Handout 6

Second Try: Jordan measure

- **Definition of Jordan measure.** The main caveat of elementary measure is that it only allows us to measure elementary sets, which is a fairly restrictive family of sets. Building on the old intuition (going back at least to Archimedes) we can lower bound (respectively upper bound) the measure of a set by approximating it from within (respectively without) by an elementary set, i.e. if A and B are elementary and $A \subset E \subset B$, then the measure of E (if it exists) should be sandwiched between that of A and B.

  Definition 0.1 (Jordan measure)**.** Let $E \subset \mathbb{R}^d$ be a bounded set.

  - The *Jordan inner measure* $\underline{m}_J(E)$ of E is defined as
  $$\underline{m}_J(E) = \sup_{A \subset E, A \text{ elementary}} m(A).$$
  Here $m(A)$ is the elementary measure of A.
  - The *Jordan outer measure* $\overline{m}_J(E)$ of E is defined as
  $$\overline{m}_J(E) = \inf_{A \supset E, A \text{ elementary}} m(A).$$
  - If $\underline{m}_J(E) = \overline{m}_J(E)$, we say that E is Jordan measurable, and call the common value $m(E)$ (the Jordan measure of E).
  By convention, we do not consider unbounded sets to be Jordan measurable.

Q1) Assume that $E \subset \mathbb{R}^d$ is bounded. Show that the following are equivalent:

a) E is Jordan measurable.

b) For every $\epsilon > 0$, there exists elementary sets A and B such that $A \subset E \subset B$ and $m(B \setminus A) \leq \epsilon$.

c) For every $\epsilon > 0$, there exists an elementary set A such that $\overline{m}_J(E \Delta A) \leq \epsilon$.

Q2) Deduce that every elementary set E is Jordan measurable and that its Jordan measure is the same as its elementary measure. In particular, $m(\emptyset) = 0$.

- **Properties of Jordan measure** Let E, F be Jordan measurable sets.

Q3) Clearly $m(E) \geq 0$. Show that

(a) Show that $E \cup F, E \cap F, E \setminus F$, and $E \Delta F$ are all Jordan measurable.

(b) (Finite additivity) If E and F are disjoint, then $m(E \cup F) = m(E) + m(F)$.

(c) (Monotonicity) If $E \subset F$, then $m(E) \leq m(F)$.

(d) (Finite subadditivity) $m(E \cup F) \leq m(E) + m(F)$.

(e) (Translation invariance) for any $x \in \mathbb{R}^d$, $m(E + x) = m(E)$.

- **Some Jordan measurable sets.** Let B be a closed box of $\mathbb{R}^d$ and $f : B \to \mathbb{R}$ a continuous function.

Q5) Show that the graph $\{(x, f(x)) : x \in B\} \subset \mathbb{R}^{d+1}$ is Jordan measurable in $\mathbb{R}^{d+1}$ and that it has Jordan measure 0. *Hint: Use that f is uniformly continuous.*

Q6) Show that the set $\{(x, t) : x \in B, 0 \leq t \leq f(x)\} \subset \mathbb{R}^{d+1}$ is Jordan measurable.

From this we conclude that some familiar sets like triangles in $\mathbb{R}^2$ and balls in $\mathbb{R}^d$ are Jordan measurable.

MATH 395 Notes

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Exercise 1.

Proof. Let's go!

(a $\implies$ b) Fix a Jordan mesaurable set E and some $\varepsilon > 0$. By definition of suprema and infima there exist elementary sets $A \subseteq E$ and $E \subseteq B$ such that:

$$m(E) - \frac{\varepsilon}{2} < m(A) \leq m(E) \leq m(B) < m(E) + \frac{\varepsilon}{2}$$

Note now that $A \cup (B \setminus A) = B$ since $A \subseteq B$ and so since this union is disjoint:

$$m(B) = m(A) + m(B \setminus A) < m(E) + \frac{\varepsilon}{2}$$

So then:

$$m(B \setminus A) < m(E) - m(A) + \frac{\varepsilon}{2} < \varepsilon$$

Great! Thus c) holds.

(b $\implies$ c) Fix a set E satisfying the condition in (b). Now fix $\varepsilon > 0$. There must be elementary sets $A \subseteq E \subseteq B$ so that $m(B \setminus A) \leq \varepsilon$. Note that:

$$E \triangle A = (E \setminus A) \cup (A \setminus E) = E \setminus A$$

So then note that $E \setminus A \subseteq B \setminus A$ since $E \subseteq B$. Also $B \setminus A$ is an elementary set, so we must have by definition of infimum that:

$$\overline{m}_J(E \triangle A) \leq m(B \setminus A) \leq \varepsilon$$

And so we are done!

(b $\implies$ a) Fix some set E satisfying (b). In order to show that $\overline{m}_J(E) = \underline{m}_J(E)$ we will show that for all $\varepsilon > 0$ we have $|\overline{m}_J(E) - \underline{m}_J(E)| \leq \varepsilon$. Fix some $\varepsilon > 0$, then there exists elementary sets $A \subseteq E \subseteq B$ so that $m(B \setminus A) \leq \varepsilon$. Note that we must have by previous work and definitions that:

$$\begin{aligned} m(A) &\leq m(B) \\ m(B) &= m(A) + m(B \setminus A) \\ \overline{m}_J(E) &\leq m(B) \\ \underline{m}_J(E) &\geq m(A) \\ \overline{m}_J(E) - \underline{m}_J(E) &\leq m(B) - m(A) \\ &= m(B \setminus A) \leq \varepsilon \end{aligned}$$

Now note that for every elementary sets C_1 and C_2 with $C_1 \subseteq E \subseteq C_2$ we must have $m(C_1) \leq m(C_2)$. This shows by 295 that:

$$\overline{m}_J(E) = \sup_{\substack{C \subseteq E \\ C \text{ elementary}}} m(C) \leq \inf_{\substack{C \supseteq E \\ C \text{ elementary}}} m(C) \leq \underline{m}_J(E)$$

Therefore we have that:

$$|\overline{m}_J(E) - \underline{m}_J(E)| = \overline{m}_J(E) - \underline{m}_J(E) \leq \varepsilon$$

Taking $\varepsilon \to 0$ we know that the outer Jordan measure agrees with the inner Jordan measure and so E is Jordan measurable.

(c $\implies$ b) Fix some set E satisfying (c). Now fix some $\varepsilon > 0$. There exists some elementary set A with $\overline{m}_J(E \triangle A) \leq \frac{\varepsilon}{2} < \varepsilon$. Therefore by definition of infima there must be some elementary set B so that $E \triangle A \subseteq B$ and:

$$\overline{m}_J(E \triangle A) \leq m(B) < \varepsilon$$

Now note that $E \setminus A \subseteq B$, and so $E \subseteq A \cup B$. Set $D := A \cup B$. Now consider $C := A \setminus B$ and note that:

$$A \setminus B \subseteq A \setminus (E \triangle A) = A \setminus ((A \cup E) \setminus (A \cap E)) = A \setminus (A \cup E) \cup (A \cap E) = A \cap E$$

And therefore $C \subseteq E$. We then note that:

$$D \setminus C = (A \cup B) \setminus (A \setminus B) = B$$

So we know in particular that $m(D \setminus C) = m(B) < \varepsilon$. Since D and C are elementary sets we must have that E satisfies (b).

Exercise 2.

Proof. Fix some elementary set E. We show that E satisfies (b) from Exercise 1 and so E is Jordan measurable. Fix some $\varepsilon > 0$ and note that $E \subseteq E \subseteq E$ and furthermore:

$$m(E \setminus E) = m(\emptyset) = 0 < \varepsilon$$

So we know that E is jordan measurable. We now only to show that the Jordan measure of E agrees with the elementary measure of E. To do this we calulate $\underline{m}_J(E)$. Fix some $A \subseteq E$ with A elementary, by previous homework $m(A) \leq m(E)$ so the elementary measure of E is an upper bound on the set defining $\underline{m}_J(E)$. Furthermore, this upper bound belongs to the set defining $\underline{m}_J(E)$ since $E \subseteq E$ and E is elementary. Therefore it is a maximum for that set, and is thus the supremum.

This gives us that the Jordan measure of E, which is equal to the Jordan inner measure is also equal to the elementary measure of E just as desired. Perfect!

Exercise 1c.

Proof. Let's go! Fix E and F as Jordan measurable sets.

- **TODO**
- **TODO**
- **TODO**
- **TODO**

This gives us exactly what we want.