OPERADS AND SHEAF COHOMOLOGY

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Abstract. I explain how to construct E_∞ cochain algebras for computing classical sheaf cohomology and, in principle, hypercohomology, and I explain how not to construct E_∞ cochain algebras for computing motivic cohomology.

Contents

Introduction

In this short expository paper, I explain how to construct E_∞ cochain algebras for computing classical sheaf cohomology and how not to construct E_∞ cochain algebras for computing motivic cohomology.

There are two brief preliminary sections. In the first, I recall relevant details about the classical Eilenberg-Zilber theorem. In the second, I recall an elementary categorical construction of endomorphism operads of functors. It has many applications, implicit or explicit, and is well-known to the experts. I show in §3 how these ideas combine to give the Eilenberg-Zilber operad $\mathscr{Z}$ that acts on the cochains of simplicial sets. This is based on Hinich and Schechtman [7] or, for a particularly clear exposition that I take as a model, Mandell [13, §5]. I give a leisurely pedantic variant that is intended to facilitate adaptation to the sheaf theoretic context.

In §4, I show how the same ideas work in the Čech approach to sheaf cohomology, explaining in modern terms ideas that were already understood by Godement [3]. In §5, I show in principle how to extend the idea to hypercohomology. After recalling how to define "singular chains" of presheaves in §6, I point out a way not to carry out this idea in motivic cohomology in §7, which unfortunately contradicts [8].

I will not repeat the definitions of operads and operad actions, for which I refer the reader to [9, I§§1-2] or [16]. However, to give motivation, let me recall some basic consequences of having a structure of an E_∞ algebra on a cochain complex A

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of modules over a commutative ring R. It is immediate that $H^*(A)$ is a (graded) commutative algebra, but the added structure on the cochains has stronger implications. The first of the following two results is fleshed out in [9, I.7.2] and [17, 6.1], by reference to [14], and the second is proven in [9, II.1.5].

Theorem 0.1. *Let $R = \mathbb{F}_p$. Then $H^*(A)$ admits Steenrod operations Sq^s of degree s if $p = 2$ or P^s of degree $2s(p-1)$ if $p > 2$ that generalize the pth power operation and satisfy the Cartan formula and Adem relations.*

Theorem 0.2. *Let R be a field of characteristic 0. Then A is quasi-isomorphic as an E_∞ algebra to a commutative DGA.*

By a DGA, we mean a differential graded R-algebra, with R understood, and of course commutativity is meant in the graded sense.

Our results in §§4,5 give many examples. In particular, these results apply to the classical Čech cochain complexes associated to presheaves of commutative R-algebras on a space (or scheme) X. Theorem 0.2 allows application of the methods of rational homotopy theory to sheaf cohomology, but that is not a new idea. There are more direct ways to construct a commutative DGA for computing the cohomology of a cosimplicial rational commutative DGA, which are already implicit in Weil's proof of the deRham theorem. They are exploited explicitly by Hain [4, 5.2] and Navarro Aznar [24], following up work of Morgan [23]. In those papers, rational homotopy theory is applied to the study of mixed Hodge structures.

As a parenthetical advertisement, another application of such methods, where at present the full strength of Theorem 0.2 is needed to obtain the relevant commutative DGA's, appears in [9, IV]. There it is applied to study one approach to mixed Tate motives. As another parenthetical advertisement, one relevant to positive characteristic, Mandell [12] has used E_∞ algebras to give an algebraization of p-adic homotopy theory analogous to the Quillen-Sullivan algebraization of rational homotopy theory. Via our observations, his methods may eventually have applications in algebraic geometry.

This note is the first in a sequence of three. Here, we only use operads and we focus on classical situations in algebraic topology and algebraic geometry. In [18], we describe a different conceptual context, discussing "caterads" and algebras over caterads. As we explain in [19], it is expected that this will give the right context in which to describe the general formal structure that is present on the motivic cochain alebras of [20, 25].

I am happy to thank Roy Joshua for getting me to start thinking about motivic cochains. I thank Mike Mandell and Vladimir Voevodsky for helping me straighten out my ideas. I especially thank Chuck Weibel for a careful reading which led to a more reader friendly reorganization.

1. The Eilenberg-Zilber theorem

To clarify details and philosophy, we recall the classical comparison [2] between tensor products of chains and chains of tensor products of simplicial Abelian groups A and B. To avoid later use of the same letter to denote the chain functor defined on different categories, we follow Mac Lane [10, VIII§5] and let K denote the chain functor from the category $\Delta^{\mathrm{op}}\mathrm{Ab}$ of simplicial Abelian groups to the category Ch of chain complexes. Thus $(KA)_n = A_n$, with differential $d = \sum (-1)^i d_i$.

Everything in this paper works equally well using either K or the normalized chain functor K_N. The latter is defined by $K_NA = KA/DA$, where $(DA)_n$ is the subgroup of A_n generated by the degenerate n-simplices; DA is a a subcomplex of KA whose homology is zero, and the quotient map $KA \longrightarrow K_NA$ is a natural chain homotopy equivalence [10, 6.1]. Normalized chains are substantially more convenient in algebraic topology, but the reasons will not concern us here. When considering sheaves, passage to quotients may be inappropriate, and it is then better to think in terms of unnormalized chains.

Recall that $(A\otimes B)_n = A_n \otimes B_n$. To compare chain complexes of tensor products with tensor products of chain complexes, we have the Alexander-Whitney map

$$f\colon K(A\otimes B) \longrightarrow KA \otimes KB$$

and the shuffle, or Eilenberg-Mac Lane, map

$$g\colon KA\otimes KB \longrightarrow K(A\otimes B).$$

Both are natural quasi-isomorphisms and are the identity map in degree 0.

The former is specified by

$$f(x\otimes y) = \sum \tilde{\partial}^{n-i}x \otimes {\partial_0}^i y \tag{1.1}$$

for $x \in A_n$ and $y \in B_n$, where $\tilde{\partial}$ denotes the last face operator. It is associative [10, VIII.8.7]. It is obviously not commutative, but it is chain homotopy commutative.

The shuffle map, which will not become relevant until §6, is specified by

$$g(x\otimes y) = \sum_{(\mu,\nu)} (-1)^{\sigma(\mu,\nu)}(s_{\nu_q}\cdots s_{\nu_1} \otimes s_{\mu_p}\cdots s_{\mu_1} y) \tag{1.2}$$

for $x \in A_p$ and $y \in B_q$. The sum runs over all (p,q)-shuffle permutations (μ,ν), and $\sigma(\mu,\nu)$ is the sign of the permutation. The shuffle map is unital, associative, and commutative, *with no chain homotopies required*. Note that the unit of the $\otimes$-product in Δ^{op}Ab is $\mathbb{Z}_\bullet$, the constant simplicial Abelian group at $\mathbb{Z}$, while the unit of Ch is the chain complex given by $\mathbb{Z}$ in degree 0. We have a unit chain homotopy equivalence $\mathbb{Z} \longrightarrow K\mathbb{Z}_\bullet$, which becomes the identity map on passage to $K_N\mathbb{Z}_\bullet$. Explicitly, commutativity says that the following diagram commutes.

$$\begin{array}{ccc} KA\otimes KB & \xrightarrow{g} & K(A\otimes B) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle Kt} \\ KB\otimes KA & \xrightarrow{g} & K(B\otimes A) \end{array} \tag{1.3}$$

Here t on the right is transposition (no sign) and τ on the left is the graded transposition $\tau(x\otimes y) = (-1)^{pq}y\otimes x$, where x is of degree p and y is of degree q.

Formally, the properties of the shuffle map imply the following conceptual result. Recall that a functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ between symmetric monoidal categories is said to be *lax symmetric monoidal* if there is a map $\lambda\colon \kappa_{\mathscr{B}} \longrightarrow F(\kappa_{\mathscr{A}})$ of unit objects and a natural map

$$\iota : FX \otimes_{\mathscr{B}} FY \longrightarrow F(X \otimes_{\mathscr{A}} Y)$$

such that all coherence diagrams relating F to the unit, associativity, and commutativity isomorphisms of $\mathscr{A}$ and $\mathscr{B}$ commute; see [11, XI§2]; F is *strong symmetric monoidal* if λ and ι are isomorphisms. These formal notions are important to us since a lax symmetric monoidal functor F carries an operad $\mathscr{O}$ in $\mathscr{A}$ to an operad

$F\mathscr{O}$ in $\mathscr{B}$ and carries an $\mathscr{O}$-algebra A to an $F\mathscr{O}$-algebra FA in $\mathscr{B}$. This principle is in fact one of the main virtues of the definition of an operad.

Proposition 1.4. *The functors K and K_N from simplicial Abelian groups to chain complexes are lax symmetric monoidal via the shuffle map.*

Let $\mathbb{Z}_\bullet$ also denote the free Abelian group functor from the category $\Delta^{\mathrm{op}}\mathrm{Sets}$ of simplicial sets to the category $\Delta^{\mathrm{op}}\mathrm{Ab}$. Since we prefer to focus on normalized chains, we define the chain complex of a simplicial set X by letting $C_*(X) = K_N\mathbb{Z}_\bullet X$. We record a triviality.

Proposition 1.5. *The functor $\mathbb{Z}_\bullet\colon \Delta^{op}Sets \longrightarrow \Delta^{op}Ab$ is strong symmetric monoidal, hence the functor C_* is lax symmetric monoidal.*

Proof. This holds since $\mathbb{Z}_\bullet(*)$ is constant at $\mathbb{Z}$ and $\mathbb{Z}_\bullet(X \times Y) \cong \mathbb{Z}_\bullet X \otimes \mathbb{Z}_\bullet Y$ for simplicial sets X and Y. □

Applying K_N, it follows that

$$C_*(X \times Y) = K_N\mathbb{Z}_\bullet(X \times Y) \cong K_N(\mathbb{Z}_\bullet X \otimes \mathbb{Z}_\bullet Y). \tag{1.6}$$

Thus the discussion above applies to compare $C_*(X \times Y)$ with $C_*(X) \otimes C_*(Y)$.

We could apply the discussion of this section equally well starting with the category $\mathscr{M}_R$ of modules over a commutative ring R, rather than with Ab. Of course, the resulting normalized chains $C_*(X; R)$ can be identified with $C_*X \otimes R$.

2. Endomorphism operads of functors

Let $\mathscr{C}$ be any closed symmetric monoidal category with unit object κ, product $\otimes$, and internal hom functor Hom, so that we have the adjunction

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

For definiteness, the reader should think of $\mathscr{C}$ as the category $\mathrm{Ch}\mathscr{M}_R$ of cochain complexes over a commutative ring R, with $\kappa = R$. Let $\varepsilon : \mathrm{Hom}(X, Y) \otimes X \longrightarrow Y$ denote the counit of the adjunction and let $\tau : X \otimes Y \longrightarrow Y \otimes X$ denote the commutativity isomorphism for $\otimes$. The adjoint of

$$(\varepsilon \otimes \varepsilon) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) : \mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(X', Y') \otimes X \otimes X' \longrightarrow Y \otimes Y',$$

gives a natural $\otimes$-product pairing

$$\mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(X', Y') \longrightarrow \mathrm{Hom}(X \otimes X', Y \otimes Y').$$

We assume that $\mathscr{C}$ is complete (has all limits). Let $\mathscr{D}$ be any small category, such as the category Δ whose covariant and contravariant functors are cosimplicial and simplicial objects, respectively.

For a pair of covariant functors $X, Y : \mathscr{D} \longrightarrow \mathscr{C}$, we define $\mathrm{Hom}_{\mathscr{D}}(X, Y)$ to be the equalizer in $\mathscr{C}$ displayed in the diagram

$$\mathrm{Hom}_{\mathscr{D}}(X, Y) \longrightarrow \prod_{d \in \mathscr{D}} \mathrm{Hom}(X_d, Y_d) \rightrightarrows \prod_{\alpha : d \to e} \mathrm{Hom}(X_d, Y_e), \tag{2.1}$$

where the second product runs over all morphisms α of $\mathscr{D}$. Here X_d denotes the object that X assigns to d. Writing $X_\alpha : X_d \longrightarrow X_e$ for the morphism that X assigns to $\alpha : d \to e$, the parallel arrows send (f_d) to $(f_e \circ X_\alpha)$ and to $(Y_\alpha \circ f_d)$. Thus the αth components of the parallel arrows are the composites of the projections to the eth or the dth component followed by

$$\mathrm{Hom}(X_\alpha, \mathrm{id}) : \mathrm{Hom}(X_e, Y_e) \longrightarrow \mathrm{Hom}(X_d, Y_e)$$

or

$$\mathrm{Hom}(\mathrm{id}, Y_\alpha) : \mathrm{Hom}(X_d, Y_d) \longrightarrow \mathrm{Hom}(X_d, Y_e).$$

This is just a pedantically precise internal hom analogue of the definition of the set of natural transformations from X to Y.

We define the tensor product over $\mathscr{D}$ of a contravariant functor $X : \mathscr{D} \longrightarrow \mathscr{C}$, written $d \longmapsto X^d$ on objects, and a covariant functor $Y : \mathscr{D} \longrightarrow \mathscr{C}$ to be the coequalizer displayed in the analogous diagram

$$\text{(2.2)} \qquad \textstyle\coprod_{\alpha: d \to e} X^d \otimes Y_e \rightrightarrows \coprod_{d \in \mathscr{D}} X^d \otimes Y_d \longrightarrow X \otimes_{\mathscr{D}} Y.$$

The parallel arrows are given by the maps

$$X^\alpha \otimes \mathrm{id} : X^e \otimes Y_d \longrightarrow X^d \otimes Y_d \quad \text{and} \quad \mathrm{id} \otimes Y_\alpha : X^e \otimes Y_d \longrightarrow X^e \otimes Y_e.$$

Construction 2.3. Fix a covariant functor $\Lambda : \mathscr{D} \longrightarrow \mathscr{C}$. We have the diagonal power functor Λ^j. It sends d to the j-fold $\otimes$-power $(\Lambda_d)^j$. By convention, the 0-fold power of Λ is the constant functor at κ. We define the *endomorphism operad* $\mathrm{End}(\Lambda)$ of the functor Λ by setting

$$\mathrm{End}(\Lambda)(j) = \mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^j).$$

The unit $\eta : \kappa \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda)$ is the adjoint of the identity and the right action of the symmetric group Σ_j on $\mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^j)$ is induced from the permutation action of Σ_j on the functor Λ^j. The product maps

$$\gamma : \mathrm{End}_\Lambda(k) \otimes \mathrm{End}_\Lambda(j_1) \otimes \cdots \otimes \mathrm{End}_\Lambda(j_k) \longrightarrow \mathrm{End}_\Lambda(j),$$

where $j = j_1 + \cdots + j_k$, are the composites

$$\begin{array}{c}
\mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^k) \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^{j_1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^{j_k}) \\
\downarrow {\scriptstyle \mathrm{id} \otimes k\text{-fold } \otimes\text{-product}} \\
\mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^k) \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda^k, \Lambda^j) \\
\downarrow {\scriptstyle \tau} \\
\mathrm{Hom}_{\mathscr{D}}(\Lambda^k, \Lambda^j) \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^k) \\
\downarrow {\scriptstyle \text{composition}} \\
\mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^j).
\end{array}$$

The verifications of the defining equivariance, unit, and associativity conditions for an operad (see [9, 16]) are immediate.

Remark 2.4. The appearance of the commutativity isomorphism τ in the definition of γ introduces appropriate signs in graded situations. In terms of elements of internal Hom objects, γ is given by

$$\gamma(g; f_1, \cdots, f_k) = (-1)^{pq}(f_1 \otimes \cdots \otimes f_k) \circ g,$$

where p is the sum of the degrees of the f_i and q is the degree of g.

Remark 2.5. Endomorphism operads are special cases of the "functor operads" of McClure and Smith [22], in which powers of a given functor (our Λ^j) are replaced by a sequence of functors $\Lambda(j)$ that are related by suitable natural transformations.

The purpose of defining an operad $\mathscr{O}$ is to define $\mathscr{O}$-algebras (see [9, 16]). Such algebras are given by a suitably equivariant, unital, and associative sequence of maps $\mathscr{O}(j) \otimes X^j \longrightarrow X$. The maps γ generalize in a way that leads to this.

Construction 2.6. For covariant functors $A_i : \mathscr{D} \longrightarrow \mathscr{C}$, $1 \leq i \leq j$, we have the diagonal $\otimes$-product $A_1 \otimes \cdots \otimes A_j : \mathscr{D} \longrightarrow \mathscr{C}$ that sends d to $A_{1,d} \otimes \cdots \otimes A_{j,d}$. In analogy with the definition of γ above,we define

$$\xi : \mathrm{End}(\Lambda)(j) \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda, A_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda, A_j) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\Lambda, A_1 \otimes \cdots \otimes A_j)$$

to be the composite

$$\begin{array}{c}
\mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^j) \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda, A_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda, A_j) \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes j\text{-fold } \otimes\text{-product}} \\
\mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^j) \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda^j, A_1 \otimes \cdots \otimes A_j) \\
\Big\downarrow {\scriptstyle \tau} \\
\mathrm{Hom}_{\mathscr{D}}(\Lambda^j, A_1 \otimes \cdots \otimes A_j) \otimes \mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^j) \\
\Big\downarrow {\scriptstyle \mathrm{composition}} \\
\mathrm{Hom}_{\mathscr{D}}(\Lambda, A_1 \otimes \cdots \otimes A_j).
\end{array}$$

We call these maps ξ *generalized Alexander-Whitney maps*.

3. The Eilenberg-Zilber operad

Specializing the construction of the previous section, we take $\mathscr{D}$ to be the category Δ of finite sets $\mathbf{n}$ and monotonic maps, so that a simplicial object is a contravariant functor defined on Δ and a cosimplicial object is a covariant functor defined on Δ. As is usual, we often use bullets $^\bullet$ and $_\bullet$ to denote cosimplicial or simplicial variables, but we note that this is opposite to the conventional use of superscripts and subscripts in the previous section.

Fix a commutative ring R and let $\mathscr{M}_R$ be the category of R-modules. We take $\mathscr{C}$ in the previous section to be the category $\mathrm{Ch}\mathscr{M}_R$ of differential $\mathbb{Z}$-graded R-modules, or R-cochain complexes, with differentials raising degree by 1. We write $X_n = X^{-n}$ and so regard cochain complexes as chain complexes, and conversely, whenever convenient. The internal hom functor is given by letting $\mathrm{Hom}^n_R(X, Y)$ be the product over q of the $\mathrm{Hom}_R(X^q, Y^{q+n})$. The differential on an element (f_q) of $\mathrm{Hom}^n_R(X, Y)$ is given by $d(f_q) = d \circ f_q + (-1)^{q+1} f_{q+1} \circ d$. Let Δ^n denote the standard simplicial n-simplex, that is, the represented functor $\Delta(-, \mathbf{n})$. As n varies, the Δ^n give a cosimplicial simplicial set $\Delta^\bullet$.

Definition 3.1. Define $\Lambda : \Delta \longrightarrow \mathrm{Ch}\mathscr{M}_R$ to be the functor that sends $\mathbf{n}$ to the normalized chain complex $C_*(\Delta^n, R)$. That is, $\Lambda = C_*(\Delta^\bullet, R)$. Define the *Eilenberg-Zilber operad* to be the operad $\mathscr{Z} = \mathrm{End}(\Lambda)$ in $\mathrm{Ch}\mathscr{M}_R$.

Let $\mathscr{C}om$ denote the commutativity operad in $\mathrm{Ch}\mathscr{M}_R$. Each $\mathscr{C}om(j) = R$, with trivial action by Σ_j; the unit and structure maps of $\mathscr{C}om$ are identity maps. The $\mathscr{C}om$-algebras are exactly the commutative DGA's. Define a map $\varepsilon \colon \mathscr{Z} \longrightarrow \mathscr{C}om$ of operads by identifying each $C_*(\Delta^0, R)^j$ with R and restricting maps in $\mathscr{Z}(j)$ to cosimplicial level zero, where they give elements of $R = \mathrm{Hom}_R(R, R)$. The method of acyclic models introduced by Eilenberg and Zilber [2] implies that the Eilenberg-Zilber operad is *acyclic*, which means that the following result holds.

Proposition 3.2. *For $j \geq 0$, $\varepsilon\colon \mathscr{Z}(j) \longrightarrow \mathscr{C}om(j) = R$ is a quasi-isomorphism.*

An E_∞ *operad* in $\mathrm{Ch}\mathscr{M}_R$ is an acyclic operad $\mathscr{E}$ such that each $\mathscr{E}(j)$ is an $R[\Sigma_j]$-free resolution of R. Thus, in addition to being acyclic, it is required that $\mathscr{E}(j)_n = 0$ for $n < 0$ and that Σ_j act freely on each $\mathscr{E}(j)_n$. These conditions are not satisfied by $\mathscr{Z}$, but we have the following result.

Proposition 3.3. *There is an E_∞ operad $\mathscr{E}$ and a quasi-isomorphism of operads $\alpha\colon \mathscr{E} \longrightarrow \mathscr{Z}$.*

A quick direct proof is given in [13, §4]. More sophisticated proofs use the fact that the category of operads admits a model structure [1, 6]. McClure and Smith [21] have constructed a combinatorially explicit E_∞ approximation $\alpha\colon \mathscr{E} \longrightarrow \mathscr{Z}$.

An E_∞ *algebra* over R is a cochain complex together with an action of an E_∞ operad. Note that we are implicitly regrading a chain E_∞ operad cohomologically to make sense of this. For purposes of analogy, we recall the proof that the cochains of a simplicial set form a $\mathscr{Z}$-algebra and thus, by pullback along α, an E_∞ algebra. To this end, we give a slightly unorthodox description of the chains and cochains of simplicial sets. Embed $\mathscr{M}_R$ in $\mathrm{Ch}\mathscr{M}_R$ by regarding R-modules as (co)chain complexes concentrated in degree zero, with differential zero.

Definition 3.4. Let X be a simplicial set. Define a simplicial R-module $R_\bullet X$ by applying the free R-module functor in each degree, so that $R_n X$ is the free R-module generated by the n-simplices of X. Define a cosimplicial R-module $R^\bullet X$ by $R^\bullet X = \mathrm{Hom}_R(R_\bullet X, R)$. Thus $R^n X = R^{X_n}$ is the cartesian product of copies of R indexed by the n-simplices of X. Regarding each $R_n X$ and $R^n X$ as a (co)chain complex concentrated in degree zero and with differential zero, redefine

$$C_*(X, R) = \Lambda \otimes_{\Delta^{\mathrm{op}}} R_\bullet X \tag{3.5}$$

and define

$$C^*(X, R) = \mathrm{Hom}_\Delta(\Lambda, R^\bullet X). \tag{3.6}$$

To see that these definitions agree with the usual ones, observe that an easy Yoneda argument shows that, in degree n, the right side of (3.5) is the free R-module on the basis $\{i_n \otimes x_n\}$, where i_n is the fundamental class of the usual $C_n(\Delta^n, R)$ and x_n runs through the nondegenerate n-simplices of X. Expressed in this basis, the differential is given by

$$d(i_n \otimes x_n) = d(i_n) \otimes x_n = \sum_{q=0}^{n} (-1)^q i_{n-1} \otimes d_q x_n$$

since $d_q i_n = \delta_q(i_{n-1})$ for the appropriate face map $\delta_q : \mathbf{n-1} \longrightarrow \mathbf{n}$ in Δ. This identifies $C_*(X, R)$ with the usual normalized chains of X, as defined in §1. Similarly, since the functor $\mathrm{Hom}_R(-, R)$ converts coequalizers to equalizers, we have a canonical natural isomorphism of cochain complexes

$$\mathrm{Hom}_R(\Lambda \otimes_{\Delta^{\mathrm{op}}} R_\bullet X, R) \cong \mathrm{Hom}_\Delta(\Lambda, \mathrm{Hom}_R(R_\bullet X, R)). \tag{3.7}$$

Using our first identification, this identifies $C^*(X, R)$ with the usual normalized cochains of X, namely $\mathrm{Hom}_R(C_*(X; R), R)$.

We need the following analogue of Proposition 1.5.

Proposition 3.8. *The functor $R^\bullet$ from simplicial sets to cosimplicial R-modules is lax symmetric monoidal.*

Proof. The unit for the $\otimes$-product of cosimplicial R-modules is the constant cosimplicial R-module $R^\bullet$ at R, which is $R^\bullet(*)$. For simplicial sets X and Y, define

$$\iota : R^\bullet X \otimes R^\bullet Y \longrightarrow R^\bullet(X \times Y)$$

by $\iota(f \otimes g)(x, y) = f(x)g(y)$ for $f : X_n \to R$, $g : Y_n \to R$, $x \in X_n$, and $y \in Y_n$. Clearly ι is associative, commutative, and unital. □

Theorem 3.9. *The cochain functor $C^*(-, R)$ on simplicial sets takes values in the category of $\mathscr{Z}$-algebras.*

Proof. Define

$$\iota : (R^\bullet X)^j \longrightarrow R^\bullet(X^j)$$

to be the identity if $j = 0$ (where both sides are constant at R) or if $j = 1$ and to be the iterate of ι if $j \geq 2$. Here the power on the left side is the tensor power of cosimplicial R-modules, while X^j on the right denotes the j-fold cartesian power. Then we have the composites

$$\begin{array}{c} \mathrm{Hom}_\Delta(\Lambda, \Lambda^j) \otimes \mathrm{Hom}_\Delta(\Lambda, R^\bullet X)^j \\ \Big\downarrow \xi \\ \mathrm{Hom}_\Delta(\Lambda, (R^\bullet X)^j) \\ \Big\downarrow \mathrm{Hom}_\Delta(\mathrm{id}, \iota) \\ \mathrm{Hom}_\Delta(\Lambda, R^\bullet(X^j)) \\ \Big\downarrow \mathrm{Hom}_\Delta(\mathrm{id}, R^\bullet(\triangle)) \\ \mathrm{Hom}_\Delta(\Lambda, R^\bullet X). \end{array} \tag{3.10}$$

The Alexander-Whitney map ξ is given by Construction 2.6; the map $\triangle : X \longrightarrow X^j$ is the projection $X \longrightarrow *$ if $j = 0$, the identity if $j = 1$, and the iterated diagonal map if $j \geq 2$. With these structure maps, denoted

$$\theta = \theta_j : \mathscr{Z}(j) \otimes C^*(X, R)^j \longrightarrow C^*(X, R), \tag{3.11}$$

$C^*(X, R)$ is an algebra over $\mathscr{Z}$. □

We give some discussion to make this more concrete. The composites

$$C_*(\Delta^n, R) \xrightarrow{\triangle_*} C_*(\Delta^n \times \Delta^n, R) \xrightarrow{f} C_*(\Delta^n, R) \otimes C_*(\Delta^n, R)$$

specify a zero cycle in the chain complex $\mathscr{Z}(2) = \mathrm{Hom}_\Delta(\Lambda, \Lambda^2)$, and the operad $\mathscr{Z}$ encodes the collection of all natural chain maps of the same general form. Restricted to this cycle, the action map θ_2 recovers the usual cup product on $C^*(X; R)$. The acyclicity of $\mathscr{Z}$ directly implies that the resulting product on $H^*(X; R)$ is associative, commutative, and unital. Note that the geometric diagonal map $\triangle$ is cocommutative. The non-commutativity on the cochain level comes from the non-commutativity of the Alexander-Whitney map f, or of any other comparison map in this direction. The $\cup_1$-product, which gives the chain homotopy commutativity of $C^*(X; R)$, and the higher $\cup_i$-products are given similarly by elements of degree

i in $\mathscr{Z}(2)$. These are the source of the classical mod 2 Steenrod operations when $R = \mathbb{F}_2$. Similar, but less explicit, structure in $\mathscr{Z}(p)$ gives the Steenrod operations when $R = \mathbb{F}_p$ for an odd prime p.

We abstract the structures used in this example. We started in the cartesian monoidal category $\mathscr{T} = \Delta^{op}$Sets of simplicial sets; the cartesian monoidal structure gave rise to the diagonal maps $\triangle$. We had a lax monoidal functor $R^\bullet$ from $\mathscr{T}$ to the category of cosimplicial R-modules. Via the embedding $\mathscr{M}_R \longrightarrow \mathrm{Ch}\mathscr{M}_R$, we regarded $R^\bullet$ as taking values in cosimplicial cochain complexes.

Theorem 3.12. *Let $\mathscr{T}$ be a cartesian monoidal category and let $F^\bullet$ be a lax symmetric monoidal functor from $\mathscr{T}$ to the category of cosimplicial R-modules. For $X \in \mathscr{T}$, define the cochains of X with coefficients in $F^\bullet$ by*

$$C^*(X, F^\bullet) = \mathrm{Hom}_\Delta(\Lambda, F^\bullet X).$$

Then $C^*(X, F^\bullet)$ is naturally an algebra over the operad $\mathscr{Z}$ with structure maps θ defined as in (3.10), but with $R^\bullet$ replaced by $F^\bullet$.

4. Classical sheaf cohomology

Using Čech cochains, we show here that Theorem 3.12 adapts to give E_∞ cochains that compute sheaf cohomology. This should not be a new result, although there seems to be no exposition in the literature. In fact, although the language of operads came almost fifteen years later [15], the basic idea was already understood by Godement [3, p.v] in 1958. To quote him, "... la théorie multiplicative des faisceaux n'est qu'un cas particulier de la théorie générale concernant les complexes <<simpliciaux>>; elle montre que toute notion reposant exclusivement sur l'existence d'une structure simpliciale s'étend automatiquement à la théorie des faisceaux; en particulier, il est clair dès maintenant que *les opérations de Steenrod peuvent se définir en théorie des faisceaux*." However, he left an exposition to "le second tome de cet ouvrage", which unfortunately never appeared. As he realized, his canonical flasque resolutions could be used instead of the Čech construction. Only the simplicial structure is relevant.

For simplicity of notation and clarity of exposition, we shall work over a topological space X rather than in a Grothendieck site. The argument adapts without difficulty to the latter situation. Let $\mathscr{U}$ be an open cover of X indexed on a totally ordered set I. Define a simplicial set $\mathscr{U}_\bullet$ by letting the set $\mathscr{U}_n$ of n-simplices be the set of ordered $(n+1)$-tuples $S = \{U_{i_0}, \ldots, U_{i_n}\}$ (possibly with repeats) of sets in $\mathscr{U}$ that have non-empty intersection, denoted U_S. The qth face operator deletes the qth set, and the qth degeneracy operator repeats the qth set. Given a presheaf $\mathscr{F}$ of R-modules on X, we define a cosimplicial R-module $\mathscr{F}^\bullet_\mathscr{U}$ by setting

$$\mathscr{F}^n_\mathscr{U} = \prod_{S \in \mathscr{U}_n} \mathscr{F}(U_S). \tag{4.1}$$

The cofaces and codegeneracies are induced by restriction maps associated in the evident fashion to the faces and degeneracies of $\mathscr{U}_\bullet$. If $\mathscr{F}$ is constant at R, this is the obvious analogue of the construction $R^\bullet$ on simplicial sets. As there, we regard $\mathscr{F}^\bullet_\mathscr{U}$ as a cosimplicial cochain complex, and we define the Čech cochain complex by

$$\check{\mathrm{C}}^*(\mathscr{U}, \mathscr{F}) = \mathrm{Hom}_\Delta(\Lambda, \mathscr{F}^\bullet_\mathscr{U}). \tag{4.2}$$

Because we are using normalized chains in the definition of Λ, we are restricting to the subproduct in (4.1) with coordinates zero when S contains repeated subsets, and we are imposing the expected differential. Up to language, this is precisely the standard definition of Čech cochains; compare [3, 5.1] or [5, III§4].

A refinement of a cover $\mathscr{U}$ indexed on an ordered set I is a cover $\mathscr{V}$ indexed on an ordered set J together with a function $\alpha\colon J \longrightarrow I$ such that $V_j \subset U_{\alpha(j)}$. A refinement induces a map $\alpha_\bullet\colon \mathscr{V}_\bullet \longrightarrow \mathscr{U}_\bullet$ of simplicial sets, a map $\alpha^\bullet\colon \mathscr{F}^\bullet_{\mathscr{U}} \longrightarrow \mathscr{F}^\bullet_{\mathscr{V}}$ of cosimplicial R-modules, and thus a map $\alpha^*\colon \check{\mathrm{C}}^*(\mathscr{U},\mathscr{F}) \longrightarrow \check{\mathrm{C}}^*(\mathscr{V},\mathscr{F})$ of Čech cochains. Two choices of α lead to chain homotopic maps [3, 5.7.1]. Regarding the set of coverings of X as partially ordered under refinement, we can pass to colimits. Alternatively, by choosing an ordering of the points of X and restricting attention to covers indexed on X and satisfying $x \in U_x$, we can pass to colimits canonically. Either way, passing to colimits over covers and refinements, we obtain a colimit cosimplicial R-module $\mathscr{F}^\bullet$, and we define cochains by

$$\check{\mathrm{C}}^*(X,\mathscr{F}) = \mathrm{Hom}_\Delta(\Lambda,\mathscr{F}^\bullet). \tag{4.3}$$

See [3, 5.8] for details and variants "with support". The essential starting point for Čech cohomology is the fact that this Čech cochain functor transforms exact sequences of presheaves to exact sequences of cochain complexes [3, 5.8.1].

However, for us, the essential point is that, again taking $\mathscr{C} = \mathrm{Ch}\mathscr{M}_R$ and $\mathscr{D} = \Delta$, we are again in a context where the generalized Alexander-Whitney maps ξ of Construction 2.6 are defined. We need analogues of the maps ι and $F^\bullet(\triangle)$ that were used to obtain action maps in Theorem 3.9. For the first, we have the following analogue of Proposition 3.8.

Proposition 4.4. *The functors $(-)^\bullet_{\mathscr{U}}$ and $(-)^\bullet$ from the category of presheaves of R-modules over X to the category of cosimplicial R-modules are lax symmetric monoidal.*

Proof. The $\otimes$-product of presheaves $\mathscr{F}$ and $\mathscr{G}$ is the "diagonal presheaf" $\mathscr{F}\otimes\mathscr{G}$ on X defined by

$$(\mathscr{F}\otimes\mathscr{G})(U) = \mathscr{F}(U)\otimes\mathscr{G}(U), \tag{4.5}$$

where the tensor product on the right is just the ordinary tensor product of R-modules. Define

$$\iota\colon \mathscr{F}^\bullet_{\mathscr{U}}\otimes\mathscr{G}^\bullet_{\mathscr{U}} \longrightarrow (\mathscr{F}\otimes\mathscr{G})^\bullet_{\mathscr{U}} \tag{4.6}$$

by the evident projections on diagonal factors

$$\Big(\prod_{S\in\mathscr{U}_n}\mathscr{F}(U_S)\Big)\otimes\Big(\prod_{T\in\mathscr{U}_n}\mathscr{G}(U_T)\Big) \longrightarrow \prod_{S\in\mathscr{U}_n}(\mathscr{F}(U_S)\otimes\mathscr{G}(U_S)). \tag{4.7}$$

As n varies, these clearly give a map of cosimplicial R-modules, and the required associativity, commutativity, and unity of ι are also clear. □

Since we are interested in multiplicative structures, we must assume such a structure on $\mathscr{F}$, and that will give us the required analogue of the map $F^\bullet(\triangle)$.

Theorem 4.8. *Let $\mathscr{F}$ be a presheaf of commutative R-algebras on X. Then $\check{C}^*(\mathscr{U},\mathscr{F})$ and $\check{C}^*(X,\mathscr{F})$ are $\mathscr{Z}$-algebras.*

Proof. The structure maps

$$\theta = \theta_j\colon \mathscr{Z}(j)\otimes\check{\mathrm{C}}^*(\mathscr{U},\mathscr{F})^j \longrightarrow \check{\mathrm{C}}^*(\mathscr{U},\mathscr{F})$$

are the composites

$$\begin{array}{c}\mathrm{Hom}_\Delta(\Lambda,\Lambda^j)\otimes \mathrm{Hom}_\Delta(\Lambda,\mathscr{F}^\bullet_{\mathscr{U}})^j\\ \Big\downarrow \xi\\ \mathrm{Hom}_\Delta(\Lambda,(\mathscr{F}^\bullet_{\mathscr{U}})^j)\\ \Big\downarrow \mathrm{Hom}_\Delta(\mathrm{id},\iota)\\ \mathrm{Hom}_\Delta(\Lambda,(\mathscr{F}^j)^\bullet_{\mathscr{U}})\\ \Big\downarrow \mathrm{Hom}_\Delta(\mathrm{id},\phi)\\ \mathrm{Hom}_\Delta(\Lambda,\mathscr{F}^\bullet_{\mathscr{U}}).\end{array}$$

Here $\iota\colon (\mathscr{F}^\bullet_{\mathscr{U}})^j \longrightarrow (\mathscr{F}^j)^\bullet_{\mathscr{U}}$ is the identity when $j = 0$, where both sides are the constant presheaf at R, and when $j = 1$; when $j \geq 2$, ι is obtained by iterating the map ι of the previous proof. Similarly, ϕ is the unit map when $j = 0$ and the identity map when $j = 1$; when $j \geq 2$, ϕ is induced by the iterated product $\mathscr{F}^j \longrightarrow \mathscr{F}$. The structure maps for $\check{\mathrm{C}}^*(X,\mathscr{F})$ are obtained by passage to colimits over covers $\mathscr{U}$. □

Remark 4.9. Since we used normalized chains in defining Λ, we have $C_{pq-i}(\Delta^q) = 0$ if $pq - i > q$. Tracing through the definition of ξ and comparing with the general algebraic definition of Steenrod operations [14, 5.1, 5.2], we see that the Steenrod operations derived on $\check{\mathrm{H}}^*(X,\mathscr{F})$ when $R = \mathbb{F}_p$ satisfy $P^s = 0$ for $s < 0$, something which is not true in all algebraic situations where Steenrod operations are defined. For example, it is not true in hypercohomology. The proof that $P^0 = \mathrm{Id}$ in the cohomology of simplicial sets is special to that situation [15, 8.1]. Since P^0 in degree zero is the pth power, or Frobenius, one does not have $P^0 = \mathrm{Id}$ in Čech cohomology. Rather, P^0 is the Frobenius operator obtained by applying the pth power in the $\mathbb{F}_p$-algebras $\mathscr{F}(U)$ to all coordinates of representative cochains of cohomology classes. All of the other basic properties familiar from algebraic topology do hold [9, I.7.2].

5. Hypercohomology

With the exposition above, our results on operad actions appear to be unnatural specializations of more general results. We have been using the evident embedding $\mathscr{M}_R \longrightarrow \mathrm{Ch}\mathscr{M}_R$, but it is more natural to start out with cosimplicial R-cochain complexes rather than just with cosimplicial R-modules. The larger category is the natural domain for the "totalization functor" $\mathrm{Hom}_\Delta(\Lambda,-)$, which is usually denoted "Tot". Theorems 3.12 and 4.8 generalize to give the following results.

Theorem 5.1. *Let $\mathscr{T}$ be a cartesian monoidal category and let $F^\bullet$ be a lax symmetric monoidal functor from $\mathscr{T}$ to the category of cosimplicial R-cochain complexes. For $X \in \mathscr{T}$, define the cochains of X with coefficients in $F^\bullet$ by*

$$C^*(X,F^\bullet) = \mathrm{Hom}_\Delta(\Lambda, F^\bullet X).$$

Then $C^*(X,F^\bullet)$ is naturally an algebra over the operad $\mathscr{Z}$ with structure maps θ defined as in (3.10), but with $R^\bullet$ replaced by $F^\bullet$.

Here, rather than ask for a functor $F^\bullet$ on a cartesian monoidal category, one can modify the argument to obtain the following more general variant.

Theorem 5.2. *Let $F^\bullet$ be a cosimplicial commutative DGA. Then* $\mathrm{Hom}_\Delta(\Lambda, F^\bullet)$ *is an algebra over the operad $\mathscr{Z}$ with structure maps the composites*

$$\mathrm{Hom}_\Delta(\Lambda, \Lambda^j) \otimes \mathrm{Hom}_\Delta(\Lambda, F^\bullet)^j \xrightarrow{\xi} \mathrm{Hom}_\Delta(\Lambda, (F^\bullet)^j) \xrightarrow{\mathrm{Hom}_\Delta(\mathrm{id},\phi)} \mathrm{Hom}_\Delta(\Lambda, F^\bullet),$$

where $\phi\colon (F^\bullet)^j \longrightarrow F^\bullet$ is the unit map if $j = 0$, the identity if $j = 1$, and the iterated product of $F^\bullet$ if $j \geq 2$.

Remark 5.3. When R is a field of characteristic zero, the conclusion of the previous result can be strengthened to obtain quasi-isomorphic commutative DGA's, either by quotation of [9, II.5.1] or by the methods of rational homotopy theory [4, 23, 24].

Similarly, in the previous section, we can start with a presheaf $\mathscr{F}$ of cochain complexes on a space X, define a cosimplicial cochain complex $\mathscr{F}^\bullet_{\mathscr{U}}$ associated to an open cover $\mathscr{U}$ as in (4.1), and define Čech hypercochain complexes by

$$\check{\mathrm{C}}^*(\mathscr{U}, \mathscr{F}) = \mathrm{Hom}_\Delta(\Lambda, \mathscr{F}^\bullet_{\mathscr{U}}). \tag{5.4}$$

Unravelling the notation shows that this agrees with the usual definition of Čech hypercochains. With no changes in the constructions and arguments, we obtain the following generalization of Theorem 4.8. In fact, this can also be thought of as a special case of Theorem 5.2 since the Čech construction gives a presheaf of cosimplicial commutative DGA's to which that result applies.

Theorem 5.5. *Let $\mathscr{F}$ be a presheaf of commutative DGA's on X. Then $\check{C}^*(\mathscr{U}, \mathscr{F})$ and $\check{C}^*(X, \mathscr{F})$ are $\mathscr{Z}$-algebras, with action maps defined as in Theorem 4.8.*

As pointed out to me by Nori, this applies, for example, to deRham cohomology in positive characteristic. However, a problem with the previous two results is that, in practice, we may not encounter commutative DGA's in nature. Rather, we may encounter only E_∞ algebras. We can generalize these results to that situation. The tensor product of operads $\mathscr{O}$ and $\mathscr{P}$ is specified by $(\mathscr{O} \otimes \mathscr{P})(j) = \mathscr{O}(j) \otimes \mathscr{P}(j)$, with the evident structure maps determined by those of $\mathscr{O}$ and $\mathscr{P}$. This simple construction has been used since the introduction of operads [15] to combine operad actions that have possibly different "geometric" origins.

Theorem 5.6. *Let $\mathscr{O}$ be an operad and let $F^\bullet$ be a cosimplicial $\mathscr{O}$-algebra with structure maps θ. Then* $\mathrm{Hom}_\Delta(\Lambda, F^\bullet)$ *is an algebra over the operad $\mathscr{O} \otimes \mathscr{Z}$. The action maps are the composites*

$$\begin{array}{c}
\mathscr{O}(j) \otimes \mathrm{Hom}_\Delta(\Lambda, \Lambda^j) \otimes \mathrm{Hom}_\Delta(\Lambda, F^\bullet)^j \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\xi} \\
\mathscr{O}(j) \otimes \mathrm{Hom}_\Delta(\Lambda, (F^\bullet)^j) \\
\Big\downarrow{\scriptstyle \zeta} \\
\mathrm{Hom}_\Delta(\Lambda, \mathscr{O}(j) \otimes (F^\bullet)^j) \\
\Big\downarrow{\scriptstyle \mathrm{Hom}_\Delta(\mathrm{id},\theta)} \\
\mathrm{Hom}_\Delta(\Lambda, F^\bullet).
\end{array}$$

Here ζ is the evident map of cochain complexes induced from the map

$$\zeta\colon X \otimes \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(Y, X \otimes Z) \tag{5.7}$$

defined for R-modules X, Y, and Z by $\zeta(x \otimes f)(y) = x \otimes f(y)$.

Theorem 5.8. *Let $\mathscr{O}$ be an operad and let $\mathscr{F}$ be a presheaf of $\mathscr{O}$-algebras on a space X with action maps θ. Then $\check{C}^*(\mathscr{U}, \mathscr{F})$ and $\check{C}^*(X, \mathscr{F})$ are $\mathscr{O} \otimes \mathscr{Z}$-algebras. The action maps*

$$\theta\colon \mathscr{O}(j) \otimes \mathscr{Z}(j) \otimes \check{C}^*(\mathscr{U}, \mathscr{F})^j \longrightarrow \check{C}^*(\mathscr{U}, \mathscr{F})$$

are the composites

$$\begin{array}{c} \mathscr{O}(j) \otimes \mathrm{Hom}_\Delta(\Lambda, \Lambda^j) \otimes \mathrm{Hom}_\Delta(\Lambda, \mathscr{F}^\bullet_\mathscr{U})^j \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes \xi} \\ \mathscr{O}(j) \otimes \mathrm{Hom}_\Delta(\Lambda, (\mathscr{F}^\bullet_\mathscr{U})^j) \\ \Big\downarrow {\scriptstyle \zeta} \\ \mathrm{Hom}_\Delta(\Lambda, \mathscr{O}(j) \otimes (\mathscr{F}^\bullet_\mathscr{U})^j) \\ \Big\downarrow {\scriptstyle \mathrm{Hom}_\Delta(\mathrm{id}, \theta)} \\ \mathrm{Hom}_\Delta(\Lambda, \mathscr{F}^\bullet_\mathscr{U}). \end{array}$$

Here, in interpreting the action θ of $\mathscr{O}$ on $\mathscr{F}^\bullet_\mathscr{U}$, we are using (4.1) and the observation that finite cartesian products of $\mathscr{O}$-algebras are $\mathscr{O}$-algebras.

We record the following variant of Proposition 3.3, which is again a special case of the results of [13, §4] and also follows from the model theoretic work of [1, 6].

Proposition 5.9. *If $\mathscr{O}$ is acyclic, there is an E_∞ operad $\mathscr{E}$ and a quasi-isomorphism of operads $\alpha\colon \mathscr{E} \longrightarrow \mathscr{O} \otimes \mathscr{Z}$. Therefore $\mathscr{O} \otimes \mathscr{Z}$-algebras are $\mathscr{E}$-algebras and thus E_∞ algebras by pullback along α.*

Remark 5.10. Let $\mathscr{S}$ be a site. Then, modifying the Čech construction to deal with covers $\mathscr{U}$ of objects X in the site, everything in this and the previous section adapts to the Čech cochain complexes of X with coefficients in sheaves on $\mathscr{S}$ of the specified algebraic types. One replaces the intersections appearing in (4.1) with finite limits, and one observes that finite limits of $\mathscr{O}$-algebras are $\mathscr{O}$-algebras.

6. Presheaf singular chains

Let $\mathscr{S} = \mathrm{Sm}/k$ be the category of smooth separated schemes of finite type over a field k and let $\mathrm{Pre}(\mathscr{S})$ be the category of presheaves on $\mathscr{S}$. We have the standard cosimplicial object $\Delta^\bullet$ in $\mathscr{S}$. Its nth scheme is

$$\Delta^n = \mathrm{Spec}(k[t_0, \ldots, t_n]/(\Sigma t_i - 1)),$$

and its faces and degeneracies are precisely analogous to the faces and degeneracies relating the simplicial or topological simplices Δ^n; see [25, pp. 150, 245].

Definition 6.1. For a presheaf $\mathscr{F}$ on $\mathscr{S}$, define a simplicial presheaf $\mathscr{F}_\bullet$ by letting

$$\mathscr{F}_n(X) = \mathscr{F}(X \times \Delta^n)$$

for $X \in \mathscr{S}$, with faces and degeneracies induced by those of $\Delta^\bullet$. If $\mathscr{F}$ is Abelian, then $\mathscr{F}_\bullet$ is a simplicial Abelian presheaf.

Proposition 6.2. *The functors*

$$(-)_\bullet : \mathit{Pre}(\mathscr{S}) \longrightarrow \Delta^{op}\mathit{Pre}(\mathscr{S}) \quad \text{and} \quad (-)_\bullet : \mathit{AbPre}(\mathscr{S}) \longrightarrow \Delta^{op}\mathit{AbPre}(\mathscr{S})$$

are strong symmetric monoidal.

Proof. This is just the observation that $(-)_\bullet$ takes unit objects to unit objects and preserves products. The latter holds since

$$(\mathscr{F}\times\mathscr{G})_n(X) = (\mathscr{F}\times\mathscr{G})(X\times\Delta^n) = \mathscr{F}(X\times\Delta^n)\times\mathscr{G}(X\times\Delta^n) = \mathscr{F}_n(X)\times\mathscr{G}_n(X)$$

for presheaves $\mathscr{F}$ and $\mathscr{G}$ and

$$(\mathscr{F}\otimes\mathscr{G})_n(X) = (\mathscr{F}\otimes\mathscr{G})(X\times\Delta^n) = \mathscr{F}(X\times\Delta^n)\otimes\mathscr{G}(X\times\Delta^n) = \mathscr{F}_n(X)\otimes\mathscr{G}_n(X)$$

for Abelian presheaves $\mathscr{F}$ and $\mathscr{G}$. □

We use the chain functor K (or K_N) of §1 to define "singular chains".

Definition 6.3. Define the *chain presheaf* $C_*(\mathscr{F})$ of an Abelian presheaf $\mathscr{F}$ by $C_*(\mathscr{F}) = K\mathscr{F}_\bullet$. Tensoring with an Abelian group A, we obtain the chain presheaf $C_*(\mathscr{F}, A)$ with coefficients in A.

Heading towards motivic cochains, let $\mathscr{F} = \{\mathscr{F}(q)\}$ be a sequence of Abelian presheaves, and suppose given natural external pairings

$$\phi\colon \mathscr{F}(q)(X)\otimes\mathscr{F}(r)(Y) \longrightarrow \mathscr{F}(q+r)(X\times Y)$$

for $X, Y \in \mathscr{S}$. These specialize to give external products

$$\mathscr{F}(q)(X\times\Delta^n)\otimes\mathscr{F}(r)(X\times\Delta^n) \longrightarrow \mathscr{F}(q+r)(X\times\Delta^n\times X\times\Delta^n).$$

Pulling back along the diagonal of $X\times\Delta^n$, this gives an internal product

$$\phi\colon \mathscr{F}(q)_\bullet\otimes\mathscr{F}(r)_\bullet \longrightarrow \mathscr{F}(q+r)_\bullet \tag{6.4}$$

of simplicial Abelian presheaves. Passing to chains and composing with the shuffle map g, we obtain the product map of presheaves of chain complexes

$$C_*(\mathscr{F}(q))\otimes C_*(\mathscr{F}(r)) \xrightarrow{\ g\ } C_*(\mathscr{F}(q)\otimes\mathscr{F}(r)) \xrightarrow{\ \phi\ } C_*(\mathscr{F}(q+r)). \tag{6.5}$$

Choosing $\mathscr{F}$ appropriately and then reindexing cohomologically with a shift of grading, this is how the products on motivic cochains are defined formally [20, 3.10]. The chain level product is not commutative because the external pairing ϕ is not commutative. Naively, the problem of constructing E_∞ motivic cochain algebras can be viewed as the problem of finding an acyclic operad $\mathscr{O}$ that acts on $\mathscr{F}$ in such a manner that the product (6.5) is given by a choice of a zero cycle in $\mathscr{O}(2)$. As I will explain in [18, 19], such a naive formulation seems to impede real understanding. However, it is plausible that it can be carried out, and then Proposition 5.8 will apply to give E_∞ motivic cochains for the hypercohomology groups that define motivic cohomology.

7. A misleading endomorphism operad

We describe one way not to proceed. Return to the context of §1, starting with a closed symmetric monoidal category $\mathscr{C}$, a small category $\mathscr{D}$, and a functor $\Lambda\colon \mathscr{D} \longrightarrow \mathscr{C}$. We have the endomorphism operad $\mathrm{End}(\Lambda)$ in $\mathscr{C}$.

Recall that a commutative monoid in $\mathscr{C}$ is an object with a commutative, associative, and unital product. For any $\mathscr{C}$, there is an operad $\mathscr{N}$ (or $\mathscr{C}om$) in $\mathscr{C}$ such that an $\mathscr{N}$-algebra is a commutative monoid. Each $\mathscr{N}(j)$ is the unit object of $\mathscr{C}$, with trivial action by Σ_j; the unit and structure maps are identity maps.

Now take $\mathscr{C}$ to be *cartesian* closed, so that its product is the categorical product, and let $*$ denote its unit object. For each $d \in \mathscr{D}$, we have the iterated diagonal map $\triangle\colon \Lambda_d \longrightarrow \Lambda_d^j$, which can be viewed as a map $* \longrightarrow \mathrm{Hom}(\Lambda_d, \Lambda_d^j)$. The naturality of diagonal maps implies that these maps are the coordinates of a map $* \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\Lambda, \Lambda^j) = \mathrm{End}(\Lambda)(j)$. Since composites of products of diagonal maps are diagonal maps, these maps specify a map $\triangle\colon \mathscr{N} \longrightarrow \mathrm{End}(\Lambda)$ of operads in $\mathscr{C}$.

Lemma 7.1. *For any cartesian closed category $\mathscr{C}$ and any functor $\Lambda\colon \mathscr{D} \longrightarrow \mathscr{C}$, an algebra over* $\mathrm{End}(\Lambda)$ *is a commutative monoid in $\mathscr{C}$ by pullback along $\triangle$.*

Now return to the notations of the previous section, letting $\mathscr{S} = \mathrm{Sm}/k$.

Construction 7.2. We construct an operad $\mathscr{O}$ in the category $\mathrm{ChPre}(\mathscr{S})$ of presheaves of chain complexes on $\mathscr{S}$, together with a map $\triangle_*\colon \mathscr{C}om \longrightarrow \mathscr{O}$, where $\mathscr{C}om$ denotes the commutativity operad in $\mathrm{ChPre}(\mathscr{S})$. Regarding schemes as representable presheaves, start with the endomorphism operad $\mathrm{End}(\Delta^\bullet)$ in $\mathrm{Pre}(\mathscr{S})$. Consider the following three functors.

$$\mathrm{Pre}(\mathscr{S}) \xrightarrow{(-)_\bullet} \Delta^{\mathrm{op}}\mathrm{Pre}(\mathscr{S}) \xrightarrow{\mathbb{Z}_\bullet} \Delta^{\mathrm{op}}\mathrm{AbPre}(\mathscr{S}) \xrightarrow{K} \mathrm{ChPre}(\mathscr{S})$$

By Propositions 1.4, 1.5, and 6.2, these functors are all lax symmetric monoidal and therefore take operads to operads. Writing C_* for the composite of these three functors, define

$$\mathscr{O} = C_*(\mathrm{End}(\Delta^\bullet)). \tag{7.3}$$

As above, if $\mathscr{N}$ is the commutativity operad in $\mathrm{Pre}(\mathscr{S})$, we have the map of operads $\triangle\colon \mathscr{N} \longrightarrow \mathrm{End}(\Delta^\bullet)$. Using normalized chains K_N, $C_*(\mathscr{N}) = \mathscr{C}om$ since it is the operad whose jth presheaf is constant at the chain complex $\mathbb{Z}$; using unnormalized chains K, there is an evident map $\mathscr{C}om \longrightarrow C_*(\mathscr{N})$. Either way, applying the composite functor C_* to $\triangle$, we obtain a map of operads $\triangle_*\colon \mathscr{C}om \longrightarrow \mathscr{O}$.

Since $\mathscr{C}om$-algebras are presheaves of commutative DGA's, the following observation is immediate.

Proposition 7.4. *By pullback along $\triangle_*$, an $\mathscr{O}$-algebra is a presheaf of commutative DGA's over $\mathbb{Z}$.*

The following result holds but, since it seems to have no applications, we omit the proof. It is an easy application of [25, 4.1] or [20, 2.18].

Proposition 7.5. *The operad $\mathscr{O}$ is acyclic.*

Scholium 7.6. An unsuccessful attempt to construct E_∞ motivic cochains was given in [8]. On close inspection, one finds that the acyclic operad intended there is in fact the operad $\mathscr{O}$ of (7.3), or rather its evident cubical variant. Since its algebras are presheaves of commutative DGA's and since the motivic cochain algebras are not commutative, this operad cannot act on motivic cochains.

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