MATH 297
Problem Sessions

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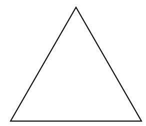
Contents

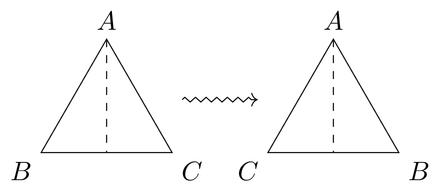
I. An Introduction to Group Theory

I.1. Definitions and Examples

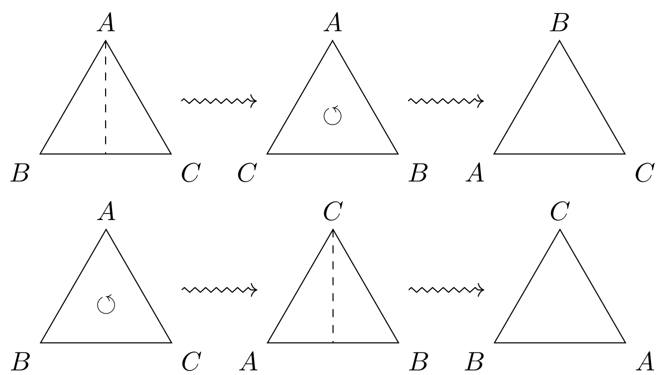
Broadly, group theory is the systematic study of different ways that objects can be symmetric. For example, we can consider the symmetries of an equilateral triangle lying in a plane under what are called "rigid motions" of the plane.



Intuitively, the symmetries express some geometric "same-ness" in the shape. However, we can view a symmetry as *acting* on the shape by envisioning it as a way to pick up the triangle and set it back down so that it looks exactly the same. Let's label the vertices to make this more clear:



With this in mind, we can combine two symmetries by first doing one, and then the other. For example, we can combine a reflection with a rotation in two ways like so:



We can notice that this gives us another symmetry, and in fact it gives us different symmetries. The first is reflection about the line coming from the vertex C, and the second is reflection about the line coming from vertex B.

For convenience we let the "do nothing" motion be a symmetry of the triangle as well (think of this as similar to why zero is an integer). Finally, we notice that we can "undo" a symmetry. For reflections we just flip again, and for rotations we just rotate the other direction twice. With these properties in mind, we can define a group:

Definition I.1.1

A *group* is a collection of elements which acts like symmetries. Concretely, a group G is a set equipped with an operation $\star : G \times G \to G$ which satisfies the following properties:

- The operation is associative, for any $a, b, c \in G$, we have $a \star (b \star c) = (a \star b) \star c$
- There is an identity element $1_G \in G$ so that $1_G \star g = g = g \star 1_G$ for all $g \in G$.
- For every element $g \in G$ there is an element $g^{-1} \in G$ so that $g \star g^{-1} = g^{-1} \star g = 1_G$. We often call g and g^{-1} *inverses*

Think about how this captures the abstract notion of symmetry!

Example I.1.1

The collection of all integers $\mathbb{Z}$ is a group with the operation of $+$.

Non-Example I.1.2

The collection of all integers $\mathbb{Z}$ is not a group with the operation of $\cdot$, and the collection of all natural numbers $\mathbb{N}$ is not a group with the operation of $+$.

Example I.1.3

There is a group of symmetries of a triangle. We denote it as $D_3 = \{r_0, r_1, r_2, s_0, s_1, s_2\}$. Intuitively we can imagine r_k as the rotation k times counterclockwise, and s_0, s_1, and s_2 are reflections about the bottom right, bottom left, and top vertex respectively.

The multiplication is given by the following table:

$\bullet$	r_0	r_1	r_2	s_0	s_1	s_2
r_0	r_0	r_1	r_2	s_0	s_1	s_2
r_1	r_1	r_2	r_0	s_1	s_2	s_0
r_2	r_2	r_0	r_1	s_2	s_0	s_1
s_0	s_0	s_2	s_1	r_0	r_2	r_1
s_1	s_1	s_0	s_2	r_1	r_0	r_2
s_2	s_2	s_1	s_0	r_2	r_1	r_0

As you can see, explicitly writing out groups can be extremely painful, and checking the axioms on such groups can be especially painful. This makes it a good idea to find other less combinatorial definitions of groups in theory. However, when doing computations, especially by computer, these explicit definitions are extremely useful.

Problem I-4

Come up with some examples of groups on your own

Problem I-5

Prove that the identity of a group is unique

Problem I-6

Prove that the inverse of an element g in a group G is unique

Problem I-7

Something to think about: Can there be two different groups on the same set S?

Definition I.1.2

Fix two groups G and H. A group homomorphism $\varphi : G \to H$ is a function from G to H such that for any two elements $g_1, g_2 \in G$ we have that:

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

Though this definition seems fairly abstract. It comes up in practice often, similar to how a linear maps definition seems very abstract at first

Example I.1.8

As we would expect, the identity $\mathrm{Id}_G : G \to G$ is always a group homomorphism.

Exercise I.1.9

Show that a group homomorphism $\varphi : G \to H$ satisfies the following algebraic properties:

- If 1_G and 1_H are the identity elements of G and H respectively, then $\varphi(1_G) = 1_H$.
- For any $g \in G$ we have that $\varphi(g^{-1}) = (\varphi(g))^{-1}$.
- If $\varphi : G \to H$ is an invertible function, then its inverse $\varphi^{-1} : H \to G$ is also a group homomorphism. Contrast this to what can go wrong for continuous or differentiable functions.

Exercise I.1.10

Show that the composition of any two group homomorphisms $\varphi : G \to H$ and $\psi : H \to K$ is a group homomorphism $\psi \circ \varphi : G \to K$.

Definition I.1.3
Two groups G and H are said to be isomorphic provided that there are group homomorphisms $\varphi : G \to H$ and $\psi : H \to G$ such that both $\psi \circ \varphi = \mathrm{Id}_G$ and $\varphi \circ \psi = \mathrm{Id}_H$.
In light of the previous two exercises, it suffices to find a bijective group homomorphism $\varphi : G \to H$.

Example I.1.11
Given any group G and any fixed element $h \in G$ there is a group homomorphism $\varphi : G \to G$ called conjugation by h

$$\varphi(g) = hgh^{-1}$$

Verify that this is indeed a group homomorphism.

Definition I.1.4
The kernel of a group homomorphism $\varphi : G \to H$ is the set of all elements $g \in G$ such that $\varphi(g) = 1_H$, where 1_H is the identity of H. That is:

$$\ker \varphi = \{g \in G \mid \varphi(g) = 1_H\} = \varphi^{-1}(\{1_H\})$$

The idea of this should be familiar from linear algebra.

Exercise I.1.12
Show that a group homomorphism $\varphi : G \to H$ is injective if and only if $\ker \varphi = \{1_G\}$. We say in this case that the kernel is trivial

Definition I.1.5
Given any set X, there is a group $S(X)$ called the symmetric group on X associated to it. The elements of $S(X)$ are invertible functions $f : X \to X$, and the group operation is composition.
In particular, we let $S_n := S(\{1, \ldots, n\})$ for any natural number n. We call this group the symmetric group on n elements

Exercise I.1.13
Show that if there is a bijection between two sets X and Y, then there is an isomorphism between $S(X)$ and $S(Y)$ which is "naturally" induced by this bijection.

I.2. Subgroups and Quotient Groups

Definition I.2.1
A subgroup H of a group G is a set $H \subseteq G$ such that the operation $G \times G \to G$ from the group structure can be restricted to an operation $H \times H \to H$, and H is a group under this operation. Equivalently, the following three conditions hold:

- For any $h_1, h_2 \in H$, the element $h_1 h_2 \in H$. That is H is closed under the group operation
- H contains the identity element 1_G of G.
- For every $h \in H$, we have that $h^{-1} \in H$.

Example I.2.1
The even integers $2\mathbb{Z}$ form a subgroup of $\mathbb{Z}$ with addition. More generally for any $n \in \mathbb{Z}$ we have that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ with addition. These subgroups are extremely important.

Exercise I.2.2
Find a subgroup of S_3 that is isomorphic to the group of symmetries of the triangle, that is D_3. A general theorem is that any finite group is isomorphic to a subgroup of a symmetric group. If you're interested, look up Cayley's Theorem.

Definition I.2.2
Fix some subgroup H of G. For any element $g \in G$ we have the following sets:

$$gH = \{gh \mid h \in H\} \qquad Hg = \{hg \mid h \in H\}$$

These are called left and right cosets of H respectively.

Definition I.2.3
A subgroup $H \subseteq G$ is called normal provided that for every $g \in G$ we have $gH = Hg$. Equivalently, H is called normal provided that for every $g \in G$ we have $gHg^{-1} = H$.

Example I.2.3
For any group G, the trivial subgroup $\{1_G\}$ and G itself are both normal subgroups of G

Example I.2.4
The subgroups $n\mathbb{Z}$ of $\mathbb{Z}$ are all normal

Example I.2.5
Given any group homomorphism $\varphi : G \to H$, the kernel $\ker \varphi$ is a normal subgroup of G

Exercise I.2.6
Verify the above examples!!!

Non-Example I.2.7
Consider the group S_3, and single out the subgroup H consisting of the identity function following "three-cycles"

$$f : \{1,2,3\} \to \{1,2,3\}$$
$$1 \xmapsto{f} 2$$
$$2 \xmapsto{f} 3$$
$$3 \xmapsto{f} 1$$
$$g : \{1,2,3\} \to \{1,2,3\}$$
$$1 \xmapsto{g} 3$$
$$3 \xmapsto{g} 2$$
$$2 \xmapsto{g} 1$$

For convenience, we write $f = (123)$ and $g = (132)$. H is not a normal subgroup of S_3.

Definition I.2.4
Given a normal subgroup H of a group G, we define the following set:

$$G/H = \{gH \mid g \in G\}$$

This set, when equipped with a particular operation, is called the quotient group of G by H. The operation on G/H is given by:

$$(g_1H)(g_2H) = (g_1g_2)H$$

Exercise I.2.8
Check that the above operation is well-defined, and makes G/H into a group.

Remark I.2.1
Try to think about why we require that H is normal.

Exercise I.2.9

Fix some $n \in \mathbb{Z}$. Verify that the group operation on $\mathbb{Z}/n\mathbb{Z}$ is addition modulo n, and that there are n elements in $\mathbb{Z}/n\mathbb{Z}$. Often we abuse notation and identify $\mathbb{Z}/n\mathbb{Z}$ with the group $\{0, \ldots, n-1\}$ with the operation of addition modulo n. Providing an isomorphism between these groups makes this abuse of notation less ugly ☺.

Exercise I.2.10

We often think of the group G/H as the group we get from G by "identifying" all the elements in H with the identity. Think about this intuition and try and explain it to someone else. This intuition will come up again in topology when we talk about quotient spaces.

Exercise I.2.11

Consider an $m \times n$ matrix A and view it as a linear function $A : \mathbb{R}^n \to \mathbb{R}^m$. Now note that this makes A into a group homomorphism with respect to the $+$ operation on $\mathbb{R}^n$ and $\mathbb{R}^m$. What is $\mathbb{R}^n / \ker A$? Is it isomorphic to anything familiar from linear algebra? *Hint:* The rank-nullity theorem is a beautiful thing, and vector space isomorphisms are also group isomorphisms.

II. An Introduction to Topology

II.1. Metric Spaces

The motivation of topology is to extend the notions of continuity and "fuzzy-ness" into more general settings. The first way we can do this is by defining a concept called a "metric space." Intuitively, we imagine a metric space as a space in which we can measure distances between points.

Definition II.1.1

Consider a set X equipped with a function $d : X \times X \to [0, \infty)$. We read $d(x, y)$ for $x, y \in X$ as "the distance between x and y in X." With this in mind, we require a few common sense notions about distance. When these axioms are satisfies we say that (X, d) is a metric space:

- $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- d satisfies the triangle inequality. That is, for $x, y, z \in X$ we have that:

$$d(x, z) \overset{\triangle}{\leq} d(x, y) + d(y, z)$$

  Intuitively, this reads that the distance between x and z must always be less than or equal to the distance between x and y plus the distance from y to z. We encounter this inequality every day in our lives when we travel between points. I choose in my writing to use $\overset{\triangle}{\leq}$ instead of $\leq$ to emphasize when I'm using the triangle inequality, as I find it helps me keep my thinking straight.

In cases where the metric is clear, we will be lazy and refer to X itself as the metric space. However, the data of the function d is just as much apart of the data of a metric space as the set.

Example II.1.1

The first example of a metric space we've encountered is $\mathbb{R}$. We define $d(x, y) = |x - y|$, and we see that all the axioms will be satisfied from previous work.

Example II.1.2

The second, and one of the most common, examples is $\mathbb{R}^n$. We define $d(x, y)$ for points $x, y \in \mathbb{R}^n$ by using the standard Euclidean distance motivated by the pythagorean theorem:

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

It is a good exercise to verify the axioms of a metric space for this metric. A hint for the triangle inequality: You can verify that $\|v + w\| \leq \|v\| + \|w\|$ using the Cauchy-Schwarz inequality.

Example II.1.3

To stress that the metric is an inherent part of the structure. Consider the following metric d defined on any set X:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

This is called the discrete metric on X. It is fairly clear that $\mathbb{R}$ equipped with the discrete metric "looks" very different from $\mathbb{R}$ equipped with the Euclidean etric

We now can use this idea to define a continuous function $f : X \to Y$ for two metric space X and Y. In order to provide a variety of nice definitions, we will first define two concpets: balls and open sets:

Definition II.1.2

Consider a metric space (X, d). For $x \in X$ and $\varepsilon > 0$ we define the open ball $B_\varepsilon(x)$ as follows, motivating it by the idea of a ball of a certain radius in $\mathbb{R}^n$:

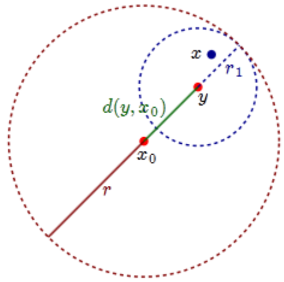
$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$$

That is, the ball of radius ε around x consists of all those points of distance less than ε from x.

We then define the concept of an open set, which we should envision as a set where you can draw a small ball around every point which remains in the set. Thus, a set $U \subseteq X$ is called an open set provided that for every $x \in X$ there is an $\varepsilon > 0$ so that $B_\varepsilon(x) \subseteq U$.

Exercise II.1.4

Verify that an open ball is indeed an open set. Use the following picture for motivation, where x_0 is the center of our ball, and y is a point in the ball of radius $r > 0$ around x_0.



Exercise II.1.5

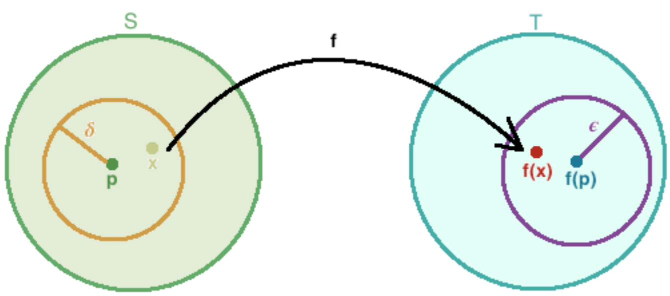
Give an intuitive definition of open for yourself. Learn to visualize the concept.

We now define continuity in three different and equivalent ways:

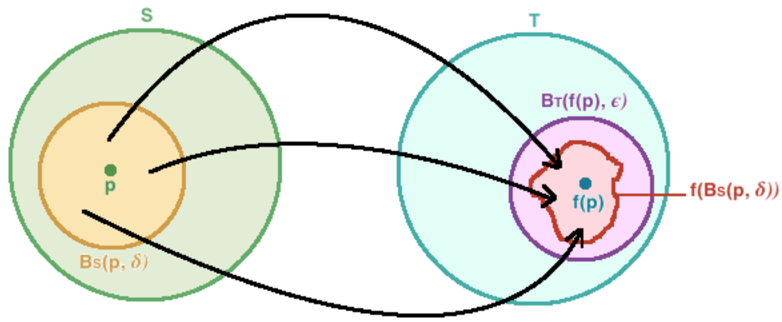
Definition II.1.3

Consider two metric space (X, d_X) and (Y, d_Y). We say that a function $f : X \to Y$ is continuous provided that it satisfies the following three equivalent conditions

- For every point $x_0 \in X$ and every $\varepsilon > 0$ there is some $\delta > 0$ so that if $d_X(x_0, x) < \delta$ then $d_Y(f(x_0), f(x)) < \varepsilon$. Here's a nice picture explaining this "greek" (Sarah Koch's words) definition:



- For every point $x_0 \in X$ and every ball $B_\varepsilon(f(x_0))$ there is some ball $B_\delta(x_0)$ which is mapped entirely into $B_\varepsilon(f(x_0))$ by f. That is, we have $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$. Here's another good picture!



- For every open subset $U \subseteq Y$ we have that $f^{-1}(U)$ is an open subset of X.

Personally, I prefer the third definition, and it will in fact be how we generalize beyond metric spaces later. The graphics are courtesy of MathOnline, whose notes can be found at this link and which are licensed under Creative Commons.

Exercise II.1.6
Draw a picture illustrating the third condition on open sets in the definition

Example II.1.7
The identity function $\mathrm{Id}_X : X \to X$ is always continuous when we equip X with the same metric in the domain and codomain. We see this from the third definition since $\mathrm{Id}_X^{-1}(U) = U$ for every set U, and so if $U \subseteq X$ is open then of course $\mathrm{Id}_X^{-1}(U)$ is open.

Example II.1.8
Consider any real number $r \in \mathbb{R}$. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x) = r \cdot x$ is continuous. We see this by the first definition. Fix some $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ and set $\delta := \frac{\varepsilon}{|r|+1}$.

> **Remark II.1.1**
> A common trick in analysis is to divide by $|r| + 1$ when we intuitively want to divide by r, since r may be negative or zero.

Then we may calculate that if $d(x_0, x) < \frac{\varepsilon}{|r|+1}$ then:

$$d(f(x_0), f(x)) = \|rx_0 - rx\| = |r| \cdot \|x_0 - x\| \leq \varepsilon \cdot \frac{|r|}{|r|+1} < \varepsilon$$

And so f is continuous.

Exercise II.1.9
Give an intuitive description of what continuity means, particularly focus on describing continuity for functions $\mathbb{R} \to \mathbb{R}$ when $\mathbb{R}$ is equipped with the standard metric $d(x, y) = |x - y|$.

Exercise II.1.10
Verify that the above three descriptions of continuous given in the definition are all equivalent

II.2. Some Basic Definitions

Definition II.2.1
Consider some set X, and some collection of subsets of X, $\mathscr{T} \subseteq P(X)$. For convenience, we call members $U \subseteq X$ of $\mathscr{T}$ open sets. $\mathscr{T}$ makes X into a topological space with topology $\mathscr{T}$ provided the following conditions are met:

- The empty set $\emptyset$ and X are both open ($\emptyset, X \in \mathscr{T}$)
- Arbitrary unions of open sets are open. Aka, for any collection $\mathcal{U} \subseteq \mathscr{T}$ we have that $\bigcup_{U \in \mathcal{U}} U \in \mathscr{T}$.
- Finite intersections of open sets are open. Aka for $U_1, \ldots, U_n \in \mathscr{T}$ we have that $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$.

By abuse of notation, we will often refer to X itself as a topological space. We often say that U is an open neighborhood of a point $x \in X$ provided that U is an open set and $x \in U$.

Exercise II.2.1

Verify that given a metric space (X, d) we have a topology $\mathscr{T}_d$ called the topology on X generated by the metric d:

$$\mathscr{T}_d = \{U \in P(X) \mid \forall x \in U \ \ \exists r > 0 \text{ such that } B_r(x) \subseteq U\}$$

Note that this gives the same definition of open as discussed previously.

Example II.2.2

The topology on a finite dimensional normed vector space V generated by the metric $d(v, w) = \|v - w\|$ is often called the Euclidean topology on V

Example II.2.3

Trivially, the set $\mathscr{T}_{\text{indiscrete}} = \{\emptyset, X\}$ always defines a topology on a set X. This is called the indiscrete topology on X.

Example II.2.4

Dually, the discrete metric generates a topology $\mathscr{T}_{\text{discrete}} = P(X)$ on any set X called the discrete topology on X

Definition II.2.2

A subset K of a topological space X is called closed provided that $X \setminus K$ is open.

Exercise II.2.5

Show that closed subsets of a topological space satisfy the following:

- The empty set and X are both closed.
- Arbitrary intersections of closed sets are closed
- Finite unions of closed sets are closed.

Example II.2.6

Closed intervals in $\mathbb{R}$ are closed (why?)

Definition II.2.3

A function $f : X \to Y$ between two topological spaces is continuous provided that for any open set $U \subseteq Y$ we have that $f^{-1}(U) \subseteq X$ is open. Equivalently, for any closed set $K \subseteq Y$ we have that $f^{-1}(K) \subseteq X$ is closed.

Definition II.2.4

We say that a sequence $(x_n) \in X$ converges to a point $x \in X$ provided that for every open neighborhood U of x there exists some $N \in \mathbb{N}$ so that for $n \geq N$ we have $x_n \in U$.

Exercise II.2.7

Verify that in a metric space (or in an inner product space), this is the usual definition of convergence.

Exercise II.2.8

Show that if a sequence $(x_n) \in K$ contained in a closed set $K \subseteq X$ converges to some point $x \in X$, then $x \in K$. Conclude that a closed and bounded subset of $\mathbb{R}$ contains its supremum and its infimum. Closed and bounded subsets of $\mathbb{R}$ will be extremely useful for us.

Remark II.2.1

Unfortunately, life can go very badly. Limits of a sequence are not unique in every topology. Show that any sequence $(x_n) \in X$ converges to every point $x \in X$ in the indiscrete topology.

Definition II.2.5

A topological space X is called Hausdorff provided that for any two distinct points $x, y \in X$, there are disjoint open neighborhoods U of x and V of y. Intuitively, we think of this as saying that x and y can be "separated" by open sets

Exercise II.2.9

Verify that limits are unique when they exist in a Hausdorff space.

Definition II.2.6

A subset $A \subseteq X$ of a topological space X can be given a topology, called the subspace topology. This topology is defined by calling a subset $U_A \subseteq A$ open provided that U_A may be written as $U \cap A$ for some open set $U \subseteq X$.

Definition II.2.7

A space X is called disconnected provided that it can be written as $X = U \cup V$ for two nonempty, disjoint, open sets $U, V \subseteq X$. We call a space connected provided that it is not disconnected

Definition II.2.8

Given two points $x, y \in X$, a path from x to y is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We call a space X path-connected provided that there is a path between any two points in X.

Exercise II.2.10

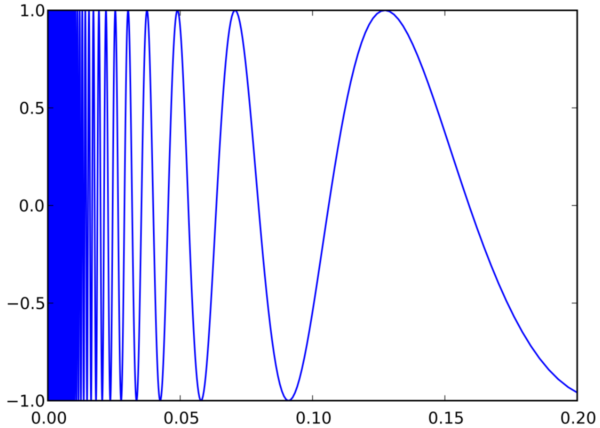
Think about why these are the "right" definition geometrically. Think about what it means for weird spaces (e.g. the discrete/indiscrete topologies). This type of thinking will eventually allow you to visualize weird topologies. For example, the discrete topology looks like a lot of isolated points far away from each other, and the indiscrete topology looks like a lot of points infinitesimally close to each other.

Remark II.2.2

The Intermediate Value Theorem (Theorem II.2.2) will somehow show us that all path-connected spaces are connected (why?). The converse does not hold. This should be weird! A good example is the topologist's sine curve, given below:

$$X = \{(x, \sin 1/x) \mid x > 0\} \cup \{(0, y) \mid 0 \leq y \leq 1\}$$

This space X is connected but it is not path-connected. Here is a picture:



Discuss why this space is not path-connected ☺

Theorem II.2.1 (Hard)
A subspace $A \subseteq \mathbb{R}$ is connected in the Euclidean topology if and only if A is an interval

Theorem II.2.2 (Intermediate Value Theorem)
Here is the statement of the intermediate value theorem in its most natural setting. Given a continuous function $f : X \to Y$ between a connected topological space X and some topological space Y, we have that $f(X)$ is a connected subspace of Y. We also have that if X is path-connected then $f(X)$ is path-connected (this is actually easier to show)

Exercise II.2.11
Interpret and internalize this theorem geometrically

Exercise II.2.12
Verify the corollary, that for any continuous function $f : X \to Y$ and any connected subspace $A \subseteq X$, we have that $f(A)$ is connected.

II.3. Quotient Spaces: The Gluening

Definition II.3.1
Consider some topological space X and some equivalence relation $\sim$. The quotient space $X/{\sim}$ is the set of all equivalence classes of X under the equivalence relation, equipped with the following topology. Let $q : X \to X/{\sim}$ be the map which acts on elements of x as $x \overset{q}{\mapsto} [x]_\sim$, where $[x]_\sim$ is the equivalence class of x under $\sim$. Then $U \subseteq X/{\sim}$ is open provided that $q^{-1}(U)$ is open.
Consider any subspace A of X, the quotient space X/A is the quotient space $X/{\sim}$ where the equivalence relation $\sim$ is given by $x \sim x$ for all $x \in X$, and $a \sim b$ for all $a, b \in A$. Intuitively, we think of X/A as identifying all of A to a single point.

Example II.3.1
Consider the interval $[0, 1]$ with the subspace topology from $\mathbb{R}$. Then define $S^1 := [0, 1]/\{0, 1\}$. Intuitively, we should imagine this space as an interval with its endpoints glued together. What do you think this space should be? Can you provide a homeomorphism of this space to some subspace of $\mathbb{R}^d$ for $d \in \mathbb{N}$?

Exercise II.3.2
Show that the quotient map $q : X \to X/{\sim}$ is always continuous. This should be trivial from the definitions.

Exercise II.3.3
Show that if X is connected, then $X/{\sim}$ is connected. Similarly, use it to show that if X is path-connected, then $X/{\sim}$ is path-connected.

Example II.3.4
Consider the unit disk $D^2 = \{x \in \mathbb{R}^2 \mid |x| \le 1\}$. Let S^1 be the unit circle, aka the boundary of this disk (why did I use the same notation as in the previous example here?)
What space is D^2/S^1?

Example II.3.5
Consider an equivalence relation $\sim$ on $\mathbb{R}$ defined as $x \sim y$ provided that $x - y \in \mathbb{Q}$. What is the topology on the space $\mathbb{R}/{\sim}$? How many points does it have?

Exercise II.3.6
Is $\mathbb{R}/{\sim}$ from the previous example Hausdorff?

Proposition II.3.1 (Universal Property of Quotient Spaces)

Here are two special properties that the quotient space satisfies. First, a function $f : X/\sim \to Y$ is continuous if and only if $f \circ q : X \to Y$ is continuous.

Second, the universal property of quotient spaces says that given a continuous unction $g : X \to Y$ such that whenever $x \sim x'$ we have $g(x) = g(x')$, there exists a unique continuous function $\widetilde{g} : X/\sim \to Y$ such that the diagram below commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q"'] & Y \\ X/\sim \arrow[ur, dashed, "\exists !", "\widetilde{g}"'] & \end{tikzcd}$$

Exercise II.3.7

Use the universal property of quotient spaces in order to show that $S^1 = [0,1]/\{0,1\}$ is homeomorphic to $\{x \in \mathbb{R}^2 \mid |x| = 1\}$. You are free to use polar coordinates and similar constructions.

Definition II.3.2

Let X and Y be topological spaces. We equip the cartesian product $X \times Y$ with a topology called the product topology. The open sets $\mathcal{O}$ of $X \times Y$ are given as follows, where $\{U_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ are open subsets o X and Y respectively:

$$\mathcal{O} = \bigcup_{i \in I} U_i \times V_i$$

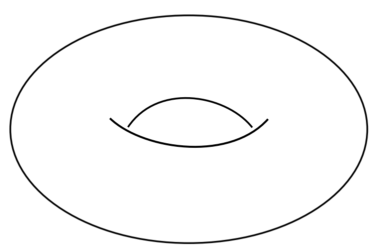
In other words, open subsets of the product are given as unions of basic open sets $U \times V$, or U open in X and V open in Y.

Example II.3.8

The standard topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the product topology.

Example II.3.9

The torus, $T^2 = S^1 \times S^1$ is a common example of a product space. Think about why this is the familiar space we know and love–the surface of a doughnut:



Exercise II.3.10

Tell me what all of the following shapes are. The colors edges indicate which sides get glued together by the equivalence relation, and the direction indicates how they get glued together. For example, the second figure is the square $[0,1]^2$ with the equivalence relation $(t, 0) \sim (1-t, 1)$ and the fourth figure is the square with the equivalence relation $(t, 0) \sim (t, 1)$.

Different colored edges indicate that these are not glued together ☺