) Consider the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as defined above. Why is this the same as the standard topology on $\mathbb{R}^2$ defined in class.

- We saw in class that the interval $[0, 1)$ is not open in $\mathbb{R}$, but is open relative to the half-line $[0, \infty)$ (taking the usual metric on $\mathbb{R}$ and $[0, \infty)]$). Let us try to formalize and generalize this.

  Let (X, d) be a metric space and $Y \subset X$. Y is a metric space itself, by restricting the metric d to $Y \times Y$.

Q4) Let $E \subset Y$. We say that E is open relative to Y if it is open in the metric space (Y, d). Untangle what this definition means in terms of $N_\delta(p)$ neighborhood of a point $p \in E$. Deduce that if there is an open subset G of X, then $G \cap Y$ is open relative to Y.

Q5) Show that E is open relative to Y if and only if there exists an open subset G of X such that $E = G \cap Y$.

Q6) Compactness on the other hand behaves better. Suppose that $K \subset Y \subset X$. Then K is compact relative to X if and only if it is compact relative to Y.

Remark: As such, we always need to specify the ambient space when we talk about open/closed sets (that's why we always say "E is an open subset of X"), but we can make statements like "K is compact (or a compact metric space)" without the need to specify the ambient space.

MATH 395 Notes

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Exercise 1. *Prove Q1*

Proof. Let's go!

- Note that $\emptyset \in \overline{\mathfrak{B}}$ is in particular an element of the set $\mathcal{T}$. Likewise $X \in \mathcal{T}$
- Consider any collection $\{U_\alpha\}_{\alpha \in A}$ where each U_α is an element of $\mathcal{T}$. Then for each α there are basis sets $\{\overline{B}_i\}_{i \in I_\alpha} \subseteq \overline{\mathfrak{B}}$ so that:

  $$U_\alpha = \bigcup_{i \in I_\alpha} \overline{B}_i$$

  Therefore we have that:

  $$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} \bigcup_{i \in I_\alpha} \overline{B}_i = \bigcup_{i \in \bigcup_{\alpha \in A} I_\alpha} \overline{B}_i$$

  And therefore by definition of $\mathcal{T}$ we know the union of the $\{U_\alpha\}$ is an element of $\mathcal{T}$.
- Consider any finite collection $U_1, \dots, U_n$ in $\mathcal{T}$. For each $1 \leq i \leq n$ there are basis sets $\{\overline{B}_\alpha\}_{\alpha \in A_i}$ each in $\overline{\mathfrak{B}}$. If any of the $\overline{B}_\alpha$ for $\alpha \in A_i$ are the empty set then they don't effect U_i, and if any of them are the whole space then that $U_i = X$ and it doesn't effect the whole intersection.

Thus we can assume that there exists $\{B_j\}_{1\leq j\leq m_\alpha}$ in $\mathfrak{B}$ such that:

$$\begin{aligned}\overline{B}_\alpha &= \bigcap_{j=1}^{m_\alpha} B_j \\ U_i &= \bigcup_{\alpha\in A_i} \overline{B}_\alpha \\ &= \bigcup_{\alpha\in A_i}\bigcap_{j=1}^{m_\alpha} B_j\end{aligned}$$

Therefore we can write by Demorgan:

$$\begin{aligned}\bigcap_{i=1}^{n} U_i &= \bigcap_{i=1}^{n}\bigcup_{\alpha\in A_i}\bigcap_{j=1}^{m_\alpha} B_j \\ &= \bigcup_{(\alpha_1,\ldots,\alpha_n)\in\prod_{i=1}^{n} A_i}\bigcap_{i=1}^{n}\bigcap_{j=1}^{m_{\alpha_i}} B_j\end{aligned}$$

And since the finite intersection of finite intersections is a finite intersection we win, this is open.

Exercise 2. *Show Q2*

Proof. Fix a topology $\mathbb{T}$ on X which contains each element of $\mathfrak{B}$. Fix some open set $U \in \mathcal{T}$. Then we know there is some collection $\{\overline{B}_\alpha\}_{\alpha\in A}$ each in $\overline{\mathfrak{B}}$ such that:

$$U = \bigcup_{\alpha\in A} \overline{B}_\alpha$$

Thus we merely just need to show that $\overline{\mathfrak{B}} \subseteq \mathbb{T}$ since $\mathbb{T}$ is closed under arbitrary unions:

- We know that $\emptyset$ and X are elements of $\mathbb{T}$ since $\mathbb{T}$ is a topology
- In the other case for $\overline{B} \in \overline{\mathfrak{B}}$ we have that for some $B_1, \ldots, B_n$ in $\mathfrak{B}$ that:

$$\overline{B} = \bigcap_{i=1}^{n} B_i$$

Since $\mathbb{T}$ contains each B_i and it is closed under finite intersection we then know that $\overline{B}$ is in $\mathbb{T}$ as desired.

Thus we win! We have that $\mathbb{T} \subseteq \mathcal{T}$.

Exercise 3. *Show Q3. That is show the product topology on $\mathbb{R}^2$ agrees with the Euclidean topology on $\mathbb{R}^2$.*

Proof. Call the product topology $\mathcal{T}_\pi$ and the Euclidean topology $\mathcal{T}_\mathcal{E}$. We proceed by two-way containment.

($\subseteq$) We know by $Q2$ that to show $\mathcal{T}_\pi \subseteq \mathcal{T}_\mathcal{E}$ it suffices to show that each cylindrical set is an open set in the Euclidean topology. There are two cases:

- Suppose that U is open in $\mathbb{R}$. We must show that $U \times \mathbb{R}$ is open in $\mathbb{R}^2$ with the Euclidean topology. Fix $(x, y) \in U \times \mathbb{R}$. Then $x \in U$, so there exists some $\varepsilon > 0$ so that $N_\varepsilon(x) \subseteq U$. We claim that $N_\varepsilon(x, y) \subseteq U \times R$. Fix $(v, w) \in N_\varepsilon(x, y)$. Then we know that:

$$\begin{aligned} d(x, v) &= |x - v| = \sqrt{(x - v)^2} \\ &\leq \sqrt{(x - v)^2 + (y - w)^2} = d((x, y), (v, w)) < \varepsilon \end{aligned}$$

  Therefore $v \in N_\varepsilon(x) \subseteq U$. Since $v \in U$ and $w \in \mathbb{R}$ we know that $(v, w) \in U \times \mathbb{R}$ as desired.

- Suppose that U is open in $\mathbb{R}$. We must show that $\mathbb{R} \times U$ is open in $\mathbb{R}^2$ with the Euclidean topology. Fix $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then $y \in U$, so there exists some $\varepsilon > 0$ so that $N_\varepsilon(y) \subseteq U$. We claim that $N_\varepsilon(x, y) \subseteq U \times R$. Fix $(v, w) \in N_\varepsilon(x, y)$. Then we know that:

$$\begin{aligned} d(y, w) &= |y - w| = \sqrt{(y - w)^2} \\ &\leq \sqrt{(x - v)^2 + (y - w)^2} = d((x, y), (v, w)) < \varepsilon \end{aligned}$$

  Therefore $w \in N_\varepsilon(y) \subseteq U$. Since $w \in U$ and $v \in \mathbb{R}$ we know that $(v, w) \in \mathbb{R} \times U$ as desired.

($\supseteq$) Fix some open set $U \subseteq \mathbb{R}^2$ with the Euclidean topology. Fix some $(x, y) \in U$. Then there is an $\varepsilon > 0$ so that $N_\varepsilon(x, y) \subseteq U$. Then set $\delta := \frac{\varepsilon}{\sqrt{2}}$. Consider then this open set in the product topology:

$$V_{(x,y)} = (N_\delta(x) \times \mathbb{R}) \cap (\mathbb{R} \times N_\delta(y)) = N_\delta(x) \times N_\delta(y)$$

It is clear that $(x, y) \in V_{(x,y)}$. Now take $(a, b) \in V_{(x,y)}$. We then know that $|a - x| < \frac{\varepsilon}{\sqrt{2}}$ and $|y - b| < \frac{\varepsilon}{\sqrt{2}}$. We then must have the following:

$$\begin{aligned}
(a-x)^2 &< \frac{\varepsilon^2}{2} \\
(b-y)^2 &< \frac{\varepsilon^2}{2} \\
(a-x)^2 + (b-y)^2 &< \varepsilon^2 \\
d((a,b),(x,y)) &< \varepsilon
\end{aligned}$$

Therefore $(a, b) \in N_\varepsilon(x, y) \subseteq U$. This shows that $V_{(x,y)} \subseteq U$. This lets us write that:

$$U = \bigcup_{(x,y) \in U} V_{(x,y)}$$

Thus since $\mathcal{T}_\pi$ is a topology and each $V_{(x,y)}$ is open in $\mathcal{T}_\pi$ we win! We have that U is open in $\mathcal{T}_\pi$.

With this we win!