

Handout 6

Second Try: Jordan measure

- **Definition of Jordan measure.** The main caveat of elementary measure is that it only allows us to measure elementary sets, which is a fairly restrictive family of sets. Building on the old intuition (going back at least to Archimedes) we can lower bound (respectively upper bound) the measure of a set by approximating it from within (respectively without) by an elementary set, i.e. if A and B are elementary and $A \subset E \subset B$, then the measure of E (if it exists) should be sandwiched between that of A and B .

Definition 0.1 (Jordan measure). Let $E \subset \mathbb{R}^d$ be a bounded set.

- The *Jordan inner measure* $\underline{m}_J(E)$ of E is defined as

$$\underline{m}_J(E) = \sup_{A \subset E, A \text{ elementary}} m(A).$$

Here $m(A)$ is the elementary measure of A .

- The *Jordan outer measure* $\overline{m}_J(E)$ of E is defined as

$$\overline{m}_J(E) = \inf_{A \supset E, A \text{ elementary}} m(A).$$

- If $\underline{m}_J(E) = \overline{m}_J(E)$, we say that E is Jordan measurable, and call the common value $m(E)$ (the Jordan measure of E).

By convention, we do not consider unbounded sets to be Jordan measurable.

Q1) Assume that $E \subset \mathbb{R}^d$ is bounded. Show that the following are equivalent:

- E is Jordan measurable.

- b) For every $\epsilon > 0$, there exists elementary sets A and B such that $A \subset E \subset B$ and $m(B \setminus A) \leq \epsilon$.
- c) For every $\epsilon > 0$, there exists an elementary set A such that $\overline{m}_J(E \Delta A) \leq \epsilon$.

Q2) Deduce that every elementary set E is Jordan measurable and that its Jordan measure is the same as its elementary measure. In particular, $m(\emptyset) = 0$.

• **Properties of Jordan measure** Let E, F be Jordan measurable sets.

Q3) Clearly $m(E) \geq 0$. Show that

- (a) Show that $E \cup F, E \cap F, E \setminus F$, and $E \Delta F$ are all Jordan measurable.
- (b) (Finite additivity) If E and F are disjoint, then $m(E \cup F) = m(E) + m(F)$.
- (c) (Monotonicity) If $E \subset F$, then $m(E) \leq m(F)$.
- (d) (Finite subadditivity) $m(E \cup F) \leq m(E) + m(F)$.
- (e) (Translation invariance) for any $x \in \mathbb{R}^d$, $m(E + x) = m(E)$.

• **Some Jordan measurable sets.** Let B be a closed box of \mathbb{R}^d and $f : B \rightarrow \mathbb{R}$ a continuous function.

Q5) Show that the graph $\{(x, f(x)) : x \in B\} \subset \mathbb{R}^{d+1}$ is Jordan measurable in \mathbb{R}^{d+1} and that it has Jordan measure 0. *Hint: Use that f is uniformly continuous.*

Q6) Show that the set $\{(x, t) : x \in B, 0 \leq t \leq f(x)\} \subset \mathbb{R}^{d+1}$ is Jordan measurable.

From this we conclude that some familiar sets like triangles in \mathbb{R}^2 and balls in \mathbb{R}^d are Jordan measurable.