MATH 395 Notes

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Theorem (Fubini)**.** *Given a box $Q = A \times B$ where $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^\ell$ are boxes. Let $f : Q \to \mathbb{R}$ be a bounded function, and we write it as $f(x, y)$ where $x \in A$ and $y \in B$.*

If f is integrable over Q, then the functions:

$$x \mapsto \underline{\int_B} f(x,y)\,\mathrm{d}y \qquad\qquad x \mapsto \overline{\int_B} f(x,y)\,\mathrm{d}y$$

are both integrable over A, and furthermore:

$$\int_Q f = \int_A \underline{\int_B} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_A \overline{\int_B} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

Remark. The drawback is that the iterated integrals are in terms of lower and upper integrals rather than having $\int_B f(x,y)\,\mathrm{d}y$. We cannot guarantee that these agree

Corrolary. *With the same assumptions as above, there holds the following:*

a) $\int_B f(x,y)\,\mathrm{d}y$ *exists for almost every $x \in A$, that is, it exists except on a set of Lebesgue measure zero. In other words $x \mapsto \int_B f(x,y)\,\mathrm{d}y$ is defined for all $x \in A \setminus N$ where N has Lebesgue measure zero.*

b) If we further assume that $\int_B f(x,y)\,\mathrm{d}y$ exists for all $x \in A$, then we have Fubini's Theorem as we would like it:

$$\int_Q f = \int_A \int_B f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

c) Let $Q = I_1 \times \cdots \times I_n$ where $I_k = [a_i, b_i] \subseteq \mathbb{R}$. Then if $f : Q \to \mathbb{R}$ is continuous *then:*

$$\int_Q f = \int_{I_1} \cdots \int_{I_n} f(x_1, \dots, x_n)\,\mathrm{d}x_n \cdots \mathrm{d}x_1$$

Proof of Corollary. By Fubini's:

$$\int_Q f = \int_A \underline{\int_B} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_A \overline{\int_B} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

$$0 = \int_A \left(\underbrace{\overline{\int_B} f(x,y)\,\mathrm{d}y - \underline{\int_B} f(x,y)\,\mathrm{d}y}_{\geq 0 \text{ and integrable}} \right) \mathrm{d}x$$

Therefore by previous work:

$$\overline{\int_B} f(x,y)\,\mathrm{d}y - \underline{\int_B} f(x,y) = 0$$

except possibly on a set of measure zero. This gives part a).

Part b) is exactly from Fubini's theorem, and part c) follows because continuous functions are always integrable.

Proof of Fubini. Let us define the following:

$$\underline{I}(x) = \underline{\int_B} f(x,y)\,\mathrm{d}y \qquad\qquad \overline{I}(x) = \overline{\int_B} f(x,y)\,\mathrm{d}y$$

We need to show that if $\int_Q f$ exists then $\underline{I}(x)$ and $\overline{I}(x)$ are both integrable over A, and their integrals are both $\int_Q f$.

Let P be any partition of Q and write $P = (P_A, P_B)$ are partitions of A and B. Any sub-box R determined by P can be written as $R = R_A \times R_B$ where R_A and R_B are sub-boxes of A and B determined by P_A and P_B respectively.

Now note that for any $x \in R_A$:

$$m_R(f) = \inf_R f(x,y) \leq \inf_{y \in R_B} f(x,y)$$

$$m_R(f) \leq m_{R_B}(f(x,-))$$

Multiplying by $v(R_B)$ and summing over all sub-boxes R_B we get for every $x \in R_A$:

$$\begin{aligned}\sum_{R_B} m_R(f) \cdot v(R_B) &\leq \sum_{R_B} m_{R_B}(f(x,-))v(R_B) \\ &= L(f(x,-), P_B) \leq \underline{I}(x)\end{aligned}$$

Then if we take the infimum over $x \in R_A$ we obtain:

$$\sum_{R_B} m_R(f)v(R_B) \leq m_{R_A}(\underline{I})$$

We then multiply by $v(R_A)$ and sum over all such R_A:

$$\begin{aligned}\sum_{R_A, R_B} m_R(f)v(R_B)v(R_A) &\leq \sum_{R_A} m_{R_A}(\underline{I})v(R_A) \\ L(f,P) &\leq L(\underline{I}, P_A)\end{aligned}$$

An exactly similar argument establishes that:

$$U(f,P) \geq U(\overline{I}, P_A)$$

Given these two inequalities, we will be able to finish the proof. Note that because $\underline{I} \leq \overline{I}$ we have:

$$\begin{aligned}L(f,P) \leq L(\underline{I}, P_A) \leq U(\underline{I}, P_A) \leq U(\overline{I}, P_A) \leq U(f,P) \\ L(f,P) \leq L(\underline{I}, P_A) \leq L(\overline{I}, P_A) \leq U(\overline{I}, P_A) \leq U(f,P)\end{aligned}$$

These inqualities hold for any partition P. Let $\varepsilon > 0$ be arbitrary and choose P so that $U(f,P) - L(f,P) < \varepsilon$. Therefore from the above inequalities and a squeezing argument:

$$\begin{aligned}U(\underline{I}, P_A) - L(\underline{I}, P_A) < \varepsilon \\ U(\overline{I}, P_A) - L(\overline{I}, P_A) < \varepsilon\end{aligned}$$

This gives that $\underline{I}$ and $\overline{I}$ are both integrable on A. Now we get that:

$$
\begin{gathered}
L(f,P) \leq \int_Q f \leq U(f,P) \\
L(f,P) \leq L(\underline{I},P) \leq \int_A \underline{I} \leq U(\underline{I},P) \leq U(f,P) \\
L(f,P) \leq L(\overline{I},P) \leq \int_A \overline{I} \leq U(\overline{I},P) \leq U(f,P)
\end{gathered}
$$

Therefore we get that:

$$
\left|\int_Q f - \int_A \underline{I}\right| < \varepsilon \qquad\qquad \left|\int_Q f - \int_A \overline{I}\right| < \varepsilon
$$

And so since $\varepsilon > 0$ was chosen arbitrarily, we must have that:

$$
\int_Q f = \int_A \underline{I} = \int_A \overline{I}
$$

Which is exactly what we wanted to show!

Integral over a bounded set

Up until now, we have been integrating functions on boxes. What if we want to integrate a function over a region $S \subseteq \mathbb{R}^n$ that is not a box.

Definition. *Let $S \subseteq \mathbb{R}^n$ be a bounded set and suppose that $f : S \to \mathbb{R}$ is a bounded function. We define $f_S(x) = f(x)$ when $x \in S$ and $f_S(x) = 0$ when $x \notin S$. Then f_S is defined on all of $\mathbb{R}^n$, that is $f_S : \mathbb{R}^n \to \mathbb{R}$*

Choose a box Q which contains S, then we define the integral of f over S as:

$$\int_S f(x)\,\mathrm{d}x = \int_Q f_S(x)\,\mathrm{d}x$$

provided that the integral on the right hand side exists.

For this definition to make sense, we should get the same answer if we change the box Q. This is guaranteed by the following lemma:

Lemma. *Let Q and Q' be two boxes in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function that is supported inside $Q \cap Q'$. That is $f = 0$ outside $Q \cap Q'$.*

$$\int_Q f = \int_{Q'} f$$

Included in the statement is that f is integrable over Q if and only if f is integrable over Q'.

Proof. Let's go!

Case 1) Suppose that $Q \subseteq Q'$. Then f is supported in Q.

Note that f is integrable on Q if and only if the set of discontinuities of f in Q has Lebesgue measure zero, we call this set $\mathscr{D}$. But wait! The set of discontinuities of f in Q', which we call $\mathscr{D}'$, is equal to $\mathscr{D} \cup A$, where $A \subseteq \partial Q$, because f is constant on $Q' \setminus Q$. Since ∂Q has Lebesgue measure zero, and so A has Lebesgue measure zero, we know $\mathscr{D}'$ has Lebesgue measure zero if and only if $\mathscr{D}$ has Lebesgue measure zero. Therefore:

$$f \text{ is integrable over } Q' \iff f \text{ is integrable over } Q$$

Now let P be a partition of Q' and let $\widetilde{P}$ be the refinement of P obtained from P by adding in the endpoints of Q. Then Q is the union of some sub-boxes determined by $\widetilde{P}$. Write $Q = \bigcup_{B \in \mathscr{S}}$ where $\mathscr{S}$ is the family of sub-boxes determined by $\widetilde{P}$ such that $B \subseteq Q$.

Now if B is determined by $\widetilde{P}$ and $B \notin \mathscr{S}$ then $f(x) = 0$ for some $x \in B$.

Therefor if $B \notin \mathscr{S}$ then $m_B(f) \leq 0 \leq M_B(f)$. Therefore:

$$L(f, P) \leq L(f, \widetilde{P}) \leq \sum_{B \in \mathscr{S}} m_B(f) v(B) \leq \int_Q f$$
$$U(f, P) \geq U(f, \widetilde{P}) \geq \sum_{B \in \mathscr{S}} M_B(f) v(B) \geq \int_Q f$$

This holds for any P. Taking suprema and infima in P:

$$\int_{Q'} f = \sup L(f, P) \leq \int_Q f$$
$$\int_{Q'} f = \inf U(f, P) \geq \int_Q f$$

And therefore $\int_{Q'} f = \int_Q f$

Case 2) Pick Q'' to be a sufficiently large box containing both Q and Q'. Then:

$$\int_Q f = \int_{Q''} f = \int_{Q'} f$$

Just by applying Case 1 twice, and of course existence of these integrals if and only if one of them exists.