Higher categories for engineers

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1 Spaces

1.1 In set theory there are two kinds of propositions. If X is a set then we can say that $x \in X$, and for $x, y \in X$ we can say that $x = y$. Equality being the archetype of an equivalence relation, it has the following properties.

(i) (reflexivity) $x = x$ for all $x \in X$.

(ii) (transitivity) $x = y$ and $y = z$ implies $x = z$ for $x, y, z \in X$.

(iii) (symmetry) $x = y$ implies $y = x$ for $x, y \in X$.

We would like to banish the notion of equality, or rather relax it to the notion of equivalence or isomorphism. In this new paradigm, equivalence is no longer a property of a pair of elements, but rather a structure. On a first pass this looks as follows.

We introduce a new datum called an isomorphism $x \tilde{\to} y$ for $x, y \in X$, and write $\mathrm{Iso}_X(x, y)$ for the totality of isomorphisms $x \tilde{\to} y$, which for now form a set. The axiom of transitivity becomes an additional structure, namely a composition law

$$\begin{aligned} \mathrm{Iso}_X(y, z) \times \mathrm{Iso}_X(x, y) &\longrightarrow \mathrm{Iso}_X(x, z) \\ (g, f) &\mapsto g \circ f. \end{aligned}$$

The necessary axioms then become:

(i) (identity) for any $x \in X$ there exists $\mathrm{id}_x \in \mathrm{Iso}_X(x, x)$ satisfying $f \circ \mathrm{id}_x = f$ and $\mathrm{id}_x \circ g = g$ for all isomorphisms $f : x \tilde{\to} y$ and $g : z \tilde{\to} x$.

(ii) (invertibility) for any isomorphism $f : x \tilde{\to} y$ in X there exists $f^{-1} : y \tilde{\to} x$ such that $f^{-1} \circ f = \mathrm{id}_x$ and $f \circ f^{-1} = \mathrm{id}_y$.

(iii) (associativity) for any isomorphisms $f : x \tilde{\to} y$, $g : y \tilde{\to} z$, and $h : z \tilde{\to} w$ in X, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Clearly identity and invertibility are replacements for reflexivity and symmetry respectively, while associativity is something new.

We have thus arrived at the notion of a 1-groupoid. As soon as we make this definition, we notice a problem: the axioms impose equations (now between isomorphisms), which was supposed to be forbidden. This can be remedied by postulating that $\mathrm{Iso}_X(x, y)$ itself has the structure of a 1-groupoid for all $x, y \in X$, meaning we allow 2-isomorphisms between the previously existing (1-)isomorphisms. The composition law

$$\mathrm{Iso}_X(y, z) \times \mathrm{Iso}_X(x, y) \longrightarrow \mathrm{Iso}_X(x, z)$$

for $x, y, z \in X$ must then have the structure of a morphism of 1-groupoids, i.e. a functor of 1-categories in the usual sense. Moreover, we specify certain 2-isomorphisms:

- for all 1-isomorphisms $f : x \to y$ and $g : z \to x$ we have 2-isomorphisms $\iota_{f,x} : f \circ \mathrm{id}_x \tilde{\to} f$ and $\iota_{x,g} : \mathrm{id}_x \circ g \tilde{\to} g$;
- for all 1-isomorphisms $f : x \to y$, $g : y \to z$, and $h : z \to w$, we have a 2-isomorphism $\alpha_{h,g,f} : (h \circ g) \circ f \tilde{\to} h \circ (g \circ f)$.

These 2-isomorphisms are subject to various coherence conditions, which we do not list, as this task already becomes somewhat tedious. To give the flavor of these conditions, we mention the so-called pentagon axiom, which says that for 1-isomorphisms $f : x \to y$, $g : y \to z$, $h : z \to w$, and $i : w \to v$, the following diagram in $\mathrm{Iso}_X(x, v)$ commutes:

$$\begin{array}{ccccc} & & ((i \circ h) \circ g) \circ f & & \\ & \swarrow^{\alpha_{i,h,g} * \mathrm{id}_f} & & \searrow^{\alpha_{i \circ h, g, f}} & \\ (i \circ (h \circ g)) \circ f & & & & (i \circ h) \circ (g \circ f) \\ \downarrow{\scriptstyle \alpha_{i, h \circ g, f}} & & & & \downarrow{\scriptstyle \alpha_{i,h,g \circ f}} \\ i \circ ((h \circ g) \circ f) & & \xrightarrow{\mathrm{id}_i * \alpha_{h,g,f}} & & i \circ (h \circ (g \circ f)). \end{array}$$

Here $*$ denotes the "horizontal" composition of 2-isomorphisms, which is induced by functoriality of the composition law for 1-isomorphisms.

It turns out that the invertibility of a 1-isomorphism $f : x \to y$ is a property rather than an additional structure, in the sense that arbitrary 2-isomorphisms $f^{-1} \circ f \tilde{\to} \mathrm{id}_x$ and $f \circ f^{-1} \tilde{\to} \mathrm{id}_y$ can be modified to satisfy the appropriate coherence conditions.

The resulting notion is that of a 2-groupoid. Once again we find ourselves in the same quandary, only one level higher: the coherence conditions we have imposed are equations between 2-isomorphisms. The obvious remedy is to introduce 3-groupoids by endowing each $\mathrm{Iso}_X(x, y)$ with the structure of a 2-groupoid, but spelling out the requisite structures and coherence conditions is a daunting combinatorial problem. Moreover, once we finish this arduous task our difficulties become still greater, requiring us to climb a never-ending hierarchy to define n-groupoids for every nonnegative integer n.

1.2 Having dug ourselves into an ever-deepening hole, we change stategies and try to imagine how a hypothetical notion of ∞-groupoid might look. Although this might seem like a huge leap given the difficulties of defining n-groupoids for a fixed n, we are aided by the following heuristic due to Grothendieck.

Postulate 1.2.1 (Homotopy hypothesis)**.** The notion of ∞-groupoid is equivalent to that of a space in the sense of homotopy theory.

We interpret this hypothesis as a requirement on any reasonable definition of ∞-groupoid. In what follows, the word "space" will always be taken in the sense of homotopy theory rather than point-set topology: although topological spaces can serve as a model of the former notion, it is conceptually distinct. For example, an isomorphism of spaces is a homotopy equivalence rather than a homeomorphism, and an isomorphism class of spaces is what is usually referred to as a homotopy type.

To help understand and justify the homotopy hypothesis, we summarize in the following table how the internal structure of a space mirrors that of a higher groupoid.

spaces	∞-groupoids
point	object
path	1-morphism
concatenation of paths	composition of 1-morphisms
constant path	identity 1-morphism
reversed path	inverse 1-morphism
path homotopy	2-morphism
homotopy between path homotopies	3-morphism
$\vdots$	$\vdots$

Various concrete models of spaces are known to homotopy theorists. One possibility, which allows for a literal interpretation of terms like "path" and thereby tracks closely with our intuition about spaces, is to use a well-behaved class of topological spaces such as CW complexes. A disadvantage of this approach is that it involves the real numbers, and spaces are really combinatorial or algebraic rather than analytic objects. So we would like to avoid invoking the continuum.

Another option is to use simplicial sets, or more precisely those satisfying the Kan condition. These so-called Kan complexes are purely combinatorial objects, and have the convenient feature that the Kan condition can be relaxed to include ∞-categories which are not groupoids.

In these notes we will attempt to work in a "model-independent" way. Hopefully the reader is willing to accept that for applications in algebraic geometry and homological algebra, the particularities of the model one uses for spaces are largely irrelevant. Instead, one need only learn the general syntax, meaning which sentences and operations are allowed, along with some basic facts. Of course, to do serious work in homotopy theory or higher category theory, it is not always possible to avoid explicit models.

1.3 Given a space X and points $x, y \in X$, we can form a space $\mathrm{Iso}_X(x, y)$ called the *path space*. A point in $\mathrm{Iso}_X(x, y)$ is a 1-isomorphism or path $x \tilde{\to} y$ in X, an isomorphism in $\mathrm{Iso}_X(x, y)$ is a 2-isomorphism or path homotopy in X, etc. In the case $x = y$ we write $\Omega(X, x) := \mathrm{Iso}_X(x, x)$ for the *loop space* of X at x, or sometimes just ΩX if the point $x \in X$ is understood.

Any set can be viewed as a space where the only isomorphisms are the identity morphisms. Conversely, given a space X we can consider the set $\pi_0 X$ of *connected components*, a.k.a. isomorphism classes. If $\pi_0 X$ consists of a single element then we say that X is *connected*.

For any spaces X and Y we can form their product $X \times Y$. Its points are pairs (x, y) where $x \in X$ and $y \in Y$, paths are pairs consisting of a path in X and a path in Y, and more generally n-isomorphisms are pairs of n-isomorphisms in X and Y. We have $\pi_0(X \times Y) = \pi_0(X) \times \pi_0(Y)$, and if $(x_1, y_1), (x_2, y_2) \in X \times Y$ then

$$\mathrm{Iso}_{X \times Y}((x_1, y_1), (x_2, y_2)) = \mathrm{Iso}_X(x_1, y_1) \times \mathrm{Iso}_Y(x_2, y_2),$$

and in particular $\Omega(X \times Y, (x, y)) = \Omega(X, x) \times \Omega(Y, y)$.

A space X gives rise to a 1-groupoid $\pi_{\leq 1} X$ in the following way. The objects of $\pi_{\leq 1} X$ are the points of X, and for $x, y \in X$ the isomorphisms $x \tilde{\to} y$ in $\pi_{\leq 1} X$ are given by $\pi_0 \, \mathrm{Iso}_X(x, y)$. Concatenation of paths in X satisfies the identity, associativity, and invertibility axioms up to homotopy, which means $\pi_{\leq 1} X$ has the structure of a 1-groupoid as claimed. In particular, for any $x \in X$ we can consider the *fundamental group* $\pi_1(X, x) := \pi_0 \Omega(X, x)$.

The loop space $\Omega(X, x)$ has a canonical point, namely the constant or identity path. Thus we may consider the iterated loop space $\Omega^n(X, x)$ for any $n \geq 0$, where by convention $\Omega^0(X, x) := X$. This allows us to define the n^{th} *homotopy group* $\pi_n(X, x) := \pi_0 \Omega^n(X, x)$. Note that for $n \geq 2$ we can view $\pi_n(X, x)$ as a group object in groups, so by the standard Eckmann-Hilton argument the group structure on $\pi_n(X, x)$ is abelian.

Once we have a working notion of space or ∞-groupoid, we may therefore define an n-groupoid to be a space X such that $\pi_i(X, x) = 1$ for all $x \in X$ and all $i > n$.

1.4 Given spaces X and Y, we have the notion of morphism (a.k.a. continuous map) $f : X \to Y$. A morphism of spaces sends points to points and n-isomorphisms to n-isomorphisms, while respecting the composition structure in a homotopy coherent manner. More generally we can form the *mapping space* $\mathrm{Map}(X, Y)$ whose points are maps $f : X \to Y$, paths are homotopies between maps, etc. For any spaces X, Y, and Z, we have a morphism

$$\mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(X, Z)$$

which encodes the composition law. Composition satisfies natural identity and associativity conditions up to coherent homotopy (we will elaborate on this when we discuss ∞-categories). In particular we have a well-defined 1-category $\mathrm{Ho}(\mathrm{Spc})$ called the *homotopy category of spaces*, whose objects are spaces and morphisms $X \to Y$ are given by $\pi_0 \mathrm{Map}(X, Y)$.

A map of spaces $f : X \to Y$ is called a *homotopy equivalence* or just an *isomorphism* if it is an isomorphism in $\mathrm{Ho}(\mathrm{Spc})$, i.e. there exists a map $g : Y \to X$ and homotopies $g \circ f \xrightarrow{\sim} \mathrm{id}_X$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_Y$. A space is called *discrete* if it is homotopy equivalent to a set, or *contractible* if it is homotopy equivalent to the singleton set pt.

A map $f : X \to Y$ is a homotopy equivalence if and only if the induced map $\pi_0 f : \pi_0 X \to \pi_0 Y$ is bijective and $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $x \in X$ and all $n \geq 1$. In particular, a space X is discrete if and only if $\pi_n(X, x) = 1$ for all $x \in X$ and all $n \geq 1$, and X is contractible if and only if X is discrete and $\pi_0 X = \mathrm{pt}$.

1.5 Given maps of spaces $f : X \to Z$ and $g : Y \to Z$, we can form their *fiber product* $X \times_Z Y$. A point of $X \times_Z Y$ consists of a point $x \in X$, a point $y \in Y$, and a path $f(x) \xrightarrow{\sim} g(y)$ in Z. More generally, an n-morphism in $X \times_Z Y$ consists of n-morphisms α in X and β in Y together with an $(n+1)$-morphism $f(\alpha) \xrightarrow{\sim} g(\beta)$ in Z. In particular we have a canonical map

$$\pi_0(X \times_Z Y) \longrightarrow \pi_0(X) \times_{\pi_0(Z)} \pi_0(Y)$$

which is surjective but generally not injective.

For example, we have $X \times_{\mathrm{pt}} Y = X \times Y$. Another important special case is the *fiber* of a map of spaces $f : X \to Y$ over a point $y \in Y$, defined by $f^{-1}(y) := X \times_Y \{y\}$. For example, given points $x_1, x_2 \in X$ we have $\{x_1\} \times_X \{x_2\} = \mathrm{Iso}_X(x_1, x_2)$, and when $x_1 = x_2$ this becomes $\{x\} \times_X \{x\} = \Omega(X, x)$.

The formation of fiber products is natural in the sense that a diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & Z_1 & \xleftarrow[g_1]{} & Y_1 \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ X_2 & \xrightarrow{f_2} & Z_2 & \xleftarrow[g_2]{} & Y_2 \end{array}$$

together with homotopies $c \circ f_1 \xrightarrow{\sim} f_2 \circ a$ and $c \circ g_1 \xrightarrow{\sim} g_2 \circ b$ determines a map $X_1 \times_{Z_1} Y_1 \to X_2 \times_{Z_2} Y_2$. The particular instance of this when $X_1 = Y_1 = Z_1$ and f_1, g_1 are the identity gives the universal property of the fiber product. Another example is that a map $f : X \to Y$ determines a map $\Omega(X, x) \to \Omega(Y, f(x))$ for any $x \in X$.

Suppose we are given a fiber square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ \mathrm{pt} & \xrightarrow{z} & Z, \end{array}$$

meaning we are given a homotopy from $g \circ f$ to the constant map $X \to Z$ with value z and the induced map $X \to g^{-1}(z)$ is a homotopy equivalence. Then a choice of point $x \in X$, which is the same datum as a point $y = f(x) \in Y$ and a homotopy $g(y) \tilde{\to} z$, determines a map

$$\partial : \Omega(Z, z) = \{z\} \times_Z \{z\} \longrightarrow Y \times_Z \{z\} = X.$$

Looping repeatedly, we get a map $\Omega^{n+1}(Z, z) \to \Omega^n(X, x)$ for all $n \geq 0$. In particular, applying π_0 we obtain the "boundary maps" in a sequence

$$\cdots \longrightarrow \pi_{n+1}(Z, z) \longrightarrow \pi_n(X, x) \longrightarrow \pi_n(Y, y) \longrightarrow \pi_n(Z, z) \longrightarrow \cdots .$$

In fact, this sequence is exact: recall that a sequence of pointed sets

$$(S, s) \xrightarrow{f} (T, t) \xrightarrow{g} (U, u)$$

is called exact at (T, t) if $g \circ f : S \to U$ is the constant map with value u and the resulting map $S \to g^{-1}(u)$ is a bijection, generalizing the usual notion of exactness for sequences of groups. It follows that a map $f : X \to Y$ is a homotopy equivalence if and only if $f^{-1}(y)$ is contractible for all $y \in Y$.

Note that the exactness assertion reduces by induction to the claim that

$$\begin{array}{ccc} \Omega(X, x) & \longrightarrow & \Omega(Y, y) \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \Omega(Z, z) \end{array}$$

is a fiber sequence, and that the sequence

$$\cdots \longrightarrow \pi_1(Y, y) \longrightarrow \pi_1(Z, z) \longrightarrow \pi_0 X \longrightarrow \pi_0 Y \longrightarrow \pi_0 Z$$

is exact. The latter assertion at least can be verified in a model-independent fashion: let's illustrate this by proving exactness at $\pi_0 X$. Suppose we are given a point $x' \in X$ such that $f(x')$ lies in the connected component of y. Identifying $X \tilde{\to} g^{-1}(y)$, can think of x' as a pair (y', α) where $y' \in Y$ and $\alpha : g(y') \tilde{\to} z$. In these terms $f(x') = y'$, so by assumption there exists a path $\beta : y \tilde{\to} y'$. Thus we obtain a loop

$$\gamma : z = g(y) \xrightarrow{g(\beta)} g(y') \xrightarrow{\alpha} z,$$

which represents an element of $\pi_1(Z, z)$. By the construction of γ we see that α lifts canonically to a path $\partial(\gamma) \tilde{\to} x'$, so the connected component of X containing x' lies in the image of $\pi_1(Z, z)$ as desired.

2 Categories

2.1 Now we begin the study of $(\infty,1)$-categories. One can say that $(\infty,1)$-categories are to ordinary categories as spaces are to sets: in particular, in an $(\infty,1)$-category the morphisms from one object to another form a space.

To put these objects in context, we should explain heuristically what an (n,k)-category is for $n,k \geq 0$. The idea is that m-morphisms are allowed for $m \leq n$, and are required to be isomorphisms if $m > k$. In particular (n,k)-categories coincide with (n,n)-categories for all $k > n$, so we may as well assume that $k \leq n$.

For small values of n and k, these are familiar objects. A $(0,0)$-category is just a set. A $(1,0)$-category is a 1-groupoid, and a $(1,1)$-category is an ordinary category. A $(2,2)$-category is a 2-category in the usual sense.

We have been studying spaces or ∞-groupoids, which are $(\infty,0)$-categories. This means that n-morphisms are allowed for all $n \geq 0$, but are required to be invertible for $n > 0$. So an $(\infty,1)$-category should be like a space, except with 1-morphisms not necessarily invertible. A prototypical example is the $(\infty,1)$-category of spaces, which we will denote by Spc.

There are various models of $(\infty,1)$-categories available. A particularly convenient one is the theory of quasicategories, which are simplicial sets satisfying a weakened form of the Kan condition. Below we will give an intrinsic realization of the $(\infty,1)$-category of $(\infty,1)$-categories as a full subcategory of simplicial spaces.

2.2 The *simplex category* Δ is a $(1,1)$-category which will play a central role in what follows. Recall that it consists of finite nonempty linearly ordered sets and weakly order-preserving maps. An equivalent "skeletal" full subcategory has objects $[n] := \{0,1,\cdots,n\}$ for $n \geq 0$.

As a warm-up, let us describe the $(1,1)$-category $\mathrm{Cat}_{\mathrm{ordn}}$ of ordinary or $(1,1)$-categories as a full subcategory of the $(1,1)$-category of simplicial sets

$$\mathrm{Set}^{\Delta^{\mathrm{op}}} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}).$$

For a given simplicial set E, we will use the notation $E^{(n)}$ for its value on $[n]$.

The *nerve* functor

$$N : \mathrm{Cat}_{\mathrm{ordn}} \longrightarrow \mathrm{Set}^{\Delta^{\mathrm{op}}}$$

is defined as follows. Recall that to specify a simplicial set, it suffices to describe for each n its value on $[n]$, the face maps $d_0,\cdots,d_n : [n-1] \to [n]$, and the degeneracy maps $s_0,\cdots,s_n : [n+1] \to [n]$ (of course, one must also check that certain relations are satisfied). Here $d_i : [n-1] \to [n]$ is the injection whose image does not contain i, and $s_i : [n+1] \to [n]$ is the surjection which hits i twice. Given a $(1,1)$-category $\mathscr{C}$, the 0-simplices $N(\mathscr{C})^{(0)}$ consist of objects of $\mathscr{C}$ (so we tacitly assume that $\mathscr{C}$ is small). The 1-simplices $N(\mathscr{C})^{(1)}$ consist of morphisms in $\mathscr{C}$, and more generally the n-simplices $N(\mathscr{C})^{(n)}$ consist of strings of composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} c_{n-1} \xrightarrow{f_n} c_n$$

of length n. The face map $d_i : N(\mathscr{C})^{(n)} \to N(\mathscr{C})^{(n-1)}$ applied to such a string forgets c_i, and for $0 < i < n$ composes $f_{i+1} \circ f_i$. The degeneracy map $s_i : N(\mathscr{C})^{(n)} \to N(\mathscr{C})^{(n+1)}$ replaces c_i with $\mathrm{id}_{c_i} : c_i \to c_i$. We leave it to the reader to verify that the necessary relations are satisfied and to extend N to a functor on $\mathrm{Cat}_{\mathrm{ordn}}$.

Let E be a simplicial set, fix $n_1, n_2 \geq 1$, and put $n := n_1 + n_2$. Consider the pushout square

$$\begin{array}{ccc} [0] & \longrightarrow & [n_1] \\ \downarrow & & \downarrow \\ [n_2] & \longrightarrow & [n] \end{array}$$

in Δ, where $[0] \to [n_1]$ and $[0] \to [n_2]$ send $0 \mapsto n_1$ and $0 \mapsto 0$ respectively, while the maps $[n_1] \to [n]$ and $[n_2] \to [n]$ send $i \mapsto i$ and $i \mapsto i + n_1$ respectively. This gives rise to a square of sets

$$\begin{array}{ccc} E^{(n)} & \longrightarrow & E^{(n_1)} \\ \downarrow & & \downarrow \\ E^{(n_2)} & \longrightarrow & E^{(0)} \end{array} \tag{2.1}$$

Proposition 2.2.1. *The functor N is fully faithful. Its essential image consists of simplicial sets E satisfying the* Segal condition*, which says that for all $n_1, n_2 \geq 1$ the square (2.1) is cartesian.*

We can also characterize groupoids by their nerves. Consider the square

$$\begin{array}{ccc} [0] & \longrightarrow & [1] \\ \downarrow & & \downarrow \\ [1] & \longrightarrow & [2] \end{array}$$

in Δ, where both maps $[0] \to [1]$ send $0 \mapsto 0$, one map $[1] \to [2]$ sends $i \mapsto i$, and the other map sends $0 \mapsto 0$, $1 \mapsto 2$. This gives rise to a square

$$\begin{array}{ccc} E^{(2)} & \longrightarrow & E^{(1)} \\ \downarrow & & \downarrow \\ E^{(1)} & \longrightarrow & E^{(0)}. \end{array} \tag{2.2}$$

Proposition 2.2.2. *A $(1,1)$-category $\mathscr{C}$ is a groupoid if and only if $N(\mathscr{C})$ has the property that (2.2) is cartesian.*

2.3 Loosely speaking, an $(\infty,1)$-category $\mathscr{C}$ is like a $(1,1)$-category, but the morphisms $\mathrm{Hom}_{\mathscr{C}}(c_1, c_2)$ between two objects forms a space, and the composition law

$$\mathrm{Hom}_{\mathscr{C}}(c_2, c_3) \times \mathrm{Hom}_{\mathscr{C}}(c_1, c_2) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(c_1, c_3)$$

is associative with identity up to coherent homotopy.

In particular, we can extract from $\mathscr{C}$ a $(1,1)$-category $\mathrm{Ho}(\mathscr{C})$ whose objects are the objects of $\mathscr{C}$, and whose morphisms are 1-morphisms of $\mathscr{C}$ modulo 2-isomorphism. Note that if $\mathscr{C}$ is actually a space, then $\mathrm{Ho}(\mathscr{C}) = \pi_{\leq 1}\mathscr{C}$ in the previously introduced notation. Conversely, we call $\mathscr{C}$ *ordinary* if $\mathrm{Hom}_{\mathscr{C}}(c_1, c_2)$ is discrete for all objects c_1, c_2 in $\mathscr{C}$. An ordinary $(\infty,1)$-category is the same thing as a $(1,1)$-category.

We can use the terminology of n-morphisms for $n \geq 1$ in an $(\infty,1)$-category $\mathscr{C}$: a 0-morphism is an object, and for $n \geq 1$ an n-morphism is an $(n-1)$-morphism in $\mathrm{Hom}_{\mathscr{C}}(c_1, c_2)$ for some objects c_1, c_2 of $\mathscr{C}$. In particular n-morphisms in $\mathscr{C}$ are invertible for $n > 1$.

A full subcategory $\mathscr{C}_0$ of $\mathscr{C}$ is determined by which objects of $\mathscr{C}$ belong to it: for any objects c_1, c_2 in $\mathscr{C}_0$ we have

$$\mathrm{Hom}_{\mathscr{C}_0}(c_1, c_2) = \mathrm{Hom}_{\mathscr{C}}(c_1, c_2).$$

Note that a full subcategory of $\mathscr{C}$ is the same datum as a full subcategory of $\mathrm{Ho}(\mathscr{C})$.

For any two $(\infty, 1)$-categories $\mathscr{C}$ and $\mathscr{D}$ we can consider their product $\mathscr{C} \times \mathscr{D}$, whose n-morphisms are pairs consisting of an n-morphism of $\mathscr{C}$ and an n-morphism of $\mathscr{D}$.

Another fundamental construction attaches a space $\mathscr{C}^{\mathrm{grpd}}$ to any $(\infty, 1)$-category $\mathscr{C}$, whose 1-morphisms are the invertible 1-morphisms of $\mathscr{C}$, and whose n-morphisms for $n \neq 1$ are the same as those of $\mathscr{C}$.

2.4 We also grant ourselves the notion of a functor of $(\infty, 1)$-categories $F : \mathscr{C} \to \mathscr{D}$, which sends n-morphisms of $\mathscr{C}$ to n-morphisms of $\mathscr{D}$, in a way which respects the composition laws up to coherent homotopy. In particular we require that F respect the structure of mapping spaces, i.e. for any objects c_1, c_2 in $\mathscr{C}$ we have a map of spaces

$$\mathrm{Hom}_{\mathscr{C}}(c_1, c_2) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(c_1), F(c_2)).$$

If this map is an isomorphism for all c_1, c_2, we say that F is *fully faithful*.

A functor of $(\infty, 1)$-categories $F : \mathscr{C} \to \mathscr{D}$ induces a functor of $(1, 1)$-categories $\mathrm{Ho}(F) : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$. We call F *essentially surjective*, respectively *conservative*, if $\mathrm{Ho}(F)$ is so.

For any $(\infty, 1)$-categories $\mathscr{C}$ and $\mathscr{D}$, functors assemble into an $(\infty, 1)$-category $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$. Moreover, we have a composition law

$$\mathrm{Fun}(\mathscr{D}, \mathscr{E}) \times \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \longrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{E})$$

is which is associative with identity up to coherent isomorphism. This is part of the structure of $(\infty, 2)$-category on the totality of $(\infty, 1)$-categories, but we will not pursue this now. We will, however, describe it as an $(\infty, 1)$-category.

2.5 Write $\mathrm{Spc}^{\Delta^{\mathrm{op}}} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Spc})$ for the $(\infty, 1)$-category of simplicial spaces. We call a simplicial space E a *Segal space* if the square (2.1) in Spc is cartesian for all $n_1, n_2 \geq 1$. Observe that we can extract a $(1, 1)$-category from any such E with objects $\pi_0(E^{(0)})$ and morphisms $\pi_0(E^{(1)})$, and in particular we can consider the space $(E^{(1)})^{\mathrm{invrt}}$ consisting of those components of $E^{(1)}$ which are invertible in $\pi_0(E^{(1)})$. In particular, the degeneracy map $E^{(0)} \to E^{(1)}$ factors through $(E^{(1)})^{\mathrm{invrt}}$.

Then Cat, the $(\infty, 1)$-category of $(\infty, 1)$-categories, is (equivalent to, or defined to be, according to one's taste) the full subcategory of $\mathrm{Spc}^{\Delta^{\mathrm{op}}}$ consisting of Segal spaces E satisfying the following *completeness* (or *univalence*) condition: the degeneracy map

$$E^{(0)} \longrightarrow (E^{(1)})^{\mathrm{invrt}}$$

is an isomorphism. For an $(\infty, 1)$-category $\mathscr{C}$, the corresponding complete Segal space E satisfies

$$E^{(n)} = \mathrm{Fun}([n], \mathscr{C})^{\mathrm{grpd}}$$

where we view $[n]$ as a $(1, 1)$-category in the usual way.

There is a canonical fully faithful functor $\mathrm{Spc} \to \mathrm{Cat}$ which sends a space X to the constant simplicial space with value X. A complete Segal space E comes from a space if the degeneracy map $E^{(0)} \to E^{(1)}$ is an isomorphism.

A functor $F : \mathscr{C} \to \mathscr{D}$ of $(\infty, 1)$-categories is called an *equivalence* if it is so in Ho(Cat). It is known that F is an equivalence if and only if it is fully faithful and essentially surjective.

We can describe some of the previously introduced constructions with an $(\infty, 1)$-category $\mathscr{C}$ in terms of the corresponding complete Segal space E. The space $\mathscr{C}^{\mathrm{grpd}}$ is just $E^{(0)}$. If we view two objects c_1, c_2 of $\mathscr{C}$ as points in $E^{(0)}$, the mapping space is the fiber

$$\mathrm{Hom}_{\mathscr{C}}(c_1, c_2) = E^{(1)} \underset{E^{(0)} \times E^{(0)}}{\times} \{(c_1, c_2)\}.$$

Ordinary categories are those $(\infty, 1)$-categories whose complete Segal space takes values in 1-groupoids. Note that this is *not* compatible with our earlier realization of ordinary categories as Segal spaces valued in sets.

Observe that $\mathrm{Spc}^{\Delta^{\mathrm{op}}}$ has a canonical involutive autoequivalence given by precomposition with the involution $\mathrm{rev} : \Delta \to \Delta$. The latter is the identity on objects, and defined on a morphism $\alpha : [n_1] \to [n_2]$ by

$$\mathrm{rev}(\alpha) : [n_1] \xrightarrow{\sim} [n_1]^{\mathrm{op}} \xrightarrow{\alpha} [n_2]^{\mathrm{op}} \xrightarrow{\sim} [n_2],$$

where we used the canonical isorphism $[n] \tilde{\to} [n]^{\mathrm{op}}$ which sends $i \mapsto n - i$. The induced involution of $\mathrm{Spc}^{\Delta^{\mathrm{op}}}$ preserves Cat, where we denote it by $\mathscr{C} \mapsto \mathscr{C}^{\mathrm{op}}$.