CRYPTIC NOTES ON GROUP COHOMOLOGY

Much more detail given in the talks. These are just reminder notes.

1. Categories and functors

Definition 1.1. Categories: data: objects, morphisms, composition, identity morphisms. Properties: associativity and unit conditions for composition.

Examples 1.2. Sets, groups, abelian groups, rings, fields, (left) R-modules, G-modules $= \mathbb{Z}[G]$-modules. Categories with one object: monoids and groups.

Definition 1.3. Functors $F\colon \mathscr{C} \longrightarrow \mathscr{D}$: data: Objects to objects, morphisms to morphisms; $\mathscr{C}(X,Y) \mapsto \mathscr{D}(FX,FY)$. Properties: $F(g\circ f) = F(g)\circ F(f)$; $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Examples 1.4. Forgetful (underlying thing) functors: Groups to sets, rings to abelian groups, R-modules to abelian groups, spaces to sets, unit group functor from rings to groups.

Examples 1.5. Free functors: sets to groups, sets to abelian groups, sets S to R-modules $R[S]$, sets to commutative rings, groups to rings (group ring $R[G]$).

Examples 1.6. Inclusion functors: abelian groups to groups, commutative rings to rings

Products of categories $\mathscr{C}\times\mathscr{D}$: pairs of objects and morphisms.

Definition 1.7. Tensor product: R a ring; left, ${}_R\mathscr{M}$, and right, $\mathscr{M}_R$, R-modules:

$$\otimes_R\colon \mathscr{M}_R \times_R \mathscr{M} \longrightarrow \mathscr{A}b.$$

$\mathscr{A}b$ = abelian groups = $\mathbb{Z}$-modules. If R is commutative, can identify ${}_R\mathscr{M}$ with $\mathscr{M}_R$ and get

$$\otimes_R\colon \mathscr{M}_R \times \mathscr{M}_R \longrightarrow \mathscr{M}_R.$$

Contravariant functors and opposite categories $\mathscr{C}^{op}$: same objects, but now $\mathscr{C}^{op}(X,Y) = \mathscr{C}(Y,X)$, obvious composition. $F\colon \mathscr{C}^{op} \longrightarrow \mathscr{D}, \mathscr{C}(Y,X) \mapsto \mathscr{D}(FX,FY)$, $F(g\circ f) = Ff\circ Fg$.

Definition 1.8. Hom functors. R a ring:

$$\mathrm{Hom}_R(-,-)\colon ({}_R\mathscr{M})^{op} \times_R \mathscr{M} \longrightarrow \mathscr{A}b.$$

R a commutative ring:

$$\mathrm{Hom}_R(-,-)\colon (\mathscr{M}_R)^{op} \times \mathscr{M}_R \longrightarrow \mathscr{M}_R.$$

G a group, G-module $= \mathbb{Z}[G]$-module, abbreviate $G\mathscr{M}$ for the category:

$$\mathrm{Hom}_G(-,-) = \mathrm{Hom}_{\mathbb{Z}[G]}(-,-)\colon (G\mathscr{M})^{op} \times G\mathscr{M} \longrightarrow G\mathscr{M}$$

Definition 1.9. Natural transformation: $\eta\colon F \longrightarrow G$, $F, G\colon \mathscr{C} \longrightarrow \mathscr{D}$: maps $\eta_X\colon FX \longrightarrow GX$ for all $X \in \mathscr{C}$ such that the following diagram commutes in $\mathscr{D}$ for all maps $f\colon X \longrightarrow Y$ in $\mathscr{C}$:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \eta_X}\big\downarrow & & \big\downarrow{\scriptstyle \eta_Y} \\ GX & \xrightarrow[Gf]{} & GY. \end{array}$$

Examples 1.10. $\rho\colon \mathrm{Id} \longrightarrow (-)^{*}*$ on vector spaces over a field K. For a vector space V and a linear map $T\colon V \longrightarrow K$, $\rho_V\colon V \longrightarrow V**$ is given by $\rho_V(v)(T) = T(v)$ for a vector $v \in V$ and linear map $T \in V^*$.

2. Begin revisit of $H^n\colon G\mathscr{M} \longrightarrow \mathscr{A}b$

Let $S_n = (G - \{e\})^n$. This is just a set, the empty set if $n = 0$. We started with functions $f\colon S_n \longrightarrow M$, where M is a G-module. Call the abelian group of such functions $C^n(G; M)$, the cochains of G with coefficients in M. We defined subgroups of cocycles, $Z^n(G; M)$, and subgroups of coboundaries, $B^n(G; M)$:

$$B^n(G; M) \subset Z^n(G; M) \subset C^n(G; M).$$

These are all functors G-mod $\longrightarrow \mathscr{A}b$. Kind of unnatural to think of functions. For any set S and abelian group M

$$\mathrm{Hom}_{Sets}(S, M) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S], M).$$

A function f uniquely determines a homomorphism $\tilde{f}$ via

$$\tilde{f}\Big(\sum_i n_i s_i\Big) = \sum_i n_i f(s_i).$$

So our cochains can be thought of as homomorphisms $\mathbb{Z}[S_n] \longrightarrow M$ rather than functions $S_n \longrightarrow M$.

Still looks strange, there is that $gf(-)$ term and then there are other terms $f(-)$ in the condition for a cochain to be a cocycle. Here is another free functor, from abelian groups to $\mathbb{Z}[G]$-modules. It is the tensor product (over $\mathbb{Z}$) $\mathbb{Z}[G] \otimes A$. Explicitly, it is the sum of copies of A indexed by the elements of G. Its elements are linear combinations $\sum_g a_g g$, where $a_g \in A$ and all but finitely many $a_g = 0$. For $h \in G$, $h \sum_g a_g g = \sum_g a_g hg$. For G-modules M, we now have an isomorphism of abelian groups

$$\mathrm{Hom}_{\mathbb{Z}}(A, M) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes A, M).$$

For a homomorphism $f\colon A \longrightarrow M$ of abelian groups, there is a unique extension of f to a homomorphism of $\tilde{f}\colon \mathbb{Z}[G] \otimes A \longrightarrow M$ of G-modules, given explicitly by

$$\tilde{f}\Big(\sum_g a_g g\Big) = \sum_g g f(a_g).$$

We define a homomorphism $\eta\colon A \longrightarrow \mathbb{Z}[G] \otimes A$ by $\eta(a) = ae$, and then $\tilde{f}$ is the unique G-homomorphism $\mathbb{Z}[G] \otimes A \longrightarrow M$ such that $\tilde{f} \circ \eta = f$.

3. Chains, cochains, homology, cohomology

Now we can begin to make more sense out of the definition of cohomology.

Define a chain complex X of R-modules to be a sequence of homomorphisms of R-modules

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow 0$$

such that $d_n \circ d_{n+1} = 0$. We often abbreviate d_n to d and write $d^2 = 0$. We define the cycles, boundaries, and homology of X by

$$Z_n(X) = \operatorname{Ker}(d_n)$$

$$B_n(X) = \operatorname{Im}(d_{n+1})$$

$$H_n(X) = Z_n(X)/B_n(X).$$

$H_n(X)$ measures how far away X is from being exact at the nth spot. These are functors from chain complexes to abelian groups. They are often written as Z_*, B_*, H_*; then they are functors from chain complexes to graded abelian groups, that is, sequences of abelian groups. Here a map $f\colon X \longrightarrow Y$ of chain complexes is a sequence of homomorphisms $f_n\colon X_n \longrightarrow Y_n$ such that the following diagram commutes.

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & X_{n+1} & \xrightarrow{d_{n+1}} & X_n & \xrightarrow{d_n} & X_{n-1} & \longrightarrow \cdots \longrightarrow & X_0 & \longrightarrow 0 \\
& \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_0} & \\
\cdots \longrightarrow & Y_{n+1} & \xrightarrow[d_{n+1}]{} & Y_n & \xrightarrow[d_n]{} & Y_{n-1} & \longrightarrow \cdots \longrightarrow & Y_0 & \longrightarrow 0
\end{array}$$

Can "dualize" all of these definitions.

Define a cochain complex X of R-modules to be a sequence of homomorphisms of R-modules

$$0 \longrightarrow X^0 \xrightarrow{\delta^0} \cdots \longrightarrow X^{n-1} \xrightarrow{\delta^{n-1}} X^n \xrightarrow{\delta^n} X^{n+1} \longrightarrow \cdots \longrightarrow$$

such that $\delta^n \circ \delta^{n-1} = 0$. We often abbreviate δ^n to δ and write $\delta^2 = 0$. We define the cocycles, coboundaries, and cohomology of X by

$$Z^n(X) = \operatorname{Ker}(\delta^n)$$

$$B^n(X) = \operatorname{Im}(\delta^{n-1})$$

$$H^n(X) = Z^n(X)/B^n(X)$$

These are functors, as for chain complexes, where a map of cochains is defined in the evident way.

If we keep on going instead of stopping at X_0 or X^0, allowing $\mathbb{Z}$-graded chain and cochain complexes, then the notions are mathematically "the same". Starting from a chain complex X_* we can define a cochain complex X^* by $X^{-n} = X_n$ and $\delta^{-n} = d_n$, and vice versa.

4. The bar construction

The bar construction: for a group G, get a chain complex $B(G)$ of FREE $\mathbb{Z}[G]$-modules, with maps d_n of G-modules, and a map $\varepsilon : B_0(G) \to \mathbb{Z}$ of G-modules such that the following sequence is exact:

$$\cdots \longrightarrow B_{n+1}(G) \xrightarrow{d_{n+1}} B_n(G) \xrightarrow{d_n} B_{n-1}(G) \longrightarrow \cdots \longrightarrow B_0[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

We write $B_n(G) = \mathbb{Z}[G] \otimes \bar{B}_n(G)$, where $\bar{B}_n(G)$ is the abelian group $\mathbb{Z}[S_n]$.

We think of $\mathbb{Z}$ as a chain complex with all terms 0 except $X_0 = \mathbb{Z}$, so all $d_n = 0$, and then we think of ε as a map of chain complexes which is 0 in degrees $n \neq 0$.

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & B_{n+1}(G) & \xrightarrow{d_{n+1}} & B_n(G) & \xrightarrow{d_n} & B_{n-1}(G) & \longrightarrow & \cdots & \longrightarrow & B_0[G] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle \varepsilon} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

We must define d_n and we must have $d_n(gx) = gd_n(x)$ for $g \in G$ and $x \in \bar{B}_n(G)$. With the original Eilenberg-Mac Lane "bar" notation $[g_1|\cdots|g_n]$ for elements of $\bar{B}_n(G)$,

$$\begin{aligned}
d_n[g_1|\cdots|g_n] &= g_1[g_2|\cdots|g_n] \\
&\quad + \sum_{1\le i\le n-1} (-1)^i [g_1|\cdots|g_{i-1}|g_i g_{i+1}|g_{i+2}|\cdots|g_n] \\
&\quad + (-1)^n [g_1|\cdots|g_{n-1}].
\end{aligned}$$

Here $B_0(G) = \mathbb{Z}[G]$ and $\varepsilon(g)[\ ] = 1$ for $g \in G$; $d_1[g] = g[\ ] - [\ ]$. We define $\eta\colon \mathbb{Z} \longrightarrow B_*(G)$, a map of chain complexes, by letting $\eta = 0$ in degrees $n \neq 0$ and by $\eta(1) = [\ ]$. Then $\varepsilon \circ \eta = \mathrm{id}\colon \mathbb{Z} \longrightarrow \mathbb{Z}$. Why is our original sequence exact?

5. Chain homotopies and chain homotopy equivalence

Suppose we have two chain complexes X and Y and two maps of chain complexes $f, g\colon X \longrightarrow Y$, all of R-modules. We say that f is homotopic to g if there are maps of R-modules

$$s_n : X_n \longrightarrow Y_{n+1}$$

such that

$$d_{n+1}s_n + s_{n-1}d_n = f_n - g_n$$

for $n \geq 0$, where $s_{-1} = 0$ by convention It follows that the induced maps of homology f_* and g_* from $H_n(X)$ to $H_n(Y)$ are equal for all n. Here f_* sends the homology class $[x]$ of a cycle x to the homology class $[f(x)]$ of the cycle $f(x)$. Since $df(x) = fd(x) = 0$, f takes cycles to cycles. Similarly, since $fd(x) = df(x)$, f takes boundaries to boundaries. Therefore f_* is well-defined. Given the chain homotopy s and a cycle $x \in X_n$, $d_{n+1}s_n(x) = f_n(x) - g_n(x)$ since $d_n(x) = 0$, and this says that $f_* = g_*\colon H_n(X) \longrightarrow H_n(Y)$.

Two chain complexes X and Y are chain homotopy equivalent if there are maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Then $f_*\colon H_*(X) \longrightarrow H_*(Y)$ and $g_*\colon H_*(Y) \longrightarrow H_*(X)$ are inverse isomorphisms of homology groups.

6. The chain homotopy for the bar construction

Define a homomorphism of abelian groups (NOT of G-modules)

$$s_n \colon B_n(G) \longrightarrow B_{n+1}(G)$$

by

$$s_n(g[g_1|\cdots|g_n]) = [g|g_1|\cdots|g_n].$$

Then

$$(d_1 s_0 + s_{-1}d_0)(g[\,]) = g[\,] - \varepsilon(g)[\,] = g[\,] - \eta\varepsilon(g[\,])$$

and

$$d_{n+1}s_n + s_{n-1}d_n = \mathrm{id} \colon B_n(G) \longrightarrow B_n(G)$$

if $n > 0$: the first term of $d_{n+1}s_n$ gives you back what you start with, and all the rest of the terms in $d_{n+1}s_n + s_{n-1}d_n$ cancel in pairs because of our choice of signs; it is a fun exercise to see that this is true. That proves the following result.

Theorem 6.1. *s is a chain homotopy between* id *and $\eta \circ \varepsilon$ mapping $B(G)$ to itself.*

Since $\varepsilon \circ \eta = \mathrm{id}$ on the chain complex $\mathbb{Z}$, this has the following implication.

Corollary 6.2. *$B(G)$ and $\mathbb{Z}$ are chain homotopy equivalent via η and ε.*

In particular, $H_n(B(G)) = 0$ for $n > 0$ and $\varepsilon_* \colon H_0(B(G)) \longrightarrow \mathbf{Z}$ is an isomorphism. Therefore our original chain complex is exact.

A comparison of definitions now shows that we have the following interpretation of $H^*(G; M)$.

Theorem 6.3. $\mathrm{Hom}_{\mathbb{Z}[G]}(B(G), M)$ *is a cochain complex whose cohomology is $H^*(G; M)$.*

This is exactly the definition we first gave, but it is now reinterpreted a bit more conceptually, heading towards a truly conceptual definition.

7. Free, projective, and injective modules

The functor $\mathrm{Hom}_R(-,-)$ is left exact but not right exact. This means two things. For any short exact sequence of (left, say) R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

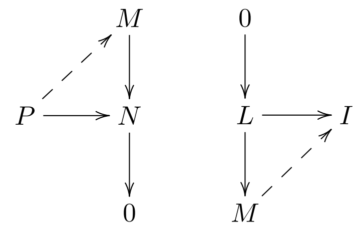
and any R-modules P and I, the following sequences are exact, up to and not including the dotted arrow at the end.

$$0 \longrightarrow \mathrm{Hom}_R(P, L) \longrightarrow \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N) - - > 0$$

$$0 \longrightarrow \mathrm{Hom}_R(N, I) \longrightarrow \mathrm{Hom}_R(M, I) \longrightarrow \mathrm{Hom}_R(L, I) - - > 0$$

Proof is an exercise (partly done in the talk), but the thing to focus on is that non-exactness at the end. Take $R = \mathbb{Z}$, for example. Consider the epimorphism $\mathbb{Z} \longrightarrow \mathbb{Z}/(n) \longrightarrow 0$. The identity map is an element of $\mathrm{Hom}_R(\mathbb{Z}/(n), \mathbb{Z}/(n))$ that is not the image of any element of $\mathrm{Hom}((n), \mathbb{Z})$ since there are no non-zero homomorphisms $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}$. Similarly, consider the monomorphism $0 \longrightarrow (n) \longrightarrow \mathbb{Z}$. The identity map is an element of $\mathrm{Hom}_R((n), (n))$ that is not the image of any element of $\mathrm{Hom}(\mathbb{Z}, (n))$, since if there were such an f, $f(1) = mn$ would have to be divisible by n.

Definition 7.1. An R-module P is projective if $\mathrm{Hom}_R(P, -)$ preserves epimorphisms. An R-module I is injective if $\mathrm{Hom}_R(-, I)$ converts monomorphisms to epimorphisms.



Lemma 7.2. *An R-module P is projective if and only if it is a direct summand of a free R-module. Every module is a quotient of a projective (indeed, a free) R-module.*

Proof. Easy and done in class. Free implies projective is immediate by freeness. Direct summand of free implies projective follows. Projective implies direct summand by choosing an epimorphism $F \longrightarrow P$ and lifting the identity map of P. $\square$

Lemma 7.3. *Every module is a submodule of an injective R-module.*

Sketch proof. Much harder since no obvious characterization of injectives. Baer's criterion: I is injective if and only if for every ideal J, every map $J \longrightarrow I$ extends to a map $R \longrightarrow I$. This implies that injective abelian groups are the same as a divisible abelian groups, and that leads to a proof for $\mathbb{Z}$-modules: An abelian group A is a quotient $\mathbb{Z}[S]/K$, and thus embeds in $\mathbb{Q}[S]/K$, which is divisible and hence injective. Now let M be an R-module, embed the abelian group M in a divisible abelian group D. Have a composite inclusion of R-modules

$$0 \longrightarrow M \xrightarrow{i} \mathrm{Hom}_{\mathbb{Z}}(R, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, D),$$

Here $i(m)(r) = rm$. On the middle and right, $(sf)(r) = f(rs)$ for $r, s \in R$ and $f\colon R \longrightarrow M$. A "change of rings" argument shows that $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module because D is an injective $\mathbb{Z}$-module. See Tor − Ext notes for details. $\square$

8. Projective and injective resolutions

9. The axiomatic definition of $H^*(G; M)$

10. Cyclic group calculations

11. Natural transformations and chain homotopies are homotopies