DUAL SPACES

In this handout we assume all vector spaces are over the field k.

1 Some Set Theoretical Thing

Definition 1.1. Let X, Y be two sets, the *function set* (this is not a standard term though), denoted $\mathrm{Fun}(X, Y)$, or Y^X, is just the set of all functions from X to Y.

Exercise 1.2. Show that $[2]^X$ is just the power set of X.

Exercise 1.3. Show that if X, Y are finite sets, then $|\mathrm{Fun}(X, Y)| = |Y|^{|X|}$. (This is why we use the exponential notation.)

Example 1.4. Composition of functions can be easily described as a function

$$\circ : \mathrm{Fun}(Y, Z) \times \mathrm{Fun}(X, Y) \to \mathrm{Fun}(X, Z)$$

Exercise 1.5. Let X, Y, Z be sets, define a function

$$\Phi : \mathrm{Fun}(X, \mathrm{Fun}(Y, Z)) \to \mathrm{Fun}(X \times Y, Z)$$

as

$$(\Phi(f))(x, y) = (f(x))(y), \forall f \in \mathrm{Fun}(X, \mathrm{Fun}(Y, Z))$$

First, figure out what I meant in the definition above... Then show that Φ is a bijection. Note that this is the same as saying that

$$\left(Z^Y\right)^X = Z^{(X \times Y)}$$

(Again, this is why we use the exponential notation.)

2 Hom Set

Definition 2.1. Let V, W be two vector spaces. We denote $\mathrm{Hom}(V, W)$ to be the set of all homomorphisms from V to W.

Exercise 2.2. Show that $\mathrm{Hom}(V, W)$ is a vector space with the operations defined as

$$(\varphi + \psi)(v) = \varphi(v) + \psi(v)$$
$$(\lambda\varphi)(v) = s\varphi(v)$$

Exercise 2.3. Show that $\mathrm{Hom}(k, V) \cong V$.

Exercise 2.4. Show that $\dim(V, W) = \dim V \dim W$.

3 Dual Spaces

Definition 3.1. Let V be a vector space. The *dual space* of V, denoted V^*, is defined as

$$V^* = \operatorname{Hom}(V, k)$$

An element in V^* is also called a *linear functional* on V. For $v \in V, \varphi \in V^*$, for convenience, sometimes we use the bracket notation $\langle \varphi, v \rangle$ to denote $\varphi(v)$.

Exercise 3.2. Assume V has a basis $\mathcal{B}_V = \{e_1, e_2, \cdots, e_n\}$. Define $\varphi_1, \varphi_2, \cdots, \varphi_n \in V^*$ as

$$\langle \varphi_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Show that $\mathcal{B}_V^* = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ is a basis of V^*. In particular, $\dim V^* = \dim V$. We call $\mathcal{B}_V^*$ the dual basis of $\mathcal{B}_V$.

Since $\dim V^* = \dim V$, there exists an isomorphism between V and V^*. However, V and V^* are **not** *canonically isomorphic*. That is, the isomorphism does not preserve the vector space structures.

Exercise 3.3. Define a function $J : V \to (V^*)^*$ by

$$\langle J(v), \varphi \rangle = \langle \varphi, v \rangle$$

First, figure out what I meant in the definition above... Then show that J is an injective homomorpshism. Moreover, show that J is an isomorphism if V is finite dimensional.

Exercise 3.4. Let V, W, U be vector spaces. Show that

$$\operatorname{Hom}(V, W^* \otimes U) \cong \operatorname{Hom}(V \otimes W, U)$$

In particular, $\operatorname{Hom}(W, U) \cong W^* \otimes U$. (Hint: It uses the same idea as Exercise 1.5.)

Exercise 3.5. Let V, W be finite dimensional vector spaces with bases $\mathcal{B}_V, \mathcal{B}_W$. Recall Exercise that we have dual bases $\mathcal{B}_V^*, \mathcal{B}_W^*$. Now given a homomorphism $f : V \to W$. Define the dual of f, denoted f^*, by

$$\begin{aligned} f^* :& W^* \to V^* \\ f^*(\psi) &= \psi \circ f \end{aligned}$$

Show that f^* is a homomorphism. Also, show that the following always holds.

$$\langle f^*(\psi), v \rangle = \langle \psi, f(v) \rangle$$

Assume $\operatorname{Rep}_{\mathcal{B}_V, \mathcal{B}_W}(f) = M$. What is $\operatorname{Rep}_{\mathcal{B}_W^*, \mathcal{B}_V^*}(f^*)$ in terms of M?

Exercise 3.6. Let $f : V \to W, g : W \to U$. Show that

$$(g \circ f)^* = f^* \circ g^*$$