MATH 395 Notes

Faye Jackson

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Continue proving the equality of mixed partials

Theorem. *If $f \in C^2(A)$ where $A \subseteq \mathbb{R}^d$ then for each $x_0 \in A$ we have:*

$$\frac{\partial^2 f}{\partial x_k \partial x_j}(x_0) = \frac{\partial^2 f}{\partial x_j \partial x_k}(x_0)$$

Corrolary. *Equality of mixed partials of order r when $f \in C^r(A)$.*

Proof. We began by reducing to the case where $d = 2$, since in general all variables different from k, j are frozen when taking these partial derivatives. Thus assume $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is C^2. Instead of referring to x_1, x_2 we'll refer to x, y.

Now lets consider our intuition. We know that $\frac{\partial f}{\partial x}$ measures $\frac{\Delta_x f}{h} = \frac{f(x_0+h,y)-f(x_0,y)}{h}$. And then:

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} \approx \frac{\Delta_y \Delta_x f}{hk} &= \frac{\Delta_x f(x, y+h) - \Delta_x f(x,y)}{hk} \\ &= \frac{1}{hk}\left[f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x,y)\right]\end{aligned}$$

Similarly:

$$\begin{aligned}\frac{\partial f}{\partial y} \approx \frac{\Delta_y f}{k} &= \frac{f(x, y+k) - f(x,y)}{k} \\ \frac{\partial^2 f}{\partial x \partial y} \approx \frac{\Delta_x \Delta_y f}{hk} &= \frac{\Delta_y f(x+k, y) - \Delta_y f(x,y)}{hk} \\ &= \frac{1}{hk}\left[f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y)\right]\end{aligned}$$

Notice that $\Delta_y \Delta_x f = \Delta_x \Delta_y f$. Thus the equality of this discrete version of the partials that we expect the partials to be the same.

Now for the real proof. Let $(x_0, y_0) \in A$ and Q be the rectangle with vertices (x_0, y_0), (x_0+h, y_0), (x_0, y_0+k), $(x_0+h, y+k)$. Since A is open, we can take h and k to be small enough so that $Q \subseteq A$. Now let:

$$G(h,k) = f(x_0+h, y_0+k) - f(x_0+k, y_0) - f(x_0, y_0+k) + f(x_0, y_0)$$

We will show that:

$$G(h,k) = hk\frac{\partial^2 f}{\partial x \partial y}(p) = hk\frac{\partial^2 f}{\partial y \partial x}(q)$$

For some $p, q \in Q$. To show the first equality. Let us use $G(h,k) = \Delta_y \Delta_x f$ and let $\phi(y) = f(x_0+h, y) - f(x_0, y)$ for y between y_0 and y_0+k. We know that ϕ is continuous on $[y_0, y_0+k]$ since f itself is. Also ϕ is differentiable on (y_0, y_0+k) since f is C^1. Therefore by the Mean Value Theorem there exists a $y_\star$ between y_0 and y_0+k so that:

$$\phi(y_0+k) - \phi(y_0) = \phi'(y_\star)k$$

Notice then that:

$$\begin{aligned}
G(h,k) &= \phi(y_0+k) - \phi(y_0) \\
\phi'(y) &= \frac{\partial f}{\partial y}(x_0+h, y) - \frac{\partial f}{\partial y}(x_0, y) \\
G(h,k) &= k\left[\frac{\partial f}{\partial y}(x_0+h, y_\star) - \frac{\partial f}{\partial y}(x_0, y_\star)\right]
\end{aligned}$$

Now we know that $\frac{\partial f}{\partial y}(x, y_\star)$ is continuous on the closed interval between x_0 and x_0+h and differentiable on the open interval. By the MVT there is a $x_\star$ between x_0 and x_0+h so that:

$$G(h,k) = kh\frac{\partial^2 f}{\partial x \partial y}(x_\star, y_\star) = k\left[\frac{\partial f}{\partial y}(x_0+h, y_\star) - \frac{\partial f}{\partial y}(x_0, y_\star)\right]$$

Note that $(x_\star, y_\star) \in Q$ so we have the first equality. To show the other equality, we argue similarly using the fact that $G(h,k) = \Delta_x \Delta_y f$. More precisely instead of ϕ above we introduce:

$$\psi(x) = f(x, y_0+k) - f(x, y_0)$$

By MVT we can get a $x_\heartsuit$ such that:

$$G(h,k) = \psi(x_0+h) - \psi(x_0) = h\psi'(x_\heartsuit)$$
$$G(h,k) = h\left[\frac{\partial f}{\partial x}(x_\heartsuit, y_0+k) - \frac{\partial f}{\partial x}(x_\heartsuit, y_0)\right]$$

By applying the mean value theorem again we get $y_\heartsuit$ between y_0 and y_0+k we get:

$$G(h,k) = hk\frac{\partial^2 f}{\partial y \partial x}(x_\heartsuit, y_\heartsuit)$$

This is exactly the same moves as in the proof for x. Then:

$$\frac{G(h,k)}{hk} = \frac{\partial^2 f}{\partial x \partial y}(x_\star, y_\star) = \frac{\partial^2}{\partial y \partial x}(x_\heartsuit, y_\heartsuit)$$

By letting $h,k \to 0$ both $(x_\star, y_\star) = p \to (x_0, y_0)$ and $(x_\heartsuit, y_\heartsuit) = q \to (x_0, y_0)$. By continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at (x_0, y_0) we obtain the desired equality that:

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

The Chain Rule and Taylor's Formula in Higher Dimensions

Recall. For $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ such that $f(A) \subseteq B$ we have $g \circ f : A \subseteq \mathbb{R} \to \mathbb{R}$. We have:

$$\frac{\mathrm{d}}{\mathrm{d}x}[g \circ f](x) = g'(f(x)) \cdot f'(x)$$

provided that $f'(x)$ and $g'(f(x))$.

Theorem (Chain Rule)**.** *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ and suppose that $f : A \subseteq \mathbb{R}^m$ and $g : B \subseteq \mathbb{R}^k$. with $f(A) \subseteq B$. Suppose that x_0 is an interior point of A and $y_0 = f(x_0)$ is an interior point of B. Furthermore suppose that f is differentiable at x_0 and g is differentiable at y_0. Then $g \circ f$ is differentiable at x_0 and:*

$$D[g \circ f](x_0) = Dg(y_0) \circ Df(x_0) = Dg(f(x_0)) \cdot Df(x_0)$$

Proof. Since y_0 is an interior point of B there exists a $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq B$. Since f is continuous at x_0 there exists a $\delta > 0$ so that $f(B(x_0, \delta)) \subseteq B(y_0, \varepsilon)$. So we can define $g \circ f : B(x_0, \delta \to \mathbb{R}^k$. Let $\|h\| < \delta$ for $h \in \mathbb{R}^n$ and define:

$$R_f(h) = \frac{f(x_0 + h) - f(x_0) - Df(x_0) \cdot h}{\|h\|} \qquad (h \neq 0)$$

$$R_f(h) = 0 \qquad (h = 0)$$

By differentiability of f at x_0 we have $R_f(h) \to 0$ as $\|h\| \to 0$. Similarly if $\|k\| < \varepsilon$ and $k \in \mathbb{R}^m$ we define:

$$R_g(k) = \frac{g(y_0 + k) - g(y_0) - Dg(y_0) \cdot k}{\|k\|} \qquad (k \neq 0)$$

$$R_g(k) = 0 \qquad (k = 0)$$

By differentiability we know that $R_g(k) \to 0$ as $\|k\| \to 0$. To show that $g \circ f$ is differentiable at x_0 we must show that there exists an $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k)$ such that:

$$R_{g \circ f}(h) = \frac{[g \circ f](x_0 + h) - [g \circ f](x_0) - Ah}{\|h\|} \to 0 \text{ as } \|h\| \to 0$$

Rewrite as the following:

$$\begin{aligned}
[g \circ f](x_0 + h) - [g \circ f](x_0) &= g(f(x_0 + h)) - g(f(x_0)) \\
&= g(f(x_0) + f(x_0 + h) - f(x_0)) - g(f(x_0)) \\
&= g(y_0 + k) - g(y_0)
\end{aligned}$$

Where we call $k = f(x_0 + h) - f(x_0)$. From $R_g(k)$ we know that for any $k \in \mathbb{R}^m$:

$$g(y_0 + k) - g(y_0) = Dg(y_0) \cdot k + \|k\| R_g(k)$$

Furthermore $k = f(x_0 + h) - f(x_0) = Df(x_0) + \|h\| R_f(h)$. Therefore:

$$\begin{aligned}
g(y_0 + k) - g(y_0) &= Dg(y_0)[Df(x_0)h + \|h\| R_f(h)] + \|k\| R_g(k) \\
&= Dg(y_0)Df(x_0) \cdot h + \|h\| D_g(y_0) R_f(h) + \|k\| R_g(k)
\end{aligned}$$

Set $A = Dg(y_0) \cdot Df(x_0)$ This gives that for $h \neq 0$ that:

$$\begin{aligned}
R_{g \circ f}(h) &= \frac{[g \circ f](x_0 + h) - [g \circ f](x_0) - Ah}{\|h\|} \\
&= Dg(y_0) R_f(h) + \frac{\|k\|}{\|h\|} R_g(k)
\end{aligned}$$

We know that $R_f(h) \to 0$ as $\|h\| \to 0$. It remains to show that $\frac{\|k\|}{\|h\|} R_g(k) \to 0$ as $\|h\| \to 0$. We know that:

$$\begin{aligned}
\|k\| &= \|Df(x_0) \cdot h + \|h\| R_f(h)\| \\
&\leq \|Df(x_0) \cdot h\| + \|h\| \|R_f(h)\| \\
&\leq C\|h\| + \|h\| \leq (C + 1)\|h\|
\end{aligned}$$

This follows since $\|R_f(h)\| \leq 1$ if $\|h\|$ is small enough. Also we kow since $Df(x_0)$ is linear we know $\|Df(x_0) \cdot h\| \leq C\|h\|$ for some constant C by 296 / linear algebra. Therefore:

$$\left\| \frac{\|k\|}{\|h\|} R_g(k) \right\| \leq (C + 1) \frac{\|h\|}{\|h\|} \|R_g(k)\| \leq (C + 1)\|R_g(k)\|$$

Therefore as $\|h\| \to 0$ we know that $\|k\| \to 0$ since $\|k\| \leq (C + 1)\|h\|$ and hence $R_g(k) \to 0$. Therefore $\frac{\|k\|}{\|h\|} R_g(k) \to 0$ as $h \to 0$ and so this finishes the proof.

Taylor's Theorem in several variables

Recall the multi-index notation from last time.

Lemma (The multinomial lemma)**.** *For any $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and any positive integer k we have:*

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha$$