1 Preliminaries

1.1 Kac-Moody and Affine Lie Algebras

Kac-Moody algebras are a class of infinite dimensional Lie algebras that has a similar representation theory to that of semi-simple Lie algebras. Namely, admitting a root space decomposition, an associated (ind) Lie group, a casimir element, Borel-Weil-Bott etc. This can be done using a generalized Cartan matrix (i.e. dropping the positive definite requirement).

Affine Kac-Moody algebras are a subclass of KM algebras associated to semi-simple Lie algebras, and are defined as a central extension \( \hat{g} \) (or its polynomial counterpart \( \hat{g}^{\text{pol}} \)) of the loop algebra \( g((t)) := g \otimes \mathbb{C}((t)) \) (resp. \( g[t^{\pm 1}] \)), with Lie bracket

\[
[X \otimes f, Y \otimes g] = [X, Y] \otimes fg
\]

The polynomial Kac-Moody algebras can be defined as an abstract Kac-Moody algebra using a Cartan matrix: Start with a semi-simple Lie algebra \( g \) with a Cartan matrix \( A = (a_{ij})_{1 \leq i, j \leq n} \). Add a zero row and column \( \hat{A} = (a_{ij})_{0 \leq i, j \leq n} \) with \( a_{0,0} = 2, a_{0,j} = -\alpha_j (\hat{\theta}), a_{j,0} = -\theta (\hat{\alpha}_j), j > 0 \). \( \hat{g}^{\text{pol}} \) is the Lie algebra associated to \( \hat{A} \). By completing with respect to the \( t \)-adic topology, we can get the entire affine Lie algebra \( \hat{g} \).

A second definition can be given by explicitly describing the central extension. This allows us to relate representation theory of \( g \) to that of \( \hat{g} \) as well as to that of the loop group \( LG \).

In this talk we’ll use the second point of view.

**Definition 1.** Let \( g \) be a semi-simple Lie algebra, and \( \kappa \) an invariant symmetric bilinear form on \( g \). The affine Lie algebra associated to \( (g, \kappa) \) is the central extension

\[
0 \to \mathbb{C}1 \to \hat{g}_\kappa \to g((t)) \to 0
\]

with 1 central and

\[
[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \kappa(X, Y) \text{Res}_{t=0} f dg \cdot 1
\]

Such \( \kappa \)'s are classified by \( H^2(g, \mathbb{C}) \) and can be shown to be one dimensional. In particular, fixing \( \kappa_0 \) the normalized Killing form with \( \kappa_0(\hat{\alpha}, \hat{\alpha}) = 2 \) for any long root \( \alpha \), any other \( \kappa \) will be a scalar multiple of \( \kappa_0 \). The critical form \( \kappa_c \) is defined as

\[
\kappa_c = -\frac{1}{2} \text{Kil}
\]
Definition 2. A \( \hat{\mathfrak{g}} \) representation is called smooth if it is killed by \( t^{-N} \mathfrak{g}[[t]] \) for \( N \gg 0 \). Let \( \hat{\mathfrak{g}} \) be the category of smooth \( \hat{\mathfrak{g}} \)-modules on which \( c \) acts as the identity. Equivalently, defining

\[
U_\kappa (\hat{\mathfrak{g}}) := U (\hat{\mathfrak{g}}) / (1 - 1)
\]

this category is the category of modules over the completion \( \overline{U}_\kappa (\hat{\mathfrak{g}}) \) in the \( t \)-adic topology.

\( \kappa \) is called the level of the representation. One of the main goals of today’s talk is to explain the special role of representations of critical level.

2 Finite Dimensional Case

Let \( G \) be a semi-simple simply connected Lie algebra, \( \mathfrak{g} = \text{Lie} G \), \( B \subset G \) a Borel subgroup. \( G \) acts on \( \mathfrak{g} \)-mod by \( V \mapsto \text{Ad}_g V \). We want to understand the categories

\[
\text{Rep} G = \mathfrak{g} \text{-mod}^G \subset \mathfrak{g} \text{-mod}^B \subset \mathfrak{g} \text{-mod}
\]

The first category is the simplest: It is a semi-simple category with simple objects \( V_\lambda \) for each dominant weight \( \lambda \). i.e.

\[
\text{Rep} G \simeq \bigoplus_{\lambda \in P^+} \text{Vect}_\lambda
\]

We want an analogous decomposition for the other two.

2.1 The center

For a DG-category \( \mathcal{C} \), the center \( Z (\mathcal{C}) \) is by definition endomorphisms of the identity functor. It is a commutative algebra. For \( x : T \to \text{Spec} Z (\mathcal{C}) \), let \( I \subset Z (\mathcal{C}) \) be the corresponding ideal. Define \( \mathcal{C}_x \) the full subcategory where the action of \( Z (\mathcal{C}) \) factors through \( Z (\mathcal{C}) / I \). If \( \mathcal{C} = A \)-mod, we have an equivalence \( Z (A \text{-mod}) \sim \to Z (A) \) given by restriction.

Let \( A = U (\mathfrak{g}) \). Choose a basis \( J_a; a = 1, \ldots, \dim \mathfrak{g} \) for \( \mathfrak{g} \) and a dual basis \( J^a \) with respect to \( \kappa_0 \). The Casimir element

\[
\Omega_1 = \frac{1}{2} \sum_a J^a J_a
\]

lies in the center. The general form of the center is described via the Harish-Chandra homomorphism

\[
Z (\mathfrak{g}) \sim \to U (\mathfrak{h})^W \subset U (\mathfrak{h})
\]

given by restriction of \( U (\mathfrak{g}) \to U (\mathfrak{g}) / U (\mathfrak{g}) \cdot \mathfrak{n} \simeq U (\mathfrak{g}) \otimes_{U (\mathfrak{h})} U (\mathfrak{h}) \).

\( U (\mathfrak{h})^W \) is a polynomial algebra \( \mathbb{C} [\Omega_i], i = 1, \ldots, \text{rank} \mathfrak{g} \), with \( P_i \in U (\mathfrak{g})_{d_i+1} \), \( d_i \) the \( i \)-th exponent.

Thus \( \text{Spec} Z (\mathfrak{g}) = \text{Spec} Z (\mathfrak{g} \text{-mod}) \) is a finite dimensional affine space. Under certain integrability conditions, \( \mathfrak{g} \text{-mod} \) will decompose into direct sum of the fibers over different points of this affine space. This is described in the next section.
2.2 Category $\mathcal{O}$

Let $\mathcal{O} = \mathfrak{g} \text{-mod}^{N,ss} \subset \mathfrak{g} \text{-mod}^N$ be the subcategory where $H$ acts semi-simply.

Let $\lambda \in \mathfrak{h}^*$. We get a map

$$\chi : Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{h})^W \rightarrow U(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C}$$

$\chi$ is a well defined element of $\mathfrak{h}^*//W$. Denote $\varpi : \mathfrak{h}^* \rightarrow \mathfrak{h}^*//W$ the projection, so $\chi = \varpi(\lambda)$.

Define $\mathcal{O}_\chi \subset \mathcal{O}$ the subcategory where the action of the center factors through the formal neighborhood of the closed point corresponding to $\chi$.

We then have a decomposition

$$\mathcal{O} \simeq \bigoplus_{\chi \in \text{Spec } Z(\mathfrak{g})} \mathcal{O}_\chi \, .$$

The blocks generically have a simple description: For $\lambda - \rho$ dominant and anti-dominant, $\mathcal{O}_{\varpi(\lambda)}$ is a direct sum of $|W|$-categories, each equivalent to Vect and generated by the Verna module $M_{w,\lambda}$. For $\lambda = -\rho$, it is a single copy of Vect generated by $M_\lambda$. For dominant integral $\lambda$, we can get nontrivial extensions between $M_{w,\lambda}$ and $M_{w',\lambda}$.

If we add the requirement that $\mathfrak{h}$ acts with integral eigenvalues, we get the category $\mathfrak{g} \text{-mod}^B$. Thus the latter has a similar decomposition but with $\lambda$ dominant and integral. Other blocks can be described as $\mathfrak{g} \text{-mod}^{\lambda,B}$ - $B$-integrable $\lambda$-twisted representations.

2.3 Flag Variety

We can use the geometry of the flag variety $G/B$ to construct elements of $\mathcal{O}$: We have an action of $G$ on $G/B$. Taking derivatives, we get a map $\mathfrak{g} \rightarrow \text{Vect } (G/B)$. Let $\lambda$ be a regular dominant weight. We have an extension

$$0 \rightarrow \mathcal{O}_{G/B} \rightarrow \mathcal{D}_{\leq 1,\lambda}(G/B) \rightarrow \text{Vect } (G/B) \rightarrow 0$$

and thus a map

$$U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}),\varpi(\lambda)} \mathbb{C} := U_{\varpi(\lambda)}(\mathfrak{g}) \rightarrow \mathcal{D}_\lambda(G/B)$$

from $\mathfrak{g}$ to twisted differential operators on $G/B$ (defined as the quotient $\mathcal{D}(G)/\mathcal{D}(G) \cdot (\xi - \lambda(\xi); \xi \in \mathfrak{b})$).

**Theorem 1.** (Beilinson-Bernstein Localization) The induces functor

$$\text{Loc} : U_{\varpi(\lambda)}(\mathfrak{g}) \text{-mod} \rightarrow \mathcal{D}_\lambda \text{-mod } (G/B)$$

is an equivalence, with an inverse given by global sections $\Gamma_\lambda$.

We get an induced functor

$$\mathcal{D}_\lambda \text{-mod } (G/B)^B \rightarrow \mathcal{O}_{\varpi(\lambda)}$$

For a Schubert cell $Nw \subset G/B$, let $\delta(Nw) := i_{Nw*} \mathcal{O}_{Nw}$. This is an $N$-invariant $\mathcal{D}_\lambda$-module, and so $\Gamma_\lambda(G/B, \delta(Nw)) \in \mathcal{O}_{\varpi(\lambda)}$. This is a twisted Verma module. This name is justified by the following:
Proposition 1. For \( w = w_0 \), \( \Gamma (\delta (wN)) = \Gamma (\text{Fun} (N^-)) \simeq M_\lambda^\vee \). For \( w_0 = 1 \), \( \Gamma (\delta (Nw)) = \Gamma (\delta_1) \simeq M_\lambda \)

If we further restrict to \( \mathcal{D}_\lambda - \text{mod} (G/B)^G \), we get integrable modules: Indeed, if \( \lambda \) is integral dominant \( \mathcal{D}_\lambda (G/B) \simeq \mathcal{D} (O (\lambda)) \). We have \( \omega_{G/B} \in \mathcal{D} - \text{mod} (G/B)^G \) and by Borel-Weil-Bott:

\[
\Gamma \left( \frac{G}{B}, O (\lambda) \otimes \omega_{G/B} \right) \simeq \Gamma \left( \frac{G}{B}, O (-\lambda) \right) \simeq V_\lambda
\]

3 Vacuum and integrable representations

We want to generalize the above to the affine case. That is, a description of the center, that will give a decomposition of some affine category \( \mathcal{O} \). We start by describing the flag variety. Unlike the finite case, there are several natural choices for a Borel subgroup. Several unequivalent choices are \( G (O) \), \( G (\mathbb{C}[t^{-1}]) \), \( N (K) T (O) \), \( I \), where \( I \) is the preimage of \( B \) under the evaluation map \( G (O) \to G \). The first will correspond to Weyl / Verma modules, the second to the corresponding contragradient modules, and the last to Wakimoto modules. We'll focus on the first: In that case the flag variety is \( \text{Gr}_{G/B} = G (O) \subset G (K) \). This is a maximal parabolic subgroup of \( \hat{G} \) regarded as an abstract Kac-Moody algebra. We have a global section functor

\[
\Gamma^{\text{IndCoh}} : \mathcal{D}_\kappa - \text{mod} (\text{Gr}_{G}) \to \hat{g}_\kappa - \text{mod}
\]

and thus a map

\[
\mathcal{D}_\kappa - \text{mod} (\text{Gr}_{G})^{G(O)} \to \hat{g}_\kappa - \text{mod}^{G(O)}
\]

Taking \( \delta_1 \in \mathcal{D}_\kappa - \text{mod} (\text{Gr}_{G})^{G(O)} \), we get

\[
\mathbb{V}_\kappa := \Gamma_\kappa (\text{Gr}, \delta_1) \in \hat{g}_\kappa - \text{mod}^{G(O)}
\]

This is the vacuum module of level \( \kappa \). Explicitly:

\[
\mathbb{V}_\kappa \simeq \text{Ind}_{L^+g \boxtimes 1}^\hat{g} C
\]

where \( L^+g \) acts by zero and \( 1 \) acts by the identity. If we replace \( C \) with any other \( V_\lambda \), we get the Weyl module \( \mathbb{V}_\kappa^\lambda \).

As for integrable modules: We have \( \text{Pic} (\text{Gr}_{G}) \simeq \mathbb{Z} O (1) \). If \( \kappa = k \cdot \kappa_0 \) is positive definite and integral, \( \mathcal{D}_\kappa = \mathcal{D} (O (k)) \), and:

**Theorem 2. (Affine Borel-Weil-Bott)**

\[
\mathbb{V}_{\kappa_0}^{\text{int}} := \Gamma (\text{Gr}_{G}, O (k) \otimes \omega_{\text{Gr}_{G}}) \simeq \Gamma (\text{Gr}_{G}, O (-k))
\]

is an integrable module concentrated in degree 0, and is isomorphic to the unique maximal quotient of \( \mathbb{V}_{\kappa_0} \).

**Remark.** In a similar way as above we can construct integrable modules \( \mathbb{V}_{\kappa_0}^{\lambda,\text{int}} \) for any dominant integral \( \lambda \) satisfying \( \lambda \left( \hat{\theta} \right) \leq \ell \). We have a fully faithful embedding

\[
\hat{g}_\kappa - \text{mod}^{G(K)} \hookrightarrow g - \text{mod}^G
\]

sending \( \mathbb{V}_{\kappa_0}^{\lambda,\text{int}} \) to \( V_\lambda \). In particular, this is a semi-simple category with finitely many simple objects.
4 The Center

4.1 The Casimir element

We’re interested in the center $Z(\hat{\mathfrak{g}}_\kappa) := Z(\hat{\mathfrak{g}}_\kappa\text{-mod}) \simeq Z(U_\kappa(\hat{\mathfrak{g}}))$.

A first attempt will be to construct the loop of the Casimir element: For an element $X \in \mathfrak{g}$, let $X_n = X \otimes t^n$ and $X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n-1}$ (for now $z^{-n-1}$ is just an index). Then we would like to define

$$\Omega^{\text{naive}}_1(z) = \frac{1}{2} \sum_a J^a(z) J_a(z)$$

The problem is that this is not even an element of $U_\kappa(\hat{\mathfrak{g}})$: Take for example $\mathfrak{g} = \mathfrak{sl}_2$, so that $\Omega_1 = \frac{1}{2} (e f + f e + \frac{1}{2} h^2)$. Then

$$\Omega^{\text{naive}}_1(z) = \frac{1}{2} \left( e(z) f(z) + f(z) e(z) + \frac{1}{2} h(z)^2 \right)$$

Take e.g. the first term:

$$e(z) f(z) = \sum_N \left( \sum_{n+m=N} e_n f_m \right) z^{-N-1}$$

but we only allow $m \to \infty$.

4.2 Fields and OPE

The solution for that comes from realizing $X(z)$ as an actual family of operators parametrized by points on a curve. Then interpret the product as a product of operators. We’ll only allow operators satisfying a locality conditions - those are called fields. The coefficients of $z^n$ will then be elements of universal enveloping algebra, and in fact will generate a Lie-subalgebra.

Since we want our operators to vary along a curve, or at least a formal disk, we need them to be $\text{Aut}(D)$-invariant. Thus we need an element of $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$. We’ll thus take the vacuum $V_\kappa$, i.e. $X(z) \in \text{End} V_\kappa((z))$.

Definition 3. A collection of operators $\mathcal{F} = \{X(z) \in \text{End} V_\kappa((z))\}$ is a set of fields if they satisfy locality: For $X, Y \in \mathcal{F}$, $X(z) Y(w)$ is well defined as an element of $\text{End} V_\kappa((z_1))((z_2))$. We then require that when taking the limit $z \to w$, we get a decomposition

$$\lim_{z \to w} X(z) Y(w) =: X(w) Y(w) + [X(z), Y(w)]$$

with $X(z) Y(w)$: regular in $z - w$ and $[X(z), Y(w)]$ is a sum of delta functions supported on the diagonal $z - w$. That is, it has a Taylor expansion with respect to $(z-w)^{-1}$.

Given two fields $X(z), Y(z)$, we define their product to be $X(z) Y(z)$ : - this is the normally ordered product, and is again a field. The commutator $[X^1(z_1), X^2(z_2)]$ is the chiral bracket that will be discussed later.
**Theorem 3.** (State-Field correspondence) For $X_{(-1)} \in t^{-1}\mathfrak{g} \subset \mathbb{V}_\kappa$, let $X(z) = X_{(-1)}(z) = \sum_n X(n)z^{-n-1}$. By repeatedly applying normally ordered product, define $X(z)$ for any $X \in \mathbb{V}_\kappa$. Then the collection $\{X(z) : X \in \mathbb{V}_\kappa\}$ is a collection of fields on $\mathbb{V}_\kappa$.

We’ll need the following consequence of locality:

**Proposition 2.** For any $X, Y \in \mathbb{V}_\kappa$,

$$[X(z), Y(w)] = \sum_{n \geq 0} (X_n \cdot Y)(w) \cdot (z - w)^{-n-1}$$

The above formula gives a Lie algebra structure on the Fourier coefficients, which is compatible with the Lie algebra structure on the universal enveloping algebra. In other words, if we denote by $L(\mathbb{V}_\kappa)$ the completion of the span of all Fourier coefficients, that is the completion of symbols $A_{[m]}$, $A \in \mathbb{V}_\kappa$, $m \in \mathbb{Z}$ with respect to the topology defined by $m \to \infty$, we have a Lie algebra embedding

$$L(\mathbb{V}_\kappa) \to \overline{U}_\kappa(\hat{\mathfrak{g}})$$

**Definition 4.** $\mathfrak{z}_\kappa(\hat{\mathfrak{g}}) := \text{End}_{\mathfrak{z}_\kappa}(\mathbb{V}_\kappa) \simeq (\mathbb{V}_\kappa)^{\mathfrak{z}_\kappa}$

So we have a map $Z_\kappa(\hat{\mathfrak{g}}) \to \mathfrak{z}_\kappa(\hat{\mathfrak{g}})$. We’ll start by constructing an element of $\mathfrak{z}_\kappa(\hat{\mathfrak{g}})$, and then show it lifts to $Z_\kappa(\hat{\mathfrak{g}})$.

### 4.3 Segal-Sugawara operator

We can now define the analog for the Casimir element: It will just be the field corresponding to the usual Casimir element (realized as an element of $\mathfrak{g} \otimes t^{-1}$)

$$\Omega_1(z) = \frac{1}{2} \sum_a : J^a(z) J_a(z) :$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2$, the first term will be

$$e(z) f(z) = \sum_N \left( \sum_{n+m=N, n<0} e_n f_m + \sum_{n+m=N, n\geq 0} f_m e_n \right) z^{-N-1}$$

**Proposition 3.** For any $J_b$, we have

$$[J^b(z), \Omega_1(z)] = \frac{\kappa - \kappa_c}{\kappa_0} J^b(z_2) \cdot (z_1 - z_2)^{-2}$$

In particular, all its Fourier coefficients $\Omega_{1,(m)}$ are in the center of $\text{End}(\mathbb{V}_\kappa)$ precisely if $\kappa = \kappa_c$.

**Proof.** By [2] we have

$$[J^b(z), \Omega_1(z)] = \sum_{n \geq 0} (J^n_b \cdot \Omega_1)(z_1) \cdot (z_2 - z_1)^{-n-1}$$
Since $\Omega$ is of degree 2, any term with $n > 2$ will vanish. We’re left with $n = 0, 1, 2$.

For $n = 2$:

$$J_2^b \cdot \frac{1}{2} \sum_a J^a_{-1} J_{a,-1} = \frac{1}{2} \sum_a \left( J^a_{-1} J_2^b + [J_2^b, J^a_{-1}] \right) J_{a,-1}$$

$$= \frac{1}{2} \sum_a J^a_{-1} [J_2^b, J_{a,-1}] + [J_2^b, J^a_{-1}] J_{a,-1}$$

$$= \frac{1}{2} \sum_a J^a_{-1} [J_2^b, J_a] + [J_2^b, J^a] J_{a,-1}$$

$$= \frac{1}{2} \sum_a \kappa ([J_2^b, J^a], J_{a,-1}) = \frac{1}{2} \sum_a \kappa (J_2^b, [J^a, J_{a,-1}]) \quad (4.1)$$

By choosing an orthogonal basis we can see that the last term equals zero.

For $n = 1$:

$$J_1^b \cdot \frac{1}{2} \sum_a J^a_{-1} J_{a,-1} = \frac{1}{2} \sum_a \left( J^a_{-1} J_1^b + [J_1^b, J^a]_0 + \kappa (J_1^b, J^a) \right) J_{a,-1}$$

$$= \frac{1}{2} \sum_a J^a_{-1} \kappa (J_1^b, J_a) + [[J_1^b, J^a], J_{a,-1}] + \kappa (J_1^b, J^a) J_{a,-1} \quad (4.2)$$

$$= \sum_a \kappa (J_1^b, J_a) J_{a,-1} + \frac{1}{2} \sum_a [[J_1^b, J^a], J_{a,-1}]$$

The first term is the decomposition of $J^b$ by the basis $J^a_{-1}$, but multiplied by $\frac{\kappa}{\kappa - \kappa_c}$. The second term is exactly the adjoint action of the Casimir element on $J^b$, thus equals $\frac{1}{2} \kappa \frac{\kappa}{\kappa - \kappa_c} = -\frac{\kappa}{\kappa_c}$. By a similar argument we can show the $n = 0$-th term vanishes, and so overall we have exactly the expression in the proposition.

**Corollary 1.** $\Omega_{1,(m)} \in \mathfrak{z}_{\kappa_c} (\hat{\mathfrak{g}})$ for all $m$. Furthermore, since $J^a_n$ generates the entire universal enveloping algebra, we get $\Omega_{1,(m)} \in Z_{\kappa_c} (\hat{\mathfrak{g}})$.

For non-critical level, define $\tilde{\Omega}_m = \frac{\kappa}{\kappa - \kappa_c} \Omega_m$. By the above, we see that

$$[\tilde{\Omega}_m, J^a_n] = -n J^a_n$$

That is - $\tilde{\Omega}_1$ is the grading operator. In particular we get:

**Corollary 2.** For non-critical level the center $\mathfrak{z}_{\kappa} (\hat{\mathfrak{g}})$ is trivial.

**Proof.** By induction on the length, one can show that an element is central iff it is killed by all $A_n, A \in \mathfrak{V}_{\kappa}, n \geq 0$. In particular, $\tilde{\Omega}_1$ will act by zero. But this is the grading operator, so the central element must be the highest weight vector. \qed

In fact, one can show that the same is true for the entire center: For non-critical level $Z_{\kappa} (\hat{\mathfrak{g}})$ is trivial.
Corollary 3. For $g = sl_2$, $\mathfrak{z}_{\kappa_c}(\hat{g}) \simeq \mathbb{C}[\Omega_{1,(m)}]_{m \leq -2}$, and $Z_{\kappa_c}(\hat{g}) \simeq \mathbb{C}[\Omega_{1,(m)}]_{m \in \mathbb{Z}}$.

Proof. (sketch) For $m > -2$, $\Omega_{1,(m)}$ will act by zero on the highest weight vector, thus as an element of End $V_{\kappa}$ it is the zero operator. We’ve already constructed a map $\mathbb{C}[\Omega_{1,(m)}]_{m \leq -2} \rightarrow \mathfrak{z}_{\kappa_c}(\hat{g})$. To show it is an isomorphism, one shows it is an isomorphism on the associated graded. The latter is just the jet space $gr \mathfrak{z}_{\kappa_c}(\hat{g}) \simeq JZ(\hat{g}) \simeq J\mathbb{C}[\Omega_1] \simeq \mathbb{C}[\Omega_{1,(m)}]$. By the commutation relation any element $\Omega_{1,(m)}$, $m \in \mathbb{Z}$ will also commute with any $J_n$, and so we get a map $\mathbb{C}[\Omega_{1,(m)}] \rightarrow Z_{\kappa_c}(\hat{g})$, and again by a similar (but more involved) argument we can relate the associated graded of the latter with $LZ(\hat{g})$, hence this map is an isomorphism. 

For a general $g$, we have $\Omega_n \in Z(\hat{g})$ for $n = 1, \ldots, \text{rk } g$. One can indeed show that a similar construction gives the entire center:

Theorem 4. $\mathfrak{z}_{\kappa_c}(\hat{g}) \simeq \mathbb{C}[\Omega_{n,(m)}]_{n=1,\ldots,\text{rk } g, m \leq -2}$ where $\Omega_{n,(m)}$ is the $m$-th Fourier coefficient of $\Omega_n(z)$. The center $Z_{\kappa_c}(\hat{g})$ is isomorphic to the completed topological algebra generated by all Fourier coefficients $\Omega_{n,m}, m \in \mathbb{Z}$.

However, an explicit description of those elements is very difficult, and the proof is rather indirect. Frenkel-Feigin’s original proof used a ”free field realization” technique - embedding the vertex algebra $V_{\kappa_c}$ into a vertex algebra associated to a commutative Lie algebra. Next week we’ll see a shorted, geometric proof.