MATH 395 Notes

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Theorem. *Let E be a subset of a metric space X. Then the following are equivalent:*

1) E is compact

2) E is sequentially compact

3) E is complete and bounded.

We've already seen that in matric spaces compactness implies sequential compactness. It remains to show:

(a) Sequential compactness implies compactness

(b) Sequential compactness implies totally bounded and complete

(c) Totally bounded and complete implies sequentially compact

We will prove (b) and (c) first and then (a). In fact, the proof of the theorem follows from the following three lemmas

Lemma 1. *A sequentially compact subset E of X is totally bounded and complete*

Lemma 2. *A totally bounded and complete subset E of X is sequentially compact*

Lemma 3. *A sequentially compact subset of a metric space is compact*

Proof of Lemma 1, Totally Bounded. Note that if $E = \emptyset$ then we are done. Thus let $E \neq \emptyset$ for the duration of this proof.

Let E be sequentially compact. To show it is totally bounded, fix an $\varepsilon > 0$.

Claim. *Let $A \subseteq E$ be a set of points of mutual distance $\geq \varepsilon$. Then A has to be finite*

Proof of claim. Suppose that A were infinite. Then we get a sequence of points $(x_n) \in A$ such that $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m$. But this means that no subsequence of (x_n) is Cauchy, and therefore no subsequence of (x_n) is convergent, violating sequential compactness.

Now let $p_1 \in E$ be arbitrary. If possible we pick $p_2 \in E$ such that $d(p_2, p_1) \geq \varepsilon$. If this is not possible then we stop. Then we pick $p_3 \in E$ such that $d(p_1, p_3) \geq \varepsilon$ and $d(p_2, p_3) \geq \varepsilon$. If this is not possible we stop

Now having picked $p_1, \ldots, p_n$ in this way such that $d(p_i, p_j) \geq \varepsilon$ for all $1 \leq i \neq j \leq n$, we pick $p_{n+1} \in E$ such that $d(p_{n+1}, p_i) \geq \varepsilon$ for all $1 \leq j \leq n$. If this is not possible, then $E \subseteq \bigcup_{i=1}^{n} N_\varepsilon(p_j)$ and we are done.

The claim above tells us that we cannot continue this process forever, and thus it must end after n steps for some $n \in \mathbb{N}$. Therefore E is totally bounded

Proof of Lemma 1, Completeness. Let (x_n) be a Cauchy sequence in E. Since E is sequentially compact there is a convergent sequence (x_{n_k}) such that x_{n_k} converges to some $p \in E$ as k goes to infinity. Now let $\varepsilon > 0$, then there is some $N \in \mathbb{N}$ large enough so that for $k > N$ and $n > N$ we know that:

$$\begin{aligned} d(x_n, x_{n_k}) &< \frac{\varepsilon}{2} \\ d(x_{n_k}, p) &< \frac{\varepsilon}{2} \\ d(x_n, p) &\overset{\triangle}{\leq} d(x_n, x_{n_k}) + d(x_{n_k}, p) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus (x_n) converges to $p \in E$. Therefore E is complete!

Proof of Lemma 2. Assume that E is totally bounded and complete. Let (x_n) be a sequence in E. We want to show that it has a convergent subsequence. If the set of all $\{x_n\}$ is finite, then we can find a constant subsequence and we are done. Assume that $\{x_n\}$ is infinite.

Since E is totally bounded, one can cover E with finitely many $\frac{1}{2}$-neighborhoods. One of these neighborhoods must contain infinitely many (x_n) by the pigeonhole principle. Thus we may call this resulting subsequence $\left(x_n^{(1)}\right)$

Now cover E with finitely many $\frac{1}{2^2}$-neighborhoods. One of these neighborhoods contains infinitely many of the $\left(x_n^{(1)}\right)$ by the pigeonhole principle. This gives a subsequence $\left(x_n^{(2)}\right)$ of $\left(x_n^{(1)}\right)$ completely contained in a $\frac{1}{2^2}$-neighborhood. This is also a subsequence of (x_n) of course.

Inductively, we can define a successivesubsequence $\left(x_n^{(k)}\right)$ such that $\left(x_n^{(k)}\right)$ is a subsequence of $\left(x_n^{(k-1)}\right)$ and $\left(x_n^{(k)}\right)$ is contained in a ball of radius $\frac{1}{2^k}$.

Now set $a_n = x_n^{(n)}$. This is a subsequence of (x_n) that satisfies:

$$d(a_n, a_m) = d\left(x_n^{(n)}, x_m^{(m)}\right)$$

If $m \geq n$ then $\left(x_p^{(m)}\right)$ is a subeequence of $\left(x_p^{(n)}\right)$ and $\left(x_p^{(n)}\right)$ is contained in a ball of radius $\frac{1}{2^n}$ with some center, say c for concreteness. Thus:

$$\begin{aligned} d\left(x_n^{(n)}, x_m^{(m)}\right) &\overset{\triangle}{\leq} d\left(x_n^{(n)}, c\right) + d\left(x_m^{(m)}, c\right) \\ &< \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}} \end{aligned}$$

Of course we can swap the role of n and m and so we always have:

$$d(a_n, a_m) \leq \frac{1}{2^{\min(n,m)-1}}$$

With this established it is clear that (a_n) is cauchy. By completeness of E, we know (a_n) converges to a point $p \in E$ as desired. Therefore (x_n) has a convergent subsequence

Lemma 4 (3')**.** *Let $E \subseteq X$ be sequentially compact. Let $\{G_\alpha\}_{\alpha \in A}$ be an open cover of E. Then there exists an $\varepsilon > 0$ such that every ball of radius ε and center $p \in E$ is contained in one of G_α for some $\alpha \in A$.*

Proof. Suppose the statement is not true. Then for any integer $n \geq 1$ there exists a $p_n \in E$ such that $N_{\frac{1}{n}}(p_n)$ is not contained in any of the $\{G_\alpha\}_{\alpha \in A}$. By sequential compactness, (p_n) has a convergent subsequence (p_{n_k}) converging to some $p \in E$.

Since $p \in E$ there exists a α_0 such that $p \in G_{\alpha_0}$, and so there is some $\delta > 0$ so that $N_\delta(p) \subseteq G_{\alpha_0}$.

Since $p_{n_k} \to p$, we may pick n_k large enough so that:

$$d\left(p_{n_k}, p\right) < \frac{\delta}{2} \qquad\qquad \frac{1}{n_k} < \frac{\delta}{2}$$

But then fixing $x \in N_{\frac{1}{n_k}}(p_{n_k})$ we have:

$$d(x,p) \overset{\triangle}{\leq} d\left(x, p_{n_k}\right) + d\left(p_{n_k}, p\right) \qquad\qquad < \frac{\delta}{2} + \frac{\delta}{2} < \delta$$

And so $x \in N_\delta(p) \subseteq G_{\alpha_0}$. This shows that $N_{\frac{1}{n_k}}(p_{n_k}) \subseteq G_{\alpha_0}$. Oops! ☺

Proof of Lemma 3. Suppose that E is sequentially compact. Now let $\{G_\alpha\}$ be any open cover of E. By Lemma 4 (3'), there exists an $\varepsilon > 0$ such that any ε-neighborhood of a point in E is contained in one of the G_α. Since sequentially compact implies totally bounded, E can be covered by finitely many ε-neighborhoods.

That is there is a list $p_1, \ldots, p_N \in E$ such that:

$$E \subseteq \bigcup_{j=1}^{N} N_\varepsilon(p_j)$$

Now for each p_j with $1 \leq j \leq N$ there exists some α_j such that $N_\varepsilon(p_j) \subseteq G_{\alpha_j}$ by construction of ε by lemma 3'. Therefore:

$$E \subseteq \bigcup_{j=1}^{N} G_{\alpha_j}$$

Thus, E is comapct as desired.