ABSTRACTS: 2014 SUMMER VIGRE REU

All concepts mentioned in the abstracts will be carefully defined

APPRENTICE PROGRAM, weeks 1–5

Laci Babai

Linear algebra and discrete structures

The course will develop the usual topics of linear algebra and illustrate them on (often striking) applications to discrete structures. Emphasis will be on creative problem solving and discovery. The basic topics include determinants, linear transformations, the characteristic polynomial, Euclidean spaces, orthogonalization, the Spectral Theorem, Singular Value Decomposition. Application areas to be highlighted include spectral graph theory (expansion, quasirandom graphs, Shannon capacity), random walks, clustering high-dimensional data, extremal set theory, and more.

FULL PROGRAM, weeks 1–8, ALL WELCOME

Probability

Gregory Lawler, weeks 1-2

Week 1: Random Walk and the Heat Equation

Two closely related topics are random walk and heat flow. One sees this by considering heat as consisting of a large (infinite?) number of "heat particles" all moving randomly and independently. In the first week, I will present some of the mathematics that makes this rigorous. I will focus on random walk on the integer lattice and use this to give a discrete model for heat flow. Analysis of this flow will use linear algebra.

Week 2: Some challenging models in random walk

I will discuss some more advanced problems in random walk that are topics of current mathematical research including:

— self-avoiding walks (SAW), a model for polymer chains that is still a very challenging mathematical problem

— spanning trees of graphs and an algorithm that uses random walk to select a "uniform spanning tree", that is, a spanning tree from the uniform distribution on all spanning trees.

Depending on time, I may have a chance to discuss how complex analysis is used to study these problems in two dimensions.

The Week 1 lectures are NOT a prerequisite for the Week 2 lectures.

Antonio Auffinger, weeks 3-8

Subadditive processes and an introduction to Ergodic Theory.

The goal of this course is to present some topics which are accessible to advanced undergraduates yet are active areas of research in probability. We will first devote our time to introduce in a pedestrian way some important theorems in Ergodic theory. Later on we will focus all our attention to subadditive processes, processes that satisfy the relation $T_{0,n} \leq T_{0,m} + Tm, n$ for $0 < m < n$. We will see that this simple property is shared by some important and fascinating models in probability theory. Some (most, actually!) of these models have 50+ year-old conjectures that we will describe in detail. There will be homework and problem sets.

Geometry and Topology

Dominic Dotterrer, weeks 1-3

The Spectral Geometry of Graphs

Every solution to the wave equation ($d^2f/dt^2 = \Delta f$) on a finite graph can be written as a linear combination of solitons ($\Delta f = \lambda f$) The *energy* of the overtones, λ, encode a great deal of geometric information about the graph.

What can listening to the harmonics of the graph tell us? We will explore ways in which the energy of the overtones quantify geometric and topological complexity of the graph.

Once you are convinced of the geometric value of high energy graphs, we will look for ex- tremal examples of such graphs. Using covering theory, we will prove a universal asymptotic upper bound to the energy of k-regular graphs.

Tentative Schedule (9 hours)

(1) Graphs, e.g. Cayley graphs, Random graphs
(2) The Graph Laplacian and the meaning of $Kerd$
(3) The Spectral Theorem
(4) Cheeger's Inequality
(5) The Existence of Expander Graphs
(6) Kolmogorov's Topological Embeddings of Graphs
(7) Metric Embeddings of Graphs
(8) Spectrum of the k-regular tree
(9) The Alon-Boppana theorem

If there is interest, I may teach an additional, *bonus*, session scheduled for a Friday after- noon on the topic of Kirchoff's Matrix Tree Theorem.

Prerequisites: Basic concepts of linear algebra, such as linear transformation, linear independence, rank, kernel.

Sebastian Hensel, weeks 4-6

In the second half of this course we look at a completely different application of the topology and geometry of graphs (and simplicial complexes). The slogan is this: if a group acts nicely on a space we understand, then we can learn something about the algebra of the group.

We'll explore this theme with various examples (both concrete and abstract). To give a brief sampling:

(1) basic techniques that allow to find finite generating sets and presentations from an action on a graph (or simplicial complex)
(2) studying the free group in more detail, showing algebraic properties (like residual finiteness, the structure of subgroups...) by simple topological arguments.
(3) relating actions of any group G on trees to free splittings of G (Bass-Serre theory)
(4) and if there is time and interest we can look at a (slightly more complicated) real research example: mapping class group actions on curve graphs.

Number theory

Matthew Emerton, Paul Herman, and Davide Reduzzi, weeks 1-3

Number Theory: Primes of the form $x^2 + ny^2$

In 1640 Fermat stated in a letter to Mersenne that an odd prime number p can be written as the sum of two squares if and only if p is congruent to 1 modulo 4. Later he formulated analogous statements characterizing primes of the form x^2+2y^2 and $x^2 + 3y^2$ via congruences. More than one hundred years after their formulation, Fermat's claims were proved using descent and special cases of quadratic reciprocity by Euler, who also conjectured a characterization (that he could not prove) of primes of the form x^2+5y^2. We will discuss methods from algebraic number theory that allow us to prove these theorems by studying orders in quadratic imaginary number fields and class groups.

Logic

Denis Hirschfeldt, week 1

Computability and definability

Every mathematician knows that if $2+2 = 5$ then Bertrand Russell is the pope. Russell is credited with having given a proof of that fact in a lecture, though from the point of view of classical logic, no such proof is needed, since a false statement implies every statement. Contrapositively, every statement implies a given true statement. But we are often interested in questions of implication and nonimplication between true statements. We have all heard our teachers and colleagues say things like "Theorems A and B are equivalent." or "Theorem C does not just follow from Theorem D." There is also a well-established practice of showing that a given theorem can be proved without using certain methods. These are often crucial things to understand about an area of mathematics, and can also help us make connections between different areas of mathematics.

Computability theory and proof theory can both be used to analyze, and hence compare, the strength of theorems and constructions. For example, when we have a principle such as "Every infinite binary tree has an infinite path", we can ask how difficult it is to compute such a path from a given tree. We can also ask how much axiomatic power is necessary to prove that this principle holds. The first kind of question leads to the program of Computable Mathematics. One version of the second kind of question leads to the program of Reverse Mathematics. I will give an introduction to these research programs, and discuss how close connection between computability and definability yields a fruitful interplay between them.

Maryanthe Malliaris, week 2

p, t, and model theory

Abstract: This course will be about the recent solution, due to Malliaris and Shelah, of the oldest problem on cardinal invariants of the continuum – whether "$p = t$" – via model theory. The course will introduce the problem and, time permitting, give a few key ideas of the proof. This is an opportunity to hear about very new research.

Prerequisites: Mathematical maturity. It would be helpful to know a little about the infinite cardinals: that, as Cantor showed, the continuum is uncountable, and that Hilbert's first problem asks whether it is the first uncountable cardinal. Previous exposure to logic is useful, but not required.

Analysis

Baoping Liu, first 2 or 3 weeks

Topic: equilibria in Nonlinear Systems

Abstract: We will start with simple examples in ordinary differential equation to introduce the idea of equilibria, and discuss their stability in proper sense. Then we will move on to discuss briefly the corresponding concepts and ideas in the context of partial differential equations.

Potpourri

Peter May and Henry Chan, weeks 1-8

Finite spaces and larger contexts

There is a fascinating and little known theory of finite topological spaces. We will present lots of the basic theory and how it relates to partially ordered sets (posets), classical simplicial complexes, finite groups, and categories.

For example, we shall show how to describe a space with $2n + 2$ points that to the eyes of algebraic topology is "just the same" as the n-sphere S^n. For another example, we shall reinterpret an interesting unsolved problem in finite group theory in terms of finite topological spaces. We shall go through a proof due to past REU participants of an unexpected result that they themselves discovered, and we shall give quite a few problems and questions. We will go slowly enough that all can follow (promise!!!), and we will introduce some more classical material to give context. However, most of the material is sure to be new to even the most advanced students.

A book in progress is available on line:
http://math.uchicago.edu/ may/FINITE/FINITEBOOK/FiniteAugBOOK.pdf
(You are offered a $1.00 reward for each typo you find.)