MATH 395 Notes

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October 7, 2020

Theorem (Taylor's Theorem)**.** *Let G be open and convex. Let $f : G \to \mathbb{C}$ be C^{k+1} and $a \in G$. Then:*

$$f(x) = R_{a,k}(x) + \sum_{|\alpha| \le k} \frac{\partial^\alpha f(a)}{\alpha!}(x-a)^\alpha$$
$$R_{a,k}(x) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f}{\alpha!}(c)(x-a)^\alpha$$

Where c is on the line segment connecting a and x

Continued Proof of Taylor's Theorem. We'll fix some $x_0 \in G$. Then set $\phi(t) = f(a + t(x_0 - a))$ where $\phi : [0,1] \to \mathbb{C}$. By taylor's theorem in one dimension:

$$f(x_0) = \phi(1) = R_{0,k}(1) \sum_{p=0}^{k} \frac{\phi^{(p)}(0)}{p!} 1^p$$
$$R_{0,k}(1) = \frac{\phi^{(k+1)}(\theta)}{(k+1)!} 1^{k+1}$$

For some $0 \le \theta \le 1$. We need a formula for $\phi^{(p)}(t)$. Let $u = (x_0 - a)$. Then by the chain rule:

$$\phi'(t) = Df(a+tu) \cdot u = D_u f(a+tu) = \left(\sum_{k=1}^{n} u_k \frac{\partial f}{\partial x_k} \right)(a+tu)$$

So then if we call $f_1 = D_u f$ then we have that:

$$\begin{aligned}\phi''(t) &= D_u f_1(a+tu) = [D_u(D_u f)](a+tu)\\ &= D_u^2 f(a+tu) = \left(\sum_{k=1}^{n} u_k \frac{\partial}{\partial x_k}\right)^2 f(a+tu)\end{aligned}$$

So then by induction we can obtain that:

$$\phi^{(j)}(x) = D_u^j f(a+tu) = \left(\sum_{k=1}^{n} u_k \frac{\partial}{\partial x_j}\right)^j f(a+tu)$$

Where this holds for $0 \leq j \leq k+1$, since f is differentiable $k+1$ times. And so for $0 \leq j \leq k$ we have:

$$\begin{aligned}\phi^{(j)}(0) &= D_u^j f(a) = \left(\sum_{k=1}^{n} u_k \frac{\partial}{\partial x_k}\right)^j f(a)\\ \phi^{(k+1)}(\theta) &= D_u^{k+1} f(a) = \left(\sum_{k=1}^{n} u_k \frac{\partial}{\partial x_k}\right)^{k+1} f(a+\theta u)\end{aligned}$$

Consider that as operators we can show—using linearity—similarly to how we showed the multinomial lemma, we have:

$$\left(\sum_{k=1}^{n} u_k \frac{\partial}{\partial x_k}\right)^p = \sum_{|\alpha|=p} \frac{p!}{\alpha!} u^\alpha \partial^\alpha$$

This gives us that:

$$\begin{aligned}\phi^{(p)}(0) &= \sum_{|\alpha|=p} \frac{p!}{\alpha!} u^\alpha \partial^\alpha f(a)\\ \phi^{(k+1)}(\theta) &= \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} u^\alpha \partial^\alpha f(a+\theta u)\end{aligned}$$

Set $c = a + \theta u$ which is on the line segment between a and x_0, so then we must have that:

$$
\begin{aligned}
f(x_0) = \phi(1) &= \frac{\phi^{(k+1)}}{(k+1)!} + \sum_{p=0}^{k} \frac{\phi^{(p)}(a)}{p!} \\
&= \frac{1}{(k+1)!} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} u^\alpha \partial^\alpha(c) \\
&\quad + \sum_{p=0}^{k} \frac{1}{p!} \left(p! \sum_{|\alpha|=p} \frac{1}{\alpha!} u^\alpha \partial^\alpha f(a) \right) \\
&= \sum_{|\alpha|\leq k} \frac{\partial^\alpha f(a)}{\alpha!} u^\alpha + \sum_{|\alpha|=k+1} \frac{\partial^\alpha(c)}{\alpha!} u^\alpha
\end{aligned}
$$

This is exactly what we want to show!

Example. Let $f(x, y) = \sin(x^2 + y)$ where $f : \mathbb{R}^2 \to \mathbb{R}$. Find the 3rd degree polynomial that best approximates f near $(0, 0)$.

This is simply:

$$
P(x, y) = \sum_{|\alpha|\leq 3} \frac{\partial^\alpha f(0)}{\alpha!} (x, y)^\alpha
$$

Let's go!

- For $|\alpha| = 0$ we have $\alpha = (0, 0)$ and so $\partial^\alpha f(0) = f(0) = 0$, and $\alpha! = 1$.
- For $|\alpha| = 1$ then $\alpha = (1, 0)$ or $\alpha = (0, 1)$. Call these α_x and α_y respectively, in either case $\alpha_x! = \alpha_y! = 1$ and then:

$$
\begin{aligned}
\partial^{\alpha_x} f(0) &= \frac{\partial f}{\partial x}(0) = 2x \cdot \cos(x^2 + y)\Big|_0 = 0 \\
\partial^{\alpha_y} f(0) &= \frac{\partial f}{\partial y}(0) = \cos(x^2 + yy^2)\Big|_0 = 1
\end{aligned}
$$

- For $|\alpha = 2|$ we have $\alpha_{xx} = (2, 0)$ where $\alpha_{x,x}! = 2$ and $\alpha_{xy} = (1, 1)$ and $\alpha_{xy}! = 1$.

And then $\alpha_{yy} = (0, 2)$ where $\alpha_{yy}! = 2$. Now:

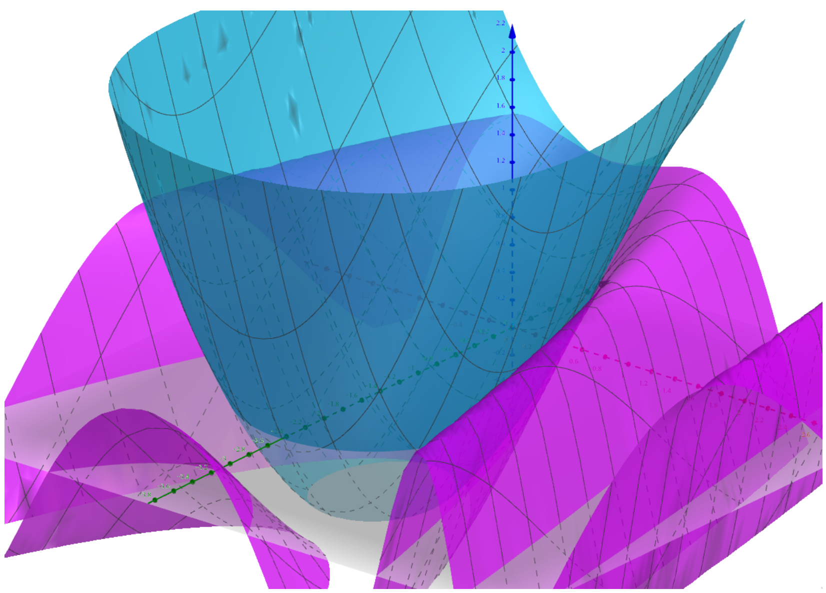
$$\begin{aligned}
\partial^{\alpha_{xx}} f(0) &= \frac{\partial^2 f}{\partial x^2}(0) = 2\cos(x^2+y) - 4x^2\sin(x^2+y)\Big|_0 = 2 \\
\partial^{\alpha_{xy}} &= \frac{\partial^2 f}{\partial x \partial y}(0) = -2x\sin(x^2+y)\Big|_0 = 0 \\
\partial^{\alpha_{yy}} f(0) &= \frac{\partial^2 f}{\partial y^2}(0) = -\sin(x^2+y)\Big|_0 = 0
\end{aligned}$$

- We omit the case where $|\alpha| = 3$ because we cannot deal. WTF

So then:

$$\begin{aligned}
P(x,y) &= \frac{\partial^{(0,1)} f(0)}{(0,1)!}(x,y)^{(0,1)} + \frac{\partial^{(2,0)} f(0)}{(2,0)!}(x,y)^{(2,0)} + \frac{\partial^{(0,3)} f(0)}{(0,3)!}(x,y)^{(0,3)} \\
&= y + \frac{2}{2}x^2 - \frac{1}{6}y^3 = x^2 + y - \frac{1}{6}y^3
\end{aligned}$$

In the following picture. The blue is our polynomial and the purple is f:



Cool!

Inverse Function Theorem

The inverse function theorem gives a necessary and sufficient condition for a function $f : \mathbb{R}^n \to \mathbb{R}^n$ to be locally invertible with a C^1 inverse.

Definition. *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where A is open, and let $x_0 \in A$. We say that f is locally invertible around x_0 provided that there is some open neighborhood U of x_0 so that $f\big|_U : U \to f(U)$ is one-to-one, and $f(U)$ is open in $\mathbb{R}^m$ This defines an inverse function $g : f(U) \to U$.*

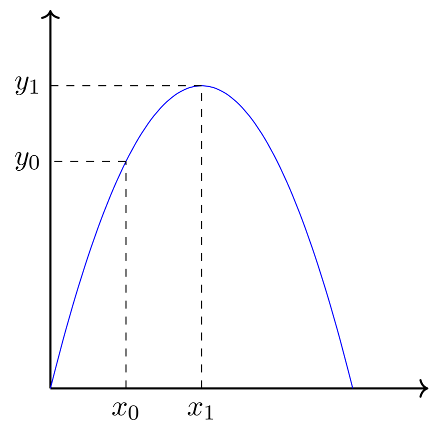
- *We say that f is a local homeomorphism around x_0 provided that both f and g are continuous.*
- *We say that f is a local diffeomorphism around x_0 provided that both f and g are differentiable.*
- *We say that f is a local C^r-diffeomorphism for $r \geq 1$ provided that both f and g are C^r-functions.*
- *We say that f is a locally invertible (resp. homeomorphism, diffeomorphism, C^r-diffeomorphism) provided that it is locally invertible (resp.) around every $x_0 \in A$.*

Remark. Soon we will give an example that is a local diffeomorphism on an open set A but is not a diffeomorphism of A.

Our goal is to find a condition for a function to be a local diffeomorphism. This is easy in one dimension.

The Key Idea

Key Figure



Being a local diffeomorphism neat x is equivalent to being able to express x as a function of y. This means that the graph of $y = f(x)$ can also be regarded as a function $x = g(y)$. This can be done when $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0) \neq 0$. If $\frac{\mathrm{d}f}{\mathrm{d}x}(x_1) = 0$, we might get multiple intersections of lines parallel to the x-axis neat $y = f(x_1)$, which means that the graph cannot define a function $x = g(y)$. The inverse function theorem will generalize this intuition to higher dimensions.

Necessity that $Df(x_0)$ is invertible

Proposition. *Suppose that $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ where A is open. Let $x_0 \in A$ and suppose f is differentiable in A. Assume that f is a local diffeomorphism around x_0 and suppose $g : \mathscr{O} \subseteq \mathbb{R}^n \to B(x_0, \delta)$ where $\mathscr{O}$ is open containing $y = f(x_0)$ is the inverse function. Then $Df(x_0)$ is invertible and:*

$$Dg(y_0) = [Df(x_0)]^{-1}$$

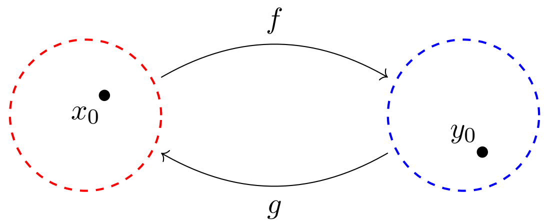
Proof. Consider that:

$$g \circ f : B(x_0, \delta) \to B(x_0, \delta)$$

And $(g \circ f)(x) = x$. Deriving both sides and using the chain rule:

$$Dg(f(x_0))Df(x_0) = Dg(y_0)Df(x_0) = \mathrm{Id}$$

And so $Df(x_0)$ is invertible and $Dg(y_0) = [Df(x_0)]^{-1}$.



Remark. The above proposition shows us that we cannot have a local diffeomorphism as defined from $A \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

One can ask if this is also the case for local homeomorphism. The answer is yes. However, the proof is more involved and uses tools from algebraic topology (Brouwer's invariance of domain theorem)