Handout 10

Where we are right now?

- **Lebesgue outer measure:** We modify the notion of Jordan outer measure by replacing the finite union of boxes by a countable union of boxes, i.e.

$$m^*(E) = \inf_{E \subset \cup_{j=1}^{\infty} B_j} \sum_{j=1}^{\infty} |B_j|$$

  where the union above is taken over boxes $B_j \subset \mathbb{R}^d$. We saw last time that this is smaller than the Jordan outer measure and that the boxes above can be taken to be open or closed. We also saw that any countable set has zero Lebesgue outer measure.

- **Lebesgue measurability** A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing E such that $m^*(U \setminus E) \leq \epsilon$. If E is measurable, we refer to $m(E) = m^*(E)$ as the Lebesgue measure of E.

  We saw last time some properties of this definition:

  - Show that $m^*(\emptyset) = 0$.
  - (Monotonicity) Show that if $E \subset F \subset \mathbb{R}^d$, then $m^*(E) \leq m^*(F)$.
  - (Countable subadditivity) If $E_1, E_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of sets, then $m^*\left(\cup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

  A natural question is whether one has that an additivity property for the outer measure: namely that if E, F are <u>disjoint</u> sets then $m^*(E \cup F) = m^*(E) + m^*(F)$? While this turns out to be correct for some sets E and F (to be called Lebesgue-measurable sets), we already saw at the start of our discussion of measures that this cannot hold for general sets (cf. the Banach-Tarski paradox). The enemy here is that we might have the two sets E and F too intertwined or entangled together which can cause the additivity property to fail.

Q1) Show that if $\mathrm{dist}(E, F) > 0$, then $m^*(E \cup F) = m^*(E) + m^*(F)$.

Q2) Show that if E is an elementary set, then $m^*(E) = m(E)$ where $m(E)$ is the elementary measure of E defined before.

Q3) Conclude that if E is any bounded set, then $\underline{m}(E) \leq m^*(E) \leq \overline{m}(E)$ where $\underline{m}(E)$ and $\overline{m}(E)$ are the inner and outer Jordan measures of E.

Q4) Construct a bounded open subset U of $\mathbb{R}$ that is not Jordan measurable. *Hint: Start with an enumeration of the rationals in* $[0, 1]$ *and create an open set whose Lebesgue outer-measure is arbitrarily small but the Jordan outer measure is* ≥ 1.

MATH 395 Notes

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Exercise 1. *Show that if* $\operatorname{dist}(E, F) > 0$ *then* $m^*(E \cup F) = m^*(E) + m^*(F)$.

Proof. We already have that $m^*(E \cup F) \leq m^*(E) + m^*(F)$. We now use the property of greatest lower bound to prove that $m^*(E \cup F) \geq m^*(E) + m^*(F)$. To do so, we will first prove a lemma:

Lemma. *For any sets* E *and* F *with* $\operatorname{dist}(E, F) > 0$ *and any box* B *we have that there is a finite collection of disjoint sub-boxes* $B_1, \ldots, B_N$ *covering* B *such that each* B_i *intersects at most one of* E *and* F.

Proof. Let $\varepsilon := \operatorname{dist}(E, F) > 0$. Now since $\varepsilon > 0$ we know that we can split B into sub-boxes $B_1, \ldots, B_N$ each of diameter less than ε. Then consider that for any i and any two points $X, y \in B_i$ we have:

$$d(x, y) \leq \operatorname{diam}(B_i) < \varepsilon = \operatorname{dist}(E, F)$$

We then may say that we cannot have $x \in E$ and $y \in F$, since if we did then we would have:

$$d(x, y) \leq \operatorname{diam}(B_i) < \varepsilon = \operatorname{dist}(E, F) \leq d(x, y)$$

Which is a contradiction. Therefore B_i intersects at most one of E and F.

Fix some countable collection $B_1, B_2, \ldots$ which covers $E \cup F$. We wish to show that $m^*(E) + m^*(F)$ is a lower bound for these, that is:

$$m^*(E) + m^*(F) \leq \sum_{i=1}^{\infty} B_i$$

Now for each B_i we use the lemma to split it into disjoint sub-boxes $B_{i1}, \ldots, B_{iN_i}$ covering B such that each box B_{ij} intersects at most one of E and F. In particular we can split this up into disjoint collections of a countbale covering of E and a countable covering of F. Then by infimums:

$$\begin{aligned}
\sum_{i=1}^{\infty} B_i &= \sum_{i=1}^{\infty} \sum_{j=1}^{N_i} B_{ij} \\
&\geq \sum_{i=1}^{\infty} \sum_{\substack{j=1 \\ B_{ij} \cap E \neq \emptyset}}^{\infty} B_{ij} + \sum_{i=1}^{\infty} \sum_{\substack{j=1 \\ B_{ij} \cap F \neq \emptyset}}^{\infty} B_{ij} \\
&\geq m^*(E) + m^*(F)
\end{aligned}$$

Taking the infimum on the left hand side we see that:

$$m^*(E \cup F) \geq m^*(E) + m^*(F)$$

And therefore since we already have the other direction of the inequality by finite subadditivity we have $m^*(E \cup F) = m^*(E) + m^*(F)$ just as desired! Great!

Exercise 2. *Show that if E is an elementary set, then $m^*(E) = m(E)$ where $m(E)$ is the elementary measure of E defined before*

Proof. We want to only work with closed elementary sets. To do this we need a lemma:

Lemma. *For any elementary set E we have that $m^*(E)$ and $m^*(\overline{E})$.*

Proof. This is not too difficult. First note since $E \subseteq \overline{E}$ we have by monotonicity that $m^*(E) \leq m^*(\overline{E})$.

Now we wish to show that $m^*(E) \geq m^*(\overline{E})$. Note by finite sub-additivity we know:

$$m^*(\overline{E}) = m^*(E \cup \partial E) = m^*(E) + m^*(\partial E)$$

But wait! We know by previous IBL work that:

$$0 \leq m^*(\partial E) \leq \overline{m}_J(\partial E) = 0$$

Since we have previously shown that the Jordan measure of the boundary of a Jordan measurable set is zero, and E is elementary so it is Jordan measurable. But then

TODO

Now write $\overline{E}$, which must be an elementary set, as a finite union of disjoint boxes $E_1, \ldots, E_n$ by definition of an elementary set. Then note that the collection $E_1, \ldots, E_n$ covers $\overline{E}$, and so by definition of the Lebesgue outer measure as an infimum:

$$m^*(\overline{E}) \leq \sum_{j=1}^{n} |E_j| = m(\overline{E})$$

We now simply need to show the other inequality. To do so, it suffices to show that $m(\overline{E})$ is a lower bound for the set which defines $m^*(\overline{E})$ by the definition of infimum. By last homework, it suffices to consider countable coverings by open boxes.

Fix some countable collection of open boxes $B_1, B_2, \ldots$ which covers $\overline{E}$. Now consider that $\overline{E}$ is compact since elementary sets are bounded. Therefore there is a finite subcollection $B_1, \ldots, B_N$ which covers $\overline{E}$. By finite sub-additivity of the elementary measure:

$$m(\overline{E}) \leq \sum_{j=1}^{N} m(B_j) \leq \sum_{j=1}^{\infty} |B_j|$$

And therefore taking an infimum on the right hand side:

$$m(\overline{E}) \leq m^*(\overline{E})$$

But wait! Then by the lemma and previous work on elementary measure we have:

$$m^*(E) = m^*(\overline{E}) = m(\overline{E}) = m(E)$$

Great! This is exactly what we wanted to show!!! ☺

Exercise 3. ***TODO***

Proof. **TODO**

Exercise 4. ***TODO***

Proof. **TODO**