

**Riemannian Geometry:
Taught by André Neves**

May 15, 2024

Faye Jackson

CONTENTS

I. The Basic Objects	2
I.1. Standing Notation	2
I.2. Logistics / Motivation	2
I.3. Riemannian Manifolds, Definitions and Examples	3
I.4. Connections	6
I.5. Geodesics	11
II. Curvature	24
II.1. The Riemannian Curvature Tensor	24
III. Parallel Transport	36
III.1. Definitions	36
III.2. Jacobi Fields	38
IV. Hopf-Rinow Theorem	42
V. Variations of Energy	50
VI. Beyond do Carmo	61
VI.1. The Geometric Aesthetic Principle, Organizing the Course	61
VI.2. Bochner Technique	62
VII. Exam Review	71
VII.1. 2019 Final	71
VII.2. The Take Home Final	71
References	74
Todo list	74

I. The Basic Objects

I.1. Standing Notation

We'll adopt some standard Notation, which we'll record here

- (M^n, g) denotes a Riemannian n -manifold with metric g .
- $D_p f$ for $f : M \rightarrow N$ denotes the derivative $D_p f : T_p M \rightarrow T_{f(p)} N$.
- $\frac{\partial}{\partial x_i}$ on M denotes the push-forward of $\frac{\partial}{\partial x_i}$ under some chart $\varphi : U \rightarrow M$ on M , where $U \subseteq \mathbb{R}^n$.
- $\mathfrak{X}(M)$ denotes the collection of all smooth vector fields on M .
- For X a smooth vector field on M , f a function on M , $X(f), df(X)$ both denote the directional derivative of f in the direction of X . Note this can be evaluated at points.

I.2. Logistics / Motivation

Quote from André Neves:

I think Riemannian Geometry is the most beautiful subject in mathematics.

The hard part is the notation. There are two ways to do Riemannian geometry

- The intrinsic approach from do Carmo (see [**docarmo**]), where no charts, but computations are very hard.
- Do everything in charts and make computations explicit.

The first will be what happens in lecture, and the second will happen in homework. First some logistics

- Midterm: April 11th, in class
- Final: Last class, in class.

The basic object in Riemannian metric is the pair (M^n, g) a Riemannian n -dimensional manifold. There are many geometric invariants of M^n . Such as

- Volume
- Diameter
- The Curvature tensor $R(g)$.

The first two invariants tell you nothing topologically, since you can just scale the metric. The last invariant is the most interesting one. So here's the central question of Riemannian geometry

Given $R(g)$ (or points of it) what can one say about (M^n, g) .

There are important physical motivations for this from general relativity. Say (M^4, g) is space-time, with a Lorentzian metric. Let T be the stress energy tensor, which encodes the mass of the universe

and its location. Then the Einstein Equation is

$$\text{Ric}(g) - \frac{S(g)}{2}g = T,$$

where $\text{Ric}(g)$ is the Ricci curvature, and $S(g)$ is the Scalar curvature. Then we want to solve for the metric g ! This will tell us the geodesics, which in physics correspond to how particles/planets/black holes/everything moves in spacetime.

I.3. Riemannian Manifolds, Definitions and Examples

Definition I.3.1

A Riemannian manifold is a pair (M^n, g) with data

- (i) M^n is an n -dimensional smooth manifold (Hausdorff, second countable)
- (ii) For all $p \in M$, $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ so that g_p is bilinear, symmetric, and $g_p(X, X) \geq 0$, with equality if and only if $X = 0$.

with the extra property that g is smooth. Explicitly, this is the condition that, if $\varphi : U \rightarrow M$, $U \subseteq \mathbb{R}^n$, is a chart then

$$x \mapsto g_{\varphi(x)} \left(D_x \varphi \frac{\partial}{\partial x_i}, D_x \varphi \frac{\partial}{\partial x_j} \right),$$

is a smooth function of x . We call this function $g_{ij}(x)$ when the chart is clear (or if we write this we're implying there's a chart). Note that the matrix $(g_{ij}(x))_{i,j=1}^n$ determines $g_{\varphi(x)}$ because $T_p M$ is generated by $\left(\frac{\partial}{\partial x_i} \right)_{i=1}^n$

We take some additional notation: We identify $D_x \varphi \frac{\partial}{\partial x_i}$ with a vector field $\frac{\partial}{\partial x_i}$ on $\varphi(U) \subseteq M$.

Theorem I.3.1 (Metrics are Cheap)

Every smooth manifold M admits a Riemannian metric, and the induced topology is the same.

Proof Sketch. Do things in charts, and then paste them together with partitions of unity. 

Example I.3.1

Lets do some basic examples.

- (1) Euclidean Space: Let $M^n = \mathbb{R}^n$ and take

$$(g_0)_p : T_p \mathbb{R}^n \times T_p \mathbb{R}^n \rightarrow \mathbb{R}$$

$$(\vec{x}, \vec{y}) \mapsto \vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i$$

where $\vec{x} = \sum_i x_i \frac{\partial}{\partial x_i}$ and similarly for $\vec{y}$.

- (2) The Unit Sphere: $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$. Consider that

Exercise I.3.2

$$T_p S^n \simeq \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x} \cdot p = 0\}.$$

This seems a bit idiotic, since it's obvious from the picture. But as you'll recall from differential topology, the definition of a tangent space is quite weird, e.g.

$$T_p M = \{\alpha'(0) : \mathcal{O}(M, p) \rightarrow \mathbb{R} \mid \alpha \text{ is a curve in } M \text{ with } \alpha(0) = p\},$$

and $\mathcal{O}(M, p)$ is the collection smooth functions locally defined at p . With that exercise out of the way, we can define the induced metric

$$\begin{aligned} g_{S^n, p} : T_p S^n \times T_p S^n &\rightarrow \mathbb{R} \\ (\vec{x}, \vec{y}) &\mapsto \vec{x} \cdot \vec{y}. \end{aligned}$$

(3) Hyperbolic Space: Take $M^n = B^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$ and consider

$$\begin{aligned} \text{hyp}_p : T_p B^n \times T_p B^n &\rightarrow \mathbb{R} \\ (\vec{x}, \vec{y}) &\mapsto \frac{4}{(1 - |p|^2)^2} \vec{x} \cdot \vec{y}. \end{aligned}$$

We use the following notation, to make things nice

$$\text{hyp} = \frac{4}{(1 - |p|^2)^2} g_0,$$

where g_0 is the standard Euclidean metric.

Now we want to find out what things we should consider the same. Of course in Riemannian geometry, two spheres which have different “shapes” should be considered different, even if they are homeomorphic. To do this, we'll consider the notion of isometry, which we need some definitions to define properly.

Definition I.3.2 (Pullback Metric)

Let $\varphi : N \rightarrow M$ be an immersion, i.e., $D_p \varphi : T_p N \rightarrow T_{\varphi(p)} M$ is injective. Then a metric g on M induces a metric on N defined as

$$\begin{aligned} (\varphi^* g)_p : T_p N \times T_p N &\rightarrow \mathbb{R} \\ (\vec{x}, \vec{y}) &\mapsto g_{\varphi(p)}(D_p \varphi \cdot \vec{x}, D_p \varphi \cdot \vec{y}), \end{aligned}$$

called the pullback metric on N . We take immersions so that $\varphi^* g$ is everywhere non-degenerate.

Remark I.3.1

$\varphi : \mathbb{C} \rightarrow \mathbb{C}$ where $\varphi(z) = z^2$ is not allowed to pullback a metric. If we did so then the “metric” we pull back would be identically zero at $0 \in \mathbb{C}$. This expresses a kind of singularity at 0 for the metric.

Definition I.3.3 (Isometry)

An isometry $\varphi : (M, g) \rightarrow (N, h)$ is a diffeomorphism $\varphi : M \rightarrow N$ so that $\varphi^*h = g$. We call $\text{Isom}(M, g)$ the group of isometries of (M, g) , this is a group via the chain rule.

Note: in particular, for $\varphi : M \rightarrow M$ a diffeomorphism, (M, g) a Riemannian manifold, we have that $(M, \varphi^*g), (M, g)$ are isometric. This tells you that a lot of things might look different but are actually the same.

Névé takes a clear water bottle and begins to shake it: in charts (namely our eyes) this looks extremely different, but the pullback metrics are all the same, because shaking the water bottle is a diffeomorphism (in fact isotopically trivial)...note these might not be the same as the induced metric from being in Euclidean space...tricky to keep track of these things.

Example I.3.3

Lets give another example not to let your eyes deceive you. The extrinsic curvature of $S^1 \subseteq \mathbb{R}^2$ might look like its curved. But the intrinsic curvature is actually flat. Let g_{S^1} be the induced metric from $\mathbb{R}^2$ on S^1 , and g_0 the flat metric on $\mathbb{R}$, then

$$\begin{aligned} \exp : (\mathbb{R}, g_0) &\rightarrow (S^1, g_{S^1}) \\ \theta &\mapsto (\cos \theta, \sin \theta) \end{aligned}$$

is a local isometry. Lets check this. To do this we need to compute the pullback metric. It suffices to compute on $\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}$, so

$$\begin{aligned} g_0 \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right) &= 1 \\ (\exp^* g_{S^1}) \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right) &= g_{\mathbb{R}^2} \left(D_{\theta} \exp \cdot \frac{\partial}{\partial \theta}, D_{\theta} \exp \cdot \frac{\partial}{\partial \theta} \right). \end{aligned}$$

Now we compute that

$$D \exp = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$$

So now we have

$$\begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \cdot \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} = 1.$$

, so these are the same...wacky.

Now we should construct a few more examples. We can do this by looking at orbit spaces

Definition I.3.4

Suppose $G < \text{Isom}(M, g)$ acts properly discontinuously on M^n . Then consider M^n/G where

$x \sim y$ if there exists $\varphi \in G$ with $\varphi(x) = y$. Then we take

$$g_{[p]} : T_{[p]}(M^n/G) \times T_{[p]}(M^n/G) \rightarrow \mathbb{R}$$

$$(\vec{x}, \vec{y}) \mapsto g_p(\vec{x}, \vec{y}).$$

as the orbit metric on M^n/G .

Exercise I.3.4

Check M^n/G is a smooth manifold and $\pi : M^n \rightarrow M^n/G$ is a local diffeomorphism. Then check that $g_{[p]}$ as above is well-defined on M/G , and that it pulls back to g on M^n via π .

Remark I.3.2

A generic metric g satisfies, $\text{Isom}(M, g) = \{\text{Id}\}$.

Example I.3.5

Consider $\text{Isom}(\mathbb{R}^n) = \{\text{Rigid Motions}\}$ with $\mathbb{Z}^n < \text{Isom}(\mathbb{R}^n)$. Where $\varphi_v : x \mapsto x + v$ for $v \in \mathbb{Z}^n$. Then $\mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$ the n -torus.

Example I.3.6

Consider the spherical case, (p, q) coprime in $\mathbb{N}$. We can then define, taking $S^3 \subseteq \mathbb{C}^2$ as

$$\varphi : S^3 \rightarrow S^3$$

$$(z, w) \mapsto (e^{2\pi i/p} z, e^{2\pi i q/p} w).$$

One can compute that $\varphi \in \text{Isom}(S^3, g_{S^3})$. Then take the group $G = \{\text{Id}, \varphi, \dots, \varphi^{p-1}\} \cong \mathbb{Z}/p\mathbb{Z}$. Then $S^3/G = L_{p,q}$ is called the Lens Space for S^3 . This is an orientable Riemannian 3-manifold which is locally isometric to a sphere but which is not homeomorphic to the sphere.

Fact: $\pi_1(L_{p,q}) = G$. Relevant Question: If we have (p, q) and (p, q') , are $L_{p,q}, L_{p,q'}$ diffeomorphic? The answer is no. Neves says due to Reidemeister torsion.

Example I.3.7

Finally we have $\Gamma < \text{Isom}(\mathbb{H}^n, \text{hyp})$ acting properly discontinuously, where $\mathbb{H}^n$ is hyperbolic n -space. See Figure 1 for an example where Γ is hyperbolic reflections in $\mathbb{H}^2$ via the Poincaré disk model, and $\mathbb{H}^2/\Gamma$ is a genus two surface.

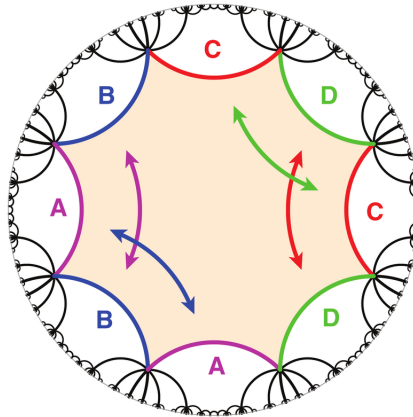
I.4. Connections

Notation: Let $\mathfrak{X}(M)$ denote the collection of all smooth vector fields on M .

Definition I.4.1

We define a connection to be a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ with the properties

- (i) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$, $\nabla_{X+Z}(Y) = \nabla_X Y + \nabla_Z Y$.

FIGURE 1. Fundamental Domain for a Genus Two Surface in $\mathbb{H}^2$

- (ii) For all $f \in C^\infty(M)$ we have $\nabla_{fX}Y = f\nabla_XY$.
- (iii) For all $f \in C^\infty(M)$, we have $\nabla_X(fY) = f\nabla_XY + X(f)Y$. Here $X(f)$ denotes the directional derivative of f in the direction of X .

Example I.4.1

For $(\mathbb{R}^n, g_0)$ we have $\mathfrak{X}(M) = \{X : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ as smooth maps. We can consider a connection}$

$$\nabla_XY := \sum_{i,j=1}^n a_i \frac{\partial}{\partial x_i}(b_j) \frac{\partial}{\partial x_j} = (X(b_1), \dots, X(b_n)).$$

where $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$.

Note: We'll sometimes write $\langle v, w \rangle$ for $g_p(v, w)$ where $v, w \in T_pM$. Or for vector fields X, Y we may write $\langle X, Y \rangle_p$ to mean $g_p(X_p, Y_p)$. Last time, we introduced connections, which can be thought of as maps

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

satisfying certain properties or more generally, for any vector bundle $E \rightarrow M$ as

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E),$$

where $\Gamma(E)$ is the space of sections of E . Lets talk about the example of a sphere to start out today.

Example I.4.2

Let S^n be the unit n -sphere with induced metric g_{S^n} . Now

$$\mathfrak{X}(S^n) = \{X : S^n \rightarrow \mathbb{R}^{n+1} \mid X(p) \cdot p = 0, \forall p \in S^n\}.$$

If we have some $X : S^n \rightarrow \mathbb{R}^{n+1}$, then we can consider the orthogonal projection $X^T : S^n \rightarrow \mathbb{R}^{n+1}$ given by $X^T(p) = X(p) - (X(p) \cdot p)p$. This is a vector field $X^T \in \mathfrak{X}(S^n)$.

As an example, take $X(p) = (0, 1, 0)$. Then $X^T = (0, 1, 0) - ((0, 1, 0) \cdot p)p$. This is zero at $(0, -1, 0)$ and $(0, 1, 0)$.

We can now define a connection as

$$\begin{aligned} \nabla^{S^n} : \mathfrak{X}(S^n) \times \mathfrak{X}(S^n) &\rightarrow \mathfrak{X}(S^n) \\ (X, T) &\mapsto (\nabla_{X^{ext}}^{\mathbb{R}^{n+1}} Y^{ext})^T, \end{aligned}$$

where X^{ext}, Y^{ext} are extensions of X, Y to all of $\mathbb{R}^{n+1}$.

Exercise I.4.3

Check this is well-defined and gives a connection on S^n .

Now, as with anything in Riemannian Geometry, we should give an interpretation of the connection in coordinates. Let $\varphi : U \rightarrow M$ be a chart, and define Γ_{ij}^k on this chart via

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

This allows us to differentiate any two vector fields. Setting $X = a^i \frac{\partial}{\partial x_i}$ and $Y = b^j \frac{\partial}{\partial x_j}$ in Einstein notation for, we have by the axioms on a connection that

$$\begin{aligned} \nabla_X Y &= \sum_{i=1}^n a^i \nabla_{\partial/\partial x_i} \left(b^j \frac{\partial}{\partial x_j} \right) = a^i b^j \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} + a^i \frac{\partial}{\partial x_i} (b^j) \frac{\partial}{\partial x_j} \\ &= a^i b^j \Gamma_{ij}^k \frac{\partial}{\partial x_k} + X(b^j) \frac{\partial}{\partial x_j}. \end{aligned}$$

The left expression in this sum is determined pointwise, while the right hand side depends on the extension of Y to a neighborhood of the point.

Why do we care about connections?

Because we want to study the *acceleration* of curves.

Let $\gamma : I \rightarrow M$ be a smooth curve and Y a vector field defined only on γ . To be precise, we're taking a section Y of the pullback bundle γ^*TM , or for every time t , a vector $Y(t) : T_{\gamma(t)}M$. We wish to define the quantity

$$\frac{DY}{dt}(t),$$

sometimes called $\nabla_{\gamma'(t)} Y(t)$ or $Y'(t)$ when the curve is understood. But we have a real problem... If we have a curve γ so that $\gamma(0) = \gamma(1)$, then $Y(0), Y(1)$ can have different values, and we might not be able to define an extension to the manifold...

Here's the trick, at least when $\gamma'(t_0) \neq 0$.

Definition I.4.2

Let $\gamma'(t_0) \neq 0$. Then there is a small neighborhood of t_0 won which γ is injective, and we can extend γ', Y around this small neighborhood and define

$$\frac{DY}{dt}(t_0) = (\nabla_{\gamma'^{ext}} Y^{ext})(\gamma(t_0)).$$

We say Y is parallel along γ if $\frac{DY}{dt} = 0$.

Definition I.4.3

Let (M^n, g) be a Riemannian manifold. Then ∇ a connection is called a Levi-Civita connection provided that

- (i) $\nabla_X Y = \nabla_Y X + [X, Y]$ (torsion-free).
- (ii) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ (leibniz rule/compatibility with the metric)

Remark I.4.1

In general, restricting the Levi-Civita connection on (M^m, g) to a submanifold $(N^n, g) \subseteq (M^m, g)$ does not define a connection on N^n , let alone the Levi-Civita connection. Consider the case of the sphere. However, the case of the sphere does suggest a way to make this work in general with projections...

Theorem I.4.1 (Levi + Civita)

Let (M^n, g) be a Riemannian manifold, then there exists a unique Levi-Civita connection on (M^n, g) .

There are two approaches to proving this theorem. Both use the fact that the vector field $\nabla_X Y$ is determined by the values of $\langle \nabla_X Y, Z \rangle$ for every vector field Z . One is intrinsic, defining a formula for the connection using the Lie derivative, and the other is extrinsic, using charts and the Christoffel symbols. We give both, and the first can be found in [docarmo]

Intrinsic Proof. We'll use the axioms to find a formula for $g(\nabla_X Y, Z)$, referred to as the Koszul formula. We may write using the Leibniz rule that

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

The first term is what we want... but the right hand side is pesky. Permuting X, Y, Z we get three equations

$$\begin{aligned} X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y(g(Z, X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Z(g(X, Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{aligned}$$

Now we see that

$$X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) = g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X). \quad (1)$$

But now, luckily, we're in business, since we can apply that the connection is torsion free, so

$$\nabla_X Y + \nabla_Y X = 2\nabla_X Y - [X, Y]$$

$$\nabla_X Z - \nabla_Z X = [X, Z]$$

$$\nabla_Y Z - \nabla_Z Y = [Y, Z].$$

Therefore the right hand side of eq. (1) is

$$2g(\nabla_X Y, Z) - g([X, Y], Z) + g([X, Z], Y) + g([Y, Z], X).$$

Rearranging this, we find that

$$g(\nabla_X Y, Z) = \frac{1}{2} [X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y)] \quad (2)$$

This formula is called the Koszul formula. This shows the Levi-Civita connection is unique, since the right hand side does not involve ∇ , and it is a good exercise in computation to show the right hand side satisfies the desired properties of the Levi-Civita connection. 

Example I.4.4

As an application of the Koszul Formula, let's compute on $\mathbb{R}^n$ the Christoffel symbols in standard coordinates. This actually follows immediately... why? Well recall that $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ for all i, j , and that

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

We can then deduce that,

$$\Gamma_{ij}^k = g\left(\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = 0,$$

just by looking at the relevant terms in the Koszul Formula.

Proof using Christoffel Symbols. This proof in some sense proceeds the exact same way as the first. But in fact we can remove some terms, since in local coordinates $\varphi : U \rightarrow M$ we have $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$. For convenience, let $\partial_i = \frac{\partial}{\partial x_i}$. Similarly, write $\nabla_i = \nabla_{\partial_i}$. We then write, reinterpret the compatibility with the metric to give

$$\partial_i g_{jk} = \partial_i g(\partial_j, \partial_k) = g(\nabla_i \partial_j, \partial_k) + g(\partial_j, \nabla_i \partial_k)$$

$$= g(\Gamma_{ij}^\ell \partial_\ell, \partial_k) + g(\partial_j, \Gamma_{ik}^\ell \partial_\ell) = \Gamma_{ij}^\ell g_{\ell k} + \Gamma_{ik}^\ell g_{j\ell}.$$

where we've adopted Einstein notation to clear things up a bit. Now we can consider, using that we're torsion-free, that

$$(\Gamma_{jk}^\ell - \Gamma_{kj}^\ell) \partial_\ell = \nabla_j \partial_k - \nabla_k \partial_j = [\partial_j, \partial_k] = 0.$$

Hence $\Gamma_{jk}^\ell = \Gamma_{kj}^\ell$. We now mimic the proof of the Koszul formula. We have three expressions

$$\partial_i g_{jk} = \Gamma_{ij}^\ell g_{\ell k} + \Gamma_{ik}^\ell g_{j\ell}$$

$$\partial_j g_{ki} = \Gamma_{jk}^\ell g_{\ell i} + \Gamma_{ji}^\ell g_{k\ell}$$

$$\partial_k g_{ij} = \Gamma_{ki}^\ell g_{\ell j} + \Gamma_{kj}^\ell g_{i\ell}.$$

Adding and subtracting these as above, together with this symmetry $\Gamma_{jk}^\ell = \Gamma_{kj}^\ell$ along with $g_{ij} = g_{ji}$, gives

$$\begin{aligned} \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} &= \Gamma_{ij}^\ell g_{\ell k} + \cancel{\Gamma_{ik}^\ell g_{j\ell}} \\ &\quad + \cancel{\Gamma_{jk}^\ell g_{\ell i}} + \Gamma_{ji}^\ell g_{k\ell} \\ &\quad - \cancel{\Gamma_{ki}^\ell g_{\ell j}} + \cancel{\Gamma_{kj}^\ell g_{i\ell}} \\ &= 2\Gamma_{ij}^\ell g_{k\ell} \end{aligned}$$

Thus, letting $(g^{k\ell})$ denote the inverse matrix of (g_{ij}) we find that,

$$\begin{aligned} g^{k\ell} \Gamma_{ij}^\ell g_{k\ell} &= \frac{1}{2} g^{k\ell} (\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}) \\ \Gamma_{ij}^\ell &= \frac{1}{2} g^{k\ell} (\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}). \end{aligned}$$

Great! 

Warning: The above $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right]$ does not hold for a local frame, only for local frames coming from coordinates. Try to construct a counterexample in hyperbolic space (hint, cook up an orthonormal frame on the imaginary axis and extend it out). Prove your counterexample is correct.

I.5. Geodesics

We're now ready to talk about geodesics. Let (M^n, g) be a Riemannian manifold, and ∇ the Levi-Civita connection.

Definition I.5.1

If $\gamma : I \rightarrow \mathbb{R}$ is a smooth curve, it is called a geodesic if $\frac{D\gamma'(t)}{dt} = 0$ (sometimes written $\nabla_{\gamma'(t)} \gamma'(t) = 0$ or $\gamma''(t) = 0$). That is γ' is parallel along γ .

In other words: a geodesic is a curve which has no *acceleration* from the perspective of the Riemannian manifold/metric.

Lemma I.5.1

If γ is a geodesic, then it is parameterized by arc length, that is it has constant speed.

Proof. We just use the Leibniz rule with the Levi-Civita connection (which is actually inherited on the pullback levi-civita)

$$\begin{aligned} \frac{d}{dt} |\gamma'(t)|^2 &= \frac{d}{dt} g(\gamma'(t), \gamma'(t)) \\ &= g\left(\frac{D\gamma'(t)}{dt}, \gamma'\right) + g\left(\gamma', \frac{D\gamma'}{dt}\right) \\ &= 0. \end{aligned}$$



Remark I.5.1

Every geodesic can be parameterized to have constant unit speed.

Now that we have the definition of geodesics under our belt, we should explore what geodesics look like in $\mathbb{R}^n, S^n, \mathbb{H}^n$!

Example I.5.1

Consider $(\mathbb{R}^n, g_0)$. Take some curve γ , we will show that γ is a geodesic if and only if γ is a straight line. Let

$$\begin{aligned} \gamma(t) &= (x_1(t), \dots, x_n(t)) \\ \gamma'(t) &= (x'_1(t), \dots, x'_n(t)). \end{aligned}$$

Or in other words $\gamma'(t) = \sum x'_i(t) \frac{\partial}{\partial x_i}$. Now, using the Leibniz rule and that $\frac{D}{dt} \frac{\partial}{\partial x_i} = 0$ (by the computation of Christoffel symbols we've done before), we have

$$\frac{D\gamma'(t)}{dt} = \sum_i x''_i(t) \frac{\partial}{\partial x_i} = (x''_1(t), \dots, x''_n(t)).$$

So this is zero if and only if $\gamma(t) = \gamma(0) + t\gamma'(0)$ for all t .

Example I.5.2

Consider (S^n, g^{S^n}) . Let $\gamma : I \subseteq \mathbb{R} \rightarrow S^n \subseteq \mathbb{R}^{n+1}$. We'll define $\gamma(t)$ by

$$\gamma(t) = \cos(t)p + \sin(t)u,$$

where $u \in T_p S^n$ and $\|u\| = 1$. Well $\gamma \cdot \gamma = 1$, so $\gamma(t) \in S^n$ to start.

Claim

γ is a geodesic.

We compute that $\gamma'(t) = -\sin(t)p + \cos(t)u$, so $\gamma''(t) = -\cos(t)p - \sin(t)u$. We then see that

$$\begin{aligned}\frac{D\gamma'(t)}{dt} &= (\gamma''(t))^T = (-\cos(t)p - \sin(t)u)^T \\ &= (-\gamma(t))^T = -\gamma(t) - (-\gamma(t) \cdot \gamma(t))\gamma(t) \\ &= 0 >\end{aligned}$$

Hence γ is a geodesic!

Example I.5.3

Consider B^n with the hyperbolic metric $\text{hyp} = \frac{4}{(1-|x|^2)^2}g_0$. Now we could consider the straight line $\gamma(t) = (0, 0, \dots, t)$, $t \in (-1, 1)$. But then

$$|\gamma'(t)| = \text{hyp}(\gamma'(t), \gamma'(t)) = \frac{4}{(1-t^2)^2},$$

is not parameterized by arc length, and so this can't be a geodesic. Instead we'll reparameterize

$$\begin{aligned}\gamma : \mathbb{R} &\rightarrow (B^n, \text{hyp}) \\ t &\mapsto (0, \dots, 0, \tanh t).\end{aligned}$$

We now compute that, since $\tanh'(t) = \text{sech}^2(t)$,

$$\begin{aligned}|\gamma'(t)| &= \langle \gamma'(t), \gamma'(t) \rangle \\ &= \frac{4}{(1 - \tanh^2 t)^2} \text{sech}^4(t) = 4,\end{aligned}$$

where $1 - \tanh^2 t = \text{sech}^2 t$ is a hyperbolic trig Pythagorean identity $\cosh^2 t - \sinh^2 t = 1$.

So now how do you check γ is a geodesic??? There are two ways

Method 1) Compute Γ_{ij}^k and then $\frac{D\gamma'}{dt}$ explicitly.

Method 2) Show that γ is fixed by an isometry which flips normal vectors, which will imply γ is a geodesic.

Lets employ Method 2. For convenience consider $n = 3$. $p : (x, y, z) \mapsto (-x, -y, z)$. We see that

(a) We can compute

$$p^*(\text{hyp}) = \frac{4}{(1-|p(x)|^2)^2}p^*(g_0) = \frac{4}{(1-|x|^2)^2}g_0 = \text{hyp},$$

and so p is an isometry.

(b) $p(\gamma(t)) = \gamma(t)$, as the last coordinate is fixed.

(c) $dp|_{\gamma(t)} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$.

And then we can see that these respectively imply that

- (a) $dp \cdot \nabla_{\gamma'} \gamma' = \nabla_{dp \cdot \gamma'} dp(\gamma')$
- (b) $dp(\gamma') = \gamma'$, since $p(\gamma) = \gamma$. Thus $dP(\nabla_{\gamma'} \gamma') = \nabla_{\gamma'} \gamma'$.
- (c) Thus the first two coordinates of γ' are zero, and hence $\nabla_{\gamma'} \gamma' = \lambda \gamma'$. One can then check that since γ is parameterized by arc length (see homework) that $\nabla_{\gamma'} \gamma' = 0$.

An interesting theorem of this flavor

Theorem I.5.2 (Liouville Theorem)

$\text{Isom}(B^n, \text{hyp}) = \text{Conformal}(B^n, g_0)$. Here $\text{Conformal}(B^n)$ is the space of conformal mappings, i.e., those which preserve angles. Explicitly this can be given as

$$\{f : B^n \rightarrow B^n \text{ diffeomorphism} \mid f^*(g_0) = h \cdot g_0\}$$

for some smooth $h : B^n \rightarrow \mathbb{R}$.

Okay, so back to geodesics. There are three pictures to keep in mind.

$$\mathbb{R}^2$$

$$S^2$$

$$(B^2, \text{hyp})$$

With probability one, geodesics intersect | geodesics always intersect | geodesics almost never intersect. These are flat, positive, and negative curvature respectively. We'll understand this distinction more formally at a later date. For now, we should understand the theoretical background about geodesics more.

Theorem I.5.3

Fix $p \in M^n$. Then there exists $\varepsilon_1, \delta_1 > 0$, U a neighborhood of p so that if

$$V_{\varepsilon_1} = \{(x, v) \mid x \in U, v \in T_x M, |v| < \varepsilon_1\},$$

where $t \mapsto \gamma(t, (x, v))$ is the unique geodesic with $\gamma(0, (x, v)) = x$ and $\gamma'(0, (x, v)) = v$. Aka the unique geodesic passing through x with velocity v .

Remark I.5.2

Homogeneity and scaling. You can trade off velocity with the time interval where a geodesic is defined. So say $t \mapsto c(t)$ for $t \in (-s, s)$ is a geodesic. Then we can take $\tilde{c} : t \mapsto c(\lambda t)$ for $t \in (-s/\lambda, s/\lambda)$ is a geodesic with $\tilde{c}(0) = c(0)$ and $\tilde{c}'(0) = \lambda c'(0)$.

Uniqueness (to be proven in the theorem) then implies that $\gamma(t, (x, v)) = \gamma(t/\lambda, (x, \lambda v))$. Because of this we denote

$$\exp_\lambda(tv) := \gamma(t, (x, v)).$$

This map is called the Exponential map.

Proof of Theorem. The idea of the proof in one line is that $\frac{D\gamma'(t)}{dt} = 0$ is a system of $2n$ 2nd order ODEs, and so it has local existence and uniqueness once we specify the $2n$ coordinates $\gamma(0), \gamma'(0)$.

More explicitly, let $\varphi : U \rightarrow M$ be a chart near p , then we write

$$\begin{aligned}\gamma(t) &= (x_1(t), \dots, x_n(t)) \\ \gamma'(t) &= \sum_{i=1}^n x'_i(t) \frac{\partial}{\partial x_i} \\ \frac{D\gamma'(t)}{dt} &= \frac{D}{dt} \left(\sum_i x'_i(t) \frac{\partial}{\partial x_i} \right) = \sum_i x''_i(t) \frac{\partial}{\partial x_i} + \sum_i x'_i(t) \frac{D}{dt} \left(\frac{\partial}{\partial x_i} \right) \\ &= \sum_{i=1}^n x''_i(t) \frac{\partial}{\partial x_i} + \sum_{i=1}^n x'_i(t) \nabla_{\gamma'(t)} \left(\frac{\partial}{\partial x_i} \right) \\ &= \sum_{k=1}^n (x''_k(t) + x'_i(t) x'_j(t) \Gamma_{ij}^k) \frac{\partial}{\partial x_k}.\end{aligned}$$

Therefore $\frac{D\gamma'(t)}{dt} = 0$, $\gamma(0) = x$, $\gamma'(0) = v$ if and only if

$$\begin{aligned}x''_k(t) + \sum_{i,j} x'_i x'_j \Gamma_{ij}^k(\gamma(t)) &= 0 \\ x_k(0) &= x_k \\ x'_k(0) &= v_k.\end{aligned}$$

Thus by ODE theory, there exists V a neighborhood of 0 , ε, δ small, so that $\gamma : (-\delta, \delta) \times V \times B_\varepsilon(0) \rightarrow U$ is smooth and $t \mapsto \gamma(t, (x, v))$ satisfies the above. Furthermore the existence is unique. 

Remark I.5.3

By making ε_1 smaller, we can assume $\delta_1 = 2$ by homogeneity and so we get a smooth map

$$\begin{aligned}\exp_p : B_{\varepsilon_1}(0) &\rightarrow M \\ v &\mapsto \exp_p(v) = \gamma(1, (t, v)).\end{aligned}$$

Remark I.5.4

In fact this ODE is extremely special, and many dynamicists use that the geodesics is what's called a Hamiltonian flow. We won't talk about this much in this class. If you want to know more though, talk to Amie Wilkinson.

Proposition I.5.4

For all $p \in M$, there exists a neighborhood U of p and an $\varepsilon > 0$ so that

- (i) For all $q \in U$, $\exp_q : B_\varepsilon(0) \rightarrow M$ is a diffeomorphism onto its image
- (ii) For all $q_1, q_2 \in U$, there exists a unique $v \in B_\varepsilon(0) \subseteq T_{q_1} M$ so that $q_2 = \exp_{q_1}(v)$.

U is called a normal neighborhood.

Proof. Fix a chart $\varphi : V_1 \rightarrow M$ so that $\varphi(0) = p$. Now we can give a function

$$F : V_1 \times B_\delta(0) \rightarrow M \times M$$

$$(x, v) \mapsto (\varphi(x), \exp_{\varphi(x)} \exp_{\varphi(x)}(d\varphi_x \cdot v)).$$

Under this setup

Exercise I.5.4

The statement of the theorem is equivalent to giving $V_2 \subseteq V_1$, $\varepsilon < \delta$ so that $F : V_2 \times B_\varepsilon(0)$ is a diffeomorphism onto its image.

Under this regime, we can just apply inverse function theorem. Since φ gives a diffeomorphism, we'll just write x for $\varphi(x)$ and v for $d\varphi_x \cdot v$. Forgive us. With this

$$DF_{(0,0)} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow T_p M \times T_p M$$

$$DF_{(0,0)} \left(\frac{\partial}{\partial x_i}, 0 \right) = \frac{\partial}{\partial x_i} (F(x, 0)) = \frac{\partial}{\partial x_i} (x, x) = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right).$$

$$DF_{(0,0)}(0, v) = \frac{d}{dt} F(0, tv) \Big|_{t=0} = \frac{d}{dt} (p, \exp_p(tv)) \Big|_{t=0} = (0, v).$$



For today:

- Show that geodesics minimize distances to nearby points.
- Show that closed Riemannian manifolds have closed geodesics, at least if $\pi_1(M) \neq 0$.

Before this, Neves needs to cross his Ts and dot his Is. Aka do the necessary technical things. Namely the Gauss Lemma

Lemma I.5.5 (Gauss Lemma)

Fix $p \in M$, $\exp_p : B_\varepsilon(0) \rightarrow M$ (we'll omit the p sometimes, since it's already here). Then we have for all $v, w \in B_\varepsilon(0) \subseteq T_p M$

$$g_{\exp(v)}(d(\exp)_v \cdot v, d(\exp)_v(w)) = g_p(v, w)$$

Content: We get a nice expression for $\exp^* g$ in radial coordinates $(r, \theta) \in (0, \infty) \times S^{n-1}$. Well we know $T_p M$ has a metric g_p as a vector space, which is then Euclidean. In coordinates this can be rewritten as

$$g_p = dr^2 + r^2 g_{S^{n-1}}$$

We also have $\exp^* g$ is another metric on $B_\varepsilon(0) \subseteq T_p M$. Gauss lemma implies (and we'll check this)

$$(\exp^* g)_{r,\theta} = dr^2 + h_{r,\theta},$$

where $h_{r,\theta}$ is a metric on S^{n-1} (aka has no dr terms, taking a vector orthogonal to the sphere gives just the dr^2 term above).

Remark I.5.5

Lets get straight what's going on here. So $d(\exp)_v$ should be a map $T_v(T_p M) \rightarrow T_{\exp(v)} M$. It turns out we can identify $T_v(T_p M) = T_p M$ in this case, because we're thinking of $B_\varepsilon(0) \subseteq T_p M \cong \mathbb{R}^n$. That's why it makes sense to apply this to v, w .

Another thing to get straight. g_p defines a bilinear form $T_p M \times T_p M \rightarrow \mathbb{R}$. Now how do we interpret this as a metric on $T_p M$? Well actually we can identify for any $x \in T_p M$, $v, w \in T_x(T_p M) \cong T_p M$, and define $(g_p)_x(v, w) = g_p(v, w)$. How does the identification $T_x(T_p M) \cong T_p M$ work? Well if you wanna cheat, use an orthonormal basis on $T_p M$ and show this doesn't depend on the choice.

Then g_p becomes a metric on $T_p M$. Another way to view this is that we can take an isomorphism $\mathbb{R}^n \rightarrow T_p M$ via an orthonormal basis, and push forward the Euclidean metric on $\mathbb{R}^n$ (it turns out this gives the same answer regardless of orthonormal basis). As Neves says, this is sort of walking through the forest of axioms as Snow White and being attacked by thorns... you just shouldn't.

Proof. Divide into cases and conquer... by linearity we can just take w a multiple of v or w orthogonal to v .

- (1) First case, let $w = \lambda v$, and set $\gamma(t) = \exp(tv)$ (a geodesic through p at velocity v). Then we see that

$$\begin{aligned} d(\exp)_v \cdot (\lambda v) &= \lambda d(\exp)_v \cdot v = \lambda \frac{d}{dt} \exp(v + tv) \Big|_{t=0} \\ &= \lambda \frac{d}{dt} \exp(v(1+t)) \Big|_{t=0} = \lambda \gamma'(1). \end{aligned}$$

So in this case

$$\begin{aligned} g_{\exp(v)}(d \exp_v \cdot v, d \exp_v \cdot w) &= \lambda g_{\exp(v)}(\gamma'(1), \gamma'(1)) \\ &= \lambda g_p(\gamma'(0), \gamma'(0)) = g_p(v, \lambda v) = g_p(v, w), \end{aligned}$$

where we used that geodesics are parameterized by arc-length, so $|\gamma'(0)|^2 = |\gamma'(1)|^2$.

- (2) Second case, $w \perp v$ (they're orthogonal) so $g_p(v, w) = 0$. We know $g_p(v, w) = 0$. We must show that

$$g_{\exp(v)}(d \exp_v \cdot v, d \exp_v \cdot w) = 0.$$

Without loss of generality, scale w so that $|w| = |v|$. Consider some path $v(s)$ in $B_\varepsilon(0)$ so that

- (i) $v(0) = v$.

(ii) $|v(s)| = |v|$.

(iii) $v'(0) = w$.

Explicitly, $v(s) = \cos(s)v + \sin(s)w$. Now consider the map

$$F : (-\delta, \delta) \times (0, 2) \rightarrow M$$

$$F(s, t) = \exp(tv(s))$$

We see then that

$$\left. \frac{\partial F}{\partial t} \right|_{(0,1)} = \left. \frac{d}{dt} \exp((1+t)v) \right|_{t=0} = d \exp_v \cdot v$$

$$\left. \frac{\partial F}{\partial s} \right|_{(0,1)} = \left. \frac{d}{ds} \exp(v(s)) \right|_{s=0} = d \exp_v \cdot w,$$

the right hand side of the second equation above follows since $v(s)$ passes through v at time $s = 0$ with velocity w , so this is the definition of $d \exp_v$. Great! Now we just need to ask if

$$g \left(\frac{\partial F}{\partial s}(0, 1), \frac{\partial F}{\partial t}(0, 1) \right) = 0.$$

...okay? How does that help! Well we now have a parameter to vary. Consider

$$a(t) := g_{F(0,t)} \left(\frac{\partial F}{\partial s}(0, t), \frac{\partial F}{\partial t}(0, 1) \right).$$

Now we need to show $a(1) = 0$, and we can evaluate

$$a(0) = g_p \left(\left. \frac{d}{ds} \exp(0 \cdot v(s)) \right|_{s=0}, \gamma'(0) \right) = g_p(0, v) = 0.$$

Awesome!!! Now we just need to show this is constant...

Claim

$$a'(t) = 0 \text{ for all } t.$$

Time to motherfucking compute,

$$\begin{aligned} \frac{d}{dt} a(t) &= \frac{d}{dt} g \left(\frac{\partial F}{\partial s}(0, t), \frac{\partial F}{\partial t}(0, t) \right) \\ &= g \left(\nabla_{\frac{\partial F}{\partial t}(0,t)} \frac{\partial F}{\partial s}(0, t), \frac{\partial F}{\partial t}(0, t) \right) + g \left(\frac{\partial F}{\partial s}(0, t), \nabla_{\frac{\partial F}{\partial t}(0,t)} \frac{\partial F}{\partial t}(0, t) \right). \end{aligned}$$

Whoof. Well, we know the right piece is zero, because $F(0, t) = \gamma(t)$ is a geodesic, so $\nabla_{\gamma'} \gamma' = 0$. Now for the next part we have

$$\begin{aligned} \nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial s} &= \nabla_{\frac{\partial F}{\partial s}} \frac{\partial F}{\partial t} + \left[\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right] \\ &= \nabla_{\frac{\partial F}{\partial s}} \frac{\partial F}{\partial t} + F_* \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = \nabla_{\frac{\partial F}{\partial s}} \frac{\partial F}{\partial t} \end{aligned}$$

where we use that the Lie bracket of coordinates is zero, and the Lie bracket is natural (aka commutes with pullback/pushforward). Hence we're looking for

$$\begin{aligned} g\left(\nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right) &= g\left(\nabla_{\frac{\partial F}{\partial s}} \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right) \\ &= \frac{1}{2} \cdot \frac{d}{ds} g_{F(t,s)}\left(\frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\right) \Big|_{(0,t)}. \end{aligned}$$

Great! Now letting $\gamma_s(t) = \exp(tv(s))$, this is a geodesic, and

$$\begin{aligned} \frac{\partial F}{\partial t}(s,t) &= \frac{d}{dt} \exp(tv(s)) = \gamma'_s(t) \\ g_{F(s,t)}\left(\frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\right) &= g_{F(s,t)}(\gamma'_s(t), \gamma'_s(t)) = g_{F(s,0)}(\gamma'_s(0), \gamma'_s(0)). \end{aligned}$$

Great! Now $\gamma'_s(0) = v(s)$, and so we see that $g_{F(s,0)}(\gamma'_s(0), \gamma'_s(0)) = |v(s)|^2 = |v|^2$, by how we chose v . This gives finally that

$$\frac{d}{ds} g_{F(s,t)}\left(\frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\right) \Big|_{(0,t)} = \frac{d}{ds} |v(s)|^2 \Big|_{(0,t)} = \frac{d}{ds} |v|^2 \Big|_{(0,t)} = 0.$$

Incredible! This gives us the result. But at what cost...



Last time; Given $x \in M$, we found U a normal neighborhood so that there exists an $\varepsilon > 0$ so that

- (i) $\exp_x : B_\varepsilon(0) \subseteq T_x M \rightarrow M$ is a diffeomorphism onto its image.
- (ii) $U \subseteq \exp_x(B_\varepsilon(0))$.

Remark I.5.6

If M is compact, then ε can be chosen independently of $p \in M$.

Now we'll move onto the length-minimizing properties of geodesics. This is the motivation for what a geodesic is in some sense, so it better be something we talk about!

Definition I.5.2

Let $c : [a, b] \rightarrow M$ be a C^1 curve, then the length of c is

$$\text{length}(c) = \int_a^b |c'(t)| dt.$$

If c is piecewise differentiable (continuous, C^1 except at finitely many points), we can define its length in the same way, since $|c'(t)|$ is defined everywhere except finitely many points.

Theorem I.5.6

Fix $x, y \in U$. Then we have

- (i) There exists a unique $\gamma_{x,y} : [0, 1] \rightarrow M$ a geodesic with $\text{length}(\gamma_{x,y}) < \varepsilon$ and $\gamma_{x,y}(0) = x, \gamma_{x,y}(1) = y$.
- (ii) If $\sigma : [0, 1] \rightarrow M$ is a C^1 curve connecting x to y , then $\text{length}(\sigma) \geq \text{length}(\gamma_{x,y})$.
- (iii) If σ above is piecewise differentiable, connecting x to y and $\text{length}(\sigma) = \text{length}(\gamma_{x,y})$. Then in fact $\sigma([0, 1]) = \gamma_{x,y}([0, 1])$.

Proof. Let's go!

- (i) $y \in U \subseteq \exp_x(B_\varepsilon(0))$ and so $y = \exp_x(v)$. We see that $\gamma_{x,y}(t) = \exp_x(tv)$ is a geodesic, and we compute that

$$\text{length}(\gamma_{x,y}) = \int_0^1 |\gamma'_{x,y}(t)| dt = |\gamma'_{x,y}(0)| = |v| < \varepsilon.$$

Now we need to prove uniqueness. Suppose $c : [0, 1] \rightarrow M$ is a geodesic connecting x to y with $\text{length}(c) < \varepsilon$. For t small we have $c(t) = \exp_x(tw)$. Hence $|w| = \text{length}(c) < \varepsilon$. Then

$$\exp_x(v) = y = c(1) = \exp_x(w).$$

Then $v = w$, and we win (paste together local uniqueness).

- (ii) First case, $\sigma([0, 1]) \subseteq \exp_x(B_\varepsilon(0))$. Then since $\exp$ is a local diffeomorphism, we can write $\sigma(t) = \exp_x(r(t)w(t))$ for $|w(t)| = |v|$ for all t . Note $w(1) = v$, $r(1) = 1$, and $r(0) = 0$. Now we compute by linearity that

$$\begin{aligned} \sigma' &= d\exp_{rw} \cdot (r'w + rw') \\ &= \frac{r'}{r} d\exp_{rw} \cdot rw + r d\exp_{rw} w'. \end{aligned}$$

Huh, looks like Gauss lemma! So now we can go

$$\begin{aligned} |\sigma'(t)|^2 &= \left(\frac{r'}{r}\right)^2 |d\exp_{rw}(rw)|^2 + 2r' \langle d\exp_{rw}(rw), d\exp_{rw}(w') \rangle + r^2 |d\exp_{rw} w'|^2 \\ &\geq (r')^2 |w|^2 + 2r'r \langle w, w' \rangle. \end{aligned}$$

Now we just need to examine $\langle w, w' \rangle \dots$ We'll finish on Tuesday!



Last time; Given $p \in U$, there exists a neighborhood U of p , $\varepsilon = \varepsilon_p > 0$ so that

- (i) For all $x \in U$, $\exp_x : B_\varepsilon(0) \rightarrow M$ is a diffeomorphism.
- (ii) $U \subseteq \exp_x(B_\varepsilon(0))$ for all $x \in U$.

Theorem I.5.7

We have the following, with $\varepsilon = \varepsilon_p$ above,

- (1) For every $y \in \exp_x(B_\varepsilon(0))$, there exists a unique $\gamma_{x,y} : [0, 1] \rightarrow M$ a geodesic with $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ with $\text{length}(\gamma_{x,y}) < \varepsilon$.
- (2) If $c : (0, 1) \rightarrow M$ is a C^1 curve connecting x to y , then $\text{length}(c) \geq \text{length}(\gamma_{x,y})$.
- (3) If $c : [0, 1] \rightarrow M$ is piecewise C^1 connecting x to y and $\text{length}(c) = \text{length}(\gamma_{x,y})$, implying $\gamma_{x,y}([0, 1]) = c([0, 1])$. Thus, after reparameterization, we can assume $c = \gamma_{x,y}$.

We now have some small technical things to cover.

Definition I.5.3

Define

$$d(p, q) := \inf\{\text{length}(\gamma) \mid \gamma \text{ is a } C^1 \text{ curve connecting } p \text{ to } q\},$$

Exercise I.5.5

Check the following technical points

- (a) $d(p, q)$ is a distance function.
- (b) If $r \leq \varepsilon_p$, then in fact $B_r(y) := \{x \mid d(x, y) < r\} = \exp_y(B_r(0))$ for all $y \in U$.
- (c) If M is closed (compact, no boundary), there exists some $\bar{\varepsilon} > 0$, so that $\bar{\varepsilon} \leq \varepsilon_p$ for all $p \in M$. I.e., $\exp_x : B_{\bar{\varepsilon}}(0) \rightarrow M$ is a diffeomorphism onto its image for all $x \in M$.
- (d) If $x_i \rightarrow x, y_i \rightarrow y$, then $\gamma_{x_i, y_i} \rightarrow \gamma_{x, y}$ (smoothly, a reasonable sense).

Question: Does every closed Riemannian manifold (M^n, g) have closed geodesics?

Example I.5.6

On the round sphere S^n , any great circle is a closed geodesic. Similarly, on a flat torus $(\mathbb{R}^n/\mathbb{Z}^n, g_0)$ you can find a closed geodesic (ex, any line of rational slope). In contrast to the case of the sphere, there are non-closed geodesics (ex, any line of irrational slope).

Example I.5.7

Consider an ellipsoid (see Figure 2). The axes of symmetry give closed geodesics (since they are fixed by an isometry) However, it is unclear if a small perturbation of the ellipsoid has closed geodesics.

Theorem I.5.8 (Hadamard)

If (M^n, g) is closed with $\pi_1(M) \neq 0$, then (M^n, g) has at least one closed geodesic.

Proof. The idea is to find a closed geodesic inside of a non-trivial homotopy class (and hence it won't be the point). Let $c : S^1 \rightarrow M$ be a non-trivial curve. Without loss of generality we can homotope c to be a smooth curve.

$$\ell = \inf\{\text{length}(\gamma) : \gamma : S^1 \rightarrow M, \text{ piecewise } C^1 \text{ freely homotopic to } c, |\gamma'(\theta)| = |\gamma'(0)|\}.$$

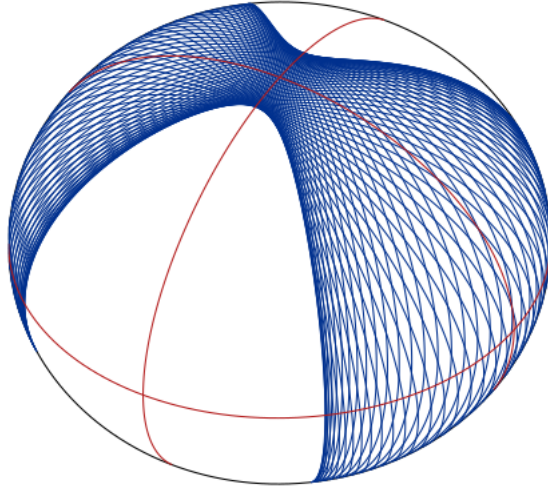


FIGURE 2. Geodesics on an Ellipsoid

Basic idea now is to find a minimizer, which we hope to end up being a geodesic. In this case,

$$\text{length}(\gamma) = \int_{S^1} |\gamma'(\theta)| d\theta = 2\pi |\gamma'(0)| >$$

Great! So now what is the approach. Pick γ_i lying in this set so that $\text{length}(\gamma_i) \rightarrow \ell$.

Hope: γ_i converges to a geodesic γ as $i \rightarrow \infty$. In fact this might not happen (e.g. because geodesics in this homotopy class are not unique, see torus). However, we can pass to a subsequence to get into the situation we like.

We could show this convergence using hard functional analysis (e.g. PDEs). But thankfully, using normal neighborhoods, we can reduce this to a finite dimensional problem. To do this and to take limits of curves, we need a regularization process. Divide S^1 as follows

$$\{\theta_0, \theta_1, \dots, \theta_N = \theta_0\} \in S^1,$$

so that the distance between θ_j, θ_{j+1} is $\frac{\pi \bar{\varepsilon}}{2\ell}$ for each j (except maybe the last one is smaller... but who truly cares). Without loss of generality, also assume $\text{length}(\gamma_u) < 2\ell$. Then

$$\begin{aligned} \text{length}(\gamma_i[\theta_j, \theta_{j+1}]) &= |\gamma_i'(0)| \frac{\pi \bar{\varepsilon}}{2\ell} \\ &= \frac{\text{length}(\gamma_i)}{2\pi} \cdot \frac{\pi \bar{\varepsilon}}{2\ell} \\ &< \frac{2\ell}{2} \cdot \frac{\bar{\varepsilon}}{2\ell} < \bar{\varepsilon}. \end{aligned}$$

Great! This allows us to use the big theorem to connect $\gamma_i(\theta_j)$ and $\gamma_i(\theta_{j+1})$. Namely, we can replace $\gamma_i[\theta_j, \theta_{j+1}]$ by the unique geodesic between $\gamma_i(\theta_j), \gamma_i(\theta_{j+1})$, since this cannot increase the distance. Call this replacement $\hat{\gamma}_i$, once we parameterize by arc-length.

Claim

If we take $\bar{\varepsilon}$ small enough, then $\hat{\gamma}_i \sim \gamma_i$. Why? Well see your homework!

So now we have $\hat{\gamma}_i$ with:

- $|\hat{\gamma}'_i(0)| = |\hat{\gamma}'_i(\theta)|$ for all $\theta \in S^1$.
- $\hat{\gamma}_i$ is a geodesic when restricted to each $[\theta_j, \theta_{j+1}]$.
- $\text{length}(\hat{\gamma}_i) \rightarrow \ell$, since

$$\text{length}(\gamma_i) \geq \text{length}(\hat{\gamma}_i) \geq \ell,$$

Now pass to a subsequence so that $\hat{\gamma}_k(\theta_j) \rightarrow p_j$ for all $j = 0, \dots, N-1$. Now in fact $d(p_j, p_{j+1}) < \bar{\varepsilon}$. We can then consider the curve σ connecting $p_0, p_1, \dots, p_N$ by piecewise geodesics. Aka $\sigma|_{[\theta_j, \theta_{j+1}]}$ is the geodesic connecting p_j, p_{j+1} . Parameterize by arc-length so that $|\sigma'(0)| = |\sigma'(\theta)|$ for all $\theta \in S^1$.

There are now three things to check

- (1) σ is not a point.
- (2) $\text{length}(\sigma) = \ell$.
- (3) σ is a geodesic.

Lets go!

- (1) We check that $\sigma \sim c$, which is nontrivial in $\pi_1(M)$ (and hence after change of basepoint is nontrivial, since this is free homotopy). Well $\hat{\gamma}_i|_{[\theta_j, \theta_{j+1}]} \rightarrow \sigma|_{[\theta_j, \theta_{j+1}]}$ by the continuity properties of connecting geodesics. Hence $\hat{\gamma}_i \rightarrow \sigma$. Moving all the homotopies as well, $\sigma \sim c$.
- (2) By the previous part, we also have $\text{length}(\hat{\gamma}_i) \rightarrow \text{length}(\sigma) = \ell$.
- (3) If σ were a piecewise geodesic, i.e., it had a kink, we could use a homework problem to reduce length. Namely, in a small neighborhood around the kink, replace the kink by a straight line between the intersections with that neighborhood.

This would make σ have smaller length but then we would not have $\text{length}(\sigma) = \ell$.



So for $\pi_1(M) \neq 0$, we have a nontrivial closed geodesic. Birkhoff (according to Nevés the first serious American mathematician) handled the hard case

Theorem I.5.9 (Birkhoff, '20s)

Every (S^2, g) (any metric g) has a closed geodesic.

Theorem I.5.10 (Lusternik-Fet)

Every closed Riemannian manifold (M^n, g) has a closed geodesic.

Question: Does every (M^n, g) have infinitely many distinct closed geodesics.

Theorem I.5.11

Every (S^2, g) has infinitely many closed geodesics.

Theorem I.5.12 (Rademacher)

Fix M^n . For a generic metric there will be infinitely many closed geodesics.

Theorem I.5.13 (Mromell-Meyer)

“Most” manifolds M^n have infinitely many closed geodesics for every metric.

Conjecture I.5.14

Every (S^3, g) has infinitely many closed geodesics.

II. Curvature

II.1. The Riemannian Curvature Tensor

Definition II.1.1

A map

$$T : \underbrace{\mathfrak{X}(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r \text{ times}} \rightarrow C^\infty(M)$$

is an r -tensor if for all $f \in C^\infty(M)$, $X_1, \dots, X_r \in \mathfrak{X}(M)$, we have

$$T(X_1, \dots, fX_j, \dots, X_r) = fT(X_1, \dots, X_r).$$

Example II.1.1

We have that

- (1) $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ where $(X, Y) \mapsto g(X, Y)$ is a tensor.
- (2) $\nabla Y : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ given by

$$(X, Z) \mapsto g(\nabla_X Y, Z)$$

is a tensor.

Remark II.1.1

For a tensor T , $T(X_1, \dots, X_r)(p)$ only depends on $X_1(p), \dots, X_r(p)$. One can show this using bump functions.

Non-Example II.1.2

Consider $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ given by

$$(X, Y, Z) \mapsto g(\nabla_X Y, Z),$$

this is not a tensor because it depends on the value of Y in a neighborhood.

Definition II.1.2

Let (M^n, g) be a Riemannian manifold. Given $X, Y \in \mathfrak{X}(M)$ consider the map

$$R(X, Y) : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

$$R(X, Y)(Z) := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]} Z.$$

This is called the Riemannian curvature operator.

Remark II.1.2

A naive explanation of curvature (being naive is good sometimes): the curvature tensor measures how much derivatives do not commute.

Lemma II.1.1

$$R(X, Y)(fZ) = fR(X, Y)(Z).$$

Proof. Lets compute folks

$$\begin{aligned} \nabla_X(\nabla_Y(fZ)) &= \nabla_X(Y(f)Z + f\nabla_Y Z) \\ &= X(Y(f))Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f\nabla_X \nabla_Y Z \\ \nabla_Y(\nabla_X(fZ)) &= Y(X(f))Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + f\nabla_Y \nabla_X Z. \end{aligned}$$

Thus

$$\begin{aligned} \nabla_X(\nabla_Y(fZ)) - \nabla_Y(\nabla_X(fZ)) &= X(Y(f))Z - Y(X(f))Z + f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z \\ &= [X, Y](f) \cdot Z + f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z. \end{aligned}$$

Furthermore

$$\nabla_{[X, Y]}(fZ) = [X, Y](f) \cdot Z + f\nabla_{[X, Y]}(Z).$$

Subtracting again we get

$$R(X, Y)(fZ) = f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - f\nabla_{[X, Y]}(Z) = f \cdot R(X, Y)(Z).$$

Perfect!



This operator is so important because using it we can form a tensor

Definition II.1.3

The Riemannian curvature tensor of a metric g is the map

$$\mathcal{R}(g) : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$$

$$(X, Y, Z, W) \mapsto g(R(X, Y)(Z), W).$$

This map is a tensor by using the lemma above and a bit of work.

So how should we understand this coordinate? Well of course we should work in coordinates. Let $\varphi : U \rightarrow M$ be a chart with $\varphi(0) = p$, and let ∂_{x_i} be shorthand notation for $d\varphi\left(\frac{\partial}{\partial x_i}\right)$. Using that coordinate vector fields commute (i.e. have zero Lie derivative), we get

$$R(\partial_{x_i}, \partial_{x_j})\partial_{x_k} = \nabla_{\partial_{x_i}}(\nabla_{\partial_{x_j}}\partial_{x_k}) - \nabla_{\partial_{x_j}}\nabla_{\partial_{x_i}}\partial_{x_k}.$$

Now assuming the ∂_{x_i} are orthonormal at a point, we get

$$\nabla_{\partial_{x_i}}\partial_{x_k} = \sum_s \Gamma_{ik}^s \partial_{x_s}.$$

Then we have to run a long computation

$$\begin{aligned} R(\partial_{x_i}, \partial_{x_j})\partial_{x_k} &= \nabla_{\partial_{x_i}}\left(\sum_s \Gamma_{jk}^s \partial_{x_s}\right) - \nabla_{\partial_{x_j}}\left(\sum_s \Gamma_{ik}^s \partial_{x_s}\right) \\ &= \sum_s \partial_{x_i} \Gamma_{jk}^s \partial_{x_s} + \sum_s \Gamma_{jk}^s \nabla_{\partial_{x_i}} \partial_{x_s} - \sum_s \partial_{x_j} \Gamma_{ik}^s \partial_{x_s} - \sum_s \Gamma_{ik}^s \nabla_{\partial_{x_j}} \partial_{x_s} \\ &= \sum_s (\partial_{x_i} \Gamma_{jk}^s - \partial_{x_j} \Gamma_{ik}^s) \partial_{x_s} + \sum_{s,q} (\Gamma_{jk}^s \Gamma_{is}^q - \Gamma_{ik}^s \Gamma_{js}^q) \partial_{x_q}. \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{R}_{ijk\ell}(x) &:= \mathcal{R}(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{x_\ell}) \\ &= \sum_s (\partial_{x_i} \Gamma_{jk}^s - \partial_{x_j} \Gamma_{ik}^s) g_{s\ell} + \sum_{s,q} (\Gamma_{jk}^s \Gamma_{is}^q - \Gamma_{ik}^s \Gamma_{js}^q) g_{q\ell}. \end{aligned}$$

Now lets make a huge simplifying assumption. Assume that $\varphi : U \rightarrow U$ is a normal chart and we evaluate at p (note we're using tensoriality!). Since this is normal $g_{ij}(0) = \delta_{ij}$ and $\partial_{x_k} g_{ij}(0) = 0$ for all i, j, k . Hence $\Gamma_{ij}^\ell(0) = 0$ for all i, j, ℓ .

This causes many terms to vanish, and we get

$$\mathcal{R}_{ijk\ell}(0) = \partial_{x_i} \Gamma_{jk}^\ell(0) - \partial_{x_j} \Gamma_{ik}^\ell(0).$$

Now lets see the symmetries of R .

Proposition II.1.2 (The Curvature Identities)

We have the following identities. The first one is called the first Bianchi Identity

(1) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$, for all $X, Y, Z \in \mathfrak{X}(M)$. This is equivalent to

$$\mathcal{R}(X, Y, Z, W) + \mathcal{R}(Y, Z, X, W) + \mathcal{R}(Z, X, Y, W) = 0,$$

for all X, Y, Z, W .

(2) $\mathcal{R}(X, Y, Z, W) = -\mathcal{R}(Y, X, Z, W)$.

(3) $\mathcal{R}(X, Y, Z, W) = -\mathcal{R}(X, Y, W, Z)$

$$(4) \mathcal{R}(X, Y, Z, W) = \mathcal{R}(Z, W, X, Y).$$

Proof. Because $R(X, Y)Z$ is tensorial (and so depends only on a point / is linear), it suffices to check with $X = \partial_{x_i}, Y = \partial_{x_j}, Z = \partial_{x_k}, W = \partial_{x_\ell}$. By tensoriality it only depends on the point as well, so we can use normal coordinates.

(1) Lets do the first one:

$$R(\partial_{x_i}, \partial_{x_j})\partial_{x_k}(0) = \sum_{\ell} (\partial_{x_i}\Gamma_{jk}^{\ell}(0) - \partial_{x_j}\Gamma_{ik}^{\ell}(0))\partial_{x_\ell}.$$

Now we cyclically permute the terms $\partial_{x_i}\Gamma_{jk}^{\ell}(0)$ and $\partial_{x_j}\Gamma_{ik}^{\ell}(0)$. Since Γ_{jk}^{ℓ} is symmetric in j, k , the terms will cancel. More explicitly

$$\begin{aligned} & \cancel{\partial_{x_i}\Gamma_{jk}^{\ell}(0)} - \cancel{\partial_{x_j}\Gamma_{ik}^{\ell}(0)} \\ & \cancel{\partial_{x_j}\Gamma_{ki}^{\ell}(0)} - \cancel{\partial_{x_k}\Gamma_{ji}^{\ell}(0)} \\ & \cancel{\partial_{x_k}\Gamma_{ij}^{\ell}(0)} - \cancel{\partial_{x_i}\Gamma_{kj}^{\ell}(0)}. \end{aligned}$$

(2) Now for the second one, there are two ways to see this. First

$$\begin{aligned} R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \\ R(Y, X)Z &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y, X]} Z, \end{aligned}$$

where we now note that $[Y, X] = -[X, Y]$ we can just add this. You could also see this from

$$\mathcal{R}_{ijk\ell} = \partial_{x_i}\Gamma_{jk}^{\ell}(0) - \partial_{x_j}\Gamma_{ik}^{\ell}(0),$$

and exchange i, j .

(3) First we need a mini-claim

Claim

$\mathcal{R}(X, Y, Z, Z) = 0$ for all X, Y, Z implies the claim that $\mathcal{R}(X, Y, Z, W) = -\mathcal{R}(X, Y, W, Z)$.

Well, we see that

$$\begin{aligned} 0 &= \mathcal{R}(X, Y, Z + W, Z + W) = \mathcal{R}(X, Y, Z, Z) + \mathcal{R}(X, Y, Z, W) + \mathcal{R}(X, Y, W, Z) + \mathcal{R}(X, Y, W, W) \\ &= \mathcal{R}(X, Y, Z, W) + \mathcal{R}(X, Y, W, Z). \end{aligned}$$

just as desired.

Great! Now we just need to check $\mathcal{R}(X, Y, Z, Z) = 0$. Lets go for it. It is no longer clear we can check this in charts with normal coordinates, since there may be cross terms coming from Z . Thankfully, we're still in business. We can perform an orthogonal change of coordinates on $T_p M$ so that $Z(p)$ is a scalar multiple of a coordinate vector field.

Then the expression $\mathcal{R}(X, Y, \partial_{x_k}, \partial_{x_k})$ is linear in X, Y and so we can choose to write X, Y in coordinate charts and win. We'll use the compatibility of the metric and the connection:

$$\begin{aligned}
 \mathcal{R}_{ijkk}(0) &= \langle \nabla_{\partial_{x_i}} (\nabla_{\partial_{x_j}} \partial_{x_k} - \nabla_{\partial_{x_k}} (\nabla_{\partial_{x_i}} \partial_{x_k}, \partial_{x_k}) \rangle \\
 &= \partial_{x_i} \langle \nabla_{\partial_{x_j}} \partial_{x_k}, \partial_{x_k} \rangle - \langle \nabla_{\partial_{x_j}} \partial_{x_k}, \nabla_{\partial_{x_i}} \partial_{x_k} \rangle \\
 &\quad - \partial_{x_j} \langle \nabla_{\partial_{x_i}} \partial_{x_k}, \partial_{x_k} \rangle - \langle \nabla_{\partial_{x_i}} \partial_{x_k}, \nabla_{\partial_{x_j}} \partial_{x_k} \rangle \\
 &= \partial_{x_i} \langle \nabla_{\partial_{x_j}} \partial_{x_k}, \partial_{x_k} \rangle - \partial_{x_j} \langle \nabla_{\partial_{x_i}} \partial_{x_k}, \partial_{x_k} \rangle \\
 &= \partial_{x_i} \partial_{x_j} \frac{\langle \partial_{x_k}, \partial_{x_k} \rangle}{2} - \partial_{x_j} \partial_{x_i} \frac{\langle \partial_{x_k}, \partial_{x_k} \rangle}{2} \\
 &= [\partial_{x_i}, \partial_{x_j}] \frac{\langle \partial_{x_k}, \partial_{x_k} \rangle}{2} = 0.
 \end{aligned}$$

Great! That was painful!

- (4) Now for the last one it suffices to see $\mathcal{R}_{ijkl}\mathcal{R}_{klij}$. We start doing cyclic permutations and using the other identities

$$\mathcal{R}_{ijkl} + \mathcal{R}_{jkil} + \mathcal{R}_{kijl} = 0$$

$$\mathcal{R}_{jkli} + \mathcal{R}_{klji} + \mathcal{R}_{ljki} = 0$$

$$\mathcal{R}_{klij} + \mathcal{R}_{likj} + \mathcal{R}_{iklj} = 0$$

$$\mathcal{R}_{lijk} + \mathcal{R}_{ijlk} + \mathcal{R}_{jlik} = 0.$$

Now we notice some terms cancel because of the other laws

$$\cancel{\mathcal{R}_{ijkl}} + \cancel{\mathcal{R}_{jkil}} + \mathcal{R}_{kijl} = 0$$

$$\cancel{\mathcal{R}_{jkli}} + \cancel{\mathcal{R}_{klji}} + \mathcal{R}_{ljki} = 0$$

$$\cancel{\mathcal{R}_{klij}} + \cancel{\mathcal{R}_{likj}} + \mathcal{R}_{iklj} = 0$$

$$\cancel{\mathcal{R}_{lijk}} + \cancel{\mathcal{R}_{ijlk}} + \mathcal{R}_{jlik} = 0.$$

Also $\mathcal{R}_{ljki} = \mathcal{R}_{jlik}$ with two swaps. So adding these together reduces to

$$2\mathcal{R}_{kijl} + 2\mathcal{R}_{ljki} = 0$$

$$\mathcal{R}_{kijl} - \mathcal{R}_{jlik} = 0,$$

which is exactly what we wanted to prove with different letters.

Perfect!



Now back to tensors.

Definition II.1.4

If T is an r -tensor, then ∇T is a $(r+1)$ -tensor defined as, e.g. for $r = 3$, there are two

notations $\nabla T(X, Y, Z, W)$ and $(\nabla_X T)(Y, Z, W)$. They both denote

$$\nabla T(X, Y, Z, W) = (\nabla_X T)(Y, Z, W) := X(T(Y, Z, W)) - T(\nabla_X Y, Z, W) - T(Y, \nabla_X Z, W) - T(Y, Z, \nabla_X W)$$

One must prove that this is a tensor.

Exercise II.1.3

Check that this is an $(r+1)$ -tensor, at least when $r = 3$.

Exercise II.1.4

Check that metric compatibility of the Levi-Civita connection is exactly the statement that $\nabla g = 0$.

So what is the Second Bianchi Identity?

Proposition II.1.3 (Second Bianchi Identity)

We have the following

$$(\nabla_X \mathcal{R})(Y, Z, W, T) + (\nabla_Y \mathcal{R})(Z, X, W, T) + (\nabla_Z \mathcal{R})(X, Y, W, T) = 0$$

Proof. It suffices to check on normal coordinates at a point, since these are all tensors. In these coordinates we have $\nabla_{\partial_{x_i}} \partial_{x_j}(0) = 0$, since the Christoffel symbols vanish at 0. Thus,

$$(\nabla_{\partial_{x_i}} \mathcal{R})(\partial_{x_j}, \partial_{x_k}, \partial_{x_\ell}, \partial_{x_s})(0) = \partial_{x_i}(\mathcal{R}(\partial_{x_j}, \partial_{x_k}, \partial_{x_\ell}, \partial_{x_s}))(0).$$

We can't use the nice expression for $\mathcal{R}_{ijkl}$ because that only holds at zero. So we have to use the bad expression and differentiate.

$$\mathcal{R}(\partial_{x_j}, \partial_{x_k}, \partial_{x_\ell}, \partial_{x_s}) = \sum_u (\partial_{x_j} \Gamma_{k\ell}^u - \partial_{x_k} \Gamma_{j\ell}^u) g_{us} + \sum_{u,q} (\Gamma_{k\ell}^u \Gamma_{ju}^q - \Gamma_{j\ell}^u \Gamma_{ku}^q) g_{qs}.$$

Now $\Gamma_{k\ell}^u(0) = 0$, so its squares have vanishing derivative. Similarly g_{qs} has vanishing derivative. Thus there's actually only one term to worry about, namely

$$\begin{aligned} (\partial_{x_i} \mathcal{R})(\partial_{x_j}, \partial_{x_k}, \partial_{x_\ell}, \partial_{x_s})(0) &= \sum_u \partial_{x_i} \partial_{x_j} \Gamma_{k\ell}^u(0) - \partial_{x_i} \partial_{x_k} \Gamma_{j\ell}^u(0) \cdot g_{us}(0) \\ &= \partial_{x_i} \partial_{x_j} \Gamma_{k\ell}^s(0) - \partial_{x_i} \partial_{x_k} \Gamma_{j\ell}^s(0) \end{aligned}$$

Now in the theorem we're cyclically permuting X, Y, Z . Denoting these as $a_{ijk} - a_{ikj}$, taking the cyclic permutations of all of these will yield

$$a_{ijk} - a_{ikj}$$

$$a_{jki} - a_{jik}$$

$$a_{kij} - a_{kji}.$$

Now since $\partial_i \partial_j f - \partial_j \partial_i f = 0$ for all i, j, f , we have $a_{ijk} - a_{jik} = 0$. Thus these terms cancel as

$$\begin{aligned} & \cancel{a_{ijk}} - \cancel{a_{ikj}} \\ & \cancel{a_{jki}} - \cancel{a_{jik}} \\ & \cancel{a_{kij}} - \cancel{a_{kji}}. \end{aligned}$$



Last Time: (M^n, g) a Riemannian manifold, we defined the Riemannian curvature tensor

$$\mathcal{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M).$$

We proved the 1st/2nd Bianchi identities and the basic symmetries

$$\mathcal{R}(X, Y, Z, W) = -\mathcal{R}(Y, X, Z, W) = \mathcal{R}(Y, X, W, Z) = \mathcal{R}(W, Z, Y, X)$$

Example II.1.5

$(\mathbb{R}^n, g_0)$. In this case $\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = 0$ for all i, j at all $x \in \mathbb{R}^n$, and so $\mathcal{R}(X, Y, Z, W) = 0$ for all vector fields X, Y, Z, W . We might also write this as $\mathcal{R}(g_0) = 0$.

Comment: Fix p , can we choose coordinates so that $g_{ij} = \delta_{ij}$ on U ? If so, then $\mathcal{R} = 0$ on U . We will see that all compact manifolds except one family (the flat tori) don't have this about each point.

We'll compute the curvature tensor by the end of class for (S^n, g_{S^n}) and on (B^n, hyp) , $\text{hyp} = \frac{4}{(1-|x|^2)^2} g_0$. Everyone else calls this $(\mathbb{H}^n, \text{hyp})$. The unit ball model is nice because you get such an explicit description of the metric.

Definition II.1.5

Fix $x \in M$, $P \subseteq T_p M$ a 2-plane. Then the sectional curvature of P at x is

$$K_x(P) := \mathcal{R}(E_1, E_2, E_2, E_1),$$

where E_1, E_2 is an orthonormal basis for P .

Exercise II.1.6

Show this definition of $K_x(P)$ is independent of the basis.

Lemma II.1.4 (Sectional Curvatures determine $\mathcal{R}$)

Suppose $\mathcal{R}_1, \mathcal{R}_2 : T_x M \times T_x M \times T_x M \times T_x M \rightarrow \mathbb{R}$ are both multilinear maps with the same symmetries as the curvature tensor. Namely, the 1st Bianchi identity, and the basic symmetries. If $K_1(P) = K_2(P)$ for all 2-planes $P \subseteq T_x M$, then $\mathcal{R}_1 = \mathcal{R}_2$.

Corollary II.1.5

Suppose that $K_x(P) = K(x)$ for all $x \in M$ and all 2-planes $P \subseteq T_x M$. Then we must have

that

$$\mathcal{R}(X, Y, Z, W) = K(x)(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)).$$

Proof of Corollary. Define the tensor $T(X, Y, Z, W) = K(x)(g(X, W)g(Y, Z) - g(X, Z)g(Y, W))$. Check that it has the symmetries of the curvature tensor and that this has sectional curvatures all $K(x)$. 

Corollary II.1.6

If (M^2, g) is a surface, then the only 2-plane is $K_x(T_x M) =: K(x)$. Hence

$$\mathcal{R}(X, Y, Z, W) = K(x)(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle).$$

The function $K : M \rightarrow \mathbb{R}$ is called the Gaussian Curvature

There are two interesting ways of defining curvature of a surface classically

- We have (M^2, g) , compute $K(x)$, which is a complicated expression depending on $g_{ij}, \partial_k g_{ij}$ and $\partial_k \partial_\ell g_{ij}$.
- When $(M^2, g) \hookrightarrow (\mathbb{R}^3, g_0)$ is an isometric embedding. Compute the principal curvatures λ_1, λ_2 of two curves passing through a point. Then we can compute $K_{\text{prin}}(x) = \lambda_1(x)\lambda_2(x)$. These are computed by intersecting the surface with carefully chosen planes in $\mathbb{R}^3$

Notice: This includes information about how M^2 sits inside of $\mathbb{R}^3$.

A priori, we do not expect these to be the same. $\lambda_1(x)$ will change if we change how M^2 sits in space. Consider a curving a piece of paper. The principal curvature $\lambda_1(x)$ can change from 0 to positive.

Theorem II.1.7 (Gauss's Theorem Egregium)

The product $K_{\text{prin}}(x) = \lambda_1(x)\lambda_2(x)$ is intrinsic, and in fact equal to $K(x)$. This theorem is absolutely incredible (egregium means awesome in Latin). It is the *birth of Riemannian geometry*.

Proof of Lemma II.1.4. Check the following: If $P = \text{span}(u, v)$, u, v not necessarily orthogonal, then we have

$$|u \wedge v|^2 := |u|^2 |v|^2 - \langle u, v \rangle^2$$

$$K_x(P) = \frac{R(u, v, v, u)}{|u \wedge v|^2},$$

This is an easy check using properties of tensors. Thus, we have $\mathcal{R}_1(u, v, v, u) = \mathcal{R}_2(u, v, v, u)$ for all $u, v \in T_x M$, by the assumption that $K_1(P) = K_2(P)$ for all $P \subseteq T_x M$.

The basic trick is to “mix” the tangent planes. Here’s what we do, let $\{e_j\}$ be a system of orthonormal vectors, and look at for j, s, k, m (not necessarily distinct), the following function.

$$f_i(\alpha, \beta) = \mathcal{R}_i(e_j + \alpha e_s, e_k + \beta e_m, e_k + \beta e_m, e_j + \alpha e_s)$$

We want to show $\mathcal{R}_1(e_j, e_k, e_m, e_s) = \mathcal{R}(e_j, e_k, e_m, e_s)$ (or some permutation). We know that $f_1(\alpha, \beta) = f_2(\alpha, \beta)$ for all α, β by assumption. But wait! This means the derivatives agree!

$$\partial_{\alpha, \beta}^2 f_1(0, 0) = \partial_{\alpha, \beta}^2 f_2(0, 0).$$

With a computation (use multilinearity) we achieve

$$\partial_{\alpha, \beta}^2 f_i(0, 0) = 2\mathcal{R}_i(e_s, e_k, e_m, e_j) + 2\mathcal{R}_i(e_s, e_m, e_k, e_j).$$

The only terms which will appear in the computation are those involving both an α and a β , and each only once. Otherwise the derivative at zero will be zero. So we see

$$\begin{aligned} \partial_{\alpha, \beta}^2 f_i(\alpha, \beta) &= \partial_{\alpha, \beta}^2 (\alpha\beta \mathcal{R}_i(e_j, e_k, e_m, e_s) + \alpha\beta \mathcal{R}_i(e_s, e_m, e_k, e_j) \\ &\quad + \alpha\beta \mathcal{R}_i(e_j, e_m, e_k, e_s) + \alpha\beta \mathcal{R}_i(e_s, e_k, e_m, e_j)). \\ &= 2\mathcal{R}_i(e_s, e_k, e_m, e_j) + 2\mathcal{R}_i(e_s, e_m, e_k, e_j). \end{aligned}$$

Where we’ve used the assumed curvature symmetries and the multilinearity of the tensor.

Defining

$$T(e_s, e_k, e_m, e_j) := \mathcal{R}_1(e_s, e_k, e_m, e_j) - \mathcal{R}_2(e_s, e_k, e_m, e_j).$$

Then the equality of $\partial_{\alpha, \beta}^2 f_i(\beta)$ gives

$$T(e_s, e_k, e_m, e_j) = T(e_m, e_s, e_k, e_j).$$

Swapping s and k check $T(e_m, e_s, e_k, e_j) = T(e_k, e_m, e_s, e_j)$.

Using the First Bianchi identity

$$T(e_s, e_k, e_m, e_j) + T(e_k, e_m, e_s, e_j) + T(e_m, e_s, e_k, e_j) = 0$$

$$3T(e_s, e_k, e_m, e_j) = 0,$$

and so $T = 0$, which is exactly what we wanted to prove.



Claim

Let (S^n, g_{S^n}) . There exists a constant c_1 so that

$$\mathcal{R}(X, Y, Z, W) = c_1(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle).$$

In other words (S^n, g_{S^n}) has constant sectional curvature. Look at the north pole N and e_1, e_2 orthonormal at N . Now consider some 2-plane P at a point $x \in S^n$.

Check we can find $A : S^n \rightarrow S^n$ (namely $A \in O(n+1)$) so that A is in $\text{Isom}(S^n)$, so that $A(N) = x$ and $\text{span}(A(e_1, e_2)) = P$. Then we'll have

$$K_N(\text{span}(e_1, e_2)) = K_x(P).$$

First check: You can do this for $x = N$ for any two plane P , then just use transitivity of $O(n+1)$ on the sphere.

Later on we'll show $c_1 = 1$.

Using the same ideas, we can get a similar result about hyperbolic space.

Exercise II.1.7

Check that $\text{Isom}(\mathbb{H}^n) = \text{Conf}(B^n)$ are such that Given any $x, y \in \mathbb{H}^n$, $P_1 \subseteq T_x \mathbb{H}^n$, $P_2 \subseteq T_y \mathbb{H}^n$ both 2-planes, there exists an isometry $T \in \text{Isom}(\mathbb{H}^n)$ so that $T(x) = y$ and $dT_x(P_1) = P_2$.

Thus

$$\mathcal{R}(X, Y, Z, W) = c_{-1}(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle).$$

In other words, hyperbolic space has constant sectional curvature. Later we'll show $c_{-1} = -1$.

Goal: Understand $\mathcal{R}$. But this is way way too hard. A matrix is like a square, a 3-tensor is like a cube, a 4-tensor is like a tesseract. A tesseract of numbers with symmetries... and symmetries on the derivatives. No way. So instead we simplify by taking traces.

Definition II.1.6

We define the Ricci Curvature as $\text{Ric} : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ where

$$\text{Ric}(X, Y)(x) = \sum_{i=1}^n \mathcal{R}(X, E_i, E_i, Y)(x),$$

where E_i is an orthonormal basis of $T_x M$. It is a simple check to see from the symmetries of the curvature tensor that

$$\text{Ric}(X, Y) = \text{Ric}(Y, X).$$

In dimensions two and three it turns out that this nails it. But even it is too complicated when in higher dimensions, so we take *another* trace.

Definition II.1.7

We define the Scalar Curvature as $S(g) : M \rightarrow \mathbb{R}$ as a smooth function

$$S(g)(x) := \sum_{j=1}^n \text{Ric}(E_j, E_j)(x) = \sum_{i,j=1}^n \mathcal{R}(E_j, E_i, E_i, E_j)(x).$$

We should think of this as averaging (well summing, but $\frac{1}{n(n-1)}$) the sectional curvatures of all of the two-planes at x .

The Einstein Equation, from general relativity, is

$$\text{Ric}(g) - \frac{S(g)}{2}g - \lambda = T,$$

where T is something called the energy-matter tensor.

Interesting story about this equation. It's actually Hilbert's equation first. Einstein worked out the story for special relativity in Minkowski spacetime. He then gave lectures at Göttingham, trying to figure out what the general equation would be. But he couldn't work it out. He knew it should be second order (aka depending on curvatures)

Shortly after these lectures, Hilbert (who was at Göttingham at the time) sent a letter to Einstein about how he found the correct equation from first principles. Namely using that it should have geometric meaning, he calculated the critical points of the Hilbert-Einstein functional $g \mapsto \int_M S(g) \, dV$ as $\text{Ric}(g) - \frac{S(g)}{2}g = 0$.

Einstein then sent a letter back, saying that he had come up with the equation the day before getting this letter. They both said they were submitting papers about it. But Hilbert's got held up in refereeing... possibly by Einstein. So Einstein got the credit.

Motivated by this

THE QUESTION OF RIEMANNIAN GEOMETRY: Given M^n , can you find a metric g with sectional curvature, Ricci curvature, or scalar curvature to be constant, > 0 , or < 0 .

THE SECOND QUESTION: Suppose that $T = 0$ above, and $\lambda = 0$. Then $\text{Ric} - \frac{S(g)}{2}g = 0$. Fact: In this case $\text{Ric}(g) = \lambda g$ and $S(g) = 2\lambda$. Metrics which satisfy $\text{Ric}(g) = \lambda g$ are called Einstein.

Yau got his fields medal for solving the Calabi conjecture. Which essentially boils down to showing that

$$\{[x : y : z : w] \mid x^4 + y^4 + z^4 + w^4 = 0\}$$

has a metric with $\text{Ric} = 0$.

Thurston got his fields medal for the following. He showed that a surface bundle over S^1 whose gluing map is Pseudo-Anosov admits a metric with $\text{Ric}(g) = -2g$.

Likewise, Perelmans proof of the Poincaré conjecture goes through showing that if $\pi_1(M^3) = 0$, then there exists a metric g with $\text{Ric}(g) = 2g$. Cartan's theorem, which we'll prove, shows that this implies the Poincaré conjecture.

Last Time: Curvature

- Riemannian curvature tensor, $\mathcal{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$.
- Too complicated, so take the trace $\text{Ric}(g)$, Ricci Curvature, defined by

$$\text{Ric}(g) : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)(X, Y) \mapsto \sum_{i=1}^n R(X, E_i, E_i, Y)$$

for E_i an orthonormal basis.

- Again too complicated, so take $S(g)$, the scalar curvature, defined by

$$S(g) : M \rightarrow \mathbb{R}$$

$$p \mapsto \sum_{j=1}^n \text{Ric}(g)(E_j, E_j)(p),$$

for E_j an orthonormal basis.

If $\dim M^n = 2$, aka a surface, these all express the same information. If $K(p) = K(T_p M)$ is the sectional curvature at $p \in M$ then

$$\mathcal{R}(X, Y, Z, W)(p) = K(p)(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle)(p)$$

$$\text{Ric}(g)(X, Y)(p) = K(p)\langle X(p), Y(p) \rangle$$

$$S(g)(p) = 2K(p).$$

Great! We should recall what this will mean in the coordinates for the metric

Curvatures/Metrics	Linear Model
Metric g	$(g_{ij})_{i,j=1}^n \begin{pmatrix} g_{11} & \cdots & g_{n1} \\ \vdots & \ddots & \vdots \\ g_{1n} & \cdots & g_{nn} \end{pmatrix}$
$\mathcal{R}, \mathcal{R}_{ijkl}$	$\partial_i \partial_j g_{kl}, \text{Hess } g$
$(\text{Ric})_{ij}$	$-\frac{1}{2} \Delta g_{ij}$ (Laplacian), $\sum_k \partial_k \partial_k g_{ij}$
$S(g)$	$\Delta \text{tr } g.$

Exercise II.1.8

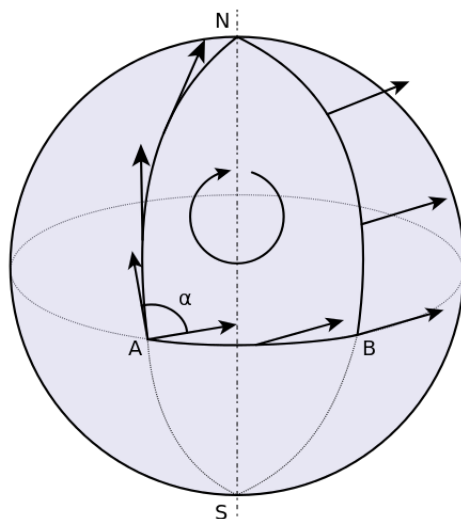
$$\nabla g = 0, \nabla^2 g = 0.$$

There is a formal sense in which the above is “nonsense.” Namely, $\Delta g = 0$, since $\Delta g = \text{tr } \nabla^2 g = 0$. However, it is not true that Δg is given by Δg_{ij} in coordinates. It is a painful exercise in deeply understanding the Levi-Civita applied to a tensor to see why these are not the same.

Remark II.1.3

Hamilton in '82 was working on the heat equation for $u : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, the heat is described by $\partial_t u = \Delta u$, where $\{u_t\} : \Omega \rightarrow \mathbb{R}$ is a family of heat functions on a domain Ω .

Hamilton thought something similar must be true for metrics. At first he tried $\partial_t g_t = \Delta g$, but this is trivial. So instead, he wrote down $\partial_t g_t = -2 \text{Ric}(g_t)$, thinking of the linear model. This flow became useful in solving the Poincaré conjecture via Ricci Flow due to Perelman.

FIGURE 3. Parallel Trasnport along a loop on S^2 **Definition II.1.8**

g is called Einstein if $\text{Ric}(g) = \lambda g$ for $\lambda \in \mathbb{R}$. Likewise g is said to have constant scalar curvature if $S(g) = \lambda \in \mathbb{R}$.

Most special metrics have one of these properties. We have that

$$\text{constant sectional curvature} \xrightarrow{\quad} \text{Einstein} \xrightarrow{\quad} \text{constant scalar curvature}$$

$\xleftarrow{n=3}$ $\xleftarrow{n=2}$

The converses only hold in the special cases of $n = 3, n = 2$.

III. Parallel Transport

III.1. Definitions

Definition III.1.1

Let $\gamma : (0, 1) \rightarrow M$ be a smooth curve. Take $X \in T_{\gamma(0)}M$, we say $X(t) \in T_{\gamma(t)}M$ is parallel (or parallel transported) if

$$\frac{DX}{dt} = \nabla_{\gamma'(t)} X(t) = 0.$$

For an example of parallel transport, see Figure 3. This picture contains both a warning and a wonderful miracle. The vector after parallel transporting around the loop is not the vector we saw originally, this is a phenomenon called holonomy. Thus crucially $X(t)$ depends on t instead of $\gamma(t)$. We can have $\gamma(0) = \gamma(1)$ but $X(0) \neq X(1)$.

Remark III.1.1

If $X(0)$ is parallel to $\gamma'(0)$, then $X(t)$ may not be parallel to $\gamma'(t)$ in general. Consider a

circle $\gamma(t) = (\cos(t), \sin(t)) \subseteq \mathbb{R}^2$ and $\gamma'(0) = (0, 1) \in T_{\gamma(0)}\mathbb{R}^2$. Then the parallel transport with respect to $\nabla^{\mathbb{R}^2}$ is just constantly $X(t) = (0, 1) \in T_{\gamma(t)}$. This is not parallel to $\gamma'(t)$.

Notice that if you parallel transport with respect to ∇^{S^1} , then you get exactly $\gamma'(t)$. The parallel transport critically depends on both the metric, the curve, and the initial vector.

In general, the parallel transport of $X(0)$ parallel to $\gamma'(0)$ along γ continues to be parallel to γ if γ is a geodesic. Consider parallel transporting along a great circle of sphere.

Lemma III.1.1 (Parallel Transport Makes Sense)

Let γ be a smooth curve, for all $v \in T_{\gamma(0)}M$, there exists a unique $X(t) \in T_{\gamma(t)}M$ so that $\frac{DX}{dt} = 0$, $X(0) = v$.

Proof. Write $\{v_i(t)\}_{i=1}^n$ so that $\text{span}(v_i(t)) = T_{\gamma(t)}$ and $t \mapsto v_i(t)$ is smooth. Then

$$\begin{aligned} X(t) &:= \sum_{i=1}^n a_i(t)v_i(t) \\ \frac{DX}{dt}(t) &= \sum_{i=1}^n a'_i(t)v_i(t) + a_i(t)\frac{Dv_i}{dt}(t) \\ \frac{Dv_i}{dt}(t) &= \sum_{j=1}^n b_{ij}(t)v_j(t) \\ \frac{DX}{dt}(t) &= \sum_{i=1}^n a'_i(t)v_i(t) + \sum_{i=1}^n \sum_{j=1}^n a_i(t)b_{ij}(t)v_j(t) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n a'_i(t)b_{ij}(t) + a_i(t)b_{ij}(t) \right) v_j(t). \end{aligned}$$

Thus

$$\frac{DX}{dt} = 0 \iff a'_j(t) + \sum_{i=1}^n a_i(t)b_{ij}(t) = 0,$$

for all $j = 1, \dots, n$. This is a Linear system of ODEs, thus there is a unique solution for all time given $a_j(0)$ for $j = 1, \dots, n$. 

Using parallel transport, we can give a classical interpretation of curvature. Let u, v be an orthonormal basis of T_pM , and let $v(t)$ trace out a box in T_pM . Then let $\alpha_t = \exp_p(v(t))$. We then can define

$$P_t : T_pM \rightarrow T_pM$$

$$P_t(X) = \text{parallel transport along } \alpha_t \text{ following direction}$$

If $M^n = \mathbb{R}^n$, g Euclidean, then $P_t = \text{Id}$.

It is a long computation to see that $P_t(X) = X + t^2 R(u, v)X + O(t^3)$. Or in other words

$$R(u, v) \simeq \frac{P_t - \text{Id}}{t^2}.$$

For an example of this parallel transport map, see again ??.

III.2. Jacobi Fields

Our next aim is to somehow measure how geodesics spread out. Let $v, w \in T_p M$, and consider the map

$$F : [0, 1] \times (-\varepsilon, \varepsilon) \rightarrow M$$

$$F(t, s) = \exp_p(t(v + sw))$$

When $s = 0$, this is just the geodesic $\exp_p(tv)$. For any fixed s , in fact $t \mapsto \exp_p(t(v + sw))$ is a geodesic. Let $\gamma(t) := F(t, 0)$. We now call

$$\mathcal{J}(t) := \frac{\partial F}{\partial s}(t, 0)$$

We can also consider

$$\mathcal{J}'(t) = \frac{D\mathcal{J}}{dt} = \nabla_{\gamma'} \mathcal{J}$$

$$\mathcal{J}''(t) = \frac{D\mathcal{J}'}{dt} = \nabla_{\gamma'} \mathcal{J}'.$$

These actually have a super special relationship

Proposition III.2.1

For such a vector field $\mathcal{J}$ along γ ,

- (i) $\mathcal{J}(0) = 0, \mathcal{J}'(0) = w$.
- (ii) $\mathcal{J}'' + R(\mathcal{J}, \gamma')\gamma' = 0$ for all t . This equation is called the Jacobi equation, and solutions to this equation are called Jacobi vector fields.

Proof of (i). We see that $F(0, s) = \exp_p(0) = p$ for all s . Hence $\mathcal{J}(0) = \frac{\partial F}{\partial s}(0, 0) = 0$. Similarly, we see that

$$\mathcal{J}(t) = \frac{\partial F}{\partial s}(t, 0) = d(\exp_p)_{tv}(tw) = t d(\exp_p)_{tv}(w).$$

We must differentiate with respect to t , so

$$\mathcal{J}'(t) = d(\exp_p)_{tv}(w) + t \frac{d}{dt} (\text{complicated}).$$

Computing at $t = 0$, we get

$$\mathcal{J}'(t) = d(\exp_p)_0(w) + 0 \cdot (\text{complicated}) = w,$$

since $d(\exp_p)_0 = \text{Id}$.



Now let's switch to a general map $F : [0, 1] \times (-\varepsilon, \varepsilon) \rightarrow M$ (not necessarily of the form above). We're going to do something important, and think carefully about curvature. We know that

$$\left[\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right] = \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0.$$

Hence for all vector fields X ,

$$\nabla_{\partial F/\partial s} \nabla_{\partial F/\partial t} X = \nabla_{\partial F/\partial t} \nabla_{\partial F/\partial s} X + R\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right) X.$$

Likewise, we have

$$\nabla_{\partial F/\partial t} \frac{\partial F}{\partial s} = \nabla_{\partial F/\partial s} \frac{\partial F}{\partial t},$$

since the Levi-Civita connection is torsion free.

Proof of (ii). We know since $t \mapsto F(s, t) = \exp_p(t(v + sw))$ is a geodesic for any fixed t , hence

$$\nabla_{\partial F/\partial t} \frac{\partial F}{\partial t} = 0.$$

Therefore

$$\nabla_{\partial F/\partial s} \nabla_{\partial F/\partial t} \frac{\partial F}{\partial t} = 0.$$

Great! Now we just commute things

$$\begin{aligned} \nabla_{\partial F/\partial t} \nabla_{\partial F/\partial s} \frac{\partial F}{\partial t} + R\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right) \frac{\partial F}{\partial t} &= 0 \\ \nabla_{\partial F/\partial t} \nabla_{\partial F/\partial t} \frac{\partial F}{\partial s} + R\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right) \frac{\partial F}{\partial t} &= 0. \end{aligned}$$

At $s = 0$, we have $\frac{\partial F}{\partial t}(t, 0) = \gamma'(t)$ and $\frac{\partial F}{\partial s}(t, 0) = \mathcal{J}(t)$. Hence

$$\begin{aligned} \nabla_{\gamma'} \nabla_{\gamma'} \mathcal{J} + R(\mathcal{J}, \gamma') \gamma' &= 0 \\ \mathcal{J}'' + R(\mathcal{J}, \gamma') \gamma' &= 0. \end{aligned}$$



Lemma III.2.2

Fix $p \in M$ and a geodesic $\gamma : [0, \ell] \rightarrow M$ a geodesic. Then for every pair of vectors $v, w \in T_p M$ there exists a unique vector field $\mathcal{J}(t) \in T_{\gamma(t)} M$ so that

$$\mathcal{J}''(t) + R(\mathcal{J}, \gamma') \gamma' = 0,$$

and $\mathcal{J}(0) = v, \mathcal{J}'(0) = w$.

Proof. $\mathcal{J}'' + R(\mathcal{J}, \gamma') \gamma'$ is a second order linear system of ODEs and so we have a unique solution given $\mathcal{J}(0), \mathcal{J}'(0)$. If you want more details, see do Carmo.



Last Time: We rederived the Jacobi equation. For $\gamma : (0, \ell) \rightarrow (M^n, g)$ a geodesic, we said a vector field $\mathcal{J}$ along γ is a Jacobi vector field if

$$\mathcal{J}''(t) + R(\mathcal{J}(t), \gamma'(t))\gamma'(t) = 0$$

for all t . This is called the Jacobi equation. We cited do Carmo (see [docarmo]) to see that given $v, w \in T_{\gamma(0)}M$, there exists a unique Jacobi vector field $\mathcal{J}$ with $\mathcal{J}(0) = v$, $\mathcal{J}'(0) = w$.

Corollary III.2.3

If $\mathcal{J}(0) = 0$, then $\mathcal{J}(t) = \frac{\partial F}{\partial s}(t, 0)$ where

$$\begin{aligned} F : (0, \ell) \times (-\varepsilon, \varepsilon) &\rightarrow M \\ F(t, s) &= \exp_{\gamma(0)}(t(\gamma'(0) + s\mathcal{J}'(0))). \end{aligned}$$

In fact,

$$\mathcal{J}(t) = d(\exp_{\gamma(0)})_{t\gamma'(0)} \cdot (t\mathcal{J}'(0))$$

Remark III.2.1

If $\mathcal{J}(0) = \gamma'(0)$ and $\mathcal{J}'(0) = 0$ then $\mathcal{J}(t) = \gamma'(t)$. Likewise, if $\mathcal{J}(0) = 0$ and $\mathcal{J}'(0) = \gamma'(0)$ then $\mathcal{J}(t) = t\gamma'(t)$.

We're going to use the Jacobi equation to compute the curvature of the sphere (S^n, g_{S^n}) . We determined that it has constant sectional curvature by symmetry. We could write out the metric explicitly, but we know the geodesics, so let's use that information!

Example III.2.1

Take $u, w \in T_N S^n$ for N the north pole, $u \perp w$, $|u| = |w| = 1$. We write

$$F(t, s) = \exp_N(t(u + sw)) = \cos(t) \cdot N + \sin(t) \cdot \frac{u + sw}{\sqrt{1 + s^2}}.$$

We can then just differentiate with respect to s ,

$$\mathcal{J}(t) = \frac{\partial F}{\partial s}(t, 0) = \sin(t)w$$

$$\mathcal{J}'(t) = \cos(t)w$$

$$\mathcal{J}''(t) = -\sin(t)w.$$

Therefore $\mathcal{J}'' + \mathcal{J} = 0$. And thus

$$R(\mathcal{J}, \gamma')\gamma' = \mathcal{J}.$$

Therefore, we have that

$$\sin(t)R(w, \gamma')\gamma' = \sin(t)w$$

$$R(w, \gamma'(0))\gamma'(0) = w$$

$$R(w, u)u = w.$$

Hence

$$\mathcal{R}(w, u, u, w) = g(w, w) = 1,$$

and so the sectional curvature of S^n is 1.

The same argument on $(\mathbb{H}^n, g_{\mathbb{H}^n})$ would show sectional curvature is -1 . In general, we have the following form for the Jacobi equation.

Exercise III.2.2

When (M^n, g) has constant sectional curvature K , when $\mathcal{J}(0) = 0, \mathcal{J}'(0) \perp \gamma'(0)$ the Jacobi equation becomes

$$\mathcal{J}'' + K\mathcal{J} = 0.$$

Check this.

Given the exercise, we can just solve this equation! Pick $w \in T_{\gamma(0)}M$, $w \perp \gamma'(0)$ where γ is a geodesic. Let $w(t)$ be the parallel transport of w along γ (note $|w(t)| = |w|$ for all t). Then for $\mathcal{J}(0) = 0, \mathcal{J}'(0) = w$, we have

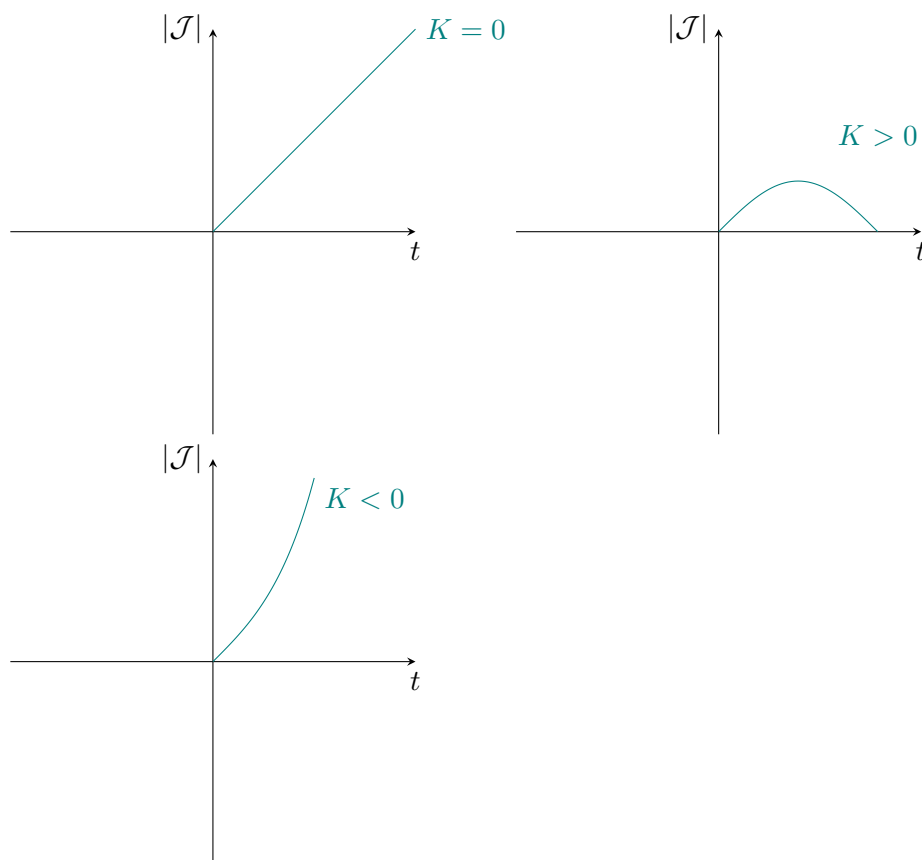
$$\mathcal{J}(t) = \begin{cases} \frac{\sin(t\sqrt{K})}{\sqrt{K}}w(t) & \text{if } K > 0 \\ tw(t) & \text{if } K = 0 \\ \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}}w(t) & \text{if } K < 0 \end{cases}$$

One should check for this choice, $\mathcal{J}'' + K\mathcal{J} = 0$ and $\mathcal{J}(0) = 0, \mathcal{J}'(0) = w$. Thus by uniqueness, this is the Jacobi field with those initial conditions.

This provides an incredible interpretation of the curvature in terms of geodesics. We'll put this together with a bit of information.

- $|\mathcal{J}(t)|$ measures the rate of spread of geodesics.

This gives us three great cases



In general, if $\mathcal{J}(0) = 0$, $\mathcal{J}'(0) = w$ (perpendicular to $\gamma'(0)$), then

$$|\mathcal{J}(t)| = t \left(1 - \frac{t}{3} \mathcal{R}(w, \gamma'(0), \gamma'(0), w) + O(t^2) \right),$$

for t small (see [docarmo]).

IV. Hopf-Rinow Theorem

Definition IV.0.1

We call (M^n, g) geodesically complete if for all $p \in M^n$, the map $\exp_p : T_p M \rightarrow M$ is well-defined, i.e. for all $v \in T_p M$, geodesic γ with $\gamma'(0) = v$, $\gamma(0) = p$ exists for all $t \geq 0$.

Recall IV.0.1

We have a distance function $d : M \times M \rightarrow (0, +\infty)$, with

$$d(p, q) = \inf \{ \text{length}(\gamma) \mid \gamma \text{ is a } C^1 \text{ curve connecting } p \text{ to } q \}.$$

d is a distance, i.e. triangle inequality, symmetry, and $d(p, q) \geq 0$ with equality if and only if $p = q$.

Exercise IV.0.2

Check that the topology induced by d is the topology on M . Hence $x \mapsto d(x, p)$ is in $C^0(M)$ for all $p \in M$.

Theorem IV.0.1 (Hopf-Rinow)

Fix $p \in M$, then

- (1) The following are equivalent
 - (a) $\exp_p$ is geodesically complete.
 - (b) $\exp_p : T_p M \rightarrow M$ is well-defined.
 - (c) Closed and bounded sets are compact.
 - (d) Cauchy sequences converge (i.e., (M^n, d) is a complete metric space).
- (2) If (M^n, g) is geodesically complete, then given any $p, q \in M$, there exists a geodesic γ connecting p to q with $d(p, q) = \text{length}(\gamma)$.

Proof of (2). Fix $p, q \in M$ with $d(p, q) = r$. Pick $\delta > 0$ small so that $B := B_\delta(p)$ is contained in a normal neighborhood (i.e. geodesics are unique inside of $B_\delta(p)$). Now pick a point $y \in \partial B$ so that

$$d(q, y) = \min_{y' \in \partial B} d(q, y').$$

The natural candidate now is to pick $v \in T_p M, |v| = 1$ so that $\exp_p(\delta v) = y$ and consider $\gamma(t) = \exp_p(tv)$. Now here's the relevant question: If $r = d(p, q)$, is $\gamma(r) = q$?

Consider the set

$$A = \{t \in [0, r] \mid d(q, \gamma(t)) = r - t\}.$$

We know that $0 \in A$. We want to show that $\max A = r$ (note that A is closed/bounded, so it has a maximum). This will tell us that $d(q, \gamma(r)) = 0$, so $q = \gamma(r)$.

How will we do this? Well assume $t_0 \in A, t_0 < r$. We'll show that there is a δ' so that $t_0 + \delta' \in A$. Great! Choose δ' so that $B' := B_{\delta'}(\gamma(t_0))$ is contained in a normal neighborhood of $\gamma(t_0)$. Pick $z \in \partial B'$ which minimizes the distance to q for a point on $\partial B'$.

Claim

$z = \gamma(t_0 + \delta')$. From this the result will follow.

The key identity is that $d(\gamma(t_0), q) = \delta' + d(z, q)$. Why? Well any curve from $\gamma(t_0)$ to q must pass through $\partial B'$, and hence any curve has length at least $\delta' + d(z', q)$ for some $z' \in \partial B'$. Thus

$$d(\gamma(t_0), q) \geq \delta' + \min_{z' \in \partial B'} d(z', q),$$

and by the triangle inequality we get the opposite inequality. Thus $d(\gamma(t_0), q) = \delta' + d(z, q)$. This implies that

$$\begin{aligned} r - t_0 &= \delta' + d(z, q) \\ d(z, q) &= r - (t_0 + \delta'). \end{aligned}$$

Great! We now have

$$\begin{aligned} d(p, z) &\leq d(p, \gamma(t_0)) + d(\gamma(t_0), z) \leq t_0 + \delta' \\ d(p, q) &\leq d(p, z) + d(z, q) \\ d(p, z) &\geq d(p, q) + d(z, q) = r - (r - t_0 + \delta') \\ &= t_0 + \delta' \end{aligned}$$

Great! But then $d(p, z) = t_0 + \delta'$. Now consider the broken curve c following γ from p to $\gamma(t_0)$ and a small geodesic from $\gamma(t_0)$ to z . Then since $d(p, z) = \text{length}(c)$.

Hence c is a geodesic, initially agreeing with γ , and so $c = \gamma$. This implies that $z = \gamma(t_0 + \delta')$.

Great! We now have by the argument above that $d(z, q) = r - (t_0 + \delta')$ and $z = \gamma(t_0 + \delta')$, so $t_0 + \delta' \in A$. 

Notice that the proof of (2) only relied on the fact that $\exp_p$ was well-defined.

Proof of (1). It is clear that (a) $\implies$ (b) by definition. Lets complete the circle.

- We show (b) $\implies$ (c). Take Ω to be a closed and bounded set. Thus $\Omega \subseteq B_r(q)$. By increasing the radius, we can take $\Omega \subseteq B_R(p)$ (take $R = d(p, q) + r + 1$). But wait!

$$\exp_p^{-1}(B_R(p)) \subseteq B_R(0)$$

We use part (2) to show this inclusion, noting that this only used well-definedness of $\exp_p$. Thus this set is bounded, and so $\exp^{-1}(\Omega)$ is closed and bounded, hence compact in $T_p M \cong \mathbb{R}^n$.

Thus $\Omega = \exp(\exp^{-1}(\Omega))$ is compact.

- (c) $\implies$ (d) is general topology. Suppose we have a Cauchy sequence (p_n) , and let $C = \{p_n\}$. We see that C is closed and bounded, and hence compact by assumption, and so p_n has a convergent subsequence (by sequential compactness in a metric space). It is a simple real analysis exercise that a Cauchy sequence with a convergent subsequence actually converges itself.
- (d) $\implies$ (a). Let $\gamma : [0, t_0] \rightarrow M$ be some geodesic. Consider $d(\gamma(t), \gamma(s)) \leq |t - s|$. Thus we can take a sequence $t_n \rightarrow t_0$ and $\gamma(t_n)$ will be Cauchy, and thus converge to some p by assumption.

Taking a normal neighborhood of p of size δ , we see that there is a geodesic connecting $\gamma(t_n)$ to p for p large enough, and this geodesic must agree with γ for some time. This implies that γ is defined for time $t_0 + \delta$, and we continue to extend this way.



We wrote before that if $\mathcal{J}(0) = 0$, then

$$\mathcal{J}(t) = d(\exp_p)_{t\gamma'(0)}(t\mathcal{J}'(0)).$$

Definition IV.0.2

If γ is a geodesic connecting p to q , we say that $\gamma(t_0)$ is a conjugate point to p if there exists a nonzero Jacobi vector field $\mathcal{J}$ along γ with $\mathcal{J}(0) = 0 = \mathcal{J}(t_0)$.

This mirrors the behavior of the north/south pole. A conjugate point is equivalent to a critical point for the exponential map. I.e.,

$$\gamma(t_0) \text{ is a conjugate point to } p \iff \ker d(\exp_p)_{t_0\gamma'(0)} \neq 0.$$

Last time: Hopf-Rinow and its proof.

Theorem IV.0.2 (Hadamard)

If (M^n, g) is simply connected, complete, and has non-positive sectional curvature then for all $p \in M$ $\exp_p : T_p M \rightarrow M$ is a diffeomorphism. In particular, M is diffeomorphic to $\mathbb{R}^n$.

This fits the mold of the central theme of geometry: Given that a manifold admits a metric satisfying certain properties, can you restrict the topological type of the manifold.

One sentence proof: “Manifolds with non-positive sectional curvature have no conjugate points”. Recall that p, q are conjugate along a geodesic γ if there exists a nontrivial $\mathcal{J}$ a jacobi field such that $\mathcal{J}(p) = 0, \mathcal{J}(q) = 0$. The canonical example is the north pole being conjugate to the south pole on the sphere.

Here’s a hard theorem in a similar theme, which was a conjecture of Hopf (and proved by him when $n = 2$).

Theorem IV.0.3 (Burago-Ivanov, 1994)

If g a metric on $T^n = S^1 \times \cdots \times S^1$ has no conjugate points then it’s flat.

Proof of Hadamard’s Theorem. First we show that $\exp : T_p M \rightarrow M$ is a local diffeomorphism. It suffices to show by the inverse function theorem that for all $v \in T_p M$, $\ker(d \exp)_v = 0$. Let $w \in T_p M$ and suppose $w \neq 0$. We want to show that $(d \exp)_v \cdot w \neq 0$. Lets set some notation / assumptions.

- Let $\gamma(t) = \exp(tv)$, and $\mathcal{J}(t) = (d \exp)_{tv}(tw)$.
- This implies that $\mathcal{J}$ is a Jacobi vector field and $\mathcal{J}(0) = 0, \mathcal{J}'(0) = w$.

We want to show that $\mathcal{J}(1) \neq 0$. In other words, that this is not a conjugate point. Set $f(t) = |\mathcal{J}(t)|^2$, we know $f(0) = 0$, and we differentiate to obtain

$$f'(t) = 2\langle \mathcal{J}'(t), \mathcal{J}(t) \rangle f''(t) = 2\langle \mathcal{J}''(t), \mathcal{J}(t) \rangle + 2\langle \mathcal{J}'(t), \mathcal{J}'(t) \rangle.$$

Now we apply the Jacobi equation

$$\mathcal{J}'' + R(\mathcal{J}, \gamma')\gamma' = 0.$$

Hence

$$\begin{aligned} f''(t) &= -2\langle R(\mathcal{J}, \gamma')\gamma', \mathcal{J} \rangle + 2\langle \mathcal{J}', \mathcal{J}' \rangle \\ &\geq 2\langle \mathcal{J}', \mathcal{J}' \rangle. \end{aligned}$$

This works because M has non-positive sectional curvature, so the first term is positive.

But wait! This means f is convex and $f(t) \geq 0$, $f(0) = 0$. Thus $f(t) = 0$ for all t or $f(t) > 0$ for all $t > 0$. The first part is impossible, because if $\mathcal{J}(t) = 0$ for all t , this contradicts $\mathcal{J}'(t) = w \neq 0$.

Great! This tells us that $\exp : T_p M \rightarrow M$ is a local diffeomorphism. There a proof in do Carmo that in fact this is a covering map.

Covering Map part

Because M is simply connected by assumption, we find that $\exp$ is a global diffeomorphism. 

Here's some related ideas.

Lemma IV.0.4

Any $F : L \rightarrow N$ which is proper and a local diffeomorphism, then it's a covering map

Exercise IV.0.3

On your homework, there exists $F : \mathbb{C} \rightarrow \mathbb{C}$, surjective, local diffeomorphism, but not injective.

Now let's explore some consequences of Hadamard's theorem.

Corollary IV.0.5

We have the following

- (1) If (M^n, g) is closed and has sectional curvature ≤ 0 , its universal cover is diffeomorphic to $\mathbb{R}^n$.
- (2) In particular, the n -sphere S^n , $\mathbb{CP}^n$, $T^m \times S^n$, none of these admit metrics with non-positive sectional curvature for $n \geq 2$.
- (3) Thus, if M^n has non-positive sectional curvature, then $\pi_k(M^n) = 0$ if $k \geq 2$.

Here's a fun consequence of (3). Take $\mathbb{H}^3/\Gamma_1, \mathbb{H}^3/\Gamma_2$ two closed hyperbolic manifolds. Then $\mathbb{H}^3/\Gamma_1 \# \mathbb{H}^3/\Gamma_2$ the connect sum, then there is no metric of non-positive sectional curvature on this space. Namely, there is a homotopically non-trivial sphere introduced by the connect sum.

Lets detail some more interesting properties.

- (1) On S^2 of course we cannot put a metric with $K(g) \leq 0$, because $\int_{S^2} K(g) dA = 4\pi$. For surfaces you can always nail things with Gauss-Bonnet.
- (2) On the other hand, on any M^n closed with $n \geq 3$, there exists g with $S(g) = -1$ (Yanabe problem, for the scalar curvature).
- (3) Gau-Yau showed on S^3 there's a metric with $\text{Ric}(g) < 0$, and someone else extended to every 3-manifold. Idea is that on S^3 – knot is a hyperbolic manifold, i.e. admits a metric with constant sectional curvature -1 . Then give an explicit description of a metric on the torus that agrees with this on the boundary.

Theorem IV.0.6 (Cartan)

Let (M^n, g) be simply connected, complete, and have constant sectional curvature $K \equiv -1, 0, 1$ then (M^n, g) is isometric to $(\mathbb{H}^n, g_{\mathbb{H}^2})$ ($\mathbb{R}^n$, Euclidean) or (S^n, g_{S^n}) .

Proof when $K = -1$. Let $K \equiv -1$, and fix $p \in M^n, O \in \mathbb{H}^n$. Then $(T_p M^n, g)$ is isometric to $(T_O \mathbb{H}^n, g_{\mathbb{H}^n})$, we'll identify them via this isometry, so $T_p M^n = T_O \mathbb{H}^n$, with $g_p = g_{\mathbb{H}^n}|_O$.

We have two maps $\exp : T_O \mathbb{H}^n \rightarrow \mathbb{H}^n$ and $\overline{\exp} : T_p M^n \rightarrow M$. For $x \in \mathbb{H}^n$, let

$$\bar{x} = F(x) = \overline{\exp} \circ \exp^{-1}(x).$$

We will show that F is an isometry. We know it is a diffeomorphism from Hadamard's theorem. Thus it suffices to check F is a local isometry.

So we must show, for $Y_1, Y_2 \in T_x \mathbb{H}^n$, that

$$g(dF_x(Y_1), dF_x(Y_2)) = g_{\mathbb{H}^n}(Y_1, Y_2).$$

Let $\gamma(t) = \exp(tv)$, $\bar{\gamma}(t) = \overline{\exp}(tv)$ where $\exp(v) = x$. Without loss of generality we can assume $Y_i \perp \gamma'(0)$ for $i = 1, 2$ (Use Gauss lemma otherwise).

Set $Y_i(t)$ parallel transport of Y_i along γ with $Y_i(1) = Y_i$, and similarly $\bar{Y}_i(t)$ parallel transport of $Y_i(0)$ along $\bar{\gamma}$.

Let $\mathcal{J}_i(t)$ be defined by

$$\mathcal{J}_i(t) := d\exp_{tv}(tY_i(0)) = \sinh tY_i(t)$$

$$\bar{\mathcal{J}}_i(t) := d\overline{\exp}_{tv}(t\bar{Y}_i(0)) = \sinh t\bar{Y}_i(t).$$

This comes from the fact that (M^n, g) has sectional curvature -1 , so these satisfy the same Jacobi equation. We see that, setting $t = 1$ above,

$$dF_x(Y_i) = d\overline{\exp}_v(d(\exp^{-1})_x(Y_i)) = d\overline{\exp}_v \frac{Y_i(0)}{\sinh 1} = \bar{Y}_i(1).$$

Therefore,

$$g(dF_x(Y_1), dF_x(Y_2)) = g(\overline{Y_1}(1), \overline{Y_2}(1)) = g(\overline{Y_1}(0), \overline{Y_2}(0)) = g(Y_1(0), Y_2(0)).$$

Hence

$$g(dF_x(Y_1), dF_x(Y_2)) = g_p(Y_1(0), Y_2(0)) = \text{hyp}(Y_1(0), Y_2(0)) = \text{hyp}(Y_1(1), Y_2(1)) = \text{hyp}(Y_1, Y_2).$$

Perfect! This completes the proof! 

Last time we proved the Cartan theorem for $K \equiv -1$. The same proof works for $K \equiv 0$, since Hadamard's theorem still applies. We just turn the hyperbolic sines $\sinh t$ into t to check it is a local isometry.

Proof for $K \equiv 1$. The same map

$$F(x) = \overline{\exp} \circ i \circ \exp_{NP}^{-1}(x)$$

is a local isometry, where i is an isometry of $T_{NP}S^n$ and $T_{\overline{NP}}M$. But this only holds when we take $S^n - \{SP\}$, that is without the south pole. The problem is that $\exp_{NP} : T_{NP}S^n \rightarrow S^n$ is not a diffeomorphism, and so there is no inverse.

So instead we'll consider two maps

$$F : S^n - \{SP\} \rightarrow M^n$$

$$F(x) = \overline{\exp} \circ \exp_{NP}^{-1}(x)$$

Let p be some point which is not the north pole or south pole. Take $\bar{p} = F(p)$, and NZ its antipodal point (portugal and new zealand!), then set

$$G : S^n - \{NZ\} \rightarrow M^n$$

$$G(x) = \exp_{\bar{p}} \circ dF_p \circ \exp_p^{-1}(x).$$

The same arguments from last time show that F, G are local isometries. Check that $G(p) = F(p)$, $dG_p = dF_p$. The midterm problem then shows that $G = F$ on $S^n \setminus \{SP, NZ\}$.

Thus we can extend F to a map $F : S^n \rightarrow M^n$ by $F(SP) = G(SP)$. It is smooth and a local isometry. A smooth local diffeomorphism between compact spaces is a covering map. Hence, by simply connectedness, F is an isometry. 

Now lets explore some powerful corollaries of this theorem.

Corollary IV.0.7

We have that

- (1) If (M^n, g) is closed and smooth and has constant sectional curvature, then (M^n, g) is isometric to $(\mathbb{R}^n, \mathbb{Z}^n, \text{flat})$, $(\mathbb{H}^n/\Gamma, \text{hyp})$, or $(S^n/\Gamma, \text{round})$. Here Γ is discrete acting freely and properly discontinuously.

These spaces are called space forms.

- (2) If (M^{2n}, g) has sectional curvature $K \equiv 1$, then $(M^{2n}, g) = (S^{2n}, \text{round})$ or $(\mathbb{RP}^{2n}, \text{round})$.

Proof. The first part is a direct corollary of Cartan's theorem, by lifting to the universal cover. For the second piece, take $M^{2n} = S^{2n}/\Gamma$, $\Gamma < \text{Isom}(S^{2n}) = O(2n+1)$.

Now for $A \in \Gamma$, we see that A must have a real eigenvalue $\lambda \in \mathbb{R}$. Why? Well A is an odd-dimensional matrix, so has odd degree characteristic polynomial. Furthermore, since A is orthogonal, $\lambda = 1$ or $\lambda = -1$. If $\lambda = 1$, then A has a fixed point, so $A = \text{Id}$ since Γ acts freely on the sphere.

Likewise, if $\lambda = -1$, then A^2 has a fixed point, so $A^2 = \text{Id}$. This will imply that A is the antipodal map. 

Example IV.0.4

There is no Γ acting on S^{2n} besides $\{\text{Id}\}$, $\{\text{Id}, A\}$ where A is the antipodal map. In contrast, for S^{2n+1} there are many spaces. E.g. for S^3 take

$$L : S^3 \rightarrow S^3$$

$$L(z, w) = (e^{2\pi i/p} z, e^{2\pi i q/p} w)$$

where p, q are coprime. This gives an action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 . The quotient is called a lens space $L(p, q)$. Similar constructions give lens spaces for S^{2n+1} .

Classification of orientable closed manifolds is one of the most important problems in mathematics. It's not hard to move from this to a classification for $n = 2$.

Theorem IV.0.8 (Uniformization)

Let Σ^2 be a closed orientable surface. Pick a metric g on Σ^2 . One can find a $\mu \in C^\infty(\Sigma)$ so that $K(e^{2\mu}g) \equiv \text{const} = 2\pi\chi(\Sigma)$ and $\text{vol}(\Sigma) = 1$.

This amounts to scaling to volume 1 and then solving the equation.

$$\Delta_g \mu + K(g) \equiv e^{2\mu} 2\pi\chi(\Sigma).$$

As a consequence of Cartan, $\Sigma = \mathbb{H}^2/\Gamma, T^2, S^2$.

The much much harder theorem, but of course wildly interesting, is

Theorem IV.0.9 (Geometrization - Perelman/Hamilton)

The geometrization for 3-manifolds. The hard case is showing that if M^3 is closed orientable and $\pi_1(M^3) = 0$ then $M^3 \simeq S^3$. Let (M^3, g) with g any metric. Find $(g_t)_{t \geq 0}$ be a family of

metrics so that

$$\partial_t g_t = -2 \operatorname{Ric}(g_t) + \frac{2}{3} S(g_t) \cdot g_t.$$

If $g_\infty = \lim_{t \rightarrow \infty} g_t$. Now of course $-2 \operatorname{Ric}(g_\infty) + \frac{2}{3} S(g_\infty) g_\infty = 0$.

Hence $\operatorname{Ric}(g_\infty) = \frac{1}{3} S(g_\infty) g_\infty$, and the midterm implies $\operatorname{Ric}(g_\infty) = \lambda g_\infty$. Again the midterm implies sectional curvature of g_∞ is constant. Then Cartan's theorem implies the result.

Unfortunately, for $n \geq 4$, the situation gets extremely difficult.

Theorem IV.0.10 (For $n = 4$)

Any finitely presented group is the fundamental group of some smooth closed 4-manifold. In fact, we can specify the finitely presented group. Essentially this means there is no classification for $n \geq 4$, since finitely presented groups are not classifiable.

Freedman tells us that if $\pi_1(M^4) = 0$, then M^4 is classified based on intersection form up to homeomorphism. On the other hand, Donaldson tells us that there are simply connected 4-manifolds which admit not differentiable structure.

The hard part here is. If you have $M^4 \simeq S^4$, then is M^4 diffeomorphic to a 4-sphere? Freedman tells us that M^4 is homeomorphic to S^4 , but it is not known if it is diffeomorphic. It is known that there are exotic $\mathbb{R}^4$ s, and there are exotic S^7 s (and higher)... but this case is not really known.

On the other hand, Smale classied $\pi_1(M^5) = 0$ up to diffeomorphism. Similarly, for dimensions $n \geq 5$, there are some successful cases are

- M^n has $S(g) > 0$.
- For M^{2n} if M^{2n} is Kahler.

V. Variations of Energy

Let (M^n, g) be complete, and fix $p, q \in M$. We can consider

$$\Omega_{p,q} = \{\gamma : (0, T) \rightarrow M \mid \gamma(0) = p, \gamma(T) = q, \gamma \text{ is piecewise } C^1\}.$$

Note: A space itself is worthless. A space with functions is a goldmine. So

$$L : \Omega_{p,q} \rightarrow (0, +\infty)$$

$$L(\gamma) = \int_0^T \|\gamma'(t)\| dt.$$

Unfortunately, if $L(\gamma) \leq L(\sigma)$ for all $\sigma \in \Omega_{p,q}$ does not imply that γ is a geodesic. For example, we can reparameterize a geodesic so that $\|\gamma'(t)\|$ is not constant.

There are two options for how to deal with this

- Change the definition of geodesic in some manner so that it is invariant under reparameterization. This turns out to be the correct thing to do for minimal surfaces.
- Look for another functional, namely the *energy*.

Definition V.0.1

We define the energy of a curve by

$$E : \Omega_{p,q} \rightarrow (0, +\infty)$$

$$E(\gamma) = \int_0^T \|\gamma'(t)\|^2 dt.$$

Our goal: Show that if $E(\gamma) \leq E(\sigma)$ for all $\sigma \in \Omega_{p,q}$, then γ is a geodesic.

One line proof: “the differential of E , $d_\gamma E$, is $-2\nabla_{\gamma'}\gamma'$, and so if γ is a local minimum, then $d_\gamma E = 0$.”

So now we have to justify the differential of E , since $\Omega_{p,q}$ is clearly not a manifold. This will take us some setup. Given $\gamma \in \Omega_{p,q}$ we define a tangent space

$$T_\gamma \Omega_{p,q} := \{V \text{ piecewise } C^1 \text{ v. field on } \gamma \mid V(0) = V(T) = 0, \text{ and if } \gamma \text{ is diff. on } (a, b) \text{ then so is } V\}.$$

This implies that for $V \in T_\gamma \Omega_{p,q}$ that $\nabla_{\gamma'} V(t)$ is well-defined.

Why is this a good idea of tangent space? Well, if we pick a $V \in T_\gamma \Omega_{p,q}$ we can take

$$F : (0, T) : (-\varepsilon, \varepsilon) \rightarrow M$$

$$F(t, s) = \exp_{\gamma(t)}(sV(t)).$$

Now lets check the obvious things

- (1) Set $\gamma_s : (0, L) \rightarrow M$, $\gamma_s(t) = F(s, t)$. We see that $\gamma_0 = \gamma$ and $\gamma_s \in \Omega_{p,q}$ for all $|s| < \varepsilon$.
- (2) Thus $s \rightarrow \gamma_s$ is a path in $\Omega_{p,q}$ passing through γ . Furthermore,

$$\left. \frac{d}{ds} \gamma_s(t) \right|_{s=0} = V(t).$$

Call $\mathcal{S}_\gamma$ the set of $t \in (0, T)$ where γ is not differentiable. Notably $\mathcal{S}_\gamma$ is finite.

Lemma V.0.1

Well, now we have the right thing to do!

$$\left. \frac{d}{ds} E(\gamma_s) \right|_{s=0} = -2 \int_0^T \langle V, \nabla_{\gamma'} \gamma' \rangle dt - 2 \sum_{\bar{t} \in \mathcal{S}_\gamma} \langle V(\bar{t}), \gamma'(\bar{t}_+) - \gamma'(\bar{t}_-) \rangle.$$

The $\gamma'(\bar{t}_+)$ here denotes a derivative from the right hand side. There's a more concise way to write this, namely

$$d_\gamma E = -2\nabla_{\gamma'} \gamma' - 2 \sum_{\bar{t} \in \mathcal{S}} (\gamma'(\bar{t}_+) - \gamma'(\bar{t}_-)) \delta_{\bar{t}}$$

Proof. For simplicity, we'll assume $\mathcal{S} = \{\bar{t}\}$, that is there is a single point of discontinuity. We then see that

$$\begin{aligned} \frac{d}{ds} E(\gamma_s) \Big|_{s=0} &= \frac{d}{ds} \int_0^T |\gamma'_s(t)|^2 dt = \frac{d}{ds} \int_0^T \left| \frac{\partial}{\partial t} F(t, s) \right|^2 dt. \\ &= 2 \int_0^T \left\langle \frac{\partial}{\partial s} \frac{\partial}{\partial t} F(t, s), \frac{\partial F}{\partial t}(t, s) \right\rangle dt \Big|_{s=0} \\ &= 2 \int_0^T \left\langle \frac{\partial}{\partial t} \frac{\partial}{\partial s} F(t, 0), \gamma'(t) \right\rangle dt \end{aligned}$$

We must now apply compatibility of the connection with the metric

$$2 \int_0^T \left\langle \frac{\partial}{\partial t} \frac{\partial}{\partial s} F(t, 0), \gamma'(t) \right\rangle dt = 2 \int_0^T \partial_t \langle \partial_s F(t, 0), \gamma'(t) \rangle - \langle \partial_s F(t, 0), \nabla_{\gamma'(t)} \gamma'(t) \rangle dt.$$

Now we split this up from 0 to $\bar{t}$. Note that $\partial_s F(t, 0) = V(t)$. Hence

$$\begin{aligned} &2 \int_0^{\bar{t}} \partial_t \langle V(t), \gamma'(t) \rangle dt + 2 \int_{\bar{t}}^T \partial_t \langle V(t), \gamma'(t) \rangle dt \\ &= 2(\langle V(\bar{t}), \gamma'(\bar{t}_-) \rangle) - 2\langle V(0), \gamma'(0) \rangle + 2\langle V(T), \gamma'(T) \rangle - 2\langle V(\bar{t}), \gamma'(\bar{t}_+) \rangle. \end{aligned}$$

The middle terms are zero, and we obtain

$$\frac{d}{ds} E(\gamma_s) \Big|_{s=0} = -2 \int_0^T \langle \partial_s F(t, 0), \nabla_{\gamma'(t)} \gamma'(t) \rangle dt + 2(\langle V(\bar{t}), \gamma'(\bar{t}_-) \rangle) - 2\langle V(\bar{t}), \gamma'(\bar{t}_+) \rangle.$$

This matches the desired formula, since $\partial_s F(t, 0) = V(t)$. 

Lets work on a complete Riemannian manifold M .

Last Time: We defined the energy functional

$$\begin{aligned} E : \Omega_{p,q} &\rightarrow [0, +\infty) \\ E(\gamma) &= \int_0^T |\gamma'(t)|^2 dt. \end{aligned}$$

Here we said

$$T_\gamma \Omega_{p,q} := \{\text{vector fields along } \gamma \text{ piecewise } C^1, V(0) = V(T) = 0\}.$$

We then had the first variation of energy

Proposition V.0.2 (1st variation formula)

$\gamma \in \Omega_{p,q}$, $V \in T_\gamma \Omega_{p,q}$ with

$$dE_\gamma(V) := \frac{d}{ds} E(\gamma_s) \Big|_{s=0} = -2 \int_0^T \langle V, \nabla_{\gamma'} \gamma' \rangle dt - 2 \sum_{t \in \mathcal{S}} \langle V(t), \gamma'(t^+) - \gamma'(t^-) \rangle$$

where $\mathcal{S}$ is the singular points of γ , $\#\mathcal{S} < \infty$, and $\gamma_s : (0, T) \rightarrow M$, $\gamma_s \in \Omega_{p,q}$ with $\gamma_0 = \gamma$ and $s \mapsto \gamma_s(t)$ smooth and $\left. \frac{d}{ds} \gamma_s(t) \right|_{s=0} = V(t)$. Explicitly we can take

$$\gamma_s(t) = \exp_{\gamma(t)}(sV(t)).$$

This is called the Euler-Lagrange equation for the energy.

Lemma V.0.3 (Critical points of E are geodesics)

Let $\gamma \in \Omega_{p,q}$. Then γ is a geodesic if $dE_\gamma(V) = 0$ for all $V \in T_\gamma \Omega_{p,q}$.

Proof. The forward direction is clear, as $\nabla_{\gamma'} \gamma' = 0$ and γ is smooth. Lets do the converse. Pick $\phi \in C^1(0, T)$ where $\phi(t) = 0$ if $t \in \mathcal{S}$. Then define

$$V(t) := \phi(t) \nabla_{\gamma'(t)} \gamma'(t) \in T_\gamma \Omega_{p,q}.$$

We then obtain

$$0 = dE_\gamma(V) = -2 \int_0^L \phi |\nabla_{\gamma'} \gamma'|^2 dt.$$

This holds for any fuction ϕ which is zero at finitely many points. Since $\nabla_{\gamma'} \gamma'$ is continuous except for these finitely many points, this implies that $\nabla_{\gamma'} \gamma' = 0$ everywhere.

Now we must rule out curves which are not smooth. Now choose $V \in T_\gamma \Omega_{p,q}$ so that $V(t) = \gamma'(t^+) - \gamma'(t^-)$ at $t \in \mathcal{S}$. We can do this continuously, just smoothly extending V along the rest of γ . Then

$$0 = dE_\gamma(V) = -2 \sum_{t \in \mathcal{S}} |\gamma'(t^+) - \gamma'(t^-)|^2.$$

Hence $\gamma'(t^+) = \gamma'(t^-)$ for each $t \in \mathcal{S}$, and hence γ is C^1 . 

Corollary V.0.4 (Minimizers of E are Minimizers of Length)

Let $\gamma \in \Omega_{p,q}$, then $E(\gamma) \leq E(\sigma)$ for all $\sigma \in \Omega_{p,q}$ if and only if γ is a length minimizing geodesic.

Proof. Lets do the backwards direction first. Let γ be a length-minimizing geodesic. Necessarily

$$|\gamma'(t)| = \frac{d(p, q)}{T}$$

for all t , since we're parameterizing from $[0, T]$ and $|\gamma'(t)|$ is constant with $\text{length}(\gamma) = d(p, q)$. Now pick $\sigma \in \Omega_{p,q}$ and write

$$E(\gamma) = \int_0^T |\gamma'(t)|^2 dt = \frac{d(p, q)^2}{T} \leq \frac{\left(\int_0^T |\sigma'(t)| dt \right)^2}{T}.$$

Now we apply Cauchy-Schwartz,

$$\left(\int_0^T |\sigma'(t)| dt \right)^2 \leq \left(\sqrt{\int_0^T dt} \right)^2 \left(\sqrt{\int_0^T |\sigma'(t)|^2 dt} \right)^2 = TE(\sigma).$$

Hence we obtain

$$E(\gamma) \leq \frac{\left(\int_0^T |\sigma'(t)| dt \right)^2}{T} \leq E(\sigma).$$

Great!

Now lets do the forward direction. Let $E(\gamma) \leq E(\sigma)$ for all $\sigma \in \Omega_{p,q}$.

Claim

This implies the first variation $dE_\gamma = 0$, and hence

Notably, for γ_s as before, we have that $s \mapsto E(\gamma_s)$ is minimized at zero, so by 1-variable calculus $\frac{d}{ds}E(\gamma_s)\Big|_{s=0} = 0$. Therefore $dE_\gamma(V) = 0$ for $V(t) = \frac{d}{ds}\gamma_s(t)\Big|_{s=0}$.

Now we must show that if $\text{length}(\gamma) > \text{length}(\tilde{\gamma})$ for some geodesic $\tilde{\gamma} \in \Omega_{p,q}$. Repeating the argument above yields that $E(\tilde{\gamma}) < E(\gamma)$ which is a contradiction. 

To distinguish the length-minimizing geodesics from other geodesics, we have to do the second variational formula.

Remark V.0.1

There is always a way to go up in energy, just wiggle more! Thus the only situations are local minimums or saddle points.

Proposition V.0.5 (Second Variation Formula)

Let γ be a geodesic in $\Omega_{p,q}$. Since we need second variation we'll have

$$T_\gamma \Omega_{p,q} = \{C^2 \text{ vector fields along } \gamma, V(0) = V(T) = 0\}.$$

We look at

$$D^2E_\gamma(V, V) := \frac{d^2}{ds^2}E(\gamma_s)\Big|_{s=0} = -2 \int_0^T \langle V, V'' + R(V, \gamma')\gamma' \rangle dt,$$

here we write $\mathfrak{J}(V) = V'' + R(V, \gamma')\gamma'$ called the Jacobi operator. So then

$$D^2E_\gamma(V, V) = -2 \int_0^T \langle V, \mathfrak{J}(V) \rangle dt.$$

“If $dE_\gamma = -2\nabla_{\gamma'}\gamma'$ then $D^2E_\gamma = -2\mathfrak{J}$.”

Proof. Lets compute... Let $F(t, s) = \exp_{\gamma(t)}(sV(t))$ so that $\gamma_s(t) = F(t, s)$. Then we write

$$\begin{aligned} \frac{d}{ds}E(\gamma_s) &= \frac{d}{ds} \int_0^T |\partial_t F|^2 = 2 \int_0^T \langle \partial_s \partial_t F, \partial_t F \rangle dt \\ &= 2 \int_0^T \langle \partial_t \partial_s F, \partial_t F \rangle dt \\ &= 2 \int_0^T \partial_t \langle \partial_s F, \partial_t F \rangle - \langle \partial_s F, \nabla_{\partial_t F} \partial_t F \rangle dt. \end{aligned}$$

Note that $F(0, s) = \gamma(0)$ for all s and $F(T, s) = \gamma(1)$ for all s . Hence $\partial_s F(0, s) = \partial_s F(T, s) = 0$. Thus the fundamental theorem of calculus will make the first term vanish, so that

$$\frac{d}{ds}E(\gamma_s) = -2 \int_0^T \langle \partial_s F, \nabla_{\partial_t F} \partial_t F \rangle dt.$$

Great! Now take another derivative!

$$\left. \frac{d}{ds} \frac{d}{ds} E(\gamma_s) \right|_{s=0} = -2 \int_0^T \langle \nabla_{\partial_s F} \partial_s F, \nabla_{\partial_t F} \partial_t F \rangle + \langle \partial_s F, \nabla_{\partial_s F} \nabla_{\partial_t F} \partial_t F \rangle dt.$$

Notice now that at $s = 0$, $\partial_t F = \gamma'$ and $\partial_t = \gamma'$. Hence the first term vanishes and we obtain by the definition of curvature that

$$\begin{aligned} \left. \frac{d^2}{ds^2} E(\gamma_s) \right|_{s=0} &= -2 \int_0^T \langle \partial_s F, \nabla_{\partial_s F} \nabla_{\partial_t F} \partial_t F \rangle dt \\ &= -2 \int_0^T \langle \partial_s F, \nabla_{\partial_t F} \nabla_{\partial_s F} \partial_t F \rangle + \langle \partial_s F, R(\partial_s F, \partial_t F) \partial_t F \rangle dt \\ &= -2 \int_0^T \langle \partial_s F, \nabla_{\partial_t F} \nabla_{\partial_t F} \partial_s F \rangle + \langle \partial_s F, R(\partial_s F, \partial_t F) \partial_t F \rangle dt. \end{aligned}$$

Now when $s = 0$ we have

$$\left. \frac{d^2}{ds^2} E(\gamma_s) \right|_{s=0} = -2 \int_0^T \langle V, V'' + R(V, \gamma') \gamma' \rangle dt.$$



Theorem V.0.6 (Hadamard)

Suppose we have a closed manifold (M^n, g) with sectional curvatures ≤ 0 , then closed geodesics are unique in their homotopy class. Furthermore, there is a geodesic in each homotopy class. Hence

$$\{\text{closed geodesics in } M^n\} \simeq \{\text{conjugacy classes in } \pi_1(M)\}.$$

Proof. One line proof: “The energy is a convex function if sectional curvatures ≤ 0 , and so there is only one critical point”.

Suppose we have a homotopy $H : S^1 \times [0, 1] \rightarrow M$ a free homotopy between closed geodesics $\gamma_0, \gamma_1 : S^1 \rightarrow M$.

Claim

There exists $G : S^1 \times [0, 1] \rightarrow M$ homotopic to H relative to ∂ where $s \mapsto G(\theta, s)$ is a geodesic for all θ (the “vertical” component of the cylinders).

Lift H to the universal cover as $\tilde{H} : \mathbb{R} \times [0, 1] \rightarrow \tilde{M}$. Since $H(\theta + 2\pi, s) = H(\theta, s)$ there must be some deck transformation ϕ lying in $\text{Isom}(\tilde{M}, g)$ such that

$$\tilde{H}(\theta + n2\pi, s) = \phi^n(\tilde{H}(\theta, s)).$$

Great! Now set $\tilde{G} : \mathbb{R} \times [0, 1] \rightarrow \tilde{M}$ with $s \mapsto G(\theta, s)$ to be the unique geodesic connecting $\tilde{H}(\theta, 0)$ to $\tilde{H}(\theta, 1)$. Uniqueness of geodesics (on simply connected spaces with non-positive sectional curvature) and ϕ being an isometry implies that

$$\tilde{G}(\theta + n2\pi, s) = \phi^n(\tilde{G}(\theta, s)).$$

for all s .

HW Problem: Set $\gamma_s(\theta) = G(\theta, s)$ and check that

$$\frac{d^2}{ds^2} E(\gamma_s) = -2 \int_0^{2\pi} \langle V, V'' + R(V, \gamma'_s) \gamma'_s \rangle d\theta,$$

where $V(\theta, s) = \partial_s \gamma_s(\theta)$. Well lets take a look at this

$$\begin{aligned} \frac{d^2}{ds^2} E &= -2 \int_0^{2\pi} \langle V, V'' \rangle dt - 2 \int_0^{2\pi} \mathcal{R}(V, \gamma'_s, \gamma'_s, V) d\theta \\ &= 2 \int_0^{2\pi} \left| \frac{DV}{d\theta} \right|^2 - 2 \int_0^{2\pi} \mathcal{R}(V, \gamma'_s, \gamma'_s, V) d\theta \geq 0, \end{aligned}$$

by integrating by parts and applying that the sectional curvature is negative. Thus $f(s) = E(\gamma_s)$ is convex, $f'(0) = f'(s) = 0$. Therefore $f(s) = f(0) = f(1)$ for all s .

Therefore $f''(s) = 0$ for all s . But wait, we can assume $\gamma_1(S^1) \neq \gamma_0(S^1)$ for contradiction. Then there must exist some $\bar{\theta} \in S^1$ so that $\partial_s G(\bar{\theta}, s)$ is not in the span of $\partial_\theta G(\bar{\theta}, 0)$. In other words, if $\partial_s G(\theta, s)$ is collinear to $G(\theta, 0)$ for all s , then γ_1 is just a reparameterization of γ_0 .

Thus

$$R(V(\bar{\theta}, 0), \gamma'_0(\bar{\theta}), \gamma'_0(\bar{\theta}), V(\bar{\theta}, 0)) < 0,$$

and so we have $f''(0) > 0$. Great!

Proof. Hence,



Last Time: We derived the second variation of energy as

$$\begin{aligned} D^2 E_\gamma(V, V) &= 2 \int_0^T -\langle V, V'' \rangle - \langle V, R(V, \gamma') \gamma' \rangle dt \\ &= 2 \int_0^T |V'|^2 - \mathcal{R}(V, \gamma', \gamma', V) dt. \end{aligned}$$

Last time we also used an assumption of negative curvature to show that geodesics are unique in their homotopy class. This time, we will make a positivity assumption and use the second variation to get interesting results.

Theorem V.0.7 (Bonnet-Myers)

Suppose (M^n, g) is complete with $\text{Ric}(g) \geq \frac{n-1}{r}g$, then $\text{diam}(M^n, g) \leq \pi r$.

To make this precise, we should think about the inequality $\text{Ric}(g) \geq \frac{n-1}{r}g$. Here we give.

Remark V.0.2

We say that $\text{Ric}(g) \geq \alpha g$ if for every vector field X ,

$$\text{Ric}(g)(X, X) \geq \alpha |X|^2.$$

We also define

$$\text{diam}(M^n, g) = \sup_{p, q \in M} d(p, q)$$

Corollary V.0.8

If (M^n, g) has $\text{Ric}(g) \geq \varepsilon g$ for $\varepsilon > 0$ then $\pi_1(M)$ is finite.

Proof of Corollary. Consider $(\widetilde{M}, \widetilde{g})$ the universal cover with the induced metric. Note that these are locally isometric, and so $\widetilde{M}$ satisfies Bonnet-Myers. Hence $\text{diam}(\widetilde{M}, \widetilde{g})$ has finite diameter, and hence is compact (since it will be complete). Thus the deck 

Lets build our list of examples some.

Example V.0.1

What are some closed simply connected manifolds with small diameter?

Example	Simply Connected	Admits Positive Ricci Curvature?
S^n	✓	✓
$S^n \times S^m$	✓	✓
$\mathbb{CP}^n$	✓	✓
K3	✓	×
$\mathbb{RP}^2$	×	✓
S^3/Γ	×	✓
$\mathbb{RP}^3 \# \mathbb{RP}^3$	×	×

Remark V.0.3

It's possible that $\text{Ric}(g) > 0$ may have $\text{diam}(M, g) = +\infty$. For example the paraboloid $z = x^2 + y^2$. Essentially at ∞ you get closer and closer to flat. If M is compact, then $\text{Ric}(g) > 0$ implies $\text{Ric}(g) \geq \varepsilon g$ for some $\varepsilon > 0$.

Corollary V.0.9

Any manifold of the form $T^k \times M^n$ for M^n closed has NO metric with positive Ric curvature.

Proof of Bonnet-Myers. Without loss of generality, assume $r = 1$, and check scaling properties of diameter/Ricci/metric. Suppose $\text{diam}(M^n, g) > \pi$. Then there exists $p, q \in M$ and $\gamma : [0, \ell] \rightarrow M$ a geodesic connecting p to q so that

$$d(p, q) = \text{length}(\gamma) = \ell > \pi.$$

Now lets do Bonnet's proof.

Claim

We can prove the theorem assuming that $\text{sectional}(g) \geq 1$, which is stronger.

We want to find $V \in T_\gamma \Omega_{p,q}$ so that $D^2 E_\gamma(V, V) < 0$. This is a contradiction because γ is energy-minimizing (being length-minimizing). Pick $e \perp \gamma'(0)$, $|e| = 1$ and set $e(t)$ to be the parallel transport. Set $V(t) = \sin\left(\frac{t\pi}{\ell}\right) e(t)$. We now obtain

$$D^2 E_\gamma(V, V) = \int_0^\ell |V'|^2 - \mathcal{R}(V, \gamma', \gamma', V) dt = \int_0^\ell |V'|^2 - \sin^2\left(\frac{\pi t}{2}\right) \mathcal{R}(e, \gamma', \gamma', e) dt.$$

Now notice that $e(t), \gamma'(t)$ both have norm 1 and are perpendicular. Thus this is the sectional curvature of this plane. By the strong assumption on sectional curvatures. We have then that $\mathcal{R}(e, \gamma', \gamma', e) \geq 1$ so

$$D^2 E_\gamma(V, V) \leq \int_0^\ell |V'|^2 - \sin^2\left(\frac{\pi t}{2}\right) dt.$$

We also compute that

$$V' = \frac{\pi}{\ell} \cos\left(\frac{t\pi}{\ell}\right) e(t),$$

using that $e'(t) = 0$. Plugging these both in gives

$$\begin{aligned} D^2 E_\gamma(V, V) &\leq 2 \int_0^\ell \frac{\pi^2}{\ell^2} \cos^2\left(\frac{\pi t}{\ell}\right) - \sin^2\left(\frac{\pi t}{\ell}\right) dt \\ &= 2 \left(\frac{\pi^2}{\ell^2} - 1 \right) \cdot \int_0^\ell \cos^2\left(\frac{\pi t}{\ell}\right) dt, \end{aligned}$$

since $\sin^2, \cos^2$ take the same values on $[0, \pi]$, just shifted by $\pi/2$. Since $\ell > \pi$, we see that $D^2E_\gamma(V, V) < 0$, and so we win!

Now we'll go over Myer's contribution, which shows that $\text{Ric}(g) \geq (n-1)g$ is sufficient. Choose $e_i \perp \gamma'(0)$ so that $\{e_1, \dots, e_{n-1}, \gamma'(0)\}$ is an orthonormal basis. As before take $e_i(t)$ the parallel transport and

$$V_i(t) = \sin\left(\frac{\pi t}{\ell}\right) e_i(t).$$

We see then that, if γ is length-minimizing, then

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n-1} D^2E_\gamma(V_i, V_i) = 2 \sum_{i=1}^{n-1} \int_0^\ell |V_i'|^2 - \sin^2\left(\frac{\pi t}{\ell}\right) \mathcal{R}(e_i, \gamma', \gamma', e_i) dt \\ &= 2 \int_0^\ell \sum_{i=1}^{n-1} |V_i'|^2 - \sin^2\left(\frac{\pi t}{\ell}\right) \sum_{i=1}^{n-1} \mathcal{R}(e_i, \gamma', \gamma', e_i) dt \\ &= 2 \int_0^\ell (n-1) \left(\frac{\pi}{\ell}\right)^2 \cos^2\left(\frac{\pi t}{\ell}\right) - \sin^2\left(\frac{\pi t}{\ell}\right) \text{Ric}(\gamma', \gamma') dt. \end{aligned}$$

The assumption is that $\text{Ric}(\gamma', \gamma') \geq (n-1)|\gamma'|^2 = (n-1)$ and so

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n-1} D^2E_\gamma(V_i, V_i) \leq 2 \int_0^\ell (n-1) \left(\frac{\pi}{\ell}\right)^2 \cos^2\left(\frac{\pi t}{\ell}\right) - (n-1) \sin^2\left(\frac{\pi t}{\ell}\right) dt \\ &\leq 2(n-1) \left(\left(\frac{\pi}{\ell}\right)^2 - 1\right) \int_0^\ell \cos^2\left(\frac{\pi t}{\ell}\right) dt < 0. \end{aligned}$$

This is a contradiction, and so we win!



Now the next question is incredible. Is this problem *rigid*. In other words,

Theorem V.0.10 (Cheng)

If $\text{Ric}(g) \geq (n-1)g$ and $\text{diam}(M^n, g) = \pi$, then (M^n, g) is isometric to (S^n, g_{S^n}) .

Theorem V.0.11 (Synge-Weinstein)

If (M^n, g) is closed and has positive sectional curvature then

- (i) If n is even and M^n is orientable this implies $\pi_1(M^n) = 0$.
- (ii) If n is odd this implies that M^n is orientable.

Proof of (i). Assume that n is even and orientable. For contradiction, suppose that $\pi_1(M^n) \neq 0$. Then there exists $\gamma : S^1 \rightarrow M$ a closed geodesic which is length-minimizing (in its homotopy class) and thus energy-minimizing. Thus $D^2E_\gamma(V, V) \geq 0$ for all V . We'll cook up a vector field so that $D^2E_\gamma(V, V) < 0$.

We'll use the orientation and the dimensionality assumption to cook up the vector using parallel transport. The difficulty is making sure the vector field is well-defined when we wrap around. Let

$$\begin{aligned}\{\gamma'(0)\}^\perp &:= \{v \in T_{\gamma(0)}M \mid V \perp \gamma'(0)\} \\ P : \{\gamma'(0)\}^\perp &\rightarrow \{\gamma'(0)\}^\perp \\ P(v) &= \text{parallel transport along } \gamma|_{[0,2\pi]}.\end{aligned}$$

Now here's the crucial step.

Claim

There exists a $v \neq 0$ in $\{\gamma'(0)\}^\perp$ so that $P(v) = v$.

P is an isometry and $\dim\{\gamma'(0)\}^\perp = n - 1$ is odd. Note that P is orientation preserving. We can see this because if we define

$$P_t : T_{\gamma(0)}M \rightarrow T_{\gamma(t)}M$$

to be parallel transport to $\gamma(t)$. Then $\text{sgn } P_t \in \{\pm 1\}$ depending on if P_t is orientation preserving, and this is continuous. $\text{sgn } P_0 = 1$, and so $\text{sgn } P_t = 1$ for all t by connectedness.

Now $P \in O_+(n - 1)$, $\det P = 1$, and so P has an odd number of real eigenvalues. These are all either 1 or -1 , and their product is the determinant which is one. Hence at least one eigenvalue is one, which gives the desired v .

Now if $P(v) = v$ and $V(t)$ is the parallel transport of v along γ , then $t \mapsto V(t)$ is well-defined along γ , and the 2nd variation formula holds

$$0 \leq D^2 E_\gamma(V, V) = \int_0^{2\pi} |V'|^2 - \mathcal{R}(V, \gamma', \gamma', V) dt = -2 \int_0^{2\pi} \mathcal{R}(V, \gamma', \gamma', V) dt < 0.$$

Perfect! This is our contradiction!



Remark V.0.4

In the proof above we had to be careful that $V(0)$ and $V(2\pi)$ were the same. This is because in our proof of the second variation formula, we considered curves.

$$\gamma_s(t) = \exp_{\gamma(t)}(sV(t))$$

If $V(0) \neq V(2\pi)$ then this curve γ_s is not a closed curve, and hence not “competitor” as it won't be freely homotopic to γ . Hence the proof of the second variation formula we did before will not work.

Corollary V.0.12

$\mathbb{RP}^2 \times \mathbb{RP}^2$ has no metric with positive sectional curvature.

Proof. Suppose $\mathbb{RP}^2 \times \mathbb{RP}^2$ admitted such a metric. Let $\widetilde{M}$ be the orientable 2-covering of $\mathbb{RP}^2 \times \mathbb{RP}^2$. Then $\dim \widetilde{M} = 4$, orientable, and has sectional curvature > 0 . However, $\pi_1(\widetilde{M})$ is an index two subgroup of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence it is not zero. Put another way, if $\pi_1 \widetilde{M}$ were 0, then $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2)$ would be $\mathbb{Z}/2\mathbb{Z}$ because $\widetilde{M}$ would be the universal cover.

This contradicts Synge-Weinstein's theorem!



Most Famous Problem in Geometry (Hopf Conjecture): $S^2 \times S^2$ has no metric with positive sectional curvature.

Remark V.0.5

The proof of Synge-Weinstein feels spiritually like it would work for Bonnet-Myers. However, we needed to arrange for $V(0) = V(\ell) = 0$ for Bonnet-Myers. This is because our curve γ was a length-minimizer *for paths between p, q* . Thus our choice of V cannot leave this space.

In contrast, in the proof of Synge-Weinstein, we know that our curve γ is a length-minimizer *in the entire homotopy class*. Thus our choice of V can make γ_s leave the space of paths from $\gamma(0)$ to itself, but must stay

- (1) Closed Curves
- (2) In the same homotopy class.

If we have (1), then (2) follows since γ_s is close enough to γ , and close curves are homotopic in Riemannian geometry.

Conjecture V.0.13 (Hopf Conjecture Strong)

If (M^4, g) has $\sec g > 0$ then M^4 is isometric to S^4 or $\mathbb{CP}^2$.

Theorem V.0.14 (Schoen/Brendel, Differentiable Sphere Theorem)

If (M^n, g) with $\frac{1}{4} < \sec(g) \leq 1$ then M^n is diffeomorphic to S^n/Γ .

VI. Beyond do Carmo

VI.1. The Geometric Aesthetic Principle, Organizing the Course

Meta Theorem: Assume a geometric condition (G) then a topological property (T) holds. In the chart below assume that M is always closed. Let $\text{sect}(g)$ be the sectional curvature as well.

Theorem	Geometry (G)	Topology (T)
Hadamard	$\text{sect}(g) \leq 0$	Universal cover $\cong_{\text{diff.}} \mathbb{R}^n$
Bonnet-Myers	$\text{Ric}(g) > 0$	$\pi_1(M)$ is finite
Synge-Weinstein	$\text{sect}(g) > 0$, n even, M^n orient.	$\pi_1 M = 0$.
★ Preissman	$\text{sect}(g) < 0$	$G < \pi_1 M$, abelian $\implies G \cong \mathbb{Z}$
★ Schoen-Yau	$S(g) > 0$	$F : M^n \rightarrow T^n \implies \deg F = 0$.

We have proved the theorems without $\star$ in front of them.

Strategy: Consider some space X ($\dim X = +\infty$) and a functional $E : X \rightarrow (0, +\infty)$, this is where the talent lies. The core of the argument is

$$(G) \implies \text{critical points of } E \text{ have some property (P)}$$

$$\text{Failure of (T)} \implies \text{There exists a critical point of } E \text{ without property (P).}$$

Now lets give the same list, but this time with the functionals and properties

Theorem	X	E	(P)
Hadamard	$\Omega_{p,q}$	$E(\gamma) = \int \gamma' ^2$	No conj. pts along critical pts
Bonnet-Myers	$\Omega_{p,q}$	$E(\gamma) = \int \gamma' ^2$	Crit. pts, length $> \pi$ are unstable.
Synge-Weinstein	$\Omega = \{\text{all closed loops}\}$	$E(\gamma) = \int \gamma' ^2$	No local minimums
$\star$ Preissman	$\{\text{all } f : T^2 \rightarrow M^n\}$	$E(f) = \int_{T^2} df ^2$	Crit. pts have one-dim. image
$\star$ Schoen-Yau	$\{\text{all hyp-surf. } \Sigma^{n-1} \subseteq M^n\}$	$E(\Sigma) = \text{area}(\Sigma)$	Crit. pts aren't local mins

There's one more important technique in this general ideas.

VI.2. Bochner Technique

We've now left the world of do Carmo. For a reference look at Chapter 4c on Riemannian Geometry, see Gallot-Hulin-Lafontaine [**gallot**]. Let M^n be closed, and recall the de Rham cohomology

$$\Omega^p(M) = \{p\text{-differential forms}\},$$

we see $\Omega^0(M) = C^\infty(M)$ and we have an exact sequence.

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0.$$

We know that $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ satisfies $d \circ d = 0$, which implies that $\text{im}(d|_{\Omega^{p-1}(M)}) \subseteq \ker(d|_{\Omega^p(M)})$.

We recall that

$$H_{dR}^p(M) := \frac{\ker(d|_{\Omega^p(M)})}{\text{im}(d|_{\Omega^{p-1}(M)})}.$$

I.e., if we have α_1, α_2 with $d\alpha_1 = d\alpha_2 = 0$, then we say $\alpha_1 \sim \alpha_2$ if $\alpha_1 - \alpha_2 = d\beta$.

Theorem VI.2.1 (de Rham)

We have that $H_{dR}^p(M) \simeq H^d(M, \mathbb{R})$, and in this case $H^d(M, \mathbb{R}) = H_d(M, \mathbb{R})^*$, the dual vector space. It gives the isomorphism as

$$I : H_{dR}^d(M) \rightarrow \text{Hom}(H_d(M, \mathbb{R}), \mathbb{R}) = H^d(M, \mathbb{R})$$

$$[\omega] \mapsto \left(I[\omega](\Sigma) = \int_{\Sigma} \omega \right)$$

We're going to find a “best” representative for $[\omega] \in H_{dR}^d(M)$ using a metric g on M . This metric induces a dot product on p -forms

$$\langle, \rangle : \Omega^p(M) \times \Omega^p(M) \rightarrow C^\infty(M)$$

$$\langle \alpha, \beta \rangle_x := p! \sum_{1 \leq i_1, \dots, i_p \leq n} \alpha(e_{i_1}, \dots, e_{i_p}) \beta(e_{i_1}, \dots, e_{i_p}),$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of $T_x M$.

Exercise VI.2.1

If $\alpha, \beta \in \Omega^1(M)$, we can write $\alpha^\# \in TM, \beta^\# \in TM$ so that $g(\alpha^\#, X) = \alpha(X), g(\beta^\#, X) = \beta(X)$. Then one can check that

$$\langle \alpha, \beta \rangle = g(\alpha^\#, \beta^\#).$$

We also get an inner product

$$(\cdot, \cdot) : \Omega^p(M) \times \Omega^p(M) \rightarrow \mathbb{R}$$

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle(x) \, d \operatorname{vol}_g(x).$$

Now $(\Omega^p(M), (\cdot, \cdot))$ is “almost” a Hilbert space. We'll treat it like a Hilbert space, since we're not analysts.

If we have $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ we have a well-defined adjoint map $\delta : \Omega^{p+1}(M) \rightarrow \Omega^p(M)$ defined by

$$(\delta \alpha, \beta) = (\alpha, d\beta).$$

Alright! Lets compute this in a particular example! First we have to recall some tensor calculus

Recall VI.2.2

Recall: If T is an R -tensor then ∇T is an $(R+1)$ -tensor. We use the notation $\nabla_X T(\dots) = \nabla T(X, \dots)$. Some properties are

- $\nabla_X(fT)(\dots) = X(f)T + f\nabla_X T(\dots)$.

Now for $\alpha \in \Omega^1(M)$ we can consider $\nabla \alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ and

$$\operatorname{div} \alpha := \operatorname{tr} \nabla \alpha = \sum_{i=1}^n \nabla_{e_i} \alpha(e_i) = \sum_{i=1}^n \nabla \alpha(e_i, e_i).$$

Lemma VI.2.2

Let $\alpha \in \Omega^1(M)$. We will compute $\delta \alpha = -\operatorname{div} \alpha$.

Proof. Lets first consider for any $f \in C^\infty(M)$ that

$$\operatorname{div}(f\alpha) = \sum_{i=1}^n \nabla(f\alpha)(e_i, e_i)$$

$$\begin{aligned}
&= \sum_{i=1}^n e_i(f) \alpha(e_i) + f \sum_{i=1}^n \nabla \alpha(e_i, e_i) \\
&= \sum_{i=1}^n df(e_i) \alpha(e_i) + f \operatorname{div} \alpha \\
&= \langle df, \alpha \rangle + f \operatorname{div} \alpha.
\end{aligned}$$

HW: $\int_M \operatorname{div}(X) d \operatorname{vol} = 0$ from do Carmo. So now we can integrate both sides over M and obtain

$$\begin{aligned}
0 &= \int \operatorname{div}(f \alpha) d \operatorname{vol}_g = \int \langle df, \alpha \rangle d \operatorname{vol}_g + \int f \operatorname{div} \alpha d \operatorname{vol}_g \\
&= (df, \alpha) + (f, \operatorname{div} \alpha).
\end{aligned}$$

Hence

$$(df, \alpha) = (f, -\operatorname{div} \alpha),$$

for all $f \in C^\infty(M)$. Therefore $\delta \alpha = -\operatorname{div} \alpha$ by uniqueness of the adjoint. 

Lets compute the adjoint for 2-forms as well

Lemma VI.2.3

Let $\omega \in \Omega^2(M)$. Then $\delta \omega_x(e_i) = -2 \sum_{j=1}^n \nabla \omega_x(e_j, e_j, e_i)$, where e_i is an orthonormal basis of $T_x M$ and $\nabla_{e_i} e_j(x) = 0$ (normal coords). This is also sometimes called the divergence.

Proof. Check for yourself (perhaps in coords): for $\alpha \in \Omega^1(M)$, we have

$$d\alpha(X, Y) = \frac{1}{2} (\nabla_X \alpha(Y) - \nabla_Y \alpha(X)).$$

Now we want to show that $(\delta \omega, \alpha) = (\omega, d\alpha)$ for all $\alpha \in \Omega^1(M)$ where $\delta \omega$ by abuse of notation is the sum above. Set $\beta \in \Omega^1(M)$ to be

$$\beta(X) = \sum_{i=1}^n \omega(e_i, X) \alpha(e_i)$$

Now lets take $\operatorname{div}(\beta)$! We have

$$\begin{aligned}
\operatorname{div}(\beta) &= \sum_{j=1}^n \nabla \beta(e_j, e_j) = \sum_j e_j(\beta(e_j)) - \sum_j \beta(\nabla_{e_j} e_j) \\
&= \sum_{i,j} e_j(\omega(e_i, e_j) \alpha(e_i)) = \sum_{i,j} e_j(\omega(e_i, e_j)) \alpha(e_i) + \omega(e_i, e_j) e_j(\alpha(e_i)) \\
&= \sum_{i,j} \nabla \omega(e_j, e_i, e_j) \alpha(e_i) + \sum_{i,j} \omega(e_i, e_j) \nabla \alpha(e_j, e_i) \\
&= - \sum_{i,j} \nabla \omega(e_j, e_j, e_i) \alpha(e_i) + \sum_{i < j} \omega(e_i, e_j) (\nabla \alpha(e_j, e_i) - \nabla \alpha(e_i, e_j)).
\end{aligned}$$

This is really good! It says

$$\begin{aligned}
 \operatorname{div}(\beta) &= \sum_{i=1}^n \frac{\delta\omega(e_i)}{2} \alpha(e_i) + 2 \sum_{i < j} \sum_{i < j} \omega(e_i, e_j) d\alpha(e_j, e_i) \\
 &= \frac{1}{2} \langle \delta\omega, \alpha \rangle - \sum_{i,j} \omega(e_i, e_j) d\alpha(e_i, e_j) \\
 &= \frac{1}{2} \langle \delta\omega, \alpha \rangle - \frac{1}{2} \langle \omega, d\alpha \rangle.
 \end{aligned}$$

Perfect! Integrating both sides yields that

$$0 = \langle \delta\omega, \alpha \rangle - \langle \omega, d\alpha \rangle,$$

just as desired! 

Definition VI.2.1 (Hodge-Laplacian)

We define the Hodge-Laplacian as $\Delta_H : \Omega^p(M) \rightarrow \Omega^p(M)$ defined by

$$\Delta_H(\omega) = -(\mathrm{d}\delta + \delta\mathrm{d})(\omega).$$

This operator is clearly self-adjoint, i.e. $(\Delta_H\alpha, \beta) = (\alpha, \Delta_H\beta)$. A form ω is called harmonic provided that $\Delta_H\omega = 0$.

$$\mathcal{H}^p(M) := \{\omega \in \Omega^p(M) \mid \Delta_H\omega = 0\}.$$

Theorem VI.2.4 (Hodge)

$$\mathcal{H}^p(M) \simeq H_{dR}^p(M) \simeq H^p(M, \mathbb{R}).$$

Proof. Consider the map $F : \mathcal{H}^p(M) \rightarrow H_{dR}^p(M)$. This map is given by

$$F(\alpha) = [\alpha]$$

Goal: F is bijective and well-defined. To see that it is well-defined, we prove that

Claim

$$\Delta_H\alpha = 0 \iff \mathrm{d}\alpha = \delta\alpha = 0.$$

The converse is obvious. For the first direction, we compute that

$$\begin{aligned}
 (\Delta_H\alpha, \alpha) &= -(\mathrm{d}\delta\alpha + \delta\mathrm{d}\alpha, \alpha, \alpha) \\
 &= -(\mathrm{d}\delta\alpha, \alpha) - (\delta\mathrm{d}\alpha, \alpha) \\
 &= -(\delta\alpha, \delta\alpha) - (\mathrm{d}\alpha, \mathrm{d}\alpha) = -|\delta\alpha|^2 - |\mathrm{d}\alpha|^2.
 \end{aligned}$$

Because the left hand side is zero, we know $\delta\alpha = \mathrm{d}\alpha = 0$.

Ok, so now we'll show $\ker F = 0$. Let $F(\alpha) = 0$, meaning that $\alpha = d\varphi$. We now evaluate

$$|\alpha|^2 = (\alpha, \alpha) = (\alpha, d\varphi) = (\delta\alpha, \varphi) = 0,$$

because α is harmonic, hence $\alpha = 0$.

Next Time: We'll prove F is surjective. 

Last Time: We introduced for (M^n, g) closed an inner product $(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle d\text{vol}$ on $\alpha, \beta \in \Omega^p(M)$. We then considered the exterior derivative and its adjoint

$$\Omega^p(M) \xrightleftharpoons[\delta]{d} \Omega^{p+1}(M).$$

We then defined the Hodge Laplacian $\Delta_H : \Omega^p(M) \rightarrow \Omega^p(M)$ via

$$\Delta_H = -(\delta d + d\delta).$$

We then defined the harmonic forms as

$$\mathcal{H}^p(M) = \{\omega \in \Omega^p(M) \mid \Delta_H \omega = 0\}.$$

We also showed that $\omega \in \mathcal{H}^p(M)$ if and only if $d\omega = 0, \delta\omega = 0$.

Theorem VI.2.5 (Hodge)

The map

$$J : \mathcal{H}^p(M) \rightarrow H_{dR}^p(M) \simeq H^p(M; \mathbb{R})$$

$$J(\omega) = [\omega]$$

is an isomorphism

We showed last time that J is well-defined and injective.

Remark VI.2.1

$\mathcal{H}^p(M)$ belongs to the world of elliptic PDEs, and $H^p(M; \mathbb{R})$ (singular cohomology) belongs to topologists.

Proof of Hodge Theorem modulo PDEs. First Principle: We have a map $\Delta_H : \Omega^p(M) \rightarrow \Omega^p(M)$ which is self-adjoint. We have rank + nullity as

$$\text{im } \Delta_H = (\ker \Delta_H)^\perp.$$

This is also sometimes called Fredholm alternative. The condition is that $\text{Id} - \Delta_H$ is a compact operator (recall Functional Analysis). This requires some PDE theory about a priori estimates for solutions to a PDE. We'll skip that.

Why this is rank+nullity for the unenlightened, since Neves won't have you leave class without knowing this: For linear maps $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\text{im } A = (\ker A)^\perp$ implies $\mathbb{R}^n = \text{im } A \oplus \ker A$. Thus $n = \text{rank } A + \text{null } A$.

Pick $[\beta] \in H_{dR}^p(M)$. We want to find $\varphi : \Omega^{p-1}(M)$ so that

$$\Delta_H(\beta + d\varphi) = 0.$$

Because then $\beta + d\varphi \in H^p(M)$ and $J(\beta + d\varphi) = [\beta]$. So how do we find: φ ? Well lets first solve the equation

$$\Delta_H \varphi = \delta \beta.$$

This is equivalent to $\delta \beta \perp \ker \Delta_H$ by the Fredholm alternative. So pick $\sigma \in \ker \Delta_H$, we have

$$(\delta \beta, \sigma) = (\beta, d\sigma) = (\beta, 0) = 0.$$

Great! Now lets see that $\Delta_H \varphi = \delta \beta$ is enough. We note that $\Delta_H \circ d = d \circ \Delta_H$, so we compute

$$\begin{aligned} \Delta_H(\beta + d\varphi) &= \Delta_H \beta + \Delta_H d\varphi = (-d\delta\beta - \delta d\beta) + d\Delta_H \varphi \\ &= -d\delta\beta + d\Delta_H \varphi = d(\Delta_H \varphi - \delta\beta) = d0 = 0. \end{aligned}$$

Perfect! This completes the proof. 

Theorem VI.2.6

Assume (M^n, g) is closed. Then

- (i) $\text{Ric}(g) > 0$ implies $H^1(M; \mathbb{R}) = 0$, i.e. $b_1(M) = 0$.
- (ii) $\text{Ric}(g) \geq 0$ implies $\dim H^1(M, \mathbb{R}) \leq n$ i.e., $b_1(M) \leq n$ and $b_1(M) = n$ if and only if $(M^n, g) \simeq_{\text{isom}} (T^n, \text{flat metric})$.

Note: By Hurewicz $H_1(M; \mathbb{Z}) = (\pi_1(M))^{ab}$, the first statement follows by Bonnet-Myers, since $\pi_1(M)$ is finite in this case.

Conjecture VI.2.7 (Milnor, disproved by Naber)

If $\text{Ric}(g) \geq 0$ then $\pi_1(M)$ is finitely generated, even if M is not compact. Unfortunately, there is a counterexample in dimension 7! This was done by Naber.

The content of today's class is the Bochner formula, which will allow us to prove the theorem above.

Recall VI.2.3

For $\alpha \in \Omega^1(M)$ if and only if $\alpha^\#$ is a vector field. We have two Laplacians, $\Delta_H \alpha$, and

$$\Delta \alpha = \text{tr}_{12} \nabla \nabla \alpha = \sum_{i=1}^n \nabla \nabla \alpha(e_i, e_i, -) = \sum_i \nabla_{e_i, e_i}^2 \alpha.$$

Proposition VI.2.8 (Bochner Formula)

Let $\alpha \in \Omega^1(M)$. Then we have

$$\Delta(\alpha) = \Delta_H(\alpha) + \text{Ric}(\alpha^\#)$$

where $\text{Ric}(\alpha^\#)(X) := \text{Ric}(X, \alpha^\#)$.

Proof. Last time, we computed for $\lambda \in \Omega^1(M)$ that

$$\begin{aligned} \delta\lambda &= -\text{div } \lambda = -\sum_{i=1}^n \nabla\lambda(e_i, e_i) \\ d\lambda(X, Y) &= \frac{1}{2}(\nabla_X\lambda(Y) - \nabla_Y\lambda(X)). \end{aligned}$$

We also computed for $\omega \in \Omega^2(M)$ that

$$\delta\omega = -2\text{div } \omega = -2\sum_{i=1}^n \nabla\omega(e_i, e_i, -)$$

Lets start with $\Delta_H\alpha$, we have, where $X = \partial_{x_k}, e_i = \partial_{x_i}$ are normal coordinates $\nabla_{\partial_{x_i}}\partial_{x_k} = 0$:

$$\begin{aligned} \Delta_H\alpha &= -(d\delta\alpha + \delta d\alpha) \\ d\delta\alpha(X) &= -d(\text{div } \alpha)(X) = -X(\text{div } \alpha) = -X\left(\sum_{i=1}^n \nabla\alpha(e_i, e_i)\right) \\ &= -\sum_{i=1}^n \nabla\nabla\alpha(X, e_i, e_i) = -\text{tr}_{2,3} \nabla\nabla(\alpha)(X). \end{aligned}$$

The last part applies the Leibniz rule for derivatives of a tensor, with $\nabla_X e_i = 0$. We use the expression for $d\alpha$ as an antisymmetrization to compute

$$\begin{aligned} \delta d\alpha &= -2\text{div}(d\alpha) = -2\text{tr}_{1,2} \nabla d\alpha = -2\sum_{i=1}^n \nabla d\alpha(e_i, e_i, -) \\ &= -\sum_{i=1}^n \nabla\nabla\alpha(e_i, e_i, -) - \nabla\nabla\alpha(e_i, -, e_i) \\ &= -\text{tr}_{1,2} \nabla\nabla\alpha + \text{tr}_{1,3} \nabla\nabla\alpha \\ &= -\Delta\alpha + \text{tr}_{1,3} \nabla\nabla\alpha. \end{aligned}$$

Hence we have that

$$\Delta_H\alpha = \Delta\alpha + \text{tr}_{2,3} \nabla\nabla\alpha - \text{tr}_{1,3} \nabla\nabla\alpha.$$

We're almost there! Now its clear where curvature is gonna happen, we're gonna commute the derivatives to promote the 2 to a 1! Lets compute this at a normal vector ∂_k , with $e_i = \partial_i$ at p .

$$\begin{aligned}
\text{tr}_{2,3} \nabla \nabla \alpha(\partial_k) - \text{tr}_{1,3} \nabla \nabla \alpha(\partial_k) &= \sum_{i=1}^n \nabla \nabla \alpha(\partial_k, \partial_i, \partial_i) - \nabla \nabla \alpha(\partial_i, \partial_k, \partial_i) \\
&= \sum_{i=1}^n \partial_k(\partial_i(\alpha(\partial_i))) - \partial_i(\partial_k(\alpha(\partial_i))) \\
&= \sum_{i=1}^n \langle \nabla_{\partial_k} \nabla_{\partial_i} \alpha^\#, \partial_i \rangle - \langle \nabla_{\partial_i} \nabla_{\partial_k} \alpha^\#, \partial_i \rangle \\
&= \sum_i \mathcal{R}(\partial_k, \partial_i, \alpha^\#, \partial_i) \\
&= - \sum_i \mathcal{R}(\partial_k, \partial_i, \partial_i, \alpha^\#) \\
&= - \text{Ric}(\alpha^\#)(\partial_k).
\end{aligned}$$



Proof of Theorem (i). We first show that

Claim

For $\alpha \in \Omega^1(M)$ we have that

$$\Delta |\alpha|^2 = 2\langle \Delta \alpha, \alpha \rangle + 2|\nabla \alpha|^2.$$

We see that

$$\begin{aligned}
\Delta |\alpha|^2 &= \sum_i \nabla_{e_i} \nabla_{e_i} |\alpha|^2 = 2 \sum_i \nabla_{e_i} \langle \nabla_{e_i} \alpha, \alpha \rangle \\
&= 2 \sum_i \langle \nabla_{e_i} \nabla_{e_i} \alpha, \alpha \rangle + 2 \sum_i \langle \nabla_{e_i} \alpha, \nabla_{e_i} \alpha \rangle = 2\langle \Delta \alpha, \alpha \rangle + 2|\nabla \alpha|^2.
\end{aligned}$$

Now suppose $H^1(M; \mathbb{R}) \neq 0$. Then pick $\alpha \in \mathcal{H}^1(M)$ a non-zero harmonic form. We then have that

$$\Delta \alpha = \Delta_H \alpha + \text{Ric}(\alpha^\#) = \text{Ric}(\alpha^\#).$$

So, we find that

$$\begin{aligned}
\Delta |\alpha|^2 &= 2\langle \Delta \alpha, \alpha \rangle + 2|\nabla \alpha|^2 \\
&= 2\langle \text{Ric}(\alpha^\#), \alpha \rangle + 2|\nabla \alpha|^2 \\
&= 2 \text{Ric}(\alpha^\#, \alpha^\#) + 2|\nabla \alpha|^2.
\end{aligned}$$

We can integrate, and using Stokes / divergence theorem

$$0 = \int_M \Delta |\alpha|^2 \, d \text{vol} = 2 \int_M \text{Ric}(\alpha^\#, \alpha^\#) + |\nabla \alpha|^2 \, d \text{vol}. \quad (\star)$$

if $\text{Ric} > 0$, the right hand side is > 0 . Contradiction!



Proof of Theorem (ii). If $\alpha \in \mathcal{H}^1(M)$ then applying Equation $(\star)$, which still holds in this case, we have

$$0 \geq \int_M |\nabla \alpha|^2 \, d\text{vol} = \int_M |\nabla \alpha^\#|^2 \, d\text{vol}.$$

This implies that $\nabla \alpha^\# = 0$. Fix $p \in M$ and consider the map

$$\begin{aligned} F : \mathcal{H}^1(M) &\rightarrow T_p M \\ \alpha &\mapsto \alpha^\#(p). \end{aligned}$$

This is a linear map. We must make sure that F is injective, as the Hodge theorem will then imply that $\dim H^1(M; \mathbb{R}) = \dim \mathcal{H}^1(M) \leq \dim T_p M = n$. Intuitively, we know that $\nabla \alpha^\# = 0$ captures the derivative, and so we should expect $\alpha^\#$ is constant (determined by a point).

Well, suppose $F(\alpha) = \alpha^\#(p) = 0$. Pick q and γ a curve connecting p to q . Then

$$\frac{d}{dt} |\alpha^\#(\gamma(t))|^2 = 2 \left\langle \frac{D}{dt} \alpha^\#, \alpha^\# \right\rangle = 0,$$

because $\nabla \alpha^\# = 0$. Thus $|\alpha^\#(q)|^2 = 0$. Now say $b_1(M) = n$, so $\dim \mathcal{H}^1(M) = n$. This implies F is an isomorphism, so we can port over an orthonormal basis for $T_p M$. We obtain then $\{e_1, \dots, e_n\} = \{\alpha_1^\#, \dots, \alpha_n^\#\}$ all parallel and linearly independent with $g(e_i, e_j) = \delta_{ij}$.

Then $\{e_1, \dots, e_n\}$ will be orthonormal everywhere with $\mathcal{R}(e_i, e_j, e_k, e_\ell) = 0$ so $M^n = \mathbb{R}^n / \Gamma$, with the flat metric. Because $b_1(M) = n$, it amounts to having Γ be a lattice.



General culture, so you don't look ignorant during tea conversations: We have the cotangent bundle

$$\begin{array}{c} T^*M \\ \downarrow \\ M \end{array}$$

with $\{\text{sections of } T^*M\} = \Omega^1(M)$. We have two natural maps $\Delta : \Omega^1(M) \rightarrow \Omega^1(M)$ and $\Delta_H : \Omega^1(M) \rightarrow \Omega^1(M)$. We showed they were related by the Ricci curvature $\Delta = \Delta_H + \text{Ric}$.

There's another bundle, the spin bundle $\text{Spin}(M)$ given as

$$\begin{array}{c} \text{Spin}(M) \\ \downarrow \\ M \end{array}$$

Where $\{\text{sections of } \text{Spin}(M)\} = S(M) = \{\text{spinors}\}$. There's a Bochner formula for spinors as

$$\begin{aligned} \Delta : S(M) &\rightarrow S(M) \\ \not{D}^2 : S(M) &\rightarrow S(M) \end{aligned}$$

$$\Delta = \mathcal{B}^2 + \frac{S(g)}{4}.$$

VII. Exam Review

VII.1. 2019 Final

Today we'll review some previous prelim / final problems. Let's go!

Problem 1:

- (1) 3 Distinct 3-Manifolds all with same universal cover and all admit metric of positive sectional curvature

$S^3/(\mathbb{Z}/p\mathbb{Z})$ via the lens actions (distinct via homology/ π_1)

- (2) Same but universal covers are all distinct

$S^3, S^2 \times S^1, (S^2 \times S^1) \# (S^2 \times S^1)$

Problem 2

- (i) Show that (M^n, g) closed with positive Ricci must have conjugate points

$\text{Ric} \geq \varepsilon g$ implies the universal cover $\widetilde{M}^n$ is closed and has $\text{Ric} \geq \varepsilon g$. If there are no conjugate points, then the exponential map $\exp_p : \mathbb{R}^n \rightarrow M^n$ would be a local diffeomorphism (argument from Hadamard). Furthermore it would be surjective. The argument from do Carmo would show this is a covering map. But this is impossible, as $\widetilde{M}^n$ is the universal cover.

- (ii) Can we change positive Ricci to non-negative Ricci in the question above? Explain or give a counterexample.

No, counterexample: the torus.

Problem 4

- (1) Draw a negatively curved non-compact surface S in $\mathbb{R}^3$ with $\pi_1(S) \neq 0$ but there's no closed geodesic.

A pseudo-sphere, aka the surface of revolution from the tractrix (aka pseudosphere).

VII.2. The Take Home Final

Problem 6

- (i) Give an example of $\{M_k\}_{k \in \mathbb{N}}$ closed manifolds with $M_K \not\cong M_j$ if $k \neq j$ and no M_k has an Einstein metric (i.e. $\text{Ric}(g) = \lambda g$).

$\Sigma_g \times S^1$ where Σ_g is a surface of genus g . If this admitted an Einstein metric, it would admit a constant sectional curvature metric (being dimension 3, see midterm). Hence it would be one of $S^3/\Gamma, \mathbb{R}^3/\Gamma, \mathbb{H}^3/\Gamma$. The universal cover of $\Sigma_g \times S^1$ is $\mathbb{H}^2 \times \mathbb{R}$, hence we have to rule out the latter two cases.

For the second case, the only closed quotient of $\mathbb{R}^3$ is T^3 , and $\pi_1(T^3) \neq \pi_1(\Sigma_g \times S^1)$. For the last part, we see that $\Gamma < \text{Isom}^+(\mathbb{H}^3)$ with $\mathbb{H}^3/\Gamma$ compact is centerless, but $\pi_1(\Sigma_g \times S^1)$ has a center (the $\mathbb{Z}$ from $\pi_1(S^1)$).

Problem 1:

- (a) Consider the ellipsoid $E = \{(x, y, z) \mid x^2 + 2y^2 + 3z^2 = 1\}$. Show that $\gamma = \{(x, y, 0) \mid x^2 + 2y^2 = 1\}$ is a geodesic

There's an orientation-reversing isometry which fixes γ .

- (b) Find $a, b, c > 0$ so that the ellipsoid

$$E_{a,b,c} = \{(x, y, z) \mid ax^2 + by^2 + cz^2 = 1\}$$

which does not have constant Gaussian curvature but has infinitely distinct simple closed geodesics.

Take $a = b = 1, c = 2$. Then the rotational symmetry will give us lots and lots of geodesics.

- (c) Show that for all $a, b, c > 0$ any two simple closed geodesics must intersect.

Note that $K(g) > 0$. Suppose γ_1, γ_2 are simple closed geodesics which don't intersect. The region Ω between them is diffeomorphic to an annulus. If we apply Gauss-Bonnet with boundary we would have

$$0 < \int_{\Omega} K(g) \, dA = 2\pi\chi(\Omega) + \int_{\partial\Omega} \langle \vec{\kappa}_{\gamma_i(s)}, \eta \rangle = 0,$$

since geodesics do not have curvature from the perspective of the manifold.

Problem 2: Let $M = \mathbb{RP}^3 \# \mathbb{RP}^3$.

- (a) Does this admit a metric of positive Ricci curvature?

The universal cover is $S^2 \times \mathbb{R}$, which is not compact. You can also compute that $H_1(\mathbb{RP}^3 \# \mathbb{RP}^3)$ is infinite, and so $\pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3)$ is infinite.

- (b) Does this admit a metric g with $\text{sect } g \geq 0$?

Yes. Notice that $\mathbb{RP}^3 \# \mathbb{RP}^3$ is the same as $(S^2/\mathbb{R})/\{T^n\}_{n \in \mathbb{Z}}$ where $T : S^2 \times \mathbb{R} \rightarrow S^2 \times \mathbb{R}$ via the isometry $(x, t) \mapsto (-x, t + 1)$. Thus we can take the product metric with round/flat.

Problem 4: Take (M^n, g) closed and X a killing vector field. I.e., $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathfrak{X}(M)$. Show that $\{\phi_t\}_{t \in \mathbb{Z}}$ where $\frac{d}{dt}\phi_t(x) = X(\phi_t(x))$ are all isometries.

We must show that $(\phi_t)^*g = g$. In other words, we must show

$$g((d\phi_t)(Y), (d\phi_t)(Z)) = g(Y, Z).$$

Then take the derivative in t to show this is constant.

References

- [Ahl53] Lars V Ahlfors. “Complex analysis: an introduction to the theory of analytic functions of one complex variable”. In: *New York, London* 177 (1953).
- [Gam03] Theodore Gamelin. *Complex analysis*. Springer Science & Business Media, 2003.

TODO LIST

 Covering Map part	46
---	----