MATH 395 Notes

Faye Jackson

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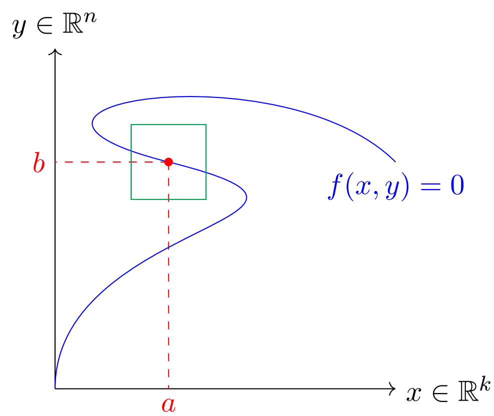
Note: No class on Friday

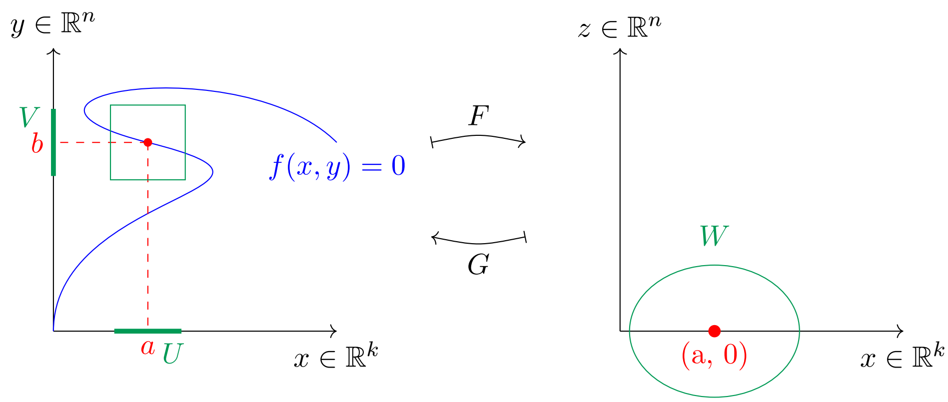
The Proof of the Implicit Function Theorem

Continued Proof of the Implicit Function Theorem. We had an $A \subseteq \mathbb{R}^{k+n}$ and an $f : A \to \mathbb{R}^n$ of class C^r with $r \geq 1$. We also had $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b)$ is nonsingular. We model this with the picture:



We constructed a neighborhood B around a, a neighborhood V around b, and a function $g : B \to V$ satisfying $g(a) = b$ and $f(x, y) = 0$ if and only if $y = g(x)$ for $(x, y) \in B \times V$. We did this with the following steps:

1) We defined an auxiliary function $F(x, y) = (x, f(x, y)) : A \to \mathbb{R}^{k+n}$. We showed that $DF(a, b)$ is invertible since $\frac{\partial f}{\partial y}(a, b)$ is invertible. We then applied the Inverse Function Theorem. This gave us the following picture



We then showed the inverse function $G(x, z)$ must be given as $(x, h(x, z))$ where $h \in C^r$.

2) We then defined g with a neighborhood $B \subseteq U$ such that $B \times \{0\} \subseteq W$. We then defined $g : B \to V$ as $g(x) = h(x, 0)$. This satisfies the desired conditions.

3) We showed the Uniqueness of g. We supposed that $g' : B \to V$ was another continuous function such that $g'(a) = b$ and $f(x, g'(x)) = 0$. We defined $S = \{x \in B \mid g'(x) = g(x)\}$. We want to show that $S = B$. Using the connectedness of B we simply need to show that S is a nonempty subset of B that is both open and closed in B.

   S is clearly nonempty since $g'(a) = g(a)$, and thus $a \in S$. We know S is closed since g, g' are both continuous, and we can rewrite $S = (g - g')^{-1}(\{0\})$. It remains to show that S is open

Let's show this! Let $x_0 \in S$, then $g'(x_0) = g(x_0) \in V$ is open. There must exist a neighborhood B' of x_0 such that $g'(B') \subseteq V$ using the fact that g' is continuous. But then:

$$f(x, g'(x)) = 0 \qquad\qquad x \in B' \subseteq B \qquad\qquad g'(x) \in V$$

But then this must mean that:

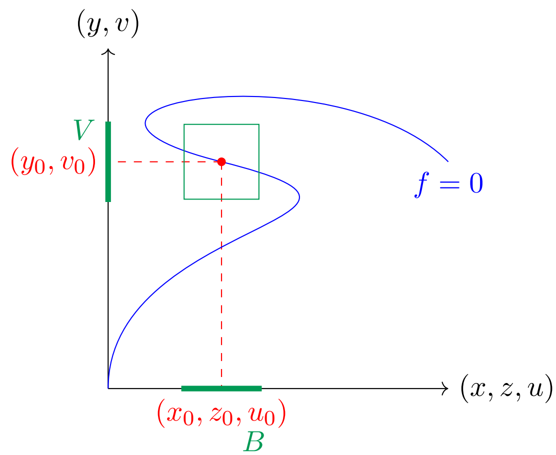
$$F(x, g'(x)) = (x, f(x, g'(x))) = (x, 0)$$
$$(x, g'(x)) = G(x, 0) = (x, h(x, 0)) = (x, g(x))$$

This of course implies that $g'(x) = g(x)$ for all $x \in B'$. Therefore S is open in B, and we win!!!! Yay ☺

How to Apply the Implicit Function Theorem

Suppose that $f : A \subseteq \mathbb{R}^5 \to \mathbb{R}^2$ is a function in C^r and the equation $f(x, y, z, u, v) = 0$ gives us two equations in five unknowns, and thus by dimension counting, the solution set is a set parameterized in three variables. We expect (under appropriate conditions) that we can solve for two of the variables in terms of the others.

Suppose one wishes to solve for (y, v) in terms of (x, z, u) near a point $(x_0, y_0, z_0, u_0, v_0 = 0$. All we need to check is that $\frac{\partial f}{\partial(y,v)}$ is nonsingular at $(x_0, y_0, z_0, u_0, v_0)$. The implicit function theorem then tells us that we can write $y = \phi(x, z, u)$ and $v = \psi(x, z, u)$



Moreover by implicit differentiation:

$$\frac{\partial(\phi, \psi)}{\partial(x, z, u)}(x_0, z_0, u_0) = -\left[\frac{\partial f}{\partial(y, v)(x_0, y_0, z_0, u_0, v_0)}\right]^{-1} \frac{\partial f}{\partial x}(x, g(x))$$

Example. Show that the system of equations:

$$x^3 - y^3 + z^2 = 0$$
$$z\cos(\pi x) + \sin(\pi y) = 0$$

admits a one-parameter family of solutions around the point $(1, 1, 0)$

Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by:

$$f(x, y, z) = \begin{pmatrix} x^3 - y^3 + z^2 \\ z\cos\pi x + \sin\pi y \end{pmatrix}$$

Then $f(1,1,0)=0$ and:

$$
\begin{aligned}
Df &= \begin{pmatrix} 3x^2 & -3y^2 & 2z \\ -\pi z \sin \pi x & \pi \cos \pi y & \cos \pi x \end{pmatrix} \\
Df(1,1,0) &= \begin{pmatrix} 3 & - & 0 \\ 0 & -\pi & -1 \end{pmatrix} = \frac{\partial f}{\partial(x,y,z)} \\
\frac{\partial f}{\partial(x,z)} &= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

This is of course a non-singular matrix, and so we can solve for (x,z) in terms of y near the point $y=1$. That is there are functions $\phi, \psi : B \to \mathbb{R}^2$ where B is an open neighborhood of $y=1$ such that $f(\phi(y), y, \psi(y)) = 0$ for all $y \in B$.

In other words, the solution set near $(1,1,0)$ is a one-parameter family of solutions. We will later find out that this means it is a "manifold of dimension one"

With this we have essentially finished differentiation!

Riemann Integration in Higher Dimensions

Definition of the integral

The purpose os this section is to generalize the notion of the Riemann integral to higher dimesnions

Definition. *We will use some concepts from our Friday sections*

1) Recall that we defined a box $B \subseteq \mathbb{R}^n$ to be the Cartesian product of n intervals $B = I_1 \times I_2 \times \cdots \times I_n$. Generally $I_1, \ldots, I_n$ can be closed, open, or half open.

However, in what follows, there will be no loss of generality in considering only closed boxes. Thus to simplify notation, we will assume that all boxes are closed unless stated otherwise

Given $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$ we set:

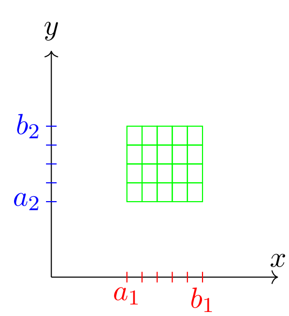
$$m(B) := \prod_{i=1}^{n} (b_i - a_i)$$

2) Partitions

$(n = 1)$ *Given an interval $I = [a, b]$ a partition of $[a, b]$ is a finite collection P of points $x_0 = a < x_1 < x_2 < \cdots < x_k = b$. Each $[x_{i-i}, x_i]$ has length $\Delta x_i = x_i - x_{i-1}$. We define the mesh (or norm) of P as:*

$$\|P\| = \max_{1 \leq i \leq k} \Delta x_i$$

$(n \geq 1)$ *Given a box $B = I_1 \times \cdots \times I_n$, a partition P of B is an n-tuple $(P_1, \ldots, P_n)$ such that P_j is a partition of I_j for each j.*



Each partition P_j decomposes I_j into sub-intervals $I_j^{(1)}, \dots, I_j^{(k_j)}$ with disjoint interiors This gives a decomposition of B into sub-boxes of the form $J_1 \times \cdots J_n$ where $J_j \in \{I_j^{(1)}, \dots, I_j^{(k_j)}\}$.

Notice that the sub-boxes can only intersect at the boundary, that is they have disjoint interiors. The mesh of a partition $P = (P_1, \dots, P_n)$ is $\|P\| = \max_{1 \leq j \leq n} \|P_j\|$.

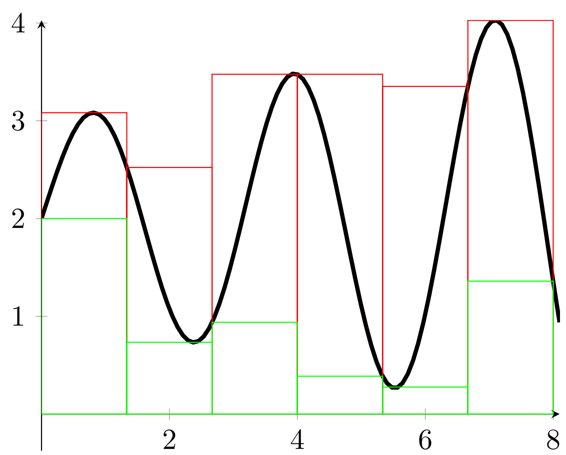
3) *We now define Lower and upper sums. Let B be a box and $f : B \to \mathbb{R}$ be bounded. Let P be a partition of B and denote by $B_1, \dots, B_N$ the resulting subboxes. Let*

$$m_{B_j}(f) := \inf_{x \in B_j} f(x)$$
$$M_{B_i}(f) := \sup_{x \in B_j} f(x)$$

Then we may define the lower and upper sums respectively as:

$$L(f, P) = \sum_{\ell=1}^{N} m_{B_\ell}(f) \cdot v(B_\ell)$$
$$U(f, P) = \sum_{\ell=1}^{N} M_{B_\ell}(f) \cdot v(B_\ell)$$

In one dimension if $f \geq 0$ then $L(f, P)$ is the sum of the green rectangles inscribed by the region under the curve, and $U(f, P)$ is the area of the red rectangles circumscribed by the region under the curve



4) *We define now the Refinement of a partition Let B be a box and le $tP =$*

$(P_1, \ldots, P_n)$ and $Q = (Q_1, \ldots, Q_n)$ be two partitions of B. We say that Q is a refinement of P is $P_j \subseteq Q_j$ for every j.

Given two partitions $P = (P_1, \ldots, P_n)$ and $P' = (P'_1, \ldots, P'_n)$ the common refinment is $Q = (P_1 \cup P'_1, \ldots, P_n \cup P'_n)$.

Lemma. *Refining a partition increases lower sums and decreases upper sums. In other words, let P be a partition of a box B and $f : B \to \mathbb{R}$ be bounded. If Q is a refinement of P, then:*

$$L(f, P) \leq L(f, Q) \qquad\qquad U(f, Q) \leq U(f, P)$$

Before proving this lemma, let us state a corollary

Corrolary. *Let B be a box and $f : B \to \mathbb{R}$ be a bounded function. If P and P' are any two partitions of B, then $L(f, P) \leq U(f, P')$.*

Proof of corollary. Clearly for any partition we have $L(f, Q) \leq U(f, Q)$. Le tQ be the common refinement of P and P' and use the lemma to see that:

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P')$$

Great!