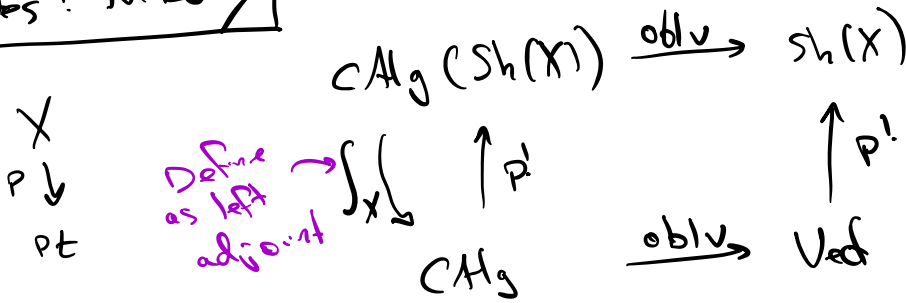


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# Vendler duality on the Ran Space

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Goal: Prove " $\prod_{x \in X} \Gamma(\text{BG}, k)$ "  $\xrightarrow{\sim}$   $\Gamma(\text{Bun}_G, k)$ .

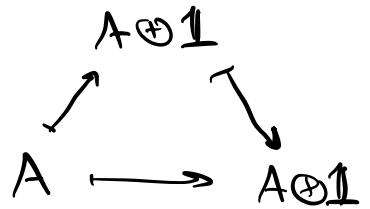
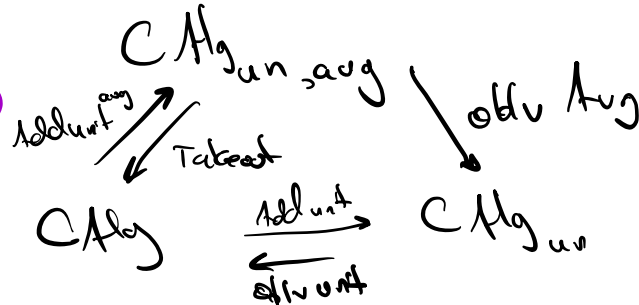


What is  $\int_X$ ?

E.g. For  $X = \{1, \dots, n\}$ ,  $\int_X (A_1, \dots, A_n) \longrightarrow B$   
 $\parallel$   
 $\bigotimes_{x \in X} A_x$

For  $X = \{1, \dots, n\}$ , what is unital  $\int_X$ ?

In fact, (AddUnit<sup>aug</sup>, Takeout) are equivalences!



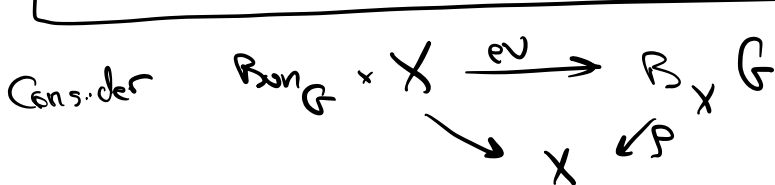
$$\Rightarrow \int_X (A_1, \dots, A_n) = \bigoplus_{\emptyset \neq S \subseteq X} \left( \bigotimes_{i \in S} A_i \right)$$

Recall Com Fact Alg  $\xrightleftharpoons[\text{Fact}]{(-)^!}$   $\text{CAlg}(\text{Sh}(X))$

$$\text{Fact}(A)_{\mathbb{I}} \cong \bigotimes_{i \in \mathbb{I}} A_i$$

Prop  $\int_X A \cong \Gamma_c(\text{Ran}, \text{Fact}(A))$

computes unital version of  $\int_X$ .



$$\text{Fact}(P_* P^* \omega_X) = B$$

$$\Rightarrow P_* P^* \omega_X \longrightarrow \Gamma(\text{Bun}_G, k) \otimes \omega_X$$

$$\Rightarrow \text{Have map } \Gamma_c(\text{Ran}, B) \longrightarrow \Gamma(\text{Bun}_G, k)$$

Thm B. This map is isomorphism.

Previously,

$$\begin{array}{ccc} & \Gamma_{G, \text{Ran}} & \xrightarrow{\pi} \text{Bun}_G \times \text{Ran} \\ & \downarrow P & \downarrow P^* \\ & \text{Ran} & \end{array}$$

$$\Rightarrow A = P_* \omega_{\Gamma_{G, \text{Ran}}} \longrightarrow \Gamma_c(\text{Bun}_G, \omega) \otimes \omega_{\text{Ran}}$$

Thm A  $\Gamma_c(\text{Ran}, A) \xrightarrow{\sim} \Gamma_c(\text{Bun}_G, \omega)$  (proved last time!)  
by homological contractibility of  $\mathcal{J}$ .

What mechanism is responsible for

$$\Gamma_c(\text{Ran}, A) \overset{!}{\simeq} \Gamma_c(\text{Ran}, B)?$$

as vector spaces/k!

Answer: Koszul duality!

Pointwise,  $A$  encodes the commutative algebra  $C^*(BG, k) \in \mathbb{E}_2\text{-alg}$ .

The  $\mathbb{E}_2$ -Koszul dual coalg is

$$C^*(\Omega^2 BG, k) \simeq C^*(\Omega G, k) \in \mathbb{E}_2\text{-coalg}$$

The coalg structure comes from  $\mathbb{E}_2$ -structure on  $\Omega^2 BG$ .

And recall,  $C_*(\Omega G, k) \simeq B_x$ .

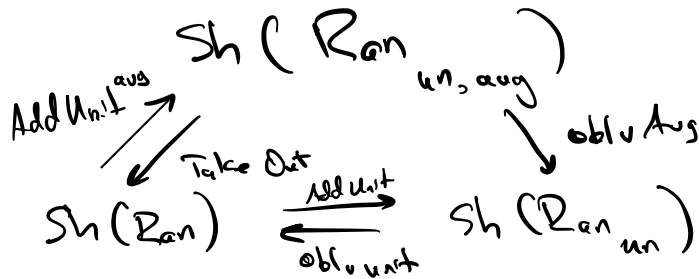
Problem: This procedure views  $A$  &  $B$  as unital  $\mathbb{E}_2$ -alg,  
while the Ran space is inherently about non-unital  
 $\mathbb{E}_2$ -alg.

$$(\int_X A)^\vee \cong \int_X (KD_{\mathbb{R}^2} (A)^\vee) \quad \text{vs.} \quad B \cong KD_{\mathbb{R}^2}^{\text{un}} (A)^\vee$$

taken in non-unital sense.

add unit, Apply KD, then remove unit.

- Applying  $KD_{\text{non-un}}$  to a unital algebra produces zero!



•  $(\text{Add Unit}, \text{oblv Unit}), (\text{Add Unit}, \text{Take Out})$  adjoint pairs

•  $\text{Add Unit} \cong \text{oblv Aug} \circ \text{Add Unit}^{\text{aug}}$

•  $\text{Take Out} \circ \text{Add Unit}^{\text{aug}} \cong \mathbb{1}$  ( $\text{Add Unit}^{\text{aug}}$  fully faithful)

•  $\Gamma(\text{Ran}, -) \cong \Gamma_c(\text{Ran}, \text{oblv Unit} \circ \text{Add Unit}(-))$

there exists  $A_{\text{red}}, B_{\text{red}} \in \text{Sh}(\text{Ran})$  s.t.

$$A^\circ = \text{oblv Unit} \circ \text{oblv Aug} \circ \text{Add Unit}^{\text{aug}}(A_{\text{red}}), \text{ some } A_{\text{red}}$$

$$B^\circ = \text{---} \text{---} \text{---} (B_{\text{red}})$$

$$A^\circ = \text{Fib}(w_{\text{Ran}} \rightarrow A) \quad (A = A_{\text{red}} \oplus k)$$

This implies  $\Gamma_c(\text{Ran}, A^\circ) \cong \Gamma_c(\text{Ran}, A_{\text{red}})$  & similar for B.

↑ Insertion of vacuum does not change chiral locality.

Recall lax prestack :  $y : \text{Aff Sch}^{\text{op}} \rightarrow \infty\text{-Cat}$  (For us, leads in 1-cat.)

• Any category is constant lax prestack

•  $S \mapsto \text{Sh}(S)$  is lax prestack  $(-)^!$

We define  $\text{Sh}(y) = \text{Maps}(y, \text{Sh}(-))$

$$S \begin{array}{c} \xrightarrow{f} \\ \downarrow i \\ \xrightarrow{g} \end{array} y \rightsquigarrow f' : \mathcal{F} \rightarrow g' : \mathcal{G}$$

Ex  $\text{Sh}(0 \rightarrow 1) \cong \{ V_0 \xrightarrow{\alpha} V_1 \}$

$i_0^! (V_0 \rightarrow V_1) \cong V_0$        $(i_0)_! (W) = (W \xrightarrow{\text{id}} W)$

$i_1^! (V_0 \rightarrow V_1) \cong V_1$        $(i_1)_! (W) = (0 \rightarrow W)$

We are interested in

$\text{Ran} : S \mapsto \{ \text{fin. nonempty subsets of } X(S) \}$

$\text{Ran}_{\text{un}} : S \mapsto \{ \text{poset of fin. nonempty subsets of } X(S) \}$

Have map  $\text{Ran} \xrightarrow{i} \text{Ran}_{\text{un}}$ , & define  $\text{Oblv Unit} = i^!$ .

Prop  $\text{Oblv Unit}$  has left adjoint  $\text{Add Unit}$ .

$\text{Fact}(A)_{\mathbf{I}} \cong \bigotimes_{i \in \mathbf{I}} A_i$ . If  $A$  unital, have  $k \rightarrow A$ .

$$\text{Fact}(A)_{\mathbf{J}} \rightarrow \text{Fact}(A)_{\mathbf{I}} \quad \forall \mathbf{J} \subseteq \mathbf{I}.$$

We expect:  $\text{Add Unit}(\text{Fact}(A))_{\mathbf{I}} \cong \bigotimes_{\mathbf{I}} (\mathbb{1} \oplus A_i)$

$$\cong \bigoplus_{\emptyset \neq \mathbf{J} \subseteq \mathbf{I}} \left( \bigotimes_{\mathbf{J}} A_i \right)$$

$$\text{Add Unit}(\mathcal{F})_{\mathbf{I}} = \bigoplus_{\emptyset \neq \mathbf{J} \subseteq \mathbf{I}} \mathcal{F}_{\mathbf{J}}$$

Proof of Prop

$$\text{Ran} \rightarrow S \rightarrow \left\{ \begin{array}{l} J=I \text{ two subsets of } X(S) \\ \text{morphisms: } \begin{array}{c} J \cong I \\ J \cong H' \end{array} \end{array} \right\}$$



$pr_{big}$  is pseudo proper (fibers are points of proper scheme), so  $(pr_{big})_!$  is well-defined.

$\sigma = \text{left-adjoint to } pr_{small}$ , so  $(pr_{small})_! = \text{left adjoint to } \sigma_!$ .

$$\Rightarrow \text{Add Unit} \cong (pr_{big})_!, (pr_{small})_!$$

Suppose  $A$  unital & augmented. The idea is that  $A$  is modeled

$$b_1: I \cong K, \quad \text{Fact}(A)_{I \cong K} \cong \bigotimes_K A/A^+ \oplus \bigotimes_{I \cong K} A$$

$$\text{Ran}_{un, aug}: S \rightarrow \left\{ \begin{array}{l} \text{poset of pairs } I \geq K \\ \text{of fn. subsets of } X(S) \end{array} \right\}$$

Use this to motivate the def. of Add Unit.

$$\begin{array}{c} I \geq K \\ \cong \\ I' \geq K' \end{array}$$

$$\begin{array}{ccc} \text{Ran}_{un} & \xrightarrow{\pi} & \text{Ran}_{un, aug} \\ I & \mapsto & I \geq \emptyset \end{array}$$

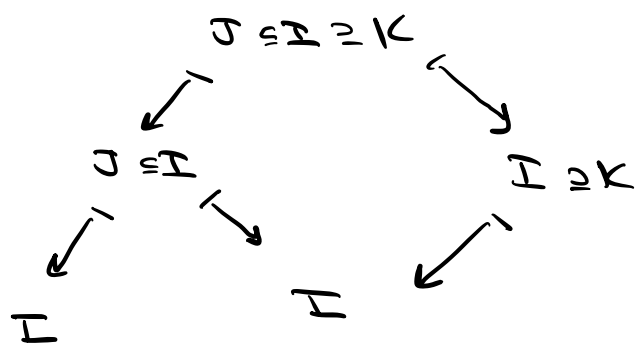
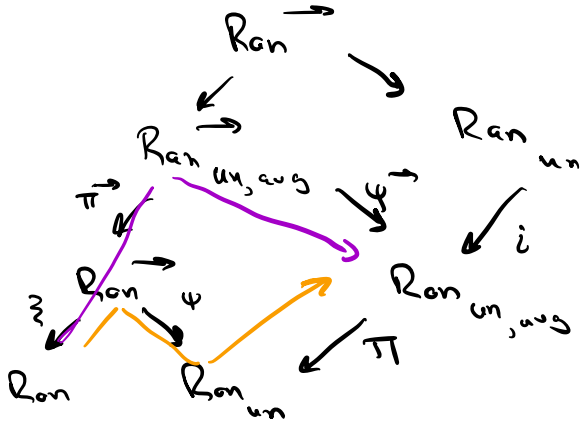
$$\text{Old } v \text{ Aug} = \pi!$$

The following was skipped in the talk

Write  $\text{Add Unit}^{\text{aug}}(\text{Fact}(A))_{I \geq K} \cong \bigoplus_{0 \neq J \subseteq I \subseteq K} (\bigotimes_J A)$

$\cong \bigoplus_{0 \neq J \subseteq I} (\bigotimes_J A) / \bigoplus_{J \subseteq I, J \cap K \neq \emptyset} (\bigotimes_J A)$

Consider  $\text{Ran}_{\text{un, aug}} = \{ J \subseteq I \geq K : J \cap K \neq \emptyset, I, K \text{ growing} \}$



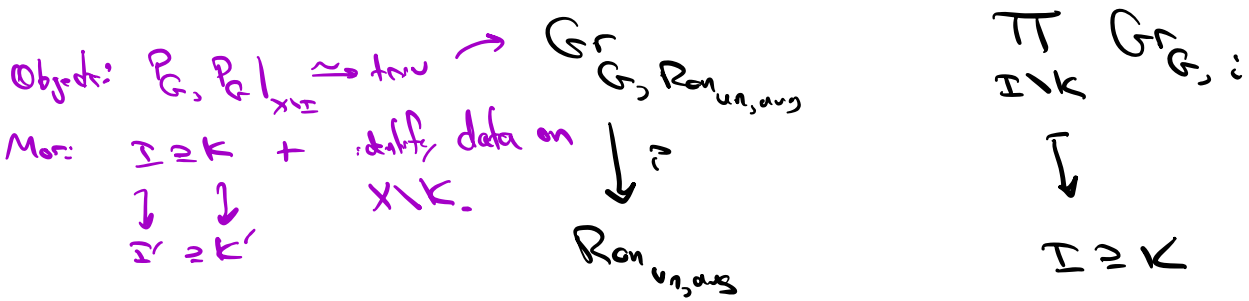
Define  $\text{Add Unit}^{\text{aug}} = \text{Fib}(\underbrace{\psi^! \cdot \pi^! \cdot \xi^!}_{\text{purple}} \rightarrow \underbrace{\pi^! \cdot \psi^! \cdot \xi^!}_{\text{orange}})$

$\text{obv Aug} = c^!$

$\Rightarrow \text{obv Aug} \circ \text{Add Unit}^{\text{aug}} = \text{Add Unit}$

Now we construct  $A_{\text{aug, un}} = \text{TakeOut}(A_{\text{red}})$

Consider the prestack



Then  $A_{\text{aug, un}} = \text{Fib}(P_! \omega \rightarrow \omega_{\text{Ran}_{\text{un, aug}}})$

$\Rightarrow \text{obv Unit} \circ \text{obv Aug} \cong A^0 = \text{Fib}(P_! \omega_{\text{Gr}_G} \rightarrow \omega_{\text{Ran}})$

End Bonus material

$Y$  prestack.  $\Delta_Y = \text{pseudo proper}$ .

A pairing between  $\mathcal{F}, G \in \text{Sh}(Y)$  is a map

$$\mathcal{F} \otimes G \rightarrow \Delta, \omega_Y$$

$R_Y \text{ def,}$  the Verdier dual of  $\mathcal{F}$  is object that universally pairs w/  $\mathcal{F}$ , i.e.  $\mathcal{F} \otimes D\mathcal{F} \rightarrow \Delta, \omega$

$$\text{Maps}(G, D\mathcal{F}) = \text{Maps}(\mathcal{F} \otimes G, \Delta, \omega)$$

Note:  $\cdot D : \text{Sh}(Y)^{op} \rightarrow \text{Sh}(Y)$  takes objects in  $\text{Sh}(Y)$  to limits in  $\text{Sh}(Y)$   
 $\cdot$  for  $Y$  scheme:  $\mathcal{F}, G \in \text{Sh}(Y)$  compactly supported

$$\begin{aligned} \text{Maps}(G, D\mathcal{F}) &= \text{Maps}(D D G, D\mathcal{F}) \\ &= \Gamma(Y, D G \otimes D\mathcal{F}) \end{aligned}$$

$$\begin{aligned} \text{Maps}(\mathcal{F} \otimes G, \Delta, \omega) &= \text{Maps}(\mathcal{F}^* \otimes G, \omega) \\ &= \Gamma(Y, D(\mathcal{F}^* \otimes G)). \end{aligned}$$

In general,  $f: Y \rightarrow Z$  pseudo proper, there's a canonical

$$\text{map } f_! D \rightarrow D f_!$$

$f$  functor pseudo proper if for any  $S \rightarrow Z$  the base change  $g_S \rightarrow S$  is a fibration of proper morphisms.

Prop If  $Y \rightarrow Z$  functor pseudo proper,  $\Delta_Y, \Delta_Z$  also, then  $f_! D \xrightarrow{\sim} D f_!$

$y = \text{colim } Y_a, \Delta_y$  Entirely pseudo proper

$$\begin{aligned} D\mathcal{F} &\simeq D(\text{colim } (ins_a), (ins_a)! \mathcal{F}) \\ &= \text{lim } (ins_a), D(ins_a)! \mathcal{F} \end{aligned}$$

Can use this to show  $D\omega_{\mathbb{R}^n} = 0$  ✓ (constant sheaves on  $X^{\text{ét}}$  grows to  $\infty$  - coherent dim)

Certainly  $\Gamma_c(\mathbb{R}^n, D\mathcal{F}) \neq \Gamma_c(\mathbb{R}^n, \mathcal{F})^\vee$  (eg  $\omega_{\mathbb{R}^n}$ )

However:  
Prop Suppose  $\mathcal{F} \in \text{Sh}(\mathbb{R}^n)$  has property: for all  $k$ , one has  $\mathcal{F}|_{\mathbb{R}^n}$  in perverse cohom  $\text{deg} \leq -n-k$  for  $n \gg 0$ .

Then  $\Gamma_c(\mathbb{R}^n, D\mathcal{F}) \xrightarrow{\sim} \Gamma_c(\mathbb{R}^n, \mathcal{F})^\vee$ .

$A_{\text{red}}$  factorizable & fibres at  $x \in X$  are  $\Gamma_c(G_{r_G}, \omega)_{\text{red}}$ .

$G$  semi-simple + s.c.  $\Rightarrow H_0 = 0, H_1 = 0$

So  $\Gamma_c(G_{r_G}, \omega_{G_{r_G}})_{\text{red}}$  lives in cohom  $\leq -2$ .

End talk, rest is leftover

Remains to explain  $\boxed{B_{\text{red}} = D_{\mathbb{R}^n} A_{\text{red}}}$

Constructing this map is natural; we do it pointwise.

Ignoring problems: Problem is étale local, so take  $X = \mathbb{P}^1, \mathcal{L} = \mathcal{O}(-1)$

then LHS  $\Gamma_{\text{red}}(BG, k) \simeq \Gamma_{c, \text{red}}(BG, \omega)^\vee$

RHS = ?



Step 1  $i_! D_{\mathbb{R}^n} \mathcal{F} \simeq (i_! \mathcal{F})^\vee \simeq \Gamma(\mathbb{R}^n, \mathcal{F})^\vee$

This follows from contraction principle in formula for  $D_{\mathbb{R}^n}$

$\Rightarrow i_! D_{\mathbb{R}^n} A_{\text{red}} \simeq \Gamma_c(\mathbb{R}^n, i_* A_{\text{red}})^\vee, \quad i: \mathbb{R}^n \rightarrow \mathbb{R}^n$

Step 2 Take Out Union!  $\text{AddUnit}^{\text{aug}}(\mathcal{F}) \simeq i_* i^! \mathcal{F}$ ,

where  $\text{union}_\infty: \mathbb{R}^n_{u_1, \dots, u_n} \rightarrow \mathbb{R}^n_{u_1, \dots, u_n}$

In fact,  $\text{union}_\infty! \cdot \text{AddUnit}^{\text{aug}}(\mathcal{F}) \simeq \text{AddUnit}^{\text{aug}}(i_* i^! \mathcal{F})$

$\forall \mathcal{F} \in \text{Sh}(\mathbb{R}^n)$

So, we get

$\Gamma_c(\mathbb{R}^n, i_! A_{\text{red}})^\vee \simeq \Gamma_c(\mathbb{R}^n, \text{TakeOut} \cdot \text{Union}_\infty! \cdot \text{AddUnit}^{\text{aug}}(A_{\text{red}}))^\vee$   
 $\simeq \Gamma_c(\mathbb{R}^n, \text{ObvUnit} \cdot \text{ObvAug} \cdot \text{union}_\infty! \cdot \text{AddUnit}^{\text{aug}}(A_{\text{red}}))^\vee$   
 $\simeq \Gamma_c(\mathbb{R}^n \times \text{Gr}_{\mathbb{R}^n, \mathbb{R}^n_{u_1, \dots, u_n}})^\vee$

Step 3  $\mathbb{R}^n \times \text{Gr}_{\mathbb{R}^n, \mathbb{R}^n_{u_1, \dots, u_n}} \xrightarrow{\pi} \text{Bun}_G(U \subseteq X) \xrightarrow{\text{Ev}_0} BG$   
*classifies bundles on U that extend to X.*  
*is homotopically contractible.*

Step 4  $\pi$  is contractible.

We can check this after base change to

$\text{Bun}_G \mathbb{P}^1 \rightarrow \text{Bun}_G(\mathbb{A}^1 \subset \mathbb{P}^1)$ , i.e. for the morphism

$\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\text{Gr}_{\mathbb{R}^n, \mathbb{R}^n}} \text{Gr}_{\mathbb{R}^n, \mathbb{R}^n} \xrightarrow{\text{UHC}} \text{Bun}_G$

$\downarrow \text{UHC} \quad \downarrow \text{UHC}$

$\mathbb{R}^n_{u_1, \dots, u_n} \times \mathbb{R}^n_{u_1, \dots, u_n} \xrightarrow{\text{Gr}_{\mathbb{R}^n, \mathbb{R}^n_{u_1, \dots, u_n}}} \text{Gr}_{\mathbb{R}^n, \mathbb{R}^n_{u_1, \dots, u_n}} \xrightarrow{\text{UHC}} \text{Bun}_G$

*UHC = universally homotopically contractible*