# MATH 395 Notes

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## Characterization of Riemann Integrability

**Definition.** *Let $A \subseteq \mathbb{R}^n$. We say that $A$ has Lebesgue measure zero in $\mathbb{R}^n$ if for every $\varepsilon > 0$ there exists a covering of $A$ by a countable collection $B_1, B_2, \ldots$ of boxes such that:*

$$\sum_{j=1}^{\infty} v(B_j) < \varepsilon$$

*We'll call this $\ell$-measure zero for convenience.*

**Proposition.** *Some properties of measure-zero sets:*

*a) If $B \subseteq A$ and $A$ has $\ell$-measure zero, then $B$ has $\ell$-measure zero*

*b) If $A = \bigcup_{j=1}^{\infty} A_j$ and $A_j$ has $\ell$-measure zero for all $j$, then $A$ has $\ell$-measure zero.*

*c) A set $A$ has $\ell$-measure zero if and only if for every $\varepsilon > 0$ there exists a covering of $A$ by a countable collection of open boxes $B_1, B_2, \ldots$ such that:*

$$\sum_{j=1}^{\infty} v(B_j) < \varepsilon$$

*Aka, we may replace the boxes in the definition by open boxes*

*d) If $B$ is a box, then $\partial B$ has $\ell$-measure zero*

*e) If $v(B) \neq 0$ then $B$ does not have $\ell$-measure zero*

*Proof.* Let's go!

a) (a) is direct

b) Fix some $\varepsilon > 0$. Then since $A_j$ has $\ell$-measure zero there are boxes $B_{j1}, B_{j2}, \dots$ such that:

$$A_j \subseteq \bigcup_{k=1}^{\infty} B_{jk}$$
$$\sum_{k=1}^{\infty} v(B_{jk}) < \frac{\varepsilon}{2^j}$$

And then:

$$A \subseteq \bigcup_{j=1}^{\infty} \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} B_{jk}$$
$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} v(B_{jk}) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

Therefore $A$ has $\ell$-measure zero.

c) The converse direction is immediate. We handle the forward direction. Let $A \subseteq \mathbb{R}^n$ have $\ell$-measure zero. Fix $\varepsilon > 0$. We know that there is a collection of boxes $B_1, B_2 \dots$ such that:

$$A \subseteq \bigcup_{j=1}^{\infty} B_j$$
$$\sum_{j=1}^{\infty} v(B_j) < \frac{\varepsilon}{2^{n+1}}$$

Then for each $B_j$ with $v(B_j) \neq 0$, let $\widetilde{B}_j$ be the open box that is obtained from $B_j$ by dilating it (around its center), by a factor of 2. If $v(B_j) = 0$ then let $\widetilde{B}_j$

be an open box containing $B_j$ with $v(\widetilde{B}_j) < \frac{\varepsilon}{2^{j+1}}$. Then clearly:

$$A \subseteq \bigcup_{j=1}^{\infty} B_j \subseteq \bigcup_{j=1}^{\infty} \widetilde{B}_j$$
$$\sum_{j=1}^{\infty} v(\widetilde{B}_j) = \sum_{\substack{j=1 \\ v(B_j)=0}}^{\infty} v(\widetilde{B}_j) + 2^n \sum_{\substack{j=1 \\ v(B_j)\neq 0}}^{\infty} v(B_j)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Great! Thus $A$ has $\ell$-measure zero.

d) Let $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then $\partial B$ is the union of the faces of $B$ given by:

$$[a_1, b_1] \times [a_{j-1}, b_{j-1}] \times \xi_j \times [a_{j+1}, b_{j+1}] \times \cdots \times [a_n, b_n]$$

Where $1 \leq j \leq n$ and $\xi_j \in \{a_j, b_j\}$. Let us denote this ace by $F_j$. Then:

$$F_j \subseteq B_j = [a_1, b_1] \times \cdots \times [\xi_j - \delta/2, \xi_j + \delta/2] \times \cdots \times [a_n, b_n]$$
$$v(B_j) = \delta \prod_{i \neq j} b_j - a_j$$

We can make this arbitrarily small by choosing $\delta$ to be small, and so $F_j$ has $\ell$-measure zero, showing that $\partial B$ has $\ell$-measure zero by part (b).

e) Now suppose that $v(B) \neq 0$ and $B$ has $\ell$-measure zero for the sake of ontradiction. We know that $\overline{B} = B \cup \partial B$ and so by part (b) we know that $\overline{B}$ has $\ell$-measure zero, and also $v(\overline{B}) \neq 0$ since $B \subseteq \overline{B}$. Now take $\varepsilon = \frac{1}{2} v(\overline{B})$ and let $B_1, B_2, \ldots$ be a countable collection of open boxes such that:

$$\overline{B} \subseteq \bigcup_{i=1}^{\infty} B_j$$
$$\sum_{j=1}^{\infty} v(B_j) < \varepsilon$$

Since $\overline{B}$ is compact, there exists a finite subcollection, say $B_1, \ldots, B_k$ such taht

$\overline{B}$ is a subset of $B_1 \cup \cdots \cup B_k$. Then:

$$v(\overline{B}) \leq \sum_{j=1}^{k} v(B_j) < \varepsilon = \frac{1}{2} v(\overline{B})$$

Since $v(\overline{B}) \neq 0$ this gives a contradiction!

Great!

**Example.** The set of rational numbers in $[0, 1]$ has $\ell$-measure zero, because it is the countable union of singletons, and every singleton has $\ell$-measure zero. Recall that this set is <u>not</u> Jordan measurable.

**Theorem** (Characterization of Riemann integrability)**.** *Let $B \subseteq \mathbb{R}^d$ be a box and $f : B \to \mathbb{R}$ be a bounded function. Let $\mathscr{D}$ be the set of points in $B$ at which $f$ is discontinuous. Then $f$ is Riemann integrable on $B$ if and only if $\mathscr{D}$ has $\ell$-measure zero.*

**Example.** Consider the following function:

$$f(x) : [0, 1] \to \mathbb{R}$$
$$x \xrightarrow{f} \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then $\mathscr{D} = [0, 1]$, which does not have $\ell$-measure zero. Therefore $f$ is not Riemann integrable.

*Proof.* Choose $M$ such that $|f(x)| \leq M$ for all $x \in B$:

($\Leftarrow$) Suppose that the set $\mathscr{D}$ has $\ell$-measure zero. Let $\varepsilon > 0$ be given. We shall exhibit a partitioon $P$ of $B$ such that:

$$U(f, P) - L(f, P) \leq C\varepsilon$$

where $C$ is a constant independent of $\varepsilon$ and $P$. By the Riemann criterion, this implies that $f$ is Riemann integrable. Since $\mathscr{D}$ has $\ell$-measure zero. There exists open boxes $B_1, B_2, \ldots$ such that:

$$\mathscr{D} \subseteq \bigcup_{j=1}^{\infty} B_j$$

$$\sum_{j=1}^{\infty} v(B_j) < \varepsilon$$

For each $x \notin \mathscr{D}$, $f$ is continuous at $x$, and so there exists an open box $Q_x$ centered at $x$ such that:

$$\left|f(y) - f(y')\right| < \varepsilon \qquad (\forall y, y' \in Q_x \cap B)$$

Let $C_x = (Q_x \cap B)^o$ which is a box. The collection $\{B_j\}$ and $\{C_x\}$ is an open cover of $B$ which is compact. Therefore there exists a finite subcover:

$$B_1 \cup \cdots \cup B_p \cup C_{x_1} \cup \cdots \cup C_{x_q}$$

Rename $C_\ell := C_{x_\ell}$. We have thus obtained that:

$$\begin{aligned} B &= \left(\bigcup_{k=1}^{p} B_k\right) \cup \left(\bigcup_{\ell=1}^{q} C_\ell\right) \\ &\sum_{k=1}^{p} v(B_k) < \varepsilon \\ &y, y' \in C_\ell \implies \left|f(y) - f(y')\right| < \varepsilon \end{aligned}$$

Let $P$ be the partition of $B$ that contains all of the endpoints of the component intervals of the boxes $\{B_k\}$ and $\{Q_\ell\}$. Then each $B_k$ and each $Q_\ell$ is the union of sub-boxes is the union of sub-boxes determined by $P$.

We split the sub-boxes $R$ determined by $P$ into two groups, which we will call $\mathscr{R}_1$ and $\mathscr{R}_2$. $\mathscr{R}_1$ is the sub-boxes that are contained in $B_k$ for some $1 \leq k \leq p$, then $\mathscr{R}_2$ are the sub-boxes contained in $Q_\ell$ for some $1 \leq \ell \leq q$.

We then estimate:

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{R} [M_R(f) - m_R(f)] \cdot v(R) \\
&\le \sum_{R \in \mathscr{R}_1} [M_R(f) - m_R(f)] \cdot v(R) + \sum_{R \in \mathscr{R}_2} [M_R(f) - m_R(f)] \cdot v(R) \\
&\le \sum_{R \in \mathscr{R}_1} 2M \cdot v(R) + \sum_{R \in \mathscr{R}_2} \varepsilon \cdot v(R) \\
&\le 2M \cdot \sum_{R \in \mathscr{R}_1} v(R) + \varepsilon \cdot \sum_{R \in \mathscr{R}_2} v(R) \\
&\le 2M \cdot \sum_{k=1}^{p} \sum_{\substack{R \in \mathscr{R}_1 \\ R \subseteq B_k}} v(R) + \varepsilon \cdot \sum_{R} v(R) \\
&= 2M \cdot \sum_{k=1}^{p} v(B_k) + \varepsilon \cdot v(B) \\
&< (2M + v(B)) \cdot \varepsilon = C \cdot \varepsilon
\end{aligned}$$

And this finishes this part of the proof!

($\Rightarrow$) We now show that if $f$ is integrable then $\mathscr{D}$ has $\ell$-measure zero. We need to introduce the notion of the oscillation of a function at a point:

**Definition.** *With $g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ bounded and for $x \in A$ we define the oscillation of $g$ at $x$:*

$$\begin{aligned}
\operatorname{osc}_\delta g(x) &:= \sup_{y, y' \in A \cap B(x,\delta)} [g(y) - g(y')] \\
\operatorname{osc} g(x) &:= \inf_{\delta > 0} \operatorname{osc}_\delta g(x)
\end{aligned}$$

**Exercise.** *Show the following properties of the oscillation function:*

*a)* $\operatorname{osc}_\delta g(x) = \sup_{B(x,\delta) \cap A)} g - \inf_{B(x,\delta) \cap A} g \ge 0$.

*b)* $\operatorname{osc}_\delta g(x)$ *is increasing in* $\delta$, *i.e. if* $\delta < \delta'$ *then* $\operatorname{osc}_\delta g(x) \le \operatorname{osc}_{\delta'} g(x)$. *This follows because the supremum over a smaller set is smaller than the supremum over a bigger set*

*c) Then we have that* $\operatorname{osc} g(x) = \lim_{\delta \to 0} \operatorname{osc}_\delta g(x)$.

*d)* $f$ *is continuous at* $x$ *if and only if* $\operatorname{osc} f(x) = 0$.

The rest of this direction will be done in next section