2/21/24

Verdier duality on the Ran Space

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Goal: Prove "$\bigotimes_{x \in X} \Gamma(BG, k)$" $\xrightarrow{\sim} \Gamma(\mathrm{Bun}_G, k)$.

$$\begin{tikzcd} X \arrow[d, "p"] \\ pt \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CAlg}(\mathrm{Sh}(X)) \arrow[r, "\mathrm{oblv}"] \arrow[d, shift right, "\int_X"'] & \mathrm{Sh}(X) \\ \mathrm{CAlg} \arrow[u, shift right, "p^!"'] \arrow[r, "\mathrm{oblv}"] & \mathrm{Vect} \arrow[u, "p^!"'] \end{tikzcd}$$

Define as left adjoint

What is $\int_X$?

E.g. For $X = \{1, \dots, n\}$, $\int_X (A_1, \dots, A_n) \longrightarrow B$

$$\int_X (A_1, \dots, A_n) = \bigotimes_{x \in X} A_x$$

For $X = \{1, \dots, n\}$, what is nonunital $\int_x$?

In fact, $(\mathrm{AddUnit}^{\mathrm{aug}}, \mathrm{Takeout})$ are equivalences!

$$\begin{tikzcd} & \mathrm{CAlg}_{\mathrm{un,aug}} \arrow[dl, shift left, "\mathrm{Takeout}"] \arrow[dr, "\mathrm{oblv\ Aug}"] & \\ \mathrm{CAlg} \arrow[ur, shift left, "\mathrm{AddUnit}^{\mathrm{aug}}"] \arrow[rr, shift left, "\mathrm{AddUnit}"] & & \mathrm{CAlg}_{\mathrm{un}} \arrow[ll, shift left, "\mathrm{oblv\ unit}"] \end{tikzcd} \qquad \begin{tikzcd} & A \oplus \mathbb{1} \arrow[dr, mapsto] & \\ A \arrow[ur, mapsto] \arrow[rr] & & A \oplus \mathbb{1} \end{tikzcd}$$

$$\Rightarrow \int_X (A_1, \dots, A_n) = \bigoplus_{\emptyset \neq S \subseteq X} \left(\bigotimes_{i \in S} A_i \right)$$

Recall $\mathrm{ComFactAlg} \underset{\mathrm{Fact}}{\overset{(-)^!}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Sh}(X))$, $\sim$

$$\mathrm{Fact}(A)_I \simeq \bigotimes_{i \in I} A_i$$

$$\boxed{\text{Prop } \int_X A \simeq \Gamma_c(\mathrm{Ran}, \mathrm{Fact}(A))}$$

computes unital version of $\int_X$.

Consider
$$\begin{tikzcd} \mathrm{Bun}_G \times X \arrow[r, "\mathrm{ev}"] \arrow[dr] & B_X G \arrow[d, "p"] \\ & X \end{tikzcd}$$

$$\mathrm{Fact}(p_* p^* \omega_X) = B$$

$$\Rightarrow p_* p^* \omega_X \longrightarrow \Gamma(\mathrm{Bun}_G, k) \otimes \omega_X$$

$\Rightarrow$ Have map $\quad \Gamma_c(\mathrm{Ran}, B) \longrightarrow \Gamma(\mathrm{Bun}_G, k)$

__Thm B.__ This map is isomorphism.

Previously,

$$\begin{tikzcd} \mathrm{Gr}_{G,\mathrm{Ran}} \arrow[rr, "\pi"] \arrow[dr, "p"'] & & \mathrm{Bun}_G \times \mathrm{Ran} \arrow[dl, "pr"] \\ & \mathrm{Ran} & \end{tikzcd}$$

$$\Rightarrow A = p_! \, \omega_{\mathrm{Gr}_{G,\mathrm{Ran}}} \longrightarrow \Gamma_c(\mathrm{Bun}_G, \omega) \otimes \omega_{\mathrm{Ran}}$$

__Thm A__ $\quad \Gamma_c(\mathrm{Ran}, A) \xrightarrow{\sim} \Gamma_c(\mathrm{Bun}_G, \omega)$ (proved last time!) by homological contractibility of Ω.

What mechanism is responsible for

$$\Gamma_c(\mathrm{Ran}, A)^\vee \simeq \Gamma_c(\mathrm{Ran}, B) \, ?$$

as vector spaces$/k$!

__Answer:__ Koszul duality!

Viewed as $\in$ Spc.

Pointwise, A encodes the commutative algebra $C^*(BG, k) \in \mathbb{E}_2$-alg.

The $\mathbb{E}_2$-Koszul dual coalg is

$$C^*(\Omega^2 BG, k) \simeq C^*(\Omega G, k) \in \mathbb{E}_2\text{-coalg}$$

The coalg structure comes from $\mathbb{E}_2$-structure on $\Omega^2 BG$.

And recall, $\quad C_*(\Omega G, k) \simeq B_x$.

__Problem:__ This procedure views A & B as unital $\mathbb{E}_2$-alg, while the Ran space is inherently about __non unital__ $\mathbb{E}_2$-alg.

$$\left(\int_X A\right)^\vee \simeq \int_X \left(KD_{\mathbb{E}_2}(A)^\vee\right) \quad \text{vs.} \quad B \simeq KD^{un}_{\mathbb{E}_2}(A)^\vee$$

taken in non-unital sense

$\underbrace{}$ add unit, Apply KD, then remove unit.

- Applying $KD_{\text{non-un}}$ to a unital algebra produces zero!

$$\begin{tikzcd}
 & Sh(Ran_{un,aug}) \arrow[dl, shift left, "\text{Take Out}"] \arrow[dr, "\text{oblv Aug}"] & \\
Sh(Ran) \arrow[ur, shift left, "\text{AddUnit}^{aug}"] \arrow[rr, shift left, "\text{Add Unit}"] & & Sh(Ran_{un}) \arrow[ll, shift left, "\text{oblv unit}"]
\end{tikzcd}$$

- (Add Unit, oblv Unit), (Add Unit, Take Out) adjoint pairs
- $\text{Add Unit} \simeq \text{oblv Aug} \circ \text{AddUnit}^{aug}$
- $\text{Take Out} \circ \text{AddUnit}^{aug} \xleftarrow{\sim} \mathbb{1}$ (AddUnit^{aug} fully faithful)
- $\Gamma(Ran, -) \xrightarrow{\sim} \Gamma_c(Ran, \text{oblv Unit} \circ \text{AddUnit}(-))$

Insertion of vacuum does not change chiral homology.

there exists $A_{red}, B_{red} \in Sh(Ran)$ s.t.

$A^\circ = \text{oblv Unit} \circ \text{oblv Aug} \circ \text{Add unit}^{aug}(A_{red})$, some A_{red}.

$B^\circ =$ " ——— ——— " (B_{red})

$A^\circ = \text{cofib}(\omega_{Ran} \longrightarrow A)$ $\quad (A = A_{red} \oplus k)$

This implies $\Gamma_c(Ran, A^\circ) \simeq \Gamma_c(Ran, A_{red})$ & similarly for B.

Break

Recall lax prestack: $\mathcal{Y} : \mathrm{AffSch}^{op} \longrightarrow \infty\text{-Cat}$ (For us, lands in 1-cat.)

- Any category is constant lax prestack
- $S \longmapsto Sh(S)$ is lax prestack $(-)^!$

We define $Sh(\mathcal{Y}) = \mathrm{Maps}(\mathcal{Y}, Sh(-))$

$$S \overset{f}{\underset{g}{\rightrightarrows}} \mathcal{Y} \rightsquigarrow f^! \mathcal{F} \longrightarrow g^! \mathcal{G}$$

Ex $\quad Sh(0 \to 1) \simeq \{ V_0 \xrightarrow{\alpha} V_1 \}$

$i_0^!(V_0 \to V_1) \simeq V_0 \qquad (i_0)_!(W) = (W \xrightarrow{id} W)$

$i_1^!(V_0 \to V_1) \simeq V_1 \qquad (i_1)_!(W) = (0 \to W)$

We are interested in

$\mathrm{Ran}: S \longmapsto \{$fin. nonempty subsets of $X(S)\}$

$\mathrm{Ran}_{un}: S \longmapsto \{$poset of fin. nonempty subsets of $X(S)\}$

Have map $\mathrm{Ran} \xrightarrow{i} \mathrm{Ran}_{un}$, & define $\mathrm{OblvUnit} = i^!$.

Prop OblvUnit has left adjoint AddUnit.

$$\mathrm{Fact}(A)_I \simeq \bigotimes_{i \in I} A_i . \quad \text{If } A \text{ unital, have } k \to A.$$

$$\mathrm{Fact}(A)_J \longrightarrow \mathrm{Fact}(A)_I \quad \forall\, J \subseteq I.$$

We expect: $\mathrm{AddUnit}(\mathrm{Fact}(A))_I \simeq \bigotimes_I (\mathbb{1} \oplus A_i)$

$$\simeq \bigoplus_{\emptyset \neq J \subseteq I} \left(\bigotimes_J A_j \right)$$

$$\mathrm{AddUnit}(\mathcal{F})_I = \bigoplus_{\emptyset \neq J \subseteq I} \mathcal{F}_J$$

Proof of Prop

$$\overrightarrow{\mathrm{Ran}} : S \longmapsto \left\{ \begin{array}{l} J \subseteq I \quad \text{two subsets of } X(S), \\ \text{morphismes:} \quad \begin{array}{ccc} J & \subseteq & I \\ \| & & \downarrow \\ J & \subseteq & I' \end{array} \end{array} \right\}$$

$$\begin{tikzcd} & \overrightarrow{\mathrm{Ran}} \arrow[dl, shift left, "\mathrm{pr}^{\mathrm{small}}"] \arrow[dr, "\mathrm{pr}^{\mathrm{big}}"] & \\ \mathrm{Ran} \arrow[ur, shift left, "\delta"] & & \mathrm{Ran} \end{tikzcd} \qquad \begin{tikzcd} (J=J) & (J\subseteq I) \arrow[d, mapsto] \arrow[dr, mapsto] & \\ J \arrow[u, mapsto] & J & I. \end{tikzcd}$$

$\mathrm{pr}^{\mathrm{big}}$ is pseudo proper (fibers are colimits of proper schemes), so $(\mathrm{pr}^{\mathrm{big}})_!$ is well-defined.

δ = left-adjoint to $\mathrm{pr}^{\mathrm{small}}$, so $(\mathrm{pr}^{\mathrm{small}})^!$ is left adjoint to $\delta^!$.

$$\Longrightarrow \mathrm{AddUnit} \simeq (\mathrm{pr}^{\mathrm{big}})_! \, (\mathrm{pr}^{\mathrm{small}})^! \qquad \blacksquare$$

Suppose A unital & augmented. The idea is that A is modeled by: $I \supseteq K$,

$$\boxed{\mathrm{Fact}(A)_{I \supseteq K} \simeq \bigotimes_{K} A/A^{+} \otimes \bigotimes_{I \setminus K} A.}$$

$$\mathrm{Ran}_{\mathrm{un,aug}} : S \longrightarrow \left\{ \begin{array}{l} \text{poset of pairs } I \supseteq K \\ \text{of fin. subsets of } X(S) \end{array} \right\}.$$

$$\begin{array}{ccc} I & \supseteq & K \\ \cap & & \cap \\ I' & \supseteq & K' \end{array}$$

Use this to motivate the def. of $\mathrm{AddUnit}^{\mathrm{aug}}$.

$$\mathrm{Ran}_{\mathrm{un}} \xrightarrow{\pi} \mathrm{Ran}_{\mathrm{un,aug}} \qquad \mathrm{Obliv\,Aug} = \pi^!$$

$$I \longmapsto I \supseteq \emptyset$$

——— the following was skipped in the talk ———

Write $\mathrm{AddUnit}^{\mathrm{aug}}(\mathrm{Fact}(A))_{I\supseteq K} \simeq \bigoplus_{\emptyset\neq J\subseteq I\setminus K} \left(\bigotimes_J A\right)$

$$\simeq \bigoplus_{\emptyset\neq J\subseteq I} \left(\bigotimes_J A\right) \Big/ \bigoplus_{\substack{J\subseteq I \\ J\cap K\neq\emptyset}} \left(\bigotimes_J A\right)$$

Consider $\mathrm{Ran}^{\to}_{\mathrm{un,aug}} = \{ J\subseteq I\supseteq K : J\cap K\neq\emptyset,\ I, K \text{ growing}$

$$\begin{tikzcd}
 & & & \mathrm{Ran}^{\to} \arrow[dl] \arrow[dr] & \\
 & & \mathrm{Ran}^{\to}_{\mathrm{un,aug}} \arrow[dl, "\pi^{\to}"'] \arrow[dr, "\psi^{\to}"] & & \mathrm{Ran}_{\mathrm{un}} \arrow[dl, "i"] \\
 & \mathrm{Ran}^{\to} \arrow[dl, "\xi"'] \arrow[dr, "\psi"] & & \mathrm{Ran}_{\mathrm{un,aug}} \arrow[dl, "\pi"] & \\
\mathrm{Ran} & & \mathrm{Ran}_{\mathrm{un}} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & J\subseteq I\supseteq K \arrow[dl] \arrow[dr] & \\
 & J\subseteq I \arrow[dl] \arrow[dr] & & I\supseteq K \arrow[dl] \\
I & & I &
\end{tikzcd}$$

Define $\mathrm{AddUnit}^{\mathrm{aug}} = \mathrm{cofib}\left(\psi^{\to}_! \circ \pi^{\to !} \circ \xi^! \longrightarrow \pi^! \circ \psi_! \circ \xi^! \right)$

$\mathrm{OblvAug} = i^!$

$\Rightarrow \mathrm{oblvAug}\circ \mathrm{AddUnit}^{\mathrm{aug}} = \mathrm{AddUnit}$

Now we construct $A_{\mathrm{aug,un}} = \mathrm{TakeOut}(A_{\mathrm{red}})$

Consider the prestack

Object: $\mathcal{P}_G,\ \mathcal{P}_G|_{X\setminus I} \simeq \mathrm{triv}$

Mor: $I\supseteq K + $ defin. data on $X\setminus K$.
$\downarrow \quad \downarrow$
$I' \supseteq K'$

$$\begin{tikzcd}
\mathrm{Gr}^{\to}_{G,\mathrm{Ran}_{\mathrm{un,aug}}} \arrow[d, "?"] & \prod_{I\setminus K} \mathrm{Gr}_{G,i} \arrow[d] \\
\mathrm{Ran}_{\mathrm{un,aug}} & I\supseteq K
\end{tikzcd}$$

Then $A_{\mathrm{aug,un}} = \mathrm{Fib}\left(p_!\,\omega \longrightarrow \omega_{\mathrm{Ran}_{\mathrm{un,aug}}}\right)$

$\Rightarrow \mathrm{oblvUnit}\circ \mathrm{oblvAug} \simeq A^{\circ} = \mathrm{Fib}\left(p_!\,\omega_{\mathrm{Gr}_G} \longrightarrow \omega_{\mathrm{Ran}}\right)$

End Bonus material

$\mathcal{Y}$ prestack. $\Delta_{\mathcal{Y}}$ = pseudo proper.

A pairing between $\mathcal{F}, \mathcal{G} \in Sh(\mathcal{Y})$ is a map

$$\mathcal{F} \boxtimes \mathcal{G} \longrightarrow \Delta_! \omega_{\mathcal{Y}}$$

By def, the Verdier dual of $\mathcal{F}$ is object that universally pairs w/ $\mathcal{F}$, i.e. $\mathcal{F} \boxtimes \mathbb{D}\mathcal{F} \longrightarrow \Delta_! \omega$

$$\mathrm{Maps}(\mathcal{G}, \mathbb{D}\mathcal{F}) = \mathrm{Maps}(\mathcal{F} \boxtimes \mathcal{G}, \Delta_! \omega)$$

Note: · $\mathbb{D} : Sh(\mathcal{Y})^{op} \to Sh(\mathcal{Y})$ takes colimits in $Sh(\mathcal{Y})$ to limits in $Sh(\mathcal{Y})$

· for $\mathcal{Y}$ scheme: $\mathcal{F}, \mathcal{G} \in Sh(\mathcal{Y})^c$ w/ compactly support

$$\mathrm{Maps}(\mathcal{G}, \text{"}\mathbb{D}\text{"}\mathcal{F}) = \mathrm{Maps}(\mathbb{D}\mathbb{D}\mathcal{G}, \text{"}\mathbb{D}\text{"}\mathcal{F})$$
$$= \Gamma(\mathcal{Y}, \mathbb{D}\mathcal{G} \overset{!}{\otimes} \text{"}\mathbb{D}\text{"}\mathcal{F})$$

$$\mathrm{Maps}(\mathcal{F} \boxtimes \mathcal{G}, \Delta_* \omega) = \mathrm{Maps}(\mathcal{F} \overset{*}{\otimes} \mathcal{G}, \omega)$$
$$= \Gamma(\mathcal{Y}, \mathbb{D}(\mathcal{F} \overset{*}{\otimes} \mathcal{G})).$$

In general, $f: \mathcal{Y} \to \mathcal{Z}$ pseudo proper, there's a canonical map $f_! \mathbb{D} \to \mathbb{D} f_!$.

f finitary pseudo-proper if for any $S \to \mathcal{Z}$, the base change $\mathcal{Y}_S \to S$ = finite colim of proper morphism.

<u>Prop</u> If $\mathcal{Y} \to \mathcal{Z}$ finitary pseudo-proper, $\Delta_{\mathcal{Y}}, \Delta_{\mathcal{Z}}$ also, then $f_! \mathbb{D} \xrightarrow{\sim} \mathbb{D} f_!$

$\mathcal{Y} = \operatorname{colim} Y_\alpha$, $\Delta_{\mathcal{Y}}$ finitary pseudo proper

$$\mathbb{D}\mathcal{F} \simeq \mathbb{D}(\operatorname{colim} (\mathrm{ins}_\alpha)_! (\mathrm{ins}_\alpha)^! \mathcal{F})$$

$$= \lim (\mathrm{ins}_\alpha)_! \mathbb{D} (\mathrm{ins}_\alpha)^! \mathcal{F}$$

Can use this to show $\mathbb{D}\omega_{\mathrm{Ran}} = 0$ ✓ (constant sheaves on X^I grows to ∞ -cohom. dim)

Certainly $\Gamma_c(\mathrm{Ran}, \mathbb{D}\mathcal{F}) \neq \Gamma_c(\mathrm{Ran}, \mathcal{F})^\vee$ (e.g. ω_{Ran})

However:

__Prop__ Suppose $\mathcal{F} \in Sh(\mathrm{Ran})$ has property: for all k, one has $\mathcal{F}|_{\overset{\circ}{X}{}^n}$ in perverse cohom deg $\leq -n-k$ for $n \gg 0$.

__Then__ $\Gamma_c(\mathrm{Ran}, \mathbb{D}\mathcal{F}) \xrightarrow{\sim} \Gamma_c(\mathrm{Ran}, \mathcal{F})^\vee$.

A_{red} Factorizable & fibers at $x \in X$ are $\Gamma_c(\mathrm{Gr}_G, \omega)_{red}$.

G semisimple + s.c. $\Rightarrow H_0 = 0$, $H_1 = 0$

So $\Gamma_c(\mathrm{Gr}_G, \omega_{\mathrm{Gr}_G})_{red}$ lives in cohom ≤ -2.

End talk, rest is leftover

Remains to explain $\boxed{B_{red} = \mathbb{D}_{\mathrm{Ran}} A_{red}}$

Constructing this map is nontrivial; we do it pointwise.

Ignoring problems: Problem is etale local, so take $X = \mathbb{P}^1$, $x = 0$

Then LHS $= \Gamma_{red}(BG, k) \simeq \Gamma_{c,red}(BG, \omega)^\vee$

RHS $= ?$

<u>Step 1</u> $\quad i_0^! \, \mathbb{D}_{Ran} \mathcal{F} \simeq (i_0^! \mathcal{F})^\vee \simeq \Gamma(Ran_G, \mathcal{F})^\vee$

This follows from contraction principle + formula for $\mathbb{D}_{Ran}$.

$$\Longrightarrow \quad i_0^! \, \mathbb{D}_{Ran} A_{red} \simeq \Gamma_c(Ran, j_* A_{red})^\vee, \quad j: Ran_U \to Ran_X$$

<u>Step 2</u> $\quad TakeOut \circ union_\infty^! \, AddUnit^{aug}(\mathcal{F}) \simeq j_* j^! \mathcal{F}$,

where $union_\infty : Ran_{un,aug} \longrightarrow Ran_{un,aug}$

In fact, $union_\infty^! \circ AddUnit^{aug}(\mathcal{F}) \simeq AddUnit^{aug}(j_* j^! \mathcal{F})$

$\forall \mathcal{F} \in Sh(Ran)$

So, we get

$$\Gamma_c(Ran, i_0 A_{red})^\vee \simeq \Gamma_c(Ran, TakeOut \circ Union_\infty^! \circ AddUnit^{aug}(A_{red}))^\vee$$

$$\simeq \Gamma_c(Ran, Oblv\,Unit \circ Oblv\,Aug \circ union_\infty^! \circ AddUnit^{aug}(A_{red}))^\vee$$

$$\simeq \Gamma_c(Ran \times_{Ran_{un,aug}} Gr_{G, Ran_{un,aug}})^\vee$$

<u>Step 3</u> $\quad Ran \times_{Ran_{un,aug}} Gr_{G, Ran_{un,aug}} \xrightarrow{\pi} Bun_G(U \subseteq X) \xrightarrow{ev_0} BG$

classifies bundles on U that extend to X.

Is homologically contractible.

<u>Step 4</u> $\quad \pi$ is contractible.

We can check this after base change to

$Bun_G \mathbb{P}^1 \longrightarrow Bun_G(\mathbb{A}^1 \subset \mathbb{P}^1)$, i.e. for the morphism

$$\begin{tikzcd}
Ran \times_{Ran} Gr_{G,Ran} \arrow[r] \arrow[d, "UHC"] & Gr_{G,Ran} \arrow[r, "UHC"] \arrow[d, "UHC"] & Bun_G \\
Ran_{un} \times_{Ran_{un}} Gr_{G,Ran_{un}} \arrow[r, "UHC"] & Gr_{G,Ran_{un}} \arrow[ur] &
\end{tikzcd}$$

UHC = universally homologically contractible