MATH 395 Notes

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Continue Differentiability in higher dimensions

We first recalled the definition of the derivative for $\phi : \mathbb{R} \to \mathbb{R}^d$:

$$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

But we cannot divide by h if $h \in \mathbb{R}^d$. We reinterpreted the definition saying that $\phi'(x)$ exists if and only if:

$$\lim_{h \to 0} \frac{|\phi(x+h) - \phi(x) - \phi'(x)h|}{|h|} = 0$$

Reinterpreting this for $\phi : \mathbb{R}^n \to \mathbb{R}^m$ we ask for a linear transformation $D\phi(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{h \to 0} \frac{\|\phi(x+h) - \phi(x) - D\phi(x) \cdot h\|}{\|h\|}$$

This recalls the best linear approximation interpretation of the derivative. If we write:

$$\begin{aligned} \Delta\phi(h) &= \phi(x+h) - \phi(x) \\ r(h) &= \Delta\phi(h) - D\phi(x) \cdot h \end{aligned}$$

Then we ask for $\frac{\|r(h)\|}{\|h\|} \to 0$ as $h \to 0$. We write this as $\|r(h)\| = o(\|h\|)$ That is $\|r(h)\| \ll \|h\|$ as $h \to 0$.

Definition. *Let $E \subseteq \mathbb{R}^n$ be open and let $f : E \to \mathbb{R}^m$. We say that f is differentiable at $x \in E$ provided that there is a linear transformation $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$*

such that:

$$\lim_{\|h\|\to 0} \frac{\|f(x+h) - f(x) - Df(x)\cdot h\|}{\|h\|} = 0 \tag{$**$}$$

We can of course think of $Df(x)$ as an $m \times n$ matrix. If f is differentiable at every $x \in E$ we say that f is differentiable in E. In this case we have the total derivative:

$$Df : E \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$$

Remark. We have some comments

- We need x to be an interior point of E so that $x + h \in E$ for small h, so that $f(x+h)$ makes sense. When E is open this is automatic.
- The numerator in the difference quotient above is in $\mathbb{R}^m$ whereas the denominator is in $\mathbb{R}^n$.
- Defining $r(h) = f(x+h) - f(x) - Df(x)\cdot h$, we have that $r(h) = o(h)$. That is:

  $$\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = 0$$

  Note then that $Df(x)\cdot h = \mathcal{O}(h)$, that is there is a constant $C \in \mathbb{R}$ so that $\|Df(x)\cdot h\| \le C\|h\|$, but this is different than $r(h) = o(h)$.
- This definition of derivative only makes sense if $Df(x)$ is unique when it exists.

Proposition 1. Let E, f, and $x \in E$ be as in the above definition. Suppose that A_1 and A_2 are two linear transformations such that $(**)$ holds. Then $A_1 = A_2$

Proof. Let $r_j(h) = f(x+h) - f(x) - A_j h$ for $j = 1, 2$. Then we have that $\frac{\|r_j(h)\|}{\|h\|} \to 0$. Let $u \in \mathbb{R}^n$ be arbitrary and nonzero and take $h = tu$ for $t > 0$, then we can divide by $\|tu\|$ to get:

$$\begin{aligned} r_1(tu) - r_2(tu) &= (A_2 - A_1)(tu) = t(A_2 - A_1)u \\ \frac{\|(A_2 - A_1)u}{\|u\|} &= \frac{\|r_1(tu) - r_2(tu)\|}{t\|u\|} \\ &\le \frac{\|r_1(tu)\|}{\|tu\|} + \frac{\|r_2(tu)\|}{\|tu\|} \end{aligned}$$

Thus $\frac{\|(A_2-A_1)u\|}{\|u\|} \to 0$ as $t \to 0$. Therefore $(A_2 - A_1)u = 0$, so $A_1 u = A_2 u$. Note that clearly $A_1 \cdot 0 = A_2 \cdot 0$. Taking these together we know $A_1 = A_2$.

Example. Let $f(x) = a + Bx$ where $a \in \mathbb{R}^m$ and and $B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ where $f : \mathbb{R}^n \to \mathbb{R}^m$. Then to compute $Df(x)$ note that:

$$
\begin{aligned}
f(x+h) - f(x) &= Bh \\
f(x+h) - f(x) - Bh &= 0
\end{aligned}
$$

Therefore we know clearly that:

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Bh\|}{\|h\|} = 0$$

Therefore $Df(x) = B$ for any $x \in \mathbb{R}^n$.

Remark. Of course, if f is differentiable at x, then it must be continuous there. Why? Continuity is equivalent to $\|f(x+h) - f(x)\| \to 0$ as $h \to 0$. Differentiability is equivalent to $\|f(x) + f(x) - Df(x)h\| = \|r(h)\| = o(\|h\|)$. In particular this implies that:

$$
\begin{aligned}
\|f(x+h) - f(x)\| &= \|Df(x)h + r(h)\| \\
&\overset{\triangle}{\leq} \|Df(x)h\| + \|r(h)\|
\end{aligned}
$$

But both of these go to 0 as $h \to 0$. Therefore:

$$\lim_{h \to 0} \|f(x+h) - f(x)\| = 0$$

Directional and Partial Derivatives, computing the derivative

Definition. *Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}^m$. Suppose $x \in A$ and $u \in \mathbb{R}^n$ with $u \neq 0$. We define the directional derivative $D_u f(x)$ as the limit:*

$$
\begin{aligned}
D_u f(x) &:= \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} \in \mathbb{R}^m \\
D_u f(x) &:= \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} \in \mathbb{R}^m
\end{aligned}
$$

Note that this just means that:

$$D_u f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(x+tu)$$

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ bed efined as $\sin(x_1 x_2)$. Then let $u = (1, 0)$:

$$\begin{aligned} D_u f(x_1, x_2) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sin((x_1 + t)x_2) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sin(x_1 x_2 + t x_2) \\ &= (\cos(x_1 x_2 + t x_2) \cdot x_2]_{t=0} \\ &= \cos(x_1 x_2) \cdot x_2 \end{aligned}$$

Theorem. *Let $A \subseteq \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$ be differentiable at $x \in A$. THen all directional derivatives $D_u f(x)$ exist at x_0 and:*

$$D_u f(x) = Df(x) \cdot u$$

In particular $D_u f(x)$ is linear in u.

Proof. From the definition of $Df(x)$ we have for any $u \in \mathbb{R}^n \setminus \{0\}$

$$\lim_{t \to 0} \frac{\|f(x+tu) - f(x) - Df(x) \cdot tu\|}{\|tu\|} = 0$$
$$\lim_{t \to 0} \frac{\|f(x+tu) - f(x) - t \cdot (Df(x) \cdot u)\|}{\|tu\|} = 0$$

This implies that:

$$f(x+tu) - f(x) - t \cdot Df(x) \cdot u = r(tu)$$

Therefore $\frac{\|r(tu)\|}{\|tu\|} \to 0$ as $t \to 0$. Dividing by t we get that:

$$\frac{f(x+tu) - f(x)}{t} - Df(x)u = \frac{r(tu)}{t}$$

Therefore:

$$\left\| \frac{f(x+tu) - f(x)}{t} - Df(x)u \right\| = \frac{\|r(tu)\|}{\|t\|} = \|u\| \cdot \frac{\|r(tu)\|}{\|tu\|} \to 0$$

As $t \to 0$. Therefore:

$$\lim_{t \to 0} \left\| \frac{f(x+tu) - f(x)}{t} - Df(x)u \right\| = 0$$
$$D_u f(x) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} = Df(x) \cdot u$$

<u>Caution</u> We will see next time that the converse is not true. Namely, the directional derivatives might exist at x without f being differntiable at x. In that case $D_u f(x)$ might not even be a linear function of u.

Partial Derivatives

Since $D_u f(x) = Df(x) \cdot u$, we can determine $Df(x)$ by letting u range over the standard basis vectors.

Definition. *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where A is open. The j-th partial derivative of f at x is defined as:*

$$\frac{\partial f}{\partial x_j}(x) = D_{e_j} f(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x + te_j)$$

Example. When $m = 1$ we know $f : \mathbb{R}^n \to \mathbb{R}$ then:

$$\begin{aligned}\frac{\partial f}{\partial x_j}(x_1, \dots, x_n) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x_1, \dots, x_j + t_j, \dots, x_n) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=x_j} f(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_n) \\ &= \phi'(x_j)\end{aligned}$$

Where $\phi(s) = f(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_n)$. This just means that $\frac{\partial f}{\partial x_j}$ is computed by pretending that $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$ are constant and differentiating with respect to x_j.