# QUANTIZATION OF HITCHIN'S INTEGRABLE SYSTEM AND HECKE EIGENSHEAVES

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## Contents

# 0. Introduction

**0.1.** Let $X$ be a connected smooth projective curve over $\mathbb{C}$ of genus $g > 1$, $G$ a semisimple group over $\mathbb{C}$, $\mathfrak{g}$ the Lie algebra of $G$. Denote by $\mathrm{Bun}_G = \mathrm{Bun}_G(X)$ the moduli stack of $G$-bundles on $X$. In [Hit87] Hitchin defined a remarkable algebra $\mathfrak{z}^{cl} = \mathfrak{z}^{cl}(X)$ of Poisson-commuting functions on the cotangent stack of $\mathrm{Bun}_G(X)$.

**0.2.** In this note the following is shown:

(a) The Hitchin construction admits a natural quantization. Namely, we define a commutative ring $\mathfrak{z} = \mathfrak{z}(X)$ of twisted differential operators on $\mathrm{Bun}_G$ such that the symbols of operators from $\mathfrak{z}$ form exactly the ring $\mathfrak{z}^{cl}$ of Hitchin's Hamiltonians. Here "twisted" means that we consider the differential operators acting on a square root of the canonical bundle $\omega_{\mathrm{Bun}_G}$. The twist is essential: one knows that the only global untwisted differential operators on $\mathrm{Bun}_G$ are multiplications by locally constant functions.

(b) The spectrum of $\mathfrak{z}$ identifies canonically with the moduli of ${}^L\mathfrak{g}$-*opers*, which is a (Lagrangian) subspace of the moduli of irreducible $\left({}^L G\right)_{ad}$-local systems on $X$. Here ${}^L G$ is the Langlands dual of $G$, ${}^L\mathfrak{g}$ its Lie algebra, $\left({}^L G\right)_{ad}$ the adjoint group; for a brief comment on opers see 0.3.

(c) For an ${}^L\mathfrak{g}$-oper $\mathfrak{F}$ denote by $N_{\mathfrak{F}}$ the quotient of the sheaf of twisted differential operators modulo the left ideal generated by the maximal ideal $\mathfrak{m}_{\mathfrak{F}} \subset \mathfrak{z}$. This is a non-zero holonomic twisted $\mathcal{D}$-module on $\mathrm{Bun}_G$.

(d) One assigns to an ${}^L G$-oper $\mathfrak{F}$ a usual (non-twisted) $\mathcal{D}$-module $M_{\mathfrak{F}}$ on $\mathrm{Bun}_G$. If $G$ is simply connected $M_{\mathfrak{F}}$ is isomorphic to $\omega_{\mathrm{Bun}_G}^{-1/2} \otimes N_{\mathfrak{F}}$ (in the simply connected case $\omega_{\mathrm{Bun}_G}^{1/2}$ is unique and on the other hand $N_{\mathfrak{F}}$ makes sense because there is no difference between ${}^L G$-opers and ${}^L\mathfrak{g}$-opers). In general $M_{\mathfrak{F}} := \lambda_{\mathfrak{F}}^{-1} \otimes N_{\bar{\mathcal{F}}}$ where $\bar{\mathcal{F}}$ is the ${}^L\mathfrak{g}$-oper corresponding to $\mathfrak{F}$ and $\lambda_{\mathfrak{F}}$ is a certain invertible sheaf on $\mathrm{Bun}_G$ equipped with a structure of twisted $\mathcal{D}$-module (see 5.1.1). The isomorphism class of $\lambda_{\mathfrak{F}}$ depends only on the connected component of $\mathfrak{F}$ in the moduli of ${}^L G$-opers.

(e) *Main theorem:* $M_{\mathfrak{F}}$ is a Hecke eigensheaf with eigenvalue $\mathfrak{F}$ (see ???for the precise statement). In other words $M_{\mathfrak{F}}$ corresponds to the local system $\mathfrak{F}$ in the Langlands sense.

**0.3.** The notion of oper (not the name) is fairly well known (e.g., the corresponding local objects were studied in [DS85]). A $G$-oper on a smooth curve $Y$ is a $G$-local system (=$G$-bundle with connection) equipped with some extra structure (see 3.1.3). If $G = SL_n$ (so we deal with local systems of vector spaces),the oper structure is a complete flag of sub-bundles that satisfies the Griffiths transversality condition and the appropriate non-degeneracy condition at every point of $Y$. A $PSL_2$-oper is the same as a projective connection on $Y$, i.e., a Sturm-Liouville operator on $Y$ (see [Del70] ( )). By definition, a $\mathfrak{g}$-oper is an oper for the adjoint group $G_{ad}$.

If $Y$ is complete and its genus is positive then a local system may carry at most one oper structure, so we may consider opers as special local systems.

**0.4.** The global constructions and statements from 0.2 have local counterparts which play a primary role. The local version of (a), (b) is a canonical isomorphism between the spectrum of the center of the critically twisted (completed) enveloping algebra of $\mathfrak{g}((t))$ and the moduli of ${}^L\mathfrak{g}$-opers on the punctured disc $\mathrm{Spec}\,\mathbb{C}((t))$. This isomorphism was established by Feigin and Frenkel [FF92] as a specialization of a remarkable symmetry between the $W$-algebras for $\mathfrak{g}$ and ${}^L\mathfrak{g}$. We do not know if this "doubly quantized" picture can be globalized. The local version of 0.2(c), (d) essentially amounts to another construction of the Feigin-Frenkel isomorphism based on the geometry of Bruhat-Tits affine Grassmannian. Here the key role belongs to a vanishing theorem for the cohomology of certain critically twisted $\mathcal{D}$-modules (a parallel result for "less than critical" twist was proved in [KT95]).

**0.5.** This note contains only sketches of proofs of principal results. A number of technical results is stated without the proofs. A detailed exposition will be given in subsequent publications.

**0.6.** We would like to mention that E. Witten independently found the idea of 0.2(a–d) and conjectured 0.2(e). As far as we know he did not publish anything on this subject.

**0.7.** A weaker version of the results of this paper was announced in [BD96].

**0.8.** The authors are grateful to P. Deligne, V. Ginzburg, B. Feigin, and E. Frenkel for stimulating discussions. We would also like to thank the Institute for Advanced Study (Princeton) for its hospitality. Our sincere gratitude is due to R. Becker, W. Snow, D. Phares, and S. Fryntova for careful typing of the manuscript.

# 1. Differential operators on a stack

**1.1. First definitions.** A general reference for stacks is [LMB93].

**1.1.1.** Let $\mathcal{Y}$ be a smooth equidimensional algebraic stack over $\mathbb{C}$. Denote by $\Theta_{\mathcal{Y}}$ the tangent sheaf; this is a coherent sheaf on $\mathcal{Y}$. The cotangent stack $T^*\mathcal{Y} = \operatorname{Spec}\operatorname{Sym}\Theta_{\mathcal{Y}}$ need not be smooth. Neither is it true in general that $\dim T^*\mathcal{Y} = 2\dim\mathcal{Y}$ (consider, e.g., the classifying stack of an infinite algebraic group or the quotient of $sl_n$ modulo the adjoint action of $SL_n$). However one always has

$$\dim T^*\mathcal{Y} \geq 2\dim\mathcal{Y} \tag{1}$$

We say that $\mathcal{Y}$ is *good* if

$$\dim T^*\mathcal{Y} = 2\dim\mathcal{Y} \tag{2}$$

Then $T^*\mathcal{Y}$ is locally a complete intersection of pure dimension $2\dim\mathcal{Y}$. This is obvious if $\mathcal{Y} = K\backslash S$ for some smooth variety $S$ with an action of an algebraic group $K$ on it (in this case $T^*\mathcal{Y}$ is obtained from $T^*S$ by Hamiltonian reduction; see 1.2.1), and the general case is quite similar.

It is easy to show that (2) is equivalent to the following condition:

$$\operatorname{codim}\{y \in \mathcal{Y} | \dim G_y = n\} \geq n \qquad \text{for all } n > 0\,. \tag{3}$$

Here $G_y$ is the automorphism group of $y$ (recall that a point of a stack may have non-trivial symmetries). $\mathcal{Y}$ is said to be *very good* if

$$\operatorname{codim}\{y \in \mathcal{Y} | \dim G_y = n\} > n \qquad \text{for all } n > 0\,. \tag{4}$$

It is easy to see that $\mathcal{Y}$ is very good if and only if $T^*\mathcal{Y}^0$ is dense in $T^*\mathcal{Y}$ where $\mathcal{Y}^0 := \{y \in \mathcal{Y} | \dim G_y = 0\}$ is the biggest Deligne-Mumford substack of $\mathcal{Y}$. In particular if $\mathcal{Y}$ is very good then $T^*\mathcal{Y}_i$ is irreducible for every connected component $\mathcal{Y}_i$ of $\mathcal{Y}$.

*Remark* "Good" actually means "good for lazybones" (see the remark at the end of 1.1.4).

**1.1.2.** Denote by $\mathcal{Y}_{sm}$ the smooth topology of $\mathcal{Y}$ (see [LMB93, Section 6]). An object of $\mathcal{Y}_{sm}$ is a smooth 1-morphism $\pi_S : S \to \mathcal{Y}$, $S$ is a scheme. A morphism $(S, \pi_S) \to (S', \pi_{S'})$ is a pair $(\phi, \alpha)$, $\phi : S \to S'$ is a *smooth* morphism of schemes, $\alpha$ is a 2-morphism $\pi_S \xrightarrow{\sim} \pi_{S'}\phi$. We often abbreviate $(S, \pi_S)$ to $S$.

For $S \in \mathcal{Y}_{sm}$ we have the relative tangent sheaf $\Theta_{S/\mathcal{Y}}$ which is a locally free $\mathcal{O}_S$-module. It fits into a canonical exact sequence

$$\Theta_{S/\mathcal{Y}} \to \Theta_S \to \pi_S^* \Theta_{\mathcal{Y}} \to 0\,.$$

Therefore $\pi_S^* \operatorname{Sym} \Theta_{\mathcal{Y}} = \operatorname{Sym} \Theta_S / I^{cl}$ where $I^{cl} := (\operatorname{Sym} \Theta_S)\Theta_{S/\mathcal{Y}}$. The algebra $\operatorname{Sym} \Theta_S$ considered as a sheaf on the étale topology of $S$ carries the usual Poisson bracket $\{\}$. Let $\tilde{P} \subset \operatorname{Sym} \Theta_S$ be the $\{\}$-normalizer of the ideal $I^{cl}$. Set $(P_{\mathcal{Y}})_S := \tilde{P}/I^{cl}$, so $(P_{\mathcal{Y}})_S$ is the Hamiltonian reduction of $\operatorname{Sym} \Theta_S$ by $\Theta_{S/\mathcal{Y}}$. This is a sheaf of graded Poisson algebras on $S_{\acute{e}t}$. If $S \to S'$ is a morphism in $\mathcal{Y}_{sm}$ then $(P_{\mathcal{Y}})_S$ equals to the sheaf-theoretic inverse image of $(P_{\mathcal{Y}})_{S'}$. So when $S$ varies $(P_{\mathcal{Y}})_S$ form a sheaf $P_{\mathcal{Y}}$ of Poisson algebras on $\mathcal{Y}_{sm}$ called the *algebra of symbols* of $\mathcal{Y}$. The embedding of commutative algebras $P_{\mathcal{Y}} \hookrightarrow \operatorname{Sym} \Theta_{\mathcal{Y}}$ induces an isomorphism between the spaces of global sections

$$\Gamma\,(\mathcal{Y}, P_{\mathcal{Y}}) \xrightarrow{\sim} \Gamma\,(\mathcal{Y}, \operatorname{Sym} \Theta_{\mathcal{Y}}) = \Gamma(T^*\mathcal{Y}, \mathcal{O}) \tag{5}$$

**1.1.3.** For $S \in \mathcal{Y}_{sm}$ consider the sheaf of differential operators $\mathcal{D}_S$. This is a sheaf of associative algebras on $S_{\acute{e}t}$. Let $\tilde{D}_S \subset \mathcal{D}_S$ be the normalizer of the left ideal $I := \mathcal{D}_S \Theta_{S/\mathcal{Y}} \subset \mathcal{D}_S$. Set $(D_{\mathcal{Y}})_S := \tilde{D}_S / I$. This algebra acts on the $\mathcal{D}_S$-module $(\mathcal{D}_{\mathcal{Y}})_S := \mathcal{D}_S / I$ from the right; this action identifies $(D_{\mathcal{Y}})_S$ with the algebra opposite to $\underline{\operatorname{End}}_{\mathcal{D}_S}((\mathcal{D}_{\mathcal{Y}})_S)$.

For any morphism $(\phi, \alpha) : S \to S'$ in $\mathcal{Y}_{sm}$ we have the obvious isomorphism of $\mathcal{D}_S$-modules $\phi^*((\mathcal{D}_{\mathcal{Y}})_{S'}) \xrightarrow{\sim} (\mathcal{D}_{\mathcal{Y}})_S$ which identifies $(D_{\mathcal{Y}})_S$ with the sheaf-theoretic inverse image of $(D_{\mathcal{Y}})_{S'}$. Therefore $(\mathcal{D}_{\mathcal{Y}})_S$ form an $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{D}_{\mathcal{Y}}$ (actually, it is a $\mathcal{D}$-module on $\mathcal{Y}$ in the sense of 1.1.5), and $(D_{\mathcal{Y}})_S$ form a sheaf of associative algebras $D_{\mathcal{Y}}$ on $\mathcal{Y}_{sm}$ called the *sheaf* *of differential operators on* $\mathcal{Y}$. The embedding of sheaves $D_{\mathcal{Y}} \hookrightarrow \mathcal{D}_{\mathcal{Y}}$ induces an isomorphism between the spaces of global sections

$$\Gamma(\mathcal{Y}, D_{\mathcal{Y}}) \overset{\sim}{\to} \Gamma(\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}). \tag{6}$$

**1.1.4.** The $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{D}_{\mathcal{Y}}$ carries a natural filtration by degrees of the differential operators. The induced filtration on $D_{\mathcal{Y}}$ is an algebra filtration such that $\mathrm{gr}\, D_{\mathcal{Y}}$ is commutative; therefore $\mathrm{gr}\, D_{\mathcal{Y}}$ is a Poisson algebra in the usual way.

We have the obvious surjective morphism of graded $\mathcal{O}_{\mathcal{Y}}$-modules $\mathrm{Sym}\, \Theta_{\mathcal{Y}} \to \mathrm{gr}\, \mathcal{D}_{\mathcal{Y}}$. The condition (2) from 1.1.1 assures that this is an isomorphism. If this happens then the inverse isomorphism $\mathrm{gr}\, \mathcal{D}_{\mathcal{Y}} \overset{\sim}{\to} \mathrm{Sym}\, \Theta_{\mathcal{Y}}$ induces a canonical embedding of Poisson algebras

$$\sigma_{\mathcal{Y}} : \mathrm{gr}\, D_{\mathcal{Y}} \hookrightarrow P_{\mathcal{Y}} \tag{7}$$

called *the symbol map*.

*Remark* In the above exposition we made a shortcut using the technical condition (2). The true objects we should consider in 1.1.2–1.1.4 are complexes sitting in degrees $\leq 0$ (now the symbol map is always defined); the naive objects we defined are their zero cohomology. The condition (2) implies the vanishing of the other cohomology, so we need not bother about the derived categories (see 7.3.3 for the definition of the “true” $\mathcal{D}_{\mathcal{Y}}$ for an arbitrary smooth stack $\mathcal{Y}$).

**1.1.5.** $\mathcal{D}$-modules are local objects for the smooth topology, so the notion of a $\mathcal{D}$-module on a smooth stack is clear [1]. Precisely, the categories $\mathcal{M}^{\ell}(S)$ of left $\mathcal{D}$-modules on $S$, $S \in \mathcal{Y}_{sm}$, form a sheaf $\underline{\mathcal{M}}^{\ell}$ of abelian categories on $\mathcal{Y}_{sm}$ (the pull-back functors are usual pull-backs of $\mathcal{D}$-modules; they are exact since the morphisms in $\mathcal{Y}_{sm}$ are smooth). The $\mathcal{D}$-modules on $\mathcal{Y}$ are Cartesian sections of $\underline{\mathcal{M}}^{\ell}$ over $\mathcal{Y}_{sm}$; they form an abelian category $\mathcal{M}^{\ell}(\mathcal{Y})$. In other words, a $\mathcal{D}$-module on $\mathcal{Y}$ is a quasicoherent $\mathcal{O}_{\mathcal{Y}}$-module $M$ together

[1]The definition of the derived category of $\mathcal{D}$-modules is not so clear; see 7.3.

with compatible $\mathcal{D}_S$-module structures on each $\mathcal{O}_S$-module $M_S$, $S \in \mathcal{Y}_{sm}$. The usual tensor product makes $\mathcal{M}^\ell(\mathcal{Y})$ a tensor category. One defines coherent, holonomic, etc. $\mathcal{D}$-modules on $\mathcal{Y}$ in the obvious way. Note that a $\mathcal{D}$-module $M$ on $\mathcal{Y}$ defines the sheaf of associative algebras $\underline{\mathrm{End}}M$ on $\mathcal{Y}_{sm}$, $\underline{\mathrm{End}}M(S) = \underline{\mathrm{End}}M_S$.

For example, in 1.1.3 we defined the $\mathcal{D}$-module $\mathcal{D}_\mathcal{Y}$ on $\mathcal{Y}$; the algebra $D_\mathcal{Y}$ is opposite to $\underline{\mathrm{End}}\mathcal{D}_\mathcal{Y}$.

**1.1.6.** Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$ and $\lambda \in \mathbb{C}$. Any $S \in \mathcal{Y}_{sm}$ carries the line bundle $\pi_S^*\mathcal{L}$. Therefore we have the category $\mathcal{M}^\ell(S)_{\mathcal{L}^\lambda}$ of $\pi_S^*(\mathcal{L})^{\otimes\lambda}$-twisted left $\mathcal{D}$-modules (see, e.g., [BB93]). These categories form a sheaf $\underline{\mathcal{M}}^\ell_{\mathcal{L}^\lambda}$ of abelian categories on $\mathcal{Y}_{sm}$. The category $\mathcal{M}^\ell(\mathcal{Y})_{\mathcal{L}^\lambda}$ of $\mathcal{L}^{\otimes\lambda}$-twisted $\mathcal{D}$-modules on $\mathcal{Y}$ is the category of Cartesian sections of $\underline{\mathcal{M}^\ell_{\mathcal{L}^\lambda}}$. There is a canonical fully faithful embedding $\mathcal{M}^\ell(\mathcal{Y})_{\mathcal{L}^\lambda} \hookrightarrow \mathcal{M}^\ell(\mathcal{L}^\cdot)$ which identifies a $\mathcal{L}^{\otimes\lambda}$-twisted $\mathcal{D}$-module on $\mathcal{Y}$ with the $\lambda$-monodromic $\mathcal{D}$-module on $\mathcal{L}^\cdot$; here $\mathcal{L}^\cdot$ is the $\mathbb{G}_m$-torsor that corresponds to $\mathcal{L}$ (i.e., the space of $\mathcal{L}$ with zero section removed). See Section 2 from [BB93].

We leave it to the reader to define the distinguished object $\mathcal{D}_{\mathcal{Y},\mathcal{L}^\lambda} \in \mathcal{M}^\ell(\mathcal{Y})_{\mathcal{L}^\lambda}$ and the sheaf $D_{\mathcal{Y},\mathcal{L}^\lambda}$ of filtered associative algebras on $\mathcal{Y}_{sm}$. All the facts from 1.1.3–1.1.5 render to the twisted situation without changes.

**1.1.7.** In Section 5 we will need the notion of $\mathcal{D}$-module on an arbitrary (not necessarily smooth) algebraic stack locally of finite type. In the case of schemes this notion is well known (see, e.g., [Sa91]). It is local with respect to the smooth topology, so the generalization for stacks is immediate.

### 1.2. Some well-known constructions.

**1.2.1.** Let $K$ be an algebraic group acting on a smooth scheme $S$ over $\mathbb{C}$. Consider the quotient stack $\mathcal{Y} = K \setminus S$. Then $S$ is a covering of $\mathcal{Y}$ in $\mathcal{Y}_{sm}$, and $\mathcal{D}$-modules, line bundles and twisted $\mathcal{D}$-modules on $\mathcal{Y}$ are the same as the corresponding $K$-equivariant objects on $S$. The $K$-action on $T^*S$ is Hamiltonian and $T^*\mathcal{Y}$ is obtained from $T^*S$ by the Hamiltonian reduction (*i.e.*, $T^*\mathcal{Y} = K \setminus \mu^{-1}(0)$ where $\mu : T^*S \to \mathfrak{k}^*$ is the moment map, $\mathfrak{k} := \mathrm{Lie}(K)$). The Poisson structure on $\Gamma(T^*\mathcal{Y}, \mathcal{O}_{T^*\mathcal{Y}})$ is obtained by identifying it with $\Gamma(\mathcal{Y}, P_{\mathcal{Y}})$ (see 1.1.2) which can be computed using the covering $S \to \mathcal{Y}$:

$$\Gamma(\mathcal{Y}, P_{\mathcal{Y}}) = \Gamma\big(S, \tilde{P}_S/I_S^{cl}\big)^{\pi_0(K)}. \tag{8}$$

Here $\tilde{P} \subset \mathrm{Sym}\,\Theta_S$ is the $\{\}$-normalizer of the ideal $I_S^{cl} := (\mathrm{Sym}\,\Theta_S)\,\mathfrak{k}$ (and $\mathfrak{k}$ is mapped to $\Theta_S \subset \mathrm{Sym}\,\Theta_S$). According to 1.1.3

$$\Gamma(\mathcal{Y}, D_{\mathcal{Y}}) = \Gamma\big(S, \tilde{D}_S/I_S\big)^{\pi_0(K)} \tag{9}$$

where $\tilde{D}_S \subset \mathcal{D}_S$ is the normalizer of $I_S := \mathcal{D}_S \cdot \mathfrak{k}$.

The following construction of symbols, differential operators, and $\mathcal{D}$-modules on $\mathcal{Y}$ is useful.

**1.2.2.** We start with a Harish-Chandra pair $(\mathfrak{g}, K)$ (so $\mathfrak{g}$ is a Lie algebra equipped with an action of $K$, called adjoint action, and an embedding of Lie algebras $\mathfrak{k} \hookrightarrow \mathfrak{g}$ compatible with the adjoint actions of $K$). Let $\tilde{P}_{(\mathfrak{g},K)} \subset \mathrm{Sym}\,\mathfrak{g}$ be the $\{\}$-normalizer of $I_{(\mathfrak{g},K)}^{cl} := (\mathrm{Sym}\,\mathfrak{g})\mathfrak{k}$ and $\tilde{D}_{(\mathfrak{g},K)} \subset U\mathfrak{g}$ be the normalizer of $I_{(\mathfrak{g},K)} := (U\mathfrak{g})\mathfrak{k}$. Set

$$P_{(\mathfrak{g},K)} \;:=\; (\mathrm{Sym}(\mathfrak{g}/\mathfrak{k}))^K = \big(\tilde{P}_{(\mathfrak{g},K)}/I_{(\mathfrak{g},k)}^{cl}\big)^{\pi_0(K)} \tag{10}$$

$$D_{(\mathfrak{g},K)} \;:=\; (U\mathfrak{g}/(U\mathfrak{g})\mathfrak{k})^K = \big(\tilde{D}_{(\mathfrak{g},K)}/I_{(\mathfrak{g},K)}\big)^{\pi_0(K)}. \tag{11}$$

Then $P_{(\mathfrak{g},K)}$ is a Poisson algebra and $D_{(\mathfrak{g},K)}$ is an associative algebra. The standard filtration on $U\mathfrak{g}$ induces a filtration on $D_{(\mathfrak{g},K)}$ such that $\mathrm{gr}\,D_{(\mathfrak{g},K)}$ is commutative. So $\mathrm{gr}\,D_{(\mathfrak{g},K)}$ is a Poisson algebra. One has the obvious embedding of Poisson algebras $\sigma = \sigma_{(\mathfrak{g},K)} : \mathrm{gr}\,D_{(\mathfrak{g},K)} \hookrightarrow P_{(\mathfrak{g},K)}$.

The *local quantization condition* for $(\mathfrak{g}, K)$ says that

$$\sigma_{(\mathfrak{g},K)} \text{ is an isomorphism.} \tag{12}$$

*Remark* Sometimes one checks this condition as follows. Consider the obvious morphisms

$$a^{cl}:((\mathrm{Sym}\,\mathfrak{g})^{\mathfrak{g}})^{\pi_0(K)}\to P_{(\mathfrak{g},K)},\quad a:(\mathrm{Center}\,U\mathfrak{g})^{\pi_0(K)}\to D_{(\mathfrak{g},K)}. \tag{13}$$

If $a^{cl}$ is surjective, then (12) is valid (because $\mathrm{gr}\,\mathrm{Center}\,U\mathfrak{g}=(\mathrm{Sym}\,\mathfrak{g})^{\mathfrak{g}}$). Actually, if $a^{cl}$ is surjective, then $a$ is also surjective and therefore $D_{(\mathfrak{g},K)}$ is commutative.

**1.2.3.** Assume now that we are in the situation of 1.2.1 and the $K$-action on $S$ is extended to a $(\mathfrak{g},K)$-action (*i.e.*, we have a Lie algebra morphism $\mathfrak{g}\to\Theta_{S'}$ compatible with the $K$-action on $S$ in the obvious sense). Comparing (8) with (10) and (9) with (11), one sees that the morphisms $\mathrm{Sym}\,\mathfrak{g}\to\mathrm{Sym}\,\Theta_S$ and $U\mathfrak{g}\to\mathcal{D}_S$ induce canonical morphisms

$$h^{cl}:P_{(\mathfrak{g},K)}\to\Gamma(\mathcal{Y},P_{\mathcal{Y}}),\quad h:D_{(\mathfrak{g},K)}\to\Gamma(\mathcal{Y},D_{\mathcal{Y}}) \tag{14}$$

of Poisson and, respectively, filtered associative algebras.

If $\mathcal{Y}$ is good in the sense of 1.1.1 then we have the symbol map $\sigma_{\mathcal{Y}}:\mathrm{gr}\,D_{\mathcal{Y}}\hookrightarrow P_{\mathcal{Y}}$, and the above morphisms are $\sigma$-compatible: $h^{cl}\sigma_{(\mathfrak{g},K)}=\sigma_{\mathcal{Y}}\,\mathrm{gr}\,h$.

The *global quantization condition* for our data says that

(15) *$h$ is strictly compatible with filtrations.*

In other words, this means that the symbols of differential operators from $h\left(D_{(\mathfrak{g},K)}\right)$ lie in $h^{cl}\sigma_{(\mathfrak{g},K)}\,(\mathrm{gr}\,D_{(\mathfrak{g},K)})$. If both local and global quantization conditions meet then the algebra $h\left(D_{(\mathfrak{g},K)}\right)$ of differential operators is a quantization of the algebra $h^{cl}\left(P_{(\mathfrak{g},K)}\right)$ of symbols: the symbol map $\sigma_{\mathcal{Y}}$ induces an isomorphism $\mathrm{gr}\,h\left(D_{(\mathfrak{g},K)}\right)\xrightarrow{\sim}h^{cl}\left(P_{(\mathfrak{g},K)}\right)$.

*Remark* The local and global quantization conditions are in a sense complementary: the local one tells that $D_{(\mathfrak{g},K)}$ is as large as possible, while the global one means that $h\left(D_{(\mathfrak{g},K)}\right)$ is as small as possible.

**1.2.4.** Denote by $\mathcal{M}(\mathfrak{g}, K)$ the category of Harish-Chandra modules. One has the pair of adjoint functors (see, e.g., [BB93])

$$\Delta : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}^{\ell}(\mathcal{Y}), \quad \Gamma : \mathcal{M}^{\ell}(\mathcal{Y}) \to \mathcal{M}(\mathfrak{g}, K).$$

Namely, for a $\mathcal{D}$-module $M$ on $\mathcal{Y}$ the Harish-Chandra module $\Gamma(M)$ is the space of sections $\Gamma(S, M_S)$ equipped with the obvious $(\mathfrak{g}, K)$-action (e.g., $\mathfrak{g}$ acts via $\mathfrak{g} \to \Theta_S \subset \mathcal{D}_S$) and for a $(\mathfrak{g}, K)$-module $V$ the corresponding $K$-equivariant $\mathcal{D}$-module $\Delta(V)_S$ is $\mathcal{D}_S \bigotimes_{U\mathfrak{g}} V$.

For example, consider the "vacuum" Harish-Chandra module $\mathit{Vac} := U\mathfrak{g}/(U\mathfrak{g})\mathfrak{k}$. For any $V \in \mathcal{M}(\mathfrak{g}, K)$ one has $\operatorname{Hom}(\mathit{Vac}, V) = V^K$, so there is a canonical bijection $\operatorname{End}(\mathit{Vac}) \to \mathit{Vac}^K = D_{(\mathfrak{g},K)}$ (see (11)) which is actually an anti-isomorphism of algebras. One has the obvious isomorphism $\Delta(\mathit{Vac}) = \mathcal{D}_{\mathcal{Y}}$, and the map $\Delta : \operatorname{End}(\mathit{Vac}) \to \operatorname{End}(\mathcal{D}_{\mathcal{Y}}) = \Gamma(\mathcal{Y}, D_{\mathcal{Y}})^{\circ}$ coincides with the map $h$ from (14).

**1.2.5.** The above constructions have twisted versions. Namely, assume we have a central extension $(\tilde{\mathfrak{g}}, K)$ of $(\mathfrak{g}, K)$ by $\mathbb{C}$, so $\mathbb{C} \subset \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}}/\mathbb{C} = \mathfrak{g}$. Denote by $U'\mathfrak{g}$ the quotient of $U\tilde{\mathfrak{g}}$ modulo the ideal generated by the central element $\mathbf{1} - 1$, $\mathbf{1} \in \mathbb{C} \subset \tilde{\mathfrak{g}}$. This is a filtered associative algebra; one identifies $\operatorname{gr} U'\mathfrak{g}$ with $\operatorname{Sym}\mathfrak{g}$ (as Poisson algebras). We get the filtered associative algebra $D'_{(\mathfrak{g},K)} := (U'\mathfrak{g}/\,(U'\mathfrak{g})\,\mathfrak{k})^K$equipped with the embedding $\sigma : \operatorname{gr} D'_{(\mathfrak{g},K)} \hookrightarrow P_{(\mathfrak{g},K)}$. The *twisted local quantization condition* says that $\sigma$ is an isomorphism. Notice that the remark at the end of 1.2.2 is not valid in the twisted case because $\operatorname{gr}\operatorname{Center} U'\mathfrak{g}$ may not be equal to $(\operatorname{Sym}\mathfrak{g})^{\mathfrak{g}}$.

Let $\mathcal{L}$ be a line bundle on $S$. Assume that the $(\mathfrak{g}, K)$-action on $S$ lifts to a $(\tilde{\mathfrak{g}}, K)$-action on $\mathcal{L}$ such that $\mathbf{1}$ acts as multiplication by $\lambda^{-1}$ for certain $\lambda \in \mathbb{C}^*$. Equivalently, we have a $(\tilde{\mathfrak{g}}, K)$-action on $\mathcal{L}^{\cdot}$ which extends the $K$-action, is compatible with the $\mathfrak{g}$-action on $S$, and $\mathbf{1}$ acts as $-\lambda^{-1} t\partial_t \in \Theta_{\mathcal{L}^{\cdot}}$. Set $D'_{\mathcal{Y}} = D_{\mathcal{Y},\mathcal{L}^{\lambda}}$. One has the morphism of filtered associative algebras $h : D'_{(\mathfrak{g},K)} \to \Gamma\left(\mathcal{Y}, D'_{\mathcal{Y}}\right)$ such that $\sigma \operatorname{gr} h = h^{cl}\sigma$. The *twisted global quantization condition* says that $h$ is strictly compatible with filtrations.

Denote by $\mathcal{M}(\mathfrak{g},K)'$ the full subcategory of $(\tilde{\mathfrak{g}},K)$ mod that consists of those Harish-Chandra modules on which $\mathbf{1}$ acts as identity. One has the adjoint functors $\Delta,\Gamma$ between $\mathcal{M}(\mathfrak{g},K)'$ and $\mathcal{M}^{\ell}(\mathcal{Y})_{\mathcal{L}^{\lambda}}$ defined exactly as their untwisted version. Again for $\mathit{Vac}' := U'\mathfrak{g}/\left(U'\mathfrak{g}\right)\mathfrak{k}$ one has $\Delta\left(\mathit{Vac}'\right) = \mathcal{D}_{\mathcal{Y},\mathcal{L}^{\lambda}}$; the algebra $\mathrm{End}(\mathit{Vac}')$ is opposite to $D'_{(\mathfrak{g},K)}$, and $\Delta\colon \mathrm{End}\left(\mathit{Vac}'\right) \to \mathrm{End}\,\mathcal{D}_{\mathcal{Y},\mathcal{L}^{\lambda}} = \Gamma(\mathcal{Y},D'_{\mathcal{Y}})$ coincides with $h$.

**1.2.6.** *An infinite-dimensional version.* Let $K$ be an affine group scheme over $\mathbb{C}$ (so $K$ is a projective limit of algebraic groups) which acts on a scheme $S$. Assume the following condition:

(16) There exists a Zariski open covering $\{U_i\}$ of $S$ such that each $U_i$ is $K$-invariant and for certain normal group subscheme $K_i \subset K$ with $K/K_i$ of finite type $U_i$ is a principal $K_i$-bundle over a smooth scheme $T_i$ (so $T_i = K_i \backslash U_i$).

Then the *fpqc*-quotient $\mathcal{Y} = K \setminus S$ is a smooth algebraic stack (it is covered by open substacks $(K/K_i) \setminus T_i$).

Let us explain how to render 1.2.1–1.2.5 to our situation. Note that $\mathfrak{k} = \mathrm{Lie}\, K$ is a projective limit of finite dimensional Lie algebras, so it is a complete topological Lie algebra. Consider the sheaf $\Theta_S = \mathrm{Der}\,\mathcal{O}_S$ and the sheaf $\mathcal{D}_S \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O}_S)$ of Grothendieck's differential operators. These are the sheaves of complete topological Lie (respectively associative) algebras. Namely, for an affine open $U \subset S$ the bases of open subspaces in $\Gamma(U,\Theta_S)$ and $\Gamma(U,\mathcal{D}_S)$ are formed by the annihilators of finitely generated subalgebras of $\Gamma(U,\mathcal{O}_U)$. The topology on $\Theta_S$ defines the topology on $\mathrm{Sym}\,\Theta_S$; denote by $\overline{\mathrm{Sym}}\Theta_S$ the completed algebra. This is a sheaf of topological Poisson algebras. Let $I_S^{cl} \subset \overline{\mathrm{Sym}}\Theta_S$ be the closure of the ideal $(\overline{\mathrm{Sym}}\Theta_S)\mathfrak{k}$, and $\tilde{P}_S \subset \overline{\mathrm{Sym}}\Theta_S$ be its $\{\ \}$-normalizer. Similarly, let $I_S \subset \mathcal{D}_S$ be the closure of the ideal $\mathcal{D}_S \cdot \mathfrak{k}$ and $\tilde{D}_S$ be its normalizer. Then the formulas from (8), (9) remain valid.

In the definition of a Harish-Chandra pair $(\mathfrak{g}, K)$ we assume that for any $\mathrm{Ad}(K)$-invariant open subspace $\mathfrak{a} \subset k$ the action of $K$ on $\mathfrak{g}/\mathfrak{a}$ is algebraic. Then $\mathfrak{g}$ is a complete topological Lie algebra (the topology on $\mathfrak{g}$ is such that $\mathfrak{k} \subset \mathfrak{g}$ is an open embedding). The algebras $\mathrm{Sym}\,\mathfrak{g}$, $U\mathfrak{g}$ carry natural topologies defined by the open ideals $(\mathrm{Sym}\,\mathfrak{g})\mathfrak{a}$, $(U\mathfrak{g})\mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{g}$ is an open subalgebra. Denote by $\overline{\mathrm{Sym}}\mathfrak{g}$, $\bar{U}\mathfrak{g}$ the corresponding completions. Let $I^{cl}_{(\mathfrak{g},K)} \subset \overline{\mathrm{Sym}\,\mathfrak{g}}$ be the closure of the ideal $(\overline{\mathrm{Sym}}\mathfrak{g})\mathfrak{k}$ and $\tilde{P}_{(\mathfrak{g},K)}$ be its $\{\ \}$-normalizer). Similarly, we have $I_{(\mathfrak{g},K)} \subset \tilde{D}_{(\mathfrak{g},K)} \subset \bar{U}_{\mathfrak{g}}$. Now we define $P_{(\mathfrak{g},K)}$, $D_{(\mathfrak{g},K)}$ by the formulas (10), (11). The rest of 1.2.2–1.2.5 remains valid, except the remark at the end of 1.2.2. It should be modified as follows.

**1.2.7.** The algebras $\overline{\mathrm{Sym}}\mathfrak{g}$ and $\bar{U}\mathfrak{g}$ carry the usual ring filtrations $\overline{\mathrm{Sym}}_n\mathfrak{g} = \bigoplus_{0\le i\le n} \overline{\mathrm{Sym}}^i\mathfrak{g}$ and $\bar{U}_i\mathfrak{g}$; however in the infinite dimensional situation the union of the terms of these filtrations does not coincide with the whole algebras. One has the usual isomorphism $\bar{\sigma}_{\mathfrak{g}} : \mathrm{gr}_i\,\bar{U}\mathfrak{g} \xrightarrow{\sim} \overline{\mathrm{Sym}}^i\mathfrak{g}$. The same facts are true for $\overline{\mathrm{Sym}}\Theta_S$ and $\mathcal{D}_S$.

The morphisms $a^{cl}, a$ from the end of 1.2.2 extend in the obvious way to the morphisms

$$\bar{a}^{cl} : \big((\overline{\mathrm{Sym}}\mathfrak{g})^{\mathfrak{g}}\big)^{\pi_0(K)} \to P_{(\mathfrak{g},K)}, \quad \bar{a} : \big(\mathrm{Center}\,\bar{U}\mathfrak{g}\big)^{\pi_0(K)} \to D_{(\mathfrak{g},K)}. \tag{17}$$

The local quantization condition (12) from 1.2.2 and the surjectivity of $\bar{a}$ follow from the surjectivity of $\bar{a}^{cl}\bar{\sigma}_{\mathfrak{g}} : \mathrm{gr}\,\big(\mathrm{Center}\,\bar{U}\mathfrak{g}\big)^{\pi_0(K)} \to P_{(\mathfrak{g},K)}$. The same is true in the twisted situation. Note that the equality $\mathrm{gr}\,\mathrm{Center}\,\bar{U}\mathfrak{g} = (\mathrm{Sym}\,\mathfrak{g})^{\mathfrak{g}}$ is not necessarily valid (even in the non-twisted case!).

# 2. Quantization of Hitchin's Hamiltonians

**2.1. Geometry of $\mathrm{Bun}_G$.** We follow the notation of 0.1; in particular $G$ is semisimple and $X$ is a smooth projective curve of genus $g > 1$.

**2.1.1.** One knows that $\mathrm{Bun}_G$ is a smooth algebraic stack of pure dimension $(g-1)\dim G$. The set of connected components of $\mathrm{Bun}_G$ can be canonically identified (via the "first Chern class" map) with $H^2(X, \pi_1^{\mathrm{et}}(G)) = \pi_1(G)$. Here $\pi_1^{\mathrm{et}}(G)$ is the fundamental group in Grothendieck's sense and $\pi_1(G)$ is the quotient of the group of coweights of $G$ modulo the subgroup of coroots; they differ by a Tate twist: $\pi_1^{\mathrm{et}}(G) = \pi_1(G)(1)$.

For $\mathcal{F} \in \mathrm{Bun}_G$ the fiber at $\mathcal{F}$ of the tangent sheaf $\Theta = \Theta_{\mathrm{Bun}_G}$ is $H^1(X, \mathfrak{g}_{\mathcal{F}})$. Let us explain that for a $G$-module $W$ we denote by $W_{\mathcal{F}}$ the $\mathcal{F}$-twist of $W$, which is a vector bundle on $X$; we consider $\mathfrak{g}$ as a $G$-module via the adjoint action.

By definition, the canonical line bundle $\omega = \omega_{\mathrm{Bun}_G}$ is the determinant of the cotangent complex of $\mathrm{Bun}_G$ (see [LMB93]). The fiber of this complex over $\mathcal{F} \in \mathrm{Bun}_G$ is dual to $R\Gamma(X, \mathfrak{g}_{\mathcal{F}})[1]$ (see [LMB93]), so the fiber of $\omega$ over $\mathcal{F}$ is $\det R\Gamma(X, \mathfrak{g}_{\mathcal{F}})$.[2]

**2.1.2.** *Proposition.* $\mathrm{Bun}_G$ *is very good in the sense of* 1.1.1.

A proof will be given in 2.10.5. Actually, we will use the fact that $\mathrm{Bun}_G$ is good. According to 1.1 we have the sheaf of Poisson algebras $P = P_{\mathrm{Bun}_G}$ and the sheaves of twisted differential operators $D^\lambda = D_{\mathrm{Bun}_G, \omega^\lambda}$. One knows that for $\lambda \neq 1/2$ the only global sections of $D^\lambda$ are locally constant functions. In Sections 2 and 3 we will deal with $D' := D^{1/2}$; we refer to its sections as simply twisted differential operators.

**2.2. Hitchin's construction I.**

[2] The authors shouldn't forget to check that [LMB93] really contains what is claimed here!!

**2.2.1.** Set $C = C_{\mathfrak{g}} := \operatorname{Spec}(\operatorname{Sym}\mathfrak{g})^G$; this is the affine scheme quotient of $\mathfrak{g}^*$ with respect to the coadjoint action. $C$ carries a canonical action of the multiplicative group $\mathbb{G}_m$ that comes from the homotheties on $\mathfrak{g}^*$. A (non-canonical) choice of homogeneous generators $p_i \in (\operatorname{Sym}\mathfrak{g})^G$ of degrees $d_i$, $i \in I$, identifies $C$ with the coordinate space $\mathbb{C}^I$, an element $\lambda \in \mathbb{G}_m$ acts by the diagonal matrix $\left(\lambda^{d_i}\right)$.

**2.2.2.** Denote by $C_{\omega_X}$ the $\omega_X$-twist of $C$ with respect to the above $\mathbb{G}_m$-action (we consider the canonical bundle $\omega_X$ as a $\mathbb{G}_m$-torsor over $X$). This is a bundle over $X$; the above $p_i$ identify $C_{\omega_X}$ with $\prod_I \omega_X^{\otimes d_i}$. Set

$$\operatorname{Hitch}(X) = \operatorname{Hitch}_{\mathfrak{g}}(X) := \Gamma(X, C_{\omega_X}).$$

In other words, $\operatorname{Hitch}(X) = \operatorname{Mor}\left((\operatorname{Sym}^{\cdot}\mathfrak{g})^G, \Gamma\left(X, \omega_X^{\otimes \cdot}\right)\right)$ (the morphisms of graded algebras). We consider $\operatorname{Hitch}(X)$ as an algebraic variety equipped with a $\mathbb{G}_m$-action; it is non-canonically isomorphic to the vector space $\prod_I \Gamma\left(X, \omega_X^{\otimes d_i}\right)$. There is a unique point $0 \in \operatorname{Hitch}(X)$ which is fixed by the action of $\mathbb{G}_m$. Denote by $\mathfrak{z}^{cl}(X) = \mathfrak{z}^{cl}_{\mathfrak{g}}(X)$ the ring of functions on $\operatorname{Hitch}(X)$; this is a graded commutative algebra. More precisely, the grading on $\mathfrak{z}^{cl}(X)$ corresponds to the $\mathbb{G}_m$-action on $\mathfrak{z}^{cl}(X)$ opposite to that induced by the $\mathbb{G}_m$-action on $C$; so the grading on $\mathfrak{z}^{cl}(X)$ is positive.

**2.2.3.** By Serre duality and 2.1.1 the cotangent space $T^*_{\mathcal{F}}\operatorname{Bun}_G$ at $\mathcal{F} \in \operatorname{Bun}_G$ coincides with $\Gamma(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X)$. The $G$-invariant projection $\mathfrak{g}^* \to C$ yields the morphism $\mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X \to C_{\omega_X}$ and the map $p_{\mathcal{F}} : T^*_{\mathcal{F}}\operatorname{Bun}_G \to \operatorname{Hitch}(X)$. When $\mathcal{F}$ varies we get a morphism

$$p : T^*\operatorname{Bun}_G \to \operatorname{Hitch}(X)$$

or, equivalently, a morphism of graded commutative algebras

$$h^{cl}_X : \mathfrak{z}^{cl}(X) \to \Gamma\left(T^*\operatorname{Bun}_G, \mathcal{O}\right) = \Gamma\left(\operatorname{Bun}_G, P\right).$$

$p$ is called *Hitchin's fibration.*

We denote by $\mathrm{Bun}_G{}^\gamma$ the connected component of $\mathrm{Bun}_G$ corresponding to $\gamma \in \pi_1(G)$ (see 2.1.1) and by $p^\gamma$ the restriction of $p$ to $T^* \mathrm{Bun}_G^\gamma$.

**2.2.4.** *Theorem.* ([Hit87], [Fal93], [Gi97]).

(i) The image of $h_X^{cl}$ consists of Poisson-commuting functions.

(ii) $\dim \mathrm{Hitch}(X) = \dim \mathrm{Bun}_G = (g-1) \cdot \dim \mathfrak{g}$.

(iii) $p$ is flat and its fibers have pure dimension $\dim \mathrm{Bun}_G$. For each $\gamma \in \pi_1(X)$, $p^\gamma$ is surjective.

(iv) There exists a non-empty open $U \subset \mathrm{Hitch}(X)$ such that for any $\gamma \in \pi_1(G)$ the morphism $(p^\gamma)^{-1}(U) \to U$ is proper and smooth, and its fibers are connected. Actually, the fiber of $p^\gamma$ over $u \in U$ is isomorphic to the product of some abelian variety $A_u$ by the classifying stack of the center $Z \subset G$.

(v) For each $\gamma \in \pi_1(X)$ the morphism $\mathfrak{z}^{cl}(X) \to \Gamma(\mathrm{Bun}_G^\gamma, P)$ is an isomorphism. □

*Remarks*

(i) Needless to say the main contribution to Theorem 2.2.4 is that of Hitchin [Hit87].

(ii) Theorem 2.2.4 implies that $p$ is a Lagrangian fibration or, if you prefer, the Hamiltonians from $h_X^{cl}(\mathfrak{z}^{cl}(X))$ define a completely integrable system on $T^*\mathrm{Bun}_G$. We are not afraid to use these words in the context of stacks because the notion of Lagrangian fibration is birational and since $\mathrm{Bun}_G$ is very good in the sense of 1.1.1 $T^*\mathrm{Bun}_G$ has an open dense Deligne-Mumford substack $T^*\mathrm{Bun}_G^0$ which is symplectic in the obvious sense (here $\mathrm{Bun}_G^0$ is the stack of $G$-bundles with a finite automorphism group).

(iii) Hitchin gave in [Hit87] a complex-analytical proof of statement (i). We will give an algebraic proof of (i) in 2.4.3.

(iv) Hitchin's proof of (ii) is easy: according to 2.2.2 $\dim \mathrm{Hitch}(X) = \sum_i \dim \Gamma(X, {\omega_X}^{\otimes d_i})$, $\dim \Gamma(X, {\omega_X}^{\otimes d_i}) = (g-1)(2d_i - 1)$ since $g > 1$, and finally $(g-1)\sum_i(2d_i - 1) = (g-1)\dim \mathfrak{g} = \dim \mathrm{Bun}_G$.

(v) Statement (iv) for classical groups $G$ was proved by Hitchin [Hit87]. In the general case it was proved by Faltings (Theorem III.2 from [Fal93]).

(vi) Statement (v) follows from (iii) and (iv).

(vii) Some comments on the proof of (iii) will be given in 2.10.

**2.2.5.** Our aim is to solve the following quantization problem: construct a filtered commutative algebra $\mathfrak{z}(X)$ equipped with an isomorphism $\sigma_{\mathfrak{z}(X)} : \mathrm{gr}\, \mathfrak{z}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}(X)$ and a morphism of filtered algebras $h_X : \mathfrak{z}(X) \to \Gamma(\mathrm{Bun}_G, D')$ compatible with the symbol maps, i.e., such that $\sigma_{\mathrm{Bun}_G} \circ \mathrm{gr}\, h_X = h_X^{cl} \circ \sigma_{\mathfrak{z}(X)}$ (see 1.1.4 and 1.1.6 for the definition of $\sigma_{\mathrm{Bun}_G}$). Note that 2.2.4(v) implies then that for any $\gamma \in \pi_1(X)$ the map $h_X^\gamma : \mathfrak{z}(X) \to \Gamma\left(\mathrm{Bun}_G^\gamma, D'\right)$ is an isomorphism. Therefore if $G$ is simply connected then such a construction is unique, and it reduces to the claims that $\Gamma(\mathrm{Bun}_G, D')$ is a commutative algebra, and any global function on $T^*\mathrm{Bun}_G$ is a symbol of a global twisted differential operator.

We do not know how to solve this problem directly by global considerations. We will follow the quantization scheme from 1.2 starting from a local version of Hitchin's picture. Two constructions of the same solution to the above quantization problem will be given. The first one (see 2.5.5) is easier to formulate, the second one (see 2.7.4) has the advantage of being entirely canonical. To prove that the first construction really gives a solution we use the second one. It is the second construction that will provide an identification of $\mathrm{Spec}\, \mathfrak{z}(X)$ with a certain subspace of the stack of $({}^L G)_{\mathrm{ad}}$-local systems on $X$ (see 3.3.2).

**2.3. Geometry of $\mathrm{Bun}_G$ II.** Let us recall how $\mathrm{Bun}_G$ fits into the framework of 1.2.6.

**2.3.1.** Fix a point $x \in X$. Denote by $O_x$ the completed local ring of $x$ and by $K_x$ its field of fractions. Let $\mathfrak{m}_x \subset O_X$ be the maximal ideal. Set $O_x^{(n)} := \mathcal{O}_X/\mathfrak{m}_x^n$ (so $O_x = \varprojlim O_x^{(n)}$). The group $G(O_x^{(n)})$ is the group of $\mathbb{C}$-points of an affine algebraic group which we denote also as $G\big(O_x^{(n)}\big)$ by abuse of notation; $G\big(O_x^{(n)}\big)$ is the quotient of $G\big(O_x^{(n+1)}\big)$. So $G(O_x) = \varprojlim G\big(O_x^{(n)}\big)$ is an affine group scheme.

Denote by $\mathrm{Bun}_{G,nx}$ the stack of $G$-bundles on $X$ trivialized over $\operatorname{Spec} O_x^{(n)}$ (notice that the divisor $nx$ is the same as the subscheme $\operatorname{Spec} O_x^{(n)} \subset X$). This is a $G\big(O_x^{(n)}\big)$-torsor over $\mathrm{Bun}_G$. We denote a point of $\mathrm{Bun}_{G,nx}$ as $\big(\mathcal{F}, \alpha^{(n)}\big)$. We have the obvious affine projections $\mathrm{Bun}_{G,(n+1)x} \to \mathrm{Bun}_{G,nx}$. Set $\mathrm{Bun}_{G,\underline{x}} := \varprojlim \mathrm{Bun}_{G,nx}$; this is a $G(O_x)$-torsor over $\mathrm{Bun}_G$.

**2.3.2.** *Proposition.* $\mathrm{Bun}_{G,\underline{x}}$ *is a scheme. The* $G(O_x)$*-action on* $\mathrm{Bun}_{G,\underline{x}}$ *satisfies condition* (16) *from* 1.2.6. □

**2.3.3.** It is well known that the $G(O_x)$-action on $\mathrm{Bun}_{G,\underline{x}}$ extends canonically to an action of the group ind-scheme $G(K_x)$ (see 7.11.1 for the definition of ind-scheme and 7.11.2 (iv) for the definition of the ind-scheme $G(K_x)$ ). Since $\operatorname{Lie} G(K_x) = \mathfrak{g} \otimes K_x$ we have, in particular, the action of the Harish-Chandra pair $(\mathfrak{g} \otimes K_x, G(O_x))$ on $\mathrm{Bun}_{G,\underline{x}}$.

Let us recall the definition of the $G(K_x)$-action. According to 7.11.2 (iv) one has to define a $G(R\widehat{\otimes}K_x)$-action on $\mathrm{Bun}_{G,\underline{x}}(R)$ for any $\mathbb{C}$-algebra $R$. To this end we use the following theorem, which is essentially due to A.Beauville and Y.Laszlo. Set $X' := X \setminus \{x\}$.

**2.3.4.** *Theorem.* A $G$-bundle $\mathcal{F}$ on $X \otimes R$ is the same as a triple $(\mathcal{F}_1, \mathcal{F}_2, \varphi)$ where $\mathcal{F}_1$ is a $G$-bundle on $X' \otimes R$, $\mathcal{F}_2$ is a $G$-bundle on $\operatorname{Spec}(R\widehat{\otimes}O_x)$, and $\varphi$ is an isomorphism between the pullbacks of $\mathcal{F}_1$ and $\mathcal{F}_2$ to $\operatorname{Spec}(R\widehat{\otimes}K_x)$. More precisely, the functor from the category (=groupoid) of $G$-bundles $\mathcal{F}$ on $X \otimes R$ to the category of triples $(\mathcal{F}_1, \mathcal{F}_2, \varphi)$ as above defined by $\mathcal{F}_1 := \mathcal{F}|_{X' \otimes R}$, $\mathcal{F}_2 :=$ the pullback of $\mathcal{F}$ to $\operatorname{Spec}(R\widehat{\otimes}O_x)$, $\varphi := \mathrm{id}$, is an equivalence.

According to the theorem an $R$-point of $\mathrm{Bun}_{G,\underline{x}}$ is the same as a $G$-bundle on $X'\otimes R$ with a trivialization of its pullback to $\mathrm{Spec}(R\widehat{\otimes}K_x)$. So $G(R\widehat{\otimes}K_x)$ acts on $\mathrm{Bun}_{G,\underline{x}}(R)$ by changing the trivialization. Thus we get the action of $G(K_x)$ on $\mathrm{Bun}_{G,\underline{x}}$ .

The proof of Theorem 2.3.4 is based on the following theorem, which is a particular case of the main result of [BLa95].

**2.3.5.** *Theorem.* (Beauville-Laszlo). The category of flat quasi-coherent $\mathcal{O}_{X\otimes R}$-modules $\mathcal{M}$ is equivalent to the category of triples $(\mathcal{M}_1, \mathcal{M}_2, \varphi)$ where $\mathcal{M}_1$ is a flat quasi-coherent $\mathcal{O}$-module on $X'\otimes R$, $\mathcal{M}_2$ is a flat quasi-coherent $\mathcal{O}$-module on $\mathrm{Spec}(R\widehat{\otimes}O_x)$, and $\varphi$ is an isomorphism between the pullbacks of $\mathcal{M}_1$ and $\mathcal{M}_2$ to $\mathrm{Spec}(R\widehat{\otimes}K_x)$ (the functor from the first category to the second one is defined as in Theorem 2.3.4). $\mathcal{M}$ is locally free of finite rank if and only if the corresponding $\mathcal{M}_1$ and $\mathcal{M}_2$ have this property.

*Remark.* If $R$ is noetherian and the sheaves are coherent then there is a much more general "glueing theorem" due to M.Artin (Theorem 2.6 from [Ar]). But since subschemes of $G(K_x)$ are usually of infinite type we use the Beauville-Laszlo theorem, which holds without noetherian assumptions.

To deduce Theorem 2.3.4 from 2.3.5 it suffices to interpret a $G$-bundle as a tensor functor $\{G\text{-modules}\}\to\{\text{vector bundles}\}$. Or one can interpret a $G$-bundle on $X\otimes R$ as a principle $G$-bundle, i.e., a flat affine morphism $\pi:\mathcal{F}\to X\otimes R$ with an action of $G$ on $\mathcal{F}$ satisfying certain properties; then one can rewrite these data in terms of the sheaf $\mathcal{M}:=\pi_*\mathcal{O}_{\mathcal{F}}$ and apply Theorem 2.3.5.

**2.3.6.** *Remark.* Here is a direct description of the action of $\mathfrak{g}\otimes K_x$ on $\mathrm{Bun}_{G,\underline{x}}$ induced by the action of $G(K_x)$ (we will not use it in the future ???). Take $(\mathcal{F},\bar{\alpha})\in\mathrm{Bun}_{G,\underline{x}}$, $\bar{\alpha}=\varprojlim\alpha^{(n)}$. The tangent space to $\mathrm{Bun}_{G,nx}$ at $\left(\mathcal{F},\alpha^{(n)}\right)$ is $H^1(X,\mathfrak{g}_{\mathcal{F}}(-nx))$, so the fiber of $\Theta_{\mathrm{Bun}_{G,\underline{x}}}$ at $(\mathcal{F},\bar{\alpha})$ equals $\varprojlim H^1\left(X,\mathfrak{g}_{\mathcal{F}}(-nx)\right) = H^1_c(X\setminus\{x\},\mathfrak{g}_{\mathcal{F}})$. We have the usual surjection $\mathfrak{g}_{\mathcal{F}}\bigotimes_{\mathcal{O}_X}K_x\twoheadrightarrow H^1_c(X\setminus\{x\},\mathfrak{g}_{\mathcal{F}})$. Use $\bar{\alpha}$ to identify $\mathfrak{g}_{\mathcal{F}}\bigotimes_{\mathcal{O}_X}K_x$ with $\mathfrak{g}\otimes K_x$.

When $(\mathcal{F}, \bar{\alpha})$ varies one gets the map $\mathfrak{g} \otimes K_x \to \Theta_{\mathrm{Bun}_G, \underline{x}}$. Our $\mathfrak{g} \otimes K_x$-action is minus this map (???).

**2.3.7.** *Remark.* Let $D \subset X \otimes R$ be a closed subscheme finite over $\operatorname{Spec} R$ which can be locally defined by one equation (i.e., $D$ is an effective relative Cartier divisor). Denote by $\tilde{D}$ the formal neighbourhood of $D$ and let $A$ be the coordinate ring of $\tilde{D}$ (so $\tilde{D}$ is an affine formal scheme and $\operatorname{Spec} A$ is a true scheme). Then Theorems 2.3.4 and 2.3.5 remain valid if $X' \otimes R$ is replaced by $(X \otimes R) \setminus D$, $R \widehat{\otimes} O_x$ by $A$, and $\operatorname{Spec}(R \widehat{\otimes} K_x)$ by $(\operatorname{Spec} A) \setminus D$. This follows from the main theorem of [BLa95] if the normal bundle of $D$ is trivial: indeed, in this case one can construct an affine neighbourhood $U \supset D$ such that inside $U$ the subscheme $D$ is defined by a global equation $f = 0$, $f \in H^0(U, \mathcal{O}_U)$ (this is the situation considered in [BLa95]).[3] For the purposes of this work the case where the normal bundle of $D$ is trivial is enough. To treat the general case one needs a globalized version of the main theorem of [BLa95] (see 2.12). Among other things, one has to extend the morphism $\tilde{D} \to X \otimes R$ to a morphism $\operatorname{Spec} A \to X \otimes R$ (clearly such an extension is unique, but its existence has to be proved); see 2.12.

## 2.4. Hitchin's construction II.

**2.4.1.** Set $\omega_{O_x} := \lim_{\leftarrow} \omega_{{O_x}^{(n)}}$ where $\omega_{{O_x}^{(n)}}$ is the module of differentials of ${O_x}^{(n)} = O_x / m_x^n$. Denote by $\mathrm{Hitch}_x^{(n)}$ the scheme of sections of $C_{\omega_X}$ over $\operatorname{Spec} O_x^{(n)}$. This is an affine scheme with $\mathbb{G}_m$-action non-canonically isomorphic to the vector space $M/m_x^n M$, $M := \prod \omega_{O_x}^{\otimes d_i}$. Set

$$\mathrm{Hitch}_x = \mathrm{Hitch}_{\mathfrak{g}}(O_x) := \varprojlim \mathrm{Hitch}_x^{(n)} .$$

This is an affine scheme with $\mathbb{G}_m$-action non-canonically isomorphic to $M = \prod \omega_{O_x}^{\otimes d_i}$. So $\mathrm{Hitch}_x$ is the scheme of sections of $C_{\omega_X}$ over $\operatorname{Spec} O_x$.

[3] To construct $U$ and $f$ notice that for $n$ big enough there exists $\varphi_n \in H^0(X \otimes R, \mathcal{O}_{X \otimes R}(nD))$ such that $\mathcal{O}_{X \otimes R}(nD)/\mathcal{O}_{X \otimes R}((n-1)D)$ is generated by $\varphi_n$; then put $U := (X \otimes R) \setminus \{\text{the set of zeros of } \varphi_n \varphi_{n+1}\}$, $f := \varphi_n / \varphi_{n+1}$ (this construction works if the map $D \to \operatorname{Spec} R$ is surjective, which is a harmless assumption).

Denote by $\mathfrak{z}_x^{cl} = \mathfrak{z}_{\mathfrak{g}}^{cl}(O_x)$ the graded Poisson algebra $P_{(\mathfrak{g}\otimes K_x, G(O_x))} = \mathrm{Sym}(\mathfrak{g}\otimes K_x/O_x)^{G(O_x)}$ from 1.2.2. We will construct a canonical $\mathbb{G}_m$-equivariant isomorphism $\operatorname{Spec}\mathfrak{z}_x^{cl} \xrightarrow{\sim} \mathrm{Hitch}_x$ (the $\mathbb{G}_m$-action on $\mathfrak{z}_x^{cl}$ is opposite to that induced by the grading; cf. the end of 2.2.2).

The residue pairing identifies $(K_x/O_x)^*$ with $\omega_{O_x}$, so $\operatorname{Spec}\mathrm{Sym}(\mathfrak{g}\otimes K_x/O_x) = \mathfrak{g}^*\otimes\omega_{O_x}$. The projection $\mathfrak{g}^*\to C$ yields a morphism of affine schemes $\mathfrak{g}^*\otimes\omega_{O_x}\to\mathrm{Hitch}_x$. It is $G(O_x)$-invariant, so it induces a morphism $\operatorname{Spec}\mathfrak{z}_x^{cl}\to\mathrm{Hitch}_x$. To show that this is an isomorphism we have to prove that every $G(O_x)$-invariant regular function on $\mathfrak{g}^*\otimes\omega_{O_x}$ comes from a unique regular function on $\mathrm{Hitch}_x$. Clearly one can replace $\mathfrak{g}^*\otimes\omega_{O_x}$ by $\mathfrak{g}^*\otimes O_x = \mathrm{Paths}(\mathfrak{g}^*)$ and $\mathrm{Hitch}_x$ by $\mathrm{Paths}(C)$ (for a scheme $Y$ we denote by $\mathrm{Paths}(Y)$ the scheme of morphisms $\operatorname{Spec}O_x\to Y$). Regular elements of $\mathfrak{g}^*$ form an open subset $\mathfrak{g}^*_{\mathrm{reg}}$ such that $\operatorname{codim}(\mathfrak{g}^*\backslash\mathfrak{g}^*_{\mathrm{reg}})>1$. So one can replace $\mathrm{Paths}(\mathfrak{g}^*)$ by $\mathrm{Paths}(\mathfrak{g}^*_{\mathrm{reg}})$. Since the morphism $\mathfrak{g}^*_{\mathrm{reg}}\to C$ is smooth and surjective, and the action of $G$ on its fibers is transitive, we are done.

**2.4.2.** According to 1.2.2 $\mathfrak{z}_x^{cl} = P_{(\mathfrak{g}\otimes K_x, G(O_x))}$ is a Poisson algebra. Actually the Poisson bracket on $\mathfrak{z}_x^{cl}$ is zero because the morphism $\overline{a}^{cl} : (\overline{\mathrm{Sym}}(\mathfrak{g}\otimes K_x))^{\mathfrak{g}\otimes K_x} \longrightarrow \mathfrak{z}_x^{cl}$ from 1.2.7 is surjective (this follows, e.g., from the description of $\mathfrak{z}_x^{cl}$ given in 2.4.1) and $(\overline{\mathrm{Sym}}(\mathfrak{g}\otimes K_x))^{\mathfrak{g}\otimes K_x}$ is the Poisson center of $\overline{\mathrm{Sym}}(\mathfrak{g}\otimes K_x)$.

*Remark* (which may be skipped by the reader). Actually for *any* algebraic group $G$ the natural morphism $\overline{a}^{cl} : (\overline{\mathrm{Sym}}(\mathfrak{g}\otimes K_x))^{G(K_x)}\to\mathfrak{z}_x^{cl} = \mathfrak{z}_{\mathfrak{g}}^{cl}(O_x)$ is surjective and therefore the Poisson bracket on $\mathfrak{z}_x^{cl}$ is zero. The following proof is the "classical limit" of Feigin-Frenkel's arguments from [FF92], p. 200–202. Identify $O_x$ and $K_x$ with $O := \mathbb{C}[[t]]$ and $K := \mathbb{C}[[t]]$. Let $f$ be a $G(O)$-invariant regular function on $\mathfrak{g}^*\otimes O$. We have to extend it to a $G(K)$-invariant regular function $\tilde{f}$ on the ind-scheme $\mathfrak{g}^*\otimes K := \varinjlim \mathfrak{g}^*\otimes t^{-n}O$ (actually $\mathfrak{g}^*$ can be replaced by any finite dimensional $G$-module). For

$\varphi \in \mathfrak{g}^*((t))$ define $h_\varphi \in \mathbb{C}((\zeta))$ by

$$h_\varphi(\zeta) = f\left(\sum_{k=0}^{N} \varphi^{(k)}(\zeta)t^k/k!\right)$$

where $N$ is big enough ($h_\varphi$ is well-defined because there is an $m$ such that $f$ comes from a function on $\mathfrak{g}^* \otimes (O/t^m O)$). Write $h_\varphi(\zeta)$ as $\sum_n h_n(\varphi)\zeta^n$. The functions $h_n : \mathfrak{g}^* \otimes K \to \mathbb{C}$ are $G(K)$-invariant. Set $\tilde{f} := h_0$.

**2.4.3.** According to 2.3 and 1.2.6 we have the morphism

$$h_x^{cl} : \mathfrak{z}_x^{cl} \to \Gamma(\mathrm{Bun}_G, P).$$

analogous to the morphism $h^{cl}$ from 1.2.3. To compare it with $h_X^{cl}$ consider the closed embedding of affine schemes $\mathrm{Hitch}(X) \hookrightarrow \mathrm{Hitch}_x$ which assigns to a global section of $C_{\omega_X}$ its restriction to the formal neighbourhood of $x$. Let $\theta_x^{cl} : \mathfrak{z}_x^{cl} \twoheadrightarrow \mathfrak{z}^{cl}(X)$ be the corresponding surjective morphism of graded algebras. It is easy to see that

$$h_x^{cl} = h_X^{cl}\theta_x^{cl}.$$

Since the Poisson bracket on $\mathfrak{z}_x^{cl}$ is zero (see 2.4.2) and $h_x^{cl}$ is a Poisson algebra morphism the Poisson bracket on $\operatorname{Im} h_x^{cl} = \operatorname{Im} h_X^{cl}$ is also zero. So we have proved 2.2.4(i).

## 2.5. Quantization I.

**2.5.1.** Let $\widetilde{\mathfrak{g} \otimes K_x}$ be the Kac-Moody central extension of $\mathfrak{g} \otimes K_x$ by $\mathbb{C}$ defined by the cocycle $(u, v) \mapsto \mathrm{Res}_x\, c(du, v)$, $u, v \in \mathfrak{g} \otimes K_x$, where

$$c(a, b) := -\frac{1}{2}\,\mathrm{Tr}(\mathrm{ad}_a \cdot \mathrm{ad}_b)\,, \quad a, b \in \mathfrak{g}\,. \tag{18}$$

As a vector space $\widetilde{\mathfrak{g} \otimes K_x}$ equals $\mathfrak{g} \otimes K_x \oplus \mathbb{C} \cdot \mathbf{1}$. We define the *adjoint action*[4] of $G(K_x)$ on $\widetilde{\mathfrak{g} \otimes K_x}$ by assigning to $g \in G(K_x)$ the following automorphism

[4]As soon as we have a central extension of $G(K_x)$ with Lie algebra $\widetilde{\mathfrak{g} \otimes K_x}$ the action (19) becomes the true adjoint action (an automorphism of $\widetilde{\mathfrak{g} \otimes K_x}$ that acts identically on $\mathbb{C} \cdot \mathbf{1}$ and $\mathfrak{g} \otimes K_x$ is identical because $\mathrm{Hom}(\mathfrak{g} \otimes K_x, \mathbb{C}) = 0$).

of $\widetilde{\mathfrak{g}\otimes K_x}$:

$$\mathbf{1}\mapsto\mathbf{1},\quad u\mapsto gug^{-1}+\operatorname{Res}_x c(u,g^{-1}dg)\cdot\mathbf{1}\text{ for }u\in\mathfrak{g}\otimes K_x \tag{19}$$

In particular we have the Harish-Chandra pair $\left(\widetilde{\mathfrak{g}\otimes K_x},G(O_x)\right)$, which is a central extension of $(\mathfrak{g}\otimes K_x,G(O_x))$ by $\mathbb{C}$. Set

$$\mathfrak{z}_x=\mathfrak{z}_{\mathfrak{g}}(O_x):=D'_{(\mathfrak{g}\otimes K_x,G(O_x))},$$

where $D'$ has the same meaning as in 1.2.5.

**2.5.2.** *Theorem.* **(**[FF92]**).**

(i) *The algebra $\mathfrak{z}_x$ is commutative.*

(ii) *The pair $\left(\widetilde{\mathfrak{g}\otimes K_x},G(O_x)\right)$ satisfies the twisted local quantization condition (see 1.2.5). That is, the canonical morphism $\sigma_{\mathfrak{z}_x}:\operatorname{gr}\mathfrak{z}_x\to\mathfrak{z}_x^{cl}$ is an isomorphism.* $\square$

*Remark* Statement (i) of the theorem is proved in [FF92] for any algebraic group $G$ and any central extension of $\mathfrak{g}\otimes K_x$ defined by a symmetric invariant bilinear form on $\mathfrak{g}$. Moreover, it is proved in [FF92] that the $\pi_0(G(K_x))$-invariant part of the center of the completed twisted universal enveloping algebra $\overline{U}'(\mathfrak{g}\otimes K_x)$ maps *onto* $\mathfrak{z}_x$. A version of Feigin–Frenkel's proof of (i) will be given in 2.9.3–2.9.5. We have already explained the "classical limit" of their proof in the Remark at the end of 2.4.2.

**2.5.3.** The line bundle $\omega_{\mathrm{Bun}_G}$ defines a $G(O_x)$-equivariant bundle on $\mathrm{Bun}_{G,\underline{x}}$. The $(\mathfrak{g}\otimes K_x,G(O_x))$-action on $\mathrm{Bun}_{G,\underline{x}}$ lifts canonically to a $\left(\widetilde{\mathfrak{g}\otimes K_x},G(O_x)\right)$-action on this line bundle, so that $\mathbf{1}$ acts as multiplication by 2. Indeed, according to 2.1.1 $\omega_{\mathrm{Bun}_G}=f^*(\det R\Gamma)$ where $f:\mathrm{Bun}_G\to\mathrm{Bun}_{SL(\mathfrak{g})}$ is induced by the adjoint representation $G\to SL(\mathfrak{g})$ and $\det R\Gamma$ is the determinant line bundle on $\mathrm{Bun}_{SL(\mathfrak{g})}$. On the other hand, it is well known (see,e.g., [BLa94]) that the pullback of $\det R\Gamma$ to $\mathrm{Bun}_{SL_n,\underline{x}}$ is equipped with the action of the Kac–Moody extension of $sl_n(K_x)$ of level $-1$.

*Remark.* In fact, the action of this extension integrates to an action of a certain central extension of $SL_n(K_x)$ (see, e.g., [BLa94]). Therefore one gets a canonical central extension

$$0 \to \mathbb{G}_m \to \widehat{G(K_x)} \to G(K_x) \to 0 \tag{20}$$

that acts on the pullback of $\omega_{\mathrm{Bun}_G}$ to $\mathrm{Bun}_{G,\underline{x}}$ so that $\lambda \in \mathbb{G}_m$ acts as multiplication by $\lambda$. The extension $0 \to \mathbb{C} \to \widetilde{\mathfrak{g} \otimes K_x} \to \mathfrak{g} \otimes K_x \to 0$ is one half of the Lie algebra extension corresponding to (20). In Chapter 4 we will introduce a square root[5] of $\omega_{\mathrm{Bun}_G}$ (the *Pfaffian* bundle) and a central extension

$$0 \to \mathbb{G}_m \to \widetilde{G(K_x)} \to G(K_x) \to 0 \tag{21}$$

(see 4.4.8), which is a square root of (20). These square roots are more important for us than $\omega_{\mathrm{Bun}_G}$ and (20), so we will not give a precise definition of $\widehat{G(K_x)}$.

**2.5.4.** According to 2.5.3 and 1.2.5 we have a canonical morphism of filtered algebras

$$h_x : \mathfrak{z}_x \to \Gamma\left(\mathrm{Bun}_G, D'\right).$$

In 2.7.5 we will prove the following theorem.

**2.5.5.** *Theorem. Our data satisfy the twisted global quantization condition (see 1.2.5).* □

As explained in 1.2.3 since the local and global quantization conditions are satisfied we obtain a solution $\mathfrak{z}^{(x)}(X)$ to the quantization problem from 2.2.5: set $\mathfrak{z}^{(x)}(X) = h_x(\mathfrak{z}_x)$ and equip $\mathfrak{z}^{(x)}(X)$ with the filtration induced from that on $\Gamma(\mathrm{Bun}_G, D')$ (2.5.5 means that it is also induced from the filtration on $\mathfrak{z}_x$); then the symbol map identifies $\mathrm{gr}\,\mathfrak{z}^{(x)}(X)$ with $h_x^{cl}(\mathfrak{z}_x^{cl})$ and according to 2.4.3 $h_x^{cl}(\mathfrak{z}_x^{cl}) = h_X^{cl}(\mathfrak{z}_x^{cl}(X)) \simeq \mathfrak{z}^{cl}(X)$.

[5]This square root and the extension (21) depend on the choice of a square root of $\omega_X$.

The proof of Theorem 2.5.5 is based on the second construction of the solution to the quantization problem from 2.2.5; it also shows that $\mathfrak{z}^{(x)}(X)$ does not depend on $x$.

*Remark* If $G$ is simply connected then 2.5.5 follows immediately from 2.2.4(v).

### 2.6. $\mathcal{D}_X$-scheme generalities.

**2.6.1.** Let $X$ be any smooth connected algebraic variety. A $\mathcal{D}_X$-*scheme* is an $X$-scheme equipped with a flat connection along $X$. $\mathcal{D}_X$-schemes affine over $X$ are spectra of commutative $\mathcal{D}_X$-*algebras* (= quasicoherent $\mathcal{O}_X$-algebras equipped with a flat connection). The fiber of an $\mathcal{O}_X$-algebra $\mathcal{A}$ at $x \in X$ is denoted by $\mathcal{A}_x$; in particular this applies to $\mathcal{D}_X$-algebras. For a $\mathbb{C}$-algebra $C$ denote by $C_X$ the corresponding "constant" $\mathcal{D}_X$-algebra (i.e., $C_X$ is $C \otimes \mathcal{O}_X$ equipped with the obvious connection).

**2.6.2.** *Proposition.* Assume that $X$ is complete.

(i) The functor $C \rightsquigarrow C_X$ admits a left adjoint functor: for a $\mathcal{D}_X$-algebra $\mathcal{A}$ there is a $\mathbb{C}$-algebra $H_\nabla(X, \mathcal{A})$ such that

$$\operatorname{Hom}(\mathcal{A}, C_X) = \operatorname{Hom}(H_\nabla(X, A), C) \tag{22}$$

for any $\mathbb{C}$-algebra $C$.

(ii) The canonical projection $\theta_\mathcal{A} : \mathcal{A} \to H_\nabla(X, \mathcal{A})_X$ is surjective. So $H_\nabla(X, \mathcal{A})_X$ is the maximal "constant" quotient $\mathcal{D}_X$-algebra of $\mathcal{A}$. In particular for any $x \in X$ the morphism $\theta_{\mathcal{A}_x} : \mathcal{A}_x \to (H_\nabla(X, \mathcal{A})_X)_x = H_\nabla(X, \mathcal{A})$ is surjective.

*Remarks.* (i) Here algebras are not supposed to be commutative, associative, etc. We will need the proposition for commutative $\mathcal{A}$.

(ii) Suppose that $\mathcal{A}$ is commutative (abbreviation for "commutative associative unital"). Then $H_\nabla(X, \mathcal{A})$ is commutative according to statement (ii) of the proposition. If $C$ is also assumed commutative then (22) just means that $\operatorname{Spec} H_\nabla(X, \mathcal{A})$ *is the scheme of horizontal sections of* $\operatorname{Spec} \mathcal{A}$.

From the geometrical point of view it is clear that such a scheme exists and is affine: all the sections of $\operatorname{Spec}\mathcal{A}$ form an affine scheme $S$ (here we use the completeness of $X$; otherwise $S$ would be an ind-scheme, see the next Remark) and horizontal sections form a closed subscheme of $S$. The surjectivity of $\theta_{\mathcal{A}_x}$ and $\theta_{\mathcal{A}}$ means that the morphisms $\operatorname{Spec} H_\nabla(X,\mathcal{A}) \to \operatorname{Spec}\mathcal{A}_x$ and $X \times \operatorname{Spec} H_\nabla(X,\mathcal{A}) \to \operatorname{Spec}\mathcal{A}$ are closed embeddings.

(iii) If $X$ is arbitrary (not necessary complete) then $H_\nabla(X,\mathcal{A})$ defined by (22) is representable by a projective limit of algebras with respect to a directed family of surjections. So if $\mathcal{A}$ is commutative then the space of horizontal sections of $\operatorname{Spec}\mathcal{A}$ is an ind-affine ind-scheme[6].

*Proof.* (a) Denote by $\mathcal{M}(X)$ the category of $\mathcal{D}_X$-modules and by $\mathcal{M}_{\mathrm{const}}(X)$ the full subcategory of constant $\mathcal{D}_X$-modules, i.e., $\mathcal{D}_X$-modules isomorphic to $V \otimes \mathcal{O}_X$ for some vector space $V$ (actually the functor $V \mapsto V \otimes \mathcal{O}_X$ is an equivalence between the category of vector spaces and $\mathcal{M}_{\mathrm{const}}(X)$). We claim that the embedding $\mathcal{M}_{\mathrm{const}}(X) \to \mathcal{M}(X)$ has a left adjoint functor, i.e., for $\mathcal{F} \in \mathcal{M}(X)$ there is an $\mathcal{F}_\nabla \in \mathcal{M}_{\mathrm{const}}(X)$ such that $\operatorname{Hom}(\mathcal{F},\mathcal{E}) = \operatorname{Hom}(\mathcal{F}_\nabla,\mathcal{E})$ for $\mathcal{E} \in \mathcal{M}_{\mathrm{const}}(X)$. It is enough to construct $\mathcal{F}_\nabla$ for coherent $\mathcal{F}$. In this case $\mathcal{F}_\nabla := (\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{F},\mathcal{O}_X))^* \otimes \mathcal{O}_X$ (here we use that $\dim \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{F},\mathcal{O}_X) < \infty$ because $X$ is complete).

(b) Since $\mathcal{O}_X$ is an irreducible $\mathcal{D}_X$-module a $\mathcal{D}_X$-submodule of a constant $\mathcal{D}_X$-module is constant. So the natural morphism $\mathcal{F} \to \mathcal{F}_\nabla$ is surjective.

(c) If $\mathcal{A}$ is a $\mathcal{D}_X$-algebra and $\mathcal{I}$ is the ideal of $\mathcal{A}$ generated by $\operatorname{Ker}(\mathcal{A} \to \mathcal{A}_\nabla)$ then $\mathcal{A}/\mathcal{I}$ is a quotient of the constant $\mathcal{D}_X$-module $\mathcal{A}_\nabla$. So $\mathcal{A}/\mathcal{I}$ is constant, i.e., $\mathcal{A}/\mathcal{I} = H_\nabla(X,\mathcal{A}) \otimes \mathcal{O}_X$ for some vector space $H_\nabla(X,\mathcal{A})$. $\mathcal{A}/\mathcal{I}$ is a $\mathcal{D}_X$-algebra, so $H_\nabla(X,\mathcal{A})$ is an algebra. Clearly it satisfies (22). $\square$

[6] This is also clear from the geometric viewpoint. Indeed, horizontal sections form a closed subspace in the space $S_X$ of all sections. If $X$ is affine $S_X$ is certainly an ind-scheme. In the general case $X$ can be covered by open affine subschemes $U_1, ..., U_n$; then $S_X$ is a closed subspace of the product of $S_{U_i}$'s.

*Remark.* The geometrically oriented reader can consider the above Remark (ii) as a proof of the proposition for commutative algebras. However in 2.7.4 we will apply (22) in the situation where $\mathcal{A}$ is commutative while $C = \Gamma(\mathrm{Bun}_G, D')$ is not obviously commutative. Then it is enough to notice that the image of a morphism $\mathcal{A} \to C \otimes \mathcal{O}_X$ is of the form $C' \otimes \mathcal{O}_X$ (see part (b) of the proof of the proposition) and $C'$ is commutative since $\mathcal{A}$ is. One can also apply (22) for $C :=$ the subalgebra of $\Gamma(\mathrm{Bun}_G, D')$ generated by the images of the morphisms $h_x : \mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G, D')$ for all $x \in X$ (this $C$ is "obviously" commutative; see 2.9.1). Actually one can show that $\Gamma(\mathrm{Bun}_G, D')$ is commutative using 2.2.4(v) (it follows from 2.2.4(v) that for any connected component $\mathrm{Bun}_G^\gamma \subset \mathrm{Bun}_G$ and any $x \in X$ the morphism $\mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G^\gamma, D')$ induced by $h_x$ is surjective).

**2.6.3.** In this subsection all algebras are assumed commutative. The forgetful functor $\{\mathcal{D}_X\text{-algebras}\} \to \{\mathcal{O}_X\text{-algebras}\}$ has an obvious left adjoint functor $\mathcal{J}$ ($\mathcal{J}\mathcal{A}$ is the $\mathcal{D}_X$-algebra generated by the $\mathcal{O}_X$-algebra $\mathcal{A}$). We claim that $\operatorname{Spec} \mathcal{J}\mathcal{A}$ is nothing but *the scheme of* $\infty$*-jets of sections of* $\operatorname{Spec} \mathcal{A}$. In particular this means that there is a canonical one-to-one correspondence between $\mathbb{C}$-points of $\operatorname{Spec}(\mathcal{J}\mathcal{A})_x$ and sections $\operatorname{Spec} O_x \to \operatorname{Spec} \mathcal{A}$ (where $O_x$ is the formal completion of the local ring at $x$). More precisely, we have to construct a functorial bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \hat{\mathcal{B}}) \tag{23}$$

where $\mathcal{B}$ is a (quasicoherent) $\mathcal{O}_X$-algebra and $\hat{\mathcal{B}}$ is the completion of $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{B}$ with respect to the ideal $\mathrm{Ker}(\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{B} \to \mathcal{B})$. Here $\hat{\mathcal{B}}$ is equipped with the $\mathcal{O}_X$-algebra structure coming from the morphism $\mathcal{O}_X \to \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{B}$ defined by $a \mapsto a \otimes 1$. Let us temporarily drop the quasicoherence assumption in the definition of $\mathcal{D}_X$-algebra. Then $\hat{\mathcal{B}}$ is a $\mathcal{D}_X$-algebra (the connection on $\hat{\mathcal{B}}$ comes from the connection on $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{B}$ such that sections of $1 \otimes \mathcal{B}$ are horizontal). So $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \hat{\mathcal{B}}) = \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{J}\mathcal{A}, \hat{\mathcal{B}})$ and to construct (23) it is enough to construct a functorial bijection

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{R},\mathcal{B}) \longleftarrow \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{R},\hat{\mathcal{B}}) \tag{24}$$

for any $\mathcal{D}_X$-algebra $\mathcal{R}$ and $\mathcal{O}_X$-algebra $\mathcal{B}$ (i.e., to show that the functor $\mathcal{B} \mapsto \hat{\mathcal{B}}$ is *right* adjoint to the forgetful functor $\{\mathcal{D}_X\text{-algebras}\} \to \{\mathcal{O}_X\text{-algebras}\}$). The mapping (24) comes from the obvious morphism $\hat{\mathcal{B}} \to \mathcal{B}$. The reader can easily prove that (24) is bijective.

For a $\mathcal{D}_X$-algebra $\mathcal{A}$ and a $\mathbb{C}$-algebra $C$ we have

$$\operatorname{Hom}_{\mathcal{D}_X\text{-alg}}(\mathcal{J}\mathcal{A}, C\otimes\mathcal{O}_X) = \operatorname{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{A}, C\otimes\mathcal{O}_X)$$

This means that *the canonical morphism* $\operatorname{Spec}\mathcal{J}\mathcal{A} \to \operatorname{Spec}\mathcal{A}$ *identifies the ind-scheme of horizontal sections of* $\operatorname{Spec}\mathcal{J}\mathcal{A}$ *with that of (all) sections of* $\operatorname{Spec}\mathcal{A}$. If $X$ is complete then, by 2.6.2, these spaces are actually schemes.

Finally let us mention that the results of this subsection can be globalized in the obvious way. The forgetful functor $\{\mathcal{D}_X\text{-schemes}\} \to \{X\text{-schemes}\}$ has a right adjoint functor $\mathcal{J} : \{X\text{-schemes}\} \to \{\mathcal{D}_X\text{-schemes}\}$. For an $X$-scheme $Y$, $\mathcal{J}Y$ is the scheme of $\infty$-jets of sections of $Y$. For an $\mathcal{O}_X$-algebra $\mathcal{A}$ we have $\mathcal{J}\operatorname{Spec}\mathcal{A} = \operatorname{Spec}\mathcal{J}\mathcal{A}$. The canonical morphism $\mathcal{J}Y \to Y$ identifies the space[7] of horizontal sections of $\mathcal{J}Y$ with the space of (all) sections of $Y$. If $X$ is complete and $Y$ is quasiprojective then our space is a scheme.

**2.6.4.** Let $(\mathfrak{l}, P)$ be a Harish-Chandra pair in the sense of 1.2.6 (so $P$ can be any affine group scheme; we do not assume that it is of finite type[8]).

*Definition.* An $(\mathfrak{l}, P)$*-structure on* $X$ is a morphism $\pi : X^\wedge \to X$ together with an action of $(\mathfrak{l}, P)$ on $X^\wedge$ such that

(i) $X^\wedge$ is a $P$-torsor over $X$.

(ii) The action of $\mathfrak{l}$ on $X^\wedge$ is formally free and transitive, i.e., it yields an isomorphism $\mathfrak{l}\widehat{\otimes}\mathcal{O}_{X^\wedge} \xrightarrow{\sim} \Theta_{X^\wedge}$.

---

[7]In the most general situation "space" means "functor $\{\mathbb{C}\text{-algebras}\} \to \{\text{Sets}\}$".

[8]As follows from the definition below $\operatorname{Lie} P$ has finite codimension in $\mathfrak{l}$ (equal to $\dim X$).

*Remark.* Let $L$ be the group ind-scheme with $\operatorname{Lie} L = \mathfrak{l}$, $L_{\mathrm{red}} = P$ (see 7.11.2(v)). Consider the homogenuous space $P \setminus L = \operatorname{Spf} O$ where $O = O_{(\mathfrak{l},P)} = (U\mathfrak{l}/(U\mathfrak{l})\mathfrak{p})^*$. Take $x \in X$ and choose $x^\wedge \in \pi^{-1}(x)$. The map $L \to X^\wedge$, $l \mapsto lx^\wedge$, yields a morphism $\alpha_{x^\wedge} : \operatorname{Spf} O \to X$, which identifies $\operatorname{Spf} O$ with the formal neighbourhood of $x$. For $l \in L$, $a \in \operatorname{Spf} O$ one has $\alpha_{lx^\wedge}(a) = \alpha_{x^\wedge}(al)$. Note that if the action of $P$ on $O$ is faithful then $x^\wedge$ is uniquely defined by $\alpha_{x^\wedge}$.

**2.6.5.** *Example.* Set $O = O_n := \mathbb{C}[[t_1, .., t_n]]$. The group of automorphisms of the $\mathbb{C}$-algebra $O$ is naturally the group of $\mathbb{C}$-points of an affine group scheme $\operatorname{Aut}^0 O$ over $\mathbb{C}$. Denote by $\operatorname{Aut} O$ the group ind-scheme such that, for any $\mathbb{C}$-algebra $R$, $(\operatorname{Aut} O)(R)$ is the automorphism group of the topological $R$-algebra $R\widehat{\otimes} O = R[[t_1, .., t_n]]$. So $\operatorname{Aut}^0 O$ is the group subscheme of $\operatorname{Aut} O$; in fact, $\operatorname{Aut}^0 O = (\operatorname{Aut} O)_{\mathrm{red}}$. One has $\operatorname{Lie} \operatorname{Aut} O = \operatorname{Der} O$, $\operatorname{Lie} \operatorname{Aut}^0 O = \operatorname{Der}^0 O := \mathfrak{m}_O \cdot \operatorname{Der} O$. Therefore $\operatorname{Aut} O$ is the group ind-scheme that corresponds to the Harish-Chandra pair $\operatorname{Aut}^{HC} O := (\operatorname{Der} O, \operatorname{Aut}^0 O)$. By abuse of notation we will write $\operatorname{Aut} O$ instead of $\operatorname{Aut}^{HC} O$.

As explained by Gelfand and Kazhdan (see [GK], [GKF], and [BR]) any smooth variety $X$ of dimension $n$ carries a canonical[9] $\operatorname{Aut} O$-structure. The space $X^\wedge = X^\wedge_{can}$ is the space of "formal coordinate systems" on $X$. In other words, a $\mathbb{C}$-point of $X^\wedge$ is a morphism $\operatorname{Spec} O \to X$ with non-vanishing differential and an $R$-point of $X^\wedge$ is an $R$-morphism $\alpha : \operatorname{Spec}(R\widehat{\otimes} O) \to X \otimes R$ whose differential does not vanish over any point of $\operatorname{Spec} R$. The group ind-scheme $\operatorname{Aut} O$ acts on $X^\wedge$ in the obvious way, and we have the projection $\pi : X^\wedge \to X$, $\alpha \mapsto \alpha(0)$. It is easy to see that $X^\wedge$ (together with these structures) is an $\operatorname{Aut} O$-structure on $X$.

We will use the canonical $\operatorname{Aut} O_n$-structure in the case $n = 1$, i.e., when $X$ is a curve, so $O = \mathbb{C}[[t]]$. Here the group $\operatorname{Aut} O$ looks as follows. There is an epimorphism $\operatorname{Aut}^0 O \to \operatorname{Aut}(tO/t^2 O) = \mathbb{G}_m$, which

[9] In fact, an $\operatorname{Aut} O$-structure on $X$ is unique up to unique isomorphism (this follows from the Remark in 2.6.4).

we call the *standard character* of $\mathrm{Aut}^0 O$; its kernel is pro-unipotent. For a $\mathbb{C}$-algebra $R$ an automorphism of $R[[t]]$ is defined by $t \mapsto \sum_i c_i t^i$ where $c_1 \in R^*$ and $c_0$ is nilpotent. So $\mathrm{Aut}\, O$ is the union of schemes $\mathrm{Spec}\, \mathbb{C}[c_0, c_1, c_1^{-1}, c_2, c_3, \ldots]/(c_0^k)$, $k \in \mathbb{N}$. $\mathrm{Aut}^0 O$ is the group subscheme of $\mathrm{Aut}\, O$ defined by $c_0 = 0$.

Some other examples of $(\mathfrak{l}, P)$-structures may be found in **??**.

**2.6.6.** Let $X$ be a variety equipped with an $(\mathfrak{l}, P)$-structure $X^\wedge$ (we will apply the constructions below in the situation where $X$ is a curve, $\mathfrak{l} = \mathrm{Der}\, O$, $P = \mathrm{Aut}^0 O$ (or a certain covering of $\mathrm{Aut}^0 O$), $O := \mathbb{C}[[t]]$). Denote by $\mathcal{M}(X, \mathcal{O})$ the category of $\mathcal{O}$-modules on $X$, and by $\mathcal{M}^l(X)$ that of left $\mathcal{D}$-modules. For $F_X \in \mathcal{M}(X, \mathcal{O})$ its pull-back $F_{X^\wedge}$ to $X^\wedge$ is a $P$-equivariant $\mathcal{O}$-module on $X^\wedge$. If $F_X$ is actually a left $\mathcal{D}_X$-module then $F_{X^\wedge}$ is in addition $\mathfrak{l}$-equivariant (since, by 2.6.4(ii), an $\mathfrak{l}$-action on an $\mathcal{O}_{X^\wedge}$-module is the same as a flat connection). The functors $\mathcal{M}(X, \mathcal{O}) \to \{P\text{-equivariant } \mathcal{O}\text{-modules on } X^\wedge\}$, $\mathcal{M}^l(X) \to \{(\mathfrak{l}, P)\text{-equivariant } \mathcal{O}\text{-modules on } X^\wedge\}$ are equivalences of tensor categories.

One has the faithful exact tensor functors

$$\mathcal{M}(P) \longrightarrow \mathcal{M}(X, \mathcal{O}), \;\; \mathcal{M}(\mathfrak{l}, P) \longrightarrow \mathcal{M}^l(X) \tag{25}$$

which send a representation $V$ to the $\mathcal{O}_X$- or $\mathcal{D}_X$-module $V_X$ such that $V_{X^\wedge}$ equals to $V \otimes \mathcal{O}_{X^\wedge}$ (the tensor product of $P$- or $(\mathfrak{l}, P)$-modules). In other words, the $\mathcal{O}_X$-module $V_X$ is the twist of $V$ by the $P$-torsor $X^\wedge$. Therefore any algebra $A$ with $P$-action yields an $\mathcal{O}_X$-algebra $A_X$; if $A$ actually carries a $(\mathfrak{l}, P)$-action then $A_X$ is a $\mathcal{D}_X$-algebra. Similarly, any scheme $H$ with $P$-action (a $P$-scheme for short) yields an $X$-scheme $H_X$. If $H$ is actually a $(\mathfrak{l}, P)$-scheme then $H_X$ is a $\mathcal{D}_X$-scheme. One has $(\mathrm{Spec}\, A)_X = \mathrm{Spec}(A_X)$.

*Remarks.* (i) The functor $\mathcal{M}(\mathfrak{l}, P) \longrightarrow \mathcal{M}^l(X)$ coincides with the localization functor $\Delta$ for the $(\mathfrak{l}, P)$-action on $X^\wedge$ (see 1.2.4).

(ii) The functors (25) admit right adjoints which assign to an $\mathcal{O}_X$- or $\mathcal{D}_X$-module $F_X$ the vector space $\Gamma(X^\wedge, F_{X^\wedge})$ equipped with the obvious $P$-

or $(\mathfrak{l}, P)$-module structure. Same adjointness holds if you consider algebras instead of modules.

(iii) Let $C$ be a $\mathbb{C}$-algebra; consider $C$ as an $(\mathfrak{l}, P)$-algebra with trivial $\operatorname{Aut} O$-action. Then $C_X$ is the "constant" $\mathcal{D}_X$-algebra from 2.6.1.

**2.6.7.** The forgetful functor $\{(\mathfrak{l}, P)\text{-algebras}\} \to \{P\text{-algebras}\}$ admits a left adjoint (induction) functor $\mathcal{J}$. For a $P$-algebra $A$ one has a canonical isomorphism

$$(\mathcal{J}A)_X = \mathcal{J}(A_X). \tag{26}$$

Indeed, the natural $\mathcal{O}_X$-algebra morphism $A_X \to (\mathcal{J}A)_X$ induces a $\mathcal{D}_X$-algebra morphism $\mathcal{J}(A_X) \to (\mathcal{J}A)_X$. To show that it is an isomorphism use the adjointness properties of $\mathcal{J}$ and $A \mapsto A_X$ (see 2.6.3 and Remark (ii) of 2.6.6).

Here is a geometric version of the above statements. The forgetful functor $\{(\mathfrak{l}, P)\text{-schemes}\} \to \{P\text{-schemes}\}$ admits a right adjoint functor[10] $\mathcal{J}$. For a $P$-algebra $A$ one has $\mathcal{J}(\operatorname{Spec} A) = \operatorname{Spec} \mathcal{J}(A)$. For any $P$-scheme $H$ one has $(\mathcal{J}H)_X = \mathcal{J}(H_X)$.

**2.7. Quantization II.** From now on $O := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$.

**2.7.1.** Consider first the "classical" picture. The schemes $\mathrm{Hitch}_x$, $x \in X$, are fibers of the $\mathcal{D}_X$-scheme $\mathrm{Hitch} = \mathcal{J}C_{\omega_X}$ affine over $X$; denote by $\mathfrak{z}^{cl}$ the corresponding $\mathcal{D}_X$-algebra. By 2.6.3 the projection $\mathrm{Hitch} \to C_{\omega_X}$ identifies the scheme of horizontal sections of Hitch with $\mathrm{Hitch}(X)$. In other words

$$\mathfrak{z}^{cl}(X) = H_\nabla\Big(X, \mathfrak{z}^{cl}\Big),$$

and the projections $\theta_x^{cl} : \mathfrak{z}_x^{cl} \to \mathfrak{z}^{cl}(X)$ from 2.4.3 are just the canonical morphisms $\theta_{\mathfrak{z}_x^{cl}}$ from Proposition 2.6.2(ii).

[10]For affine schemes this is just a reformulation of the above statement for $P$-algebras. The general situation does not reduce immediately to the affine case (a $P$-scheme may not admit a covering by $P$-invariant affine subschemes), but the affine case is enough for our purposes.

Consider $C$ as an $\mathrm{Aut}^0 O$-scheme via the standard character $\mathrm{Aut}^0 O \to \mathbb{G}_m$ (see 2.6.5). The $X$-scheme $C_{\omega_X}$ coincides with the $X^\wedge$-twist of $C$. Therefore the isomorphism (26) induces a canonical isomorphism

$$\mathfrak{z}^{cl} = \mathfrak{z}_{\mathfrak{g}}^{cl}(O)_X$$

where $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is the $\mathrm{Aut}\, O$-algebra $\mathcal{J}(\mathrm{Sym}\,\mathfrak{g})^G$, and the $\mathrm{Aut}^0 O$-action on $(\mathrm{Sym}\,\mathfrak{g})^G$ comes from the $\mathbb{G}_m$-action opposite to that induced by the grading of $(\mathrm{Sym}\,\mathfrak{g})^G$ (cf. the end of 2.2.2).

**2.7.2.** Let us pass to the "quantum" situation. Set $\mathfrak{z}_{\mathfrak{g}}(O) := D'_{(\mathfrak{g}\otimes K, G(O))}$. This is a commutative algebra (see 2.5.2(i)). $\mathrm{Aut}\, O$ acts on $\mathfrak{z}_{\mathfrak{g}}(O)$ since $\mathfrak{z}_{\mathfrak{g}}(O)$ is the endomorphism algebra of the twisted vacuum module $Vac'$ (see 1.2.5) and $\mathrm{Aut}\, O$ acts on $Vac'$. (The latter action is characterized by two properties: it is compatible with the natural action of $\mathrm{Aut}\, O$ on $\widetilde{\mathfrak{g}\otimes K}$ and the vacuum vector is invariant; the action of $\mathrm{Aut}\, O$ on $\widetilde{\mathfrak{g}\otimes K}$ is understood in the topological sense, i.e., $\mathrm{Aut}(O\widehat{\otimes} R)$ acts on $\widetilde{\mathfrak{g}\otimes K}\widehat{\otimes} R$ for any commutative $\mathbb{C}$-algebra $R$.) Consider the $\mathcal{D}_X$-algebra

$$\mathfrak{z} = \mathfrak{z}_{\mathfrak{g}} := \mathfrak{z}_{\mathfrak{g}}(O)_X$$

corresponding to the commutative $(\mathrm{Aut}\, O)$-algebra $\mathfrak{z}_{\mathfrak{g}}(O)$ (see 2.6.5, 2.6.6). Its fibers are the algebras $\mathfrak{z}_x$ from 2.5.1. A standard argument shows that when $x \in X$ varies the morphisms $h_x$ from 2.5.4 define a morphism of $\mathcal{O}_X$-algebras $h : \mathfrak{z} \to \Gamma(\mathrm{Bun}_G, D')_X$.

**2.7.3.** *Horizontality Theorem.* $h$ is horizontal, i.e., it is a morphism of $\mathcal{D}_X$-algebras.

For a proof see 2.8.

**2.7.4.** Set

$$\mathfrak{z}(X) = \mathfrak{z}_{\mathfrak{g}}(X) := H_\nabla(X, \mathfrak{z})\,. \tag{27}$$

According to 2.6.2(i) the $\mathcal{D}_X$-algebra morphism $h$ induces a $\mathbb{C}$-algebra morphism

$$h_X : \mathfrak{z}(X) \to \Gamma(\mathrm{Bun}_G, D')$$

We are going to show that $(\mathfrak{z}(X), h_X)$ is a solution to the quantization problem from 2.2.5. Before doing this we have to define the filtration on $\mathfrak{z}(X)$ and the isomorphism $\sigma_{\mathfrak{z}(X)} : \mathrm{gr}\, \mathfrak{z}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}(X)$.

The canonical filtration on $\mathfrak{z}_{\mathfrak{g}}(O)$ is $\mathrm{Aut}\, O$-invariant and the isomorphism $\sigma_{\mathfrak{z}(O)} : \mathrm{gr}\, \mathfrak{z}_{\mathfrak{g}}(O) \overset{\sim}{\rightarrow} \mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ (see 2.5.2(ii)) is compatible with $\mathrm{Aut}\, O$-actions. Therefore $\mathfrak{z}$ carries a horizontal filtration and we have the isomorphism of $\mathcal{D}_X$-algebras

$$\sigma_{\mathfrak{z}} : \mathrm{gr}\, \mathfrak{z} \overset{\sim}{\rightarrow} \mathfrak{z}^{cl}$$

which reduces to the isomorphism $\sigma_{\mathfrak{z}_x}$ from 2.5.2(ii) at each fiber. The image of this filtration by $\theta_{\mathfrak{z}} : \mathfrak{z} \twoheadrightarrow H_{\nabla}(X, \mathfrak{z})_X = \mathfrak{z}(X)_X$ is a horizontal filtration on $\mathfrak{z}(X)_X$ which is the same as a filtration on $\mathfrak{z}(X)$. Consider the surjective morphism of graded $\mathcal{D}_X$-algebras $(\mathrm{gr}\, \theta_{\mathfrak{z}})\sigma_{\mathfrak{z}}^{-1} : \mathfrak{z}^{cl} \twoheadrightarrow \mathrm{gr}\, \mathfrak{z}(X)_X$. By adjunction (see (22)) it defines the surjective morphism of graded $\mathbb{C}$-algebras $j : \mathfrak{z}^{cl}(X) = H_{\nabla}\left(X, \mathfrak{z}^{cl}\right) \twoheadrightarrow \mathrm{gr}\, \mathfrak{z}(X)$.

Note that $h_X$ is compatible with filtrations, and we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{z}^{cl}(X) \arrow[r, hook, "h_X^{cl}"] \arrow[d, two heads, "j"'] & \Gamma(\mathrm{Bun}_G, P) \\
\mathrm{gr}\, \mathfrak{z}(X) \arrow[r, "\mathrm{gr}\, h_X"] & \mathrm{gr}\, \Gamma(\mathrm{Bun}_G, D') \arrow[u, hook, "\sigma_{\mathrm{Bun}_G}"']
\end{tikzcd} \tag{28}$$

Therefore $j$ is an isomorphism and $\mathrm{gr}\, h_X$ (hence $h_X$) is injective. Define $\sigma_{\mathfrak{z}(X)} : \mathrm{gr}\, \mathfrak{z}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}(X)$ by $\sigma_{\mathfrak{z}(X)} := j^{-1}$. The triple $(\mathfrak{z}(X), h_X, \sigma_{\mathfrak{z}(X)})$ is a solution to the quantization problem from 2.2.5.

**2.7.5.** Let us prove Theorem 2.5.5 and compare $\mathfrak{z}^{(x)}(X)$ from 2.5.5 with $\mathfrak{z}(X)$. Clearly $h_x = h_X \cdot \theta_{\mathfrak{z}_x}$ where $\theta_{\mathfrak{z}_x} : \mathfrak{z}_x \to \mathfrak{z}(X)$ was defined in Proposition 2.6.2(ii). $\theta_{\mathfrak{z}_x}$ is surjective (see 2.6.2(ii)) and strictly compatible with filtrations (see the definition of the filtration on $\mathfrak{z}(X)$ in 2.7.4). $h_X$

is injective and strictly compatible with filtrations (see the end of 2.7.4). So $h_x$ is strictly compatible with filtrations (which is precisely Theorem 2.5.5) and $h_X$ induces an isomorphism between the filtered algebras $\mathfrak{z}(X)$ and $\mathfrak{z}^{(x)}(X) := h_x(\mathfrak{z}_x)$.

**2.8. Horizontality.** In this subsection we introduce $\mathcal{D}_X$-structure on some natural moduli schemes and prove the horizontality theorem 2.7.3 modulo certain details explained in 4.4.14. The reader may skip this subsection for the moment.

In 2.8.1–2.8.2 we sketch a proof of Theorem 2.7.3. The method of 2.8.2 is slightly modified in 2.8.3. In 2.8.4–2.8.5 we explain some details and refer to 4.4.14 for the rest of them. In 2.8.6 we consider very briefly the ramified situation.

**2.8.1.** Let us construct the morphism $h$ from Theorem 2.7.3.

Recall that the construction of $h_x$ from 2.5.3–2.5.4 involves the scheme $\mathrm{Bun}_{G,\underline{x}}$, i.e., the moduli scheme of $G$-bundles on $X$ trivialized over the formal neighbourhood of $x$. It also involves the action of the Harish-Chandra pair $(\mathfrak{g}\otimes K_x, G(O_x))$ on $\mathrm{Bun}_{G,\underline{x}}$ and its lifting to the action of $(\widetilde{\mathfrak{g}\otimes K_x}, G(O_x))$ on the line bundle $\pi_x^*\omega_{\mathrm{Bun}_G}$ where $\pi_x$ is the natural morphism $\mathrm{Bun}_{G,\underline{x}} \to \mathrm{Bun}_G$. These actions come from the action of the group ind-scheme $G(K_x)$ on $\mathrm{Bun}_{G,\underline{x}}$ and its lifting to the action of a certain central extension[11] $\widehat{G(K_x)}$ on $\pi_x^*\omega_{\mathrm{Bun}_G}$.

To construct $h$ one has to organize the above objects depending on $x$ into families. One defines in the obvious way a scheme $M$ over $X$ whose fiber over $x$ equals $\mathrm{Bun}_{G,\underline{x}}$. One defines a group scheme $J(G)$ over $X$ and a group ind-scheme $J^{\mathrm{mer}}(G)$ over $X$ whose fibers over $x$ are respectively $G(O_x)$ and $G(K_x)$. $J(G)$ is the scheme of jets of functions $X \to G$ and $J^{\mathrm{mer}}(G)$ is the ind-scheme of "meromorphic jets". $J^{\mathrm{mer}}(G)$ acts on $M$. Finally one

[11]This extension was mentioned (rather than defined) in the Remark from 2.5.3. This is enough for the sketch we are giving.

defines a central extension $\widehat{J}^{\mathrm{mer}}(G)$ and its action on $\pi^*\omega_{\mathrm{Bun}_G}$ where $\pi$ is the natural morphism $M \to \mathrm{Bun}_G$. These data being defined the construction of $h : \mathfrak{z} \to \Gamma(\mathrm{Bun}_G, D')_X$ is quite similar to that of $h_x$ (see 2.5.3–2.5.4).

**2.8.2.** The crucial observation is that *there are canonical connections along $X$ on $J(G)$, $J^{\mathrm{mer}}(G)$, $\widehat{J}^{\mathrm{mer}}(G)$, $M$ and $\pi^*\omega_{\mathrm{Bun}_G}$ such that the action of $J^{\mathrm{mer}}(G)$ on $M$ and the action of $\widehat{J}^{\mathrm{mer}}(G)$ on $\pi^*\omega_{\mathrm{Bun}_G}$ are horizontal.* This implies the horizontality of $h$.

For an $X$-scheme $Y$ we denote by $\mathcal{J}Y$ the scheme of jets of sections $X \to Y$. It is well known (and more or less explained in 2.6.3) that $\mathcal{J}Y$ has a canonical connection along $X$ (i.e., $\mathcal{J}Y$ is a $\mathcal{D}_X$-scheme in the sense of 2.6.1). In particular this applies to $J(G) = \mathcal{J}(G \times X)$. If $\mathcal{F}$ is a principal $G$-bundle over $X$ then the fiber of $\pi : M \to \mathrm{Bun}_G$ over $\mathcal{F}$ equals $\mathcal{J}\mathcal{F}$, so it is equipped with a connection along $X$. One can show that these connections come from a connection along $X$ on $M$.

To define the connection on $M$ as well as the other connections it is convenient to use Grothendieck's approach [Gr68]. According to [Gr68] a connection (=integrable connection = "stratification") along $X$ on an $X$-scheme $Z$ is a collection of bijections $\varphi_{\alpha\beta} : \mathrm{Mor}_\alpha(S, Z) \xrightarrow{\sim} \mathrm{Mor}_\beta(S, Z)$ for every scheme $S$ and every pair of infinitely close "points" $\alpha, \beta : S \to X$ (here $\mathrm{Mor}_\alpha(S, Z)$ is the preimage of $\alpha$ in $\mathrm{Mor}(S, Z)$ and "infinitely close" means that the restrictions of $\alpha$ and $\beta$ to $S_{\mathrm{red}}$ coincide); the bijections $\varphi_{\alpha\beta}$ are required to be functorial with respect to $S$ and to satisfy the equation $\varphi_{\beta\gamma}\varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$.

For instance, if $Z$ is the jet scheme of a scheme $Y$ over $X$ then $\mathrm{Mor}_\alpha(S, Z) := \mathrm{Mor}_X(S'_\alpha, Y)$ where $S'_\alpha$ is the formal neighbourhood of the graph $\Gamma_\alpha \subset S \times X$ and the morphism $S'_\alpha \to X$ is induced by the projection $\mathrm{pr}_X : S \times X \to X$. It is easy to show that if $\alpha$ and $\beta$ are infinitely close then $S'_\alpha = S'_\beta$, so we obtain a connection along $X$ on $Z$. One can show that it coincides with the connection defined in 2.6.3.

The connections along $X$ on $J^{\mathrm{mer}}(G)$, $\widehat{J}^{\mathrm{mer}}(G)$, and $M$ are defined in the similar way. The horizontality of the action of $J^{\mathrm{mer}}(G)$ on $M$ and the action of $\widehat{J}^{\mathrm{mer}}(G)$ on $\pi^*\omega_{\mathrm{Bun}_G}$ easily follows from the definitions.

**2.8.3.** The method described in 2.8.2 can be modified as follows. Recall that $O := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$; $\operatorname{Aut} O$ and $X^\wedge$ were defined in 2.6.5. Set $M^\wedge = M \times_X X^\wedge$. So $M^\wedge$ is the moduli space of quadruples $(x, t_x, \mathcal{F}, \gamma_x)$ where $x \in X$, $t_x$ is a formal parameter at $x$, $\mathcal{F}$ is a $G$-torsor on $X$, $\gamma_x$ is a section of $\mathcal{F}$ over the formal neighbourhood of $x$. The group ind-scheme $G(K)$ acts on the fiber of $M^\wedge$ over any $\widehat{x} \in X^\wedge$ (indeed, this fiber coincides with $\mathrm{Bun}_{G,\underline{x}}$ where $x$ is the image of $\widehat{x}$ in $X$, so $G(K_x)$ acts on the fiber; on the other hand the formal parameter at $x$ corresponding to $\widehat{x}$ defines an isomorphism $K_x \xrightarrow{\sim} K$). Actually $G(K)$ acts on $M^\wedge$ (see 2.8.4) and the central extension $\widehat{G(K)}$ acts on $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$ where $\widehat{\pi}$ is the natural morphism $M^\wedge \to \mathrm{Bun}_G$. This action induces a morphism $\widehat{h} : \mathfrak{z}_{\mathfrak{g}}(O) \to \Gamma(X^\wedge, \mathcal{O}_{X^\wedge}) \otimes \Gamma(\mathrm{Bun}_G, D')$ (see 2.7.2 for the definition of $\mathfrak{z}_{\mathfrak{g}}(O)$).

On the other hand the action of $\operatorname{Aut} O$ on $X^\wedge$ from 2.6.5 lifts canonically to its action on $M^\wedge$ (see 2.8.4) and the sheaf $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$ . The actions of $\operatorname{Aut} O$ and $\widehat{G(K)}$ on $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$ are compatible in the obvious sense. Therefore $\widehat{h}$ is $\operatorname{Aut} O$-equivariant. So $\widehat{h}$ induces a horizontal morphism $h : \mathfrak{z} = \mathfrak{z}_{\mathfrak{g}}(O)_X \to \Gamma(\mathrm{Bun}_G, D')_X$.

**2.8.4.** To turn the sketch from 2.8.3 into a proof of Theorem 2.7.3 we first of all give a precise definition of the action of the semidirect product $\operatorname{Aut} O \ltimes G(K)$ on $M^\wedge$. Let $R$ be a $\mathbb{C}$-algebra. By definition, an $R$-point of $M^\wedge$ is a triple $(\alpha, \mathcal{F}, \gamma)$ where $\alpha : \operatorname{Spec} R\widehat{\otimes}O \to X \otimes R$ is an $R$-morphism whose differential does not vanish over any point of $\operatorname{Spec} R$, $\mathcal{F}$ is a $G$-torsor on $X \otimes R$, and $\gamma$ is a section of $\alpha^*\mathcal{F}$. Let $\Gamma_\alpha$ denote the graph of the composition $\operatorname{Spec} R \to \operatorname{Spec} R\widehat{\otimes}O \xrightarrow{\alpha} X \otimes R$ and $\alpha'$ the morphism $\operatorname{Spec} R\widehat{\otimes}K \to (X \otimes R) \setminus \Gamma_\alpha$ induced by $\alpha$. According to

Beauville and Laszlo[12] (see 2.3.7 and 2.3.4) $R$-points of $M^\wedge$ are in one-to-one correspondence with triples $(\alpha, \mathcal{F}', \gamma')$ where $\alpha$ is as above, $\mathcal{F}'$ is a $G$-torsor on $(X \otimes R) \setminus \Gamma_\alpha$, and $\gamma'$ is a section of $\alpha'^*\mathcal{F}'$ (of course, $\mathcal{F}'$ is the restriction of $\mathcal{F}$, $\gamma'$ is the restriction of $\gamma$). This interpretation shows that $G(R\widehat{\otimes}K)$ and $\mathrm{Aut}(R\widehat{\otimes}O)$ act on $M^\wedge(R)$: the action of $G(R\widehat{\otimes}K)$ changes $\gamma'$ and the action of $\mathrm{Aut}(R\widehat{\otimes}O)$ changes $\alpha$ (if $\alpha$ is replaced by $\alpha\circ\varphi$, $\varphi \in \mathrm{Aut}\,\mathrm{Spec}\, R\widehat{\otimes}O$, then $\Gamma_\alpha$ changes as a subscheme of $X \otimes R$ but not as a subset, so $(X \otimes R) \setminus \Gamma_\alpha$ remains unchanged). Thus we obtain the action of $\mathrm{Aut}\, O \ltimes G(K)$ on $M^\wedge$ mentioned in 2.8.3.

**2.8.5.** According to 2.8.4 $\mathrm{Aut}\, O$ acts on $M^\wedge$ considered as a scheme over $\mathrm{Bun}_G$. So $\mathrm{Aut}\, O$ acts on $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$. In 2.5.3 we mentioned the canonical action of $\widehat{G(K_x)}$ on the pullback of $\omega_{\mathrm{Bun}_G}$ to $\mathrm{Bun}_{G,\underline{x}}$. So $\widehat{G(K)}$ acts on the restriction of $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$ to the fiber of $M^\wedge$ over any $\widehat{x} \in X^\wedge$. As explained in 2.8.3, to finish the proof of 2.7.3 it suffices to show that

(i) the actions of $\widehat{G(K)}$ corresponding to various $\widehat{x} \in X^\wedge$ come from an (obviously unique) action of $\widehat{G(K)}$ on $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$,

(ii) this action is compatible with that of $\mathrm{Aut}\, O$.

To prove (i) and (ii) it is necessary (and almost sufficient) to define the central extension $\widehat{G(K_x)}$ and its action on the pullback of $\omega_{\mathrm{Bun}_G}$ to $\mathrm{Bun}_{G,\underline{x}}$. The interested reader can do it using, e.g., [BLa94].

Instead of proving (i) and (ii) we will prove in 4.4.14 a similar statement for a square root of $\omega_{\mathrm{Bun}_G}$ (because we need the square roots of $\omega_{\mathrm{Bun}_G}$ to formulate and prove Theorem 5.4.5, which is the main result of this work). More precisely, for any square root $\mathcal{L}$ of $\omega_X$ one defines a line bundle $\lambda'_{\mathcal{L}}$ on $\mathrm{Bun}_G$, which is essentially a square root of $\omega_{\mathrm{Bun}_G}$ (see 4.4.1). One constructs a central extension[13] $\widetilde{G(K_x)}_{\mathcal{L}}$ acting on the pullback of $\lambda'_{\mathcal{L}}$ to $\mathrm{Bun}_{G,\underline{x}}$ (see 4.4.7 – 4.4.8). The morphism $h_x : \mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G, D')$ from 2.5.4 can be

[12]The normal bundle of $\Gamma_\alpha \subset X \otimes R$ is trivial, so according to 2.3.7 one can apply the main theorem of [BLa95] rather than its globalized version.

[13]In fact, this extension is a square root of $\widehat{G(K_x)}$.

naturally defined using this action (see 4.4.12 – 4.4.13). Finally, in 4.4.14 we prove the analog of the above statements (i) and (ii) for $\lambda'_{\mathcal{L}}$, which implies the horizontality theorem 2.7.3.

**2.8.6.** Let $\Delta \subset X$ be a finite subscheme. Denote by $\mathrm{Bun}_{G,\Delta}$ the stack of $G$-bundles on $X$ trivialized over $\Delta$. Denote by $D'$ the sheaf $D_{\mathcal{Y},\mathcal{L}^\lambda}$ from 1.1.6 for $\mathcal{Y} = \mathrm{Bun}_{G,\Delta}$, $\mathcal{L} =$ the pullback of $\omega_{\mathrm{Bun}_G}$, $\lambda = 1/2$. Just as in the case $\Delta = \emptyset$ one defines a horizontal morphism $h : \mathfrak{z}_{X\setminus\Delta} \to \Gamma(\mathrm{Bun}_{G,\Delta}, D') \otimes \mathcal{O}_{X\setminus\Delta}$ where $\mathfrak{z}_{X\setminus\Delta}$ is the restriction of $\mathfrak{z}$ to $X\setminus\Delta$. $h$ induces an injection $\Gamma(N, \mathcal{O}_N) \to \Gamma(\mathrm{Bun}_{G,\Delta}, D')$ where $N = N_\Delta(G)$ is a closed subscheme of the ind-scheme $N'_\Delta(G)$ of horizontal sections of $\mathrm{Spec}\, \mathfrak{z}_{X\setminus\Delta}$.

*Problem.* Describe $N_\Delta(G)$ explicitly.

We are going to indicate the geometric objects used in the solution of the problem. Since we do not explain the details of the solution one can read the rest of this subsection without knowing the answer to the problem, which can be found in 3.8.2.

For $n \in \mathbb{Z}_+$ denote by $M_{\Delta,n}$ the stack of triples consisting of a point $x \in X$, a $G$-bundle $\mathcal{F}$ on $X$, and a trivialization of $\mathcal{F}$ over $\Delta + nx$ (here we identify finite subshemes of $X$ with effective divisors, so $\Delta + nx$ makes sense). $M_{\Delta,n}$ is an algebraic stack and $M_\Delta := \varprojlim_n M_{\Delta,n}$ is a scheme over $X$.

*Remark.* Let $M_{\Delta,x}$ be the fiber of $M_\Delta$ over $x \in X$. If $x \in X\setminus\Delta$ then $M_{\Delta,x}$ is the moduli scheme of $G$-bundles trivialized over $\Delta$ and the formal neighbourhood of $x$. If $x \in \Delta$ then $M_{\Delta,x} = M_{\Delta\setminus\{x\},x}$.

Consider the "congruence subgroup" scheme $\underline{G}_\Delta$ defined as follows: $\underline{G}_\Delta$ is a scheme flat over $X$ such that for any scheme $S$ flat over $X$

$$\mathrm{Mor}_X(S, \underline{G}_\Delta) = \{f : S \to G \text{ such that } f|_{\Delta_S} = 1\}$$

where $\Delta_S$ is the preimage of $\Delta$ in $S$. $\underline{G}_\Delta$ is a group scheme over $X$. A $G$-bundle on $X$ trivialized over $\Delta$ is the same as a $\underline{G}_\Delta$-bundle (this becomes clear if $G$-bundles and $\underline{G}_\Delta$-bundles are considered as torsors for the étale topology). So $\mathrm{Bun}_{G,\Delta}$ is the stack of $\underline{G}_\Delta$-bundles.

One can show that if $D \subset X$ is a finite subscheme and $\Delta + D$ is understood in the sense of divisors then for every scheme $S$ flat over $X$

$$\mathrm{Mor}_X(S, \underline{G}_{\Delta+D}) = \{f \in \mathrm{Mor}_X(S, \underline{G}_\Delta) \text{ such that } f|_{D_S} = 1\}$$

Therefore a $G$-bundle on $X$ trivialized over $\Delta + D$ is the same as a $\underline{G}_\Delta$-bundle trivialized over $D$. So $M_\Delta$ is the moduli scheme of triples consisting of a point $x \in X$, a $\underline{G}_\Delta$-bundle on $X$, and its trivialization over the formal neighbourhood of $x$. Now one can easily define a canonical action of $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$ on $M_\Delta$ where $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$ is the group ind-scheme of "meromorphic jets" of sections $X \to \underline{G}_\Delta$. $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$ and $M_\Delta$ are equipped with connections along $X$ and the above action is horizontal. And so on...

*Remarks*

(i) If $\Delta \neq \emptyset$ the method of 2.8.3 does not allow to avoid using group ind-schemes over $X$.

(ii) There are pitfalls connected with infinite dimensional schemes and ind-schemes like $M_\Delta$ or $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$. Here is an example. The morphism $\underline{G}_\Delta \to \underline{G} := \underline{G}_\emptyset = G \times X$ induces $f : \mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta) \to \mathcal{J}^{\mathrm{mer}}(\underline{G})$. This $f$ induces an isomorphism of the fibers over any point $x \in X$ (the fiber of $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$ over $x$ is $G(K_x)$, it does not depend on $\Delta$). But if $\Delta \neq \emptyset$ then $f$ is not an isomorphism, nor even a monomorphism.

**2.9. Commutativity of $\mathfrak{z}_{\mathfrak{g}}(O)$.** The algebras $\mathfrak{z}_{\mathfrak{g}}(O)$ and $\mathfrak{z}_x = \mathfrak{z}_{\mathfrak{g}}(O_x)$ were defined in 2.5.1 and 2.7.2 (of course they are isomorphic). Feigin and Frenkel proved in [FF92] that $\mathfrak{z}_{\mathfrak{g}}(O)$ is commutative. In this subsection we give two proofs of the commutativity of $\mathfrak{z}_{\mathfrak{g}}(O)$: the global one (see 2.9.1–2.9.2) and the local one (see 2.9.3–2.9.5). The latter is in fact a version of the original proof from [FF92].

The reader may skip this subsection for the moment. We will not use 2.9.1–2.9.2 in the rest of the paper.

**2.9.1.** Let us prove that

$$[h_x(\mathfrak{z}_x), h_y(\mathfrak{z}_y)] = 0 \tag{29}$$

(see 2.5.4 for the definition of $h_x : \mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G, D')$ ). Since $\mathfrak{z}_x$ is the fiber at $x$ of the $\mathcal{O}_X$-algebra $\mathfrak{z} = \mathfrak{z}_{\mathfrak{g}}(O)_X$ and $h_x$ comes from the $\mathcal{O}_X$-algebra morphism $h : \mathfrak{z} \to \mathcal{O}_X \otimes \Gamma(\mathrm{Bun}_G, D')$ it is enough to prove (29) for $x \neq y$. Denote by $\mathrm{Bun}_{G,\underline{x},\underline{y}}$ the moduli scheme of $G$-bundles on $X$ trivialized over the formal neighbourhoods of $x$ and $y$. $G(K_x) \times G(K_y)$ acts on $\mathrm{Bun}_{G,\underline{x},\underline{y}}$. In particular the Harish-Chandra pair $((\mathfrak{g} \otimes K_x) \times (\mathfrak{g} \otimes K_y), G(O_x) \times G(O_y))$ acts on $\mathrm{Bun}_{G,\underline{x},\underline{y}}$. This action lifts canonically to an action of $((\widetilde{\mathfrak{g} \otimes K_x}) \times (\widetilde{\mathfrak{g} \otimes K_y}), G(O_x) \times G(O_y))$ on the pullback of $\omega_{\mathrm{Bun}_G}$ to $\mathrm{Bun}_{G,\underline{x},\underline{y}}$ such that $\mathbf{1}_x \in \mathfrak{g} \otimes K_x$ and $\mathbf{1}_y \in \mathfrak{g} \otimes K_y$ act as multiplication by 2 and $G(O_x) \times G(O_y)$ acts in the obvious way. The action of $G(O_x) \times G(O_y)$ on $\mathrm{Bun}_{G,\underline{x},\underline{y}}$ satisfies condition (16) from 1.2.6 and the quotient stack equals $\mathrm{Bun}_G$. So according to 1.2.5 we have a canonical morphism $h_{x,y} : \mathfrak{z}_x \otimes \mathfrak{z}_y \to \Gamma(\mathrm{Bun}_G, D')$. Its restrictions to $\mathfrak{z}_x$ and $\mathfrak{z}_y$ are equal to $h_x$ and $h_y$. So (29) is obvious.

**2.9.2.** Let us prove the commutativity of $\mathfrak{z}_{\mathfrak{g}}(O)$. Suppose that $a \in [\mathfrak{z}_{\mathfrak{g}}(O), \mathfrak{z}_{\mathfrak{g}}(O)]$, $a \neq 0$. If $x = y$ then (29) means that $h_x(\mathfrak{z}_x)$ is commutative. So for any $X$, $x \in X$, and $f : O \xrightarrow{\sim} O_x$ one has $h_x(f_*(a)) = 0$. Let $\bar{a} \in \mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ be the principal symbol of $a$. Then for any $X$, $x$, $f$ as above one has $h_x^{cl}(f_*(\bar{a})) = 0$ (see 2.4.3 for the definition and geometric description of $h_x^{cl} : \mathfrak{z}_x^{cl} \to \Gamma(\mathrm{Bun}_G, P) = \Gamma(T^*\mathrm{Bun}_G, \mathcal{O})$. This means that $\bar{a}$ considered as a polynomial function on $\mathfrak{g}^* \otimes \omega_O$ (see 2.4.1) has the following property: for any $X$, $x$ as above, any $G$-bundle $\mathcal{F}$ on $X$ trivialized over the formal neighbourhood of $x$, and any isomorphism $O_x \xrightarrow{\sim} O$ the restriction of $\bar{a}$ to the image of the map $H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X) \to \mathfrak{g}^* \otimes \omega_{O_x} \xrightarrow{\sim} \mathfrak{g}^* \otimes \omega_O$ is zero. There is an $n$ such that $\bar{a}$ comes from a function on $\mathfrak{g}^* \otimes (\omega_O/\mathfrak{m}^n\omega_O)$ where $\mathfrak{m}$ is the maximal ideal of $O$. Choose $X$ and $x$ so that the mapping

$H^0(X,\omega_X) \to \omega_{O_x}/\mathfrak{m}_x^n\omega_{O_x}$ is surjective and let $\mathcal{F}$ be the trivial bundle. Then the map $H^0(X,\mathfrak{g}^*_{\mathcal{F}}\otimes\omega_X) \to \mathfrak{g}^*\otimes(\omega_{O_x}/\mathfrak{m}_x^n\omega_{O_x})$ is surjective and therefore $\bar{a}=0$, i.e., a contradiction.

*Remark.* Let $\mathrm{Bun}_{G,\Delta}$ be the stack of $G$-bundles on $X$ trivialized over a finite subscheme $\Delta\subset X$. To deduce from (29) the commutativity of $\mathfrak{z}_{\mathfrak{g}}(O)$ one can use the natural homomorphism from $\mathfrak{z}_x$, $x\notin\Delta$, to the ring of twisted differential operators on $\mathrm{Bun}_{G,\Delta}$. Then instead of choosing $(X,x)$ as in the above proof one can fix $(X,x)$ and take $\Delta$ big enough.

**2.9.3.** Denote by $\mathfrak{Z}$ the center of the completed twisted universal enveloping algebra $\overline{U}'(\mathfrak{g}\otimes K)$, $K:=\mathbb{C}((t))\supset\mathbb{C}[[t]]=O$. In [FF92] Feigin and Frenkel deduce the commutativity of $\mathfrak{z}_{\mathfrak{g}}(O)$ from the surjectivity of the natural homomorphism $f:\mathfrak{Z}\to\mathfrak{z}_{\mathfrak{g}}(O)$. We will present a proof of the surjectivity of $f$ which can be considered as a geometric version of the one from [FF92] and also as a "quantization" of the remark at the end of 2.4.2. The relation with [FF92] and 2.4.2 will be explained in 2.9.7 and 2.9.8.

*Remark.* In the definition of the central extension of $\mathfrak{g}\otimes K$ (see 2.5.1) and therefore in the definition of $\mathfrak{Z}$ and $\mathfrak{z}_{\mathfrak{g}}(O)$ we used the "critical" bilinear form $c$ defined by (18). In the proof of the surjectivity of $f$ one can assume that $c$ is *any* invariant symmetric bilinear form on $\mathfrak{g}$ and $\mathfrak{g}$ is *any* finite dimensional Lie algebra. On the other hand it is known that if $\mathfrak{g}$ is simple and $c$ is non-critical then the corresponding algebra $\mathfrak{z}_{\mathfrak{g}}(O)$ is trivial (see ???).

**2.9.4.** We need the interpretation of $\overline{U}':=\overline{U}'(\mathfrak{g}\otimes K)$ from [BD94]. Denote by $U'$ the non-completed twisted universal enveloping algebra of $\mathfrak{g}\otimes K$. For $n\geq 0$ let $I_n$ be the left ideal of $U'$ generated by $\mathfrak{g}\otimes\mathfrak{m}^n\subset\mathfrak{g}\otimes O\subset U'$. By definition, $\overline{U}':=\varprojlim\limits_n U'/I_n$. Let $U'_k$ be the standard filtration of $U'$ and $\overline{U}'_k$ the closure of $U'_k$ in $\overline{U}'$, i.e., $\overline{U}'_k:=\varprojlim\limits_n U'_k/I_{n,k}$, $I_{n,k}:=I_n\cap U'_k$. The main theorem of [BD94] identifies the dual space $(U'_k/I_{n,k})^*$ with a certain topological vector space $\Omega_{n,k}$. So

$$U'_k/I_{n,k} = (\Omega_{n,k})^*\,, \qquad \overline{U}'_k = (\Omega_k)^* \tag{30}$$

where $\Omega_k = \varinjlim\limits_n \Omega_{n,k}$ and $*$ denotes the topological dual.

To define $\Omega_{n,k}$ we need some notation. Denote by $O_r$ (resp. $\omega_r^O$) the completed tensor product of $r$ copies of $O$ (resp. of $\omega_O$). Set $\omega_r^K = \omega_r^O \bigotimes_{O_r} K_r$ where $K_r$ is the field of fractions of $O_r$. We identify $O_r$ with $\mathbb{C}[[t_1,\dots,t_r]]$ and write elements of $\omega_r^K$ as $f(t_1,\dots,t_r)\,dt_1\dots dt_r$ where $f$ belongs to the field of fractions of $\mathbb{C}[[t_1,\dots,t_r]]$.

*Definition.* $\Omega_{n,k}$ is the set of $(k+1)$-tuples $(w_0,\dots,w_k)$, $w_r \in (\mathfrak{g}^*)^{\otimes r}\otimes\omega_r^K$, such that

1) $w_r$ is invariant with respect to the action of the symmetric group $S_r$ ($S_r$ acts both on $(\mathfrak{g}^*)^{\otimes r}$ and $\omega_r^K$);
2) $w_r$ has poles of order $\le n$ at the hyperplanes $t_i = 0$, $1\le i\le r$, poles of order $\le 2$ at the hyperplanes $t_i = t_j$, $1 \le i < j \le r$, and no other poles;
3) if $w_r = f_r(t_1,\dots,t_r)\,dt_1\dots dt_r$, $r\ge 2$, then

$$\begin{aligned} f_r(t_1\dots,t_r) \;&=\; \frac{f_{r-2}(t_1,\dots,t_{r-2})\otimes c}{(t_{r-1}-t_r)^2} \\ &+\; \frac{\varphi^*(f_{r-1}(t_1,\dots,t_{r-1}))}{t_{r-1}-t_r} + \cdots \end{aligned} \tag{31}$$

Here $c\in\mathfrak{g}^*\otimes\mathfrak{g}^*$ is the bilinear form used in the definition of the central extension of $\mathfrak{g}\otimes K$, $\varphi^* : (\mathfrak{g}^*)^{\otimes(r-1)} \to (\mathfrak{g}^*)^{\otimes r}$ is dual to the mapping $\varphi : \mathfrak{g}^{\otimes r} \to \mathfrak{g}^{\otimes(r-1)}$ given by $\varphi(a_1\otimes\dots\otimes a_r) = a_1\otimes\dots\otimes a_{r-2}\otimes[a_{r-1},a_r]$ and the dots in (31) denote an expression which does not have a pole at the generic point of the hyperplane $t_{r-1}=t_r$.

The topology on $\Omega_{n,k}$ is induced by the embedding $\Omega_{n,k}\hookrightarrow \prod\limits_{0\le r\le k}(\mathfrak{g}^*)^{\otimes r}\otimes \Omega_r^O$ given by $(w_0,\dots,w_k)\mapsto(\eta_0,\dots,\eta_k)$, $\eta_r = \prod\limits_i t_i^n\cdot\prod\limits_{i<j}(t_i-t_j)^2\cdot w_r$.

Let us explain that in (31) we consider $f_r$ as a function with values in $(\mathfrak{g}^*)^{\otimes r}$.

We will not need the explicit formula from [BD94] for the isomorphism (30). Let us only mention that according to Proposition 5 from [BD94] the adjoint action of $\mathfrak{g}\otimes K$ on $\overline{U}'_k$ induces via (30) the following action of $\mathfrak{g}\otimes K$ on $\Omega_k$ : $a\in\mathfrak{g}\otimes K$ sends $(w_0,\dots,w_k)\in\Omega_k$ to $(0,w''_1,\dots,w''_k)$ where

$$\begin{aligned} w''_r &= \frac{1}{(r-1)\,!}\,\mathrm{Sym}\; w'_r\,,\\ w'_r(t_1,\dots,t_r) &:= (\mathrm{id}\otimes\dots\otimes\mathrm{id}\otimes\mathrm{ad}_{a(t_r)})w_r(t_1,\dots,t_r)\\ &\quad - w_{r-1}(t_1,\dots,t_{r-1})\otimes c\cdot da(t_r)\,. \end{aligned}$$

Here Sym denotes the symmetrization operator (without the factor $1/r!$), $\mathrm{ad}_{a(t_r)}\,:\,\mathfrak{g}^*\to\mathfrak{g}^*$ is the operator corresponding to $a(t_r)$ in the coadjoint representation, and $c:\,\mathfrak{g}\to\mathfrak{g}^*$ is the bilinear form of $\mathfrak{g}$.

*Remark.* Suppose that $c=0$ and $\mathfrak{g}$ is commutative. Then $U'_k/I_{n,k}=\bigoplus_{r=0}^k\mathrm{Sym}^r(\mathfrak{g}\otimes K/\mathfrak{m}^n)$ and $\Omega_{n,k}=\bigoplus_{r=0}^k\overline{\mathrm{Sym}}^r(\mathfrak{g}^*\otimes\mathfrak{m}^{-n}\omega_O)$ where $\overline{\mathrm{Sym}}^r$ denotes the completed symmetric power. The isomorphism $U'_k/I_{n,k}\xrightarrow{\sim}(\Omega_{n,k})^*$ is the identification of $\mathrm{Sym}(\mathfrak{g}\otimes K/\mathfrak{m}^n)$ with the space of polynomial functions on $\mathfrak{g}^*\otimes\mathfrak{m}^{-n}\omega_O$ used in 2.4.1 and 2.4.2.

**2.9.5.** According to 2.9.4 to prove the surjectivity of $f\,:\,\mathfrak{Z}\to\mathfrak{z}_{\mathfrak{g}}(O)$ it is enough to show that any $(\mathfrak{g}\otimes O)$-invariant continuous linear functional $l\,:\,\Omega_{0,k}\to\mathbb{C}$ can be extended to a $(\mathfrak{g}\otimes K)$-invariant continuous linear functional $\Omega_k\to\mathbb{C}$. Consider the continuous linear operator

$$T:\,\Omega_k\to\mathbb{C}((\zeta))\widehat{\otimes}\Omega_{0,k}=\Big\{\sum_{n=-\infty}^{\infty}a_n\zeta^n|a_n\in\Omega_{0,k}\,,\;a_n\to 0\text{ for }n\to-\infty\Big\}$$

defined by

$$T(w_0,\dots,w_k)=(\widehat{w}_0,\dots,\widehat{w}_k)\,,\quad \widehat{w}_r=w_r(\zeta+t_1,\dots,\zeta+t_r) \tag{32}$$

where $w_r(\zeta+t_1,\dots,\zeta+t_r)$ is considered as an element of

$$\Delta^{-1}\mathbb{C}((\zeta))[[t_1,\dots,t_n]]\,dt_1\dots dt_n=\mathbb{C}((\zeta))\widehat{\otimes}\Delta^{-1}\mathbb{C}[[t_1,\dots,t_n]]\,dt_1\dots dt_n\,,$$

$$\Delta := \prod_{1\le i<j\le r} (t_i - t_j)^2 \, .$$

If $l \in (\Omega_{0,k})^*$ let $\bar{l} : \Omega_k \to \mathbb{C}((\zeta))$ be the composition of $T : \Omega_k \to \mathbb{C}((\zeta))\widehat{\otimes}\Omega_{0,k}$ and $\mathrm{id}\otimes l : \mathbb{C}((\zeta))\widehat{\otimes}\Omega_{0,k} \to \mathbb{C}((\zeta))$. Write $\bar{l}$ as $\sum\limits_i l_i\zeta^i$, $l_i \in (\Omega_k)^*$. If $l$ is $\mathfrak{g}\otimes O$-invariant then the functionals $l_i$ are $\mathfrak{g}\otimes K$-invariant. Besides $l_0|_{\Omega_{0,k}} = l$.

*Remark.* Let $G$ be an algebraic group such that $\mathrm{Lie}\, G = \mathfrak{g}$. Then $G(K)$ acts on our central extension of $\mathfrak{g}\otimes K$ (see (19)), so it acts on $\overline{U}'_k$; moreover, $G(O)$ acts on $U'_k/I_{n,k}$. Therefore $G(K)$ acts on $\Omega_k$ and $G(O)$ acts on $\Omega_{n,k}$. In the above situation if $l$ is $G(O)$-invariant then the functionals $l_i$ are $G(K)$-invariant (see formula (24) from [BD94] for the action of $G(K)$ on $\Omega_k$). Notice that if $G$ is connected $G(K)$ is not necessarily connected, so $G(K)$-invariance does not follow immediately from $(\mathfrak{g}\otimes K)$-invariance.

**2.9.6.** Since $\bar{l}$ is continuous $l_i \to 0$ for $i \to -\infty$ (i.e., for every $n$ we have $l_{-i}(\Omega_{n,k}) = 0$ if $i$ is big enough). So the map $l \mapsto \bar{l}$ can be considered as a map from $U'_k/I_{0,k}$ to $W_k := \{\sum\limits_{i=-\infty}^{\infty} a_i\zeta^i | a_i \in \overline{U}'_k \, , a_i \to 0 \text{ for } i \to -\infty\}$. These maps define an operator

$$\Phi :\ \mathit{Vac}' \to W := \bigcup_k W_k \tag{33}$$

where $\mathit{Vac}' = U'/I_0$ is the twisted vacuum module. As explained in 2.9.5, $\Phi$ induces a map

$$\mathfrak{z}_{\mathfrak{g}}(O) \to \mathfrak{Z}\widehat{\otimes}\mathbb{C}((\zeta)) := \{\sum_{i=-\infty}^{\infty} a_i\zeta^i | a_i \in \mathfrak{Z} \, , a_i \to 0 \text{ for } i \to -\infty\} \, . \tag{34}$$

One can prove that (34) is a ring homomorphism (see ???). It is easy to see that the composition of (34) and the projection $\mathfrak{Z}\widehat{\otimes}\mathbb{C}((\zeta)) \to \mathfrak{z}_{\mathfrak{g}}(O)((\zeta))$ maps $\mathfrak{z}_{\mathfrak{g}}(O)$ to $\mathfrak{z}_{\mathfrak{g}}(O)[[\zeta]]$ and the composition $\mathfrak{z}_{\mathfrak{g}}(O) \to \mathfrak{z}_{\mathfrak{g}}(O)[[\zeta]] \xrightarrow{\zeta=0} \mathfrak{z}_{\mathfrak{g}}(O)$ is the identity.

*Remark.* Let $G$ be a connected algebraic group such that $\mathrm{Lie}\, G = \mathfrak{g}$. Then all elements of the image of (34) are $G(K)$-invariant (see the remark from 2.9.5).

**2.9.7.** One can show that (33) coincides with the operator $F: \mathit{Vac}' \to W$ constructed by Feigin and Frenkel (see the proof of Lemma 1 from [FF92]) and therefore 2.9.5 is just a version of a part of [FF92].

The definition of $F$ from [FF92] can be reformulated as follows. Set $W_k^+ := \overline{U}'_k((\zeta))$, $W_k^- := \{\sum\limits_i a_i\zeta^i \in W_k | a_{-i} = 0 \text{ for } i \text{ big enough}\}$. Define $W^\pm \subset W$ by $W^\pm = \bigcup\limits_k W_k^\pm$. $W^+$ and $W^-$ have natural algebra structures and $W$ has a natural structure of $(W^+, W^-)$-bimodule ($W$ is a left $W^+$-module and a right $W^-$-module). Consider the linear maps $\varphi^\pm: \widetilde{\mathfrak{g}\otimes K} \to W^\pm$ such that

$$\varphi^+(\mathbf{1}) = 1\,, \qquad \varphi^-(\mathbf{1}) = 0$$

and for $a \in \mathfrak{g}((t)) = \mathfrak{g}\otimes K \subset \widetilde{\mathfrak{g}\otimes K}$

$$\varphi^+(a) = a(t-\zeta) \in \mathfrak{g}((t))((\zeta))\,, \qquad \varphi^-(a) = a(t-\zeta) \in \mathfrak{g}((\zeta))((t))\,.$$

It is easy to show that $\varphi^\pm$ are Lie algebra homomorphisms. Consider the $\widetilde{\mathfrak{g}\otimes K}$-module structure on $W$ defined by $a \circ w := \varphi_+(a)w - w\varphi_-(a)$, $a \in \widetilde{\mathfrak{g}\otimes K}$, $w \in W$. Then $F: \mathit{Vac}' \to W$ is the $\widetilde{\mathfrak{g}\otimes K}$-module homomorphism that maps the vacuum vector from $\mathit{Vac}'$ to $1 \in W$.

**2.9.8.** Let us explain the relation between (34) and its classical analog from 2.4.2.

$\overline{U}'$ is equipped with the standard filtration $\overline{U}'_k$ (see 2.9.4). It induces the filtration $\mathfrak{Z}_k := \mathfrak{Z} \cap \overline{U}_k$. We identify $\operatorname{gr}_k \overline{U}' := \overline{U}'_k/\overline{U}'_{k-1}$ with the completion of $\operatorname{Sym}^k(\mathfrak{g}\otimes K)$, i.e., the space of homogeneous polynomial functions $\mathfrak{g}^*\otimes\omega_K \to \mathbb{C}$ of degree $k$ where $\omega_K := \omega_O \otimes_O K$ (a function $f$ on $\mathfrak{g}^*\otimes\omega_K$ is said to be polynomial if for every $n$ its restriction to $\mathfrak{g}^*\otimes\mathfrak{m}^{-n}$ is polynomial, i.e., comes from a polynomial function on $\mathfrak{g}^*\otimes(\mathfrak{m}^{-n}/\mathfrak{m}^N)$ for some $N$ depending on $n$). Denote by $\mathfrak{Z}^{cl}$ the algebra of $\mathfrak{g}\otimes K$-invariant polynomial functions on $\mathfrak{g}^*\otimes\omega_K$. Clearly the image of $\operatorname{gr}\mathfrak{Z}$ in $\operatorname{gr}\overline{U}'$ is contained in $\mathfrak{Z}^{cl}$.

The filtration of $\mathfrak{Z}$ induces a filtration of $\mathfrak{Z}\widehat{\otimes}\mathbb{C}((\zeta))$ and the map (34) is compatible with the filtrations. We claim that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{gr}_k\, \mathfrak{z}_{\mathfrak{g}}(O) & \longrightarrow & \mathrm{gr}_k\, \mathfrak{Z}\widehat{\otimes}\mathbb{C}((\zeta)) \\ \sigma\Big\downarrow & & \Big\downarrow \\ \mathfrak{z}_{\mathfrak{g}}^{cl}(O) & \xrightarrow{\nu} & \mathfrak{Z}^{cl}\widehat{\otimes}\mathbb{C}((\zeta)) \end{array} \tag{35}$$

Here the upper arrow is induced by (34), $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ was defined in 2.4.1, $\sigma$ is the symbol map from 1.2.5, and $\nu$ is defined by

(36)

$$\nu(f) := h(\zeta)\,, \quad (h(\zeta))(\varphi) := f(\varphi(\zeta+t))$$
$$f \in \mathfrak{z}_{\mathfrak{g}}^{cl}(O)\,,\ \varphi \in \mathfrak{g}^*\otimes\omega_K\,,\ \varphi(\zeta+t) \in \mathfrak{g}^*((\zeta))[[t]]dt = (\mathfrak{g}^*\otimes\omega_O)\widehat{\otimes}\mathbb{C}((\zeta))\,.$$

Here $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is identified with the algebra of $\mathfrak{g}\otimes O$-invariant polynomial functions on $\mathfrak{g}^*\otimes\omega_O$ (cf.2.4.1). The map $\nu$ was considered in the Remark from 2.4.2.

The commutativity of (35) follows from the commutativity of the diagram

$$\begin{array}{ccc} (U'_k/I_{n,k})^* & \xrightarrow{\sim} & \Omega_{n,k} \\ \sigma^*\Big\uparrow & & \Big\uparrow \\ (\mathrm{Sym}^k(\mathfrak{g}\otimes K/\mathfrak{g}\otimes\mathfrak{m}^n))^* & \xrightarrow{\sim} & ((\mathfrak{m}^{-n}\omega_O)^{\otimes k})^{S_k} \end{array} \tag{37}$$

Here the upper arrow is dual to (30), $\sigma : U'_k/I_{n,k} \to \mathrm{Sym}^k(\mathfrak{g}\otimes K/\mathfrak{g}\otimes\mathfrak{m}^n)$ is the symbol map, and the right vertical arrow is defined by $w \mapsto (0,\dots,0,w)$. The commutativity of (37) is an immediate consequence of the definition of (30); see [BD94].

**2.10. Geometry of $T^*\mathrm{Bun}_G$.** This subsection should be considered as an appendix; the reader may certainly skip it.

Set $\mathrm{Nilp} = \mathrm{Nilp}(G) := p^{-1}(0)$ where $p : T^*\mathrm{Bun}_G \to \mathrm{Hitch}(X)$ is the Hitchin fibration (see 2.2.3). Nilp was introduced in [La87] and [La88] under the name of *global nilpotent cone* (if $\mathcal{F}$ is a $G$-bundle on $X$ and $\eta \in T^*_{\mathcal{F}}\mathrm{Bun}_G = H^0(X, \mathfrak{g}^*_{\mathcal{F}}\otimes\omega_X)$ then $(\mathcal{F},\eta) \in \mathrm{Nilp}$ if and only if the image of $\eta$ in $H^0(X, \mathfrak{g}_{\mathcal{F}}\otimes\omega_X)$ is nilpotent).

In 2.10.1 we show that Proposition 2.2.4 (iii) easily follows from the equality

$$\dim \mathrm{Nilp} = \dim \mathrm{Bun}_G\,. \tag{38}$$

We also deduce from (38) that $\mathrm{Bun}_G$ is good in the sense of 1.1.1. The equality (38) was proved by Faltings and Ginzburg; in the particular case $G = PSL_n$ it had been proved by Laumon. In 2.10.2 we give some comments on their proofs. In 2.10.3 we discuss the set of irreducible components of Nilp. In 2.10.4 we show that Nilp is equidimensional even if the genus of $X$ equals 0 or 1 (if $g > 1$ this follows from 2.2.4 (iii)). In 2.10.5 we prove that $\mathrm{Bun}_G$ is very good in the sense of 1.1.1.

We will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using an invariant scalar product on $\mathfrak{g}$.

**2.10.1.** Assuming (38) we are going to prove 2.2.4 (iii) and show that $\mathrm{Bun}_G$ is good in the sense of 1.1.1. Let $U \subset T^*\mathrm{Bun}_G$ be the biggest open substack such that $\dim U \leq 2\dim \mathrm{Bun}_G$. (38) means that the fiber of $p : T^*\mathrm{Bun}_G \to \mathrm{Hitch}(X)$ over 0 has dimension $\dim \mathrm{Bun}_G$. Since $\dim \mathrm{Hitch}(X) = \dim \mathrm{Bun}_G$ this implies that $U \supset p^{-1}(0)$. $U$ is invariant with respect to the natural action of $\mathbb{G}_m$ on $T^*\mathrm{Bun}_G$. Therefore $U = T^*\mathrm{Bun}_G$. So $\dim T^*\mathrm{Bun}_G \leq 2\dim \mathrm{Bun}_G$. According to 1.1.1 this means that $\mathrm{Bun}_G$ is good and $T^*\mathrm{Bun}_G$ is a locally complete intersection of pure dimension $2\dim \mathrm{Bun}_G$.

For an open $V \subset T^*\mathrm{Bun}_G$ the following properties are equivalent: 1) the restriction of $p$ to $V$ is flat, 2) the fibers of this restriction have dimension $\dim \mathrm{Bun}_G$. Let $V_{\max}$ be the maximal $V$ with these properties. $V_{\max}$ is $\mathbb{G}_m$-invariant and according to (38) $V_{\max} \supset p^{-1}(0)$. So $V_{\max} = T^*\mathrm{Bun}_G$ and we have proved the first statement of 2.2.4 (iii). It implies that the image of $p^\gamma$ is open. On the other hand it is $\mathbb{G}_m$-invariant and contains 0. So $p^\gamma$ is surjective. QED.

Since Nilp contains the zero section of $T^*\mathrm{Bun}_G$ (38) follows from the inequality $\dim \mathrm{Nilp} \leq \dim \mathrm{Bun}_G$, which was obtained in [La88], [Fal93], [Gi97] as a corollary of the following theorem.

**2.10.2.** *Theorem.* **(**[La88]**,** [Fal93]**,** [Gi97]**).** Nilp is isotropic.

*Remarks*

(i) Let us explain that a subscheme $N$ of a smooth symplectic variety $M$ is said to be *isotropic* if any smooth subvariety of $N$ is isotropic. One can show that $N$ is isotropic if and only if the set of nonsingular points of $N_{\mathrm{red}}$ is isotropic. $N$ is said to be *Lagrangian* if it is isotropic and $\dim_x N = \dfrac{1}{2}\dim_x M$ for all $x \in N$. If $\mathcal{Y}$ is a smooth algebraic stack then a substack $\mathcal{N} \subset T^*\mathcal{Y}$ is said to be isotropic (resp. Lagrangian) if $\mathcal{N} \times_{\mathcal{Y}} S \subset (T^*\mathcal{Y}) \times_{\mathcal{Y}} S \subset T^*S$ is isotropic (resp. Lagrangian) for some presentation[14]
$S \to \mathcal{Y}$ (then it is true for all presentations $S \to \mathcal{Y}$).

(ii) The proofs of Theorem 2.10.2 given in [Fal93] and [Gi97] do not use the assumption $g > 1$ where $g$ is the genus of $X$. If $g > 1$ then Faltings and Ginzburg show that Nilp is Lagrangian. Their argument was explained in 2.10.1: (38) implies that Nilp has *pure* dimension $\dim \mathrm{Bun}_G$. In 2.10.1 we used the equality $\dim \mathrm{Hitch}(X) = \dim \mathrm{Bun}_G$, which holds only if $g > 1$. In fact Nilp is Lagrangian even if $g = 0, 1$ (see 2.10.4).

(iii) Since $\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$ is Lagrangian and $\mathbb{G}_m$-invariant it is a union of conormal bundles to certain reduced irreducible closed substacks of $\mathrm{Bun}_G$. For $G = PSL_n$ a description of some of these substacks was obtained by Laumon (see §§3.8–3.9 from [La88]).

(iv) Ginzburg's proof of Theorem 2.10.2 is based on the following interpretation of Nilp in terms of $\pi : \mathrm{Bun}_B \to \mathrm{Bun}_G$ where $B$ is a Borel subgroup of $G$: if $\mathcal{F} \in \mathrm{Bun}_G$, $\eta \in T^*_{\mathcal{F}}\mathrm{Bun}_G$ then $(\mathcal{F}, \eta) \in \mathrm{Nilp}$

[14]A presentation of $\mathcal{Y}$ is a smooth surjective morphism $S \to \mathcal{Y}$ where $S$ is a scheme.

if and only if there is an $\mathcal{E} \in \pi^{-1}(\mathcal{F})$ such that the image of $\eta$ in $T^*_{\mathcal{E}} \operatorname{Bun}_B$ equals 0. This interpretation enables Ginzburg to prove Theorem 2.10.2 using a simple and general argument from symplectic geometry (see §§6.5 from [Gi97]). Falting's proof of Theorem 2.10.2 is also very nice and short (see the first two paragraphs of the proof of Theorem II.5 from [Fal93]).

(v) The proof of Theorem 2.10.2 for $G = PSL_n$ given in [La88] does not work in the general case because it uses the following property of $\mathfrak{g} = sl_n$: for every nilpotent $A \in \mathfrak{g}$ there is a parabolic subgroup $P \subset G$ such that $A$ belongs to the Lie algebra of the unipotent radical $U \subset P$, the $P$-orbit of $A$ is open in $\operatorname{Lie} U$, and the centralizer of $A$ in $G$ is contained in $P$. This property holds for $\mathfrak{g} = sl_n$ (e.g., one can take for $P$ the stabilizer of the flag $0 \subset \operatorname{Ker} A \subset \operatorname{Ker} A^2 \subset \dots$) but not for an arbitrary semisimple $\mathfrak{g}$ (e.g., it does not hold if $\mathfrak{g} = sp_4$ and $A \in sp_4$ is a nilpotent operator of rank 1).

**2.10.3.** In this subsection we "describe" the set of irreducible components of Nilp.

Recall that Nilp is the stack of pairs $(\mathcal{F}, \eta)$ where $\mathcal{F}$ is a $G$-bundle on $X$ and $\eta \in H^0(X, \mathfrak{g}_{\mathcal{F}} \otimes \omega_X) = H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X)$ is nilpotent. For a nilpotent conjugacy class $C \subset \mathfrak{g}$ we have the locally closed substack $\operatorname{Nilp}_C$ parametrizing pairs $(\mathcal{F}, \eta)$ such that $\eta(x) \in C$ for generic $x \in X$.

Fix some $e \in C$ and include it into an $sl_2$-triple $\{e, f, h\}$. Let $\mathfrak{g}^k$ be the decreasing filtration of $\mathfrak{g}$ such that $[h, \mathfrak{g}^k] \subset \mathfrak{g}^k$ and $\operatorname{ad}_h$ acts on $\mathfrak{g}^k/\mathfrak{g}^{k+1}$ as multiplication by $k$. $\mathfrak{g}^k$ depend on $e$ but not on $h$ and $f$. Set $\mathfrak{p} = \mathfrak{p}_e := \mathfrak{g}^0$. $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. Let $P \subset G$ be the corresponding subgroup. We have the map $C \to G/P$ that associates to $a \in C$ the parabolic subalgebra $\mathfrak{p}_a$. Its fiber $\{a \in C | \mathfrak{p}_a = \mathfrak{p}\}$ (i.e., the $P$-orbit of $e \in C$) equals $\mathfrak{g}^2 \cap C$; this is an open subset of $\mathfrak{g}^2$. An element of $\mathfrak{g}^2$ is said to be *generic* if it belongs to $\mathfrak{g}^2 \cap C$.

Let $(\mathcal{F},\eta)\in \mathrm{Nilp}_C$, $U:=\{x\in X|\eta(x)\in C\}$. The image of $\eta\in\Gamma(U,C_{\mathcal{F}}\otimes\omega_X)$ in $\Gamma(U,(G/P)_{\mathcal{F}})$ extends to a section of $(G/P)_{\mathcal{F}}$ over $X$. So we obtain a $P$-structure on $\mathcal{F}$. In terms of this $P$-structure $\eta\in H^0(X,\mathfrak{g}^2_{\mathcal{F}}\otimes\omega_X)$ and $\eta(x)$ is generic for $x\in U$.

Denote by $Y_C$ the stack of pairs $(\mathcal{F},\eta)$ where $\mathcal{F}$ is a $P$-bundle on $X$ and $\eta\in H^0(X,\mathfrak{g}^2_{\mathcal{F}}\otimes\omega_X)$ is such that $\eta(x)$ is generic for almost all $x\in X$. For a $P$-bundle $\mathcal{F}$ let $\deg\mathcal{F}\in\mathrm{Hom}(P,\mathbb{G}_m)^*$ be the functional that associates to $\varphi: P\to\mathbb{G}_m$ the degree of the push-forward of $\mathcal{F}$ by $\varphi$. $Y_C$ is the disjoint union of open substacks $Y^u_C$, $u\in\mathrm{Hom}(P,\mathbb{G}_m)^*$, parametrizing pairs $(\mathcal{F},\eta)\in Y_C$ such that $\deg\mathcal{F}=u$. It is easy to show that for each $u\in\mathrm{Hom}(P,\mathbb{G}_m)^*$ the natural morphism $Y^u_C\to\mathrm{Nilp}_C$ is a locally closed embedding and the substacks $Y^u_C\subset\mathrm{Nilp}_C$ form a stratification of $\mathrm{Nilp}_C$.

*Lemma.*

1) $Y^u_C$ is a smooth equidimensional stack. $\dim Y^u_C\leq\dim\mathrm{Bun}_G$ .
2) Let $Y^*_C$ be the union of connected components of $Y_C$ of dimension $\dim\mathrm{Bun}_G$. Then $Y^*_C$ is the stack of pairs $(\mathcal{F},\eta)\in Y_C$ such that $\mathrm{ad}_\eta:(\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{F}}\to(\mathfrak{g}^1/\mathfrak{g}^2)_{\mathcal{F}}\otimes\omega_X$ is an isomorphism.

*Remark.* (38) follows from the lemma.

*Proof.* The deformation theory of $(\mathcal{F},\eta)\in Y^u_C$ is controled by the hypercohomology of the complex $C^\cdot$ where $C^0=\mathfrak{p}_{\mathcal{F}}=\mathfrak{g}^0_{\mathcal{F}}$, $C^1=\mathfrak{g}^2_{\mathcal{F}}\otimes\omega_X$, $C^i=0$ for $i\neq 0,1$, and the differential $d: C^0\to C^1$ equals $\mathrm{ad}_\eta$. Since $\mathrm{Coker}\, d$ has finite support $\mathbb{H}^2(X,C^\cdot)=0$. So $Y_C$ is smooth and

$$\begin{aligned}\dim_{(\mathcal{F},\eta)}Y_C&=\chi(\mathfrak{g}^2_{\mathcal{F}}\otimes\omega_X)-\chi(\mathfrak{g}^0_{\mathcal{F}})=-\chi(\mathfrak{g}_{\mathcal{F}}/\mathfrak{g}^{-1}_{\mathcal{F}})-\chi(\mathfrak{g}^0_{\mathcal{F}})\\&=-\chi(\mathfrak{g}_{\mathcal{F}})+\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})=\dim\mathrm{Bun}_G+\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})\,.\end{aligned}$$

Clearly $\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})$ depends only on $u=\deg\mathcal{F}$. The morphism $\mathrm{ad}_\eta$ : $\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}}\to(\mathfrak{g}^1/\mathfrak{g}^2)_{\mathcal{F}}\otimes\omega_X$ is injective and its cokernel $\mathcal{A}$ has finite support. So $2\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})=\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})-\chi((\mathfrak{g}^1/\mathfrak{g}^2)_{\mathcal{F}}\otimes\omega_X)=-\chi(\mathcal{A})\leq 0$ and $\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})=0$ if and only if $\mathcal{A}=0$. $\square$

Since Nilp has pure dimension $\dim \mathrm{Bun}_G$ the lemma implies that the irreducible components of Nilp are parametrized by $\coprod\limits_C \pi_0(Y_C^*)$.

$\pi_0(Y_C^*)$ can be identified with $\pi_0$ of a simpler stack $M_C$ defined as follows. Set $L = P/U$ where $U$ is the unipotent radical of $P$. $L$ acts on $V := \mathfrak{g}^2/\mathfrak{g}^3$. Denote by $D_i$ the set of $a \in V$ such that the determinant of $(\mathrm{ad}_a)^i : \mathfrak{g}^{-i}/\mathfrak{g}^{-i+1} \to \mathfrak{g}^i/\mathfrak{g}^{i+1}$ equals 0. $D_i \subset V$ is an $L$-invariant closed subset of pure codimension 1. An element of $\mathfrak{g}^2$ is generic if and only if its image in $V$ does not belong to $D_2$. Therefore $D_i \subset D_2$ for all $i$. Denote by $M_C$ the stack of pairs $(\mathcal{F}, \eta)$ where $\mathcal{F}$ is an $L$-bundle on $X$ and $\eta \in H^0(X, V_{\mathcal{F}} \otimes \omega_X)$ is such that $\eta(x) \notin D_1$ for all $x \in X$ and $\eta(x) \notin D_2$ for generic $x \in X$. It is easy to see that the natural morphism $Y_C^* \to M_C$ induces a bijection $\pi_0(Y_C^*) \to \pi_0(M_C)$.

So irreducible components of Nilp are parametrized by $\coprod\limits_C \pi_0(M_C)$. Hopefully $\pi_0(M_C)$ can be described in terms of "standard" objects associated to $C$ and $X$ . . .

*Remark.* If $G = PSL_n$ then $\mathrm{Nilp}_C$ has pure dimension $\dim \mathrm{Bun}_G$ for every nilpotent conjugacy class $C \subset sl_n$ (see [La88]). This is not true, e.g., if $G = Sp_4$ and $C$ is the set of nilpotent matrices from $sp_4$ of rank 1. Indeed, let $(\mathcal{F}, \eta) \in Y_C$ be such that $\eta \in H^0(X, \mathfrak{g}^2_{\mathcal{F}} \otimes \omega_X)$ has only simple zeros. Then it is easy to show that the morphism $Y_C \to \mathrm{Nilp}_C$ is an open embedding in a neighbourhood of $(\mathcal{F}, \eta)$. On the other hand it follows from the above lemma that if $\eta$ has a zero then the dimension of $Y_C$ at $(\mathcal{F}, \eta)$ is less than $\mathrm{Bun}_G$.

**2.10.4.** *Theorem.* Nilp is Lagrangian.

In this theorem we do not assume that $g > 1$.

*Proof.* As explained in Remark (ii) from 2.10.2 we only have to show that Nilp has pure dimension $\dim \mathrm{Bun}_G$ for $g \leq 1$.

1) Let $g = 0$. Then $\mathrm{Nilp} = T^*\mathrm{Bun}_G$. A quasicompact open substack of $\mathrm{Bun}_G$ can be represented as $H \backslash M$ where $M$ is a smooth variety and $H$ is an

algebraic group acting on $M$. Then $T^*(H\backslash M) = H\backslash N$ where $N \subset T^*M$ is the union of the conormal bundles of the orbits of $H$. Each conormal bundle has pure dimension $\dim M$ and since $g = 0$ the number of $H$-orbits is finite.

*Remark.* Essentially the same argument shows that for any smooth algebraic stack $\mathcal{Y}$ the dimension of $T^*\mathcal{Y}$ at each point is $\geq \dim \mathcal{Y}$. If $g = 0$ and $\mathcal{Y} = \mathrm{Bun}_G$ then $T^*\mathcal{Y} = \mathrm{Nilp}$ and $\dim T^*\mathcal{Y} = \dim \mathcal{Y}$ according to Theorem 2.10.2. So we have again proved Theorem 2.10.4 for $g = 0$.

2) Let $g = 1$. It is convenient to assume $G$ reductive but not necessarily semisimple (this is not really essential because Theorem 2.10.4 for reductive $G$ easily follows from the semisimple case).

Before proceeding to the proof let us recall the notions of semistability and Shatz stratification. Fix a Borel subgroup $B \subset G$ and denote by $H$ its maximal abelian quotient. Let $P \subset G$ be a parabolic subgroup containing $B$, $L$ the maximal reductive quotient of $P$, $Z$ the center of $L$. Let $\Gamma$ (resp. $\Delta$) be the set of simple roots of $G$ (resp. $L$). The embedding $Z \hookrightarrow L$ induces an isomorphism $\mathrm{Hom}(Z, \mathbb{G}_m) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{Hom}(L, \mathbb{G}_m) \otimes \mathbb{Q}$. Denote by $p$ the composition $\mathrm{Hom}(H, \mathbb{G}_m) \to \mathrm{Hom}(Z, \mathbb{G}_m) \to \mathrm{Hom}(L, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q}$. We say that $l \in \mathrm{Hom}(P, \mathbb{G}_m)^*$ is *strictly dominant* if $l(p(\alpha)) > 0$ for $\alpha \in \Gamma \backslash \Delta$.

For a $P$-bundle $\mathcal{F}$ let $\deg \mathcal{F} \in \mathrm{Hom}(P, \mathbb{G}_m)^*$ be the functional that associates to $\varphi : P \to \mathbb{G}_m$ the degree of the push-forward of $\mathcal{F}$ by $\varphi$. A $G$-bundle is said to be *semistable* if it does not come from a $P$-bundle of strictly dominant degree for any $P \neq G$. Semistable $G$-bundles form an open substack $\mathrm{Bun}_G{}^{ss} \subset \mathrm{Bun}_G$. Semistable $G$-bundles of fixed degree $d \in \mathrm{Hom}(G, \mathbb{G}_m)$ form an open substack $\mathrm{Bun}_G{}^{ss,d} \subset \mathrm{Bun}_G{}^{ss}$. If $P \subset G$ is a parabolic subgroup containing $B$ and $d \in \mathrm{Hom}(P, \mathbb{G}_m)^*$ is strictly dominant denote by $\mathrm{Shatz}_P^d$ the stack of $P$-bundles $\mathcal{F}$ of degree $d$ such that the corresponding $L$-bundle is semistable. It is known that the natural morphism $\mathrm{Shatz}_P^d \to \mathrm{Bun}_G$ is a locally closed embedding and the substacks

$\mathrm{Shatz}_P^d$ for all $P$, $d$ form a stratification of $\mathrm{Bun}_G$, which is called the *Shatz stratification*.

Denote by $\mathrm{Nilp}_P^d(G)$ (resp. $\mathrm{Nilp}^{ss}(G)$, $\mathrm{Nilp}^{ss,d}(G)$) the fibered product of $\mathrm{Nilp} = \mathrm{Nilp}(G)$ and $\mathrm{Shatz}_P^d$ (resp. ${\mathrm{Bun}_G}^{ss}$, ${\mathrm{Bun}_G}^{ss,d}$) over $\mathrm{Bun}_G$. To show that $\mathrm{Nilp}(G)$ has pure dimension $\dim \mathrm{Bun}_G = 0$ it is enough to show that $\mathrm{Nilp}_P^d(G)$ has pure dimension 0 for each $P$ and $d$. Let $L$ be the maximal reductive quotient of $P$, $\mathfrak{p} := \mathrm{Lie}\, P$, $\mathfrak{l} := \mathrm{Lie}\, L$. If $\mathcal{F}$ is a $P$-bundle of strictly dominant degree such that the corresponding $L$-bundle $\overline{\mathcal{F}}$ is semistable then $H^0(X, \mathfrak{g}_{\mathcal{F}}) = H^0(X, \mathfrak{p}_{\mathcal{F}})$, so we have the natural map $\eta \mapsto \bar{\eta}$ from $H^0(X, \mathfrak{g}_{\mathcal{F}})$ to $H^0(X, \mathfrak{l}_{\mathcal{F}})$. Define $\pi : \mathrm{Nilp}_P^d(G) \to \mathrm{Nilp}^{ss,d}(L)$ by $(\mathcal{F}, \eta) \mapsto (\overline{\mathcal{F}}, \bar{\eta})$, $\eta \in H^0(X, \mathfrak{g}_{\mathcal{F}} \otimes \omega_X) = H^0(X, \mathfrak{g}_{\mathcal{F}})$ ($\omega_X$ is trivial because $g = 1$). Using again that $g = 1$ one shows that $\pi$ is smooth and its fibers are 0-dimensional stacks. So it is enough to show that $\mathrm{Nilp}^{ss}(L)$ is of pure dimension 0.

A point of $\mathrm{Nilp}^{ss}(L)$ is a pair consisting of a semistable $L$-bundle $\mathcal{F}$ and a nilpotent $\eta \in H^0(X, \mathfrak{l}_{\mathcal{F}})$. Since $\mathfrak{l}_{\mathcal{F}}$ is a semistable vector bundle $\mathrm{ad}_\eta : \mathfrak{l}_{\mathcal{F}} \to \mathfrak{l}_{\mathcal{F}}$ has constant rank. So the conjugacy class of $\eta(x)$ does not depend on $x \in X$. For a nilpotent conjugacy class $C \subset \mathfrak{l}$ denote by $\mathrm{Nilp}_C^{ss}(L)$ the locally closed substack of $\mathrm{Nilp}^{ss}(L)$ parametrizing pairs $(\mathcal{F}, \eta)$ such that $\eta(x) \in C$. It is enough to show that $\mathrm{Nilp}_C^{ss}(L)$ has pure dimension 0 for each $C$. Let $Z(A) \subset L$ be the centralizer of some $A \in C$, $\mathfrak{z}(A) := \mathrm{Lie}\, Z(A)$. If $(\mathcal{F}, \eta) \in \mathrm{Nilp}_C^{ss}(L)$ then $\eta \in \Gamma(X, C_{\mathcal{F}}) = \Gamma(X, (G/Z(A))_{\mathcal{F}})$ defines a $Z(A)$-structure on $\mathcal{F}$. Thus we obtain an open embedding $\mathrm{Nilp}_C^{ss}(L) \hookrightarrow \mathrm{Bun}_{Z(A)}$. Finally $\mathrm{Bun}_{Z(A)}$ has pure dimension 0 because for any $Z(A)$-bundle $\mathcal{E}$ one has $\chi(\mathfrak{z}(A)_{\mathcal{E}}) = \deg \mathfrak{z}(A)_{\mathcal{E}} = 0$ (notice that since $G/Z(A) = C$ has a $G$-invariant symplectic structure the adjoint representation of $Z(A)$ has trivial determinant and therefore $\mathfrak{z}(A)_{\mathcal{E}}$ is trivial). $\square$

**2.10.5.** *Proof of Proposition 2.1.2.* We must prove that (4) holds for $\mathcal{Y} = \mathrm{Bun}_G$, i.e., $\operatorname{codim}\{\mathcal{F} \in \mathrm{Bun}_G | \dim H^0(X, \mathfrak{g}_{\mathcal{F}}) = n\} > n$ for all $n > 0$.

This is equivalent to proving that

$$\dim(A(G)\setminus A^0(G)) < \dim \mathrm{Bun}_G \tag{39}$$

where $A(G)$ is the stack of pairs $(\mathcal{F}, s)$, $\mathcal{F} \in \mathrm{Bun}_G$, $s \in H^0(X, \mathfrak{g}_{\mathcal{F}})$, and $A^0(G) \subset A(G)$ is the closed substack defined by the equation $s = 0$. Set $C := \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{g}^*)^G$. This is the affine scheme quotient of $\mathfrak{g}$ with respect to the adjoint action of $G$; in fact $C = W\backslash\mathfrak{h}$ where $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$ and $W$ is the Weyl group. The morphism $\mathfrak{g} \to C$ induces a map $H^0(X, \mathfrak{g}_{\mathcal{F}}) \to \mathrm{Mor}(X, C) = C$. So we have a canonical morphism $f : A(G) \to C = W\backslash\mathfrak{h}$. For $h \in \mathfrak{h}$ set $A_h(G) = f^{-1}(\bar{h})$ where $\bar{h} \in W\backslash\mathfrak{h}$ is the image of $h$. Set $G^h := \{g \in G | ghg^{-1} = h\}$, $\mathfrak{g}^h := \mathrm{Lie}\, G^h = \{a \in \mathfrak{g} | [a, h] = 0\}$. Denote by $\mathfrak{z}_h$ the center of $\mathfrak{g}^h$. Since $h \in \mathfrak{z}_h$ and there is a finite number of subalgebras of $\mathfrak{g}$ of the form $\mathfrak{z}_h$ (39) follows from the inequality $\dim(A_h(G)\setminus A^0(G)) < \dim \mathrm{Bun}_G - \dim \mathfrak{z}_h$. So it is enough to prove that

$$\dim A_h(G) < \dim \mathrm{Bun}_G - \dim \mathfrak{z}_h \quad \text{for} \quad h \neq 0 \tag{40}$$

$$\dim(A_0(G)\setminus A^0(G)) < \dim \mathrm{Bun}_G\,. \tag{41}$$

Denote by $Z_h$ the center of $G^h$. Let us show that (40) follows from the inequality (41) with $G$ replaced by $G^h/Z_h$. Indeed, we have the natural isomorphisms $A_0(G^h) \xrightarrow{\sim} A_h(G^h) \xrightarrow{\sim} A_h(G)$ and the obvious morphism $\varphi : A_0(G^h) \to A_0(G^h/Z_h)$. A non-empty fiber of $\varphi$ is isomorphic to $\mathrm{Bun}_{Z_h}$, so $\dim A_h(G) \leq \dim \mathrm{Bun}_{Z_h} + \dim A_0(G^h/Z_h)$. Since $\dim \mathrm{Bun}_{Z_h} = (g-1)\cdot \dim \mathfrak{z}_h$ and (41) implies that $\dim A_0(G^h/Z_h) = (g-1)\cdot\dim(\mathfrak{g}^h/\mathfrak{z}_h)$ we have $\dim A_h(G) \leq (g-1)\cdot\dim\mathfrak{g}^h = \dim\mathrm{Bun}_G - (g-1)\cdot\dim(\mathfrak{g}/\mathfrak{g}^h) \leq \dim\mathrm{Bun}_G - \dim(\mathfrak{g}/\mathfrak{g}^h)$. Finally $\dim(\mathfrak{g}/\mathfrak{g}^h) \geq 2\cdot\dim\mathfrak{z}_h > \dim\mathfrak{z}_h$ if $h \neq 0$.

To prove (41) we will show that if $Y \subset A_0(G)$ is a locally closed reduced irreducible substack then $\dim Y \leq \dim \mathrm{Bun}_G$ and $\dim Y = \dim \mathrm{Bun}_G$ only if $Y \subset A^0(G)$. For $\xi \in H^0(X, \omega_X)$ consider the morphism $m_\xi : A_0(G) \to \mathrm{Nilp}$ defined by $(\mathcal{F}, s) \mapsto (\mathcal{F}, s\xi)$, $\mathcal{F} \in \mathrm{Bun}_G$, $s \in H^0(X, \mathfrak{g}_{\mathcal{F}})$. The morphisms

$m_\xi$ define $m: A_0(G) \times H^0(X, \omega_X) \to \mathrm{Nilp}$. The image of $m$ is contained in some locally closed reduced irreducible substack $Z \subset \mathrm{Nilp}$. If $\xi \neq 0$ then $m_\xi$ induces an embedding $Y \hookrightarrow Z_\xi$ where $Z_\xi$ is the closed substack of $Z$ consisting of pairs $(\mathcal{F}, \eta) \in H^0(X, \mathfrak{g}_\mathcal{F} \otimes \omega_X)$ such that the restriction of $\eta$ to the subscheme $D_\xi := \{x \in X | \xi(x) = 0\}$ is zero. So $\dim Y \leq \dim Z_\xi \leq \dim Z \leq \dim \mathrm{Nilp} = \dim \mathrm{Bun}_G$. If $\dim Y = \dim \mathrm{Bun}_G$ then $Z_\xi = Z$ for all nonzero $\xi \in H^0(X, \omega_X)$. This means that $\eta = 0$ for all $(\mathcal{F}, \eta) \in Z$ and therefore $s = 0$ for all $(\mathcal{F}, s) \in Y$, i.e., $Y \subset A^0(G)$. $\square$

**2.11. On the stack of local systems.** Denote by $\mathcal{LS}_G$ the stack of $G$-local systems on $X$ (a $G$-local system is a $G$-bundle with a connection). Kapranov [Kap97] explained that $\mathcal{LS}_G$ has a derived version $R\mathcal{LS}_G$, which is a DG stack. Using the results of 2.10 we will show that if $g > 1$ and $G$ is semisimple then $R\mathcal{LS}_G = \mathcal{LS}_G$. We also describe the set of irreducible components of $\mathcal{LS}_G$. This section may be skipped by the reader; its results are not used in the rest of the work.

**2.11.1.** Fix $x \in X$. Denote by $\mathcal{LS}_G^x$ the stack of $G$-biundles $\mathcal{F}$ on $X$ equipped with a connection $\nabla$ having a simple pole at $x$. Denote by $\mathcal{E}$ the restriction to $\mathcal{LS}_G^x = \mathcal{LS}_G^x \times \{x\}$ of the universal $G$-bundle on $\mathcal{LS}_G^x \times X$. The residue of $\nabla$ at $x$ is a section $R \in \Gamma(\mathcal{LS}_G^x, \mathfrak{g}_\mathcal{E})$, and $\mathcal{LS}_G$ is the closed substack of $\mathcal{LS}_G^x$ defined by the equation $R = 0$. Consider the open substack $\widetilde{\mathcal{LS}}_G^x \subset \mathcal{LS}_G^x$ parametrizing pairs $(\mathcal{F}, \nabla)$ such that $\nabla : H^1(X, \mathfrak{g}_\mathcal{F}) \to H^1(X, \mathfrak{g}_\mathcal{F} \otimes \omega_X(x))$ is surjective. It is easy to see that $\widetilde{\mathcal{LS}}_G^x$ is a smooth stack of pure dimension $(2g-1) \cdot \dim G$ and $\mathcal{LS}_G \subset \widetilde{\mathcal{LS}}_G^x$.

Consider $\mathfrak{g}_\mathcal{E}$ as a stack over $\mathcal{LS}_G^x$. The sections $R, 0 \in \Gamma(\mathcal{LS}_G^x, \mathfrak{g}_\mathcal{E})$ define two closed substacks of $\mathfrak{g}_\mathcal{E}$, and $R\mathcal{LS}$ is their intersection in the derived sense while $\mathcal{LS}_G$ is their usual intersection. So the following conditions are equivalent:

1) $R\mathcal{LS}_G = \mathcal{LS}_G$;

2) $\mathcal{LS}_G$ is a locally complete intersection of pure dimension $(2g-2)\cdot\dim G$;
3) $\dim\mathcal{LS}_G \leq (2g-2)\cdot\dim G$.

The following proposition shows that these conditions are satisfied if $g>1$ and $G$ is semisimple.

**2.11.2.** *Proposition.* Suppose that $g>1$ and $G$ is reductive. Then $\mathcal{LS}_G$ is a locally complete intersection of pure dimension $(2g-2)\cdot\dim G+l$ where $l$ is the dimension of the center of $G$.

*Proof.* Let $R$ have the same meaning as in 2.11.1. Clearly $R\in\Gamma(\mathcal{LS}_G^x,[\mathfrak{g},\mathfrak{g}]_{\mathcal{E}})$, so it suffices to show that

$$\dim\mathcal{LS}_G \leq (2g-2)\cdot\dim G+l. \tag{42}$$

Denote by $G_{\mathrm{ad}}$ the quotient of $G$ by its center. Consider the projection $p:\mathcal{LS}_G\to\mathrm{Bun}_{G_{\mathrm{ad}}}$. If the fiber of $p$ over a $G_{\mathrm{ad}}$-bundle $\mathcal{F}$ is not empty then its dimension equals $\dim T^*_{\mathcal{F}}\mathrm{Bun}_{G_{\mathrm{ad}}}+l(2g-1)$, so $\dim\mathcal{LS}_G\leq\dim T^*\mathrm{Bun}_{G_{\mathrm{ad}}}+l(2g-1)$. Finally $\dim T^*\mathrm{Bun}_{G_{\mathrm{ad}}}\leq\dim G_{\mathrm{ad}}\cdot(2g-2)$ because $\mathrm{Bun}_{G_{\mathrm{ad}}}$ is good in the sense of 1.1.1 (we proved this in 2.10.1). □

**2.11.3.** Let $\mathrm{Bun}'_G\subset\mathrm{Bun}_G$ denote the preimage of the connected component of $\mathrm{Bun}_{G/[G,G]}$ containing the trivial bundle. The image of $\mathcal{LS}_G\to\mathrm{Bun}_G$ is contained in $\mathrm{Bun}'_G$.

**2.11.4.** *Proposition.* Suppose that $g>1$ and $G$ is reductive. Then the preimage in $\mathcal{LS}_G$ of every connected component of $\mathrm{Bun}'_G$ is non-empty and irreducible.

So irreducible components of $\mathcal{LS}_G$ are parametrized by

$$\mathrm{Ker}(\pi_1(G)\to\pi_1(G/[G,G]))=\pi_1([G,G]).$$

*Proof.* Consider the open substack $\mathrm{Bun}^0_{G_{\mathrm{ad}}}\subset\mathrm{Bun}_{G_{\mathrm{ad}}}$ parametrizing $G_{\mathrm{ad}}$-bundles $\mathcal{F}$ such that $H^0(X,(\mathfrak{g}_{\mathrm{ad}})_{\mathcal{F}})=0$ (this is the biggest Deligne-Mumford substack of $\mathrm{Bun}_{G_{\mathrm{ad}}}$). Denote by $\mathrm{Bun}^{0\prime}_G$ the preimage of $\mathrm{Bun}^0_{G_{\mathrm{ad}}}$ in $\mathrm{Bun}'_G$.

Let $\mathcal{LS}_G^0$ denote the preimage of $\mathrm{Bun}_{G_{\mathrm{ad}}}^0$ in $\mathcal{LS}_G$. In 2.10.5 we proved that $\mathrm{Bun}_{G_{\mathrm{ad}}}$ is very good in the sense of 1.1.1, so $\dim(T^*\mathrm{Bun}_{G_{\mathrm{ad}}} \setminus T^*\mathrm{Bun}_{G_{\mathrm{ad}}}^0) < \dim T^*\mathrm{Bun}_{G_{\mathrm{ad}}}$. The argument used in the proof of (42) shows that $\dim(\mathcal{LS}_G \setminus \mathcal{LS}_G^0) < (2g-2)\cdot\dim G + l$. Using 2.11.2 one sees that $\mathcal{LS}_G^0$ is dense in $\mathcal{LS}_G$. So it suffices to prove that the preimage in $\mathcal{LS}_G^0$ of every connected component of $\mathrm{Bun}_G^{0\prime}$ is non-empty and irreducible. This is clear because the morphism $\mathcal{LS}_G^0 \to \mathrm{Bun}_G^{0\prime}$ is a torsor[15] over $T^*\mathrm{Bun}_G^{0\prime}$. $\square$

**2.12. On the Beauville – Laszlo Theorem.** This section is, in fact, an appendix in which we explain a globalized version of the main theorem of [BLa95]. This version is used in 2.3.7 but not in an essential way. So this section can be skipped by the reader.

**2.12.1.** *Theorem.* Let $p : \widetilde{S} \to S$ be a morphism of schemes, $D \subset S$ an effective Cartier divisor. Suppose that $\widetilde{D} := p^{-1}(D)$ is a Cartier divisor in $\widetilde{S}$ and the morphism $\widetilde{D} \to D$ is an isomorphism. Set $U := S \setminus D$, $\widetilde{U} := \widetilde{S} \setminus \widetilde{D}$. Denote by $C$ the category of quasi-coherent $\mathcal{O}_S$-modules that have no non-zero local sections supported at $D$. Denote by $\widetilde{C}$ the similar category for $(\widetilde{S}, \widetilde{D})$. Denote by $C'$ the category of triples $(\mathcal{M}_1, \mathcal{M}_2, \varphi)$ where $\mathcal{M}_1$ is a quasi-coherent $\mathcal{O}_U$-module, $\mathcal{M}_2 \in \widetilde{C}$, $\varphi$ is an isomorphism between the pullbacks of $\mathcal{M}_1$ and $\mathcal{M}_2$ to $\widetilde{U}$.

1) $p^*$ maps $C$ to $\widetilde{C}$, so we have the functor $F : C \to C'$ that sends $\mathcal{M} \in C$ to $(\mathcal{M}|_U, p^*\mathcal{M}, \varphi)$ where $\varphi$ is the natural isomorphism between the pullbacks of $\mathcal{M}|_U$ and $p^*\mathcal{M}$ to $\widetilde{U}$.
2) $F : C \to C'$ is an equivalence.
3) $\mathcal{M} \in C$ is locally of finite type (resp. flat, resp. locally free of finite rank) if and only if $\mathcal{M}|_U$ and $f^*\mathcal{M}$ have this property.

[15]The torsor structure depends on the choice of an invariant scalar product on $\mathfrak{g}$.

This theorem is easily reduced to the case where $S$ and $\widetilde{S}$ are affine[16] and $D$ is *globally* defined by one equation (so $S = \operatorname{Spec} A$, $\widetilde{S} = \operatorname{Spec} \widetilde{A}$, $D = \operatorname{Spec} A/fA$, $f \in A$ is not a zero divisor). This case is treated just as in [BLa95] (in [BLa95] it is supposed that $\widetilde{A} = \widehat{A}$ :=the completion of $A$ for the $f$-adic topology, but the only properties of $\widehat{A}$ used in [BLa95] are the injectivity of $f : \widehat{A} \to \widehat{A}$ and the bijectivity of $A/fA \to \widehat{A}/f\widehat{A}$).

**2.12.2.** Let $D$ be a closed affine subscheme of a scheme $S$. Denote by $\widehat{S}$ the completion of $S$ along $D$ and by $\widehat{S}'$ the spectrum of the ring of regular functions on $\widehat{S}$ (so $\widehat{S}$ is an affine formal scheme and $\widehat{S}'$ is the corresponding true scheme). We have the morphisms $\pi : \widehat{S} \to S$ and $i : \widehat{S} \to \widehat{S}'$.

**2.12.3.** *Proposition.* There is at most one morphism $p : \widehat{S}' \to S$ such that $pi = \pi$.

*Proof.* Suppose that $\pi = p_1 i = p_2 i$ for some $p_1, p_2 : \widehat{S}' \to S$. Let $Y \subset \widehat{S}'$ be the preimage of the diagonal $\Delta \subset S \times S$ under $(p_1, p_2) : \widehat{S}' \to S \times S$. Then $Y$ is a locally closed subscheme of $\widehat{S}'$ containing the $n$-th infinitesimal neighbourhood of $D \subset \widehat{S}'$ for every $n$. So $(\bar{Y} \setminus Y) \cap D = \emptyset$ and therefore $\bar{Y} \setminus Y = \emptyset$, i.e., $Y$ is closed. A closed subscheme of $\widehat{S}'$ containing all infinitesimal neighbourhoods of $D$ equals $\widehat{S}'$. So $Y = \widehat{S}'$ and $p_1 = p_2$. $\square$

**2.12.4.** Suppose we are in the situation of 2.12.2 and $D \subset S$ is an effective Cartier divisor. If there exists $p : \widehat{S}' \to S$ such that $pi = \pi$ then $p^{-1}(D) \subset \widehat{S}'$ is a Cartier divisor and the morphism $p^{-1}(D) \to D$ is an isomorphism. So Theorem 2.12.1 is applicable.

[16]For any $x \in S$ there is an affine neighbourhood $U$ of $x$ and an open affine $\widetilde{U} \subset \widetilde{S}$ such that $\widetilde{U} \subset p^{-1}(U)$ and $\widetilde{U} \cap \widetilde{D} = p^{-1}(U) \cap \widetilde{D}$. Indeed, we can assume that $S$ is affine and $x \in D$. Let $\widetilde{U}_1 \subset S$ be an affine neighbourhood of the preimage of $x$ in $\widetilde{D}$. Then $p(\widetilde{U}_1 \cap \widetilde{D})$ is an affine neighbourhood of $x$ in $D$, so it contains $U \cap D$ for some open affine $U \subset S$ such that $x \in U$. Then $\widetilde{U} := \widetilde{U}_1 \times_S U$ has the desired properties.

**2.12.5.** Suppose we are in the situation of 2.12.2 and $S$ is quasi-separated. Then there exists $p : \widehat{S}' \to S$ such that $pi = \pi$. The proof we know is rather long. We first treat the noetherian case and then use the following fact (Deligne, private communication): for any quasi-compact quasi-separated scheme $S$ there exists an affine morphism from $S$ to some scheme of finite type over $\mathbb{Z}$.

In 2.3.7 we use the existence of $p : \widehat{S}' \to S$ for $S = X \otimes R$ where $X$ is our curve and $R$ is a $\mathbb{C}$-algebra. So the following result suffices.

**2.12.6.** *Proposition.* Suppose that in the situation of 2.12.2 $S$ is a locally closed subscheme of $\mathbb{P}^n \otimes R$ for some ring $R$. Then there exists $p : \widehat{S}' \to S$ such that $pi = \pi$.

*Proof.* We use Jouanolou's device. Let $\mathbb{P}^*$ be the projective space dual to $\mathbb{P} = \mathbb{P}^n$, $Z \subset \mathbb{P} \times \mathbb{P}^*$ the incidence correspondence, $U := (\mathbb{P} \times \mathbb{P}^*) \setminus Z$. Since the morphism $U \to \mathbb{P}$ is a torsor over some vector bundle on $\mathbb{P}$ and $\widehat{S}$ is an affine formal scheme the morphism $\widehat{S} \to \mathbb{P}$ lifts to a morphism $\widehat{S} \to U$. Since $U$ is affine $\mathrm{Mor}(\widehat{S}, U) = \mathrm{Mor}(\widehat{S}', U)$, so we get a morphism $\widehat{S}' \to U$. The composition $\widehat{S}' \to U \to \mathbb{P}$ yields a morphism $f : \widehat{S}' \to \mathbb{P} \otimes R$. The locally closed subscheme $f^{-1}(S) \subset \widehat{S}'$ contains the $n$-th infinitesimal neighbourhood of $D \subset \widehat{S}'$ for every $n$, so $f^{-1}(S) = \widehat{S}'$ (cf. 2.12.3) and $f$ induces a morphism $p : \widehat{S}' \to S \subset \mathbb{P} \otimes R$. Clearly $pi = \pi$. $\square$

*Remark.* One can also prove the proposition interpreting the morphism $\widehat{S} \to \mathbb{P}^n$ as a pair $(\mathcal{M}, \varphi)$ where $\mathcal{M}$ is an invertible sheaf on $\widehat{S}$ and $\varphi$ is an epimorphism $\mathcal{O}^{n+1} \to \mathcal{M}$. Then one shows that $(\mathcal{M}, \varphi)$ extends to a pair $(\mathcal{M}', \varphi')$ on $\widehat{S}'$. Of course, this proof is essentially equivalent to the one based on Jouanolou's device.

# 3. Opers

## 3.1. Definition and first properties.

**3.1.1.** Let $G$ be a connected reductive group over $\mathbb{C}$ with a fixed Borel subgroup $B = B_G \subset G$. Set $N = [B, B]$, so $H = B/N$ is the Cartan group. Denote by $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$, $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ the corresponding Lie algebras. $\mathfrak{g}$ carries a canonical decreasing Lie algebra filtration $\mathfrak{g}^k$ such that $\mathfrak{g}^0 = \mathfrak{b}$, $\mathfrak{g}^1 = \mathfrak{n}$, and for any $k > 0$ the weights of the action of $\mathfrak{h} = \mathrm{gr}^0\,\mathfrak{g}$ on $\mathrm{gr}^k\,\mathfrak{g}$ (resp. $\mathfrak{g}r^{-k}\mathfrak{g}$) are sums of $k$ simple positive (resp. negative) roots. In particular $\mathrm{gr}^{-1}\,\mathfrak{g} = \oplus \mathfrak{g}^\alpha$, $\alpha$ is a simple negative root. Set $Z = Z_G = \mathrm{Center} G$.

**3.1.2.** Let $X$ be any smooth (not necessarily complete) curve, $\mathfrak{F}_B$ a $B$-bundle on $X$. Denote by $\mathfrak{F}_G$ the induced $G$-torsor, so $\mathfrak{F}_B \subset \mathfrak{F}_G$. We have the corresponding twisted Lie algebras $\mathfrak{b}_\mathfrak{F} := \mathfrak{b}_{\mathfrak{F}_B}$ and $\mathfrak{g}_\mathfrak{F} := \mathfrak{g}_{\mathfrak{F}_B} = \mathfrak{g}_{\mathfrak{F}_G}$ equipped with the Lie algebra filtration $\mathfrak{g}^k_\mathfrak{F}$. Consider the sheaves of connections $\mathrm{Conn}(\mathfrak{F}_B)$, $\mathrm{Conn}(\mathfrak{F}_G)$; these are $\mathfrak{b}_\mathfrak{F} \otimes \omega_X$- and $\mathfrak{g}_\mathfrak{F} \otimes \omega_X$-torsors. We have the obvious embedding $\mathrm{Conn}(\mathfrak{F}_B) \subset \mathrm{Conn}(\mathfrak{F}_G)$. It defines the projection $c : \mathrm{Conn}(\mathfrak{F}_G) \to (\mathfrak{g}/\mathfrak{b})_\mathfrak{F} \otimes \omega_X$ such that $c^{-1}(0) = \mathrm{Conn}(\mathfrak{F}_B)$ and $c(\nabla + \nu) = c(\nabla) + \nu \mod \mathfrak{b}_\mathfrak{F} \otimes \omega_X$ for any $\nabla \in \mathrm{Conn}(\mathfrak{F}_G)$, $\nu \in \mathfrak{g}_\mathfrak{F} \otimes \omega_X$.

**3.1.3.** *Definition.* A $G$-oper on $X$ is a pair $(\mathfrak{F}_B, \nabla)$, $\nabla \in \Gamma(X, \mathrm{Conn}(\mathfrak{F}_G))$ such that

(1) $c(\nabla) \in \mathrm{gr}^{-1}\,\mathfrak{g}_\mathfrak{F} \otimes \omega_X \subset (\mathfrak{g}/\mathfrak{b})_\mathfrak{F} \otimes \omega_X$
(2) For any simple negative root $\alpha$ the $\alpha$-component $c(\nabla)^\alpha \in \Gamma(X, \mathfrak{g}^\alpha_\mathfrak{F} \otimes \omega_X)$ does not vanish at any point of $X$.

If $\mathfrak{g}$ is a semisimple Lie algebra then a $\mathfrak{g}$-*oper* is a $G_{ad}$-oper where $G_{ad}$ is the adjoint group corresponding to $\mathfrak{g}$.

We will usually consider $G$-oper as a $G$-local system $(\mathfrak{F}_G, \nabla)$ equipped with an extra *oper structure* (a $B$-flag $\mathfrak{F}_B \subset \mathfrak{F}_G$ which satisfies conditions (1) and (2) above).

$G$-opers on $X$ form a groupoid $\mathcal{O}\mathfrak{p}_G(X)$. The groupoids $\mathcal{O}\mathfrak{p}_G(X')$ for $X'$ étale over $X$ form a sheaf of groupoids $\mathcal{O}\mathfrak{p}_G$ on $X_{\text{ét}}$.

**3.1.4.** *Proposition.* Let $(\mathfrak{F}_B, \nabla)$ be a $G$-oper. Then $\operatorname{Aut}(\mathfrak{F}_B, \nabla) = Z$ if $X$ is connected. □

In particular $\mathfrak{g}$-opers have no symmetries, i.e., $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is a set and $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}$ is a sheaf of sets.

**3.1.5.** *Proposition.* Suppose that $X$ is complete and connected of genus $g > 1$. Let $(\mathfrak{F}_G, \nabla)$ be a $G$-local system on $X$ that has an oper structure. Then

(i) the oper structure on $(\mathfrak{F}_G, \nabla)$ is unique: the corresponding flag $\mathfrak{F}_B \subset \mathfrak{F}_G$ is the Harder-Narasimhan flag;
(ii) $\operatorname{Aut}(\mathfrak{F}_G, \nabla) = Z$;
(iii) $(\mathfrak{F}_G, \nabla)$ cannot be reduced to a non-trivial parabolic subgroup $P \subset G$. □

Of course ii) follows from i) and 3.1.4.

**3.1.6.** *Example.* A $GL_n$-oper can be considered as an $\mathcal{O}_X$-module $\mathcal{E}$ equipped with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \omega_X$ and a filtration $\mathcal{E} = \mathcal{E}_n \supset \mathcal{E}_{n-1} \supset \cdots \supset \mathcal{E}_0 = 0$ such that

(i) The sheaves $\operatorname{gr}_i \mathcal{E}$, $n \geq i \geq 1$, are invertible
(ii) $\nabla(\mathcal{E}_i) \subset \mathcal{E}_{i+1} \otimes \omega_X$ and for $n-1 \geq i \geq 1$ the morphism $\operatorname{gr}_i \mathcal{E} \to \operatorname{gr}_{i+1} \mathcal{E} \otimes \omega_X$ induced by $\nabla$ is an isomorphism.

One may construct $GL_n$-opers as follows. Let $\mathcal{A}, \mathcal{B}$ be invertible $\mathcal{O}_X$-modules and $\partial : \mathcal{A} \to \mathcal{B}$ a differential operator of order $n$ whose symbol $\sigma(\partial) \in \Gamma\left(X, \mathcal{B} \bigotimes \mathcal{A}^{\otimes(-1)} \bigotimes \Theta_X^{\otimes n}\right)$ has no zeros. Our $\partial$ is a section of $\mathcal{B} \bigotimes_{\mathcal{O}_X} \mathcal{D}_X \bigotimes_{\mathcal{O}_X} \mathcal{A}^{\otimes(-1)}$ or, equivalently, an $\mathcal{O}$-linear map $\mathcal{B}^{\otimes(-1)} \to \mathcal{D}_X \otimes \mathcal{A}^{\otimes(-1)}$. Let $I \subset \mathcal{D}_X \otimes \mathcal{A}^{\otimes(-1)}$ be the $\mathcal{D}_X$-sub-module generated by the image of this map. Let $\mathcal{E} := \mathcal{D}_X \otimes \mathcal{A}^{\otimes(-1)}/I$; denote by $\mathcal{E}_i$ the filtration on $\mathcal{E}$ induced by the usual filtration of $\mathcal{D}_X$ by degree of an operator. Then $\mathcal{E}$ is a $\mathcal{D}_X$-module, i.e., an $\mathcal{O}_X$-module with a connection, and the filtration $\mathcal{E}_i$ satisfies the conditions (i), (ii). Therefore $(\mathcal{E}, \{\mathcal{E}_i\}, \nabla)$ is a $GL_n$-oper. This construction defines an equivalence between the groupoid of $GL_n$-opers and that of the data $\partial : \mathcal{A} \to \mathcal{B}$ as above. The inverse functor $\Phi$ associates to $(\mathcal{E}, \{\mathcal{E}_i\}, \nabla)$ the following differential operator : $\mathcal{A} \longrightarrow \mathcal{B}, \mathcal{A} := \mathcal{E}_1^{\otimes(-1)}, \mathcal{B} := \omega_X \otimes (\mathcal{E}/\mathcal{E}_{n-1})^{\otimes(-1)}$. Consider $\mathcal{E}$ as a $\mathcal{D}_X$-module. Let $\mathcal{D}_X^{(k)} \subset \mathcal{D}_X$ be the subsheaf of operators of order $\leq k$. Then the morphism $\mathcal{D}_X^{(n-1)} \otimes_{\mathcal{O}_X} \mathcal{E}_1 \longrightarrow \mathcal{E}$ is an isomorphism and therefore the composition $\mathcal{D}_X^{(n)} \otimes_{\mathcal{O}_X} \mathcal{E}_1 \longrightarrow \mathcal{E} \xrightarrow{\sim} \mathcal{D}_X^{(n-1)} \otimes_{\mathcal{O}_X} \mathcal{E}_1$ defines a splitting of the exact sequence $0 \longrightarrow \mathcal{D}_X^{(n-1)} \otimes_{\mathcal{O}_X} \mathcal{E}_1 \longrightarrow \mathcal{D}_X^{(n)} \otimes_{\mathcal{O}_X} \mathcal{E}_1 \longrightarrow \omega_X^{\otimes(-n)} \otimes \mathcal{E}_1 \longrightarrow 0$, i.e., a morphism $\omega_X^{\otimes(-n)} \otimes \mathcal{E}_1 \longrightarrow \mathcal{D}_X^{(n)} \otimes_{\mathcal{O}_X} \mathcal{E}_1$, which is the same as a differential operator $\partial : \mathcal{A} \longrightarrow \mathcal{B}$ (notice that the isomorphisms $\mathrm{gr}_i\, \mathcal{E} \xrightarrow{\sim} \mathrm{gr}_{i+1}\, \mathcal{E} \otimes \omega_X$ induce an isomorphism $\mathcal{E}_1 \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}_{n-1}) \otimes \omega_X^{\otimes(n-1)}$, so $\omega_X^{\otimes(-n)} \otimes \mathcal{E}_1 = \omega_X^{\otimes(-1)} \otimes (\mathcal{E}/\mathcal{E}_{n-1}) = \mathcal{B}^{\otimes(-1)}$).

Applying the above functor $\Phi$ to an $SL_2$-oper one obtains a differential operator $\partial : \mathcal{A} \longrightarrow \omega_X \otimes \mathcal{A}^{\otimes(-1)}$. It is easy to show that one thus obtains an equivalence between the groupoid of $SL_2$-opers and that of pairs $(\mathcal{A}, \partial)$ consisting of an invertible sheaf $\mathcal{A}$ and a Sturm-Liouville operator $\partial : \mathcal{A} \longrightarrow \omega_X \otimes \mathcal{A}^{\otimes(-1)}$, i.e., a self-adjoint differential operator $\partial$ of order 2 whose symbol $\sigma(\partial)$ has no zeros. Notice that $\sigma(\partial)$ induces an isomorphism $\omega_X^{\otimes 2} \otimes \mathcal{A} \xrightarrow{\sim} \omega_X \otimes \mathcal{A}^{\otimes(-1)}$, so $\mathcal{A}$ is automatically a square root of $\omega_X^{\otimes(-1)}$.

If $(\mathcal{A}, \partial)$ is a Sturm-Liouville operator and $\mathcal{M}$ is a line bundle equipped with an isomorphism $\mathcal{M}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ then $\mathcal{M}$ has a canonical connection and therefore tensoring $(\mathcal{A}, \partial)$ by $\mathcal{M}$ one obtains a Sturm-Liouville operator $(\widetilde{\mathcal{A}}, \widetilde{\partial})$, $\widetilde{\mathcal{A}} = \mathcal{A} \otimes \mathcal{M}$. We say that $(\mathcal{A}, \partial)$ and $(\widetilde{\mathcal{A}}, \widetilde{\partial})$ are *equivalent*. It is easy to see that the natural map $\mathcal{O}\mathfrak{p}_{SL_2}(X) \longrightarrow \mathcal{O}\mathfrak{p}_{sl_2}(X)$ identifies $\mathcal{O}\mathfrak{p}_{sl_2}(X)$ with the set of equivalence classes of Sturm-Liouville operators.

Opers for other classical groups may be described in similar terms (in the local situation this was done in [DS85, section 8]).

**3.1.7.** Identifying $sl_2$-opers with equivalence classes of Sturm-Liouville operators (see 3.1.6) one sees that $\mathcal{O}\mathfrak{p}_{sl_2}$ is an $\omega_X^{\otimes 2}$-torsor: a section $\eta$ of $\omega_X^{\otimes 2}$ maps a Sturm-Liouville operator $\partial : \mathcal{A} \longrightarrow \mathcal{A} \otimes \omega_X^{\otimes 2}$, $\mathcal{A}^{\otimes(-2)} = \omega_X$, to $\partial - \eta$. Let us describe this action of $\omega_X^{\otimes 2}$ on $\mathcal{O}\mathfrak{p}_{sl_2}$ without using Sturm-Liouville operators.

Identify $\mathfrak{n} \subset sl_2$ with $(sl_2/\mathfrak{b})^*$ using the bilinear form $\mathrm{Tr}(AB)$ on $sl_2$. If $\mathfrak{F} = (\mathfrak{F}_B, \nabla)$ is an $sl_2$-oper then according to 3.1.3 the section $c(\nabla)$ trivializes the sheaf $(sl_2/\mathfrak{b})_{\mathfrak{F}_B} \otimes \omega_X$. So $(sl_2/\mathfrak{b})_{\mathfrak{F}_B} = \omega_X^{\otimes(-1)}$, $\mathfrak{n}_{\mathfrak{F}} = \omega_X$, and we have the embedding $\omega_X^{\otimes 2} = \mathfrak{n}_{\mathfrak{F}_B} \otimes \omega_X \hookrightarrow (sl_2)_{\mathfrak{F}_B} \otimes \omega_X$. Translating $\nabla$ by a section $\mu$ of $\omega_X^{\otimes 2} \subset (sl_2)_{\mathfrak{F}_B} \otimes \omega_X$ we get a new oper denoted by $\mathfrak{F} + \mu$. This $\omega_X^{\otimes 2}$-action on $\mathcal{O}\mathfrak{p}_{sl_2}$ coincides with the one introduced above, so it makes $\mathcal{O}\mathfrak{p}_{sl_2}$ an $\omega_X^{\otimes 2}$-torsor.

*Remark* It is well known that this torsor is trivial (even if $H^1(X, \omega_X^{\otimes 2}) \neq 0$, i.e., $g \leq 1$; Sturm-Liouville operators on $\mathbb{P}^1$ or on an elliptic curve do exist). However for families of curves $X$ this torsor may not be trivial.

**3.1.8.** In 3.1.9 we will use the following notation. Let $B_0 \subset PSL_2$ be the group of upper-triangular matrices. Set $N_0 := [B_0, B_0], \mathfrak{b}_0 := \mathrm{Lie}\, B_0, \mathfrak{n}_0 := \mathrm{Lie}\, N_0$. Identify $B_0/N_0$ with $\mathbb{G}_m$ via the adjoint action $B_0/N_0 \longrightarrow \mathrm{Aut}\, \mathfrak{n}_0 = \mathbb{G}_m$. Using the matrices $e := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $f := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ we identify $\mathfrak{n}_0$ and $sl_2/\mathfrak{b}_0$ with $\mathbb{C}$. Then for an $sl_2$-oper $\mathfrak{F} = (\mathfrak{F}_{B_0}, \nabla)$ the isomorphism $(sl_2/\mathfrak{b}_0)_{\mathfrak{F}_{B_0}} \xrightarrow{\sim} \omega_X^{\otimes(-1)}$ from 3.1.7 (or the isomorphism $\mathfrak{n}_{\mathfrak{F}_{B_0}} \xrightarrow{\sim} \omega_X$) induces an isomorphism between the push-forward of $\mathfrak{F}_{B_0}$ by $B_0 \longrightarrow B_0/N_0 = \mathbb{G}_m$ and the $\mathbb{G}_m$-torsor $\omega_X$.

**3.1.9.** For any semisimple Lie algebra $\mathfrak{g}$ we will give a rather explicit description of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$. In particular we will introduce a "canonical" structure of affine space on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ (for $\mathfrak{g} = sl_2$ it was introduced in 3.1.7).

Let $G$ be the adjoint group corresponding to $\mathfrak{g}$ and $B$ its Borel subgroup. We will use the notation from 3.1.8. Fix a principal embedding $i : sl_2 \hookrightarrow \mathfrak{g}$ such that $i(\mathfrak{b}_0) \subset \mathfrak{b}$; one has the corresponding embeddings $i_G : PSL_2 \hookrightarrow G$,

$i_B : B_0 \hookrightarrow B$. Set $V = V_{\mathfrak{g}} := \mathfrak{g}^{N_0}$. Then $\mathfrak{n}_0 \subset V \subset \mathfrak{n}$. One has the adjoint action Ad of $\mathbb{G}_m = B_0/N_0$ on $V$. Define a new $\mathbb{G}_m$-action $a$ on $V$ by $a(t)v := t\,\mathrm{Ad}(t)v$, $v \in V$, $t \in \mathbb{G}_m$.

Consider the vector bundle $V_{\omega_X}$ i.e., the $\omega_X$-twist of $V$ with respect to the $\mathbb{G}_m$-action $a$ (we consider $\omega_X$ as a $\mathbb{G}_m$-torsor on $X$). Twisting by $\omega_X$ the embedding $\mathbb{C} \xrightarrow{\sim} \mathbb{C}e = \mathfrak{n}_0 \hookrightarrow V$ we get an embedding ${\omega_X}^{\otimes 2} \hookrightarrow V_{\omega_X}$.

For any $sl_2$-oper $\mathfrak{F}_0 = (\mathfrak{F}_{B_0}, \nabla_0)$ its $i$-push-forward $i\mathfrak{F}_0 = (\mathfrak{F}_B, \nabla)$ is a $\mathfrak{g}$-oper. It follows from 3.1.8 that we have a canonical isomorphism $V_{\omega_X} = V_{\mathfrak{F}_{B_0}} \otimes \omega_X$ and therefore a canonical embedding $V_{\omega_X} \subset \mathfrak{b}_{\mathfrak{F}_0} \otimes \omega_X = \mathfrak{b}_{\mathfrak{F}_B} \otimes \omega_X$. Translating $\nabla$ by a section $\nu$ of $V_{\omega_X}$ we get a new $\mathfrak{g}$-oper denoted by $i\mathfrak{F}_0 + \nu$.

Let $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}$ be the $V_{\omega_X}$-torsor induced from the ${\omega_X}^{\otimes 2}$-torsor $\mathcal{O}\mathfrak{p}_{sl_2}$ by the embedding ${\omega_X}^{\otimes 2} \subset V_{\omega_X}$. A section of $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}$ is a pair $(\mathfrak{F}_0, \nu)$ as above, and we assume that $(\mathfrak{F}_0 + \mu, \nu) = (\mathfrak{F}_0, \mu + \nu)$ for a section $\mu$ of ${\omega_X}^{\otimes 2}$. We have a canonical map

$$\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}} \longrightarrow \mathcal{O}\mathfrak{p}_{\mathfrak{g}} \tag{43}$$

which sends $(\mathfrak{F}_0, \nu)$ to $i\mathfrak{F}_0 + \nu$.

**3.1.10.** *Proposition.* The mapping (43) is bijective. □

*Remarks*

(i) Though the bijection (43) is canonical we are not sure that it gives a reasonable description of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}$.

(ii) The space $V = V_{\mathfrak{g}}$ from 3.1.9 depends on the choice of a principal embedding $i : sl_2 \hookrightarrow \mathfrak{g}$ (for such an $i$ there is a unique Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $i(\mathfrak{b}_0)$). But any two principal embeddings $sl_2 \hookrightarrow \mathfrak{g}$ are conjugate by a unique element of $G = G_{\mathrm{ad}}$. So we can identify the $V$'s corresponding to various $i$'s and obtain a vector space (not a subspace of $\mathfrak{g}$!) canonically associated to $\mathfrak{g}$.

(iii) Let $G$ be the adjoint group corresponding to $\mathfrak{g}$, $B$ a Borel subgroup of $G$. Proposition 3.1.10 implies that for any $\mathfrak{g}$-oper $\mathfrak{F} = (\mathfrak{F}_B, \nabla)$ $\mathfrak{F}_B$ is isomorphic to a certain canonical $B$-bundle $\mathfrak{F}_B^0$ which does not depend on $\mathfrak{F}$. Actually $\mathfrak{F}^0_B$ is the push-forward of the canonical $(\mathrm{Aut}^0 O)$-bundle from 2.6.5 by a certain homomorphism $i_B\sigma\pi : \mathrm{Aut}^0 O \longrightarrow B$. Here $\pi$ is the projection $\mathrm{Aut}^0 O \longrightarrow \mathrm{Aut}(O/m^3)$ where $m$ is the maximal ideal of $O$, $\sigma$ is an isomorphism $\mathrm{Aut}(O/m^3) \xrightarrow{\sim} B_0$ where $B_0$ is a Borel subgroup of $PSL_2$, and $i_B : B_0 \longrightarrow B$ is induced by a principal embedding $PSL_2 \longrightarrow G$ ($\sigma$ and $i_B$ are unique up to a unique conjugation).

**3.1.11.** Assume that $X$ is complete. Then $G$-opers form a smooth algebraic stack which we again denote as $\mathcal{O}\mathfrak{p}_G(X)$ by abuse of notation. If $G$ is semisimple this is a Deligne-Mumford stack (see 3.1.4); if $G$ is adjoint then $\mathcal{O}\mathfrak{p}_G(X) = \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is a scheme isomorphic to the affine space $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X)$ via (43).

*Remarks*

(i) If $X$ is non-complete, then $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is an ind-scheme.

(ii) If $X$ is complete, connected, and of genus $g > 1$, then $\dim \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) = (g-1)\cdot\dim \mathfrak{g}$. Indeed, according to Proposition 3.1.10, $\dim \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) = \dim \underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X) = \dim \Gamma(X, V_{\omega_X})$ and an easy computation due to Hitchin (see Remark 4 from 2.2.4) shows that $\dim \Gamma(X, V_{\omega_X}) = (g-1)\cdot \dim \mathfrak{g}$ if $g > 1$. Actually we will see in 3.1.13 that $\Gamma(X, V_{\omega_X}) = \mathrm{Hitch}_{{}^L\mathfrak{g}}(X)$, so we can just use Hitchin's formula

$$\dim \mathrm{Hitch}_{{}^L\mathfrak{g}}(X) = (g-1)\cdot \dim {}^L\mathfrak{g} = (g-1)\cdot \dim \mathfrak{g}$$

mentioned in 2.2.4(ii).

(iii) Let $X$ be as in Remark ii and $G$ be the adjoint group corresponding to $\mathfrak{g}$. One has the obvious morphism $i : \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \longrightarrow \mathrm{LocSys}_G$ where $\mathrm{LocSys}_G$ is the stack of $G$-local systems on $X$. One can show that $G$-local systems which cannot be reduced to a non-trivial parabolic subgroup $P \subset G$ and which have no non-trivial automorphisms form an open substack $U \subset \mathrm{LocSys}_G$ which is actually a smooth variety; $U$ has a canonical symplectic structure. According to 3.1.5

$i(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)) \subset U$ and $i$ is a set-theoretical embedding. In fact $i$ is a closed embedding and $i(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X))$ is a Lagrangian subvariety of $U$. Besides, $i(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)) = \pi^{-1}(S)$ where $\pi : \mathrm{LocSys}_G \longrightarrow \mathrm{Bun}_G$ corresponds to forgetting the connection and $S \subset \mathrm{Bun}_G$ is the locally closed substack of $G$-bundles isomorphic to $\mathfrak{F}_G^0$, the $G$-bundle corresponding to the $B$-bundle $\mathfrak{F}_B^0$ introduced in Remark iii from 3.1.10 (so $S$ is the classifying stack of the unipotent group $\mathrm{Aut}\,\mathfrak{F}_G^0$).

**3.1.12.** Denote by $A_{\mathfrak{g}}(X)$ the coordinate ring of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$. We will construct a canonical filtration on $A_{\mathfrak{g}}(X)$ and a canonical isomorphism of graded algebras

$$\sigma_{A(X)} : \mathrm{gr}\, A_{\mathfrak{g}}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}_{{}^L\mathfrak{g}}(X) \tag{44}$$

where ${}^L\mathfrak{g}$ denotes the Langlands dual of $\mathfrak{g}$ and the r.h.s. of (44) was defined in 2.2.2. We give two equivalent constructions. The one from 3.1.13 is straightforward; it involves the isomorphism (43). The construction from 3.1.14 is more natural.

**3.1.13.** Using 3.1.8 we identify $A_{\mathfrak{g}}(X)$ with the coordinate ring of $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X)$. Denote by $A^{cl}_{\mathfrak{g}}(X)$ the coordinate ring of the vector space $\Gamma(X, V_{\omega_X})$ corresponding to the affine space $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X)$. Consider the $\mathbb{G}_m$-action on $A^{cl}_{\mathfrak{g}}(X)$ opposite to that induced by the $\mathbb{G}_m$-action $a$ on $V$ (see 3.1.7); the corresponding grading on $A^{cl}_{\mathfrak{g}}(X)$ is positive. It induces a canonical ring filtration on $A_{\mathfrak{g}}(X)$ and a canonical isomorphism $\mathrm{gr}\, A_{\mathfrak{g}}(X) \xrightarrow{\sim} A^{cl}_{\mathfrak{g}}(X)$.

So to define (44) it remains to construct a graded isomorphism $A^{cl}_{\mathfrak{g}}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}_{{}^L\mathfrak{g}}(X)$, which is equivalent to constructing a $\mathbb{G}_m$-equivariant isomorphism of schemes $\Gamma(X, V_{\omega_X}) \xrightarrow{\sim} \mathrm{Hitch}_{{}^L\mathfrak{g}}(X)$. According to 2.2.2 $\mathrm{Hitch}_{{}^L\mathfrak{g}}(X) := \Gamma(X, C_{\omega_X})$, $C := C_{{}^L\mathfrak{g}}$. So it suffices to construct a $\mathbb{G}_m$-equivariant isomorphism of schemes $V_{\mathfrak{g}} \xrightarrow{\sim} C_{{}^L\mathfrak{g}}$. ($V_{\mathfrak{g}}$ is equipped with the action $a$ from 3.1.7.)

According to 2.2.1 $C_{^L\mathfrak{g}} = \operatorname{Spec}(\operatorname{Sym}{}^L\mathfrak{g})^{^LG}$ where $G$ is a connected group corresponding to $\mathfrak{g}$. We can identify $(\operatorname{Sym}{}^L\mathfrak{g})^{^LG}$ with $(\operatorname{Sym}\mathfrak{g}^*)^G$ because both graded algebras are canonically isomorphic to $(\operatorname{Sym}\mathfrak{h}^*)^W$ where $W$ is the Weyl group. So $C_{^L\mathfrak{g}} = C'_{\mathfrak{g}}$ where

$$C'_{\mathfrak{g}} = \operatorname{Spec}(\operatorname{Sym}\mathfrak{g}^*)^G\,, \tag{45}$$

i.e., $C'_{\mathfrak{g}}$ is the affine scheme quotient of $\mathfrak{g}$ with respect to the adjoint action of $G$. Finally according to Theorem 0.10 from Kostant's work [Ko63] we have the canonical isomorphism $V_{\mathfrak{g}} \xrightarrow{\sim} C'_{\mathfrak{g}}$ that sends $v \in V_{\mathfrak{g}}$ to the image of $v + i(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \in \mathfrak{g}$ in $C'_{\mathfrak{g}}$. It commutes with the $\mathbb{G}_m$-actions.

**3.1.14.** Here is a more natural way to describe the canonical filtration on $A_{\mathfrak{g}}(X)$ and the isomorphism (44).

There is a standard way to identify filtered $\mathbb{C}$-algebras with graded flat $\mathbb{C}[\hbar]$-algebras (here $\deg\hbar = 1$). Namely, an algebra $A$ with an increasing filtration $\{A_i\}$ corresponds to the graded $\mathbb{C}[\hbar]$-algebra $A^\sim = \oplus A_i$, the multiplication by $\hbar$ is the embedding $A_i \hookrightarrow A_{i+1}$. Note that $A = A^\sim/(\hbar - 1)A^\sim$, $\operatorname{gr} A = A^\sim/\hbar A^\sim$. Passing to spectra we see that $\operatorname{Spec} A^\sim$ is a flat affine scheme over the line $\mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[\hbar]$, and the grading on $A^\sim$ is the same as a $\mathbb{G}_m$-action on $\operatorname{Spec} A^\sim$ compatible with the action by homotheties on $\mathbb{A}^1$. We are going to construct the scheme $\operatorname{Spec} A_{\mathfrak{g}}(X)^\sim$.

Let $\mathfrak{F}$ be a $G$-torsor on $X$. Denote by $\mathcal{E}_{\mathfrak{F}}$ the Lie algebroid of infinitesimal symmetries of $\mathfrak{F}$; we have a canonical exact sequence

$$0 \to \mathfrak{g}_{\mathfrak{F}} \to \mathcal{E}_{\mathfrak{F}} \xrightarrow{\pi} \Theta_X \to 0.$$

Recall that for $\hbar \in \mathbb{C}$ an *$\hbar$-connection on* $\mathfrak{F}$ is an $\mathcal{O}_X$-linear map $\nabla_\hbar : \Theta_X \to \mathcal{E}_{\mathfrak{F}}$ such that $\pi\nabla_\hbar = \hbar\,\mathrm{id}_{\Theta_X}$ (usual connections correspond to $\hbar = 1$). One defines a *$G - \hbar$-oper* as in 3.1.3 replacing the connection $\nabla$ by an $\hbar$-connection $\nabla_\hbar$. The above results about $G$-opers render to $G - \hbar$-opers. In particular $\mathfrak{g} - \hbar$-opers, i.e., $\hbar$-opers for the adjoint group form an affine scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g},\hbar}(X)$. For $\lambda \in \mathbb{C}^*$ we have the isomorphism of schemes

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g},\hbar}(X) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_{\mathfrak{g},\lambda\hbar}(X) \tag{46}$$

defined by $(\mathfrak{F}_B, \nabla_\hbar) \mapsto (\mathfrak{F}_B, \lambda\nabla_\hbar)$. When $\hbar$ varies $\mathcal{O}\mathfrak{p}_{\mathfrak{g},\hbar}(X)$ become fibers of an affine $\mathbb{C}[\hbar]$-scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)^\sim = \operatorname{Spec} A_{\mathfrak{g}}(X)^\sim$. Using an analog of 3.1.9–3.1.10 for $\mathfrak{g} - \hbar$-opers one shows that $A_{\mathfrak{g}}(X)^\sim$ is flat over $\mathbb{C}[h]$. The morphisms (46) define the action of $\mathbb{G}_m$ on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)^\sim$, i.e., the grading of $A_{\mathfrak{g}}(X)^\sim$. The corresponding filtration on $A_{\mathfrak{g}}(X) = A_{\mathfrak{g}}(X)^\sim/(\hbar-1)A_{\mathfrak{g}}(X)^\sim$ coincides with the filtration from 3.1.13.

To construct (44) is the same as to construct a $\mathbb{G}_m$-equivariant isomorphism between $\mathcal{O}\mathfrak{p}_{\mathfrak{g},0}(X) = \operatorname{Spec} \operatorname{gr} A_{\mathfrak{g}}(X)$ and $\operatorname{Hitch}_{^L\mathfrak{g}}(X) = \operatorname{Spec} \mathfrak{z}^{cl}_{^L\mathfrak{g}}(X)$. As explained in 3.1.11 $\operatorname{Hitch}_{^L\mathfrak{g}}(X) = \Gamma(X, C'_{\omega_X})$ where $C' = C'_{\mathfrak{g}}$ is defined by (45). We have a canonical mapping of sheaves

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g},0} \longrightarrow C'_{\omega_X} \tag{47}$$

which sends $(\mathfrak{F}_B, \nabla_0)$ to the image of $\nabla_0 \in \mathfrak{g}_{\mathfrak{F}} \otimes \omega_X$ by the projection $\mathfrak{g} \longrightarrow C'$. Theorem 0.10 and Proposition 19 from Kostant's work [Ko63] imply that (47) is a bijection. It induces the desired isomorphism $\mathcal{O}\mathfrak{p}_{\mathfrak{g},0}(X) \xrightarrow{\sim} \Gamma(X, C'_{\omega_X})$.

### 3.2. Local opers and Feigin-Frenkel isomorphism.

**3.2.1.** Let us replace $X$ by the formal disc $\operatorname{Spec} O$, $O \simeq \mathbb{C}[[t]]$. The constructions and results of 3.1 render easily to this situation. $\mathfrak{g}$-opers on $\operatorname{Spec} O$ form a scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ isomorphic to the spectrum of the polynomial ring in a countable number of variables. More precisely, the isomorphism (43) identifies $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ with an affine space corresponding to the vector space $H^0(\operatorname{Spec} O, V_{\omega_O})$, $V := V_{\mathfrak{g}}$. $G$-opers on $\operatorname{Spec} O$ form an algebraic stack $\mathcal{O}\mathfrak{p}_G(O)$ isomorphic to $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \times B(Z)$ where $B(Z)$ is the classifying stack of the center $Z \subset G$ and $\mathfrak{g} := \operatorname{Lie}(G/Z)$ (the isomorphism is not quite canonical; see (58) for a canonical description of $\mathcal{O}\mathfrak{p}_G(O)$).

Just as in the global situation (see 3.1.12–3.1.14) the coordinate ring $A_{\mathfrak{g}}(O)$ of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ carries a canonical filtration and we have a canonical isomorphism

$$\sigma_A : \operatorname{gr} A_{\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}^{cl}_{{}^L\mathfrak{g}}(O) \tag{48}$$

(see (44)). Note that $\operatorname{Aut} O$ acts on all the above objects in the obvious way. So $A_{\mathfrak{g}}(O)$ is a filtered $\operatorname{Aut} O$-algebra and $\sigma_A$ is an isomorphism of graded $\operatorname{Aut} O$-algebras.

**3.2.2.** *Theorem.* ([FF92]). There is a canonical isomorphism of filtered $\operatorname{Aut} O$-algebras

$$\varphi_O : A_{\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_{{}^L\mathfrak{g}}(O) \tag{49}$$

such that $\sigma_{\mathfrak{z}} \operatorname{gr} \varphi_O = \sigma_A$, where $\sigma_{\mathfrak{z}} : \operatorname{gr} \mathfrak{z}_{{}^L\mathfrak{g}}(O) \to \mathfrak{z}^{cl}_{{}^L\mathfrak{g}}(O)$ is the symbol map. $\square$

*Remarks*

(i) This isomorphism is uniquely determined by some extra compatibilities; see 3.6.7.

(ii) The original construction of Feigin and Frenkel is representation-theoretic and utterly mysterious (for us). A different, geometric construction is given in ???; the two constructions are compared in ???.

(iii) For $\mathfrak{g} = sl_2$ there is a simple explicit description of (49), which is essentially due to Sugawara; see ???.

### 3.3. Global version.

**3.3.1.** Let us return to the global situation, so our $X$ is a complete curve. We will construct a canonical isomorphism between the algebras $A_{\mathfrak{g}}(X)$ and $\mathfrak{z}_{{}^L\mathfrak{g}}(X)$ (the latter is defined by formula (27) from 2.7.4).

Take $x \in X$. The restriction of a global $\mathfrak{g}$-oper to $\operatorname{Spec} O_x$ defines a morphism of affine schemes

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \longrightarrow \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O_x).$$

This is a closed embedding, so we have the surjective morphism of coordinate rings

$$\theta_x^A : A_{\mathfrak{g}}(O_x) \twoheadrightarrow A_{\mathfrak{g}}(X)\,. \tag{50}$$

$\theta_x^A$ is strictly compatible with the canonical filtrations (to see this use, e.g., the isomorphism (24)).

**3.3.2.** *Theorem.* There is a unique isomorphism of filtered algebras

$$\varphi_X : A_{\mathfrak{g}}(X) \xrightarrow{\sim} \mathfrak{z}_{{}^L\mathfrak{g}}(X) \tag{51}$$

such that for any $x \in X$ the diagram

$$\begin{tikzcd} A_{\mathfrak{g}}(O_x) \arrow[r, two heads, "\theta_x^A"] \arrow[d, "\varphi_{O_x}"', "\wr"] & A_{\mathfrak{g}}(X) \arrow[d, "\wr\ \varphi_X"] \\ \mathfrak{z}_{{}^L\mathfrak{g}}(O_x) \arrow[r, two heads, "\theta_x^{\mathfrak{z}}"] & \mathfrak{z}_{{}^L\mathfrak{g}}(X) \end{tikzcd}$$

commutes (here $\varphi_{O_x}$ is the isomorphism (49) for $O = O_x$). One has $\sigma_{\mathfrak{z}(X)} \cdot \operatorname{gr} \varphi_X = \sigma_{A(X)}$ where $\sigma_{A(X)}$ is the isomorphism (44) and $\sigma_{\mathfrak{z}(X)} : \operatorname{gr} \mathfrak{z}(X) \longrightarrow \mathfrak{z}^{cl}(X)$ was defined at the end of 2.7.4.

*Proof* Since $\theta_x^A$ and $\theta_x^{\mathfrak{z}}$ are surjective and strictly compatible with filtrations it is enough to show the existence of an isomorphism $\varphi_X$ such that the diagram commutes. According to 2.6.5 we have a $\mathcal{D}_X$-algebra $\mathcal{A}_{\mathfrak{g}} := A_{\mathfrak{g}}(O)_X$ with fibers $A_{\mathfrak{g}}(O_x)$. Any global oper $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ defines a section $\gamma_{\mathfrak{F}} : X \to \operatorname{Spec} \mathcal{A}_{\mathfrak{g}}$, $\gamma_{\mathfrak{F}}(x)$ is the restriction of $\mathfrak{F}$ to $\operatorname{Spec} O_x$. The sections $\gamma_{\mathfrak{F}}$ are horizontal and this way we get an isomorphism between $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ and the scheme of horizontal sections of $\operatorname{Spec} \mathcal{A}_{\mathfrak{g}}$ (the reader who thinks that this requires a proof can find it in 3.3.3). Passing to coordinate rings we get a canonical isomorphism

$$A_{\mathfrak{g}}(X) \overset{\sim}{\to} H_\nabla(X, \mathcal{A}_{\mathfrak{g}}) \tag{52}$$

(see 2.6.2 for the definition of $H_\nabla$). On the other hand (49) yields the isomorphism of $\mathcal{D}_X$-algebras

$$\varphi : \mathcal{A}_{\mathfrak{g}} \overset{\sim}{\to} \mathfrak{z}_{L_{\mathfrak{g}}},$$

hence the isomorphism

$$H_\nabla(X, \mathcal{A}_{\mathfrak{g}}) \overset{\sim}{\to} H_\nabla(X, \mathfrak{z}_{L_{\mathfrak{g}}}) = \mathfrak{z}_{L_{\mathfrak{g}}}(X)\,. \tag{53}$$

Now $\varphi_X$ is the composition of (52) and (53). □

**3.3.3.** In this subsection (which can certainly be skipped by the reader) we prove that $\mathfrak{g}$-opers can be identified with horizontal sections of $\operatorname{Spec} \mathcal{A}_{\mathfrak{g}}$ (this identification was used in 3.3.2).

Denote by $\mathfrak{g}^+$ the set of all $a \in \mathfrak{g}^{-1}$ such that the image of $a$ in $\mathfrak{g}^\alpha$ is nonzero for any simple negative root $\alpha$ (we use the notation of 3.1.1). $\mathfrak{g}^+$ is an affine scheme. Consider the action of $\operatorname{Aut}^0 O$ on $\mathfrak{g}^+$ via the standard character $\operatorname{Aut}^0 O \to \operatorname{Aut}(tO/t^2O) = \mathbb{G}_m$. Denote by $B$ the Borel subgroup of the adjoint group corresponding to $\mathfrak{g}$. Equip $B$ with the trivial action of $\operatorname{Aut}^0 O$. Applying the functor $\mathcal{J} : \{\operatorname{Aut}^0 O\text{-schemes}\} \to \{\operatorname{Aut} O\text{-schemes}\}$ from 2.6.7 we obtain $\mathcal{J}B$ = the scheme of morphisms $\operatorname{Spec} O \to B$ and $\mathcal{J}\mathfrak{g}^+$ = the scheme of $\mathfrak{g}^+$-valued differential forms on $\operatorname{Spec} O$. The group $\mathcal{J}B$ acts on $\mathcal{J}\mathfrak{g}^+$ by gauge transformations and $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ is the quotient scheme. The action of $\mathcal{J}B$ on $\mathcal{J}\mathfrak{g}^+$ and the morphism $\mathcal{J}\mathfrak{g}^+ \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ are $\operatorname{Aut} O$-equivariant. Actually $\mathcal{J}\mathfrak{g}^+$ is a $\mathcal{J}B$-torsor over $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$. Moreover, a choice of $\eta \in \omega_O^+ := \omega_O \backslash t\omega_O$ defines its section $S_\eta \subset \mathcal{J}\mathfrak{g}^+$, $S_\eta := \eta \cdot i(f) + V \otimes \omega_O$ (here $f := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $i, V$ were defined in 3.1.9). The fact that $S_\eta$ is a section is just the local form of Proposition 3.1.10. The sections $S_\eta$ define an $\operatorname{Aut} O$-equivariant section $s : \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \times \omega_O^+ \to \mathfrak{g}^+ \times \omega_O^+$ of the induced torsor $\mathfrak{g}^+ \times \omega_O^+ \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \times \omega_O^+$.

Now consider the $\mathcal{D}_X$-schemes $(\mathcal{J}\mathfrak{g}^+)_X$, $(\mathcal{J}B)_X$, and $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X = \operatorname{Spec}\mathcal{A}_\mathfrak{g}$. Clearly $(\mathcal{J}B)_X$ is a group $\mathcal{D}_X$-scheme over $X$ and the scheme $(\mathcal{J}\mathfrak{g}^+)_X$ is a $(\mathcal{J}B)_X$-torsor over $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X$. Actually $(\mathcal{J}B)_X = \mathcal{J}(B_X)$ and $(\mathcal{J}\mathfrak{g}^+)_X$ is the scheme of jets of $\mathfrak{g}^+$-valued differential forms on $X$. Clearly $\mathcal{O}\mathfrak{p}_\mathfrak{g} = \operatorname{Sect}(\mathfrak{g}_X^+)/\operatorname{Sect}(B_X) = \operatorname{Sect}^\nabla((\mathcal{J}\mathfrak{g}^+)_X)/\operatorname{Sect}^\nabla((\mathcal{J}B)_X) \subset \operatorname{Sect}^\nabla(\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X)$. Here Sect denotes the sheaf of sections of an $X$-scheme and $\operatorname{Sect}^\nabla$ denotes the sheaf of horizontal sections of a $\mathcal{D}_X$-scheme. To show that $\mathcal{O}\mathfrak{p}_\mathfrak{g} = \operatorname{Sect}^\nabla(\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X)$ it remains to prove the surjectivity of $\operatorname{Sect}^\nabla((\mathcal{J}\mathfrak{g}^+)_X) \to \operatorname{Sect}^\nabla(\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X)$. To this end use the morphism of $\mathcal{D}_X$-schemes $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X \times (\omega_O^+)_X \to (\mathfrak{g}^+)_X$ induced by $s$ and the fact that $(\omega_O^+)_X$ (i.e., the scheme of jets of non-vanishing differential forms) has a horizontal section in a neighborhood of each point of $X$.

So we have identified $\mathcal{O}\mathfrak{p}_\mathfrak{g}(X)$ with the set of horizontal sections of $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X = \operatorname{Spec}\mathcal{A}_\mathfrak{g}$. In the same way one identifies the scheme $\mathcal{O}\mathfrak{p}_\mathfrak{g}(X)$ with the scheme of horizontal sections of $\operatorname{Spec}\mathcal{A}_\mathfrak{g}$.

*Remark* We used $s$ only to simplify the proof ???.

**3.4. $G$-opers and $\mathfrak{g}$-opers.** In this subsection we assume that $G$ is semisimple (actually the general case can be treated in a similar way; see Remark iii at the end of 3.4.2). We fix a non-zero $y_\alpha \in \mathfrak{g}^\alpha$ for each negative simple root $\alpha$. Set $G_{\text{ad}} := G/Z$, $B_{\text{ad}} := B/Z$, $H_{\text{ad}} := H/Z$ where $Z$ is the center of $G$.

**3.4.1.** There is an obvious projection $\mathcal{O}\mathfrak{p}_G(X) \longrightarrow \mathcal{O}\mathfrak{p}_\mathfrak{g}(X) := \mathcal{O}\mathfrak{p}_{G_{\text{ad}}}(X)$. We will construct a section $\mathcal{O}\mathfrak{p}_\mathfrak{g}(X) \longrightarrow \mathcal{O}\mathfrak{p}_G(X)$ depending on the choice of a square root of $\omega_X$, i.e., a line bundle $\mathcal{L}$ equipped with an isomorphism $\mathcal{L}^{\otimes 2} \xrightarrow{\sim} \omega_X$. Let $(\mathfrak{F}_{B_{\text{ad}}}, \nabla)$ be a $\mathfrak{g}$-oper. Lifting it to a $G$-oper is equivalent to lifting $\mathfrak{F}_{B_{\text{ad}}}$ to a $B$-bundle, which is equivalent to lifting $\mathfrak{F}_{H_{\text{ad}}}$ to an $H$-bundle (here $\mathfrak{F}_{H_{\text{ad}}}$ is the push-forward of $\mathfrak{F}_{B_{\text{ad}}}$ by $B_{\text{ad}} \longrightarrow H_{\text{ad}}$). In the particular case $\mathfrak{g} = sl_2$ we constructed in 3.1.8 a canonical isomorphism $\mathfrak{F}_{H_{\text{ad}}} \xrightarrow{\sim} \omega_X$; the construction from 3.1.8 used a fixed element $f \in sl_2/\mathfrak{b}_0$. Quite similarly one constructs in the general case a canonical isomorphism $\mathfrak{F}_{H_{\mathrm{ad}}} \xrightarrow{\sim} \lambda\omega_X :=$ the push-forward of $\omega_X$ by the homomorphism $\lambda : \mathbb{G}_m \longrightarrow H_{\mathrm{ad}}$ such that for any simple positive root $\alpha$, $\lambda(t)$ acts on $\mathfrak{g}^\alpha$ as multiplication by $t$ (the construction uses the elements $y_\alpha$ fixed at the beginning of 3.4). There is a unique morphism $\lambda^\# : \mathbb{G}_m \longrightarrow H$ such that

$$\lambda^\#(t) \mod Z = \lambda(t)^2 \tag{54}$$

(Indeed, $\lambda$ corresponds to the coweight $\check{\rho} :=$ the sum of fundamental coweights, and $2\check{\rho}$ belongs to the coroot lattice). We lift $\mathfrak{F}_{H_{\mathrm{ad}}} = \lambda\omega_X$ to the $H$-bundle $\lambda^\#\mathcal{L}$ where $\mathcal{L}$ is our square root of $\omega_X$.

**3.4.2.** Denote by $\omega^{1/2}(X)$ the groupoid of square roots of $\omega_X$. For a fixed $\mathcal{L} \in \omega^{1/2}(X)$ we have an equivalence

$$\Phi_{\mathcal{L}} : \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \times Z\,\mathrm{tors}(X) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_G(X) \tag{55}$$

where $Z\,\mathrm{tors}(X)$ is the groupoid of $Z$-torsors on $X$. $\Phi_{\mathcal{L}}(\mathfrak{F}, \mathcal{E})$ is defined as follows: using $\mathcal{L}$ lift $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ to a $G$-oper (see 3.4.1) and then twist this $G$-oper by $\mathcal{E}$. $\Phi_{\mathcal{L}}$ depends on $\mathcal{L}$ in the following way:

$$\Phi_{\mathcal{L}\otimes\mathcal{A}}(\mathfrak{F}, \mathcal{E}) = \Phi_{\mathcal{L}}(\mathfrak{F}, \mathcal{E} \cdot \alpha\mathcal{A})$$

Here $\mathcal{A}$ is a square root of $\mathcal{O}_X$ or, which is the same, a $\mu_2$-torsor on $X$, while $\alpha\mathcal{A}$ is the push-forward of the $\mu_2$-torsor $\mathcal{A}$ by the morphism

$$\alpha : \mu_2 \longrightarrow Z, \qquad \alpha := \lambda^\#|\mu_2 \tag{56}$$

Recall that $\lambda^\#$ is defined by (54).

*Remarks*

(i) If one considers $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ as a scheme and $\mathcal{O}\mathfrak{p}_G(X)$ and $Z\,\mathrm{tors}(X)$ as algebraic stacks then (55) becomes an isomorphism of algebraic stacks.

(ii) $\alpha$ is the restriction of “the” principal homomorphism $SL_2 \longrightarrow G$ to the center $\mu_2 \subset SL_2$.

(iii) If $G$ is reductive but not semisimple and $\mathfrak{g} := \mathrm{Lie}(G/Z)$ then one defines the section $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \longrightarrow \mathcal{O}\mathfrak{p}_G(X)$ depending on $\mathcal{L} \in \omega^{1/2}(X)$ as the composition $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \longrightarrow \mathcal{O}\mathfrak{p}_{[G,G]}(X) \longrightarrow \mathcal{O}\mathfrak{p}_G(X)$. The results of 3.4.2 remain valid if $Z\,\mathrm{tors}(X)$ is replaced by $Z^{\nabla}\,\mathrm{tors}(X)$, the groupoid of $Z$-torsors on $X$ equipped with a connection.

**3.4.3.** Here is a more natural reformulation of 3.4.2. First let us introduce a groupoid $Z\,\mathrm{tors}_\theta(X)$ ($\theta$ should remind the reader about $\theta$-characteristics, i.e., square roots of $\omega_X$). The objects of $Z\,\mathrm{tors}_\theta(\mathcal{L})$ are pairs $(\mathcal{E},\mathcal{L})$, $\mathcal{E} \in Z\,\mathrm{tors}(X)$, $\mathcal{L} \in \omega^{1/2}(X)$, but we prefer to write $\mathcal{E}\cdot\mathcal{L}$ instead of $(\mathcal{E},\mathcal{L})$. We set $\mathrm{Mor}(\mathcal{E}_1\cdot\mathcal{L}_1, \mathcal{E}_2\cdot\mathcal{L}_2) := \mathrm{Mor}(\mathcal{E}_1, \mathcal{E}_2\cdot\alpha(\mathcal{L}_2/\mathcal{L}_1))$ where $\alpha(\mathcal{L}_2/\mathcal{L}_1)$ is the push-forward of the $\mu_2$-torsor $\mathcal{L}_2/\mathcal{L}_1 := \mathcal{L}_2 \otimes \mathcal{L}_1^{\otimes(-1)}$ by the homomorphism (56). Composition of morphisms is defined in the obvious way. One can reformulate 3.4.2 as a canonical equivalence:

$$\Phi : \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \times Z\,\mathrm{tors}_\theta(X) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_G(X) \tag{57}$$

where $\Phi(\mathfrak{F}, \mathcal{L}\cdot\mathcal{E}) := \Phi_{\mathcal{L}}(\mathfrak{F},\mathcal{E})$ and $\Phi_{\mathcal{L}}$ is defined by (55).

In the local situation of 3.2.1 one has a similar canonical equivalence

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \times Z\,\mathrm{tors}_\theta(O) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_G(O) \tag{58}$$

where $Z\,\mathrm{tors}_\theta(O)$ is defined as in the global case. Of course all the objects of $Z\,\mathrm{tors}_\theta(O)$ are isomorphic to each other and the group of automorphisms of an object of $Z\,\mathrm{tors}_\theta(O)$ is $Z$. The same is true for $Z\,\mathrm{tors}(O)$. The difference between $Z\,\mathrm{tors}_\theta(O)$ and $Z\,\mathrm{tors}(O)$ becomes clear if one takes the automorphisms of $O$ into account (see 3.5.2).

**3.4.4.** To describe an "economical" version of $Z\,\mathrm{tors}_\theta(O)$ we need some abstract nonsense.

Let $Z$ be an abelian group. A *$Z$-structure* on a category $C$ is a morphism $Z \to \mathrm{Aut}\,\mathrm{id}_C$. Equivalently, a $Z$-structure on $C$ is an action of $Z$ on $\mathrm{Mor}(c_1,c_2)$, $c_1,c_2 \in \mathrm{Ob}C$, such that for any morphisms $c_1 \xrightarrow{f} c_2 \xrightarrow{g} c_3$ and any $z \in Z$ one has $z(gf) = (zg)f = g(zf)$. A *$Z$-category* is a category with a $Z$-structure. If $C$ and $C'$ are $Z$-categories then a functor $F : C \to C'$ is said to be a *$Z$-functor* if for any $c_1, c_2 \in C$ the map $\mathrm{Mor}(c_1, c_2) \to \mathrm{Mor}(F(c_1), F(c_2))$ is $Z$-equivariant. If $A \to Z$ is a morphism of abelian groups and $C$ is an $A$-category we define the *induced* $Z$-category $C \otimes_A Z$ as follows: $\mathrm{Ob}(C \otimes_A Z) = \mathrm{Ob} C$, the set of $(C \otimes_A Z)$-morphisms $c_1 \to c_2$ is $(\mathrm{Mor}_C(c_1, c_2) \times Z)/A = \{$the $Z$-set induced by the $A$-set $\mathrm{Mor}_C(c_1, c_2)\}$, and composition of morphisms in $C \otimes_A Z$ is defined in the obvious way. We have the natural $A$-functor $C \to C \otimes_A Z$ and for any $Z$-category $C'$ any $A$-functor $C \to C'$ has a unique decomposition $C \to C \otimes_A Z \xrightarrow{F} C'$ where $F$ is a $Z$-functor.

Denote by $\omega^{1/2}(O)$ the groupoid of square roots of $\omega_O$. This is a $\mu_2$-category. $Z\,\mathrm{tors}(O)$ and $Z\,\mathrm{tors}_\theta(O)$ are $Z$-categories. The canonical $\mu_2$-functor $\omega^{1/2}(O) \to Z\,\mathrm{tors}_\theta(O)$ induces an equivalence $\omega^{1/2}(O) \otimes_{\mu_2} Z \to Z\,\mathrm{tors}_\theta(O)$.

**3.4.5.** The reader may prefer the following "concrete" versions of $Z\,\mathrm{tors}_\theta(X)$ and $Z\,\mathrm{tors}_\theta(O)$. Define an exact sequence

$$0 \to Z \to \widetilde{Z} \to \mathbb{G}_m \to 0 \tag{59}$$

as the push-forward of

$$0 \to \mu_2 \to \mathbb{G}_m \xrightarrow{f} \mathbb{G}_m \to 0\,, \quad f(x) := x^2 \tag{60}$$

by the morphism (56). Denote by $\widetilde{Z}\,\mathrm{tors}_\omega(X)$ the groupoid of liftings of the $\mathbb{G}_m$-torsor $\omega_X$ to a $\widetilde{Z}$-torsor (i.e., an object of $\widetilde{Z}\,\mathrm{tors}_\omega(X)$ is a $\widetilde{Z}$-torsor on $X$ plus an isomorphism between the corresponding $\mathbb{G}_m$-torsor and $\omega_X$). The morphism from (60) to (59) induces a functor $F : \omega^{1/2}(X) \to \widetilde{Z}\,\mathrm{tors}_\omega(X)$. The functor $Z\,\mathrm{tors}_\theta(X) \to \widetilde{Z}\,\mathrm{tors}_\omega(X)$ defined by

$$\mathcal{E} \cdot \mathcal{L} \mapsto \mathcal{E} \cdot F(\mathcal{L})\,, \quad \mathcal{E} \in Z\,\mathrm{tors}(X)\,, \quad \mathcal{L} \in \omega^{1/2}(X)$$

is an equivalence.

Quite similarly one defines $\widetilde{Z}\,\mathrm{tors}_\omega(O)$ and a canonical equivalence $Z\,\mathrm{tors}_\theta(O) \xrightarrow{\sim} \widetilde{Z}\,\mathrm{tors}_\omega(O)$.

The equivalences (57) and (58) can be easily understood in terms of $\widetilde{Z}\,\mathrm{tors}_\omega(X)$ and $\widetilde{Z}\,\mathrm{tors}_\omega(O)$. Let us, e.g., construct the equivalence

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \times \widetilde{Z}\,\mathrm{tors}_\omega(X) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_G(X)\,.$$

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_2 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\lambda^\#}\downarrow & & \downarrow{\scriptstyle\lambda} & & \\ 0 & \longrightarrow & Z & \longrightarrow & H & \longrightarrow & H_{\mathrm{ad}} & \longrightarrow & 0 \end{array} \tag{61}$$

Here the upper row is (60); the lower row and the morphisms $\lambda$, $\lambda^\#$ were defined in 3.4.1. According to 3.4.1 a $G$-oper on $X$ is the same as a $\mathfrak{g}$-oper on $X$ plus a lifting of the $H_{\mathrm{ad}}$-torsor $\lambda_*(\omega_X)$ to an $H$-torsor. Such a lifting is the same as an object of $\widetilde{Z}\,\mathrm{tors}_\omega(X)$: look at the right (Cartesian) square of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \widetilde{Z} & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\lambda} & & \\ 0 & \longrightarrow & Z & \longrightarrow & H & \longrightarrow & H_{\mathrm{ad}} & \longrightarrow & 0 \end{array} \tag{62}$$

(the upper row of (62) is (59) and the lower rows of (62) and (61) are the same).

**3.4.6.** $Z\,\mathrm{tors}(X)$ is a (strictly commutative) Picard category (see Definition 1.4.2 from [Del73]) and $Z\,\mathrm{tors}_\theta(X)$ is a Torsor over $Z\,\mathrm{tors}(X)$; actually $Z\,\mathrm{tors}_\theta(X)$ is induced from the Torsor $\omega^{1/2}(X)$ over $\mu_2\,\mathrm{tors}(X)$ via the Picard functor $\mu_2\,\mathrm{tors}(X) \to Z\,\mathrm{tors}(X)$ corresponding to (56). We will use this language in §4, so let us recall the definitions.

A *Picard category* is a tensor category $\mathcal{A}$ in the sense of [De-Mi] (i.e., a symmetric=commutative monoidal category) such that all the morphisms of $\mathcal{A}$ are invertible (i.e., $\mathcal{A}$ is a groupoid) and all the objects of $\mathcal{A}$ are invertible, i.e., for every $a \in \mathrm{Ob}\,\mathcal{A}$ there is an $a' \in \mathrm{Ob}\,\mathcal{A}$ such that $a \cdot a'$ is a unit object (we denote by $\cdot$ the "tensor product" functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$; in [De-Mi] and [Del73] it is denoted respectively by $\otimes$ and $+$). Strict commutativity means that for every $a \in \operatorname{Ob} \mathcal{A}$ the commutativity isomorphism $a \otimes a \xrightarrow{\sim} a \otimes a$ is the identity.

An *Action* of a monoidal category $\mathcal{A}$ on a category $C$ is a functor $\mathcal{A} \times C \to C$ (denoted by $\cdot$) equipped with an associativity constraint, i.e., a functorial isomorphism $(a_1 \cdot a_2) \cdot c \xrightarrow{\sim} a_1 \cdot (a_2 \cdot c)$, $a_i \in \mathcal{A}$, $c \in C$, satisfying the pentagon axiom analogous to the pentagon axiom for the associativity constraint in $\mathcal{A}$ (see [Del73] and [De-Mi]); we also demand the functor $F : C \to C$ corresponding to a unit object of $\mathcal{A}$ to be fully faithful (then the isomorphism $F^2 \xrightarrow{\sim} F$ yields a canonical isomorphism $F \xrightarrow{\sim} \mathrm{id}$). This definition can be found in [Pa] and §3 from [Yet]. An $\mathcal{A}$-*Module* is a category equipped with an Action of $\mathcal{A}$. If $C$ and $\widetilde{C}$ are $\mathcal{A}$-Modules then an $\mathcal{A}$-*Module functor* $C \to \widetilde{C}$ is a functor $\Phi \colon C \to \widetilde{C}$ equipped with a functorial isomorphism $\Phi(a \cdot c) \xrightarrow{\sim} a \cdot \Phi(c)$ satisfying the natural compatibility condition (the two ways of constructing an isomorphism $\Phi((a_1 \cdot a_2) \cdot c) \xrightarrow{\sim} a_1 \cdot (a_2 \cdot \Phi(c))$ must give the same result; see [Pa], [Yet]). $\mathcal{A}$-Module functors are also called *Morphisms* of $\mathcal{A}$-Modules.

A *Torsor* over a Picard category $\mathcal{A}$ is an $\mathcal{A}$-Module such that for some $c \in \operatorname{Ob} C$ the functor $a \mapsto a \cdot c$ is an equivalence between $\mathcal{A}$ and $C$ (then this holds for all $c \in \operatorname{Ob} C$).

Let $\mathcal{A}$ and $\mathcal{B}$ be Picard categories. A *Picard functor* $\mathcal{A} \to \mathcal{B}$ is a functor $F \colon \mathcal{A} \to \mathcal{B}$ equipped with a functorial isomorphism $F(a_1 \cdot a_2) \xrightarrow{\sim} F(a_1) \cdot F(a_2)$ compatible with the commutativity and associativity constraints. Then $F$ sends a unit object of $\mathcal{A}$ to a unit object of $\mathcal{B}$, i.e., $F$ is a tensor functor in the sense of [De-Mi]. In [Del73] Picard functors are called additive functors.

Let $F \colon \mathcal{A}_1 \to \mathcal{A}_2$ be a Picard functor and $C_i$ a torsor over $\mathcal{A}_i$, $i = 1, 2$. Then $C_2$ is equipped with an Action of $\mathcal{A}_1$. In this situation $\mathcal{A}_1$-Module functors $C_1 \to C_2$ are called $F$-*affine* functors.

**Examples.** 1) Let $A$ be a commutative algebraic group. Then $A \operatorname{tors}(X)$ has a canonical structure of Picard category.

2) A morphism $A \to B$ of commutative algeraic groups induces a Picard functor $A\,\mathrm{tors}(X) \to B\,\mathrm{tors}(X)$.
3) The groupoid $\omega^{1/2}(X)$ from 3.4.2 is a Torsor over the Picard category $\mu_2\,\mathrm{tors}(X)$. The groupoid $Z\,\mathrm{tors}_\theta(X)$ from 3.4.3 is a Torsor over $Z\,\mathrm{tors}(X)$.

If $F\colon \mathcal{A} \to \mathcal{B}$ is a Picard functor between Picard categories and $C$ is a Torsor over $\mathcal{A}$ then we can form the *induced* Torsor $\mathcal{B} \cdot_{\mathcal{A}} C$ over $\mathcal{B}$. The defintion of $\mathcal{B} \cdot_{\mathcal{A}} C$ can be reconstructed by the reader from the following example (see 3.4.3): if $\mathcal{A} = \mu_2\,\mathrm{tors}(X)$, $\mathcal{B} = Z\,\mathrm{tors}(X)$, $F$ comes from (56), and $C = \omega^{1/2}(X)$ then $\mathcal{B} \cdot_{\mathcal{A}} C = Z\,\mathrm{tors}_\theta(X)$. The objects of $\mathcal{B} \cdot_{\mathcal{A}} C$ are denoted by $b \cdot c$, $b \in \mathrm{Ob}\,\mathcal{B}$, $c \in \mathrm{Ob}\,C$ (see 3.4.3).

The interested reader can formulate the universal property of $\mathcal{B} \cdot_{\mathcal{A}} C$. We need the following weaker property. Given a category $\widetilde{C}$ with an Action of $\mathcal{B}$ and an $\mathcal{A}$-Module functor $\Phi\colon C \to \widetilde{C}$ there is a natural way to construct a $\mathcal{B}$-Module functor $\Psi\colon \mathcal{B} \cdot_{\mathcal{A}} C \to \widetilde{C}$: set $\Psi(b \cdot c) := b \cdot \Phi(c)$, and define $\Psi$ on morphisms in the obvious way (i.e., the map $\mathrm{Mor}(b_1 \cdot c_1, b_2 \cdot c_2) \to \mathrm{Mor}(b_1 \cdot \Phi(c_1), b_2 \cdot \Phi(c_2))$ is the composition $\mathrm{Mor}(b_1 \cdot c_1, b_2 \cdot c_2) = \mathrm{Mor}(b_1, b_2 \cdot c_2/c_1) \to \mathrm{Mor}(b_1 \cdot \Phi(c_1), b_2 \cdot c_2/c_1 \cdot \Phi(c_1)) \xrightarrow{\sim} \mathrm{Mor}(b_1 \cdot \Phi(c_1), b_2 \cdot \Phi(c_2)))$. The isomorphism $\Psi(b_1 \cdot (b_2 \cdot c)) \xrightarrow{\sim} b_1 \cdot \Psi(b_2 \cdot c)$ is the obvious one.

We will use this construction in the following situation. Suppose we have a Picard functor $\ell\colon \mathcal{B} \to \widetilde{\mathcal{B}}$, a Torsor $\widetilde{C}$ over $\widetilde{\mathcal{B}}$, and an $\ell'$-affine functor $\Phi\colon C \to \widetilde{C}$ where $\ell'$ is the composition $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{\ell} \widetilde{\mathcal{B}}$. Then the above construction yields an $\ell$-affine functor $\mathcal{B} \cdot_{\mathcal{A}} C \to \widetilde{C}$.

**3.4.7.** Let $Z$ be an abelian group and $Z\,\mathrm{tors}$ the Picard category of $Z$-torsors (over a point). The following remarks will be used in 4.4.9.

*Remarks*

(i) A Picard functor from $Z\,\mathrm{tors}$ to a Picard category $\mathcal{A}$ is "the same as" a morphism $Z \to \mathrm{Aut}\,1_{\mathcal{A}}$ where $1_{\mathcal{A}}$ is the unit object of $\mathcal{A}$. More precisely, the natural functor from the category of Picard functors

$Z\,\mathrm{tors} \to \mathcal{A}$ to $\mathrm{Hom}(Z, \mathrm{Aut}\, 1_{\mathcal{A}})$ is an equivalence. Here the set $\mathrm{Hom}(Z, \mathrm{Aut}\, 1_{\mathcal{A}})$ is considered as a discrete category.

(ii) The previous remark remains valid if "Picard" is replaced by "monoidal".

(iii) An Action of $Z\,\mathrm{tors}$ on a category $C$ is "the same as" a $Z$-structure on $C$, i.e., a morphism $Z \to \mathrm{Aut}\,\mathrm{id}_C$ (notice that an Action of a monoidal category $\mathcal{A}$ on $C$ is the same as a monoidal functor $\mathcal{A} \to \mathrm{Funct}(C, C)$ and apply the previous remark).

(iv) Let $C_1$ and $C_2$ be Modules over $Z\,\mathrm{tors}$. According to the previous remark $C_1$ and $C_2$ are $Z$-categories in the sense of 3.4.4. It is easy to see that a ($Z\,\mathrm{tors}$)-Module functor $C_1 \to C_2$ is the same as a $Z$-functor in the sense of 3.4.4 (i.e., a functor $F : C_1 \to C_2$ has at most one structure of a ($Z\,\mathrm{tors}$)-Module functor and such a structure exists if and only if $F$ is a $Z$-functor).

(v) A Torsor over $Z\,\mathrm{tors}$ is "the same as" a $Z$-category which is $Z$-equivalent to $Z\,\mathrm{tors}$. (do we need this???)

**3.5. Local opers II.** For most of the Lie algebras $\mathfrak{g}$ (e.g., $\mathfrak{g} = sl_n$, $n > 5$) the Feigin-Frenkel isomorphism (49) is *not uniquely* determined by the properties mentioned in Theorem 3.2.2 because $A_{\mathfrak{g}}(O)$ has a lot of $\mathrm{Aut}\, O$-equivariant automorphisms inducing the identity on $\mathrm{gr}\, A_{\mathfrak{g}}(O)$; this is clear from the geometric description of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \mathrm{Spec}\, A_{\mathfrak{g}}(O)$ in 3.2.1 or from the description of $A_{\mathfrak{g}}(O)$ that will be given in 3.5.6 (see (65)–(68)). The goal of 3.5–3.6 is to formulate the property 3.6.7 that uniquely determines the Feigin-Frenkel isomorphism. This property and also 3.6.11 will be used in the proof of our main theorem 5.2.6. In 3.7 and 3.8 we explain how to extract the properties 3.6.7 and 3.6.11 from [FF92]. One may (or perhaps should) read §4 and (a large part of ?) §5 before 3.5–3.8. We certainly recommend the reader to skip 3.5.16–3.5.18 and 3.6.8–3.6.11 before 3.6.11 is used in ??.

The idea[17] of 3.5 and 3.6 is to "kill" the automorphisms of $A_{\mathfrak{g}}(O)$ and its counterpart $\mathfrak{z}_{^L\mathfrak{g}}(O)$ by equipping these algebras with certain additional structures. In the case of $A_{\mathfrak{g}}(O)$ this is the Lie algebroid $\mathfrak{a}_{\mathfrak{g}}$ from 3.5.11. Its counterpart for $\mathfrak{z}_{^L\mathfrak{g}}(O)$ is introduced in 3.6.5. The definition of $\mathfrak{a}_{\mathfrak{g}}$ is simple: this is the algebroid of infinitesimal symmetries of the tautological $G$-bundle $\mathfrak{F}_G^0$ on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$. $\mathfrak{F}_G^0$ and therefore $\mathfrak{a}_{\mathfrak{g}}$ are equipped with an action of $\operatorname{Der} O$. It turns out that the pair $(A_{\mathfrak{g}}(O), \mathfrak{F}_G^0)$ has no nontrivial $\operatorname{Der} O$-equivariant automorphisms (see 3.5.9) and this is "almost" true for $(A_{\mathfrak{g}}(O), \mathfrak{a}_{\mathfrak{g}})$ (see 3.5.13).

**3.5.1.** We have a universal family of $\mathfrak{g}$-opers on $\operatorname{Spec} O$ parametrized by the scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \operatorname{Spec} A_{\mathfrak{g}}(O)$ from 3.2.1. Fix a one-dimensional free $O$-module $\omega_O^{1/2}$ equipped with an isomorphism $\omega_O^{1/2} \otimes \omega_O^{1/2} \xrightarrow{\sim} \omega_O$ (of course $\omega_O^{1/2}$ is unique up to isomorphism). Then the above universal family lifts to a family of $G$-opers; see 3.4.1[18]. So we have a $B$-bundle $\mathfrak{F}_B$ on $\operatorname{Spec}(A_{\mathfrak{g}}(O)\widehat{\otimes} O) = \operatorname{Spec} A_{\mathfrak{g}}(O)[[t]]$ and a connection $\nabla$ along $\operatorname{Spec} O$ on the associated $G$-bundle $\mathfrak{F}_G$.

**3.5.2.** Consider the group ind-scheme $\operatorname{Aut}_2 O := \operatorname{Aut}(O, \omega_O^{1/2})$. We have a canonical exact sequence

$$0 \to \mu_2 \to \operatorname{Aut}_2 O \to \operatorname{Aut} O \to 0 \tag{63}$$

and $\operatorname{Aut}_2 O$ is connected. The exact sequence (63) and the connectedness property can be considered as another definition of $\operatorname{Aut}_2 O$. Denote by $\operatorname{Aut}_2^0 O$ the preimage of $\operatorname{Aut}^0 O$ in $\operatorname{Aut}_2 O$.

$\operatorname{Aut} O$ acts on $A_{\mathfrak{g}}(O)$ and $O$, so it acts on $\operatorname{Spec}(A_{\mathfrak{g}}(O)\widehat{\otimes} O)$. This action lifts canonically to an action of $\operatorname{Aut}_2 O$ on $(\mathfrak{F}_B, \nabla)$. $\mu_2 \subset \operatorname{Aut}_2 O$ acts on $\mathfrak{F}_B$ via the morphism (56).

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[17]Inspired by [Phys]

[18]To tell the truth we must also choose a non-zero $y_\alpha \in \mathfrak{g}^\alpha$ for each negative simple root $\alpha$ (see 3.4.1)

**3.5.3.** *Lemma.* Let $L$ be an algebraic group, $A$ an algebra equipped with an action of $\operatorname{Aut} O$. Consider the action of $\operatorname{Aut} O$ on $A\widehat{\otimes} O$ induced by its actions on $A$ and $O$. Let $i: \operatorname{Spec} A \hookrightarrow \operatorname{Spec}(A\widehat{\otimes} O)$ be the natural embedding and $\pi: \operatorname{Spec}(A\widehat{\otimes} O) \to \operatorname{Spec} A$ the projection.

1) $i^*$ is an equivalence between the category of $\operatorname{Aut}_2 O$-equivariant $L$-bundles on $\operatorname{Spec}(A\widehat{\otimes} O)$ and that of $\operatorname{Aut}_2^0 O$-equivariant $L$-bundles on $\operatorname{Spec} A$.

2) $\pi^*$ is an equivalence between the category of $\operatorname{Aut}_2 O$-equivariant $L$-bundles on $\operatorname{Spec} A$ and that of $\operatorname{Aut}_2 O$-equivariant $L$-bundles on $\operatorname{Spec}(A\widehat{\otimes} O)$ equipped with an $\operatorname{Aut}_2 O$-invariant connection along $\operatorname{Spec} O$.

3) These equivalences are compatible with the forgetful functors $\{\operatorname{Aut}_2 O$-equivariant bundles on $\operatorname{Spec} A\} \to \{\operatorname{Aut}_2^0 O$-equivariant bundles on $\operatorname{Spec} A\}$ and $\{$bundles with connection$\} \to \{$bundles$\}$. $\square$

**3.5.4.** Denote by $\mathfrak{F}_B^0$ and $\mathfrak{F}_G^0$ the restrictions of $\mathfrak{F}_B$ and $\mathfrak{F}_G$ to $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \operatorname{Spec} A_{\mathfrak{g}}(O) \subset \operatorname{Spec} A_{\mathfrak{g}}(O)\widehat{\otimes} O$. $\mathfrak{F}_B^0$ is a $B$-bundle on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ and $\mathfrak{F}_G^0$ is the corresponding $G$-bundle. $\mathfrak{F}_B^0$ is $\operatorname{Aut}_2^0 O$-equivariant and according to 3.5.3 $\mathfrak{F}_G^0$ is $\operatorname{Aut}_2 O$-equivariant. Since the connection $\nabla$ on $\mathfrak{F}_G$ does not preserve $\mathfrak{F}_B$ the action of $\operatorname{Aut}_2 O$ on $\mathfrak{F}_G^0$ does not preserve $\mathfrak{F}_B^0$. According to 3.5.3 $\mathfrak{F}_G^0$ equipped with the action of $\operatorname{Aut}_2 O$ and the $B$-structure $\mathfrak{F}_B^0 \subset \mathfrak{F}_G^0$ "remembers" the universal oper $(\mathfrak{F}_B, \nabla)$.

**3.5.5.** Denote by $F_H^0$ the $H$-bundle on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ corresponding to $\mathfrak{F}_B^0$. Since $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ is an (infinite dimensional) affine space any $H$-bundle on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ is trivial and the action of $H$ on the set of its trivializations is transitive. In particular this applies to $\mathfrak{F}_H^0$, so $\mathfrak{F}_H$ is the pullback of some $H$-bundle $F_H$ over $\operatorname{Spec} \mathbb{C}$. According to 3.4.1 $F_H$ is the pushforward of the $\mathbb{G}_m$-bundle $\omega_O^{1/2}/t\omega_O^{1/2}$ over $\operatorname{Spec} \mathbb{C}$ via the morphism $\lambda^\#: \mathbb{G}_m \to H$ defined by (54). In particular the action of $\operatorname{Aut}_2^0 O$ on $F_H$ comes from the composition

$$\operatorname{Aut}_2^0 O \to \operatorname{Aut}(\omega_O^{1/2}/t\omega_O^{1/2}) = \mathbb{G}_m \xrightarrow{\lambda^\#} H\,.$$

So the action of $\mathrm{Der}^0 O$ on $F_H$ is the sum of the "obvious" action (the one which preserves any trivialization of $F_H$) and the morphism $\mathrm{Der}^0 O \to \mathfrak{h}$ defined by $f(t) \cdot t\dfrac{d}{dt} \mapsto f(0)\check{\rho}$. Here $\check{\rho}$ is the sum of fundamental coweights.

**3.5.6.** Here is an explicit description of $A_{\mathfrak{g}}(O)$ and $\mathfrak{F}_G^0$ in the spirit of 3.1.9–3.1.10. Let $e, f \in sl_2$ be the matrices from 3.1.8. Fix a principal embedding $i : sl_2 \hookrightarrow \mathfrak{g}$ such that $i(e) \in \mathfrak{b}$. If a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ is chosen so that $i([e, f]) \in \mathfrak{h}$ then $i([e, f])$ can be identified with $2\check{\rho}$. Just as in 3.1.9 set $V := \mathrm{Ker}\ \mathrm{ad}\ i(e)$. Choose a basis $e_1, \dots, e_r \in V$ so that $e_1 = i(e)$ and all $e_j$ are eigenvectors of $\mathrm{ad}\,\check{\rho}$. In fact $[\check{\rho}, e_j] = (d_j - 1)e_j$ where $d_j$ are the degrees of "basic" invariant polynomials on $\mathfrak{g}$ (in particular $d_1 = 2$). The connection

$$\nabla_{\frac{d}{dt}} = \frac{d}{dt} + i(f) + u_1(t)e_1 + \dots u_r(t)e_r \tag{64}$$

on the trivial $G$-bundle defines a $\mathfrak{g}$-oper and according to 3.1.10 this is a bijection between $\mathfrak{g}$-opers on $\mathrm{Spec}\, O$, $O := \mathbb{C}[[t]]$, and $r$-tuples $(u_1(t), \dots, u_r(t))$, $u_j(t) \in \mathbb{C}[[t]]$. Write $u_j(t)$ as $u_{j0} + u_{j1}t + \dots$. Then $A_{\mathfrak{g}}(O)$ is the ring of polynomials in $u_{jk}$, $1 \le j \le r$, $0 \le k < \infty$. The bundles $\mathfrak{F}_B$, $\mathfrak{F}_G$, $\mathfrak{F}_B^0$, $\mathfrak{F}_G^0$ from 3.5.1 and 3.5.4 are trivial and we have trivialized them by choosing the canonical form (64) for opers.

To describe the action of $\mathrm{Der}\, O$ on $A_{\mathfrak{g}}(O)$ and $\mathfrak{F}_G^0$ introduce the standard notation $L_n := -t^{n+1}\dfrac{d}{dt} \in \mathrm{Der}\, \mathbb{C}((t))$ (so $L_n \in \mathrm{Der}\, O$ for $n \ge -1$). Set $u_j := u_{j0}$. Then

$$u_{jk} = (L_{-1})^k u_j / k! \tag{65}$$

$$L_0 u_j = d_j u_j \tag{66}$$

$$L_n u_j = 0 \quad \text{if } n > 0\,, \quad j \ne 1 \tag{67}$$

$$L_n u_1 = 0 \quad \text{if } n > 0\,,\ n \ne 2\,; \quad L_2 u_1 = -3\,. \tag{68}$$

So $A_{\mathfrak{g}}(O)$ is the commutative $(\mathrm{Der}\, O)$-algebra with generators $u_1, \dots, u_r$ and defining relations (66)–(68). Denote by $L_n^{\mathrm{hor}}$ the vector field on $\mathfrak{F}_G^0$ that comes from our trivialization of $\mathfrak{F}_G^0$ and the action of $\operatorname{Der} O$ on $A_{\mathfrak{g}}(O)$. $L_n$ acts on $\mathfrak{F}_G^0$ as $L_n^{\mathrm{hor}} + M_n$, $M_n \in \mathfrak{g} \otimes A_{\mathfrak{g}}(O)$. One can show that

$$M_0 = -\check{\rho} \tag{69}$$

$$M_1 = -i(e)\,, \quad M_n = 0 \quad \text{for } n > 1 \tag{70}$$

$$M_{-1} = i(f) + u_1 e_1 + \ldots + u_r e_r$$

Only (69) will be used in the sequel (I'm afraid we'll use at least (70)) !???!).

*Remark.* If $n \geq 0$ then $M_n \in i(\mathfrak{b}_0) \subset \mathfrak{b} \subset \mathfrak{b} \otimes A_{\mathfrak{g}}(O)$ where $\mathfrak{b}_0 := \operatorname{Lie} B_0$ and $B_0 \subset SL_2$ is the group of upper-triangular matrices. This means that we have identified the $\operatorname{Aut}_2^0 O$-equivariant bundle $\mathfrak{F}_B^0$ with the pullback of a certain $\operatorname{Aut}_2^0 O$-equivariant $B$-bundle on $\operatorname{Spec} \mathbb{C}$ and the latter comes from a certain morphism $\operatorname{Aut}_2^0 O \to B_0 \to B$ (cf. Remark (iii) from 3.1.10).

**3.5.7.** Let $A$ be an algebra equipped with an action of $\operatorname{Aut} O$. Then $\operatorname{Der} O$ acts on $A$, the action of $L_0$ on $A$ is diagonalizable, and the eigenvalues of $L_0 : A \to A$ are integers. Assume that the eigenvalues of $L_0 : A/\mathbb{C} \to A/\mathbb{C}$ are positive. Then $A = \mathbb{C} \oplus A_+$ where $A_+$ is the sum of all eigenspaces of $L_0$ in $A$ corresponding to positive eigenvalues. $A_+$ is the unique $L_0$-invariant maximal ideal of $A$. The corresponding point of $\operatorname{Spec} A$ will be denoted by 0. Since $[L_0, L_n] = -nL_n$ we have $L_{-1}A_+ \subset A_+$. Assume that

$$L_1 A_+ \subset A_+\,. \tag{71}$$

In particular (71) is satisfied if the eigenvalues of $L_0$ on $A/\mathbb{C}$ are greater than 1, e.g., for $A = A_{\mathfrak{g}}(O)$ (see (66) and (65)).

Assume that $G$ is the adjoint group corresponding to $\mathfrak{g}$. Let $\mathcal{E}$ be an $\operatorname{Aut} O$-equivariant $G$-bundle on $\operatorname{Spec} A$. The algebra $\mathbb{C}L_{-1} + \mathbb{C}L_0 + \mathbb{C}L_1 \simeq sl_2$ stabilizes $0 \in \operatorname{Spec} A$, so it acts on the fiber of $\mathcal{E}$ over 0. Thus we obtain a morphism $\sigma : sl_2 \to \mathfrak{g}$ defined up to conjugation.

*Example.* The point $0 \in \operatorname{Spec} A_{\mathfrak{g}}(O)) = \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ is the push-forward via the principal embedding $sl_2 \to \mathfrak{g}$ of the $sl_2$-oper corresponding to the Sturm-Liouville operator $(d/dt)^2$. If $A = A_{\mathfrak{g}}(O)$ and $\mathcal{E} = \mathfrak{F}_G^0$ then $\sigma$ is the principal embedding.

**3.5.8.** *Proposition.*

1) The following conditions are equivalent:
   a) the $\operatorname{Aut} O$-equivariant $G$-bundle $\mathcal{E}$ is isomorphic to $\varphi^*\mathfrak{F}_G^0$ for some $\operatorname{Aut} O$-equivariant $\varphi :\ \operatorname{Spec} A \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$;
   b) there is an $\operatorname{Aut}^0 O$-invariant $B$-structure on $\mathcal{E}$ such that the corresponding $\operatorname{Aut}^0 O$-equivariant $H$-bundle is isomorphic to the pullback of the $\operatorname{Aut}^0 O$-equivariant $H$-bundle $F_H$ on $\operatorname{Spec} \mathbb{C}$ defined in 3.5.5[19];
   c) $\sigma :\ sl_2 \to \mathfrak{g}$ is the principal embedding.
2) The morphism $\varphi$ and the isomorphism $\mathcal{E} \xrightarrow{\sim} \varphi^*\mathfrak{F}_G^0$ mentioned in a) are unique.
3) The $B$-structure mentioned in b) is unique.

*Proof.* According to 3.5.5 b) follows from a). To deduce c) from b) just look what happens over $0 \in \operatorname{Spec} A$. Let us deduce a) from b) and show that 2) follows from 3). To do this it suffices to show that if a $B$-structure $\mathcal{E}_B \subset \mathcal{E}$ with the property mentioned in b) is fixed there is exactly one way to construct $\operatorname{Aut} O$-equivariant $\varphi :\ \operatorname{Spec} A \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ and $f :\ \mathcal{E} \xrightarrow{\sim} \varphi^*\mathfrak{F}_G^0$ so that $f(\mathcal{E}_B) = \varphi^*\mathfrak{F}_B^0$. According to 3.5.3 $\mathcal{E}$ and $\mathcal{E}_B$ yield a $G$-bundle $\widetilde{\mathcal{E}}_G$ on $\operatorname{Spec}(A\widehat{\otimes}O)$ with a $B$-structure $\widetilde{\mathcal{E}}_B \subset \widetilde{\mathcal{E}}_G$, a connection $\nabla$ on $\widetilde{\mathcal{E}}_G$ along $\operatorname{Spec} O$, and an action of $\operatorname{Aut} O$ on $(\widetilde{\mathcal{E}}_G, \widetilde{\mathcal{E}}_B, \nabla)$. Now the uniqueness of $\varphi$ and $f$ is clear and to prove their existence we must show that $(\widetilde{\mathcal{E}}_G, \widetilde{\mathcal{E}}_B, \nabla)$ is a family of opers, i.e., we must prove that $c(\nabla)$ defined in 3.1.2 satisfies conditions 1 and 2 from Definition 3.1.3. In our situation $c(\nabla)$ is an $\operatorname{Aut} O$-invariant section of $(\mathfrak{g}/\mathfrak{b})_{\widetilde{\mathcal{E}}} \bigotimes_O \omega_O$ and it is enough to verify conditions 1

[19]Since $G$ is the adjoint group the action of $\operatorname{Aut}_2^0 O$ on $F_H$ factors through $\operatorname{Aut}^0 O$

and 2 for its restriction $c_0(\nabla)$ to $\operatorname{Spec} A \subset \operatorname{Spec} A\widehat{\otimes}O$. $c_0(\nabla)$ is an $\operatorname{Aut}^0 O$-invariant element of $H^0(\operatorname{Spec} A, (\mathfrak{g}/\mathfrak{b})_{\mathcal{E}_B})\otimes\omega_O/t\omega_O$. Let $\operatorname{gr}^k\mathfrak{g}$ have the same meaning as in 3.1.1. Since we know the $H$-bundle corresponding to $\mathcal{E}_B$ we see that there is an $\operatorname{Aut}^0 O$-equivariant isomorphism

$$H^0(\operatorname{Spec} A, \operatorname{gr}^k\mathfrak{g}_{\mathcal{E}_B})\otimes\omega_O/t\omega_O \xrightarrow{\sim} A\otimes(\omega_O/t\omega_O)^{\otimes(k+1)}\otimes\operatorname{gr}^k\mathfrak{g}\,. \tag{72}$$

Since $L_0$ acts on $(\omega_O/t\omega_O)^{\otimes(k+1)}$ as multiplication by $-k-1$ the $\operatorname{Aut}^0 O$-invariant part of $A\otimes(\omega_O/t\omega_O)^{\otimes(k+1)}$ equals 0 if $k<-1$ and $\mathbb{C}$ if $k=-1$. Therefore

$$c_0(\nabla)\in\operatorname{gr}^{-1}\mathfrak{g}\subset A\otimes\operatorname{gr}^{-1}\mathfrak{g}=H^0(\operatorname{Spec} A,\operatorname{gr}^{-1}\mathfrak{g}_{\mathcal{E}_B})\otimes\omega_O/t\omega_O\,.$$

So we have checked condition 1 from 3.1.3 and it remains to check condition 2 over some point of $\operatorname{Spec} A$, e.g., over $0\in\operatorname{Spec} A$. Denote by $(\widetilde{\mathcal{E}}^0_G,\widetilde{\mathcal{E}}^0_B,\nabla)$ the restriction of $(\widetilde{\mathcal{E}}_G,\widetilde{\mathcal{E}}_B,\nabla)$ to $\{0\}\times\operatorname{Spec} O\subset\operatorname{Spec}(A\widehat{\otimes}O)$. Then $\widetilde{\mathcal{E}}^0_G$ is the trivial $G$-bundle, $\nabla$ is the trivial connection, $sl_2$ acts on $(\widetilde{\mathcal{E}}^0_G,\nabla)$ via the morphism $\sigma: sl_2\to\mathfrak{g}$ mentioned in 3.5.7 and the embedding $sl_2=\mathbb{C}L_{-1}+\mathbb{C}L_0+\mathbb{C}L_1\hookrightarrow\operatorname{Der} O$, $\widetilde{\mathcal{E}}^0_B$ is invariant with respect to $sl_2$. Since $\sigma$ is the principal embedding $(\widetilde{\mathcal{E}}^0_G,\widetilde{\mathcal{E}}^0_B,\nabla)$ is the oper corresponding to $0\in\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$.

Let us prove 3). Set $\mathfrak{a}=H^0(\operatorname{Spec} A,\mathfrak{g}_{\mathcal{E}})$, $\mathfrak{a}_k:=\{a\in\mathfrak{a}|L_0a=ka\}$. If a $B$-structure on $\mathcal{E}$ is fixed then the filtration $\mathfrak{g}^k$ from 3.1.1 induces a filtration $\mathfrak{a}^k$ on $\mathfrak{a}$. If the $B$-structure has the property mentioned in b) then $\mathfrak{a}^k$ is $\operatorname{Aut}^0 O$-invariant and $\mathfrak{a}^k/\mathfrak{a}^{k+1}$ is $\operatorname{Aut}^0 O$-isomorphic to $A\otimes(\omega_O/t\omega_O)^{\otimes k}\otimes\operatorname{gr}^k\mathfrak{g}$ (see (72)). Therefore the eigenvalues of $L_0$ on $\mathfrak{a}^k/\mathfrak{a}^{k+1}$ are $\geq -k$ and the $A$-module $\mathfrak{a}^k/\mathfrak{a}^{k+1}$ is generated by its $L_0$-eigenvectors corresponding to the eigenvalue $-k$. So

$$\mathfrak{a}^k=\sum_{i\leq -k}A\mathfrak{a}_i\,. \tag{73}$$

The $B$-structure on $\mathcal{E}$ is reconstructed from the Borel subalgebra $\mathfrak{a}^0\subset\mathfrak{a}$.

It remains to deduce b) from c). Define $\mathfrak{a}^k$ by (73). Since $\mathfrak{a}$ is a free $L_0$-graded $A$-module of finite type so are $\mathfrak{a}^k/\mathfrak{a}^{k+1}$. Therefore $\mathfrak{a}^k$ defines a vector subbundle of $\mathfrak{g}_{\mathcal{E}}$. If $k = 0$ this subbundle is a Lie subalgebra, so it defines a section $s : \operatorname{Spec} A \to S_{\mathcal{E}}$ where $S$ is the scheme of subalgebras of $\mathfrak{g}$. An infinitesimal calculation shows that the morphism $G/B \to S$, $g \mapsto g\mathfrak{b}g^{-1}$, is an open embedding and since $G/B$ is projective it is also a closed embedding. According to c) $s(0) \in (G/B)_{\mathcal{E}} \subset S_{\mathcal{E}}$, so $s(\operatorname{Spec} A) \subset (G/B)_{\mathcal{E}}$ and $s$ defines a $B$-structure on $\mathcal{E}$. Clearly it is $\operatorname{Aut}^0 O$-invariant. The corresponding $\operatorname{Aut}^0 O$-equivariant $H$-bundle on $\operatorname{Spec} A$ is the pullback of some $\operatorname{Aut}^0 O$-equivariant $H$-bundle $F$ on $\operatorname{Spec} \mathbb{C}$ (this is true for *any* $\operatorname{Aut}^0 O$-equivariant $H$-bundle on $\operatorname{Spec} A$ and *any* torus $H$; indeed, one can assume that $H = \mathbb{G}_m$, interpret a $\mathbb{G}_m$-bundle as a line bundle and use the fact that a graded projective $A$-module of finite type is free). To find $F$ look what happens over $0 \in \operatorname{Spec} A$. □

*Remark.* The proof of Proposition 3.5.8 shows that if c) is satisfied then there is a unique $\operatorname{Aut}^0 O$-invariant $B$-structure on $\mathcal{E}$.

**3.5.9.** *Corollary.* If $G$ is the adjoint group then the pair $(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O), \mathfrak{F}_G^0)$ has no nontrivial $\operatorname{Aut} O$-equivariant automorphisms.

This is statement 2) of Proposition 3.5.8 for $A = A_{\mathfrak{g}}(O)$.

**3.5.10.** Recall that a *Lie algebroid* over a commutative $\mathbb{C}$-algebra $R$ is a Lie $\mathbb{C}$-algebra $\mathfrak{a}$ equipped with an $R$-module structure and a map $\varphi : \mathfrak{a} \to \operatorname{Der} R$ such that 1) $\varphi$ is a Lie algebra morphism and an $R$-module morphism, 2) for $a_1, a_2 \in \mathfrak{a}$ and $f \in R$ one has $[a_1, fa_2] = f[a_1, a_2] + v(f)a_2$, $v := \varphi(a_1)$.

*Remarks*

(i) [Ma87] and [Ma96] are standard references on Lie algebroids and Lie groupoids. See also [We] and [BB93]. In this paper we need only the definition of Lie algebroid.

(ii) Lie algebroids are also known under the name of ($\mathbb{C}$, $R$)-Lie algebras (see [R]) and under a variety of other names (see [Ma96]).

**3.5.11.** Denote by $\mathfrak{a}_{\mathfrak{g}}$ the space of (global) infinitesimal symmetries of $\mathfrak{F}_G^0$. Elements of $\mathfrak{a}_{\mathfrak{g}}$ are pairs consisting of a vector field on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \operatorname{Spec} A_{\mathfrak{g}}(O)$

(i.e., a derivation of $A_{\mathfrak{g}}(O)$) and its lifting to a $G$-invariant vector field on the principal $G$-bundle $\mathfrak{F}^0_G$. $\mathfrak{a}_{\mathfrak{g}}$ is a Lie algebroid over $A_{\mathfrak{g}}(O)$. We have a canonical exact sequence.

$$0 \to \mathfrak{g}_{\mathrm{univ}} \to \mathfrak{a}_{\mathfrak{g}} \to \operatorname{Der} A_{\mathfrak{g}}(O) \to 0$$

where $\mathfrak{g}_{\mathrm{univ}}$ is the space of global sections of the $\mathfrak{F}^0_G$-twist of $\mathfrak{g}$. Of course $\mathfrak{a}_{\mathfrak{g}}$ and $\mathfrak{g}_{\mathrm{univ}}$ do not change if $G$ is replaced by the adjoint group $G_{\mathrm{ad}}$. So $\mathfrak{a}_{\mathfrak{g}}$ and $\mathfrak{g}_{\mathrm{univ}}$ do not depend on the choice of $\omega_O^{1/2}$.

The action of $\operatorname{Der} O$ on $\mathfrak{F}^0_G$ induces a Lie algebra morphism $\operatorname{Der} O \to \mathfrak{a}_{\mathfrak{g}}$. In particular $\operatorname{Der} O$ acts on $\mathfrak{a}_{\mathfrak{g}}$.

**3.5.12.** *Lemma.* The adjoint representation of $\mathfrak{a}_{\mathfrak{g}}$ on $\mathfrak{g}_{\mathrm{univ}}$ defines an isomorphism between $\mathfrak{a}_{\mathfrak{g}}$ and the algebroid of infinitesimal symmetries of $\mathfrak{g}_{\mathrm{univ}}$. $\square$

**3.5.13.** *Proposition.* The group of $\operatorname{Der} O$-equivariant automorphisms of the pair $(A_{\mathfrak{g}}(O), \mathfrak{a}_{\mathfrak{g}})$ equals $\operatorname{Aut}\Gamma$ where $\Gamma$ is the Dynkin graph of $\mathfrak{g}$.

*Proof.* Let $G$ be the adjoint group corresponding to $\mathfrak{g}$. Denote by $L$ the group of $\operatorname{Der} O$-equivariant automorphisms of $(A_{\mathfrak{g}}(O), \mathfrak{g}_{\mathrm{univ}})$. According to 3.5.12 we have to show that $L = \operatorname{Aut}\Gamma$. We have the obvious morphisms $i : \operatorname{Aut}\Gamma = \operatorname{Aut}(G,B)/B \to L$ and $\pi : L \to \operatorname{Aut}\Gamma$ such that $\pi i = \mathrm{id}$. $\operatorname{Ker}\pi$ is the group of $\operatorname{Der} O$-equivariant automorphisms of $(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O), \mathfrak{F}^0_G)$, so $\operatorname{Ker}\pi$ is trivial according to 3.5.9. $\square$

**3.5.14.** *Proposition.* The pair $(A_{\mathfrak{g}}(O), \mathfrak{a}_{\mathfrak{g}})$ does not have nontrivial $\operatorname{Der} O$-equivariant automorphisms inducing the trivial automorphism of $\operatorname{gr} A_{\mathfrak{g}}(O)$ (gr corresponds to the filtration from 3.2.1).

*Proof.* Let $\Gamma$ be the Dynkin graph of $\mathfrak{g}$. According to 3.5.13 and (48) we have to show that the action of $\operatorname{Aut}\Gamma$ on the algebra $\mathfrak{z}^{cl}_{L_{\mathfrak{g}}}(O)$ from 2.7.1 is exact. So it suffices to show that the action of $\operatorname{Aut}\Gamma$ on $W\backslash\mathfrak{h}$ is exact ($W$ denotes the Weyl group). Let $C \subset \operatorname{Aut}\mathfrak{h}$ be the automorphism group of the root system. There is an $a \in \mathfrak{h}$ whose stabilizer in $C$ is trivial. So the action of $\operatorname{Aut}\Gamma = C/W$ on $W\backslash\mathfrak{h}$ is exact. □

**3.5.15.** We equip $\mathfrak{a}_{\mathfrak{g}}$ with the weakest translation-invariant topology such that the stabilizer of any regular function on the total space of $\mathfrak{F}_G^0$ is open (recall that $\mathfrak{a}_{\mathfrak{g}}$ acts on $\mathfrak{F}_G^0$). This is the weakest translation-invariant topology such that the $\mathfrak{a}_{\mathfrak{g}}$-centralizer of every element of $\mathfrak{g}_{\text{univ}}$ is open. So the topology is reconstructed from the Lie algebroid structure on $\mathfrak{a}_{\mathfrak{g}}$.

Clearly the canonical morphism $\operatorname{Der} O \to \mathfrak{a}_{\mathfrak{g}}$ is continuous.

**3.5.16.** Denote by $\mathfrak{a}_{\mathfrak{b}}$ the Lie algebroid of (global) infinitesimal symmetries of $\mathfrak{F}_B^0$. Let $\mathfrak{b}_{\text{univ}}$ (resp. $\mathfrak{n}_{\text{univ}}$) denote the space of global sections of the $\mathfrak{F}_B^0$-twist of $\mathfrak{b}$ (resp. $\mathfrak{n}$). There is a canonical exact sequence

$$0 \to \mathfrak{b}_{\text{univ}} \to \mathfrak{a}_{\mathfrak{b}} \to \operatorname{Der} A_{\mathfrak{g}}(O) \to 0\,.$$

$\mathfrak{a}_{\mathfrak{b}}$ is a subalgebroid of $\mathfrak{a}_{\mathfrak{g}}$; in fact $\mathfrak{a}_{\mathfrak{b}}$ is the normalizer of $\mathfrak{b}_{\text{univ}} \subset \mathfrak{a}_{\mathfrak{g}}$. The image of $\operatorname{Der}^0 O$ in $\mathfrak{a}_{\mathfrak{g}}$ is contained in $\mathfrak{a}_{\mathfrak{b}}$.

$\mathfrak{n}_{\text{univ}}$ is an ideal in $\mathfrak{a}_{\mathfrak{b}}$ and $\mathfrak{a}_{\mathfrak{b}}/\mathfrak{n}_{\text{univ}}$ is the algebroid of (global) infinitesimal symmetries of $\mathfrak{F}_H^0$. Since $\mathfrak{F}_H^0$ is trivial and its trivialization is "almost" unique (see 3.5.5) $\mathfrak{a}_{\mathfrak{b}}/\mathfrak{n}_{\text{univ}}$ is canonically isomorphic to the semidirect sum of $\operatorname{Der} A_{\mathfrak{g}}(O)$ and $A_{\mathfrak{g}}(O) \otimes \mathfrak{h}$. Denote by $\mathfrak{a}_{\mathfrak{n}}$ the preimage of $\operatorname{Der} A_{\mathfrak{g}}(O) \subset \mathfrak{a}_{\mathfrak{b}}/\mathfrak{n}_{\text{univ}}$ in $\mathfrak{a}_{\mathfrak{b}}$.

*Remark.* According to 3.5.5 the composition $\operatorname{Der}^0 O \to \mathfrak{a}_{\mathfrak{b}}/\mathfrak{n}_{\text{univ}} = \operatorname{Der} A_{\mathfrak{g}}(O) \oplus (A_{\mathfrak{g}}(O) \otimes \mathfrak{h})$ is contained in $\operatorname{Der} A_{\mathfrak{g}}(O) \oplus \mathfrak{h}$; it is equal to the sum of the natural morphism $\operatorname{Der}^0 O \to \operatorname{Der} A_{\mathfrak{g}}(O)$ and the morphism $\operatorname{Der}^0 O \to \mathfrak{h}$ such that $L_0 \mapsto -\check{\rho}$, $L_n \mapsto 0$ for $n > 0$.

**3.5.17.** We are going to describe $\mathfrak{a}_{\mathfrak{b}}$, $\mathfrak{b}_{\text{univ}}$, etc. in terms of the action of $L_0$ on $\mathfrak{a}_{\mathfrak{g}}$. The following notation will be used. If $\operatorname{Der} O$ acts on a topological vector space $V$ so that the eigenvalues of $L_0 : V \to V$ are integers denote by $V^{\leq k}$ the smallest closed subspace of $V$ containing all $v \in V$ such that $L_0 v = nv$, $n \leq k$. Set $V^{<k} := V^{\leq k-1}$. If $V$ is a topological module over

some algebra $A$ and $W$ is a subspace of $V$ we denote by $A\cdot W$ the smallest closed subspace of $V$ containing $aw$ for every $a\in A$ and $w\in W$.

**3.5.18.** *Proposition.* i) The following equalities hold:

$$\mathfrak{b}_{\mathrm{univ}}=A_{\mathfrak{g}}(O)\cdot(\mathfrak{g}_{\mathrm{univ}})^{\leq 0} \tag{74}$$

$$\mathfrak{n}_{\mathrm{univ}}=A_{\mathfrak{g}}(O)\cdot\mathfrak{g}_{\mathrm{univ}}^{<0} \tag{75}$$

$$\mathfrak{a}_{\mathfrak{b}}=A_{\mathfrak{g}}(O)\cdot(\mathfrak{a}_{\mathfrak{g}})^{\leq 0} \tag{76}$$

$$\mathfrak{a}_{\mathfrak{n}}=A_{\mathfrak{g}}(O)\cdot\mathfrak{a}_{\mathfrak{g}}^{<0} \tag{77}$$

ii) The image of the morphism

$$(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}\to A_{\mathfrak{g}}(O)(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}/A_{\mathfrak{g}}(O)\mathfrak{a}_{\mathfrak{g}}^{<0}=\mathfrak{a}_{\mathfrak{b}}/\mathfrak{a}_{\mathfrak{n}}=A_{\mathfrak{g}}(O)\otimes\mathfrak{h}$$

equals $\mathfrak{h}$, so we have a canonical isomorphism

$$(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}/(A_{\mathfrak{g}}(O)\cdot\mathfrak{a}_{\mathfrak{g}}^{<0}\cap(\mathfrak{a}_{\mathfrak{g}})^{\leq 0})\xrightarrow{\sim}\mathfrak{h} \tag{78}$$

*Proof.* i) (74)–(77) follow from (69). Or one can notice that (74) and (75) are particular cases of (73) and prove, e.g., (76) as follows. According to (74) $A_{\mathfrak{g}}(O)\cdot(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}\supset\mathfrak{b}_{\mathrm{univ}}$ and $A_{\mathfrak{g}}(O)\cdot(\operatorname{Der}A_{\mathfrak{g}}(O))^{\leq 0}=\operatorname{Der}A_{\mathfrak{g}}(O)$, so $A_{\mathfrak{g}}(O)\cdot(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}\supset\mathfrak{a}_{\mathfrak{b}}$. $A_{\mathfrak{g}}(O)\cdot(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}\subset\mathfrak{a}_{\mathfrak{b}}$ because $(\mathfrak{a}_{\mathfrak{g}}/\mathfrak{a}_{\mathfrak{b}})^{\leq 0}=(\mathfrak{g}_{\mathrm{univ}}/\mathfrak{b}_{\mathrm{univ}})^{\leq 0}=(\mathfrak{g}_{\mathrm{univ}})^{\leq 0}/(\mathfrak{b}_{\mathrm{univ}})^{\leq 0}=0$ according to (74).

ii) The image of $(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}$ in $A_{\mathfrak{g}}(O)\otimes\mathfrak{h}$ equals $(A_{\mathfrak{g}}(O)\otimes\mathfrak{h})^{\leq 0}=\mathfrak{h}$. □

### 3.6. Feigin-Frenkel isomorphism II.

**3.6.1.** Let $A$ be an associative algebra over $\mathbb{C}[h]$ flat as a $\mathbb{C}[h]$-module. Set $A_0:=A/hA$. Denote by $\mathfrak{Z}$ the center of $A_0$. If $\mathfrak{Z}=A_0$, i.e., if $A_0$ is commutative, then $\mathfrak{Z}$ is equipped with the standard Poisson bracket

$$\{z_1,z_2\}:=[\widetilde{z}_1,\widetilde{z}_2]/h \bmod h \tag{79}$$

where $z_1, z_2 \in \mathfrak{Z}$ and $\widetilde{z}_i$ is a preimage of $z_i$ in $A$. Hayashi noticed in [Ha88] that even without the assumption $\mathfrak{Z} = A_0$ (79) is a well-defined Poisson bracket on $\mathfrak{Z}$ (in particular the r.h.s. of (79) belongs to $\mathfrak{Z}$).

*Remarks*

(i) In the above situation there is a canonical Lie algebra morphism $\varphi : \mathfrak{Z} \to \operatorname{Der} A_0 / \operatorname{Int} A_0$ where $\operatorname{Int} A_0$ is the space of inner derivations. $\varphi$ is defined by $\varphi(z) = D_z$, $D_z(a) := [\widetilde{z}, \widetilde{a}]/h \bmod h$ where $\widetilde{z}, \widetilde{a} \in A$ are preimages of $z \in \mathfrak{Z}$ and $a \in A_0$. If $z' \in \mathfrak{Z}$ then $D_z(z') = \{z, z'\}$. $\operatorname{Der} A_0 / \operatorname{Int} A_0$ is a $\mathfrak{Z}$-module and $\varphi(z_1 z_2) = z_1 \varphi(z_2) + z_2 \varphi(z_1)$. So $\varphi$ induces a $\mathfrak{Z}$-module morphism $\Phi : \Omega^1_{\mathfrak{Z}} \to \operatorname{Der} A_0 / \operatorname{Int} A_0$. In fact $\Phi$ is a morphism of Lie algebroids over $\mathfrak{Z}$ (see 3.5.10 for the definition of Lie algebroid); the Lie algebroid structure on $\operatorname{Der} A_0 / \operatorname{Int} A_0$ is defined in the obvious way and the one on $\Omega^1_{\mathfrak{Z}}$ is the standard algebroid structure induced by the Poisson bracket on $\mathfrak{Z}$ (cf. [We88] ), i.e., $[dz, dz'] := d\{z, z'\}$ for $z, z' \in \mathfrak{Z}$ and the morphism $\Omega^1_{\mathfrak{Z}} \to \operatorname{Der} \mathfrak{Z}$ maps $dz$ to $\operatorname{grad} z$, $(\operatorname{grad} z)(z') := \{z, z'\}$.

(ii) The above constructions make sense if $\mathbb{C}[h]$ is replaced by $\mathbb{C}[h]/(h^3)$.

**3.6.2.** Now let $\mathfrak{g}$ be a semisimple Lie algebra and $K := \mathbb{C}((t))$. Denote by $A$ the completed universal enveloping algebra of the Lie algebra $\widetilde{\mathfrak{g} \otimes K}$ from 2.5.1, i.e., $A := \varprojlim_n (U\widetilde{\mathfrak{g} \otimes K})/J_n$ where $J_n \subset U\widetilde{\mathfrak{g} \otimes K}$ is the left ideal generated by $\mathfrak{g} \otimes t^n \mathbb{C}[[t]] \subset \mathfrak{g} \otimes K \subset \widetilde{\mathfrak{g} \otimes K}$, $n \geq 0$. Consider the $\mathbb{C}[h]$-algebra structure on $A$ defined by $ha = \mathbf{1} \cdot a - a$, $a \in A$, where $\mathbf{1} \in \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K} \subset A$. $A$ is flat over $\mathbb{C}[h]$ and $A/hA$ is the completed twisted universal enveloping algebra $\overline{U}' = \overline{U}'(\mathfrak{g} \otimes K)$ from 2.5.2 and 2.9.4. So (79) defines a Poisson bracket on the center $\mathfrak{Z}$ of $\overline{U}'$. It was introduced in [Ha88], so we call it the *Hayashi bracket.*

**3.6.3.** For an open Lie subalgebra $\mathfrak{a} \subset \mathfrak{g} \otimes O$ denote by $\mathcal{I}_\mathfrak{a}$ (resp. $\widetilde{\mathcal{I}}_\mathfrak{a}$) the closure of the left ideal of $\overline{U}'$ (resp. of $A = \overline{U}\widetilde{\mathfrak{g} \otimes K}$) generated by $\mathfrak{a} \subset \mathfrak{g} \otimes O \subset \widetilde{\mathfrak{g} \otimes K}$. Clearly $\mathcal{I}_\mathfrak{a}$ is the image of $\widetilde{\mathcal{I}}_\mathfrak{a}$ in $\overline{U}'$. Set $I_\mathfrak{a} := \mathcal{I}_\mathfrak{a} \cap \mathfrak{Z}$. We equip $\mathfrak{Z}$ with the topology induced from $\overline{U}'$. The ideals $\mathcal{I}_{\mathfrak{a}}$ (resp. $I_{\mathfrak{a}}$) form a base of neighbourhoods of zero in $\overline{U}'$ (resp. in $\mathfrak{Z}$).

**3.6.4.** *Lemma.*

(i) $\{I_{\mathfrak{a}}, I_{\mathfrak{a}}\} \subset I_{\mathfrak{a}}$.

(ii) The Hayashi bracket on $\mathfrak{Z}$ is continuous.

*Proof.* Use the fact that $A/\widetilde{\mathcal{I}}_{\mathfrak{a}}$ equipped with the $\mathbb{C}[h]$-module structure from 3.6.2 is flat. □

**3.6.5.** Set $I := I_{\mathfrak{g}\otimes O}$. The canonical morphism $\mathfrak{Z} \to \mathfrak{z}_{\mathfrak{g}}(O)$ is surjective (see 2.9.3–2.9.5) and its kernel equals $I$. So $\mathfrak{z}_{\mathfrak{g}}(O) = \mathfrak{Z}/I$.

Denote by $I^2$ the closed ideal of $\mathfrak{Z}$ generated by elements of the form $ab$ where $a, b \in I$. Then $I/I^2$ is a Lie algebroid over $\mathfrak{z}_{\mathfrak{g}}(O)$ (the commutator $I/I^2 \times I/I^2 \to I/I^2$ and the mapping $I/I^2 \to \operatorname{Der} \mathfrak{z}_{\mathfrak{g}}(O)$ are induced by the Hayashi bracket). The Lie algebra $\operatorname{Der} O$ acts on $I/I^2$ and $\mathfrak{z}_{\mathfrak{g}}(O)$. These actions are continuous ($I/I^2$ is equipped with the topology induced from $\mathfrak{Z}$ and $\mathfrak{z}_{\mathfrak{g}}(O)$ is discrete).

**3.6.6.** Let us formulate a more precise version of Theorem 3.2.2. We have the algebra $\mathfrak{z}_{\mathfrak{g}}(O)$ and the Lie algebroid $I/I^2$ over $\mathfrak{z}_{\mathfrak{g}}(O)$. On the other hand denote by ${}^L\mathfrak{g}$ the Langlands dual and consider the algebra $A_{{}^L\mathfrak{g}}(O)$ (see 3.2.1) and the Lie algebroid $\mathfrak{a}_{{}^L\mathfrak{g}}$ over it (see 3.5.11). $I/I^2$ and $\mathfrak{a}_{{}^L\mathfrak{g}}$ are equipped with topologies (see 3.6.5 and 3.5.15). The Lie algebra $\operatorname{Der} O$ acts on all these objects. $\mathfrak{z}_{\mathfrak{g}}(O)$ and $A_{{}^L\mathfrak{g}}(O)$ are equipped with filtrations (see 1.2.5 and 3.2.1), and we have the morphism $\sigma_A^{-1}\sigma_{\mathfrak{z}} : \operatorname{gr} \mathfrak{z}_{\mathfrak{g}}(O) \to \operatorname{gr} A_{{}^L\mathfrak{g}}(O)$ where $\sigma_{\mathfrak{z}} : \operatorname{gr} \mathfrak{z}_{\mathfrak{g}}(O) \to \mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is the symbol map and $\sigma_A$ is the isomorphism (48) with $\mathfrak{g}$ replaced by ${}^L\mathfrak{g}$.

**3.6.7.** *Theorem.* There is an isomorphism of filtered $\operatorname{Der} O$-algebras

$$\varphi_O : A_{{}^L\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(O) \tag{80}$$

such that $\operatorname{gr}\varphi_O^{-1} = \sigma_A^{-1}\sigma_{\mathfrak{z}}$ and $\varphi_O$ extends to a topological $\operatorname{Der} O$-equivariant isomorphism of Lie algebroids

$$\mathfrak{a}_{{}^L\mathfrak{g}} \xrightarrow{\sim} I/I^2\,. \tag{81}$$

This theorem can be extracted from [FF92] (see 3.7.12–3.7.17).

*Remark.* According to 3.5.14 the isomorphisms (80) and (81) are unique.

In 3.6.11 we will formulate an additional property of the isomorphism (81). But first we must define an analog of (78) for the algebroid $I/I^2$.

**3.6.8.** We will use the notation from 3.5.17.

*Lemma.* Set $\mathcal{I}_- := (\overline{U}')^{\leq 0} \cap \mathcal{I}_{\mathfrak{a}}$ where $\mathfrak{a} = t\mathfrak{g}[[t]]$ and $\mathcal{I}_{\mathfrak{a}}$ was defined in 3.6.3. Then $\mathcal{I}_-$ is a two-sided ideal in $(\overline{U}')^{\leq 0}$ and

$$(\overline{U}')^{\leq 0} = U\mathfrak{g} \oplus \mathcal{I}_-\,. \tag{82}$$

*Proof.* (82) is clear. Since $\mathcal{I}_-$ is a left ideal and $[\mathfrak{g}, \mathcal{I}_-] \subset \mathcal{I}_-$ (82) implies that $\mathcal{I}_-$ is a two-sided ideal. □

Define $\pi : (\overline{U}')^{\leq 0} \to U\mathfrak{g}$ to be the morphism such that $\pi(\mathcal{I}_-) = 0$ and $\pi(a) = a$ for $a \in U\mathfrak{g}$.

Here is an equivalent definition of $\pi$. Set $Vac'_{\mathfrak{a}} := \overline{U}'/\mathcal{I}_{\mathfrak{a}}$, $\mathfrak{a} = t\mathfrak{g}[[t]]$. Then $Vac'_{\mathfrak{a}}$ is a left $\overline{U}'$-module and a right $U\mathfrak{g}$-module. The eigenvalues of $L_0$ on $Vac'_{\mathfrak{a}}$ are non-negative and $\operatorname{Ker}(L_0 : Vac'_{\mathfrak{a}} \to Vac'_{\mathfrak{a}}) = U\mathfrak{g}$. So $U\mathfrak{g} \subset Vac'_{\mathfrak{a}}$ is invariant with respect to the left action of $(\overline{U}')^{\leq 0}$. The left action of $(\overline{U}')^{\leq 0}$ commutes with the right action of $U\mathfrak{g}$, so it defines a morphism $(\overline{U}')^{\leq 0} \to U\mathfrak{g}$. This is $\pi$.

**3.6.9.** Denote by $C$ the center of $U\mathfrak{g}$. Then

$$\pi(\mathfrak{Z}^{\leq 0}) \subset C\,, \qquad \pi(\mathfrak{Z}\cdot\mathfrak{Z}^{<0} \cap \mathfrak{Z}^{\leq 0}) = 0\,.$$

Let $m \subset C$ be the maximal ideal corresponding to the unit representation of $U\mathfrak{g}$. Recall that $I := \operatorname{Ker}(\mathfrak{Z} \to \mathfrak{z}_{\mathfrak{g}}(O))$. Then $\pi(I^{\leq 0}) \subset m$. Since $(I^2)^{\leq 0} \subset I^{\leq 0}\cdot I^{\leq 0} + (\mathfrak{Z}\cdot\mathfrak{Z}^{<0} \cap \mathfrak{Z}^{\leq 0})$ one has $\pi((I^2)^{\leq 0}) \subset m^2$. So $\pi$ induces a $\mathbb{C}$-linear map $d : (I/I^2)^{\leq 0} \to m/m^2$ such that $\mathfrak{z}_{\mathfrak{g}}(O)\cdot(I/I^2)^{<0} \cap (I/I^2)^{\leq 0} \subset \operatorname{Ker} d$.

*Exercise.* $\pi(\{z_1, z_2\}) = 0$ for $z_1, z_2 \in \mathfrak{Z}^{\leq 0}$ (so $d$ is a Lie algebra morphism).

**3.6.10.** Identify $C$ with the algebra of $W$-invariant polynomials on $\mathfrak{h}^*$ where $W$ is the Weyl group. Then $m$ consists of $W$-invariant polynomials on $\mathfrak{h}^*$ vanishing at $\rho :=$ the sum of fundamental weights. Since $\rho \in \mathfrak{h}^*$ is regular we can identify $m/m^2$ with $\mathfrak{h}$ by associating to a $W$-invariant polynomial from $m$ its differential at $\rho$. So we have constructed a map

$$d: (I/I^2)^{\leq 0}/(\mathfrak{z}_{\mathfrak{g}}(O) \cdot (I/I^2)^{<0} \cap (I/I^2)^{\leq 0}) \to \mathfrak{h} \tag{83}$$

**3.6.11.** *Theorem.* The diagram

$$\begin{array}{ccc} (\mathfrak{a}_{^L\mathfrak{g}})^{\leq 0}/(A_{^L\mathfrak{g}}(O) \cdot \mathfrak{a}_{^L\mathfrak{g}}^{\leq 0} \cap (\mathfrak{a}_{^L\mathfrak{g}})^{\leq 0}) & \xrightarrow{\sim} & \mathfrak{h}^* \\ \downarrow\wr & & \uparrow\wr \\ d: (I/I^2)^{\leq 0}/(\mathfrak{z}_{\mathfrak{g}}(O) \cdot (I/I^2)^{<0} \cap (I/I^2)^{\leq 0}) & \longrightarrow & \mathfrak{h} \end{array} \tag{84}$$

anticommutes. Here the upper arrow is the isomorphism (78) with $\mathfrak{g}$ replaced by $^L\mathfrak{g}$, the left one is induced by (81), and the right one comes from the scalar product (18).

This theorem can be extracted from [FF92] (see 3.8.15–3.8.22).

**3.6.12.** The reason why the "critical" scalar product (18) appears in 3.6.11 is not very serious. The reader may prefer the following point of view. Denote by $B$ the set of invariant bilinear forms on $\mathfrak{g}$. For each $b \in B$ we have the completed twisted universal enveloping algebra $\overline{U}'_b = \overline{U}'_b(\mathfrak{g} \otimes K)$ corresponding to the cocycle $(u, v) \mapsto \operatorname{Res} b(du, v)$, $u, v \in \mathfrak{g} \otimes K$ (so $\overline{U}' = \overline{U}'_c$ where $c$ is defined by (18)). One can associate to $b \in B$ a Poisson bracket $\{\ \}_b$ on $\mathfrak{Z}$ by applying the general construction from 3.6.1 to the family of algebras $\overline{U}'_{c+hb}$ depending on the parameter $h$ (the bracket from 3.6.2 corresponds to $b = c$). The Lie algebroid structure on $I/I^2$ depends on $b$. Then 3.6.7 and 3.6.11 hold for every nondegenerate $b \in B$ (notice that in (84) *both* vertical arrows depend on $b$).

**3.6.13.** In fact, the action of $\mathrm{Der}\, O$ on $I/I^2$ mentioned in 3.6.6–3.6.7 comes from a canonical morphism $\mathrm{Der}\, O \to I$, which is essentially due to Sugawara. We will explain this in 3.6.16 after a brief overview of Sugawara formulas in 3.6.14–3.6.15. These formulas also yield elements of $\mathfrak{z}_{\mathfrak{g}}(O)$; in the case $\mathfrak{g} = sl_2$ they generate $\mathfrak{z}_{\mathfrak{g}}(O)$. We remind this in 3.6.18. Both 3.6.18 and 3.6.19 are not used in the sequel (?).

**3.6.14.** In this subsection we remind the general Sugawara formulas. In 3.6.15 we remind their consequences for the critical level.

Let $A$ be the completed universal enveloping algebra of $\widetilde{\mathfrak{g} \otimes K}$. As a vector space $\widetilde{\mathfrak{g} \otimes K}$ is the direct sum of $\mathfrak{g} \otimes K$ and $\mathbb{C} = \mathbb{C} \cdot \mathbf{1}$. The Sugawara elements $\widetilde{\mathfrak{L}}_n \in A$ are defined by

$$\widetilde{\mathfrak{L}}_n := \frac{1}{2} \sum_{r+l=n} g^{\lambda\mu} : e_\lambda^{(r)} e_\mu^{(l)} : \tag{85}$$

Here $\{e_\lambda\}$ is a basis of $\mathfrak{g}$, $e_\lambda^{(r)} := e_\lambda t^r \in \mathfrak{g}((t)) = \mathfrak{g} \otimes K \subset \widetilde{\mathfrak{g} \otimes K}$, $(g^{\lambda\mu})$ is inverse to the Gram matrix $(e_\lambda, e_\mu)$ with respect to the "critical" scalar product (18) and

$$: e_\lambda^{(r)} e_\mu^{(l)} := \begin{cases} e_\lambda^{(r)} e_\mu^{(l)} & \text{if} \quad r \leq l \\ e_\mu^{(l)} e_\lambda^{(r)} & \text{if} \quad r > l \end{cases} \tag{86}$$

Of course summation over $\lambda$ and $\mu$ is implicit in (85). Clearly the infinite series (85) converges and $\widetilde{\mathfrak{L}}_n \to 0$ for $n \to \infty$.

*Remark.* If $n \neq 0$ then $: e_\lambda^{(r)} e_\mu^{(l)} :$ can be replaced in (85) by $e_\lambda^{(r)} e_\mu^{(l)}$. Indeed, since $g^{\lambda\mu}$ is symmetric $g^{\lambda\mu}[e_\lambda^{(r)}, e_\mu^{(l)}] = 0$ unless $r + l = 0$, $r \neq 0$.

The proof of the following formulas can be found[20], e.g., in Lecture 10 from [KR] and § 12.8 from [Kac90] :

$$\operatorname{ad}\widetilde{\mathfrak{L}}_n = hL_n \tag{87}$$

$$L_m(\widetilde{\mathfrak{L}}_n) = (m-n)\widetilde{\mathfrak{L}}_{m+n} + \delta_{m,-n}\cdot\frac{m^3-m}{12}\cdot(\dim\mathfrak{g})\cdot\mathbf{1}\,. \tag{88}$$

In (87) $\operatorname{ad}\widetilde{\mathfrak{L}}_n$ is an operator $A\to A$, $L_n := -t^{n+1}\frac{d}{dt}\in\operatorname{Der}K$ is also considered as an operator $A\to A$ (the Lie algebra $\operatorname{Der}K$ acts on $A$ in the obvious way), and $h$ has the same meaning as in 3.6.2, i.e., $h: A\to A$ is multiplication by $\mathbf{1}-1$.

Using (87) one can rewrite (88) in the Virasoro form:

$$[\widetilde{\mathfrak{L}}_m,\widetilde{\mathfrak{L}}_n] = h((m-n)\widetilde{\mathfrak{L}}_{m+n} + \delta_{m,-n}\cdot\frac{m^3-m}{12}\cdot(\dim\mathfrak{g})\cdot\mathbf{1})\,. \tag{89}$$

**3.6.15.** The image of $\widetilde{\mathfrak{L}}_n$ in $A/hA=\overline{U}'$ will be denoted by $\mathfrak{L}_n$. According to (87) $\mathfrak{L}_n$ belongs to the center $\mathfrak{Z}\subset\overline{U}'$ and

$$\{\mathfrak{L}_n, z\} = L_n(z), \qquad z\in\mathfrak{Z} \tag{90}$$

where $\{\ \}$ denotes the Hayashi bracket on $\mathfrak{Z}$. According to (88) and (89)

$$L_m(\mathfrak{L}_n) = (m-n)\mathfrak{L}_{m+n} + \delta_{m,-n}\cdot\frac{m^3-m}{12}\cdot\dim\mathfrak{g} \tag{91}$$

$$\{\mathfrak{L}_m,\mathfrak{L}_n\} = (m-n)\mathfrak{L}_{m+n} + \delta_{m,-n}\cdot\frac{m^3-m}{12}\cdot\dim\mathfrak{g}\,. \tag{92}$$

**3.6.16.** If $n\geq -1$ then $\mathfrak{L}_n\in I := \operatorname{Ker}(\mathfrak{Z}\to\mathfrak{z}_{\mathfrak{g}}(O))$ (indeed, a glance at (85) shows that $\mathfrak{L}_n$ annihilates the vacuum vector from $\mathit{Vac}'$). If $m,n\geq -1$ then the "Virasoro term" $\delta_{m,-n}(m^3-m)$ vanishes, so one has the continuous Lie algebra morphism $\operatorname{Der}O\to I$ defined by $L_n\mapsto\mathfrak{L}_n$, $n\geq -1$. It induces a continuous algebra morphism

$$\operatorname{Der}O\to I/I^2\,. \tag{93}$$

[20]The reader should take in account that experts in Kac – Moody algebras usually equip $\mathfrak{g}$ with the scalar product obtained by dividing (18) by minus the dual Coxeter number.

*Remark.* According to (90) the action of $\operatorname{Der} O$ on $I/I^2$ induced by (93) coincides with the action considered in 3.6.6–3.6.7.

**3.6.17.** *Lemma.* The composition of (93) and the isomorphism $I/I^2 \xrightarrow{\sim} \mathfrak{a}_{L_{\mathfrak{g}}}$ inverse to (81) is equal to the morphism $\operatorname{Der} O \to \mathfrak{a}_{L_{\mathfrak{g}}}$ from 3.5.11.

*Proof* The two morphisms $\operatorname{Der} O \to \mathfrak{a}_{L_{\mathfrak{g}}}$ induce the same action of $\operatorname{Der} O$ on $\mathfrak{a}_{L_{\mathfrak{g}}}$. So they are equal by 3.5.12. $\square$

**3.6.18.** Denote by $\overline{\mathfrak{L}}_n$ the image of $\mathfrak{L}_n$ in $\mathfrak{Z}/I = \mathfrak{z}_{\mathfrak{g}}(O)$ . If $n \geq -1$ then $\overline{\mathfrak{L}}_n = 0$. The natural morphism $\mathbb{C}[\overline{\mathfrak{L}}_{-2}, \overline{\mathfrak{L}}_{-3}, \ldots] \to \mathfrak{z}_{\mathfrak{g}}(O)$ is injective and if $\mathfrak{g} = sl_2$ it is an isomorphism. To show this it is enough to compute the principal symbol of $\overline{\mathfrak{L}}_n$ and to use the description of $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ from 2.4.1. If $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is identified with the space of $G(O)$-invariant polynomials on $\mathfrak{g}^* \otimes \omega_O$ (see 2.4.1) then the principal symbol of $\overline{\mathfrak{L}}_n$ is the polynomial $\ell_n : \mathfrak{g}^* \otimes \omega_O \to \mathbb{C}$ defined by $\ell_n(\eta) = \frac{1}{2} \operatorname{Res}(\eta, \eta) L_n$; here $(\eta, \eta) \in \omega_O^{\otimes 2}$, $L_n \in \omega_K^{\otimes(-1)}$, $(\eta, \eta) L_n \in \omega_K$, so the residue makes sense. Clearly the mapping $\mathbb{C}[\ell_{-2}, \ell_{-3}, \ldots] \to \mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is injective and if $\mathfrak{g} = sl_2$ it is an isomorphism.

For $\mathfrak{g} = sl_2$ the Feigin – Frenkel isomorphism is the unique $\operatorname{Der} O$-equivariant isomorphism $A_{L_{\mathfrak{g}}}(O) \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(O)$. An $sl_2$-oper over $\operatorname{Spec} O$ can be represented as a connection $\frac{d}{dt} + \left(\begin{smallmatrix} 0 & u \\ 1 & 0 \end{smallmatrix}\right)$, $u = u(t) = u_0 + u_1 t + \ldots$, or as a Sturm – Liouville operator $\left(\frac{d}{dt}\right)^2 - u(t) : \omega_O^{-1/2} \to \omega_O^{3/2}$. One has $A_{sl_2}(O) = \mathbb{C}[u_0, u_1, \ldots]$ and the Feigin – Frenkel isomorphism maps $u_j$ to $-2\overline{\mathfrak{L}}_{-2-j}$.

For any semisimple $\mathfrak{g}$ we gave in 3.5.6 a description of $A_{L_{\mathfrak{g}}}(O)$ as an algebra with an action of $\operatorname{Der} O$; see (64)–(68). Using the $\operatorname{Der} O$-equivariance property of the Feigin – Frenkel isomorphism one sees that if $\mathfrak{g}$ is simple then $\overline{\mathfrak{L}}_{-2-j} \in \mathfrak{z}_{\mathfrak{g}}(O)$ corresponds to $cu_{1j} \in A_{L_{\mathfrak{g}}}(O)$, $c = -(\dim \mathfrak{g})/6$ (???).

**3.6.19.** Consider the vacuum module $\mathit{Vac}_\lambda := \mathit{Vac}_A/(h - \lambda)\, \mathit{Vac}_A$, where $\mathit{Vac}_A$ is the quotient of $A$ modulo the closed left ideal generated by $\mathfrak{g} \otimes O$. In 2.9.3 we mentioned that $\operatorname{End}_A \mathit{Vac}_\lambda = \mathbb{C}$ for $\lambda \neq 0$. The following proof of this statement was told us by E. Frenkel. As explained in 2.9.3–2.9.5 any endomorphism $f : Vac_\lambda \to Vac_\lambda$ comes from some central element $z$ of $A/(h-\lambda)A$. In fact the center of $A/(h-\lambda)A$ equals $\mathbb{C}$ if $\lambda \neq 0$, but instead of proving this let us notice that $[\widetilde{\mathfrak{L}}_0, z] = 0$ and therefore $L_0(z) = 0$ (see (87)). So $[L_0, f] = 0$ where $L_0$ is considered as an operator in $Vac_\lambda$. Therefore $f$ preserves the space $\operatorname{Ker}(L_0 : Vac_\lambda \to Vac_\lambda)$, which is generated by the vacuum vector. Since the $A$-module $Vac_\lambda$ is generated by this space $f$ is a scalar operator.

### 3.7. The center and the Gelfand - Dikii bracket.

**3.7.1.** Set $Y := \operatorname{Spec} O$, $Y' := \operatorname{Spec} K$ where, as usual, $O = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$. Let $A$ be a (commutative) $\operatorname{Aut} O$-algebra. Then for any smooth curve $X$ one obtains a $\mathcal{D}_X$-algebra $A_X$ (see 2.6.5). Though $Y$ and $Y'$ are not curves in the literal sense the construction from 2.6.5 works for them (with a minor change explained below). So one gets a $\mathcal{D}_Y$-algebra $A_Y$ and a $\mathcal{D}_{Y'}$-algebra $A_{Y'}$, which is the restriction of $A_Y$ to $Y'$. The fiber of $A_Y$ at the origin $0 \in Y$ equals $A$.

Let us explain some details. The definition of $A_X$ from 2.6.5 used a certain scheme $X^\wedge$. Since $Y$ is not a curve in the literal sense the definition of $Y^\wedge$ should be modified as follows. Denote by $\Delta_n$ the $n$-th infinitesimal neighbourhood of the diagonal $\Delta \subset \operatorname{Spec} O\widehat{\otimes}O$. The morphism $\operatorname{Spec} O\widehat{\otimes}O \to \operatorname{Spec} O \otimes O = Y \times Y$ induces an embedding $\Delta_n \hookrightarrow Y \times Y$ (if $n > 0$ then $\Delta_n$ is *smaller* than the $n$-th infinitesimal neighbourhood of the diagonal $\Delta \subset Y \times Y$). Now in the definition of an $R$-point of $Y^\wedge$ one should consider only $R$-morphisms $\gamma : \operatorname{Spec} R\widehat{\otimes}O \to Y$ with the following property: for any $n$ there is an $N$ such that the morphism $\operatorname{Spec} O/t^nO \times \operatorname{Spec} O/t^nO \times \operatorname{Spec} R \to Y \times Y$ induced by $\gamma$ factors through $\Delta_N$ (then one can set $N = 2n-2$).

**3.7.2.** Sometimes we will use the section

$$Y \to Y^\wedge \tag{94}$$

corresponding to the morphism $\gamma : \operatorname{Spec} O\widehat{\otimes} O \to Y = \operatorname{Spec} O$ defined by

$$\gamma^*(t) = t \otimes 1 + 1 \otimes t\,. \tag{95}$$

The section (94) yields an isomorphism

$$A_Y \xrightarrow{\sim} A \otimes \mathcal{O}_Y\,. \tag{96}$$

Of course (94) and (96) are not canonical: they depend on the choice of a local parameter $t \in O$.

**3.7.3.** In the situation of 3.7.1 consider the functor $F : \{\mathbb{C}\text{-algebras}\} \to \{\text{Sets}\}$ such that $F(R)$ is the set of horizontal $Y'$-morphisms $\operatorname{Spec} R\widehat{\otimes} K \to \operatorname{Spec} A_{Y'}$ or, which is the same, the set of horizontal $K$-morphisms $H^0(Y', A_{Y'}) \to R\widehat{\otimes} K$. $F$ is representable by an ind-affine ind-scheme $S$ (which may be called the ind-scheme of horizontal sections of $\operatorname{Spec} A_{Y'}$). Indeed, $F$ is a closed subfunctor of the functor $R \mapsto \operatorname{Hom}(V, R\widehat{\otimes} K)$ where $V = H^0(Y', A_{Y'})$ and Hom means the set of $K$-linear maps.

Denote by $A_K$ the ring of regular functions on $S$. This is a complete topological algebra (the ideals of $A_K$ corresponding to closed subschemes of $S$ form a base of neighbourhoods of 0).

$A_K$ is equipped with an action of the group ind-scheme $\operatorname{Aut} K$ (an $R$-point of $\operatorname{Aut} K$ is an automorphism of the topological $R$-algebra $R\widehat{\otimes} K$).

The scheme of horizontal sections of $\operatorname{Spec} A_Y$ is canonically isomorphic to $\operatorname{Spec} A$ (to a horizontal section $s : Y \to \operatorname{Spec} A_Y$ one associates $s(0) \in \operatorname{Spec} A$). This is a closed subscheme of $S = \operatorname{Spec} A_K$, so we get a canonical epimorphism

$$A_K \to A\,. \tag{97}$$

Clearly it is $\operatorname{Aut} O$-equivariant.

*Example.* Suppose that $A = \mathbb{C}[u_0, u_1, u_2, \ldots]$ and $u_k = (L_{-1})^k u_0/k!$, $L_0 u_0 = d u_0$, $d \in \mathbb{Z}$ (as usual, $L_n := -t^{n+1}\frac{d}{dt} \in \operatorname{Der} O$). Then one has the obvious isomorphism $f$ between the $\mathcal{D}_Y$-scheme $\operatorname{Spec} A_Y$ and the scheme of jets of $d$-differentials on $Y$. Clearly $\operatorname{Aut} O = \operatorname{Aut} Y$ acts on both schemes by functoriality. $f$ is equivariant with respect to the group ind-scheme of $\operatorname{Aut} O$ generated by $L_0$ and $L_{-1}$. Using $f$ we identify horizontal sections of $\operatorname{Spec} A_{Y'}$ with $d$-differentials on $Y'$, i.e., sections of $\omega_{Y'}^{\otimes d}$. A $d$-differential on $Y'$ can be written as $\sum_i \widetilde{u}_i t^i (dt)^{\otimes d}$, so $A_K = \mathbb{C}[[\ldots \widetilde{u}_{-1}, \widetilde{u}_0, \widetilde{u}_1, \ldots]$ where

$$(98) \quad \mathbb{C}[[\ldots \widetilde{u}_{-1}, \widetilde{u}_0, \widetilde{u}_1, \ldots] := \varprojlim_n \mathbb{C}[\ldots \widetilde{u}_{-1}, , \widetilde{u}_0, \widetilde{u}_1, \ldots]/(u_{-n}, u_{-n-1}, \ldots)\,.$$

Clearly $L_0 \widetilde{u}_k = (d+k)\widetilde{u}_k$, $L_{-1}\widetilde{u}_k = (k+1)\widetilde{u}_{k+1}$, and the morphism (97) maps $\widetilde{u}_k$ to $u_k$ if $k \geq 0$ and to 0 if $k < 0$.

**3.7.4.** Denote by $\mathfrak{z}_{\mathfrak{g}}(K)$ the algebra $A_K$ from 3.7.3 in the particular case $A = \mathfrak{z}_{\mathfrak{g}}(O)$ (see 2.5.1 or 2.7.2 for the definition of $\mathfrak{z}_{\mathfrak{g}}(O)$). We are going to define a canonical morphism from $\mathfrak{z}_{\mathfrak{g}}(K)$ to the center $\mathfrak{Z}$ of the completed twisted universal enveloping algebra $\overline{U}' = \overline{U}'(\mathfrak{g} \otimes K)$. To this end rewrite (34) as a $K$-linear map $\mathfrak{z}_{\mathfrak{g}}(O) \otimes_O K \to \mathfrak{Z} \widehat{\otimes} K$. Using the noncanonical isomorphism $\mathfrak{z}_{\mathfrak{g}}(O)_Y \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(O) \otimes \mathcal{O}_Y$ (see (96)) one gets a map

$$(99) \qquad H^0(Y', \mathfrak{z}_{\mathfrak{g}}(O)_{Y'}) \to \mathfrak{Z} \widehat{\otimes} K\,,$$

which is easily shown to be canonical, i.e., independent of the choice of a local parameter $t \in O$ (in fact, (34) is a noncanonical version of (99); (34) depends on the choice of $t$ because (32) involves $\zeta + t$, which is nothing but the noncanonical section $Y' \to Y'^{\wedge}$ defined by (95)).

**3.7.5.** *Theorem.*

(i) The map (99) is a horizontal morphism of $K$-algebras. Therefore (99) defines a continuous morphism

$$(100) \qquad \mathfrak{z}_{\mathfrak{g}}(K) \to \mathfrak{Z}\,.$$

(ii) The composition $\mathfrak{z}_{\mathfrak{g}}(K) \to \mathfrak{Z} \to \mathfrak{z}_{\mathfrak{g}}(O)$ is the morphism (97) for $A = \mathfrak{z}_{\mathfrak{g}}(O)$.

(iii) The morphism (100) is $\operatorname{Aut} K$-equivariant.

We will not prove this theorem. In fact, the only nontrivial statement is that (99) (or equivalently (34)) is a ring homomorphism; see ???for a proof.

The natural approach to the above theorem is based on the notion of VOA (i.e., vertex operator algebra) or its geometric version introduced in [BD] under the name of *chiral algebra*.[21] In the next subsection (which can be skipped by the reader) we outline the chiral algebra approach.

**3.7.6.** A chiral algebra on a smooth curve $X$ is a (left) $\mathcal{D}_X$-module $\mathcal{A}$ equipped with a morphism

$$j_*j^!(\mathcal{A}\boxtimes\mathcal{A})\to\Delta_*\mathcal{A} \tag{101}$$

where $\Delta: X\hookrightarrow X\times X$ is the diagonal, $j:(X\times X)\setminus\Delta(X)\hookrightarrow X$. The morphism (101) should satisfy certain axioms, which will not be stated here. A chiral algebra is said to be *commutative* if (101) maps $\mathcal{A}\boxtimes\mathcal{A}$ to 0. Then (101) induces a morphism $\Delta_*(\mathcal{A}\otimes\mathcal{A})=j_*j^!(\mathcal{A}\boxtimes\mathcal{A})/\mathcal{A}\boxtimes\mathcal{A}\to\Delta_*\mathcal{A}$ or, which is the same, a morphism

$$\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\,. \tag{102}$$

In this case the chiral algebra axioms just mean that $\mathcal{A}$ equipped with the operation (102) is a commutative associative unital algebra. So a commutative chiral algebra is the same as a commutative associative unital $\mathcal{D}_X$-algebra in the sense of 2.6. On the other hand, the $\mathcal{D}_X$-module $\mathit{Vac}'_X$ corresponding to the $\operatorname{Aut}O$-module $\mathit{Vac}'$ by 2.6.5 has a natural structure of chiral algebra (see the Remark below). The map $\mathfrak{z}_{\mathfrak{g}}(O)_X\to\mathit{Vac}'_X$ induced by the embedding $\mathfrak{z}_{\mathfrak{g}}(O)\to\mathit{Vac}'$ is a chiral algebra morphism. Given a point $x\in X$ one defines a functor $\mathcal{A}\mapsto\mathcal{A}_{((x))}$ from chiral algebras to associative topological algebras. If $\mathcal{A}=A_X$ for some commutative $\operatorname{Aut}O$-algebra $A$ then $\mathcal{A}_{((x))}$ is the algebra $A_{K_x}$ from 3.7.3. If $\mathcal{A}=\mathit{Vac}'_X$ then $\mathcal{A}_{((x))}$ is the completed twisted universal enveloping algebra $\overline{U}'=\overline{U}'(\mathfrak{g}\otimes K)$. So

[21] In 2.9.4 – 2.9.5 we used some ideas of VOA theory (or chiral algebra theory).

by functoriality one gets a morphism $\mathfrak{z}_{\mathfrak{g}}(K) = \mathfrak{z}_{\mathfrak{g}}(O)_K \to \overline{U}'$. Its image is contained in $\mathfrak{Z}$ because $\mathfrak{z}_{\mathfrak{g}}(O)_X$ is the *center* of the chiral algebra $Vac'_X$.

*Remark.* Let us sketch a definition of the chiral algebra structure on $Vac'_X$. First of all, for every $n$ one constructs a $\mathcal{D}$-module $Vac'_{\mathrm{Sym}^n X}$ on $\mathrm{Sym}^n X$ (for $n=1$ one obtains $Vac'_X$). The fiber $Vac'_D$ of $Vac'_{\mathrm{Sym}^n X}$ at $D \in \mathrm{Sym}^n X$ can be described as follows. Consider $D$ as a closed subscheme of $X$ of order $n$, denote by $O_D$ the ring of functions on the formal completion of $X$ along $D$, and define $K_D$ by $\operatorname{Spec} K_D = (\operatorname{Spec} O_D) \setminus D$. One defines the central extension $\mathfrak{g} \widetilde{\otimes K_D}$ of $\mathfrak{g} \otimes K_D$ just as in the case $n=1$. $Vac'_D$ is the twisted vacuum module corresponding to the Harish-Chandra pair $(\mathfrak{g} \widetilde{\otimes K_D}, G(O_D))$ (see 1.2.5). Denote by $Vac'_{X\times X}$ the pullback of $Vac'_{\mathrm{Sym}^2 X}$ to $X \times X$. Then

$$j^! \, Vac'_{X\times X} = j^! ( Vac'_X \boxtimes Vac'_X ) \, , \tag{103}$$

$$\Delta^\dagger \, Vac'_{X\times X} = Vac'_X \tag{104}$$

where $j$ and $\Delta$ have the same meaning as in (101) and $\Delta^\dagger$ denotes the naive pullback, i.e., $\Delta^\dagger = H^1 \Delta^!$. One defines (101) to be the composition

$$j_* j^! \, Vac'_X \boxtimes Vac'_X = j_* j^! \, Vac'_{X\times X} \to j_* j^! \, Vac'_{X\times X} / Vac'_{X\times X} = \Delta_* \, Vac'_X$$

where the last equality comes from (104).

**3.7.7.** *Theorem.* (i) The morphism (100) is a topological isomorphism.

(ii) The adjoint action of $G(K)$ on $\mathfrak{Z}$ is trivial.

The proof will be given in 3.7.10. It is based on the Feigin - Frenkel theorem, so it is essential that $\mathfrak{g}$ is semisimple and the central extension of $\mathfrak{g} \otimes K$ corresponds to the "critical" scalar product (18). This was not essential for Theorem 3.7.5.

We will also prove the following statements.

**3.7.8.** *Theorem.* The map $\operatorname{gr} \mathfrak{Z} \to \mathfrak{Z}^{cl}$ defined in 2.9.8 induces a topological isomorphism $\operatorname{gr}_i \mathfrak{Z} \xrightarrow{\sim} \mathfrak{Z}^{cl}_{(i)} :=$ {the space of homogeneous polynomials from $\mathfrak{Z}^{cl}$ of degree $i$}.

**3.7.9.** *Theorem.* Denote by $\mathcal{I}_n$ the closed left ideal of $\overline{U}'$ topologically generated by $\mathfrak{g}\otimes t^n O$, $n \geq 0$. Then the ideal $I_n := \mathcal{I}_n \cap \mathfrak{Z} \subset \mathfrak{Z}$ is topologically generated by the spaces $\mathfrak{Z}_i^m$, $m < i(1-n)$, where $\mathfrak{Z}_i^m := \{z \in \mathfrak{Z}_i | L_0 z = mz\}$, $\mathfrak{Z}_i$ is the standard filtration of $\mathfrak{Z}$, and $L_0 := -t\frac{d}{dt} \in \operatorname{Der} O$.

**3.7.10.** Let us prove the above theorems. The elements of the image of (100) are $G(K)$-invariant (see the Remark from 2.9.6). So 3.7.7(ii) follows from 3.7.7(i). Let us prove 3.7.7(i), 3.7.8, and 3.7.9.

By 2.5.2 $\operatorname{gr}\mathfrak{z}_{\mathfrak{g}}(O) = \mathfrak{z}_{\mathfrak{g}}^{cl}(O)$. According to 2.4.1 $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ can be identified with the ring of $G(O)$-invariant polynomial functions on $\mathfrak{g}^* \otimes \omega_O$. Choose homogeneous generators $p_1, \ldots, p_r$ of the algebra of $G$-invariant polynomials on $\mathfrak{g}^*$ and set $d_j := \deg p_j$. Define $v_{jk} \in \mathfrak{z}_{\mathfrak{g}}^{cl}(O)$, $1 \leq j \leq r$, $0 \leq k < \infty$, by

$$p_j(\eta) = \sum_{k=0}^{\infty} v_{jk}(\eta) t^k (dt)^{d_j}\,, \quad \eta \in \mathfrak{g}^* \otimes \omega_O\,. \tag{105}$$

According to 2.4.1 the algebra $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is freely generated by $v_{jk}$. The action of $\operatorname{Der} O$ on $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is easily described. In particular $v_{jk} = (L_{-1})^k v_{j0}/k!$, $L_0 v_{j0} = d_j v_{j0}$. Lift $v_{j0} \in \mathfrak{z}_{\mathfrak{g}}^{cl}(O) = \operatorname{gr}\mathfrak{z}_{\mathfrak{g}}(O)$ to an element $u_j \in \mathfrak{z}_{\mathfrak{g}}(O)$ so that $L_0 u_j = d_j u_j$. Then the algebra $\mathfrak{z}_{\mathfrak{g}}(O)$ is freely generated by $u_{jk} := (L_{-1})^k u_j/k!$, $1 \leq j \leq r$, $0 \leq k < \infty$. Just as in the example at the end of 3.7.3 we see that $\mathfrak{z}_{\mathfrak{g}}(O)_K = \mathbb{C}[[\ldots, \widetilde{u}_{j,-1}, \widetilde{u}_{j0}, \widetilde{u}_{j1}, \ldots]$ and $L_0 \widetilde{u}_{jk} = (d_j + k)\widetilde{u}_{jk}$.

Denote by $\overline{u}_{jk}$ the image of $\widetilde{u}_{jk}$ in $\mathfrak{Z}$. By 2.9.8 $\overline{u}_{jk} \in \mathfrak{Z}_{d_j}$ and the image of $\overline{u}_{jk}$ in $\mathfrak{Z}_{(d_j)}^{cl}$ is the function $\widetilde{v}_{jk} : \mathfrak{g}^* \otimes \omega_K \to \mathbb{C}$ defined by

$$p_j(\eta) = \sum_k \widetilde{v}_{jk}(\eta) t^k (dt)^{d_j}, \quad \eta \in \mathfrak{g}^* \otimes \omega_K\,. \tag{106}$$

We have an isomorphism of topological algebras

$$\mathfrak{Z}^{cl} = \mathbb{C}[[\ldots \widetilde{v}_{j,-1}, \widetilde{v}_{j0}, \widetilde{v}_{j1}, \ldots] \tag{107}$$

because

(108) the algebra of $G(O)$-invariant polynomial functions on $\mathfrak{g}^* \otimes t^{-n}\omega_O$ is freely generated by the restrictions of $\widetilde{v}_{jk}$ for $k \geq -nd_j$ while for $k < -nd_j$ the restriction of $\widetilde{v}_{jk}$ to $\mathfrak{g}^* \otimes t^{-n}\omega_O$ equals 0.

(This statement is immediately reduced to the case $n = 0$ considered in 2.4.1). Theorem 3.7.8 follows from (107).

Now consider the morphism $f_n : \mathfrak{z}_{\mathfrak{g}}(O)_K \to \mathfrak{Z}/I_n$ where $I_n$ was defined in 3.7.9. We will show that

(109) $f_n$ is surjective and its kernel is the ideal $J_n$ topologically generated by $u_{jk}$, $k < d_j(1-n)$.

Theorems 3.7.7 and 3.7.9 follow from (109).

To prove (109) consider the composition $\overline{f}_n : \mathfrak{z}_{\mathfrak{g}}(O)_K \to \mathfrak{Z}/I_n \hookrightarrow (\overline{U}'/\mathcal{I}_n)^{G(O)}$. Equip $\overline{U}'/\mathcal{I}_n$ with the filtration induced by the standard one on $\overline{U}'$. The eigenvalues of $L_0$ on the $i$-th term of this filtration are $\geq i(1-n)$. So $\operatorname{Ker} \overline{f}_n \supset J_n$ where $J_n$ was defined in (109). Now $\operatorname{gr}(\overline{U}'/\mathcal{I}_n)^{G(O)}$ is contained in $(\operatorname{gr} \overline{U}'/\mathcal{I}_n)^{G(O)}$, i.e., the algebra of $G(O)$-invariant polynomials on $\mathfrak{g}^* \otimes t^{-n}\omega_O$. Using (108) one easily shows that the map $\mathfrak{z}_{\mathfrak{g}}(O)_K/J_n \to (\overline{U}'/\mathcal{I}_n)^{G(O)}$ induced by $\overline{f}_n$ is an isomorphism. This implies (109). We have also shown that

(110) the map $\mathfrak{Z} \to (\overline{U}'/\mathcal{I}_n)^{G(O)}$ is surjective

and therefore

(111) $$\mathfrak{Z} = (\overline{U}')^{G(O)}.$$

**3.7.11.** *Remarks*

(i) Here is another proof[22] of (111). Let $u \in (\overline{U}')^{G(O)}$. Take $h \in H(K)$ where $H \subset G$ is a fixed Cartan subgroup. Then $h^{-1}uh$ is invariant

[22]It is analogous to the proof of the fact that an integrable discrete representation of $\mathfrak{g} \otimes K$ is trivial. We are not able to use the fact itself because $\overline{U}'$ is not discrete.

with respect to a certain Borel subgroup $B_h \subset G$. So $h^{-1}uh$ is $G$-invariant (it is enough to prove this for the image of $h^{-1}uh$ in the *discrete* space $\overline{U}'/\mathcal{I}_n$ where $\mathcal{I}_n$ was defined in 3.7.9). Therefore $u$ is invariant with respect to $h\mathfrak{g}h^{-1} \subset \mathfrak{g} \otimes K$ for any $h \in H(K)$. The Lie algebra $\mathfrak{g} \otimes K$ is generated by $\mathfrak{g} \otimes O$ and $h\mathfrak{g}h^{-1}$, $h \in H(K)$. So $u \in \mathfrak{Z}$.

(ii) In fact

$$\mathfrak{Z} = (\overline{U}')^{\mathfrak{a}} \quad \text{for any open } \mathfrak{a} \subset \mathfrak{g} \otimes K\,. \tag{112}$$

Indeed, one can modify the above proof as follows. First write $u$ as an (infinite) sum of $u_\chi$, $\chi \in \mathfrak{h}^* := (\operatorname{Lie} H)^*$, $[a, u_\chi] = \chi(a)u_\chi$ for $a \in \mathfrak{h}$. Then take an $h \in H(K)$ such that the image of $h$ in $H(K)/H(O) = \{\text{the coweight lattice}\}$ is "very dominant" with respect to a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{h}$, so that $h^{-1}\mathfrak{a}h \supset [\mathfrak{b}, \mathfrak{b}]$. We see that $u_\chi = 0$ unless $\chi$ is dominant, and $h^{-1}u_0h$ is $\mathfrak{g}$-invariant. Replacing $h$ by $h^{-1}$ we see that $u = u_0$, etc.

(iii) Here is another proof of 3.7.7(ii). Consider the canonical filtration $\overline{U}'_k$ of $\overline{U}'$. It follows from (109) that the union of the spaces $\overline{U}'_k \cap \mathfrak{Z}$, $k \in \mathbb{N}$, is dense in $\mathfrak{Z}$. So it suffices to show that the action of $G(K)$ on $\overline{U}'_k \cap \mathfrak{Z}$ is trivial for every $k$. The action of $G(K)$ on $\mathfrak{Z}^{cl}$ is trivial (see (107), (106)). So the action of $G(K)$ on $\operatorname{gr} \mathfrak{Z}$ is trivial. The action of $\mathfrak{g} \otimes K$ on $\widetilde{\mathfrak{g} \otimes K}$ corresponding to the action of $G(K)$ defined by (19) is the adjoint action, and the adjoint action of $\mathfrak{g} \otimes K$ on $\mathfrak{Z}$ is trivial. So the action of $G(K)$ on $\mathfrak{Z}$ factors through $\pi_0(G(K))$. The group $\pi_0(G(K))$ is finite (see 4.5.4), so we are done.

**3.7.12.** We are going to deduce Theorem 3.6.7 from [FF92]. Denote by $A_{^L\mathfrak{g}}(O)$ the coordinate ring of $\mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(O)$ (i.e., the scheme of $^L\mathfrak{g}$-opers on $\operatorname{Spec} O$). Let $\varphi_O : A_{^L\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(O)$ be an isomorphism satisfying the conditions of 3.2.2. It induces an $\operatorname{Aut} K$-equivariant isomorphism $\varphi_K : A_{^L\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(K)$ where $A_{^L\mathfrak{g}}(K)$ is the algebra $A_K$ from 3.7.3

corresponding to $A = A_{^L\mathfrak{g}}(O)$. Recall that $A_K$ is the coordinate ring of the ind-scheme of horizontal sections of $\operatorname{Spec} A_{Y'}$, $Y' := \operatorname{Spec} K$. If $A = A_{^L\mathfrak{g}}(O)$ then $\operatorname{Spec} A_{Y'}$ is the scheme of jets of $^L\mathfrak{g}$-opers on $Y'$ and its horizontal sections are $^L\mathfrak{g}$-opers on $Y'$ (cf. 3.3.3). So $A_{^L\mathfrak{g}}(K)$ is the coordinate ring of $\mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(K) :=$ the ind-scheme of $^L\mathfrak{g}$-opers on $\operatorname{Spec} K$. It is a Poisson algebra with respect to the *Gelfand - Dikii bracket* (we remind its definition in 3.7.14). The Gelfand - Dikii bracket depends on the choice of a non-degenerate invariant bilinear form on $^L\mathfrak{g}$. We define it to be *dual* to the form (18) on $\mathfrak{g}$ (i.e., its restriction to $\mathfrak{h}^* = {}^L\mathfrak{h} \subset {}^L\mathfrak{g}$ is dual to the restriction of (18) to $\mathfrak{h}$).

By 3.7.5 and 3.7.7 we have a canonical isomorphism $\mathfrak{z}_{\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{Z}$, so $\varphi_K$ can be considered as an $\operatorname{Aut} K$-equivariant isomorphism

$$A_{^L\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{Z}\,. \tag{113}$$

$\mathfrak{Z}$ is a Poisson algebra with respect to the Hayashi bracket (see 3.6.2).

**3.7.13.** *Theorem.* [FF92]

There is an isomorphism

$$\varphi_O : A_{^L\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(O) \tag{114}$$

satisfying the conditions of 3.2.2 and such that the corresponding isomorphism (113) is compatible with the Poisson structures.

We will show (see 3.7.16) that an isomorphism (114) with the properties mentioned in the theorem satisfies the conditions of 3.6.7. So it is unique (see the Remark from 3.6.7).

*Remark.* As explained in 3.6.12, one can associate a Poisson bracket on $\mathfrak{Z}$ to *any* invariant bilinear form $B$ on $\mathfrak{g}$ (the bracket from 3.6.2 corresponds to the form (18)). If $B$ is non-degenerate one can consider the dual form on $^L\mathfrak{g}$ and the corresponding Gelfand - Dikii bracket on $A_{^L\mathfrak{g}}(K)$. The isomorphism (113) corresponding to (114) is compatible with these Poisson brackets.

**3.7.14.** Let us recall the definition of the Gelfand - Dikii bracket from [DS85]. This is a Poisson bracket on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ (i.e., a Poisson bracket on its coordinate ring $A_{\mathfrak{g}}(K)$). It depends on the choice of a non-degenerate invariant bilinear form ( , ) on $\mathfrak{g}$.

Denote by $\widetilde{\mathfrak{g}\otimes K}$ the Kac–Moody central extension of $\mathfrak{g}\otimes K$ corresponding to ( , ). As a vector space $\widetilde{\mathfrak{g}\otimes K}$ is $(\mathfrak{g}\otimes K)\oplus\mathbb{C}$ and the commutator in $\widetilde{\mathfrak{g}\otimes K}$ is defined by the 2-cocycle $\operatorname{Res}(du, v)$, $u, v\in\mathfrak{g}\otimes K$. The topological dual space $(\widetilde{\mathfrak{g}\otimes K})^*$ is an ind-scheme. The algebra of regular functions on $(\widetilde{\mathfrak{g}\otimes K})^*$ is a Poisson algebra with respect to the Kirillov bracket[23] (i.e., the unique continuous Poisson bracket such that the natural map from $\widetilde{\mathfrak{g}\otimes K}$ to the algebra of regular functions on $(\widetilde{\mathfrak{g}\otimes K})^*$ is a Lie algebra morphism). So $(\widetilde{\mathfrak{g}\otimes K})^*$ is a Poisson "manifold". Denote by $(\widetilde{\mathfrak{g}\otimes K})^*_1$ the space of continuous linear functionals $l:\widetilde{\mathfrak{g}\otimes K}\to\mathbb{C}$ such that the restriction of $l$ to the center $\mathbb{C}\subset\widetilde{\mathfrak{g}\otimes K}$ is the identity. $(\widetilde{\mathfrak{g}\otimes K})^*_1$ is a Poisson submanifold of $(\widetilde{\mathfrak{g}\otimes K})^*$.

We identify $(\widetilde{\mathfrak{g}\otimes K})^*_1$ with Conn := the ind-scheme of connections on the trivial $G$-bundle on $\operatorname{Spec} K$: to a connection $\nabla = d+\eta$, $\eta\in\mathfrak{g}\otimes\omega_K$, we associate $l\in(\widetilde{\mathfrak{g}\otimes K})^*_1$ such that for any $u\in\mathfrak{g}\otimes K\subset\widetilde{\mathfrak{g}\otimes K}$ one has $l(u)=\operatorname{Res}(u,\eta)$. It is easy to check that the gauge action of $\mathfrak{g}\otimes K$ on Conn corresponds to the coadjoint action of $\mathfrak{g}\otimes K$ on $(\widetilde{\mathfrak{g}\otimes K})^*_1$, and one defines the coadjoint action[24] of $G(K)$ on $(\widetilde{\mathfrak{g}\otimes K})^*$ so that its restriction to $(\widetilde{\mathfrak{g}\otimes K})^*_1$ corresponds to the gauge action of $G(K)$ on Conn. The action of $G(K)$ on the Poisson "manifold" $(\widetilde{\mathfrak{g}\otimes K})^*_1$ is not Hamiltonian in the literal sense, i.e., one cannot define the moment map $(\widetilde{\mathfrak{g}\otimes K})^*_1\to(\mathfrak{g}\otimes K)^*$. However one can define the moment map $(\widetilde{\mathfrak{g}\otimes K})^*_1\to(\widetilde{\mathfrak{g}\otimes K})^*$: this is the identity map.

[23] As explained in [We83] the "Kirillov bracket" was invented by Sophus Lie and then rediscovered by several people including A.A. Kirillov.

[24] It is dual to the adjoint action of $G(K)$ on $\widetilde{\mathfrak{g}\otimes K}$ defined by (19) (of course in (19) $c$ should be replaced by our bilinear form on $\mathfrak{g}$).

The point is that $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ can be obtained from Conn $= (\widetilde{\mathfrak{g}\otimes K})_1^*$ by Hamiltonian reduction (such an interpretation of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ automatically defines a Poisson bracket on $A_{\mathfrak{g}}(K)$). Fix a Borel subgroup $B \subset G_{\mathrm{ad}}$. Let $N$ be its unipotent radical, $\mathfrak{n} := \operatorname{Lie} N$. Since the restriction of the Kac-Moody cocycle to $\mathfrak{n}\otimes K$ is trivial we have the obvious splitting $\mathfrak{n}\otimes K \to \widetilde{\mathfrak{g}\otimes K}$. It is $B(K)$-equivariant and this property characterizes it uniquely. The action of $N(K)$ on Conn is Hamiltonian: the moment map $\mu :$ Conn $= (\widetilde{\mathfrak{g}\otimes K})_1^* \to (\mathfrak{n}\otimes K)^*$ is induced by the above splitting. Let $\mathrm{Char}^\star \subset (\mathfrak{n}\otimes K)^*$ be the set of *non-degenerate characters*, i.e., the set of Lie algebra morphisms $l : \mathfrak{n}\otimes K \to \mathbb{C}$ such that for each simple root $\alpha$ the restriction of $l$ to $\mathfrak{g}^\alpha \otimes K$ is nonzero. For every $l \in \mathrm{Char}^\star$ the action of $N(K)$ on $\mu^{-1}(l)$ is free and the quotient $N(K)\backslash\mu^{-1}(l)$ can be canonically identified with $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ (indeed, $\mu^{-1}(l)$ is the space of connections $\nabla = d+\eta \in$ Conn such that $\eta = \sum\limits_{\alpha\in\Gamma} J_\alpha + q$ where $q \in \mathfrak{b}\otimes\omega_K$, $\Gamma$ is the set of simple roots, and $J_\alpha = J_\alpha(l)$ is a fixed nonzero element of $\mathfrak{g}^{-\alpha}\otimes\omega_K$). So $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ is obtained from Conn by Hamiltonian reduction over $l$ with respect to the action of $N(K)$, whence we get a Poisson bracket on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. It is called the *Gelfand - Dikii bracket*. It does not depend on $l$. Indeed, for $l, l' \in \mathrm{Char}^\star$ consider the isomorphism

$$N(K)\setminus\mu^{-1}(l) \xrightarrow{\sim} N(K)\setminus\mu^{-1}(l') \tag{115}$$

that comes from the identification of both sides of (115) with $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. The (co) adjoint action of $H(K)$ on Conn $= (\widetilde{\mathfrak{g}\otimes K})_1^*$ preserves the relevant structures (i.e., the Poisson bracket on Conn, the action of $N(K)$ on Conn, and the moment map $\mu :$ Conn $\to (\mathfrak{n}\otimes K)^*$). There is a unique $h \in H(K)$ that transforms $l$ to $l'$ and (115) is induced by the action of this $h$. So (115) is a Poisson map.

*Remarks*

(i) If the bilinear form ( , ) on $\mathfrak{g}$ is multiplied by $c \in \mathbb{C}^*$ then the Poisson bracket on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ is multiplied by $c^{-1}$.

(ii) The Gelfand - Dikii bracket defined above is the "second Gelfand - Dikii bracket". The definition of the first one and an explanation of the relation with the original works by Gelfand - Dikii ([GD76], [GD78]) can be found in [DS85] (see Sections 2.3, 3.6, 3.7, 6.5, and 8 from loc. cit).

**3.7.15.** Let $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$, i.e., $\mathfrak{F} = (\mathfrak{F}_B, \nabla)$ where $\mathfrak{F}_B$ is a $B$-bundle on $\operatorname{Spec} K$ and $\nabla$ is a connection on the corresponding $G$-bundle satisfying the conditions of 3.1.3 (here $G$ is the adjoint group corresponding to $\mathfrak{g}$ and $B \subset G$ is the Borel subgroup). We are going to describe the tangent space $T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ and the cotangent space $T^*_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. Then we will write an explicit formula for $\{\varphi, \psi\}(\mathfrak{F})$, $\varphi, \psi \in A_{\mathfrak{g}}(K)$.

*Remark.* Of course $\mathfrak{F}_B$ is always trivial, so we could consider $\mathfrak{F}$ as a connection $\nabla$ in the trivial $G$-bundle (i.e., $\nabla = d + q$, $q \in \mathfrak{g} \otimes \omega_K$) modulo gauge transformations with respect to $B$.

To describe $T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ we must study infinitesimal deformations of $\mathfrak{F} = (\mathfrak{F}_B, \nabla)$. Since $\mathfrak{F}_B$ cannot be deformed all of them come from infinitesimal deformations of $\nabla$, which have the form $\nabla(\varepsilon) = \nabla + \varepsilon q$, $q \in H^0(\operatorname{Spec} K, \mathfrak{g}^{-1}_{\mathfrak{F}} \otimes \omega_K)$ (see 3.1.1 for the definition of $\mathfrak{g}^{-1}$; $\mathfrak{g}^{-1}_{\mathfrak{F}} := \mathfrak{g}^{-1}_{\mathfrak{F}_B}$ is the $\mathfrak{F}_B$-twist of $\mathfrak{g}^{-1}$). Taking in account the infinitesimal automorphisms of $\mathfrak{F}_B$ we get:

$$T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K) = H^0(\operatorname{Spec} K, \operatorname{Coker}(\nabla : \mathfrak{b}_{\mathfrak{F}} \to \mathfrak{g}^{-1}_{\mathfrak{F}} \otimes \omega_K)) . \tag{116}$$

Here is a more convenient description of the tangent space:

$$T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K) = \operatorname{Coker}(\nabla : \mathfrak{n}^K_{\mathfrak{F}} \to \mathfrak{b}^K_{\mathfrak{F}} \otimes \omega_K) \tag{117}$$

where $\mathfrak{n}^K_{\mathfrak{F}} := H^0(\operatorname{Spec} K, \mathfrak{n}_{\mathfrak{F}})$, $\mathfrak{b}^K_{\mathfrak{F}} := H^0(\operatorname{Spec} K, \mathfrak{b}_{\mathfrak{F}})$ (the natural map from the r.h.s. of (117) to the r.h.s. of (116) is an isomorphism). Using the invariant scalar product ( , ) on $\mathfrak{g}$ we identify $\mathfrak{b}^*$, $\mathfrak{n}^*$ with $\mathfrak{g}/\mathfrak{n}$, $\mathfrak{g}/\mathfrak{b}$ and get the following description of the cotangent space:

$$T^*_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K) = \{u \in \mathfrak{g}^K_{\mathfrak{F}} | \nabla(u) \in \mathfrak{b}^K_{\mathfrak{F}} \otimes \omega_K\}/\mathfrak{n}^K_{\mathfrak{F}} . \tag{118}$$

Here is an explicit formula for the Gelfand - Dikii bracket:

$$\{\varphi, \psi\}(\mathfrak{F}) = \operatorname{Res}(\nabla(d_{\mathfrak{F}}\varphi), d_{\mathfrak{F}}\psi), \qquad \varphi, \psi \in A_{\mathfrak{g}}(K)\,. \tag{119}$$

In this formula the differentials $d_{\mathfrak{F}}\varphi$ and $d_{\mathfrak{F}}\psi$ are considered as elements of the r.h.s. of (118).

**3.7.16.** *Theorem.* [25]

(i) Set $I := \operatorname{Ker}(A_{\mathfrak{g}}(K) \to A_{\mathfrak{g}}(O))$. Then $\{I, I\} \subset I$ and therefore $I/I^2$ is a Lie algebroid over $A_{\mathfrak{g}}(O)$.

(ii) There is an $\operatorname{Aut} O$-equivariant topological isomorphism of Lie algebroids

$$I/I^2 \xrightarrow{\sim} \mathfrak{a}_{\mathfrak{g}} \tag{120}$$

(see 3.5.11, 3.5.15 for the definition of $\mathfrak{a}_{\mathfrak{g}}$).

(In this theorem $I^2$ denotes the *closure* of the subspace generated by $ab$, $a \in I$, $b \in I$).

Theorem 3.6.7 follows from 3.7.13 and 3.7.16.

*Remark.* The isomorphism (120) is unique (see 3.5.13 or 3.5.14).

**3.7.17.** Let us prove Theorem 3.7.16. We keep the notation of 3.7.15. Take $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$. Here is a description of $T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ similar to (117):

$$T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \operatorname{Coker}(\nabla : \mathfrak{n}_{\mathfrak{F}}^{O} \to \mathfrak{b}_{\mathfrak{F}}^{O} \otimes \omega_O) \tag{121}$$

where $\mathfrak{n}_{\mathfrak{F}}^{O} := H^0(\operatorname{Spec} O, \mathfrak{n}_{\mathfrak{F}})$. The fiber of $I/I^2$ over $\mathfrak{F}$ is the conormal space $T_{\mathfrak{F}}^{\perp}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \subset T_{\mathfrak{F}}^{*}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. According to (121) it has the following description in terms of (118):

$$T_{\mathfrak{F}}^{\perp}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \{u \in \mathfrak{g}_{\mathfrak{F}}^{O} \,|\, \nabla(u) \in \mathfrak{b}_{\mathfrak{F}}^{O} \otimes \omega_O\}/\mathfrak{n}_{\mathfrak{F}}^{O}\,. \tag{122}$$

Now it is clear that $\{I, I\} \subset I$: if $\varphi, \psi \in I$, $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ then $d_{\mathfrak{F}}\varphi$ and $d_{\mathfrak{F}}\psi$ belong to the r.h.s. of (122) and therefore the r.h.s. of (119) equals 0.

[25]Inspired by [Phys]

The map

$$I/I^2 \to \operatorname{Der} A_{\mathfrak{g}}(O)\,, \tag{123}$$

which is a part of the algebroid structure on $I/I^2$, is defined by $\varphi \mapsto \partial_\varphi$, $\partial_\varphi(\psi) := \{\varphi, \psi\}$, $\varphi \in I$, $\psi \in A_{\mathfrak{g}}(K)/I = A_{\mathfrak{g}}(O)$. So according to (119) the map

$$T^{\perp}_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \to T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \tag{124}$$

induced by (123) is the operator

$$\nabla:\, \{u \in \mathfrak{g}^O_{\mathfrak{F}} \,|\, \nabla(u) \in \mathfrak{b}^O_{\mathfrak{F}} \otimes \omega_O\}/\mathfrak{n}^O_{\mathfrak{F}} \to (\mathfrak{b}^O_{\mathfrak{F}} \otimes \omega_O)/\nabla(\mathfrak{n}^O_{\mathfrak{F}})\,. \tag{125}$$

The algebroid structure on $I/I^2$ induces a Lie algebra structure on the kernel $\mathfrak{a}_{\mathfrak{F}}$ of the map (124). On the other hand, $\mathfrak{a}_{\mathfrak{F}}$ is the kernel of (125), i.e., $\mathfrak{a}_{\mathfrak{F}} = \{u \in \mathfrak{g}^O_{\mathfrak{F}} \,|\, \nabla(u) = 0\}/\{u \in \mathfrak{n}^O_{\mathfrak{F}} \,|\, \nabla(u) = 0\}$. Since $\{u \in \mathfrak{n}^O_{\mathfrak{F}} \,|\, \nabla(u) = 0\} = 0$ we have

$$\mathfrak{a}_{\mathfrak{F}} = \{u \in \mathfrak{g}^O_{\mathfrak{F}} \,|\, \nabla(u) = 0\}\,. \tag{126}$$

The r.h.s. of (126) is a Lie subalgebra of $\mathfrak{g}^O_{\mathfrak{F}}$.

*Lemma.* The Lie algebra structure on $\mathfrak{a}_{\mathfrak{F}}$ that comes from the algebroid structure on $I/I^2$ coincides with the one induced by (126).

*Proof.* It suffices to show that if $\varphi_1, \varphi_2 \in A_{\mathfrak{g}}(K)$ and $d_{\mathfrak{F}}\varphi_i \in \mathfrak{a}_{\mathfrak{F}}$ then

$$d_{\mathfrak{F}}\{\varphi_1, \varphi_2\} = [d_{\mathfrak{F}}\varphi_1, d_{\mathfrak{F}}\varphi_2] \tag{127}$$

(in the r.h.s. of (127) $d_{\mathfrak{F}}\varphi_i$ are considered as elements of $\mathfrak{g}^O_{\mathfrak{F}}$ via (126)). Consider a deformation $\mathfrak{F}(\varepsilon)$ of $\mathfrak{F}$, $\varepsilon^2 = 0$. Write $\mathfrak{F}$ as $(\mathfrak{F}_B, \nabla)$. Without loss of generality we can assume that $\mathfrak{F}(\varepsilon) = (\mathfrak{F}_B, \nabla + \varepsilon q)$, $q \in \mathfrak{b}^K_{\mathfrak{F}} \otimes \omega_K$. Write $d_{\mathfrak{F}(\varepsilon)}\varphi_i$ as $d_{\mathfrak{F}}\varphi_i + \varepsilon\mu_i$. Then

$$\{\varphi_1, \varphi_2\}(\mathfrak{F}(\varepsilon)) = \operatorname{Res}((\nabla + \varepsilon \operatorname{ad} q)(d_{\mathfrak{F}}\varphi_1 + \varepsilon\mu_1), d_{\mathfrak{F}}\varphi_2 + \varepsilon\mu_2) =$$

$$\varepsilon \operatorname{Res}([q, d_{\mathfrak{F}}\varphi_1], d_{\mathfrak{F}}\varphi_2) = \varepsilon \operatorname{Res}(q, [d_{\mathfrak{F}}\varphi_1, d_{\mathfrak{F}}\varphi_2])$$

(we have used that $\nabla(d_{\mathfrak{F}}\varphi_i) = 0$). The equality (127) follows. $\square$

According to the lemma the kernel $\mathfrak{a}_{\mathfrak{F}}$ of the map (124) coincides as a Lie algebra with $(\mathfrak{g}_{\mathrm{univ}})_{\mathfrak{F}}$, i.e., the fiber at $\mathfrak{F}$ of the Lie algebra $\mathfrak{g}_{\mathrm{univ}}$ from 3.5.11. The map (124)=(125) is surjective because $\nabla : \mathfrak{g}_{\mathfrak{F}}^O \to \mathfrak{g}_{\mathfrak{F}}^O \otimes \omega_O$ is surjective. It is easy to show that (121) and (122) are homeomorphisms and that the map (124) is open.

In a similar way one shows that the morphism (123) is surjective and open, and its kernel can be canonically identified with $\mathfrak{g}_{\mathrm{univ}}$ equipped with the discrete topology (the identification induces the above isomorphism $\mathfrak{a}_{\mathfrak{F}} \xrightarrow{\sim} (\mathfrak{g}_{\mathrm{univ}})_{\mathfrak{F}}$ for every $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$). Lemma 3.5.12 yields a continuous Lie algebroid morphism $f : I/I^2 \to \mathfrak{a}_{\mathfrak{g}}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g}_{\mathrm{univ}} & \longrightarrow & I/I^2 & \longrightarrow & \operatorname{Der} A_{\mathfrak{g}}(O) & \longrightarrow & 0 \\ & & {\scriptstyle\mathrm{id}}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} & & \\ 0 & \longrightarrow & \mathfrak{g}_{\mathrm{univ}} & \longrightarrow & \mathfrak{a}_{\mathfrak{g}} & \longrightarrow & \operatorname{Der} A_{\mathfrak{g}}(O) & \longrightarrow & 0 \end{array}$$

is commutative. Since the rows of the diagram are exact in the topological sense, $f$ is a topological isomorphism. Clearly $f$ is $\operatorname{Aut} O$-equivariant.

### 3.8. Singularities of opers.

**3.8.1.** Let $U$ be an open dense subset of our curve $X$. We are going to represent the ind-scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(U)$ as a union of certain closed subschemes $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ where $D$ runs through the set of finite subschemes of $X$ such that $D \cap U = \emptyset$.

According to 3.1.9 we have a canonical isomorphism $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(U) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(U)$ where $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(U)$ is the $\Gamma(U, V_{\omega_X})$-torsor induced from the $\Gamma(U, \omega_X^{\otimes 2})$-torsor $\mathcal{O}\mathfrak{p}_{sl_2}(U)$ by a certain embedding $\Gamma(U, \omega_X^{\otimes 2}) \subset \Gamma(U, V_{\omega_X})$. The definition of this embedding and of $V = V_{\mathfrak{g}}$ can be found in 3.1.9. Let us remind that $V$ is a vector space equipped with a $\mathbb{G}_m$-action (i.e., a grading) and $V_{\omega_X}$ denotes the twist of $V$ by the $\mathbb{G}_m$-torsor $\omega_X$. We have $\dim V = r := \operatorname{rank} \mathfrak{g}$ and the degrees of the graded components of $V$ are equal to the degrees $d_1, \ldots, d_r$ of "basic" invariant polynomials on $\mathfrak{g}$.

If $D$ is a finite subscheme of $X$ one has a canonical embedding $V_{\omega_X} \hookrightarrow V_{\omega_X(D)}$. Denote by $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g},D}(X)$ the $\Gamma(X, V_{\omega_X(D)})$-torsor induced by the $\Gamma(X, V_{\omega_X})$-torsor $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X)$. Clearly $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g},D}(X)$ is a closed subscheme of $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X \setminus D)$. Denote by $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ the image of $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g},D}(X)$ in $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X \setminus D)$. If $D \subset D'$ then $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X) \subset \mathcal{O}\mathfrak{p}_{\mathfrak{g},D'}(X)$ . For any open dense $U \subset X$ we have $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(U) = \bigcup\limits_{D \cap U = \emptyset} \mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$.

In 3.8.23 we will give an "intrinsic" description of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$, which does not use the isomorphism $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_{\mathfrak{g}}$. The local version of this description is given in 3.8.7 – 3.8.10.

**3.8.2.** Now we can formulate the answer to the problem from 2.8.6:

$$N_\Delta(G) = \mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g},\Delta}(X)\,. \tag{128}$$

$N_\Delta(G)$ is defined as a subscheme of an ind-scheme $N'_\Delta(G)$, which is canonically identified with $\mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g}}(X \setminus \Delta)$ via the Feigin - Frenkel isomorphism. (128) is an equality of subschemes of $\mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g}}(X \setminus \Delta)$.

We will not prove (128). A hint will be given in 3.8.6.

**3.8.3.** The definition of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ from 3.8.1 makes sense in the following local situation: $X = \operatorname{Spec} O$, $O := \mathbb{C}[[t]]$, $D = \operatorname{Spec} O/t^n O$. In this case we write $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$ instead of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$. $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$ is a closed subscheme of the ind-scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. Of course $\mathcal{O}\mathfrak{p}_{\mathfrak{g},0}(O) = \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$, $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O) \subset \mathcal{O}\mathfrak{p}_{\mathfrak{g},n+1}(O)$, and $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ is the inductive limit of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$.

According to 3.7.12 $A_{\mathfrak{g}}(K)$ is the algebra of regular functions on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. Denote by $I_n$ the ideal of $A_{\mathfrak{g}}(K)$ corresponding to $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O) \subset \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. Clearly $I_n \supset I_{n+1}$ and $I_0$ is the ideal $I$ from 3.7.16 (i). The ideals $I_n$ form a base of neighbourhoods of 0 in $A_{\mathfrak{g}}(K)$.

**3.8.4.** Here is an explicit description of $A_{\mathfrak{g}}(K)$ and $I_n$. We use the notation of 3.5.6, so $\mathfrak{g}$-opers on $\operatorname{Spec} K$ are in one-to-one correspondence with operators (64) such that $u_j(t) \in \mathbb{C}((t))$. Write $u_j(t)$ as $\sum\limits_k \widetilde{u}_{jk} t^k$. Then $A_{\mathfrak{g}}(K) = \mathbb{C}[[\ldots \widetilde{u}_{j,-1}, \widetilde{u}_{j0}, \widetilde{u}_{j1}, \ldots]$ (we use notation (98)). The ideal $I_n$ is topologically generated by $\widetilde{u}_{jk}$, $k < -d_j n$. The $u_{jk}$ from 3.5.6 are the images of $\widetilde{u}_{jk}$ in $A_{\mathfrak{g}}(O) = A_{\mathfrak{g}}(K)/I$.

It is easy to describe the action of $\operatorname{Der} K$ on $A_{\mathfrak{g}}(K)$. In particular

$$L_0 \widetilde{u}_{jk} = (d_j + k)\widetilde{u}_{jk}\,. \tag{129}$$

Just as in the global situation (see 3.1.12 – 3.1.14) the coordinate ring $A_{\mathfrak{g}}(K)$ of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ carries a canonical filtration. Its $i$-th term consists of those "polynomials" in $\widetilde{u}_{jk}$ whose weighted degree is $\leq i$, it being understood that the weight of $\widetilde{u}_{jk}$ is $d_j$.

**3.8.5.** *Proposition.* The ideal $I_n \subset A_{\mathfrak{g}}(K)$ is topologically generated by the spaces $A_i^m$, $m < i(1-n)$, where $A_i^m$ is the set of elements $a$ from the $i$-th term of the filtration of the $A_{\mathfrak{g}}(K)$ such that $L_0 a = ma$. □

The isomorphism $A_{{}^L\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{Z}$ (see (113)) preserves the filtrations and is $\operatorname{Aut} K$-equivariant. So Proposition 3.8.5 implies the following statement.

**3.8.6.** *Proposition.* The Feigin - Frenkel isomorphism $A_{{}^L\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{Z}$ maps $I_n \subset A_{{}^L\mathfrak{g}}(K)$ onto the ideal $I_n$ from 3.7.9.

This is one of the ingredients of the proof of (128).

**3.8.7.** We are going to describe $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$ in "natural" terms (without using the isomorphism (43)). Denote by $\mathfrak{g}^+$ the locally closed reduced subscheme of $\mathfrak{g}$ consisting of all $a \in \mathfrak{g}$ such that for positive roots $\alpha$ one has $a_{-\alpha} = 0$ if $\alpha$ is non-simple, $a_{-\alpha} \neq 0$ if $\alpha$ is simple ($a_{-\alpha}$ is the component of $a$ from the root subspace $\mathfrak{g}^{-\alpha}$). Then for any $\mathbb{C}$-algebra $R$ the set $\mathfrak{g}^+(R)$ consists of $a \in \mathfrak{g} \otimes R$ such that $a_{-\alpha} = 0$ for each non-simple $\alpha > 0$ and $a_{-\alpha}$ generates the $R$-module $\mathfrak{g}^{-\alpha} \otimes R$ for each simple $\alpha$.

Recall that a $\mathfrak{g}$-oper over $\operatorname{Spec} K$ is a $B(K)$-conjugacy class of operators $\frac{d}{dt} + q(t)$, $q \in \mathfrak{g}^+(K)$. Here $B$ is the Borel subgroup of the adjoint group $G$ corresponding to $\mathfrak{g}$.

**3.8.8.** *Definition.* A ($\leq n$)-singular $\mathfrak{g}$-oper on $\operatorname{Spec} O$ is a $B(O)$-conjugacy class of operators $\frac{d}{dt} + t^{-n}q(t)$, $q \in \mathfrak{g}^+(O)$.

*Remarks*

(i) The action of $B(O)$ on the set of operators $\frac{d}{dt} + t^{-n}q(t)$, $q \in \mathfrak{g}^+(O)$, is free. Indeed, the action of $B(K)$ on $\{\frac{d}{dt} + q(t) | q \in \mathfrak{g}^+(K)\}$ is free (see 3.1.4).

(ii) For $n = 0$ one obtains the usual notion of $\mathfrak{g}$-oper on $\operatorname{Spec} O$.

**3.8.9.** *Proposition.* The map $\{(\leq n)\text{-singular } \mathfrak{g}\text{-opers on } \operatorname{Spec} O\} \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ is injective. Its image equals $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$.

*Proof.* We use the notation of 3.5.6. For every $v_1, \ldots, v_r \in \mathbb{C}[[t]]$ the operator

$$\frac{d}{dt} + t^{-n}(i(f) + v_1(t)e_1 + \ldots + v_r(t)e_r) \tag{130}$$

defines a ($\leq n$)-singular $\mathfrak{g}$-oper on $\operatorname{Spec} O$. It is easy to show that this is a bijection between operators (130) and ($\leq n$)-singular $\mathfrak{g}$-opers on $\operatorname{Spec} O$. Now let us transform (130) to the "canonical form" (64) by $B(K)$-conjugation. Conjugating (130) by $t^{-n\check{\rho}}$ we obtain

$$\frac{d}{dt} + i(f) + n\check{\rho}t^{-1} + t^{-nd_1}v_1(t)e_1 + \ldots + t^{-nd_r}v_r(t)e_r\,. \tag{131}$$

To get rid of $n\check{\rho}t^{-1}$ we conjugate (131) by $\exp(-ne_1/2t)$ and obtain the operator (64) with

$$u_j(t) = t^{-nd_j}v_j(t) \quad \text{for} \quad j > 1\,,$$

$$u_1(t) = t^{-nd_1}v_1(t) + n(n-2)/4t^2\,, \quad d_1 = 2\,.$$

Clearly $v_j \in \mathbb{C}[[t]]$ if and only if $u_j \in t^{-nd_j}\mathbb{C}[[t]]$. $\square$

**3.8.10.** If points of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$ are considered as ($\leq n$)-singular $\mathfrak{g}$-opers on $\operatorname{Spec} O$ then the canonical embedding $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O) \hookrightarrow \mathcal{O}\mathfrak{p}_{\mathfrak{g},n+1}(O)$ maps the $B(O)$-conjugacy class of $\frac{d}{dt} + t^{-n}q(t)$, $q \in \mathfrak{g}^+(O)$, to the $B(O)$-conjugacy class of $t^{\check{\rho}}(\frac{d}{dt} + t^{-n}q(t))t^{-\check{\rho}}$ (it is well-defined because $t^{\check{\rho}}B(O)t^{-\check{\rho}} \subset B(O)$).

**3.8.11.** Denote by $\mathrm{Inv}(\mathfrak{g})$ the algebra of $G$-invariant polynomials on $\mathfrak{g}$. There is a canonical morphism $\mathfrak{g} \to \operatorname{Spec}\mathrm{Inv}(\mathfrak{g}) = W \setminus \mathfrak{h}$ where $W$ is the Weyl group.

Suppose one has a $(\leq 1)$-singular $\mathfrak{g}$-oper on $\operatorname{Spec} O$, i.e., a $B(O)$-conjugacy class of $\frac{d}{dt} + t^{-1}q(t)$, $q \in \mathfrak{g}^+(O)$. The image of $q(0) \in \mathfrak{g}$ in $\operatorname{Spec}\mathrm{Inv}(\mathfrak{g})$ is called the *residue* of the oper. So we have defined the *residue map*

$$\mathrm{Res}: \mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O) \to \operatorname{Spec}\mathrm{Inv}(\mathfrak{g}) = W \setminus \mathfrak{h}\,. \tag{132}$$

It is surjective. Therefore it induces an embedding

$$\mathrm{Inv}(\mathfrak{g}) \hookrightarrow A_{\mathfrak{g}}(K)/I_1 \tag{133}$$

(recall that $A_{\mathfrak{g}}(K)/I_1$ is the coordinate ring of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$; see 3.8.3).

**3.8.12.** *Proposition.* $\mathrm{Res}(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)) \subset W \setminus \mathfrak{h}$ consists of a single point, which is the image of $-\check{\rho} \in \mathfrak{h}$.

*Remark.* We prefer to forget that $-\check{\rho}$ and $\check{\rho}$ have the same image in $W \setminus \mathfrak{h}$.

*Proof.* We must compute the composition of the map $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \to \mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$ described in 3.8.10 and the map (132). If $q(t) \in \mathfrak{g}^+(O)$ then $t^{\check{\rho}}(\frac{d}{dt}+q(t))t^{-\check{\rho}} = \frac{d}{dt} + \frac{a-\check{\rho}}{t} + \{\text{something regular}\}$ where $a$ belongs to the sum of the root spaces corresponding to simple negative roots. Now $a - \check{\rho}$ and $-\check{\rho}$ have the same image in $W \setminus \mathfrak{h}$. $\square$

**3.8.13.** *Proposition.* Let $f \in A_{\mathfrak{g}}(K)/I_1$, i.e., $f$ is a regular function on $\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$. Then the following conditions are equivalent:

(i) $f \in \mathrm{Inv}(\mathfrak{g})$, where $\mathrm{Inv}(\mathfrak{g})$ is identified with its image by (133);
(ii) $f$ is $\mathrm{Aut}^0 O$-invariant;
(iii) $L_0 f = 0$.

*Proof.* Clearly (i)$\Rightarrow$(ii)$\Rightarrow$(iii). Let us deduce (i) from (iii). Consider a $(\leq 1)$-singular $\mathfrak{g}$-oper on $\operatorname{Spec} O$. This is the $B(O)$-conjugacy class of a connection $\frac{d}{dt} + t^{-1}q(t)$, $q \in \mathfrak{g}^+(O)$. If $t$ is replaced by $\lambda t$ this connection is replaced by $\frac{d}{dt} + t^{-1}q(\lambda t)$. Since $L_0 f = 0$ the value of $f$ on the connection

$\frac{d}{dt}+t^{-1}q(\lambda t)$ does not depend on $\lambda$, so it depends only on $q(0)\in\mathfrak{g}^+$ (because $\lim\limits_{\lambda\to 0} q(\lambda t)=q(0)$). It remains to use the fact that a $B$-invariant regular function on $\mathfrak{g}^+$ extends to a $G$-invariant polynomial on $\mathfrak{g}$ (see Theorem 0.10 from [Ko63]). □

**3.8.14.** *Remark.* According to 3.8.4 the algebra $A_{\mathfrak{g}}(K)/I_1$ is freely generated by $\overline{u}_{jk}$, $k\geq -d_j$, where $\overline{u}_{jk}\in A_{\mathfrak{g}}(K)/I_1$ is the image of $\widetilde{u}_{jk}\in A_{\mathfrak{g}}(K)$. By 3.8.13 and (129) $\mathrm{Inv}(\mathfrak{g})\subset A_{\mathfrak{g}}(K)/I_1$ is generated by $v_j:=\overline{u}_{j,-d_j}$. The isomorphism $\mathrm{Spec}\,\mathbb{C}[v_1,\dots,v_r]\xrightarrow{\sim}\mathrm{Spec}\,\mathrm{Inv}(\mathfrak{g})$ is the composition $\mathrm{Spec}\,\mathbb{C}[v_1,\dots,v_r]\to\mathfrak{g}\to\mathrm{Spec}\,\mathrm{Inv}(\mathfrak{g})$ where the first map equals $i(f)-\check{\rho}+v_1e_1+\ldots+v_re_r$ (we use the notation of 3.5.6).

**3.8.15.** We are going to prove Theorem 3.6.11. In 3.8.16 – 3.8.17 we will formulate a property of the Feigin - Frenkel isomorphism (113). This property reduces Theorem 3.6.11 to a certain statement (see 3.8.19), which involves only opers and the Gelfand - Dikii bracket. This statement will be proved in 3.8.20 – 3.8.22.

**3.8.16.** We will use the notation of 3.5.17. Besides, if $\mathrm{Der}\,O$ acts on a vector space $V$ we set $V^0:=\{v\in V|L_0v=0\}$.

As explained in 3.6.9, the map $\pi$ from 3.6.8 induces a morphism

$$(\mathfrak{z}/\mathfrak{z}\cdot\mathfrak{z}^{<0})^0=(\mathfrak{z}/\mathfrak{z}\cdot\mathfrak{z}^{<0})^{\leq 0}=\mathfrak{z}^{\leq 0}/(\mathfrak{z}\cdot\mathfrak{z}^{<0}\cap\mathfrak{z}^{\leq 0})\to C \tag{134}$$

where $C$ is the center of $U\mathfrak{g}$. Now (113) induces an isomorphism

$$(\mathfrak{z}/\mathfrak{z}\cdot\mathfrak{z}^{<0})^0\xrightarrow{\sim}(A_{{}^L\mathfrak{g}}(K)/I_1)^0 \tag{135}$$

because by 3.8.5 $I_1=A_{{}^L\mathfrak{g}}(K)\cdot A_{{}^L\mathfrak{g}}(K)^{<0}$. By 3.8.13 the r.h.s. of (135) equals $\mathrm{Inv}({}^L\mathfrak{g})$. So (134) and (135) yield a morphism

$$\mathrm{Inv}({}^L\mathfrak{g})\to C\,. \tag{136}$$

Denote by $\mathrm{Inv}(\mathfrak{h}^*)$ the algebra of $W$-invariant polynomials on $\mathfrak{h}^*$. Since ${}^L\mathfrak{h}=\mathfrak{h}^*$ there is a canonical isomorphism $\mathrm{Inv}({}^L\mathfrak{g})\xrightarrow{\sim}\mathrm{Inv}(\mathfrak{h}^*)$. We also have the Harish-Chandra isomorphism $C \xrightarrow{\sim} \mathrm{Inv}(\mathfrak{h}^*)$. So (136) can be considered as a map

$$\mathrm{Inv}(\mathfrak{h}^*) \to \mathrm{Inv}(\mathfrak{h}^*)\,. \tag{137}$$

**3.8.17.** *Theorem.* (E. Frenkel, private communication)

The morphism (137) maps $f \in \mathrm{Inv}(\mathfrak{h}^*)$ to $f^-$ where $f^-(\lambda) := f(-\lambda)$, $\lambda \in \mathfrak{h}^*$. □

**3.8.18.** Using 3.8.17 we can replace the mysterious lower left corner of diagram (84) by its oper analog. Diagram (143) below is obtained essentially this way. Let us define the lower arrow of (143), which is the oper analog of the map (83) constructed in 3.6.9 – 3.6.10.

According to 3.8.5

$$I_1 = A_{\mathfrak{g}}(K) \cdot A_{\mathfrak{g}}(K)^{<0}\,. \tag{138}$$

By 3.8.13 we have a canonical isomorphism

$$(A_{\mathfrak{g}}(K)/I_1)^0 \xrightarrow{\sim} \mathrm{Inv}(\mathfrak{g})\,. \tag{139}$$

For $h \in \mathfrak{h}$ denote by $m_h$ the maximal ideal of $\mathrm{Inv}(\mathfrak{g})$ consisting of polynomials vanishing at $h$. Set $m := m_{-\check{\rho}}$. By 3.8.12 the isomorphism (139) induces

$$(I/I_1)^0 \xrightarrow{\sim} m\,. \tag{140}$$

Now we obtain

$$(I/(I^2 + I_1))^0 \xrightarrow{\sim} m/m^2 \tag{141}$$

(to get (141) from (140) we use that

$$(I^2)^0 \subset (I^0)^2 + I \cdot I^{<0} \subset (I^0)^2 + A_{\mathfrak{g}}(K) \cdot A_{\mathfrak{g}}(K)^{<0} = (I^0)^2 + I_1\,;$$

see (138)).

For a regular $h \in \mathfrak{h}$ we identify $m_h/m_h^2$ with $\mathfrak{h}^*$ by assigning to a $W$-invariant polynomial on $\mathfrak{h}$ its differential at $h$. In particular for $m = m_{-\check{\rho}}$

we have $m/m^2 \xrightarrow{\sim} \mathfrak{h}^*$ (by the way, if we wrote $m$ as $m_{\check{\rho}}$ we would obtain a different isomorphism $m/m^2 \xrightarrow{\sim} \mathfrak{h}^*$).

Finally, using (138) we rewrite the l.h.s. of (141) in terms of $I/I^2$ and get an isomorphism

$$(I/I^2)^{\leq 0}/(A_{\mathfrak{g}}(O) \cdot (I/I^2)^{<0} \cap (I/I^2)^{\leq 0}) \xrightarrow{\sim} \mathfrak{h}^* . \tag{142}$$

**3.8.19.** *Proposition.* The diagram

$$\begin{tikzcd}
(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}/(A_{\mathfrak{g}}(O) \cdot \mathfrak{a}_{\mathfrak{g}}^{<0} \cap (\mathfrak{a}_{\mathfrak{g}})^{\leq 0}) \arrow[r, "\sim"] & \mathfrak{h} \arrow[d, "\wr"] \\
(I/I^2)^{\leq 0}/(A_{\mathfrak{g}}(O) \cdot (I/I^2)^{<0} \cap (I/I^2)^{\leq 0}) \arrow[u, "\wr"] \arrow[r, "\sim"] & \mathfrak{h}^*
\end{tikzcd} \tag{143}$$

commutes. Here the lower arrow is the isomorphism (142), the upper one is the isomorphism (78), the left one is induced by the isomorphism (120) (which comes from the Gelfand - Dikii bracket on $A_{\mathfrak{g}}(K)$), and the right one is induced by the invariant scalar product on $\mathfrak{g}$ used in the definition of the Gelfand - Dikii bracket.

The proposition will be proved in 3.8.20 – 3.8.22.

Theorem 3.6.11 follows from 3.8.17 and 3.8.19. The commutativity of (143) implies the *anti*commutativity of (84) because the following diagram is anticommutative:

$$\begin{tikzcd}
m_{\check{\rho}}/(m_{\check{\rho}})^2 \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & m_{-\check{\rho}}/(m_{-\check{\rho}})^2 \arrow[dl, "\sim"] \\
& \mathfrak{h}^* &
\end{tikzcd}$$

Here the upper arrow is induced by the map $f \mapsto f^-$ from 3.8.17.

**3.8.20.** We are going to formulate a lemma used in the proof of Proposition 3.8.19. Consider the composition

$$I/I^2 \to I/(I^2 + I_1) \xrightarrow{\sim} \mathfrak{a}_{\mathfrak{g}}/A_{\mathfrak{g}}(O) \cdot \mathfrak{a}_{\mathfrak{g}}^{<0} = \mathfrak{a}_{\mathfrak{g}}/\mathfrak{a}_{\mathfrak{n}} = \mathfrak{g}_{\text{univ}}/\mathfrak{n}_{\text{univ}} . \tag{144}$$

Here the second arrow comes from (120) and (138); $\mathfrak{a}_{\mathfrak{n}}$ and $\mathfrak{n}_{\text{univ}}$ were defined in 3.5.16, $\mathfrak{a}_{\mathfrak{g}}$ was defined in 3.5.11; the equality $\mathfrak{a}_{\mathfrak{n}} = A_{\mathfrak{g}}(O) \cdot \mathfrak{a}_{\mathfrak{g}}^{<0}$

was proved in 3.5.18. The fiber of $I/I^2$ over $\mathfrak{F} = (\mathfrak{F}_B, \nabla) \in \mathcal{O}\mathfrak{p}_\mathfrak{g}(O)$ equals $\{u \in \mathfrak{g}_\mathfrak{F}^O | \nabla(u) \in \mathfrak{b}_\mathfrak{F}^O \otimes \omega_O\}/\mathfrak{n}_\mathfrak{F}^O$ (see (122)) and the fiber of $\mathfrak{g}_{\text{univ}}/\mathfrak{n}_{\text{univ}}$ over $\mathfrak{F}$ equals $(\mathfrak{g}_\mathfrak{F}/\mathfrak{n}_\mathfrak{F})_0$ :=the fiber of $\mathfrak{g}_\mathfrak{F}/\mathfrak{n}_\mathfrak{F}$ at the origin $0 \in \operatorname{Spec} O$. Consider the maps

$$\varphi, \psi : \{u \in \mathfrak{g}_\mathfrak{F}^O | \nabla(u) \in \mathfrak{b}_\mathfrak{F}^O \otimes \omega_O\}/\mathfrak{n}_\mathfrak{F}^O \to (\mathfrak{g}_\mathfrak{F}/\mathfrak{n}_\mathfrak{F})_0$$

where $\varphi$ is induced by (144) and $\psi$ is evaluation at 0.

**3.8.21.** *Lemma.* $\varphi = \psi$.

*Proof.* It follows from 3.7.17 that the restrictions of $\varphi$ and $\psi$ to $\mathfrak{a}_\mathfrak{F} := \{u \in \mathfrak{g}_\mathfrak{F}^O | \nabla(u) = 0\}$ are equal. So it suffices to show that $\operatorname{Ker} \varphi \subset \operatorname{Ker} \psi$. Clearly $\operatorname{Ker} \varphi = T_\mathfrak{F}^\perp \mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$ := the conormal space to $\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$ at $\mathfrak{F}$. For any $q \in \mathfrak{b}_\mathfrak{F}^O$ the oper $\mathfrak{F}_q := (\mathfrak{F}_B, \nabla + q \cdot \frac{dt}{t})$ is $(\leq 1)$-singular. So the image of $\mathfrak{b}_\mathfrak{F}^O \otimes t^{-1}\omega_O$ in the r.h.s. of (117) is contained in the tangent space $T_\mathfrak{F}\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$. Therefore $T_\mathfrak{F}^\perp \mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O) \subset \operatorname{Ker} \psi$. □

**3.8.22.** Now let us prove 3.8.19. Since the l.h.s. of (142) equals the l.h.s. of (141) we can reformulate 3.8.19 as follows.

Let $f \in \operatorname{Inv}(\mathfrak{g})$, $f(-\check{\rho}) = 0$. Consider $f$ as an element of $A_\mathfrak{g}(K)/I_1$ (see (133)). By 3.8.12 $f \in I/I_1$. The image of $f$ in $I/(I^2 + I_1)$ can be considered as an element $\nu \in \mathfrak{g}_{\text{univ}}/\mathfrak{n}_{\text{univ}}$ (see (144)). On the other hand, let $\lambda \in \mathfrak{h}^*$ be the differential at $-\check{\rho}$ of the restriction of $f \in \operatorname{Inv}(\mathfrak{g})$ to $\mathfrak{h}$. To prove 3.8.19 we must show that $\nu$ equals the image of $\lambda$ under the composition

$$\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h} \subset \mathfrak{h} \otimes A_\mathfrak{g}(O) = \mathfrak{b}_{\text{univ}}/\mathfrak{n}_{\text{univ}} \subset \mathfrak{g}_{\text{univ}}/\mathfrak{n}_{\text{univ}}\,.$$

By 3.8.21 this is equivalent to the following statement: let $\mathfrak{F} = (\mathfrak{F}_B, \nabla) \in \mathcal{O}\mathfrak{p}_\mathfrak{g}(O)$, $q \in \mathfrak{b}_\mathfrak{F}^O$, $\mathfrak{F}_{\varepsilon q} := (\mathfrak{F}_B, \nabla + \varepsilon q \frac{dt}{t})$, then

$$\frac{d}{d\varepsilon} f(\operatorname{Res}(\mathfrak{F}_{\varepsilon q}))|_{\varepsilon=0} = \lambda(q_\mathfrak{h}(0)) \tag{145}$$

where $q_\mathfrak{h}(t) \in \mathfrak{h}[[t]]$ is the image of $q$ in $\mathfrak{b}_\mathfrak{F}^O/\mathfrak{n}_\mathfrak{F}^O = \mathfrak{h} \otimes O$. Just as in the proof of 3.8.12 one shows that $\operatorname{Res}(\mathfrak{F}_{\varepsilon q})$ equals the image of $-\check{\rho} + \varepsilon q_\mathfrak{h}(0)$ in $W \setminus \mathfrak{h}$. So (145) is clear.

**3.8.23.** In this subsection (which can certainly be skipped by the reader) we give an "intrinsic" description of the scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ from 3.8.1. It is obtained by a straightforward "globalization" of 3.8.7 – 3.8.10.

Denote by $G$ the adjoint group corresponding to $\mathfrak{g}$. Suppose we are in the situation of 3.1.2. So we have a $B$-bundle $\mathfrak{F}_B$ on $X$, the induced $G$-bundle $\mathfrak{F}_G$, and the $\mathfrak{g}_{\mathfrak{F}} \otimes \omega_X$-torsor $\mathrm{Conn}(\mathfrak{F}_G)$. Let $D$ be a finite subscheme of $X$. Denote by $\mathrm{Conn}_D(\mathfrak{F}_G)$ the $\mathfrak{g}_{\mathfrak{F}} \otimes \omega_X(D)$-torsor induced by $\mathrm{Conn}(\mathfrak{F}_G)$; so sections of $\mathrm{Conn}_D(\mathfrak{F}_G)$ are connections with $(\leq D)$-singularities. Just as in 3.1.2 one defines $c : \mathrm{Conn}_D(\mathfrak{F}_G) \to (\mathfrak{g}/\mathfrak{b})_{\mathfrak{F}} \otimes \omega_X(D)$. The notion of *$(\leq D)$-singular $\mathfrak{g}$-oper* on $X$ is defined as follows: in Definition 3.1.3 replace Conn by $\mathrm{Conn}_D$ and $\omega_X$ by $\omega_X(D)$.

If $X$ is complete then $(\leq D)$-singular $\mathfrak{g}$-opers on $X$ form a scheme. Just as in 3.8.9 one shows that the natural morphism from this scheme to $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X \setminus D)$ is a closed embedding and its image equals $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$. So one can consider $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ as the moduli scheme of $(\leq D)$-singular $\mathfrak{g}$-opers on $X$.

If $D \subset D'$ then $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X) \subset \mathcal{O}\mathfrak{p}_{\mathfrak{g},D'}(X)$, so we should have a natural way to construct a $(\leq D')$-singular $\mathfrak{g}$-oper $(\mathfrak{F}'_B, \nabla')$ from a $(\leq D)$-singular $\mathfrak{g}$-oper $(\mathfrak{F}_B, \nabla)$. Of course $(\mathfrak{F}'_B, \nabla')$ should be equipped with an isomorphism $\alpha : (\mathfrak{F}'_B, \nabla')|_{X \setminus \Delta} \xrightarrow{\sim} (\mathfrak{F}_B, \nabla)|_{X \setminus \Delta}$ where $\Delta \subset X$ is the finite subscheme such that $D' = D + \Delta$ if $D$, $D'$, $\Delta$ are considered as effective divisors. The connection $\nabla'$ is reconstructed from $\nabla$ and $\alpha$, while $(\mathfrak{F}'_B, \alpha)$ is defined by the following property (cf. 3.8.10): if $x \in \Delta$, $f$ is a local equation of $\Delta$ at $x$ and $s$ is a local section of $\mathfrak{F}_B$ at $x$ then there is a local section $s'$ of $\mathfrak{F}'_B$ at $x$ such that $\alpha(s') = \lambda(f)s$ where $\lambda : \mathbb{G}_m \to H$ is the morphism corresponding to $\check{\rho}$.

# 4. Pfaffians and all that

## 4.0. Introduction.

**4.0.1.** Consider the "normalized" canonical bundle

$$\omega^\sharp_{\mathrm{Bun}_G} := \omega_{\mathrm{Bun}_G} \otimes \omega_0^{\otimes -1} \tag{146}$$

where $\omega_0$ is the fiber of $\omega_{\mathrm{Bun}_G}$ over the point of $\mathrm{Bun}_G$ corresponding to the trivial $G$-bundle on $X$. In this section we will associate to an ${}^LG$-oper $\mathfrak{F}$ the invertible sheaf $\lambda_{\mathfrak{F}}$ on $\mathrm{Bun}_G$ mentioned in 0.2(d). $\lambda_{\mathfrak{F}}$ will be equipped with an isomorphism $\lambda_{\mathfrak{F}}^{\otimes 2n} \xrightarrow{\sim} (\omega^\sharp_{\mathrm{Bun}_G})^{\otimes n}$ for some $n \neq 0$. This isomorphism induces the twisted $\mathcal{D}$-module structure on $\lambda_{\mathfrak{F}}$ required in 0.2(d).

According to formula (57) from 3.4.3 $\mathcal{O}\mathfrak{p}_{{}^LG}(X) = \mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g}}(X) \times Z\,\mathrm{tors}_\theta(X)$ where $Z$ is the center of ${}^LG$. Actually $\lambda_{\mathfrak{F}}$ depends only on the image of $\mathfrak{F}$ in $Z\,\mathrm{tors}_\theta(X)$. So our goal is to construct a canonical functor

$$\lambda\colon\ Z\,\mathrm{tors}_\theta(X) \to \mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G) \tag{147}$$

where $\mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G)$ is the groupoid of line bundles $\mathcal{A}$ on $\mathrm{Bun}_G$ equipped with an isomorphism $\mathcal{A}^{\otimes 2n} \xrightarrow{\sim} (\omega^\sharp_{\mathrm{Bun}_G})^{\otimes n}$ for some $n \neq 0$.

**4.0.2.** The construction of (147) is quite simple if $G$ is simply connected. In this case $Z$ is trivial, so one just has to construct an object of $\mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G)$. Since $G$ is simply connected $\mathrm{Bun}_G$ is connected and simply connected (interpret a $G$-bundle on $X$ as a $G$-bundle on the $C^\infty$ manifold corresponding to $X$ equipped with a $\bar{\partial}$-connection). So the problem is to show the existence of a square root of $\omega^\sharp_{\mathrm{Bun}_G}$ (then $\mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G)$ has a unique object whose fiber over the point of $\mathrm{Bun}_G$ corresponding to the trivial $G$-bundle is trivialized). To solve this problem we use the notion of *Pfaffian.*

To any vector bundle $\mathcal{Q}$ equipped with a non-degenerate symmetric form $\mathcal{Q}\otimes\mathcal{Q} \to \omega_X$ Laszlo and Sorger associate in [La-So] its Pfaffian $\mathrm{Pf}(\mathcal{Q})$, which is a canonical square root of $\det R\Gamma(X, \mathcal{Q})$. In 4.2 we give another definition of Pfaffian presumably equivalent to the one from [La-So].

Fix $\mathcal{L} \in \omega^{1/2}(X)$ (i.e., $\mathcal{L}$ is a square root of $\omega_X$). Then the line bundle on $\mathrm{Bun}_G$ whose fiber at $\mathcal{F} \in \mathrm{Bun}_G$ equals

$$\mathrm{Pf}(\mathfrak{g}_{\mathcal{F}} \otimes \mathcal{L}) \otimes \mathrm{Pf}(\mathfrak{g} \otimes \mathcal{L})^{\otimes -1} \tag{148}$$

is a square root of $\omega^{\sharp}_{\mathrm{Bun}_G}$ (see 4.3.1 for details).

So to understand the case where $G$ is simply connected it is enough to look through 4.2 and 4.3.1. In the general case the construction of (147) is more complicated. The main point is that the square root of $\omega^{\sharp}_{\mathrm{Bun}_G}$ defined by (148) depends on $\mathcal{L} \in \omega^{1/2}(X)$.

**4.0.3.** Here is an outline of the construction of (148) for any semisimple $G$.

As explained in 3.4.6 $Z\,\mathrm{tors}_{\theta}(X)$ is a Torsor over the Picard category $Z\,\mathrm{tors}(X)$ and $\mu_\infty\,\mathrm{tors}_{\theta}(\mathrm{Bun}_G)$ is a Torsor over the Picard category

$$\mu_\infty\,\mathrm{tors}(\mathrm{Bun}_G) := \varinjlim_{n} \mu_n\,\mathrm{tors}(\mathrm{Bun}_G) \tag{149}$$

The functor (147) we are going to construct is $\ell$-affine in the sense of 3.4.6 for a certain Picard functor $\ell :\; Z\,\mathrm{tors}(X) \to \mu_\infty\,\mathrm{tors}(\mathrm{Bun}_G)$. We define $\ell$ in 4.1. The Torsor $Z\,\mathrm{tors}_{\theta}(X)$ is induced from $\omega^{1/2}(X)$ via a certain Picard functor $\mu_2\,\mathrm{tors}(X) \to Z\,\mathrm{tors}(X)$ (see 3.4.6). So to construct $\lambda$ it is enough to construct an $\ell'$-affine functor $\lambda' \colon\; \omega^{1/2}(X) \to \mu_\infty\,\mathrm{tors}_{\theta}(X)$ where $\ell'$ is the composition $\mu_2\,\mathrm{tors}(X) \to Z\,\mathrm{tors}(X) \xrightarrow{\ell} \mu_\infty\,\mathrm{tors}(\mathrm{Bun}_G)$. We define $\lambda'$ by $\mathcal{L} \mapsto \lambda'_{\mathcal{L}}$ where $\lambda'_{\mathcal{L}}$ is the line bundle on $\mathrm{Bun}_G$ whose fiber at $\mathcal{F} \in \mathrm{Bun}_G$ equals (148). The fact that $\lambda'$ is $\ell'$-affine is deduced in 4.4 from 4.3.10, which is a general statement on $SO_n$-bundles[26]. Actually in subsections 4.2 and 4.3 devoted to Pfaffians the group $G$ does not appear at all.

**4.0.4.** Each line bundle on $\mathrm{Bun}_G$ constructed in this section is equipped with the following extra structure: for every $x \in X$ a central extension of $G(K_x)$ acts on its pullback to the scheme $\mathrm{Bun}_{G,\underline{x}}$ from 2.3.1. This structure is used in 4.3. We will also need it in Chapter 5.

[26]In fact 4.3.10 is a refinement of Proposition 5.2 from [BLaSo].

## 4.1. $\mu_\infty$-torsors on $\mathrm{Bun}_G$.

**4.1.1.** Let $G$ be a connected affine algebraic group, $\Pi$ a finite abelian group, $0 \to \Pi(1) \to \tilde{G} \to G \to 0$ an extension of $G$. Our goal is to construct a canonical Picard functor $\ell : \Pi^\vee \operatorname{tors}(X) \to \mu_\infty \operatorname{tors}(\mathrm{Bun}_G)$ where $\Pi^\vee := \mathrm{Hom}(\Pi, \mu_\infty)$.

*Remark.* If $G$ is semisimple and $\tilde{G}$ is the universal covering of $G$ then $\Pi = \pi_1(G)$ and $\Pi^\vee$ is canonically isomorphic to the center $Z$ of ${}^L G$ (the isomorphism is induced by the duality between the Cartan tori of $G$ and ${}^L G$). So in this case $\ell$ is a Picard functor $Z \operatorname{tors}(X) \to \mu_\infty \operatorname{tors}(\mathrm{Bun}_G)$, as promised in 4.0.3.

We construct $\ell$ in 4.1.2–4.1.4. We "explain" the construction in 4.1.5 and slightly reformulate it in 4.1.6. In 4.1.7–4.1.9 the action of a central extension of $G(K_x)$ is considered. In 4.1.10–4.1.11 we give a description of the fundamental groupoid of $\mathrm{Bun}_G$, which clarifies the construction of torsors on $\mathrm{Bun}_G$. The reader can skip 4.1.5 and 4.1.10–4.1.11.

**4.1.2.** For $\mathcal{F} \in \mathrm{Bun}_G$ denote by $\tilde{\mathcal{F}}$ the $\Pi(1)$-gerbe on $X$ of $\tilde{G}$-liftings of $\mathcal{F}$. Its class $c(\mathcal{F})$ is the image of $cl(\mathcal{F})$ by the boundary map $H^1(X, G) \to H^2\big(X, \Pi(1)\big) = \Pi$. For a finite non-empty $S \subset X$ the gerbe $\tilde{\mathcal{F}}_{X \setminus S}$ is neutral. Therefore $|\tilde{\mathcal{F}}(X \setminus S)|$ (:= the set of isomorphism classes of objects of $\tilde{\mathcal{F}}(X \setminus S)$) is a non-empty $H^1\big(X \setminus S, \Pi(1)\big)$-torsor. Denote it by $\phi_{S,\mathcal{F}}$. When $\mathcal{F}$ varies $\phi_{S,\mathcal{F}}$ become fibers of an $H^1\big(X \setminus S, \Pi(1)\big)$-torsor $\phi_S$ over $\mathrm{Bun}_G$.

**4.1.3.** For any $x \in X$ the set $|\tilde{\mathcal{F}}(\operatorname{Spec} O_x)|$ has a single element. We use it to trivialize the $\Pi$-torsor $|\tilde{\mathcal{F}}(\operatorname{Spec} K_x)|$ (note that $\Pi = H^1\big(\operatorname{Spec} K_x, \Pi(1)\big)$). Thus the restriction to $\operatorname{Spec} K_s$, $s \in S$, defines a $\mathrm{Res}_s$-affine map $\mathrm{Res}_{s,\mathcal{F}} : \phi_{S,\mathcal{F}} \to \Pi$ where $\mathrm{Res}_s : H^1\big(X \setminus S, \Pi(1)\big) \to \Pi$ is the residue at $s$. For $c \in \Pi$ set $\Pi^S_c := \{\pi_S = (\pi_s) : \sum \pi_s = c\} \subset \Pi^S$. The map $\mathrm{Res}_{S,\mathcal{F}} := (\mathrm{Res}_{s,\mathcal{F}}) : \phi_{S,\mathcal{F}} \to \Pi^S$ has image $\Pi^S_{c(\mathcal{F})}$.

**4.1.4.** Recall that $\Pi^\vee$ is the group dual to $\Pi$, so we have a non-degenerate pairing $(\ ):\Pi\times\Pi^\vee\to\mu_\infty$ .

Let $\mathcal{E}$ be a $\Pi^\vee$-torsor on $X$. Set $\mathcal{E}_S:=\prod_{s\in S}\mathcal{E}_s=$ the set of trivializations of $\mathcal{E}$ at $S$; this is a $(\Pi^\vee)^S$-torsor. For any $e\in\mathcal{E}_S$ we have the class $cl(\mathcal{E},e)\in H^1_c(X\setminus S,\Pi^\vee)$. Denote by $\ell_{S,\mathcal{E},\mathcal{F}}$ a $\mu_\infty$-torsor equipped with a map

$$(150)\qquad (\ ,\ )_\ell:\phi_{S,\mathcal{F}}\times\mathcal{E}_S\to\ell_{S,\mathcal{E},\mathcal{F}}$$

such that for $\varphi\in\phi_{S,\mathcal{F}}$, $e=(e_s)\in\mathcal{E}_S$, $h\in H^1\big(X\setminus S,\Pi(1)\big)$, $\chi=(\chi_s)\in(\Pi^\vee)^S$ one has

$$(151)\qquad \begin{gathered}(\varphi+h,e)_\ell=(h,cl(\mathcal{E},e))_{\mathcal{P}}(\varphi,e)_\ell\\ (\varphi,\chi e)_\ell=(\operatorname{Res}_S\varphi,\chi)(\varphi,e)_\ell.\end{gathered}$$

Here $(\ ,\ )_{\mathcal{P}}:H^1\big(X\setminus S,\Pi(1)\big)\times H^1_c\big(X\setminus S,\Pi^\vee\big)\to\mu_\infty$ is the Poincaré pairing and $(\operatorname{Res}_S\varphi,\chi):=\prod_{s\in S}(\operatorname{Res}_s\varphi,\chi_s)\in\mu_\infty$. Such $(\ell_{S,\mathcal{E},\mathcal{F}},(\ )_\ell))$ exists and is unique. If $S'\supset S$ then we have obvious maps $\phi_{S,\mathcal{F}}\hookrightarrow\phi_{S',\mathcal{F}}$, $\mathcal{E}_{S'}\twoheadrightarrow\mathcal{E}_S$, and there is a unique identification of $\mu_\infty$-torsors $\ell_{S,\mathcal{E},\mathcal{F}}=\ell_{S',\mathcal{E},\mathcal{F}}$ that makes these maps mutually adjoint for $(\ ,\ )_\ell$. Thus our $\mu_\infty$-torsor is independent of $S$ and we denote it simply $\ell_{\mathcal{E},\mathcal{F}}$.

When $\mathcal{F}$ varies $\ell_{\mathcal{E},\mathcal{F}}$ become fibers of a $\mu_\infty$-torsor $\ell_{\mathcal{E}}$ over $\mathrm{Bun}_G$. The functor

$$(152)\qquad \ell=\ell^{\tilde G}:\Pi^\vee\operatorname{tors}(X)\to\mu_\infty\operatorname{tors}(\mathrm{Bun}_G),$$

$\mathcal{E}\mapsto\ell_{\mathcal{E}}$, has an obvious structure of Picard functor. The corresponding homomorphism of the automorphism groups $\Pi^\vee\to\Gamma(\mathrm{Bun}_G,\mu_\infty)$ is $\chi\mapsto(c,\chi)$.

*Remark.* In fact $\ell$ is a functor $\Pi^\vee\operatorname{tors}(X)\to\mu_m\operatorname{tors}(\mathrm{Bun}_G)$ where $m$ is the order of $\Pi$. This follows from the construction or from the fact that (152) is a Picard functor.

**4.1.5.** For an abelian group $A$ denote by $A\,\mathrm{gerbes}(X)$ the category associated to the 2-category of $A$-gerbes on $X$ (so $A\,\mathrm{gerbes}(X)$ is the groupoid whose objects are $A$-gerbes on $X$ and whose morphisms are 1-morphisms up to 2-isomorphism). In 4.1.2–4.1.4 we have in fact constructed a bi-Picard functor

$$\Pi^{\vee}\,\mathrm{tors}(X)\times\Pi(1)\,\mathrm{gerbes}(X)\to\mu_{\infty}\,\mathrm{tors} \tag{153}$$

where $\mu_\infty\,\mathrm{tors}$ denotes the category of $\mu_\infty$-torsors over a point. In this subsection (which can be skipped by the reader) we give a "scientific interpretation" of this construction.

In §1.4.11 from [Del73] Deligne associates a Picard category to a complex $K^{\cdot}$ of abelian groups such that $K^i=0$ for $i\neq 0,-1$. We denote this Picard category by $P(K^{\cdot})$. Its objects are elements of $K^0$ and a morphism from $x\in K^0$ to $y\in K^0$ is an element $f\in K^{-1}$ such that $df=y-x$.

In 4.1.4 we implicitly used the interpretation of $\Pi^{\vee}\,\mathrm{tors}(X)$ as $P(K^{\cdot}_S)$ where $K^0_S=H^1_c(X\backslash S,\Pi^{\vee})=$ the set of isomorphism classes of $\Pi^{\vee}$-torsors on $X$ trivialized over $S$, $K^{-1}_S=H^0(S,\Pi^{\vee})$. In 4.1.3 we implicitly used the interpretation of $\Pi(1)\,\mathrm{gerbes}(X)$ as $P(L^{\cdot}_S)$ where $L^0_S=H^2_S(X,\Pi(1))=\Pi^S$, $L^{-1}_S=H^1(X\backslash S,\Pi(1))$ ($L^0_S$ parametrizes $\Pi(1)$-gerbes on $X$ with a fixed object over $X\backslash S$). The construction of the bi-Picard functor (153) given in 4.1.4 uses only the canonical pairing $K^{\cdot}_S\times L^{\cdot}_S\to\mu_\infty[1]$.

For $S'\supset S$ we have canonical quasi-isomorphisms $K^{\cdot}_{S'}\to K^{\cdot}_S$ and $L^{\cdot}_S\to L^{\cdot}_{S'}$. The corresponding equivalences $P(K^{\cdot}_{S'})\to P(K^{\cdot}_S)$ and $P(L^{\cdot}_S)\to P(L^{\cdot}_{S'})$ are compatible with our identifications of $P(K^{\cdot}_S)$ and $P(K^{\cdot}_{S'})$ with $\Pi^{\vee}\,\mathrm{tors}(X)$ and also with the identifications of $P(L^{\cdot}_S)$ and $P(L^{\cdot}_{S'})$ with $\Pi(1)\,\mathrm{gerbes}(X)$. The morphism $L^{\cdot}_S\to L^{\cdot}_{S'}$ is adjoint to $K^{\cdot}_{S'}\to K^{\cdot}_S$ with respect to the pairings $K^{\cdot}_S\times L^{\cdot}_S\to\mu_\infty[1]$ and $K^{\cdot}_{S'}\times L^{\cdot}_{S'}\to\mu_\infty[1]$. Therefore (153) does not depend on $S$.

*Remarks*

(i) Instead of $K^{\cdot}_S$ and $L^{\cdot}_S$ it would be more natural to use their images in the derived category, i.e., $(\tau_{\leq 1}R\Gamma(X,\Pi^\vee))[1]$ and $(\tau_{\geq 1}R\Gamma(X,\Pi(1)))[2]$. However the usual derived category is not enough: according to §§1.4.13–1.4.14 from [Del73] the image of $K^{\cdot}$ in the derived category only gives $P(K^{\cdot})$ up to equivalence unique up to *non-unique* isomorphism. So one needs a refined version of the notion of derived category, which probably cannot be found in the literature.

(ii) From the non-degeneracy of the pairing $K^{\cdot}_S \times L^{\cdot}_S \to \mu_\infty[1]$ one can easily deduce that (153) induces an equivalence between $\Pi^\vee$ tors$(X)$ and the category of Picard functors $\Pi(1)$ gerbes$(X) \to \mu_\infty$ tors (this is a particular case of the equivalence (1.4.18.1) from [Del73]).

**4.1.6.** The definition of $\ell_{\mathcal{E}}$ from 4.1.4 can be reformulated as follows. Let $S \subset X$ be finite and non-empty. For a fixed $e \in \mathcal{E}_S$ we have the class $c = cl(\mathcal{E}, e) \in H^1_c(X\backslash S, \Pi^\vee)$ and therefore a morphism $\lambda_e : H^1(X\backslash S, \Pi(1)) \to \mu_\infty$ defined by $\lambda_e(h) = (h, c)_{\mathcal{P}}$. Denote by $\ell_{\mathcal{E},e}$ the $\lambda_e$-pushforward of the $H^1(X\backslash S, \Pi(1))$-torsor $\phi_S$ from 4.1.2. The torsors $\ell_{\mathcal{E},e}$ for various $e \in \mathcal{E}_S$ are identified as follows.

Let $\tilde{e} = \chi e$, $\chi \in (\Pi^\vee)^S$. Then $\lambda_{\tilde{e}}(h)/\lambda_e(h) = (\mathrm{Res}_S(h), \chi)$ where $\mathrm{Res}_S$ is the boundary morphism $H^1(X\backslash S, \Pi(1)) \to H^2_S(X, \Pi(1)) = \Pi^S$. So $\ell_{\mathcal{E},\tilde{e}}/\ell_{\mathcal{E},e}$ is the pushforward of the $\Pi^S$-torsor $(\mathrm{Res}_S)_*\phi_S$ via $\chi : \Pi^S \to \mu_\infty$. The map $\mathrm{Res}_{S,\mathcal{F}} : \phi_{S,\mathcal{F}} \to \Pi^S$ from 4.1.3 induces a canonical trivialization of $(\mathrm{Res}_S)_*\phi_S$ and therefore a canonical isomorphism $\ell_{\mathcal{E},e} \xrightarrow{\sim} \ell_{\mathcal{E},\tilde{e}}$. So we can identify $\ell_{\mathcal{E},e}$ for various $e \in \mathcal{E}_S$ and obtain a $\mu_\infty$-torsor on $\mathrm{Bun}_G$, which does not depend on $e \in \mathcal{E}_S$. Clearly it does not depend on $S$. This is $\ell_{\mathcal{E}}$.

**4.1.7.** Let $S \subset X$ be a non-empty finite set, $O_S := \prod\limits_{x\in S} O_x$, $K_S := \prod\limits_{x\in S} K_x$ where $O_x$ is the completed local ring of $x$ and $K_x$ is its field of fractions. Denote by $\underline{S}$ the formal neighbourhood of $S$ and by $\mathrm{Bun}_{G,\underline{S}}$ the moduli scheme of $G$-bundles on $X$ trivialized over $\underline{S}$ (in 2.3.1 we introduced $\mathrm{Bun}_{G,\underline{x}}$, which corresponds to $S = \{x\}$). One defines an action of $G(K_S)$ on $\mathrm{Bun}_{G,\underline{S}}$

extending the action of $G(O_S)$ by interpreting a $G$-bundle on $X$ as a $G$-bundle on $X \setminus S$ with a trivialization of its pullback to $\operatorname{Spec} K_S$ (see 2.3.4 and 2.3.7).

Let $\ell_{\mathcal{E}}$ be the $\mu_\infty$-torsor on $\mathrm{Bun}_G$ corresponding to a $\Pi^\vee$-torsor $\mathcal{E}$ on $X$ (see 4.1.4, 4.1.6). Denote by $\ell_{\mathcal{E}}^S$ the inverse image of $\ell_{\mathcal{E}}$ on $\mathrm{Bun}_{G,\underline{S}}$. The action of $G(O_S)$ on $\mathrm{Bun}_{G,\underline{S}}$ canonically lifts to its action on $\ell_{\mathcal{E}}^S$. We claim that a trivialization of $\mathcal{E}$ over $S$ defines an action of $G(K_S)$ on $\ell_{\mathcal{E}}^S$ extending the above action of $G(O_S)$ and compatible with the action of $G(K_S)$ on $\mathrm{Bun}_{G,\underline{S}}$. Indeed, once $e \in \mathcal{E}_S$ is chosen $\ell_{\mathcal{E}}^S$ can be identified with $\ell_{\mathcal{E},e}^S = (\lambda_e)_* \tilde{\phi}_S$ where $\tilde{\phi}_S$ is the pullback of $\phi_S$ to $\mathrm{Bun}_{G,\underline{S}}$ and $\lambda_e$ was defined in 4.1.6. $G(K_S)$ acts on $\tilde{\phi}_S$ because $\phi_{S,\mathcal{F}}$ depends only on the restriction of $\mathcal{F}$ to $X \setminus S$. So $G(K_S)$ acts on $\ell_{\mathcal{E},e}^S$.

The isomorphism $\ell_{\mathcal{E},e}^S \xrightarrow{\sim} \ell_{\mathcal{E},\tilde{e}}^S$ induced by the isomorphism $\ell_{\mathcal{E},e} \xrightarrow{\sim} \ell_{\mathcal{E},\tilde{e}}$ from 4.1.6 is *not* $G(K_S)$-equivariant. Indeed, if $\tilde{e} = \chi e$, $\chi \in (\Pi^\vee)^S$, then according to 4.1.6 $\ell_{\mathcal{E},\tilde{e}}^S / \ell_{\mathcal{E},e}^S$ is the pushforward of the $\Pi^S$-torsor $(\mathrm{Res})_* \tilde{\phi}_S$ via $\chi : \Pi^S \to \mu_\infty$. The identification $(\mathrm{Res})_* \tilde{\phi}_S = \mathrm{Bun}_{G,\underline{S}} \times \Pi^S$ from 4.1.6 becomes $G(K_S)$-equivariant if $G(K_S)$ acts on $\Pi^S$ via the boundary morphism $\varphi : G(K_S) \to H^1(\operatorname{Spec} K_S, \Pi(1)) = \Pi^S$ (we should check the sign!!!). Therefore the trivial $\mu_\infty$-torsor $\ell_{\mathcal{E},\tilde{e}}^S / \ell_{\mathcal{E},e}^S$ is equipped with a nontrivial action of $G(K_S)$: it acts by $\chi\varphi : G(K_S) \to \mu_\infty$.

So to each $e \in \mathcal{E}_S$ there corresponds an action of $G(K_S)$ on $\tilde{\phi}_S$, and if $e$ is replaced by $\chi e$, $\chi \in (\Pi^\vee)^S = \mathrm{Hom}(\Pi^S, \mu_\infty)$, then the action is multiplied by $\chi\varphi : G(K_S) \to \mu_\infty$.

*Remark.* By the way, we have proved that the coboundary map $\varphi : G(K_S) \to H^1(\operatorname{Spec} K_S, \Pi(1)) = \Pi^S$ is locally constant[27] (indeed, $G(K_S)$ acts on $(\mathrm{Res})_* \widetilde{\phi}_S$ as a group ind-scheme, so $\varphi$ is a morphism of ind-schemes, i.e., $\varphi$ is locally constant. The proof can be reformulated as follows. Without loss of generality we may assume that $S$ consists of a single point $x$. The group ind-scheme $G(K_x)$ acts on $\mathrm{Bun}_{G,\underline{x}}$ (see 2.3.3 –

[27] See also 4.5.4.

2.3.4), so it acts on $\pi_0(\mathrm{Bun}_{G,\underline{x}}) = \pi_0(\mathrm{Bun}_G)$. One has the "first Chern class" map $c: \pi_0(\mathrm{Bun}_G) \to \Pi$. It is easy to show that $c(gu) = \varphi(g)c(u)$ for $u \in \pi_0(\mathrm{Bun}_G)$, $g \in G(K_x)$ where $\varphi : G(K_x) \to H^1(K_x, \Pi(1)) = \Pi$ is the coboundary map. So $\varphi$ is locally constant.

**4.1.8.** Denote by $\widetilde{G(K_S)}_{\mathcal{E}}$ the group generated by $\mu_\infty$ and elements $\langle g, e\rangle$, $g \in G(K_S)$, $e \in \mathcal{E}_S$, with the defining relations

$$\begin{aligned} \langle g_1 g_2, e\rangle &= \langle g_1, e\rangle\langle g_2, e\rangle \\ \langle g_1, \chi e\rangle &= \chi(\varphi(g)) \cdot \langle g, e\rangle\,, \quad \chi \in (\Pi^\vee)^S = \mathrm{Hom}(\Pi^S, \mu_\infty) \\ \alpha\langle g, e\rangle &= \langle g, e\rangle\alpha\,, \quad \alpha \in \mu_\infty \end{aligned}$$

$\widetilde{G(K_S)}_{\mathcal{E}}$ is a central extension of $G(K_S)$ by $\mu_\infty$. The extension is trivial: a choice of $e \in \mathcal{E}_S$ defines a splitting

$$\sigma_e : G(K_S) \to \widetilde{G(K_S)}_{\mathcal{E}}\,, \quad g \mapsto \langle g, e\rangle\,. \tag{154}$$

It follows from 4.1.7 that $\widetilde{G(K_S)}_{\mathcal{E}}$ acts on $\ell_{\mathcal{E}}^S$ so that $\mu_\infty \subset \widetilde{G(K_S)}_{\mathcal{E}}$ acts in the obvious way and the action of $G(K_S)$ on $\ell_{\mathcal{E}}^S$ corresponding to $e \in \mathcal{E}_S$ (see 4.1.7) comes from the splitting (154).

**4.1.9.** Consider the point of $\mathrm{Bun}_{G,\underline{S}}$ corresponding to the trivial $G$-bundle on $X$ with the obvious trivialization over $\underline{S}$. Acting by $G(K_S)$ on this point one obtains a morphism $f : G(K_S) \to \mathrm{Bun}_{G,\underline{S}}$. Suppose that $G$ is semisimple. Then $f$ induces an isomorphism.

$$G(K_S)/G(A_S) \xrightarrow{\sim} \mathrm{Bun}_{G,\underline{S}} \tag{155}$$

where $A_S := H^0(X \setminus S, \mathcal{O}_X)$ (see Theorem 1.3 from [La-So] and its proof in §3 of loc.cit). It is essential that $G(K_S)$ and $G(A_S)$ are considered as group ind-schemes and $G(K_S)/G(A_S)$ as an fppf quotient, so (155) is more than a bijection between the sets of $\mathbb{C}$-points. We also have an isomorphism

$$G(O_S) \setminus G(K_S)/G(A_S) \xrightarrow{\sim} \mathrm{Bun}_G\,. \tag{156}$$

It is easy to see that the $\mu_\infty$-torsors $\ell_{\mathcal{E}}$ and $\ell_{\mathcal{E}}^S$ defined in 4.1.4 and 4.1.7 can be described as

$$\ell_{\mathcal{E}}^S = \widetilde{G(K_S)}_{\mathcal{E}}/G(A_S) \tag{157}$$

$$\ell_{\mathcal{E}} = G(O_S)\setminus\widetilde{G(K_S)}_{\mathcal{E}}/G(A_S) \tag{158}$$

where $\widetilde{G(K_S)}_{\mathcal{E}}$ is the central extension from 4.1.8. Here the embeddings $i: G(O)\to\widetilde{G(K_S)}_{\mathcal{E}}$ and $j: G(A_S)\to\widetilde{G(K_S)}_{\mathcal{E}}$ are defined by

$$i(g) = \langle g,e\rangle\,,\quad e\in\mathcal{E}_S \tag{159}$$

$$j(g) = \langle g,e\rangle\cdot(\psi(g),cl(\mathcal{E},e))_{\mathcal{P}}^{-1}\,,\quad e\in\mathcal{E}_S \tag{160}$$

(we should check the sign!!!) where $\psi$ is the boundary morphism $G(A_S)\to H^1(X\setminus S,\Pi(1))$ and $cl(\mathcal{E},e)\in H^1_c(X\setminus S,\Pi^\vee)$ is the class of $(\mathcal{E},e)$ (the r.h.s. of (159) and (160) do not depend on $e$).

*Remark.* The morphisms $\varphi: G(K_S)\to\Pi^S$ and $\psi: G(A_S)\to H^1(X\setminus S,\Pi(1))$ induce a morphism

$$\mathrm{Bun}_G = G(O_S)\setminus G(K_S)/G(A_S)\to\Pi^S/H^1(X\setminus S,\Pi(1)) \tag{161}$$

where the r.h.s. of (161) is understood as a quotient *stack*. Clearly $\ell_{\mathcal{E}}$ is the pullback of a certain $\mu_\infty$-torsor on the stack $\Pi^S/H^1(X\setminus S,\Pi(1))$.

**4.1.10.** The reader can skip the remaining part of 4.1.

Let $C$ be a groupoid. Denote by $\underline{C}$ the corresponding constant sheaf of groupoids on the category of $\mathbb{C}$-schemes equipped with the fppf topology. If the automorphism groups of objects of $C$ are finite then $\underline{C}$ is an algebraic stack. By abuse of notation we will often write $C$ instead of $\underline{C}$ (e.g., if $C$ is a set then $\underline{C}=C\times\operatorname{Spec}\mathbb{C}$ is usually identified with $C$).

**Examples.** 1) If $C$ has a single object and $G$ is its automorphism group then $\underline{C}$ is the classifying stack of $G$.

2) If $C = P(K^\cdot)$ (see 4.1.5) then $\underline{C}$ is the quotient stack of $K^0$ with respect to the action of $K^{-1}$. So according to 4.1.5 the r.h.s. of (161) is the stack corresponding to the groupoid $\Pi(1)\,\text{gerbes}(X)$.
3) If $C = A\,\text{gerbes}(X)$ then $\underline{C}$ is the sheaf of groupoids associated to the presheaf $S \mapsto A\,\text{gerbes}(X \times S)$.

Consider the groupoid $\Pi(1)\,\text{gerbes}(X)$ as an algebraic stack. In 4.1.2 we defined a canonical morphism

$$\tilde{c} : \operatorname{Bun}_G \to \Pi(1)\,\text{gerbes}(X) \tag{162}$$

that associates to a $G$-bundle $\mathcal{F}$ the $\Pi(1)$-gerbe of $\tilde{G}$-liftings of $\mathcal{F}$ (by the way, the morphism (161) defined for semisimple $G$ coincides with $\tilde{c}$). $\tilde{c}$ is a refinement of the Chern class map $c : \operatorname{Bun}_G \to H^2(X, \Pi(1)) = \Pi$; more precisely, $c$ is the composition of $\tilde{c}$ and the canonical morphism $\Pi(1)\,\text{gerbes}(X) \to H^2(X, \Pi(1)) =$ the set of isomorphism classes of $\Pi(1)\,\text{gerbes}(X)$.

The $\mu_\infty$-torsors on $\operatorname{Bun}_G$ constructed in 4.1.4 come from $\mu_\infty$-torsors on $\Pi(1)\,\text{gerbes}(X)$. The following proposition shows that if $\tilde{G}$ is the universal covering of $G$ then *any* local system on $\operatorname{Bun}_G$ comes from a unique local system on $\Pi(1)\,\text{gerbes}(X)$.

**4.1.11.** *Proposition.* Suppose that $\tilde{G}$ is the universal covering of $G$ (so $\Pi = \pi_1(G)$). Then the morphism (162) induces an equivalence between the fundamental groupoid of $\operatorname{Bun}_G$ and $\Pi(1)\,\text{gerbes}(X)$.

Let us sketch a transcendental proof (since it is transcendental we will not distinguish between $\Pi$ and $\Pi(1)$). Denote by $X^{\text{top}}$ the $C^\infty$ manifold corresponding to $X$; for a $G$-bundle $\mathcal{F}$ on $X$ denote by $\mathcal{F}^{\text{top}}$ the corresponding $G$-bundle on $X^{\text{top}}$. Consider the groupoid $\operatorname{Bun}_G^{\text{top}}$ whose objects are $G$-bundles on $X^{\text{top}}$ and morphisms are isotopy classes of $C^\infty$ isomorphisms between $G$-bundles. It is easy to show that the natural functor $\operatorname{Bun}_G^{\text{top}} \to \Pi\,\text{gerbes}(X^{\text{top}}) = \Pi\,\text{gerbes}(X)$ is an equivalence. So we must prove that for a $G$-bundle $\xi$ on $X^{\text{top}}$ the stack of $G$-bundles $\mathcal{F}$ on $X$ equipped with an isotopy class of isomorphisms $\mathcal{F}^{\mathrm{top}} \xrightarrow{\sim} \xi$ is non-empty, connected, and simply connected. This is clear if a $G$-bundle on $X$ is interpreted as a $G$-bundle on $X^{\mathrm{top}}$ equipped with a $\bar{\partial}$-connection.

*Remark.* In 4.1.2 we defined the $H^1(X \setminus S, \Pi(1))$-torsor $\phi_S \to \mathrm{Bun}_G$. If $S = \{x\}$ for some $x \in X$ then $H^1(X \setminus S, \Pi(1)) = H^1(X, \Pi(1))$, so $\phi_{\{x\}} \to \mathrm{Bun}_G$ is a $H^1(X, \Pi(1))$-torsor. Proposition 4.1.11 can be reformulated as follows: if $\tilde{G}$ is the universal covering of $G$ then the Chern class map $\pi_0(\mathrm{Bun}_G) \to \Pi$ is bijective and the restriction of $\phi_{\{x\}} \to \mathrm{Bun}_G$ to each connected component of $\mathrm{Bun}_G$ is a universal covering. This is really a reformulation because a choice of $x$ defines an equivalence.

$$\Pi(1)\,\mathrm{gerbes}(X) \xrightarrow{\sim} \Pi \times H^1(X, \Pi(1))\,\mathrm{tors} \tag{163}$$

(to a $\Pi(1)$-gerbe on $X$ one associates its class in $H^2(X, \Pi(1)) = \Pi$ and the $H^1(X, \Pi(1))$-torsor of isomorphism classes of its objects over $X \setminus \{x\}$).

**4.2. Pfaffians I.** In this subsection we assume that for $(\mathbb{Z}/2\mathbb{Z})$-graded vector spaces $A$ and $B$ the identification of $A \otimes B$ with $B \otimes A$ is defined by $a \otimes b \mapsto (-1)^{p(a)p(b)} b \otimes a$ where $p(a)$ is the parity of $a$. Following [Kn-Mu] for a vector space $V$ of dimension $n < \infty$ we consider $\det V$ as a $(\mathbb{Z}/2\mathbb{Z})$-graded space of degree $n$ mod 2.

**4.2.1.** Let $X$ be a smooth complete curve over $\mathbb{C}$. An *$\omega$-orthogonal bundle* on $X$ is a vector bundle $\mathcal{Q}$ equipped with a non-degenerate symmetric pairing $\mathcal{Q} \otimes \mathcal{Q} \to \omega_X$. Denote by $\omega$-Ort the stack of $\omega$-orthogonal bundles on $X$. There is a well known line bundle $\det R\Gamma$ on $\omega$-Ort (its fiber over $\mathcal{Q}$ is $\det R\Gamma(X, \mathcal{Q})$). Laszlo and Sorger [La-So] construct a $(\mathbb{Z}/2\mathbb{Z})$-graded line bundle on $\omega$-Ort (which they call the *Pfaffian*) and show that the tensor square of the Pfaffian is $\det R\Gamma$. For our purposes it is more convenient to use another definition of Pfaffian. Certainly it should be equivalent to the one from [La-So], but we did not check this.

We will construct a line bundle Pf on $\omega$-Ort which we call the Pfaffian; its fiber over an $\omega$-orthogonal bundle $\mathcal{Q}$ is denoted by $\mathrm{Pf}(\mathcal{Q})$. The action of $-1 \in \operatorname{Aut}\mathcal{Q}$ on $\mathrm{Pf}(\mathcal{Q})$ defines a $(\mathbb{Z}/2\mathbb{Z})$-grading on Pf. Since Pf is a line bundle, "grading" just means that there is a locally constant $p : (\omega\text{-Ort}) \to \mathbb{Z}/2\mathbb{Z}$ such that $\mathrm{Pf}(\mathcal{Q})$ has degree $p(\mathcal{Q})$. Actually $p(\mathcal{Q}) = \dim H^0(\mathcal{Q}) \mod 2$ (the fact that $\dim H^0(\mathcal{Q}) \mod 2$ is locally constant was proved by M. Atiyah and D. Mumford [At, Mu]).

For an $\omega$-orthogonal bundle $\mathcal{Q}$ denote by $\mathcal{Q}^-$ the same bundle $\mathcal{Q}$ equipped with the *opposite* pairing $\mathcal{Q} \otimes \mathcal{Q} \to \omega_X$. Set $\mathrm{Pf}^-(\mathcal{Q}) := \mathrm{Pf}(\mathcal{Q}^-)$. We will define a canonical isomorphism $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det R\Gamma$. Define isomorphisms $f_{\pm i} : \mathrm{Pf}(\mathcal{Q}) \xrightarrow{\sim} \mathrm{Pf}(\mathcal{Q}^-)$ by $f_{\pm i} := (\varphi_{\pm i})_*$ where $i = \sqrt{-1}$ and $\varphi_i : \mathcal{Q} \xrightarrow{\sim} \mathcal{Q}^-$ is multiplication by $i$. Identifying Pf and $\mathrm{Pf}^-$ by means of $f_{\pm i}$ we obtain isomorphisms $c_{\pm i} : \mathrm{Pf}^{\otimes 2} \xrightarrow{\sim} \det R\Gamma$ such that $(c_i)^{-1} c_{-i} : \mathrm{Pf}(\mathcal{Q})^{\otimes 2} \xrightarrow{\sim} \mathrm{Pf}(\mathcal{Q})^{\otimes 2}$ is multiplication by $(-1)^{p(\mathcal{Q})}$.

*Remarks*

(i) If $\mathcal{Q}$ is an $\omega$-orthogonal bundle then by Serre's duality $H^1(X, \mathcal{Q}) = (H^0(X, \mathcal{Q}))^*$, so $\det R\Gamma(X, \mathcal{Q}) = \det H^0(X, \mathcal{Q})^{\otimes 2}$. The naive definition would be $\mathrm{Pf}^?(\mathcal{Q}) := \det H^0(X, \mathcal{Q})$, but this does not make sense for families of $\mathcal{Q}$'s because $\dim H^0(X, \mathcal{Q})$ can jump.

(ii) Let $\mathcal{Q}$ be the orthogonal direct sum of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then $\det R\Gamma(X, \mathcal{Q}) = \det R\Gamma(X, \mathcal{Q}_1) \otimes \det R\Gamma(X, \mathcal{Q}_2)$. From the definitions of Pf and $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det R\Gamma$ it will be clear that there is a canonical ismorphism $\mathrm{Pf}(\mathcal{Q}) \xrightarrow{\sim} \mathrm{Pf}(\mathcal{Q}_1) \otimes \mathrm{Pf}(\mathcal{Q}_2)$ and the diagram

$$\begin{tikzcd}
\mathrm{Pf}(\mathcal{Q}) \otimes \mathrm{Pf}(\mathcal{Q}^-) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{Pf}(\mathcal{Q}_1) \otimes \mathrm{Pf}(\mathcal{Q}_1^-) \otimes \mathrm{Pf}(\mathcal{Q}_2) \otimes \mathrm{Pf}(\mathcal{Q}_2^-) \arrow[d, "\wr"] \\
\det R\Gamma(X, \mathcal{Q}) \arrow[r, "\sim"] & \det R\Gamma(X, \mathcal{Q}_1) \otimes \det R\Gamma(X, \mathcal{Q}_2)
\end{tikzcd}$$

is commutative. Therefore the isomorphisms $c_{\pm i} : \mathrm{Pf}(\mathcal{Q})^{\otimes 2} \xrightarrow{\sim} \det R\Gamma(X, \mathcal{Q})$ are compatible with decompositions $\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$.

(iii) One can define $c_\pm : \mathrm{Pf}(\mathcal{Q})^{\otimes 2} \xrightarrow{\sim} \det R\Gamma(X,\mathcal{Q})$ by $c_\pm = i^{\pm p(\mathcal{Q})^2} c_i$ where $p(\mathcal{Q})^2$ is considered as an element of $\mathbb{Z}/4\mathbb{Z}$. Then $c_\pm$ does not change if $i$ is replaced by $-i$. However $c_\pm$ do not seem to be naturalobjects, e.g., they are not compatible with decompositions $\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$ (the "error" is $(-1)^{p(\mathcal{Q}_1)p(\mathcal{Q}_2)}$).

(iv) The construction of $\mathrm{Pf}(\mathcal{Q})$ works if $\mathbb{C}$ is replaced by any field $k$ such that $\operatorname{char} k \neq 2$. The case $\operatorname{char} k = 2$ is discussed in 4.2.16.

**4.2.2.** A *Lagrangian triple* consists of an even-dimensional vector space $V$ equipped with a non-degenerate bilinear symmetric form $(\ ,\ )$ and Lagrangian (= maximal isotropic) subspaces $L_+, L_- \subset V$. If $X$ and $\mathcal{Q}$ are as in 4.2.1 and $\mathcal{Q}' \subset \mathcal{Q}$ is a subsheaf such that $H^0(X,\mathcal{Q}') = 0$ and $S := \mathrm{Supp}(\mathcal{Q}/\mathcal{Q}')$ is finite then one associates to $(\mathcal{Q},\mathcal{Q}')$ a Lagrangian triple $(V; L_+, L_-)$ as follows (cf. [Mu]):

(1) $V := H^0(X, \mathcal{Q}''/\mathcal{Q}')$ where $\mathcal{Q}'' := \underline{\mathrm{Hom}}(\mathcal{Q}', \omega_X) \supset \mathcal{Q}$;

(2) $L_+ := H^0(X, \mathcal{Q}/\mathcal{Q}') \subset V$;

(3) $L_- := H^0(X, \mathcal{Q}'') \subset V$;

(4) the bilinear form on $V$ is induced by the natural pairing $\mathcal{Q}''/\mathcal{Q}' \otimes \mathcal{Q}''/\mathcal{Q}' \to (j_*\omega_{X\setminus S})/\omega_X$ and the "sum of residues" map $H^0(X, (j_*\omega_{X\setminus S})/\omega_X) \to \mathbb{C}$ where $j$ is the embedding $X\setminus S \to X$. In this situation one can identify $R\Gamma(X,\mathcal{Q})$ with the complex

$$0 \to L_- \to V/L_+ \to 0 \tag{164}$$

concentrated in degrees 0 and 1. In particular $H^0(X,\mathcal{Q}) = L_+ \cap L_-$, $H^1(X,\mathcal{Q}) = V/(L_+ + L_-)$ and Serre's pairing between $H^0(X,\mathcal{Q}) = L_+ \cap L_-$ and $H^1(X,\mathcal{Q}) = V/(L_+ + L_-)$ is induced by the bilinear form on $V$.

**4.2.3.** For a Lagrangian triple $(V; L_+, L_-)$ set

$$\det(V; L_+, L_-) := \det L_+ \otimes \det L_- \otimes (\det V)^* . \tag{165}$$

$\det(V; L_+, L_-)$ is nothing but the determinant of the complex (164). Formula (165) defines a line bundle det on the stack of Lagrangian triples. In 4.2.4 and 4.2.8 we will construct a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle Pf on this stack and a canonical isomorphism $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det$ where $\mathrm{Pf}^-(V; L_+, L_-) := \mathrm{Pf}(V^-; L_+, L_-)$ and $V^-$ denotes $V$ equipped with the form $-( \, , \, )$. The naive "definition" would be $\mathrm{Pf}^?(V; L_+, L_-) := \det(L_+ \cap L_-)$ or $\mathrm{Pf}^?(V; L_+, L_-)^* := \det((L_+ \cap L_-)^*) = \det(V/(L_+ + L_-))$ (cf. Remark (i) from 4.2.1).

**4.2.4.** For a Lagrangian triple $(V; L_+, L_-)$ define $\mathrm{Pf}(V; L_+, L_-)$ as follows. Denote by $\mathrm{Cl}(V)$ the Clifford algebra equipped with the canonical $(\mathbb{Z}/2\mathbb{Z})$-grading ($V \subset \mathrm{Cl}(V)$ is odd). Let $M$ be an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathrm{Cl}(V)$-module (actually $M$ is irreducible even without taking the grading into account). $M$ is defined uniquely up to tensoring by a 1-dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded vector space. Set $M_{L_-} = M/L_- M$, $M^{L_+} := \{m \in M | L_+ m = 0\}$. Then $M^{L_+}$ and $M_{L_-}$ are 1-dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded spaces. We set

$$\mathrm{Pf}(V; L_+, L_-) := M^{L_+} \otimes (M_{L_-})^* . \tag{166}$$

In particular we can take $M = \mathrm{Cl}(V)/\mathrm{Cl}(V)L_+$. Then $M^{L_+} = \mathbb{C}$, so

$$\mathrm{Pf}(V; L_+, L_-)^* = \mathrm{Cl}(V)/(L_- \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_+) . \tag{167}$$

Clearly (166) or (167) defines Pf as a $(\mathbb{Z}/2\mathbb{Z})$-graded line bundle on the stack of Lagrangian triples.[28] The grading corresponds to the action of $-1 \in \mathrm{Aut}(V; L_+, L_-)$ on $\mathrm{Pf}(V; L_+, L_-)$.

If $V$ is the orthogonal direct sum of $V_1$ and $V_2$ then $\mathrm{Cl}(V)$ is the tensor product of the superalgebras $\mathrm{Cl}(V_1)$ and $\mathrm{Cl}(V_2)$. Therefore if $(V^1; L^1_+, L^1_-)$ and $(V^2; L^2_+, L^2_-)$ are Lagrangian triples one has a canonical isomorphism

$$\mathrm{Pf}(V^1 \oplus V^2; L^1_+ \oplus L^2_+, L^1_- \oplus L^2_-) = \mathrm{Pf}(V^1; L^1_+, L^1_-) \otimes \mathrm{Pf}(V^2; L^2_+, L^2_-) . \tag{168}$$

where $\oplus$ denotes the orthogonal direct sum.

[28] In other words, passing from individual Lagrangian triples to families is obvious. This principle holds for all our discussion of Pfaffians (only in the infinite-dimensional setting of 4.2.14 we explicitly consider families because this really needs some care).

$\mathrm{Pf}(V;L_+,L_-)$ is even if and only if $\dim(L_+\cap L_-)$ is even. This follows from (168) and statement (i) of the following lemma.

**4.2.5.** *Lemma.*

(i) Any Lagrangian triple $(V;L_+,L_-)$ can be represented as an orthogonal direct sum of Lagrangian triples $(V^1;L^1_+,L^1_-)$ and $(V^2;L^2_+,L^2_-)$ such that $L^1_+\cap L^1_-=0$, $L^2_+=L^2_-$.

(ii) Moreover, if a subspace $\Lambda\subset L_+$ is fixed such that $L_+=\Lambda\oplus(L_+\cap L_-)$ then one can choose the above decomposition $(V;L_+,L_-)=(V^1;L^1_+,L^1_-)\oplus(V^2;L^2_+,L^2_-)$ so that $L^1_+=\Lambda$.

*Proof*

(i) Choose a subspace $P\subset V$ such that $V=(L_+ + L_-)\oplus P$. Then set $V^2:=(L_1\cap L_2)\oplus P$, $V^1:=(V^2)^\perp$.

(ii) Choose a subspace $P\subset\Lambda^\perp$ such that $\Lambda^\perp=L_+\oplus P$ (this implies that $V=(L_+ + L_-)\oplus P$ because $\Lambda^\perp/L_+\to V/(L_+ + L_-)$ is an isomorphism). Then proceed as above. □

**4.2.6.** In this subsection (which can be skipped by the reader) we construct a canonical isomorphism between $\mathrm{Pf}(V;L_+,L_-)$ and the naive $\mathrm{Pf}^?(V;L_+,L_-)$ from 4.2.3. Recall that $\mathrm{Pf}^?(V;L_+,L_-):=\det(L_+\cap L_-)$, so $\mathrm{Pf}^?(V;L_+,L_-)^*=\det((L_+\cap L_-)^*)=\det(V/(L_+ + L_-))$, it being understood that the pairing $\det W\otimes\det W^*\to\mathbb{C}$, $W:=L_+\cap L_-$, is defined by $(e_1\wedge\ldots\wedge e_k)\otimes(e^k\wedge\ldots\wedge e^1)\mapsto 1$ where $e_1,\ldots,e_k$ is a base of $W$ and $e^1,\ldots,e^k$ is the dual base of $W^*$ (this pairing is reasonable from the "super" point of view; e.g., it is compatible with decompositions $W=W_1\oplus W_2$).

To define the isomorphism $\mathrm{Pf}(V;L_+,L_-)\xrightarrow{\sim}\mathrm{Pf}^?(V;L_+,L_-)$ we use the canonical filtration on $\mathrm{Cl}(V)$ defined by

$$\mathrm{Cl}_0(V)=\mathbb{C},\quad \mathrm{Cl}_{k+1}(V)=\mathrm{Cl}_k(V)+V\cdot\mathrm{Cl}_k(V)\,. \tag{169}$$

We have $\mathrm{Cl}_k(V)/\mathrm{Cl}_{k-1}(V)=\bigwedge^k V$. Set $r:=\dim(L_+\cap L_-)$. One has the canonical epimorphism $\varphi:\mathrm{Cl}_r(V)\to\bigwedge^r V\to\bigwedge^r(V/(L_+ +

$L_-)) = \det(V/(L_+ + L_-)) = \mathrm{Pf}^?(V; L_+, L_-)^*$. It is easy to deduce from 4.2.5(i) that the canonical mapping $\mathrm{Cl}_r(V) \to \mathrm{Cl}(V)/(L_- \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_+) = \mathrm{Pf}(V; L_+, L_-)^*$ factors through $\varphi$ and the induced map $f : \mathrm{Pf}^?(V; L_+, L_-)^* \to \mathrm{Pf}(V; L_+, L_-)^*$ is an isomorphism. $f^*$ is the desired isomorphism $\mathrm{Pf}(V; L_+, L_-) \xrightarrow{\sim} \mathrm{Pf}^?(V; L_+, L_-)$.

Here is an equivalent definition. Let $M$ be an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathrm{Cl}(V)$-module. The canonical embedding $\det(L_+ \cap L_-) \subset \bigwedge^*(L_+ \cap L_-) = \mathrm{Cl}(L_+ \cap L_-) \subset \mathrm{Cl}(V)$ induces a map $\det(L_+ \cap L_-) \otimes M_{L_+ \cap L_-} \to M^{L_+ \cap L_-}$, which is actually an isomorphism. It is easy to deduce from 4.2.5(i) that the composition $M^{L_+} \to M^{L_+ \cap L_-} \xrightarrow{\sim} \det(L_+ \cap L_-) \otimes M_{L_+ \cap L_-} \to \det(L_+ \cap L_-) \otimes M_{L_-}$ is an isomorphism. It induces an isomorphism $\mathrm{Pf}(V; L_+, L_-) := M^{L_+} \otimes (M_{L_-})^{\otimes -1} \to \det(L_+ \cap L_-) = \mathrm{Pf}^?(V; L_+, L_-)$, which is actually inverse to the one constructed above.

**4.2.7.** Before constructing the isomorphism $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det$ we will construct a canonical isomorphism

$$\mathrm{Pf}(V \oplus V^*; L_+ \oplus L_+^\perp, L_- \oplus L_-^\perp) \xrightarrow{\sim} \det(V; L_+, L_-) \tag{170}$$

where $V$ is a finite dimensional vector space without any bilinear form on it, $L_\pm \subset V$ are arbitrary subspaces and $V \oplus V^*$ is equipped with the obvious bilinear form (the l.h.s. of (170) makes sense because $L_\pm \oplus L_\pm^\perp$ is Lagrangian, the r.h.s. of (170) is defined by (165)). Set

$$M = \bigwedge V \otimes (\det L_+)^*\,, \quad \bigwedge V := \bigoplus_i \bigwedge^i V\,. \tag{171}$$

$M$ is the irreducible $\mathrm{Cl}(V \oplus V^*)$-module with $M^{L_+ \oplus L_+^\perp} = \mathbb{C}$, so according to (166) $\mathrm{Pf}(V \oplus V^*; L_+ \oplus L_+^\perp, L_- \oplus L_-^\perp) = (M_{L_- \oplus L_-^\perp})^*$. Clearly $M_{L_-} = \bigwedge(V/L_-) \otimes (\det L_+)^*$ and $M_{L_- \oplus L_-^\perp} = \det(V/L_-) \otimes (\det L_+)^* = \det(V; L_+, L_-)^*$ (see (165)). So we have constructed the isomorphism (170).

**4.2.8.** Now let $(V; L_+, L_-)$ be a Lagrangian triple. We will construct a canonical isomorphism

$$\mathrm{Pf}(V; L_+, L_-) \otimes \mathrm{Pf}(V^-; L_+, L_-) \xrightarrow{\sim} \det(V; L_+, L_-) \tag{172}$$

where $V^-$ denotes $V$ equipped with the bilinear form $-( \, , \, )$. If $W$ is a finite dimensional vector space equipped with a nondegenerate symmetric bilinear form then $(V \otimes W; L_+ \otimes W, L_- \otimes W)$ is a Lagrangian triple. (170) can be rewritten as a canonical isomorphism.

$$\det(V; L_+, L_-) \xrightarrow{\sim} \mathrm{Pf}(V \otimes H; L_+ \otimes H, L_- \otimes H) \tag{173}$$

where $H$ denotes $\mathbb{C}^2$ equipped with the bilinear form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. On the other hand (168) yields an isomorphism

$$\mathrm{Pf}(V; L_+, L_-) \otimes \mathrm{Pf}(V^-; L_+, L_-) \xrightarrow{\sim} \mathrm{Pf}(V \otimes H'; L_+ \otimes H', L_- \otimes H') \tag{174}$$

where $H'$ denotes $\mathbb{C}^2$ equipped with the bilinear form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. So an isomorphism $\varphi : H' \xrightarrow{\sim} H$ induces an isomorphism

$$\varphi_* : \mathrm{Pf}(V; L_+, L_-) \otimes \mathrm{Pf}(V^-; L_+, L_-) \xrightarrow{\sim} \det(V; L_+, L_-) \, .$$

*Lemma.* If $\psi \in \mathrm{Aut}\, H'$ then

$$(\varphi\psi)_* = (\det \psi)^n \varphi_* \, , \quad n = \dim(L_+ \cap L_-) \, . \tag{175}$$

*Proof.* $\mathrm{Aut}\, H'$ acts on the r.h.s. of (174) by some character $\chi : \mathrm{Aut}\, H' \to \mathbb{C}^*$. Any character of $\mathrm{Aut}\, H'$ is of the form $\psi \mapsto (\det \psi)^m$, $m \in \mathbb{Z}/2\mathbb{Z}$. $\chi\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = (-1)^n$, $n := \dim(L_+ \cap L_-)$, because $-1 \in \mathrm{Aut}(V; L_+, L_-)$ acts on $\mathrm{Pf}(V; L_+, L_-)$ as $(-1)^n$ (see 4.2.4). So $m = n \mod 2$. $\square$

We define (172) to be $\varphi_*$ for any $\varphi : H' \xrightarrow{\sim} H$ such that $\det \varphi = 1$.

*Remarks*

(i) (172) is compatible with decompositions of $(V; L_+, L_-)$ into orthogonal direct sums; i.e., if one has such a decomposition $(V; L_+, L_-) = (V^1; L^1_+, L^1_-) \oplus (V^2; L^2_+, L^2_-)$ then the isomorphisms

(172) for $(V; L_+, L_-)$, $(V^1; L^1_+, L^1_-)$, and $(V^2; L^2_+, L^2_-)$ are compatible with (168) and the canonical isomorphism $\det(V; L_+, L_-) = \det(V^1; L^1_+, L^1_-) \otimes \det(V^2; L^2_+, L^2_-)$.

(ii) (170) is compatible with decompositions of $(V; L_+, L_-)$ into direct sums.

**4.2.9.** In this subsection (which can be skipped by the reader) we give an equivalent construction of (172). We will use the superalgebra anti-isomorphism $* : \mathrm{Cl}(V^-) \xrightarrow{\sim} \mathrm{Cl}(V)$ identical on $V$ (for any $v_1, \dots, v_k \in V$ one has $(v_1 \dots v_k)^* = (-1)^{k(k-1)/2} v_k \dots v_1$). We also use the canonical map $\mathrm{sTr} : \mathrm{Cl}(V) = \mathrm{Cl}_n(V) \to \mathrm{Cl}_n(V)/\mathrm{Cl}_{n-1}(V) = \det V$ where $n = \dim V$ and $\mathrm{Cl}_k(V)$ is defined by (169). It has the "supertrace property"

$$\mathrm{sTr}(ab) = (-1)^{p(a)p(b)}\, \mathrm{sTr}(ba) \tag{176}$$

where $a, b \in \mathrm{Cl}(V)$ are homogeneous of degrees $p(a), p(b) \in \mathbb{Z}/2\mathbb{Z}$. Indeed, it is enough to prove (176) in the case $a \in V$, $p(ab) = n \mod 2$; then $b \in \mathrm{Cl}_{n-1}(V)$ and (176) is obvious. Or one can check that $\mathrm{sTr}(a)$ coincides up to a sign with the supertrace of the operator $a : M \to M$ where $M$ is an irreducible $\mathrm{Cl}(V)$-module.

Now consider the map

$$\det L_- \otimes \mathrm{Pf}(V; L_+, L_-)^* \otimes \det L_+ \otimes \mathrm{Pf}(V^-; L_+, L_-)^* \to \det V \tag{177}$$

defined by $a_- \otimes x \otimes a_+ \otimes y \mapsto \mathrm{sTr}(a_- x a_+ y^*)$. Here $a_\pm \in \det L_\pm \subset \Lambda^*(L_\pm) = \mathrm{Cl}(L_\pm) \subset \mathrm{Cl}(V)$, $x \in \mathrm{Pf}(V; L_+, L_-)^* = \mathrm{Cl}(V)/(L_- \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_+)$, $y^* \in \mathrm{Cl}(V)/(L_+ \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_-)$, so (177) is well-defined. It is easy to see (e.g., from 4.2.5 (i)) that (177) is an isomorphism. It induces an isomorphism

$$\mathrm{Pf}(V; L_+, L_-) \otimes \mathrm{Pf}(V^-; L_+, L_-) \xrightarrow{\sim} \det L_+ \otimes \det L_- \otimes (\det V)^* = \det(V; L_+, L_-)$$

One can show that this isomorphism equals (172).

**4.2.10.** Let $X$ and $\mathcal{Q}$ be as in 4.2.1 and $\mathcal{Q}' \subset \mathcal{Q}$ as in 4.2.2. To these data we have associated a Lagrangian triple $(V; L_+, L_-)$ such that $\det(V; L_+, L_-) = \det R\Gamma(X, \mathcal{Q})$ (see 4.2.2). Set $\mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}) := \mathrm{Pf}(V; L_+, L_-)$. According to 4.2.9 we have a canonical isomorphism $\mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}) \otimes \mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}^-) \xrightarrow{\sim} \det R\Gamma(X, \mathcal{Q})$. To define $\mathrm{Pf}(\mathcal{Q})$ it is enough to define a compatible system of isomorphisms $\mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}) \xrightarrow{\sim} \mathrm{Pf}_{\widetilde{\mathcal{Q}}'}(\mathcal{Q})$ for all pairs $(\mathcal{Q}', \widetilde{\mathcal{Q}}')$ such that $\mathcal{Q}' \subset \widetilde{\mathcal{Q}}'$. To define $\mathrm{Pf}(\mathcal{Q}) \otimes \mathrm{Pf}(\mathcal{Q}^-) \xrightarrow{\sim} \det R\Gamma(X, \mathcal{Q})$ it suffices to prove the commutativity of

$$\begin{array}{ccc} \mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}) \underset{\xrightarrow{\sim}}{\otimes} \mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}^-) & \xrightarrow{\sim} & \det R\Gamma(X, \mathcal{Q}) \\ & & \\ \mathrm{Pf}_{\widetilde{\mathcal{Q}}'}(\mathcal{Q}) \otimes \mathrm{Pf}_{\widetilde{\mathcal{Q}}'}(\mathcal{Q}^-) & \xrightarrow{\sim} & \end{array}$$

The Lagrangian triple $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ corresponding to $\widetilde{\mathcal{Q}}'$ is related to the triple $(V; L_+, L_-)$ corresponding to $\mathcal{Q}'$ as follows: if $\Lambda = H^0(X, \widetilde{\mathcal{Q}}'/\mathcal{Q}') \subset H^0(X, \mathcal{Q}/\mathcal{Q}') = L_+$ then

$$\widetilde{V} = \Lambda^\perp/\Lambda, \quad \widetilde{L}_+ = L_+/\Lambda \subset \widetilde{V}, \quad \widetilde{L}_- = L_- \cap \Lambda^\perp \hookrightarrow \widetilde{V} \tag{178}$$

(notice that $\Lambda \cap L_- = H^0(X, \widetilde{\mathcal{Q}}') = 0$). So it remains to do some linear algebra (see 4.2.11). It is easy to check that our definition of $\mathrm{Pf}(\mathcal{Q})$ and $\mathrm{Pf}(\mathcal{Q}) \otimes \mathrm{Pf}(\mathcal{Q}^-) \xrightarrow{\sim} \det R\Gamma(X, \mathcal{Q})$ makes sense for families of $\mathcal{Q}$'s.

**4.2.11.** Let $(V; L_+, L_-)$ be a Lagrangian triple, $\Lambda \subset L_+$ a subspace such that $\Lambda \cap L_- = 0$. Then $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ defined by (178) is a Lagrangian triple. In this situation we will say that $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is a *subquotient* of $(V; L_+, L_-)$. It is easy to show that a subquotient of a subquotient is again a subquotient. So we can consider the category $T$ with Lagrangian triples as objects such that a morphism from $(V; L_+, L_-)$ to $(V'; L'_+, L'_-)$ is defined to be an isomorphism between $(V; L_+, L_-)$ and a subquotient of $(V'; L'_+, L'_-)$. Consider also the category $C$ whose objects are finite complexes of finite dimensional vector spaces and morphisms are quasi-isomorphisms. Denote by $\mathbb{1}$ the category whose objects are $(\mathbb{Z}/2\mathbb{Z})$-graded 1-dimensional vector spaces and morphisms are isomorphisms preserving the grading. The complex (164) considered as an object of $C$ depends functorially on $(V; L_+, L_-) \in T$: if $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is the subquotient of $(V; L_+, L_-)$ corresponding to $\Lambda \subset L_+$ then we have the quasi-isomorphism

$$\begin{array}{ccc} L_- & \longrightarrow & V/L_+ \\ \hookrightarrow & & \hookrightarrow \\ \widetilde{L}_- & \longrightarrow & \widetilde{V}/\widetilde{L}_+ = \Lambda^\perp/L_+ \end{array}$$

Applying the functor $\det : C \to \mathbb{I}$ from [Kn-Mu] we see that $\det(V; L_+, L_-) \in \mathbb{I}$ depends functorially on $(V; L_+, L_-) \in T$. If $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is the subquotient of $(V; L_+, L_-)$ corresponding to $\Lambda \subset L_+$ then the isomorphism between $\det(V; L_+, L_-) = (\det L_+) \otimes (\det L_-) \otimes (\det V)^*$ and $\det(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-) = (\det \widetilde{L}_+) \otimes (\det \widetilde{L}_-) \otimes (\det \widetilde{V})^*$ comes from the natural isomorphisms $\det L_+ = \det \Lambda \otimes \det \widetilde{L}_+$, $\det L_- = \det \widetilde{L}_- \otimes \det(V/\Lambda^\perp)$, $\det V = \det \Lambda \otimes \det \widetilde{V} \otimes \det(V/\Lambda^\perp)$.

As explained in 4.2.10 we have to define Pf as a functor $T \to \mathbb{I}$ and to show that the isomorphism $\mathrm{Pf}(V; L_+, L_-) \otimes \mathrm{Pf}^-(V; L_+, L_-) \xrightarrow{\sim} \det(V; L_+, L_-)$ from 4.2.8 is functorial.

If $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is the subquotient of $(V; L_+, L_-)$ corresponding to $\Lambda \subset L_+$ then

$$\begin{aligned} \mathrm{Pf}(V; L_+, L_-)^* &= \mathrm{Cl}(V)/(L_- \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_+) \\ \mathrm{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)^* &= \mathrm{Cl}(\Lambda^\perp)/((L_- \cap \Lambda^\perp) \cdot \mathrm{Cl}(\Lambda^\perp) + \mathrm{Cl}(\Lambda^\perp) \cdot L_+)\,. \end{aligned}$$

So the embedding $\mathrm{Cl}(\Lambda^\perp) \to \mathrm{Cl}(V)$ induces a mapping

$$\mathrm{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)^* \to \mathrm{Pf}(V; L_+, L_-)^*\,. \tag{179}$$

This defines $\mathrm{Pf}^*$ as a functor $T \to \{(\mathbb{Z}/2\mathbb{Z})\text{-graded 1-dimensional spaces}\}$ (it is easy to see that composition corresponds to composition). It remains to show that

a) (179) is an isomorphism,

b) (179) is compatible with the pairings $\mathrm{Pf}(V;L_+,L_-)^*\otimes\mathrm{Pf}(V^-;L_+,L_-)^*\xrightarrow{\sim}\det(V;L_+,L_-)^*$ and $\mathrm{Pf}(\widetilde{V};\widetilde{L}_+,\widetilde{L}_-)^*\otimes\mathrm{Pf}(\widetilde{V}^-;L_+,L_-)^*\xrightarrow{\sim}\det(\widetilde{V};\widetilde{L}_+,\widetilde{L}_-)^*$ from 4.2.8.

b) can be checked directly and a) follows from b). One can also prove a) by reducing to the case where $(\widetilde{V};\widetilde{L}_+,\widetilde{L}_-)$ is a *maximal* subquotient, (i.e., $\Lambda\oplus(L_+\cap L_-)=L_+$) and then using 4.2.5 (ii).

**4.2.12.** Let $E$ be a vector bundle on $X$. Then $E\oplus(E^*\otimes\omega_X)$ has the obvious structure of $\omega$-orthogonal bundle. We will construct a canonical isomorphism

$$\mathrm{Pf}(E\oplus(E^*\otimes\omega_X))\xrightarrow{\sim}\det R\Gamma(X,E)\,. \tag{180}$$

Choose a subsheaf $E'\subset E$ and a locally free sheaf $E''\supset E$ so that $H^0(X,E')=0$, $H^1(X,E'')=0$, and $E''/E'$ has finite support. Set $V:=H^0(X,E''/E')$, $L_+:=H^0(X,E'/E)\subset V$, $L_-:=H^0(X,E'')\subset V$. Then $R\Gamma(X,E)$ can be identified with the complex $0\to L_-\to V/L_+\to 0$ and $\det R\Gamma(X,E)$ with $\det(V;L_+,L_-)$. On the other hand the Pfaffian of $\mathcal{Q}:=E\oplus(E^*\otimes\omega_X)$ can be computed using the subsheaf $\mathcal{Q}':=E'\oplus((E'')^*\otimes\omega_X)\subset\mathcal{Q}$. Then $\mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q})$ equals the l.h.s. of (170). So (170) yields the isomorphism (180). One checks that (180) does not depend on $E'$ and $E''$.

**4.2.13.** The notion of Lagrangian triple has a useful infinite dimensional generalization. First let us recall some basic definitions.

*Definition.* A *Tate space* is a complete topological vector space having a base of neighbourhoods of 0 consisting of commensurable vector subspaces (i.e., $\dim U_1/(U_1\cap U_2)<\infty$ for any $U_1$, $U_2$ from this base).

*Remark.* Tate spaces are implicit in his remarkable work [T]. In fact, the approach to residues on curves developed in [T] can be most naturally interpreted in terms of the canonical central extension of the endomorphism algebra of a Tate space, which is also implicit in [T]. A construction of the Tate extension can be found in 7.13.18.

Let $V$ be a Tate space. A vector subspace $P \subset V$ is *bounded* if for every open subspace $U \subset V$ there exists a finite set $\{v_1, \dots, v_n\} \subset V$ such that $P \subset U + \mathbb{C}v_1 + \dots \mathbb{C}v_n$. The *topological dual* of $V$ is the space $V^*$ of continuous linear functionals on $V$ equipped with the (linear) topology such that orthogonal complements of bounded subspaces of $V$ form a base of neighbourhoods of $0 \in V^*$. Clearly $V^*$ is a Tate space and the canonical morphism $V \to (V^*)^*$ is an isomorphism.

*Example* (coordinate Tate space). Let $I$ be a set. We say that $A, B \subset I$ are commensurable if $A \backslash (A \cap B)$ and $B \backslash (B \cap A)$ are finite. Commensurability is an equivalence relation. Suppose that an equivalence class $\mathcal{A}$ of subsets $A \subset I$ is fixed. Elements of $\mathcal{A}$ are called *semi-infinite* subsets. Denote by $\mathbb{C}((I, \mathcal{A}))$ the space of formal linear combinations $\sum\limits_i c_i e_i$ where $c_i \in \mathbb{C}$ vanish when $i \notin A$ for some semi-infinite $A$. This is a Tate vector space (the topology is defined by subspaces $\mathbb{C}[[A]] := \{\sum\limits_{i \in A} c_i e_i\}$ where $A$ is semi-infinite). The space dual to $\mathbb{C}((I, \mathcal{A}))$ is $\mathbb{C}((I, \mathcal{A}'))$ where $\mathcal{A}'$ consists of complements to subsets from $\mathcal{A}$. Any Tate vector space is isomorphic to $\mathbb{C}((I, \mathcal{A}))$ for appropriate $I$ and $\mathcal{A}$; such an isomorphism is given by the corresponding subset $\{e_i\} \subset V$ called *topological basis* of $V$.

A *c-lattice* in $V$ is an open bounded subspace. A *d-lattice*[*)] in $V$ is a discrete subspace $\Gamma \subset V$ such that $\Gamma + P = V$ for some c-lattice $P \subset V$. If $W \subset V$ is a d-lattice (resp. c-lattice) then there is a c-lattice (resp. d-lattice) $W' \subset V$ such that $V = W \oplus W'$. If $W \subset V$ is a d-lattice (resp. c-lattice) then $W^\perp \subset V^*$ is also a d-lattice (resp. c-lattice) and $(W^\perp)^\perp = W$.

A (continuous) bilinear form on a Tate space $V$ is said to be *nondegenerate* if it induces a topological isomorphism $V \to V^*$. Let $V$ be a Tate space equipped with a nondegenerate symmetric bilinear form. A subspace $L \subset V$ is *Lagrangian* if $L^\perp = L$.

---

[*)] c and d are the first letters of "compact" and "discrete".

*Definition.* A Tate Lagrangian triple consists of a Tate space $V$ equipped with a nondegenerate symmetric bilinear form, a Lagrangian c-lattice $L_+ \subset V$, and a Lagrangian d-lattice $L_- \subset V$.

*Example.* Let $\mathcal{Q}$ be an $\omega$-orthogonal bundle on $X$. If $x \in X$ let $\mathcal{Q} \otimes O_x$ (resp. $\mathcal{Q} \otimes K_x$) denote the space of global sections of the pullback of $\mathcal{Q}$ to $\operatorname{Spec} O_x$ (resp. $\operatorname{Spec} K_x$). $\mathcal{Q} \otimes K_x$ is a Tate space equipped with the nondegenerate symmetric bilinear form $\operatorname{Res}(\ ,\ )$. For every non-empty finite $S \subset X$ we have the Tate Lagrangian triple

$$V := \bigoplus_{x \in S} (\mathcal{Q} \otimes K_x)\,, \quad L_+ := \bigoplus_{x \in S} (\mathcal{Q} \otimes O_x)\,, \quad L_- := \Gamma(X \backslash S, \mathcal{Q})\,. \tag{181}$$

Let $(V; L_+, L_-)$ be a Tate Lagrangian triple. Then for any c-lattice $\Lambda \subset L_+$ such that $\Lambda \cap L_- = 0$ one has the finite-dimensional Lagrangian triple $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ defined by (178). As explained in 4.2.11 $\operatorname{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ and $\det(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ do not depend on $\Lambda$. Set $\operatorname{Pf}(V; L_+, L_-) := \operatorname{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$, $\det(V; L_+, L_-) := \det(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$. Equivalently one can define $\det(V; L_+, L_-)$ to be the determinant of the complex (164) and $\operatorname{Pf}(V; L_+, L_-)$ can be defined by (166) or (167) (the $\operatorname{Cl}(V)$-module $M$ from (166) should be assumed *discrete*, which means that $\{v \in V | vm = 0\}$ is open for every $m \in M$).

*Example.* If $(V; L_+, L_-)$ is defined by (181) then $\operatorname{Pf}(V; L_+, L_-) = \operatorname{Pf}(\mathcal{Q})$, $\det(V; L_+, L_-) = \det R\Gamma(X, \mathcal{Q})$.

The constructions from 4.2.7 and 4.2.8 make sense in the Tate situation with the following obvious changes: a) in 4.2.7 one should suppose that $L_+$ is a c-lattice and $L_-$ is a d-lattice, b) (171) should be replaced by the following formula:

$$M = \varinjlim_{U} \bigwedge (V/U) \otimes \det(L_+/U)^* \tag{182}$$

where $U$ belongs to the set of c-lattices in $L_+$. The r.h.s. of (182) is the *fermionic Fock space*, i.e., the direct sum of *semi-infinite powers* of $V$ (cf. Lecture 4 from [KR] and references therein).

*Remark.* The expression for $\mathrm{Pf}(\mathcal{Q})$ in terms of the triple (181) can be reformulated as follows. For $x \in X$ consider the abelian Lie superalgebras $\mathfrak{a}_{O_x} \subset \mathfrak{a}_{K_x}$ such that the odd component of $\mathfrak{a}_{O_x}$ (resp. $\mathfrak{a}_{K_x}$) is $\mathcal{Q} \otimes O_x$ (resp. $\mathcal{Q} \otimes K_x$) and the even components are 0. The bilinear symmetric form on $\mathcal{Q} \otimes K_x$ defines a central extension $0 \to \mathbb{C} \to \tilde{\mathfrak{a}}_{K_x} \to \mathfrak{a}_{K_x} \to 0$ with a canonical splitting over $\mathfrak{a}_{O_x}$. The Clifford algebra $\mathrm{Cl}(\mathcal{Q} \otimes K_x)$ is the twisted universal enveloping algebra $U'\mathfrak{a}_{K_x}$ and $M_x := \mathrm{Cl}(\mathcal{Q} \otimes K_x)/\,\mathrm{Cl}(\mathcal{Q} \otimes K_x) \cdot (\mathcal{Q} \otimes O_x)$ is the vacuum module over $U'\mathfrak{a}_{K_x}$. According to (167) $\mathrm{Pf}(\mathcal{Q})^*$ *is the space of coinvariants of the action of* $\Gamma(X \backslash S, \mathcal{Q})$ *on* $\underset{x \in S}{\otimes} M_x$.

**4.2.14.** In this subsection we discuss *families* of Tate Lagrangian triples. Let $R$ be a commutative ring. We define a *Tate $R$-module* to be a topological $R$-module isomorphic to $P \oplus Q^*$ where $P$ and $Q$ are (infinite) direct sums of finitely generated projective $R$-modules (a base of neighbourhoods of $0 \in P \oplus Q^*$ is formed by $M^\perp \subset Q^*$ for all possible finitely generated submodules $M \subset Q$). This bad*) definition is enough for our purposes. In fact, we mostly work with Tate $R$-modules isomorphic to $V_0 \widehat{\otimes} R$ where $V_0$ is a Tate space.

The discussion of Tate linear algebra from 4.2.13 remains valid for Tate $R$-modules if one defines the notions of c-lattice and d-lattice as follows.

*Definition.* A *c-lattice* in a Tate $R$-module $V$ is an open bounded submodule $P \subset V$ such that $V/P$ is projective. A *d-lattice* in $V$ is a submodule $\Gamma \subset V$ such that for some c-lattice $P \subset V$ one has $\Gamma \cap P = 0$ and $V/(\Gamma + P)$ is a projective module of finite type.*)

Now if $\frac{1}{2} \in R$ we can define the notion of Tate Lagrangian triple just as in 4.2.13 (of course, if $\frac{1}{2} \notin R$ one should work with quadratic forms instead of bilinear ones, which is easy). The Pfaffian of a Tate Lagrangian triple

*)A projective $R((t))$-module of finite rank is not necessarily a Tate module in the above sense. Our notion of Tate $R$-module is not local with respect to $\operatorname{Spec} R$. There are also other drawbacks.

*)Then this holds for all c-lattices $P' \subset P$.

$(V; L_+, L_-)$ over $R$ is defined as in 4.2.13 with the following minor change: to pass to the finite-dimensional Lagrangian triple $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ defined by (178) one has to assume that $\Lambda \subset L_+$ is a c-lattice such that $\Lambda \cap L_- = 0$ and $V/(\Lambda + L_-)$ is projective (these two properties are equivalent to the following one: $\Lambda^\perp + L_- = V$).

*Example.* Let $D \subset X \otimes R$ be a closed subscheme finite over $\operatorname{Spec} R$ that can be locally defined by one equation (i.e., $D$ is an effective relative Cartier divisor). Let $\mathcal{Q}$ be a vector bundle on $X \otimes R$. Suppose that the morphism $D \to \operatorname{Spec} R$ is surjective. Then

$$V := \varprojlim_m \varinjlim_n H^0(X \otimes R, \mathcal{Q}(nD)/\mathcal{Q}(-mD))$$

is a Tate $R$-module*) ,

$$L_+ := \varprojlim_m H^0(X \otimes R, \mathcal{Q}/\mathcal{Q}(-mD)) \subset V$$

is a c-lattice, and

$$L_- := H^0((X \otimes R) \setminus D, \mathcal{Q}) \subset V$$

is a d-lattice. If $\mathcal{Q}$ is an $\omega$-orthogonal bundle then $(V; L_+, L_-)$ is a Lagrangian triple and $\operatorname{Pf}(\mathcal{Q}) = \operatorname{Pf}(V; L_+, L_-)$ (cf. 4.2.13).

**4.2.15.** Denote by $\mathcal{B}$ the groupoid of finite dimensional vector spaces over $\mathbb{C}$ equipped with a nondegenerate symmetric bilinear form. In this subsection (which can be skipped by the reader) we construct canonical isomorphisms

(183)
$$\operatorname{Pf}(V \otimes W; L_+ \otimes W, L_- \otimes W) \xrightarrow{\sim} \operatorname{Pf}(V; L_+, L_-)^{\otimes \dim W} \otimes |\det W|^{\otimes p(V; L_+, L_-)},$$

(184)
$$\operatorname{Pf}(\mathcal{Q} \otimes W) \xrightarrow{\sim} \operatorname{Pf}(\mathcal{Q})^{\otimes \dim W} \otimes |\det W|^{\otimes p(\mathcal{Q})}$$

*)In fact, $V$ is isomorphic to $V_0 \widehat{\otimes} R$ for some Tate space $V_0$ over $\mathbb{C}$. Indeed, we can assume that $R$ is finitely generated over $\mathbb{C}$ and then apply 7.12.11. We need 7.12.11 in the case where $R$ is finitely generated over $\mathbb{C}$ and the projective module from 7.12.11 is a direct sum of finitely generated modules; in this case 7.12.11 follows from Serre's theorem (Theorem 1 of [Se]; see also [Ba68], ch.4, §2) and Eilenberg's lemma [Ba63].

where $W \in \mathcal{B}$, $(V; L_+, L_-)$ is a (Tate) Lagrangian triple, $\mathcal{Q}$ is an $\omega$-orthogonal bundle on $X$, $|\det W|$ is the determinant of $W$ considered as a space (not super-space!), and $p(V; L_+, L_-)$, $p(\mathcal{Q}) \in \mathbb{Z}/2\mathbb{Z}$ are the parities of $\mathrm{Pf}(V; L_+, L_-)$, $\mathrm{Pf}(\mathcal{Q})$. $|\det W|^{\otimes n}$ makes sense for $n \in \mathbb{Z}/2\mathbb{Z}$ because one has the canonical isomorphism $|\det W|^{\otimes 2} \xrightarrow{\sim} \mathbb{C}$, $(w_1 \wedge \ldots \wedge w_r)^{\otimes 2} \mapsto \det(w_i, w_j)$.

To define (183) and(184) notice that $\mathcal{B}$ is a tensor category with $\oplus$ as a tensor "product" and both sides of (183) and (184) are tensor functors from $\mathcal{B}$ to the category of 1-dimensional superspaces (to define the r.h.s. of (184) as a tensor functor rewrite it as $|\mathrm{Pf}(\mathcal{Q})|^{\otimes \dim W} \otimes (\det W)^{\otimes p(\mathcal{Q})}$ where $|\mathrm{Pf}(\mathcal{Q})|$ is obtained from $\mathrm{Pf}(\mathcal{Q})$ by changing the $(\mathbb{Z}/2\mathbb{Z})$-grading to make it even and $\det W$ is the determinant of $W$ considered as a *super*space).

We claim that *there is a unique way to define* (183) *and* (184) *as isomorphisms of tensor functors so that for* $W = (\mathbb{C}, 1)$ (183) *and* (184) *equal* id. Here 1 denotes the bilinear form $(x, y) \mapsto xy$, $x, y \in \mathbb{C}$.

To prove this apply the following lemma to the tensor functor $F$ obtained by dividing the l.h.s. of (183) or (184) by the r.h.s.

*Lemma.* Every tensor functor $F : \mathcal{B} \to \{$1-dimensional vector spaces$\}$ is isomorphic to the tensor functor $F_1$ defined by $F_1(W) = L^{\otimes \dim W}$, $L := F(\mathbb{C}, 1)$. There is a unique isomorphism $F \xrightarrow{\sim} F_1$ that induces the identity map $F(\mathbb{C}, 1) \to F_1(\mathbb{C}, 1)$.

*Proof.* For every $W \in \mathcal{B}$ the functor $F$ induces a homomorphism $f_W$ : $\mathrm{Aut}\, W \to \mathbb{C}^*$. Since $\mathrm{Aut}\, W$ is an orthogonal group $f_W(g) = (\det g)^{n(W)}$ for some $n(W) \in \mathbb{Z}/2\mathbb{Z}$. Clearly $n(W) = n$ does not depend on $W$. Set $W_1 := (\mathbb{C}, 1)$. $F$ maps the commutativity isomorphism $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) : W_1 \oplus W_1 \to W_1 \oplus W_1$ to id. So $n = 0$, i.e., $f_W$ is trivial for every $W$. The rest is clear because the semigroup $|\mathcal{B}|$ of isomorphism classes of objects of $\mathcal{B}$ is $\mathbb{Z}_+$. $\square$

*Remarks*

(i) (183) was implicitly used in 4.2.8.

(ii) We will use (183) in 4.2.16.

**4.2.16.** In this subsection (which can certainly be skipped by the reader) we explain what happens if $\mathbb{C}$ is replaced by a field $k$ of characteristic 2. In this case one must distinguish between quadratic forms (see [Bourb59], §3, n°4) and symmetric bilinear forms. In the definition of Lagrangian triple $V$ should be equipped with a nondegenerate *quadratic* form. So in the definition of $\omega$-orthogonal bundle $\mathcal{Q}$ should be equipped with a nondegenerate *quadratic* form $\mathcal{Q} \to \omega_X$ (since $k$ has characteristic 2 nondegeneracy implies that the rank of $\mathcal{Q}$ is even). The construction of $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det$ from 4.2.8 has to be modified. If $(V; L_+, L_-)$ is a Lagrangian triple and $W$ is equipped with a nondegenerate symmetric *bilinear* form then $(V \otimes W; L_+ \otimes W, L_- \otimes W)$ is a Lagrangian triple. The bilinear forms $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are not equivalent in characteristic 2, but one can use (183) for $W = H$ and $W = H'$ to construct $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det$. Finally we have to construct (183) and (184) in characteristic 2. Let us assume for simplicity that $k$ is perfect. Then the characteristic property *) of the isomorphisms (183) and (184) is formulated just as in 4.2.15, but the proof of their existence and uniqueness should be modified. The semigroup $|\mathcal{B}|$ (see the end of the proof of the lemma from 4.2.15) is no longer $\mathbb{Z}_+$; it has generators $a$ and $b$ with the defining relation $a + b = 3a$ ($a$ corresponds to the matrix (1) of order 1 and $b$ corresponds to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). So the group corresponding to $\mathcal{B}$ is $\mathbb{Z}$, which is enough.

### 4.3. Pfaffians II.

**4.3.1.** Fix an $n$-dimensional vector space $W$ over $\mathbb{C}$ and a nondegenerate symmetric bilinear form ( ) on it. To simplify notation we write $O_n$ and $SO_n$ instead of $O(W)$ and $SO(W)$.

Let $\mathcal{F}$ be an $SO_n$-torsor on $X$. The corresponding rank $n$ vector bundle $W_{\mathcal{F}}$ carries the bilinear form $(\ )_{\mathcal{F}}$, and we have a canonical isomorphism $\det W_{\mathcal{F}} = \mathcal{O}_X \otimes \det W$. Let $\mathcal{L} \in \omega^{1/2}(X)$, i.e., $\mathcal{L}$ is a square root of $\omega_X$.

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*) To formulate this property in the non-perfect case one should consider $\mathcal{B}$ as a stack rather than a groupoid.

Then $W_{\mathcal{F}} \otimes \mathcal{L}$ is an $\omega$-orthogonal bundle, so $\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L})$ makes sense (see 4.2). Consider the "normalized" Pfaffian

$$\mathrm{Pf}_{\mathcal{L},\mathcal{F}} := \mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}) \otimes \mathrm{Pf}(W \otimes \mathcal{L})^{\otimes -1} \tag{185}$$

and the "normalized" determinant

$$\nu(\mathcal{F}) := \det R\Gamma(X, W_{\mathcal{F}}) \otimes \det R\Gamma(X, \mathcal{O}_X \otimes W)^{\otimes -1}. \tag{186}$$

As explained in 4.2.1 there are canonical isomorphisms $c_{\pm i} : \mathrm{Pf}^{\otimes 2} \xrightarrow{\sim} \det R\Gamma$. Using, e.g., $c_i$ one obtains an isomorphism*)

$$\mathrm{Pf}_{\mathcal{L},\mathcal{F}}{}^{\otimes 2} \xrightarrow{\sim} \nu_{\mathcal{L}}(\mathcal{F}) \tag{187}$$

where

$$\nu_{\mathcal{L}}(\mathcal{F}) := \det R\Gamma(X, W_{\mathcal{F}} \otimes \mathcal{L}) \otimes \det R\Gamma(X, W \otimes \mathcal{L})^{\otimes -1}. \tag{188}$$

Construction 7.2 from [Del87] yields a canonical isomorphism

$$\nu_{\mathcal{L}}(\mathcal{F}) = \nu(\mathcal{F}) \otimes \left\langle \det W_{\mathcal{F}} \otimes (\det W)^{\otimes -1}, \mathcal{L} \right\rangle$$

Since $\det W_{\mathcal{F}} = \mathcal{O}_X \otimes \det W$ one has $\nu_{\mathcal{L}}(\mathcal{F}) = \nu(\mathcal{F})$ and

$$\mathrm{Pf}_{\mathcal{L},\mathcal{F}}{}^{\otimes 2} = \nu(\mathcal{F}). \tag{189}$$

When $\mathcal{F}$ varies $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}$ and $\nu(\mathcal{F})$ become fibers of line bundles on $\mathrm{Bun}_{SO_n}$ which we denote by $\mathrm{Pf}_{\mathcal{L}}$ and $\nu$.

Denote by $\nu^{1/2}(\mathrm{Bun}_{SO_n})$ the category of square roots of $\nu$. We have the functor

$$\mathrm{Pf} : \omega^{1/2}(X) \to \nu^{1/2}(\mathrm{Bun}_{SO_n}) \tag{190}$$

defined by $\mathcal{L} \mapsto \mathrm{Pf}_{\mathcal{L}}$.

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*)So the isomorphism (187)=(189) depends on the choice of a square root of -1. This dependence disappears if one multiplies (187) by $i^{\pm p(\mathcal{F})^2}$ where $p$ is the canonical map $\mathrm{Bun}_{SO_n} \to \pi_0(\mathrm{Bun}_{SO_n}) = \pi_1(SO_n) = \mathbb{Z}/2\mathbb{Z}$ and $p(\mathcal{F})^2 \in \mathbb{Z}/4\mathbb{Z}$. We prefer not to do it for the reason explained in Remark (iii) from 4.2.1.

$\omega^{1/2}(X)$ and $\nu^{1/2}(\mathrm{Bun}_{SO_n})$ are Torsors over the Picard categories $\mu_2\,\mathrm{tors}(X)$ and $\mu_2\,\mathrm{tors}(\mathrm{Bun}_{SO_n})$. We have the Picard functor $\ell^{\mathrm{Spin}}$ : $\mu_2\,\mathrm{tors}(X) \to \mu_2\,\mathrm{tors}(\mathrm{Bun}_{SO_n})$; this is the functor $\ell = \ell^{\tilde{G}}$ from 4.1 in the particular case $G = SO_n$, $\tilde{G} = \mathrm{Spin}_n$, $\Pi = \mathbb{Z}/2\mathbb{Z}$. In 4.3.8–4.3.15 we will show that the functor $\mathrm{Pf} : \omega^{1/2}(X) \to \nu^{1/2}(\mathrm{Bun}_{SO_n})$ has a canonical structure of $\ell^{\mathrm{Spin}}$-affine functor. Before doing it we show in 4.3.2–4.3.7 that for a finite $S \subset X$ the action of $SO_n(K_S)$ on $\mathrm{Bun}_{SO_n,\underline{S}}$ defined in 4.1.7 lifts to an action of a certain central extension of $SO_n(K_S)$ on the pullback of $\mathrm{Pf}_{\mathcal{L}}$ to $\mathrm{Bun}_{SO_n,\underline{S}}$. Once this action is introduced it is easy to characterize the $\ell^{\mathrm{Spin}}$-affine structure on the functor Pf essentially by the $SO_n(K_S)$-invariance property (see 4.3.8–4.3.10).

**4.3.2.** Let $V$ be a Tate space equipped with a nondegenerate symmetric bilinear form of *even type*, i.e., there exists a Lagrangian c-lattice $L \subset V$ (see 4.2.13); if $\dim V < \infty$ this means that $\dim V$ is even. Denote by $O(V)$ the group of topological automorphisms of $V$ preserving the form. Let us remind the well known construction of a canonical central extension

$$0 \to \mathbb{C}^* \to \tilde{O}(V) \to O(V) \to 0\,. \tag{191}$$

Let $M$ be an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded discrete module over the Clifford algebra $\mathrm{Cl}(V)$ (discreteness means that $\{v \in V \,|\, vm = 0\}$ is open for every $m \in M$). Then $M$ is unique up to tensoring by a 1-dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded space. So there is a natural projective representation of $O(V)$ in $M$. (191) is the extension corresponding to this representation, i.e.,

$$\tilde{O}(V) := \{(g,\varphi) \,|\, g \in O(V),\, \varphi \in \mathrm{Aut}_{\mathbb{C}} M,\, \varphi(vm) = g(v)\cdot\varphi(m) \text{ for } m \in M\}\,.$$

Clearly $\tilde{O}(V)$ does not depend on the choice of $M$ (in fact $\mathrm{Aut}_{\mathbb{C}} M$ is the group of invertible elements of the natural completion of $\mathrm{Cl}(V)$). If $(g,\varphi) \in \tilde{O}(V)$ then $\varphi$ is either even or odd. Let $\chi(g) \in \mathbb{Z}/2\mathbb{Z}$ denote the parity of $\varphi$. Then $\chi :\, O(V) \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism.

The preimages of $-1 \in O(V)$ in $\tilde{O}(V)$ are not central. Indeed, if $\varphi : M \to M$, $\varphi(m) = m$ for even $m$ and $\varphi(m) = -m$ for odd $m$ then $[-1] := (-1, \varphi) \in \tilde{O}(V)$ and

$$[-1] \cdot \tilde{g} = (-1)^{\chi(g)} \cdot \tilde{g} \cdot [-1]\,, \quad g \in O(V) \tag{192}$$

where $\tilde{g}$ denotes a preimage of $g$ in $\tilde{O}(V)$.

$O(V)$ and $\operatorname{Aut}_{\mathbb{C}} M$ have natural structures of group ind-schemes. More precisely, the functors that associate to a $\mathbb{C}$-algebra $R$ the sets $O(V \widehat{\otimes} R)$ and $\operatorname{Aut}_{\mathbb{C}}(M \otimes R)$ are ind-schemes (if $\dim V = \infty$ then they can be represented as a union of an *uncountable* filtered family of closed subschemes.) So $\tilde{O}(V)$ is a group ind-scheme.

Denote by $\operatorname{Lagr}(V)$ the set of Lagrangian c-lattices in $V$. It has a natural structure of ind-scheme: $\operatorname{Lagr}(V) = \varinjlim \operatorname{Lagr}(\Lambda^{\perp}/\Lambda)$ where $\Lambda$ belongs to the set of isotropic c-lattices in $V$ (so an $R$-point of $\operatorname{Lagr}(V)$ is a Lagrangian c-lattice in $V \widehat{\otimes} R$ in the sense of 4.2.14). Denote by $\mathcal{P} = \mathcal{P}_M$ the line bundle on $\operatorname{Lagr}(V)$ whose fiber over $L \in \operatorname{Lagr}(V)$ equals $M^L := \{m \in M \,|\, Lm = 0\}$. The action of $O(V)$ on $\operatorname{Lagr}(V)$ canonically lifts to an action of $\tilde{O}(V)$ on $\mathcal{P}$.

$\operatorname{Lagr}(V)$ has two connected components distinguished by the parity of the 1-dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded space $M^L$, $L \in \operatorname{Lagr}(V)$. The proof of this statement is easily reduced to the case where $\dim V$ is finite (and even). The same argument shows that $L_1, L_2 \in \operatorname{Lagr}(V)$ belong to the same component if and only if $\dim(L_1/(L_1 \cap L_2))$ is even. Clearly the connected components of $\operatorname{Lagr}(V)$ are invariant with respect to $g \in O(V)$ if and only if $\chi(g) = 0$. Therefore $\chi : O(V) \to \mathbb{Z}/2\mathbb{Z}$ is a morphism of group ind-schemes.

Let us prove that (191) comes from an exact sequence of group ind-schemes

$$0 \to \mathbb{G}_m \to \tilde{O}(V) \to O(V) \to 0\,. \tag{193}$$

We only have to show that the morphism $\tilde{O}(V) \to O(V)$ is a $\mathbb{G}_m$-torsor. To this end fix $L \in \operatorname{Lagr}(V)$ and set $M = \operatorname{Cl}(V)/\operatorname{Cl}(V)L$, so that the fiber of $\mathcal{P} = \mathcal{P}_M$ over $L$ equals $\mathbb{C}$. Define $f : O(V) \to \operatorname{Lagr}(V)$ by $f(g) = gL$.

Set $\mathcal{P}' := \mathcal{P} \setminus \{\text{zero section}\}$; this is a $\mathbb{G}_m$-torsor over $\mathrm{Lagr}(V)$. It is easy to show that the natural morphism $\tilde{O}(V) \to f^*\mathcal{P}'$ is an isomorphism, so $\tilde{O}(V)$ is a $\mathbb{G}_m$-torsor over $O(V)$.

*Remark.* Let $L \in \mathrm{Lagr}(V)$. Then (193) splits canonically over the stabilizer of $L$ in $O(V)$: if $g \in O(V)$, $gL = L$, then there is a unique preimage of $g$ in $\tilde{O}(V)$ that acts identically on $M^L$.

**4.3.3.** Set $O := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$. Denote by $\omega_O$ the (completed) module of differentials of $O$. Fix a square root of $\omega_O$, i.e., a 1-dimensional free $O$-module $\omega_O^{1/2}$ equipped with an isomorphism $\omega_O^{1/2} \otimes \omega_O^{1/2} \xrightarrow{\sim} \omega_O$. Let $W$ have the same meaning as in 4.3.1. We will construct a central extension of $O_n(K) := O(W \otimes K)$ considered as a group ind-scheme over $\mathbb{C}$.

Set $\omega_K^{1/2} := \omega_O^{1/2} \otimes_O K$, $\omega_K := \omega_O \otimes_O K$. Consider the Tate space $V := \omega_K^{1/2} \otimes W$. The bilinear form on $W$ induces a $K$-bilinear form $V \times V \to \omega_K$. Composing it with $\mathrm{Res} : \omega_K \to \mathbb{C}$ one gets a nondegenerate symmetric bilinear form $V \times V \to \mathbb{C}$ of even type. Restricting the extension (193) to $O_n(K) \hookrightarrow O(V)$ one gets a central extension

$$0 \to \mathbb{G}_m \to \widetilde{O_n(K)} \to O_n(K) \to 0\,. \tag{194}$$

It splits canonically over $O_n(O) \subset O_n(K)$ (use the remark at the end of 4.3.2 for $L = \omega_O^{1/2} \otimes W \subset V$). The group $\mathrm{Aut}\,\omega_O^{1/2} = \mu_2$ acts on the extension (194) preserving the splitting over $O_n(O)$.

**4.3.4.** *Lemma.* The automorphism of $\widetilde{O_n(K)}$ induced by $-1 \in \mathrm{Aut}\,\omega_O^{1/2}$ maps $\tilde{g} \in \widetilde{O_n(K)}$ to $(-1)^{\theta(g)}\tilde{g}$ where $g$ is the image of $\tilde{g}$ in $O_n(K)$ and $\theta : O_n(K) \to K^*/(K^*)^2 = \mathbb{Z}/2\mathbb{Z}$ is the spinor norm.

*Proof.* According to (192) we only have to show that $\chi(g) = \theta(g)$ for $g \in O_n(K) \subset O(V)$. According to the definition of $\theta$ (see [D71], ch. II, §7) it suffices to prove that if $g$ is the reflection with respect to the orthogonal complement of a non-isotropic $x \in K^n$ then $\chi(g)$ equals the image of $(x, x) \in K^*$ in $K^*/(K^*)^2 = \mathbb{Z}/2\mathbb{Z}$. We can assume that $x \in O^n$, $x \notin tO^n$.

$L := \omega_O^{1/2} \otimes W$ is a Lagrangian c-lattice in $V$, so $\chi(g)$ is the parity of $\dim L/(L \cap gL) = \dim O/(x,x)O$. $\square$

*Remarks*

(i) Instead of using reflections one can compute the restriction of $\chi$ to a split Cartan subgroup of $SO_n(K)$ and notice that $\chi(g) = 0$ for $g \in O_n(\mathbb{C})$.

(ii) The restriction of $\theta$ to $SO_n(K)$ is the boundary morphism

$$SO_n(K) \to H^1(K, \mu_2) = \mathbb{Z}/2\mathbb{Z} \tag{195}$$

for the exact sequence $0 \to \mu_2 \to \mathrm{Spin}_n \to SO_n \to 0$.

(iii) If $g \in O_n(K) = O(W \otimes K)$ then $\dim(W \otimes O)/((W \otimes O) \cap g(W \otimes O))$ is even if and only if $\theta(g) = 0$. This follows from the proof of Lemma 4.3.4.

**4.3.5.** Consider the restriction of the extension (194) to $SO_n(K)$:

$$0 \to \mathbb{G}_m \to \widetilde{SO_n(K)} \to SO_n(K) \to 0\,. \tag{196}$$

It splits canonically over $SO_n(O)$. The extension (196) depends on the choice of $\omega_O^{1/2}$, so one should rather write $\widetilde{SO_n(K)}_{\mathcal{C}}$ where $\mathcal{C}$ is a square root of $\omega_O$. Let $\mathcal{C}'$ be another square root of $\omega_O$, then $\mathcal{C}' = \mathcal{C} \otimes \mathcal{A}$ where $\mathcal{A}$ is a $\mu_2$-torsor over $\operatorname{Spec} O$ (or over $\operatorname{Spec} \mathbb{C}$, which is the same). Consider the (trivial) extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{G}_m$ such that $\mathcal{A}$ is the $\mu_2$-torsor of its splittings. Its pullback by (195) is a (trivial) extension

$$0 \to \mathbb{G}_m \to \widetilde{SO_n(K)}_{\mathcal{A}} \to SO_n(K) \to 0 \tag{197}$$

equipped with a splitting over $SO_n(O)$ (in 4.1.8 we have already introduced this extension in a more general situation).

Lemma 4.3.4 yields a canonical isomorphism between $\widetilde{SO_n(K)}_{\mathcal{C}'}$ and the sum of the extensions $\widetilde{SO_n(K)}_{\mathcal{C}}$ and $\widetilde{SO_n(K)}_{\mathcal{A}}$. It is compatible with the splittings over $SO_n(O)$.

**4.3.6.** Let $S$, $O_S$, and $K_S$ have the same meaning as in 4.1.7. Fix $\mathcal{L} \in \omega^{1/2}(X)$ and denote by $\omega_{K_S}^{1/2}$ the space of sections of the pullback of $\mathcal{L}$ to $\operatorname{Spec} K_S$. Then proceed as in 4.3.3: set $V := \omega_{K_S}^{1/2} \otimes W$, define the scalar product on $V$ using the "sum of residues" map $\omega_{K_S} \to \mathbb{C}$, embed $SO_n(K_S)$ into $O(V)$ and finally get a central extension

$$0 \to \mathbb{G}_m \to \widetilde{SO_n(K_S)}_{\mathcal{L}} \to SO_n(K_S) \to 0 \tag{198}$$

with a canonical splitting over $SO_n(O_S)$.

*Remark.* (198) is the "super-sum" of the extensions (196) for $K = K_x$, $x \in S$. Let us explain that if $G_i$, $i \in I$, are groups equipped with morphisms $\theta_i : G_i \to \mathbb{Z}/2\mathbb{Z}$ and $\tilde{G}_i$ are central extensions of $G_i$ by $\mathbb{G}_m$ then the super-sum of these extensions is the extension of $\bigoplus\limits_i G_i$ by $\mathbb{G}_m$ obtained from the usual sum by adding the pullback of the standard extension

$$0 \to \mathbb{G}_m \to A \to \bigoplus_{i \in I} (\mathbb{Z}/2\mathbb{Z}) \to 0$$

where $A$ is generated by $\mathbb{G}_m$ and elements $e_i$, $i \in I$, with the defining relations $e_i^2 = 1$, $ce_i = e_i c$ for $c \in \mathbb{G}_m$, $e_i e_j = (-1) \cdot e_j e_i$ for $i \neq j$. In our situation $\theta_x : SO_n(K_x) \to \mathbb{Z}/2\mathbb{Z}$ is the spinor norm.

If $\mathcal{L}, \mathcal{L}' \in \omega^{1/2}(X)$ then $\mathcal{L}' = \mathcal{L} \otimes \mathcal{E}$ where $\mathcal{E}$ is a $\mu_2$-torsor. It follows from 4.3.5 that there is a canonical isomorphism between $\widetilde{SO_n(K_S)}_{\mathcal{L}'}$ and the sum of the extensions $\widetilde{SO_n(K_S)}_{\mathcal{L}}$ and $\widetilde{SO_n(K_S)}_{\mathcal{E}}$ (see 4.1.8 for the definition of $\widetilde{SO_n(K_S)}_{\mathcal{E}}$).

**4.3.7.** In 4.3.1 we defined the line bundles $\mathrm{Pf}_{\mathcal{L}}$ on $\mathrm{Bun}_{SO_n}$, $\mathcal{L} \in \omega^{1/2}(X)$. Denote by $\mathrm{Pf}_{\mathcal{L}}^S$ the pullback of $\mathrm{Pf}_{\mathcal{L}}$ to the scheme $\mathrm{Bun}_{SO_n \underline{S}}$ defined in 4.1.7. We have the obvious action of $SO_n(O_S) \times \mathbb{G}_m$ on $\mathrm{Pf}_{\mathcal{L}}^S$ ( $\lambda \in \mathbb{G}_m$ acts as multiplication by $\lambda$). We are going to extend it to an action of $\widetilde{SO_n(K_S)}_{\mathcal{L}}$ on $\mathrm{Pf}_{\mathcal{L}}^S$ compatible with the action of $SO_n(K_S)$ on $\mathrm{Bun}_{SO_n,\underline{S}}$.

Let $u \in \mathrm{Bun}_{SO_n,\underline{S}}$, $\tilde{g} \in \widetilde{SO_n(K_S)}_{\mathcal{L}}$. Denote by $\mathcal{F}$ and $\mathcal{F}'$ the $SO(W)$-bundles corresponding to $u$ and $gu$ where $g \in SO_n(K_S)$ is the image of $\tilde{g}$. We must define an isomorphism $\mathrm{Pf}_{\mathcal{L},\mathcal{F}} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L},\mathcal{F}'}$, i.e., an isomorphism

$\mathrm{Pf}(W_{\mathcal{F}}\otimes\mathcal{L})\xrightarrow{\sim}\mathrm{Pf}(W_{\mathcal{F}'}\otimes\mathcal{L})$. According to 4.2.13 it suffices to construct an isomorphism $\mathrm{Pf}(V;L_+,L_-)\xrightarrow{\sim}\mathrm{Pf}(V;L_+,L'_-)$ where $V$ is the Tate space from 4.3.6, $L_+=\omega_{O_S}^{1/2}\otimes W\subset V$, and $L_-\,,L'_-\subset V$ are discrete Lagrangian subspaces such that $L'_-=gL_-$. According to (166) this is equivalent to constructing an isomorphism $f:(M_{L_-})^{\otimes -1}\xrightarrow{\sim}(M_{gL_-})^{\otimes -1}$. We define $f$ to be induced by the action of $\tilde{g}\in\tilde{O}(V)$ on $M$.

Attention: $\lambda\in\mathbb{G}_m\subset\widetilde{SO_n(K_S)}_{\mathcal{L}}$ acts on $\mathrm{Pf}^S_{\mathcal{L}}$ as multiplication by $\lambda^{-1}$.

**4.3.8.** As explained in 4.3.1 our goal is to define a canonical $\ell^{\mathrm{Spin}}$-affine structure on the functor (190). This means that for $\mathcal{L}\in\omega^{1/2}(X)$ and a $\mu_2$-torsor $\mathcal{E}$ on $X$ we must define an isomorphism

$$\mathrm{Pf}_{\mathcal{L}}\otimes\ell^{\mathrm{Spin}}_{\mathcal{E}}\xrightarrow{\sim}\mathrm{Pf}_{\mathcal{L}'}\,,\quad \mathcal{L}':=\mathcal{L}\otimes\mathcal{E}\,. \tag{199}$$

We must also check certain compatibility properties for the isomorphisms (199).

To simplify notation we will write $\ell_{\mathcal{E}}$ instead of $\ell^{\mathrm{Spin}}_{\mathcal{E}}$. Let $S\subset X$ be finite. In 4.1.7–4.1.8 we constructed an action of the central extension $\widetilde{SO_n(K_S)}_{\mathcal{E}}$ on $\ell^S_{\mathcal{E}}:=$ the pullback of $\ell_{\mathcal{E}}$ to $\mathrm{Bun}_{SO_n,\underline{S}}$. So it follows from 4.3.6–4.3.7 that $\widetilde{SO_n(K_S)}_{\mathcal{L}'}$ acts both on $\mathrm{Pf}^S_{\mathcal{L}}\otimes\ell^S_{\mathcal{E}}$ and $\mathrm{Pf}^S_{\mathcal{L}'}$. Recall that the fibers of both sides of (199) over the trivial $SO_n$-bundle equal $\mathbb{C}$.

**4.3.9.** *Theorem.* There is a unique isomorphism (199) such that for every $S$ the corresponding isomorphism $\mathrm{Pf}^S_{\mathcal{L}}\otimes\ell^S_{\mathcal{E}}\xrightarrow{\sim}\mathrm{Pf}^S_{\mathcal{L}'}$ is $\widetilde{SO_n(K_S)}_{\mathcal{L}'}$-equivariant and the isomorphism between the fibers over the trivial $SO_n$-bundle induced by (199) is identical.

The proof will be given in 4.3.11–4.3.13. See §5.2 from [BLaSo] for a short proof of a weaker statement.

**4.3.10.** *Proposition.* The isomorphisms (199) define an $\ell^{\mathrm{Spin}}$-affine structure on the functor $\mathrm{Pf}:\omega^{1/2}(X)\to\nu^{1/2}(\mathrm{Bun}_{SO_n})$.

The proof will be given in 4.3.15.

**4.3.11.** Let us start to prove Theorem 4.3.9. The uniqueness of (199) is clear if $n > 2$: in this case $SO_n$ is semisimple, so one has the isomorphism (155) for $G = SO_n$, $S \neq \emptyset$. If $n = 2$ the action of $SO_n(K_S)$ on $\mathrm{Bun}_{SO_n,\underline{S}}$ is not transitive, but $SO_n$ over the adeles acts transitively on $\varprojlim_S \mathrm{Bun}_{SO_n,\underline{S}}(\mathbb{C})$, which is enough for uniqueness.

While proving the existence of (199) we will assume that $n > 2$. The case $n = 2$ can be treated using the embedding $SO_2 \hookrightarrow SO_3$ and the corresponding morphism $\mathrm{Bun}_{SO_2} \to \mathrm{Bun}_{SO_3}$ or using the remark at the end of 4.3.14.

Consider the $SO_n(K_S)$-equivariant line bundle $C_S := \mathrm{Pf}^S_{\mathcal{L}} \otimes \ell^S_{\mathcal{E}} \otimes (\mathrm{Pf}^S_{\mathcal{L}'})^*$ on $\mathrm{Bun}_{SO_n,\underline{S}}$. The stabilizer of the point of $\mathrm{Bun}_{SO_n,\underline{S}}$ corresponding to the trivial $SO_n$-bundle with the obvious trivialization over $\underline{S}$ equals $SO_n(A_S)$, $A_S := H^0(X \setminus S, \mathcal{O}_X)$. So the action of $SO_n(K_S)$ on $C_S$ induces a morphism $f_S : SO_n(A_S) \to \mathbb{G}_m$. It suffices to prove that $f_S$ is trivial for all $S$ (then for $S \neq \emptyset$ one can use (155) to obtain a $SO_n(K_S)$-equivariant trivialization of $C_S$ and of course these trivializations are compatible with each other).

Denote by $\Sigma$ the scheme of finite subschemes of $X$ (so $\Sigma$ is the disjoint union of the symmetric powers of $X$). $A_S$, $O_S$, and $K_S$ make sense for a non-necessarily reduced*) $S \in \Sigma$ (e.g., $O_S$ is the ring of functions on the completion of $X$ along $S$) and the rings $A_S$, $O_S$, $K_S$ are naturally organized into families (i.e., there is an obvious way to define three ring ind-schemes over $\Sigma$ whose fibers over $S \in \Sigma$ are equal to $A_S$, $O_S$, $K_S$ respectively).

It is easy to show that *the morphisms $f_S$ form a family* (i.e., they come from a morphism of group ind-schemes over $\Sigma$). Clearly if $S \subset S'$ then the restriction of $f_{S'}$ to $SO_n(A_S)$ equals $f_S$. In 4.3.12–4.3.13 we will deduce from these two facts that $f_S = 1$.

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*)This is important when $S$ varies. For a fixed $S$ the rings $A_S$, $O_S$ and $K_S$ depend only on $S_{\mathrm{red}}$.

**4.3.12.** Let $Y$ be a separated scheme of finite type over $\mathbb{C}$ and $R$ a $\mathbb{C}$-algebra. Set $Y_{\mathrm{rat}}(R) = \varinjlim_{U} \mathrm{Mor}(U, Y)$ where the limit is over all open $U \subset X \otimes R$ such that the fiber of $U$ over any point of $\operatorname{Spec} R$ is non-empty. In other words, elements of $Y_{\mathrm{rat}}(R)$ are families of rational maps $X \to Y$ parameterized by $\operatorname{Spec} R$. The functor $Y_{\mathrm{rat}}$ is called the *space of rational maps* $X \to Y$. It is easy to show that $Y_{\mathrm{rat}}$ is a sheaf for the fppf topology, i.e., a "space" in the sense of [LMB93].

We have the spaces $Y(A_S)$, $S \in \Sigma$, which form a family (i.e., there is a natural space over $\Sigma$ whose fiber over each $S$ equals $Y(A_S)$). So a regular function on $Y_{\mathrm{rat}}$ defines a family of regular functions $f_S$ on $Y(A_S)$, $S \in \Sigma$, such that for $S \subset S'$ the pullback of $f_{S'}$ to $Y(A_S)$ equals $f_S$. It is easy to see that a function on $Y_{rat}$ is *the same* as a family of functions $f_S$ with this property.

**4.3.13.** *Proposition.* Let $G$ be a connected algebraic group.

(i) Every regular function on $G_{\mathrm{rat}}$ is constant. In particular every group morphism $G_{\mathrm{rat}} \to \mathbb{G}_m$ is trivial.

(ii) Moreover, for every $\mathbb{C}$-algebra $R$ every regular function on $G_{\mathrm{rat}} \otimes R$ is constant (i.e., an element of $R$).

*Proof.* Represent $G$ as $\bigcup_i U_i$ where $U_i$ are open sets isomorphic to $(\mathbb{A}^1 \setminus \{0\})^r \times \mathbb{A}^s$ (e.g., let $U \subset G$ be the big cell with respect to some Borel subgroup, then $G$ is covered by a finite number of sets of the form $gU$, $g \in G$). One has the open covering $G_{\mathrm{rat}} = \bigcup_i (U_i)_{\mathrm{rat}}$ and $(U_i)_{\mathrm{rat}} \cap (U_j)_{\mathrm{rat}} \neq \emptyset$. So it is enough to prove the proposition for $G = (\mathbb{G}_m)^r \times (\mathbb{G}_a)^s$. Moreover, it suffices to prove (ii) for $\mathbb{G}_a$ and $\mathbb{G}_m$.

Consider, e.g., the $\mathbb{G}_m$ case. Choose an ample line bundle $\mathcal{A}$ on $X$ and set $V_n := H^0(X, \mathcal{A}^{\otimes n})$, $V_n' := V_n \setminus \{0\}$. Define $\pi_n : V_n' \times V_n' \to (\mathbb{G}_m)_{\mathrm{rat}}$ by $(f, g) \mapsto f/g$. A regular function $\varphi$ on $(\mathbb{G}_m)_{\mathrm{rat}} \otimes R$ defines a regular function $\pi_n^* \varphi$ on $(V_n' \times V_n') \otimes R$, which is invariant with respect to the obvious action of $\mathbb{G}_m$ on $V_n' \times V_n'$. For $n$ big enough $\dim V_n > 1$ and therefore $\pi_n^* \varphi$ extends to a $\mathbb{G}_m$-invariant regular function on $(V_n \times V_n) \otimes R$, which is necessarily a constant. So $\varphi$ is constant. $\square$

**4.3.14.** This subsection is not used in the sequel (except the definition of $\mathrm{GRAS}_G$ needed in 5.3.10).

Let $G$ be a connected algebraic group. The following approach to $\mathrm{Bun}_G$ seems to be natural.

Denote by $\mathrm{GRAS}_G$ the space of $G$-torsors on $X$ equipped with a rational section. The precise definition of this space is quite similar to the definition of $Y_{\mathrm{rat}}$ from 4.3.12. We would call $\mathrm{GRAS}_G$ the *big Grassmannian* corresponding to $G$ and $X$ because for a fixed finite $S \subset X$ the space of $G$-bundles on $X$ trivialized over $X \setminus S$ can be identified with the ind-scheme $G(K_S)/G(O_S) = \prod_{x \in X} G(K_x)/G(O_x)$ (see 5.3.10), and $G(K_x)/G(O_x)$ is called the *affine Grassmannian* or *loop Grassmannian* (see 4.5 or [MV]).

The morphism $\pi : \mathrm{GRAS}_G \to \mathrm{Bun}_G$ is a $G_{\mathrm{rat}}$-torsor for the smooth topology (the existence of a section $S \to \mathrm{GRAS}_G$ for some smooth surjective morphism $S \to \mathrm{Bun}_G$ is obvious if the reductive part of $G$ equals $GL_n$, $SL_n$, or $Sp_n$; for a general $G$ one can use [DSim]).

Consider the functor

$$\pi^* : \mathrm{Vect}(\mathrm{Bun}_G) \to \mathrm{Vect}(\mathrm{GRAS}_G) \tag{200}$$

where Vect denotes the category of vector bundles. It follows from 4.3.13 that (200) is fully faithful. One can show that for any scheme $T$ every vector bundle on $G_{\mathrm{rat}} \times T$ comes from $T$. This implies that (200) is an equivalence.

*Remark.* Our construction of (199) can be interpreted as follows: we constructed an isomorphism between the pullbacks of the l.h.s. and r.h.s. of (199) to $\mathrm{GRAS}_{SO_n}$, then we used the fact that (200) is fully faithful. It was not really necessary to use the isomorphism (155). So the construction of (199) also works in the case of $SO_2$.

**4.3.15.** Let us prove Proposition 4.3.10. The isomorphisms (199) are compatible with each other (use the uniqueness statement from 4.3.9). It remains to show that the tensor square of (199) equals the composition

$$\mathrm{Pf}_{\mathcal{L}}^{\otimes 2} \xrightarrow{\sim} \nu_{\mathcal{L}} \xrightarrow{\sim} \nu \xrightarrow{\sim} \nu_{\mathcal{L}'} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}'}^{\otimes 2} \tag{201}$$

where $\nu_{\mathcal{L}}$ is defined by (188).

Fix an $SO_n$-torsor $\mathcal{F}$ on $X$ and its trivialization over $X \setminus S$ for some non-empty finite $S \subset X$. Using the trivialization we will compute the isomorphisms $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}^{\otimes 2} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}',\mathcal{F}}^{\otimes 2}$ induced by (199) and (201).

Recall that $\mathrm{Pf}_{\mathcal{L},\mathcal{F}} := \mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}) \otimes \mathrm{Pf}(W \otimes \mathcal{L})^{\otimes -1}$. According to 4.2.13

$$\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}) = \mathrm{Pf}(V; L_+, L_-), \quad \mathrm{Pf}(W \otimes \mathcal{L}) = \mathrm{Pf}(V; L_+^0, L_-)$$

where $V = \mathcal{L}_{K_S} \otimes W$, $L_- = \Gamma(X \setminus S, \mathcal{L} \otimes W)$, $L_+^0 = \mathcal{L}_{O_S} \otimes W$, and $L_+$ is the space of sections of the pullback of $W_{\mathcal{F}} \otimes \mathcal{L}$ to $\operatorname{Spec} O_S$ (we use the notation of 4.3.6). Using (166) one gets

$$\mathrm{Pf}_{\mathcal{L},\mathcal{F}} = M^{L_+} \otimes (M^{L_+^0})^* \tag{202}$$

where $M$ is an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded discrete module over $\mathrm{Cl}(V)$. $\mathrm{Pf}_{\mathcal{L}',\mathcal{F}}$ has a similar description in terms of $V'$, $L'_+$, $(L_+^0)'$, $L'_-$ where $V' = \mathcal{L}'_{K_S} \otimes W$, etc. Fix a trivialization of the $\mu_2$-torsor $\mathcal{E}$ from 4.3.8 over $S$. It yields a trivialization of $\mathcal{E}$ over $\operatorname{Spec} O_S$ and therefore an identification

$$(V', L'_+, (L_+^0)') = (V, L_+, L_+^0). \tag{203}$$

Since $L_-$ is not involved in (202) we obtain an isomorphism $\mathrm{Pf}_{\mathcal{L},\mathcal{F}} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}',\mathcal{F}}$. It is easy to show that it coincides with the one induced by (199) (notice that the trivialization of $\mathcal{F}$ over $X \setminus S$ and the trivialization of $\mathcal{E}$ over $S$ induce a trivialization of $\ell_{\mathcal{E}}^{\mathrm{Spin}}$ over $\mathcal{F}$ because the l.h.s. of (150) has a distinguished element).

Now we have to show that the isomorphism $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}^{\otimes 2} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}',\mathcal{F}}^{\otimes 2}$ induced by (201) is the identity provided $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}$ and $\mathrm{Pf}_{\mathcal{L}',\mathcal{F}}$ are identified with the r.h.s. of (202).

The trivialization of $\mathcal{F}$ over $X \setminus S$ yields an isomorphism $\nu_{\mathcal{L}}(\mathcal{F}) \xrightarrow{\sim} d(L_+^0, L_+)$ where $d(L_+^0, L_+)$ is the relative determinant, i.e., $d(L_+^0, L_+) = \det(L_+/U) \otimes \det(L_+^0/U)^{\otimes -1}$ for any c-lattice $U \subset L \cap L_+^0$. We have a similar identification $\nu_{\mathcal{L}'}(\mathcal{F}) = d((L_+^0)', L'_+)$. The isomorphism $\nu_{\mathcal{L}}(\mathcal{F}) \xrightarrow{\sim} \nu_{\mathcal{L}'}(\mathcal{F})$ from (201) is defined in [Del87] as follows. One chooses *any* isomorphism $f$ between the pullbacks of $\mathcal{L}$ and $\mathcal{L}'$ to $\operatorname{Spec} O_S$. $f$ yields an isomorphism $f_* : (V, L_+, L_+^0) \xrightarrow{\sim} (V', L'_+, (L_+^0)')$ and therefore an isomorphism $d(L_+^0, L_+) \xrightarrow{\sim} d(L'_+, (L_+^0)')$, which actually does not depend on the choice of $f$. It is convenient to define $f$ using the above trivialization of the $\mu_2$-torsor $\mathcal{E} = \mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ over $\operatorname{Spec} O_S$. Then $f_*$ coincides with (203).

Thus we have identified $\nu_{\mathcal{L}}(\mathcal{F})$ and $\nu_{\mathcal{L}'}(\mathcal{F})$ with $d(L_+^0, L_+)$ so that the isomorphism $\nu_{\mathcal{L}}(\mathcal{F}) \xrightarrow{\sim} \nu_{\mathcal{L}'}(\mathcal{F})$ from (201) becomes the identity map. We have identified both $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}$ and $\mathrm{Pf}_{\mathcal{L}',\mathcal{F}}$ with the r.h.s. of (202). It remains to show that the isomorphism (187) and its analog for $\mathcal{L}'$ induce the same isomorphism

$$(M^{L_+} \otimes (M^{L_+^0})^*)^{\otimes 2} \xrightarrow{\sim} d(L_+^0, L_+) \tag{204}$$

According to 4.2.8 and 4.2.13 the isomorphism (204) induced by (187) can be described as follows. We have the canonical isomorphism

$$N^{L_+ \otimes H} \otimes (N^{L_+^0 \otimes H})^* \xrightarrow{\sim} d(L_+^0, L) \tag{205}$$

where $N$ is an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded discrete module over the Clifford algebra $\mathrm{Cl}(V \oplus V^*) = \mathrm{Cl}(V \oplus V) = \mathrm{Cl}(V \otimes H)$ and $H$ denotes $\mathbb{C}^2$ equipped with the bilinear form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (to construct (205) take for $N$ the r.h.s. of (182)). On the other hand, $P := M \otimes M$ is an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded discrete module over $\mathrm{Cl}(V) \otimes \mathrm{Cl}(V) = \mathrm{Cl}(V \otimes H'')$ where $H''$ denotes $\mathbb{C}^2$ with the bilinear form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Rewrite the l.h.s. of (204) as $P^{L_+ \otimes H''} \otimes (P^{L_+^0 \otimes H''})^*$. So an orthogonal isomorphism $\psi : H'' \xrightarrow{\sim} H$ induces an isomorphism (204). To get the isomorphism (204) induced by (187) we must normalize $\psi$ by $\det \psi = i$ (or $-i$ ?? we should check!).

Since $L_-$ is not involved in the above description the analog of (187) for $\mathcal{L}'$ induces the same isomorphism (204), QED.

**4.3.16.** This subsection and 4.3.17 will be used in 4.4.14 (end of the proof of the horizontality theorem 2.7.3) and in the proof of Theorem 5.4.5 (which is the main result of this work). However the reader can skip them for the moment.

As usual, we set $O := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$. Fix $\mathcal{L} \in \omega^{1/2}(X)$, i.e., $\mathcal{L}$ is a square root of $\omega_X$. Fix also a square root of $\omega_O$ and denote it by $\omega_O^{1/2}$. Then one defines a 2-sheeted covering $X_2^\wedge$ of the scheme $X^\wedge$ from 2.6.5. Recall that an $R$-point of $X^\wedge$ is an $R$-morphism $\alpha : \operatorname{Spec}(R\widehat{\otimes}O) \to X \otimes R$ whose differential does not vanish over any point of $\operatorname{Spec} R$. Denote by $\mathcal{L}_R$ the pullback of $\mathcal{L}$ to $X \otimes R$. By definition, the fiber of $X_2^\wedge(R)$ over $\gamma \in X^\wedge(R)$ is the set of isomorphisms $H^0(\operatorname{Spec} R\widehat{\otimes}O, \alpha^*\mathcal{L}_R) \xrightarrow{\sim} R\widehat{\otimes}\omega_O^{1/2})$ in the groupoid of square roots of $R\widehat{\otimes}\omega_O$.

The group ind-scheme $\operatorname{Aut}_2 O := \operatorname{Aut}(O, \omega_O^{1/2})$ introduced in 3.5.2 acts on $X_2^\wedge$ by transport of structure.

Let $M$ be the scheme from 2.8.1 in the particular case $G = SO(W) = SO_n$. Denote by $M_2^\wedge$ the fiber product of $M$ and $X_2^\wedge$ over $X$ (so $M_2^\wedge$ is a 2-sheeted covering of the scheme $M^\wedge$ from 2.8.3). Then the semidirect product $\operatorname{Aut}_2 O \ltimes SO_n(K)$ acts on $M_2^\wedge$. Indeed, $M_2^\wedge$ is the fiber product of $M^\wedge$ and $X_2^\wedge$ over $X^\wedge$, and $\operatorname{Aut}_2 O \ltimes SO_n(K)$ acts on the diagram

$$\begin{array}{ccc} & & M^\wedge \\ & & \downarrow \\ X_2^\wedge & \longrightarrow & X^\wedge \end{array}$$

(the action of $\operatorname{Aut} O \ltimes SO_n(K)$ on $M^\wedge$ was defined in 2.8.4; $\operatorname{Aut}_2 O \ltimes SO_n(K)$ acts on $X_2^\wedge$ and $X^\wedge$ via its quotients $\operatorname{Aut}_2 O$ and $\operatorname{Aut} O$).

Denote by $\operatorname{Pf}_{\mathcal{L}}^\wedge$ the pullback to $M_2^\wedge$ of the line bundle $\operatorname{Pf}_{\mathcal{L}}$ on $\operatorname{Bun}_{SO_n}$ defined in 4.3.1. We will lift the action of $\operatorname{Aut}_2 O \ltimes SO_n(K)$ on $M_2^\wedge$ to an action of $\operatorname{Aut}_2 O \ltimes \widetilde{SO_n(K)}$ on $\operatorname{Pf}_{\mathcal{L}}^\wedge$, where $\widetilde{SO_n(K)}$ is the central extension

(196) corresponding to $\omega_O^{1/2}$. The action of $\operatorname{Aut}_2 O$ on $\mathrm{Pf}_{\mathcal{L}}^\wedge$ is clear because $\operatorname{Aut}_2 O$ acts on $M_2^\wedge$ considered as a scheme over $\mathrm{Bun}_{SO_n}$. On the other hand, $\widetilde{SO_n(K)}$ acts on $\mathrm{Pf}_{\mathcal{L},\widehat{x}}$ :=the restriction of $\mathrm{Pf}_{\mathcal{L}}^\wedge$ to the fiber of $M_2^\wedge$ over $\widehat{x} \in X_2^\wedge$. Indeed, this fiber equals $\mathrm{Bun}_{SO_n,\underline{x}}$ where $x$ is the image of $\widehat{x}$ in $X$, and by 4.3.7 the central extension $\widetilde{SO_n(K_x)}_{\mathcal{L}}$ acts on the pullback of $\mathrm{Pf}_{\mathcal{L}}$ to $\mathrm{Bun}_{SO_n,\underline{x}}$. This extension depends only on $\mathcal{L}_x$ :=the pullback of $\mathcal{L}$ to $\operatorname{Spec} O_x$. Since $\widehat{x}$ defines an isomorphism between $(O,\omega_O^{1/2})$ and $(O_x, H^0(\operatorname{Spec} O_x, \mathcal{L}_x))$ we get an isomorphism $\widetilde{SO_n(K_x)}_{\mathcal{L}} \xrightarrow{\sim} \widetilde{SO_n(K)}$ and therefore the desired action of $\widetilde{SO_n(K)}$.

**4.3.17.** *Proposition.*

(i) The action of $\widetilde{SO_n(K)}$ on $\mathrm{Pf}_{\mathcal{L},\widehat{x}}^\wedge$, $\widehat{x} \in X_2^\wedge$, comes from an (obviously unique) action of $\widetilde{SO_n(K)}$ on $\mathrm{Pf}_{\mathcal{L}}^\wedge$.

(ii) The actions of $\operatorname{Aut}_2 O$ and $\widetilde{SO_n(K)}$ on $\mathrm{Pf}_{\mathcal{L}}^\wedge$ define an action of $\operatorname{Aut}_2 O \ltimes \widetilde{SO_n(K)}$.

*Remark.* Statement (ii) can be interpreted in the spirit of 2.8.2: the action of $\operatorname{Aut}_2 O$ yields a connection along $X$ on $\pi^* \mathrm{Pf}_{\mathcal{L}}$ where $\pi$ is the morphism $M \to \mathrm{Bun}_G$, and the compatibility of the action of $\operatorname{Aut}_2 O$ with that of $\widetilde{SO_n(K)}$ means that the action on $\pi^* \mathrm{Pf}_{\mathcal{L}}$ of a certain central extension $\widetilde{J}^{\mathrm{mer}}(SO_n)_{\mathcal{L}}$ is horizontal.

*Proof.* To define the action of $\operatorname{Aut}_2 O \ltimes \widetilde{SO_n(K)}$ on $\mathrm{Pf}_{\mathcal{L}}^\wedge$ with the desired properties we proceed as in 4.3.7. Let $R$ be a $\mathbb{C}$-algebra. Consider an $R$-point $u$ of $M_2^\wedge$ and an $R$-point $\widetilde{g}$ of $\operatorname{Aut}_2 O \ltimes \widetilde{SO_n(K)}$. Recall that $SO_n$ is an abbreviation for $SO(W)$. Denote by $\mathcal{F}$ and $\mathcal{F}'$ the $SO(W)$-torsors on $X \otimes R$ corresponding to $u$ and $gu$ where $g$ is the image of $\widetilde{g}$ in $\operatorname{Aut}_2 O \ltimes SO_n(K)$. We have to define an isomorphism

$$\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}_R) \xrightarrow{\sim} \mathrm{Pf}(W_{\mathcal{F}'} \otimes \mathcal{L}_R) \tag{206}$$

where $\mathcal{L}_R$ is the pullback of $\mathcal{L}$ to $X \otimes R$.

Set $V := \omega_O^{1/2} \otimes_O K \otimes W$. This is a Tate space over $\mathbb{C}$ equipped with a nondegenerate symmetric bilinear form (see 4.3.3). By 4.2.14

$$\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}_R) = \mathrm{Pf}(V \widehat{\otimes} R; L_+ \widehat{\otimes} R, L_-^u) \tag{207}$$

where $L_+ := \omega_O^{1/2} \otimes W \subset V$ (so $L_+$ is a Lagrangian c-lattice in $V$) and the Lagrangian d-lattice $L_-^u \subset V \widehat{\otimes} R$ is defined as follows. The point $u \in M_2^\wedge(R)$ is a quadruple $(\alpha, \mathcal{F}, \gamma, f)$ where $\alpha$, $\mathcal{F}$, $\gamma$ have the same meaning as in 2.8.4 (in our special case $G = SO(W)$) and $f$ is an isomorphism between $H^0(\operatorname{Spec} R \widehat{\otimes} K, \alpha^* \mathcal{L}_R)$ and $R \widehat{\otimes} \omega_O^{1/2}$ in the groupoid of square roots of $R \widehat{\otimes} \omega_O$. Let $\Gamma_\alpha$ have the same meaning as in 2.8.4. Then

$$L_-^u := H^0((X \otimes R) \setminus \Gamma_\alpha, W_{\mathcal{F}} \otimes \mathcal{L}_R) \subset H^0(\operatorname{Spec} R \widehat{\otimes} K, \alpha^* W_{\mathcal{F}} \otimes \alpha^* \mathcal{L}_R) \xrightarrow[\sim]{\varphi} V \widehat{\otimes} R$$

(the isomorphism $\varphi$ is induced by $\gamma$ and $f$).

Taking (207) into account we see that constructing (206) is equivalent to defining an isomorphism

$$\mathrm{Pf}(V \widehat{\otimes} R; L_+ \otimes R, L_-^u) \xrightarrow{\sim} \mathrm{Pf}(V \widehat{\otimes} R; L_+ \otimes R, L_-^{gu})\,. \tag{208}$$

The group ind-scheme $\mathrm{Aut}_2 O \ltimes SO(W \otimes K)$ acts on $V$ in the obvious way, and it is easy to see that $L_-^{gu} = g L_-^u$. By (166) the l.h.s. of (208) is inverse to $(M \otimes R)_{L_-}$ where$M$ is the Clifford module $\mathrm{Cl}(V)/\mathrm{Cl}(V) L_+$ and $L_- := L_-^u$. So it remains to construct an isomorphism $(M \otimes R)_{L_-} \xrightarrow{\sim} (M \otimes R)_{gL_-}$. We define it to be induced by the action*) of $\widetilde{g}$ on $M \otimes R$. $\square$

## 4.4. Half-forms on $\mathrm{Bun}_G$.

**4.4.1.** Let $G$ be semisimple. Fix a $G$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. Set $n := \dim \mathfrak{g}$ and write $SO_n$ instead of $SO(\mathfrak{g})$. The adjoint representation $G \to SO(\mathfrak{g})$ induces a morphism $f : \mathrm{Bun}_G \to \mathrm{Bun}_{SO_n}$. For $\mathcal{L} \in \omega^{1/2}(X)$ set $\lambda'_{\mathcal{L}} := f^* \mathrm{Pf}_{\mathcal{L}}$ where $\mathrm{Pf}_{\mathcal{L}}$ is the line bundle

*)Recall that $g$ is an $R$-point of $\mathrm{Aut}_2 O \ltimes \widetilde{SO_n(K)} = \mathrm{Aut}_2 O \ltimes \widetilde{SO(W \otimes K)}$. By the definition of $\widetilde{SO_n(K)}$ it acts on $M$. The group ind-scheme $\mathrm{Aut}_2 O$ acts on $(V, L_+)$ and therefore on $M$.

from 4.3.1; so the fiber of $\lambda'_{\mathcal{L}}$ over $\mathcal{F}\in\mathrm{Bun}_G$ equals $\mathrm{Pf}(\mathfrak{g}_{\mathcal{F}}\otimes\mathcal{L})\otimes\mathrm{Pf}(\mathfrak{g}\otimes\mathcal{L})^{\otimes-1}$. The isomorphism (189) induces an isomorphism

$$(\lambda'_{\mathcal{L}})^{\otimes 2}=\omega^{\sharp}_{\mathrm{Bun}_G} \tag{209}$$

Here $\omega^{\sharp}_{\mathrm{Bun}_G}$ is the normalized canonical bundle (146); according to 2.1.1 the fiber of $\omega^{\sharp}_{\mathrm{Bun}_G}$ over $\mathcal{F}\in\mathrm{Bun}_G$ equals $\det R\Gamma(X,\mathfrak{g}_{\mathcal{F}})\otimes(\det R\Gamma(X,\mathfrak{g}\otimes\mathcal{O}_X))^{\otimes-1}$.

**4.4.2.** Consider the functor

$$\lambda':\omega^{1/2}(X)\to(\omega^{\sharp})^{1/2}(\mathrm{Bun}_G), \tag{210}$$

$\mathcal{L}\mapsto\lambda'_{\mathcal{L}}$. By 4.3.10 $\lambda'$ is affine with respect to the Picard functor $\tilde{\ell}':\mu_2\operatorname{tors}(X)\to\mu_2\operatorname{tors}(\mathrm{Bun}_G)$ that sends a $\mu_2$-torsor $\mathcal{E}$ on $X$ to $\tilde{\ell}'_{\mathcal{E}}:=$ the pullback to $\mathrm{Bun}_G$ of the torsor $\ell^{\mathrm{Spin}}_{\mathcal{E}}$ on $\mathrm{Bun}_{SO_n}$.

**4.4.3.** *Proposition.* $\tilde{\ell}'=\ell'$ *where* $\ell'$ *is the composition of the functor* $\mu_2\operatorname{tors}(X)\to Z\operatorname{tors}(X)$ *induced by (56) and the functor* $\ell:Z\operatorname{tors}(X)\to\mu_\infty\operatorname{tors}(\mathrm{Bun}_G)$ *constructed in 4.1.1–4.1.4. Here* $Z=\pi_1(G)^\vee$ *=the center of* ${}^LG$ *(see the Remark from 4.1.1).*

Assuming the proposition we define a canonical $\ell$-affine functor

$$\lambda:Z\operatorname{tors}_\theta(X)\to\mu_\infty\operatorname{tors}_\theta(\mathrm{Bun}_G) \tag{211}$$

by $\mathcal{E}\cdot\mathcal{L}\mapsto\lambda_{\mathcal{E}\cdot\mathcal{L}}:=\ell_{\mathcal{E}}\cdot\lambda'_{\mathcal{L}}$, $\mathcal{E}\in Z\operatorname{tors}(X)$, $\mathcal{L}\in\omega^{1/2}(X)$. (Attention: normalization problem!!!???)

To prove Proposition 4.4.3 notice that $\tilde{\ell}'$ is the functor (152) corresponding to the extension of $G$ by $\mu_2$ induced by the spinor extension of $SO(\mathfrak{g})$. Therefore $\tilde{\ell}'$ is the composition of $\ell:Z\operatorname{tors}(X)\to\mu_\infty\operatorname{tors}(\mathrm{Bun}_G)$ and the functor $\mu_2\operatorname{tors}(X)\to Z\operatorname{tors}(X)$ induced by the morphism $\mu_2\to Z=\pi_1(G)^\vee$ dual to $\pi_1(G)\to\pi_1(SO(\mathfrak{g}))=\mathbb{Z}/2\mathbb{Z}$. So it suffices to prove the following.

**4.4.4.** *Lemma.* The morphism $\pi_1(G) \to \pi_1(SO(\mathfrak{g})) = \mathbb{Z}/2\mathbb{Z}$ is dual to the morphism (56) for the group ${}^L G$.

*Proof.* We have the canonical isomorphism $f : P/P_G \xrightarrow{\sim} \mathrm{Hom}(\pi_1(G)(1), \mu_\infty)$ where $P_G$ is the group of weights of $G$ and $P$ is the group of weights of its universal covering $\widetilde{G}$; a weight $\lambda \in P$ is a character of the Cartan subgroup $\widetilde{H} \subset \widetilde{G}$ and $f(\lambda)$ is its restriction to $\pi_1(G)(1) \subset \widetilde{H}$. Let $M$ be a spinor representation of $so(\mathfrak{g})$. Then $\widetilde{G}$ acts on $M$ and $\pi_1(G)(1) \subset \tilde{G}$ acts according to some character $\chi \in \mathrm{Hom}(\pi_1(G)(1), \mu_\infty)$. According to the definition of (56) (see also the definition of $\lambda^\#$ in 3.4.1) the lemma just says that $\chi = f(\rho)$ where $\rho \in P$ is the sum of fundamental weights.

Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra. Choose a $\mathfrak{b}$-invariant flag $0 \subset V_1 \subset \ldots \subset V_n = \mathfrak{g}$ such that $\dim V_k = k$, $V_k^\perp = V_{n-k}$, and $\mathfrak{b}$ is one of the $V_k$. Let $\mathfrak{b}'$ be the stabilizer of this flag in $so(\mathfrak{g})$. This is a Borel subalgebra of $so(\mathfrak{g})$ containing $\mathfrak{b}$. Let $m \in M$ be a highest vector with respect to $\mathfrak{b}'$. Then $\mathbb{C}m$ is $\mathfrak{b}$-invariant and the corresponding character of $\mathfrak{b}$ equals one half of the sum of the positive roots, i.e., $\rho$. So $\chi = f(\rho)$. $\square$

*Remark.* According to Kostant (cf. the proof of Lemma 5.9 from [Ko61]) the $\mathfrak{g}$-module $M$ is isomorphic to the sum of $2^{[r/2]}$ copies of the irreducible $\mathfrak{g}$-module with highest wight $\rho$ (where $r$ is the rank of $\mathfrak{g}$).

**4.4.5.** Our construction of (211) slightly depends on the choice of a scalar product on $\mathfrak{g}$ (see 4.4.1). Since there are several "canonical" scalar products on $\mathfrak{g}$ the reader may prefer the following version of (211).

To simplify notation let us assume that $G$ is simple. Then the space of invariant symmetric bilinear forms on $\mathfrak{g}$ is 1-dimensional. Denote it by $\beta$. Choose a square root of $\beta$, i.e., a 1-dimensional vector space $\beta^{1/2}$ equipped with an isomorphism $\beta^{1/2} \otimes \beta^{1/2} \xrightarrow{\sim} \beta$. So $\mathfrak{g} \otimes \beta^{1/2}$ carries a *canonical* bilinear form. Consider the representation $G \to SO(\mathfrak{g} \otimes \beta^{1/2})$ and then proceed as in 4.4.1–4.4.3 (e.g., now the fiber of $\lambda'_{\mathcal{L}}$ over $\mathcal{F} \in \mathrm{Bun}_G$ equals $\mathrm{Pf}(\mathfrak{g}_{\mathcal{F}} \otimes \mathcal{L} \otimes \beta^{1/2}) \otimes \mathrm{Pf}(\mathfrak{g} \otimes \mathcal{L} \otimes \beta^{1/2})^{\otimes -1}$). The functor (211) thus obtained slightly depends on the choice of $\beta^{1/2}$. More precisely, $-1 \in \operatorname{Aut}\beta^{1/2}$ acts on $\lambda'_{\mathcal{L}}$ and therefore on $\lambda_{\mathcal{M}}$, $\mathcal{M} \in Z\operatorname{tors}_\theta(X)$, as multiplication by $(-1)^p$ where $p : \mathrm{Bun}_G \to \mathbb{Z}/2\mathbb{Z}$ is the composition

$$\mathrm{Bun}_G \to \pi_0(\mathrm{Bun}_G) = \pi_1(G) \to \pi_1(SO(\mathfrak{g})) = \mathbb{Z}/2\mathbb{Z}.$$

Do we want to consider $\lambda_{\mathcal{M}}$ as a SUPER-sheaf??!

**4.4.6.** We have associated to $\mathcal{L} \in Z\operatorname{tors}_\theta(X)$ a line bundle $\lambda_{\mathcal{L}}$ on $\mathrm{Bun}_G$ (see 4.4.1–4.4.3). For $x \in X$ denote by $\lambda_{\mathcal{L},\underline{x}}$ the pullback of $\lambda_{\mathcal{L}}$ to $\mathrm{Bun}_{G,\underline{x}}$. In 4.4.7–4.4.10 we will define a central extension $\widetilde{G(K_x)}_{\mathcal{L}}$ of $G(K_x)$ that acts on $\lambda_{\mathcal{L},\underline{x}}$. In 4.4.11–4.4.13 we consider the Lie algebra of $\widetilde{G(K_x)}_{\mathcal{L}}$.

**4.4.7.** Let $O$, $K$ and $\omega_O$ have the same meaning as in 4.3.3. Fix a square root $\mathcal{L}$ of $\omega_O$. Then we construct a central extension of group ind-schemes

$$0 \to \mathbb{G}_m \to \widetilde{G(K)}_{\mathcal{L}} \to G(K) \to 0 \tag{212}$$

as follows. $\mathcal{L}$ defines the central extension (196). Fix a non-degenerate invariant symmetric bilinear form*) on $\mathfrak{g}$ and write $SO_n$ instead of $SO(\mathfrak{g})$, $n := \dim\mathfrak{g}$. We define (212) to be the central extension of $G(K)$ *opposite* to the one induced from (196) via the adjoint representation $G \to SO(\mathfrak{g}) = SO_n$. The extension (212) splits over $G(O)$.

*Remark.* In the case $G = SO_r$ our notation is ambiguous: $\widetilde{G(K)} \neq \widetilde{SO_r(K)}$. Hopefully this ambiguity is harmless.

**4.4.8.** Let $\mathcal{L} \in \omega^{1/2}(X)$, $x \in X$. According to 4.4.7 the restriction of $\mathcal{L}$ to $\operatorname{Spec} O_x$ defines a central extension of $G(K_x)$, which will be denoted by $\widetilde{G(K_x)}_{\mathcal{L}}$. Denote by $\lambda'_{\mathcal{L},\underline{x}}$ the pullback to $\mathrm{Bun}_{G,\underline{x}}$ of the line bundle $\lambda'_{\mathcal{L}}$ from 4.4.1. It follows from 4.3.7 that the action of $G(K_x)$ on $\mathrm{Bun}_{G,\underline{x}}$ lifts to a canonical action of $\widetilde{G(K_x)}_{\mathcal{L}}$ on $\lambda'_{\mathcal{L}}$. The subgroup $\mathbb{G}_m \subset \widetilde{G(K_x)}_{\mathcal{L}}$ acts on $\lambda'_{\mathcal{L}}$ in the natural way (see the definition of $\widetilde{G(K_x)}_{\mathcal{L}}$ in 4.4.7 and the last

*)Instead of fixing the form on $\mathfrak{g}$ the reader can proceed as in 4.4.5.

sentence of 4.3.7). The action of $G(O_x) \subset \widetilde{G(K_x)}_{\mathcal{L}}$ on $\lambda'_{\mathcal{L},\underline{x}}$ is the obvious one.

**4.4.9.** In 4.4.7 we defined a functor

$$\omega^{1/2}(O) \to \{\text{central extensions of } G(K) \text{ by } \mathbb{G}_m\} \tag{213}$$

where $\omega^{1/2}(O)$ is the groupoid of square roots of $\omega_O$. The l.h.s. of (213) is a $\mu_2$-category in the sense of 3.4.4. The r.h.s. of (213) is a $Z$-category, $Z := \pi_1(G)^\vee = \mathrm{Hom}(\pi_1(G), \mathbb{G}_m)$. Indeed, the coboundary morphism*)

$$G(K) \to H^1(K, \pi_1^{\text{et}}(G)) = \pi_1(G) = Z^\vee \tag{214}$$

induces a morphism*)

$$Z \to \mathrm{Hom}(G(K), \mathbb{G}_m)\,, \tag{215}$$

i.e., a $Z$-structure on the r.h.s. of (213). Using the morphism $\mu_2 \to Z$ defined by (56) we consider the r.h.s. of (213) as a $\mu_2$-category. Then (213) is a $\mu_2$-functor (use 4.3.4, Remark (ii) from 4.3.4, and 4.4.4). So by 3.4.4 the functor (213) yields a $Z$-functor

$$Z\,\mathrm{tors}_\theta(O) \to \{\text{central extensions of } G(K) \text{ by } \mathbb{G}_m\}\,. \tag{216}$$

The central extension of $G(K)$ corresponding to $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$ by (213) will be denoted by $\widetilde{G(K)}_{\mathcal{L}}$. The extension

$$0 \to \mathbb{G}_m \to \widetilde{G(K)}_{\mathcal{L}} \to G(K) \to 0 \tag{217}$$

splits over $G(O)$.

*Remarks*

(i) According to 3.4.7 (i) the $Z$-structure on the r.h.s. of (213) yields a Picard functor

$$Z\,\mathrm{tors}(O) = Z\,\mathrm{tors} \to \{\text{central extensions of } G(K) \text{ by } \mathbb{G}_m\}\,. \tag{218}$$

*) A priori (214) is a morphism of abstract groups, but according to the Remark from 4.1.7 it is, in fact, a morphism of group ind-schemes. See also 4.5.4.

*) In fact, an isomorphism (see 4.5.4)

Explicitly, (218) is the composition of the canonical equivalence

$$\begin{aligned}&\{\text{trivial extensions of } Z^\vee \text{ by } \mathbb{G}_m\} = Z\,\text{tors}\\ &\text{an extension} \mapsto \text{the } Z\text{-torsor of its splittings}\end{aligned} \tag{219}$$

and the functor from the l.h.s. of (219) to the r.h.s. of (218) induced by (214). In other words, (218) is the functor $\mathcal{E} \mapsto \widetilde{G(K)}_{\mathcal{E}}$ from 4.1.8.

(ii) By 3.4.7 (iv) the functor (216) is affine with respect to the Picard functor (218).

**4.4.10.** Let $\mathcal{L} \in Z\,\text{tors}_\theta(X)$. According to 4.4.9 the image of $\mathcal{L}$ in $Z\,\text{tors}_\theta(O_x)$ defines a central extension of $G(K_x)$, which will be denoted by $\widetilde{G(K_x)}_{\mathcal{L}}$. Denote by $\lambda_{\mathcal{L},\underline{x}}$ the pullback of $\lambda_{\mathcal{L}}$ to $\text{Bun}_{G,\underline{x}}$. The action of $G(K_x)$ on $\text{Bun}_{G,\underline{x}}$ lifts to a canonical action of $\widetilde{G(K_x)}_{\mathcal{L}}$ on $\lambda_{\mathcal{L},\underline{x}}$ (use 4.3.7–4.3.9, 4.1.8, and the Remarks from 4.4.9). $G(O_x) \times \mathbb{G}_m \subset \widetilde{G(K_x)}_{\mathcal{L}}$ acts on $\lambda_{\mathcal{L},\underline{x}}$ in the obvious way.

**4.4.11.** *Proposition.* The Lie algebra extension corresponding to (217) is the extension

$$0 \to \mathbb{C} \to \widetilde{\mathfrak{g} \otimes K} \to \mathfrak{g} \otimes K \to 0$$

from 2.5.1.

*Proof.* The Lie algebra extension corresponding to (217) does not depend on $\mathcal{L} \in Z\,\text{tors}_\theta(O)$, so instead of (217) one can consider (212) and finally (194). So it is enough to use the Kac–Peterson–Frenkel theorem which says that the Lie algebra extension

$$0 \to \mathbb{C} \to \widetilde{o_n(K)} \to o_n(K) \to 0 \tag{220}$$

corresponding to (194) is defined by the cocycle $(u, v) \mapsto \frac{1}{2} \operatorname{Res} \operatorname{Tr}(du, v)$, $u, v \in o_n(K)$. In fact, to use [KP] or Proposition I.3.11 from [Fr81] one has to use the following characterization of $\widetilde{o_n(K)}$ (which does not involve the group $\widetilde{O_n(K)}$): let $V$ have the same meaning as in 4.3.3 and let $M$ be an irreducible discrete module over $\text{Cl}(V)$, then one has a representation of

$\widetilde{o_n(K)}$ in $M$ compatible with the action of $\widetilde{o_n(K)}$ on $\mathrm{Cl}(V)$ and such that $1 \in \mathbb{C} \subset \widetilde{o_n(K)}$ acts on $M$ identically. $\square$

**4.4.12.** Let $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{L},\underline{x}}$ have the same meaning as in 4.4.10. According to 4.4.10 and 4.4.11 the action of $\mathfrak{g} \otimes K_x$ on $\mathrm{Bun}_{G,\underline{x}}$ lifts to a canonical action of $\widetilde{\mathfrak{g} \otimes K_x}$ on $\lambda_{\mathcal{L},\underline{x}}$ whose restriction to $\mathbb{C} \times (\mathfrak{g} \otimes O_x) \subset \widetilde{\mathfrak{g} \otimes K_x}$ is the obvious one; in particular $\mathbf{1} \in \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K_x}$ acts as multiplication by 1.

$\lambda_{\mathcal{L}}$ is equipped with an isomorphism $\lambda_{\mathcal{L}}^{\otimes 2n} \xrightarrow{\sim} (\omega^{\sharp}_{\mathrm{Bun}_G})^{\otimes n}$ for some $n \neq 0$, so the sheaf of differential operators acting on $\lambda_{\mathcal{L}}$ is $D'$. Therefore according to 1.2.5 the action of $\widetilde{\mathfrak{g} \otimes K_x}$ on $\lambda_{\mathcal{L},\underline{x}}$ induces a canonical morphism

$$h_x : \mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G, D')\,.$$

Clearly $h_x$ does not depend on $\mathcal{L} \in Z\mathrm{tors}_\theta(X)$.

**4.4.13.** In this subsection we prove that the $h_x$ from 4.4.12 coincides with the $h_x$ from 2.5.4. The reader can skip this proof and simply forget the old definition of $h_x$ (it was introduced only to avoid the discussion of square roots of $\omega_{\mathrm{Bun}_G}$ in Section 2).

To prove that the two definitions of $h_x$ are equivalent it suffices to show that if $\mathcal{L}$ is a square root of $\omega_X$ then the isomorphism $\lambda_{\mathcal{L}}^{\otimes 2} \xrightarrow{\sim} \omega^{\sharp}_{\mathrm{Bun}_G}$ induces a $\widetilde{\mathfrak{g} \otimes K_x}$-equivariant isomorphism between their pullbacks to $\mathrm{Bun}_{G,\underline{x}}$. This can be proved directly, but in fact it cannot be otherwise. Indeed, the obstruction to $\widetilde{\mathfrak{g} \otimes K_x}$-equivariance is a 1-cocycle $\widetilde{\mathfrak{g} \otimes K_x} \to H^0(\mathrm{Bun}_{G,\underline{x}}, \mathcal{O})$. Since $\mathrm{Hom}(\widetilde{\mathfrak{g} \otimes K_x}, \mathbb{C}) = 0$ it is enough to show that every regular function $f$ on $\mathrm{Bun}_{G,\underline{x}}$ is locally constant. According to 2.3.1 $\mathrm{Bun}_{G,\underline{x}}$ is the inverse limit of $\mathrm{Bun}_{G,nx}$, $n \in \mathbb{N}$. Clearly $f$ comes from a regular function on $\mathrm{Bun}_{G,nx}$ for some $n$. So it suffices to prove the following lemma.

*Lemma.* Every regular function on $\mathrm{Bun}_{G,nx}$ is locally constant.

*Proof.* Choose $y \in X \backslash \{x\}$ and consider the scheme $M$ parametrizing $G$-bundles on $X$ trivialized over $nx$ and the formal neighbourhood of $y$ (here the divisor $nx$ is considered as a subscheme). $G(K_y)$ acts on $M$ and a regular function $f$ on $\mathrm{Bun}_{G,nx}$ is a $G(O_y)$-invariant element of $H^0(M, \mathcal{O}_M)$. Clearly $H^0(M, \mathcal{O}_M)$ is an integrable discrete $\mathfrak{g} \otimes K_y$-module. It is well known and very easy to prove that a $(\mathfrak{g} \otimes O_y)$-invariant element of such a module is $(\mathfrak{g} \otimes K_y)$-invariant. So $f$ is $(\mathfrak{g} \otimes K_y)$-invariant. Since the action of $\mathfrak{g} \otimes K_y$ on $M$ is (formally) transitive $f$ is locally constant. $\square$

*Remark.* The above lemma is well known. A standard way to prove it would be to represent $\mathrm{Bun}_{G,nx}$ as $\Gamma \backslash G(K_y)/G(O_y)$ for some $\Gamma \subset G(K_y)$ (see [La-So] for the case $n = 0$) and then to use the fact that a regular function on $G(K_y)/G(O_y)$ is locally constant.

**4.4.14.** Now we will finish the proof of the horizontality theorem 2.7.3 (see 2.8.3 – 2.8.5 for the beginning of the proof).

Let $M$ be the scheme over $X$ whose fiber over $x \in X$ is $\mathrm{Bun}_{G,\underline{x}}$. Fix $\mathcal{L} \in \omega^{1/2}(X)$ and $\mathcal{L}^{\mathrm{loc}} \in \omega^{1/2}(O)$ (i.e., $\mathcal{L}$ is a square root of $\omega_X$, $\mathcal{L}^{\mathrm{loc}}$ is a square root of $\omega_O$). Then one has the scheme $X_2^\wedge$ defined in 4.3.16. Denote by $M_2^\wedge$ the fiber product of $M$ and $X_2^\wedge$ over $X$. The semidirect product $\mathrm{Aut}_2 O \ltimes G(K)$ acts on $M_2^\wedge$ (cf. 4.3.16).

One has its central extension $\mathrm{Aut}_2 O \ltimes \widetilde{G(K)}$ where $\widetilde{G(K)}$ is the central extension (212) corresponding to $\mathcal{L}^{\mathrm{loc}}$ and $\mathrm{Aut}_2 O = \mathrm{Aut}(O, \mathcal{L}^{\mathrm{loc}})$ acts on $\widetilde{G(K)} = \widetilde{G(K)}_{\mathcal{L}^{\mathrm{loc}}}$ by transport of structure. Denote by $\lambda_{\mathcal{L}}^\wedge$ the pullback to $M_2^\wedge$ of the Pfaffian line bundle $\lambda'_{\mathcal{L}}$ from 4.4.1. Since $\mathrm{Aut}_2 O$ acts on $M_2^\wedge$ as on a scheme over $\mathrm{Bun}_G$ one gets the action of $\mathrm{Aut}_2 O$ on $\lambda_{\mathcal{L}}^\wedge$. On the other hand, $\widetilde{G(K)}$ acts on $\lambda_{\mathcal{L},\widehat{x}}^\wedge$ :=the restriction of $\lambda_{\mathcal{L}}^\wedge$ to the fiber of $M_2^\wedge$ over $\widehat{x} \in X_2^\wedge$. Indeed, this fiber equals $\mathrm{Bun}_{G,\underline{x}}$ where $x$ is the image of $\widehat{x}$ in $X$, and by 4.4.8 the central extension $\widetilde{G(K_x)}_{\mathcal{L}}$ acts on $\lambda'_{\mathcal{L},\underline{x}} = \lambda_{\mathcal{L},\widehat{x}}^\wedge$. This extension depends only on $\mathcal{L}_x$ :=the pullback of $\mathcal{L}$ to $\operatorname{Spec} O_x$. Since $\widehat{x}$ defines an isomorphism $(O_x, \mathcal{L}_x) \xrightarrow{\sim} (O, \mathcal{L}^{\mathrm{loc}})$ we get an isomorphism $\widetilde{G(K_x)}_{\mathcal{L}} \xrightarrow{\sim} \widetilde{G(K)}$ and therefore an action of $\widetilde{G(K)}$ on $\lambda_{\mathcal{L},\widehat{x}}^\wedge$. As explained in 2.8.5, to finish the proof of 2.7.3 it suffices to show that

i) the action of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L},\widehat{x}}$ corresponding to various $\widehat{x}\in X^\wedge_2$ come from an (obviously unique) action of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}}$,

ii) this action is compatible with that of $\operatorname{Aut}_2 O$ (i.e., we have, in fact, an action of $\operatorname{Aut}_2 O\ltimes\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}}$).

This follows immediately from 4.3.17.

**4.4.15.** In this subsection and the following one we formulate and prove a generalization of statements i) and ii) from 4.4.14, which will be used in the proof of the main result of this work (Theorem 5.4.5). The generalization is obvious ($\omega^{1/2}(X)$ is replaced by $Z\operatorname{tors}_\theta(X)$, etc.), and the reader can certainly skip these subsections for the moment.

Fix $\mathcal{L}\in Z\operatorname{tors}_\theta(X)$ and $\mathcal{L}^{\mathrm{loc}}\in Z\operatorname{tors}_\theta(O)$. Denote by $X^\wedge_Z$ the etale $Z$-covering of $X^\wedge$ such that the preimage in $X^\wedge_Z(R)$ of a point of $X^\wedge(R)$ corresponding to a morphism $\alpha:\operatorname{Spec}(R\widehat{\otimes}O)\to X$ is the set of isomorphisms $\mathcal{L}^{\mathrm{loc}}_R\xrightarrow{\sim}\alpha^*\mathcal{L}$ in the groupoid*) $Z\operatorname{tors}_\theta(R\widehat{\otimes}O)$, where $\mathcal{L}^{\mathrm{loc}}_R$ is the pullback of $\mathcal{L}^{\mathrm{loc}}$ to $\operatorname{Spec}R\widehat{\otimes}O$. The group ind-scheme $\operatorname{Aut}_Z O=\operatorname{Aut}(O,\mathcal{L}^{\mathrm{loc}})$ from 4.6.6 acts on $X^\wedge_Z$ by transport of structure. Denote by $M^\wedge_Z$ the fiber product of $M$ and $X^\wedge_Z$ over $X$. Let $\lambda^\wedge_{\mathcal{L}}$ denote the pullback to $M^\wedge_Z$ of the line bundle $\lambda_{\mathcal{L}}$ defined in 4.4.3. The semidirect product $\operatorname{Aut}_Z O\ltimes G(K)$ acts on $M^\wedge_Z$. One has its central extension $\operatorname{Aut}_Z O\ltimes\widetilde{G(K)}$, where $\widetilde{G(K)}$ is the central extension (217) corresponding to $\mathcal{L}^{\mathrm{loc}}$ and $\operatorname{Aut}_Z O=\operatorname{Aut}(O,\mathcal{L}^{\mathrm{loc}})$ acts on $\widetilde{G(K)}=\widetilde{G(K)}_{\mathcal{L}^{\mathrm{loc}}}$ by transport of structure. Let us lift the action of $\operatorname{Aut}_Z O\ltimes G(K)$ on $M^\wedge_Z$ to an action of $\operatorname{Aut}_Z O\ltimes\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}}$.

Just as in 4.4.14 one defines the action of $\operatorname{Aut}_Z O$ on $\lambda^\wedge_{\mathcal{L}}$ and the action of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L},\widehat{x}}$ :=the restriction of $\lambda^\wedge_{\mathcal{L}}$ to the fiber of $M^\wedge_Z$ over $\widehat{x}\in\widehat{X}_Z$.

**4.4.16.** *Proposition.*

(i) The actions of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L},\widehat{x}}$ corresponding to various $\widehat{x}\in X^\wedge_Z$ come from an (obviously unique) action of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}}$.

*)Here it is convenient to use the definition $Z\operatorname{tors}_\theta$ from 3.4.5

(ii) The actions of $\mathrm{Aut}_Z O$ and $\widetilde{G(K)}$ on $\lambda_{\mathcal{L}}^{\wedge}$ define an action of $\mathrm{Aut}_Z O \ltimes \widetilde{G(K)}$.

*Proof.* Represent $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ as $\mathcal{L} = \mathcal{E} \cdot \mathcal{L}_0$, $\mathcal{E} \in Z\,\mathrm{tors}(X)$, $\mathcal{L}_0 \in \omega^{1/2}(X)$. By definition, $\lambda_{\mathcal{L}} = l_{\mathcal{E}} \otimes \lambda'_{\mathcal{L}_0}$ (see 4.1.4 or 4.1.6 for the definition of the $\mu_\infty$-torsor $l_{\mathcal{E}}$ on $\mathrm{Bun}_G$).

Consider $\mathcal{L}^{\mathrm{loc}}$ as an object of $\omega^{1/2}(O)$ (this is possible because both $Z\,\mathrm{tors}_\theta(O)$ and $\omega^{1/2}(O)$ have one and only one isomorphism class of objects). Using $\mathcal{L}_0$ and $\mathcal{L}^{\mathrm{loc}}$ construct $X_2^{\wedge}$, $M_2^{\wedge}$, and $\lambda_{\mathcal{L}_0}^{\wedge}$ (see 4.4.14).

Consider $\mathcal{E}$ as a $Z$-covering $\mathcal{E} \to X$. Set $X_{\mathcal{E}}^{\wedge} := \mathcal{E} \times_X X^{\wedge}$, $M_{\mathcal{E}}^{\wedge} := \mathcal{E} \times_X M^{\wedge}$, where $X^{\wedge}$ and $M^{\wedge}$ have the same meaning as in 2.6.5 and 2.8.3. Denote by $l_{\mathcal{E}}^{\wedge}$ the pullback of $l_{\mathcal{E}}$ to $M_{\mathcal{E}}^{\wedge}$.

Set $M_{\mathcal{E},2}^{\wedge} := \mathcal{E} \times_X M_2^{\wedge}$. One has the etale coverings $M_{\mathcal{E},2}^{\wedge} \to M_2^{\wedge}$, $M_{\mathcal{E},2}^{\wedge} \to M_{\mathcal{E}}^{\wedge}$, and $p : M_{\mathcal{E},2}^{\wedge} \to M_Z^{\wedge}$. Clearly $p^*\lambda_{\mathcal{L}}^{\wedge}$ is the tensor product of the pullbacks of $l_{\mathcal{E}}^{\wedge}$ and $\lambda_{\mathcal{L}_0}^{\wedge}$ to $M_{\mathcal{E},2}^{\wedge}$. Now consider $l_{\mathcal{E}}^{\wedge}$ and $\lambda_{\mathcal{L}_0}^{\wedge}$ separately.

The semidirect product $\mathrm{Aut}\,O \ltimes G(K)$ acts on $M_{\mathcal{E}}^{\wedge}$, and the action of $\mathrm{Aut}\,O$ on $M_{\mathcal{E}}^{\wedge}$ lifts canonically to its action on $l_{\mathcal{E}}^{\wedge}$ (cf. 4.4.14 or 2.8.5). $G(K)$ acts on the restriction of $l_{\mathcal{E}}^{\wedge}$ to the fiber over each point of $X_{\mathcal{E}}^{\wedge}$ (see 4.1.7). It is easy to see that these actions come from an action of $\mathrm{Aut}\,O \ltimes G(K)$ on $l_{\mathcal{E}}^{\wedge}$. On the other hand, by 4.4.14 we have a canonical action of $\mathrm{Aut}_2\,O \ltimes \widetilde{G(K)}$ on $\lambda_{\mathcal{L}_0}^{\wedge}$.

So we get an action of $\mathrm{Aut}_2\,O \ltimes \widetilde{G(K)}$ on $p^*\lambda_{\mathcal{L}}^{\wedge}$, which is compatible with the action of $\mathrm{Aut}_2\,O$ on $\lambda_{\mathcal{L}}^{\wedge}$ and with the action of $\widetilde{G(K)}$ on $\lambda_{\mathcal{L},\widehat{x}}^{\wedge}$, $\widehat{x} \in X_Z^{\wedge}$. Since $p$ is etale and surjective the action of $\mathrm{Aut}_2\,O \ltimes \widetilde{G(K)}$ on $p^*\lambda_{\mathcal{L}}^{\wedge}$ descends to an action of $\mathrm{Aut}_2\,O \ltimes \widetilde{G(K)}$ on $\lambda_{\mathcal{L}}^{\wedge}$. Since $\mathrm{Aut}_Z\,O$ is generated by $\mathrm{Aut}_2\,O$ and $Z$ it remains to show that the action of $Z \subset \mathrm{Aut}_Z\,O$ on $\lambda_{\mathcal{L}}^{\wedge}$ is compatible with that of $\widetilde{G(K)}$. This is clear because the actions of $Z$ and $\widetilde{G(K)}$ on $\lambda_{\mathcal{L},\widehat{x}}^{\wedge}$ are compatible for every $\widehat{x} \in X_Z^{\wedge}$. $\square$

**4.5. The affine Grassmannian.** The *affine Grassmannian* $\mathcal{GR}$ is the fpqc quotient $G(K)/G(O)$ where $O = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$. In this section we recall some basic properties of $\mathcal{GR}$. In 4.6 we construct and investigate the *local Pfaffian bundle*; this is a line bundle on $\mathcal{GR}$.

The affine Grassmannian will play an essential role in the proof of our main theorem 5.2.6. However the reader can skip this section for the moment.

In 4.5.1 – ? $G$ denotes an arbitrary connected affine algebraic group. Connectedness is a harmless assumption because $G(K)/G(O) = G^0(K)/G^0(O)$ where $G^0$ is the connected component of $G$.

**4.5.1.** *Theorem.*

(i) The fpqc quotient $G(K)/G(O)$ is an ind-scheme of ind-finite type.

(ii) $G(K)/G(O)$ is formally smooth.*)

(iii) The projection $p: G(K) \to G(K)/G(O)$ admits a section locally for the Zariski topology.

(iv) $G(K)/G(O)$ is ind-proper if and only if $G$ is reductive.

(v) $G(K)$, or equivalently $G(K)/G(O)$, is reduced if and only if $\mathrm{Hom}(G, \mathbb{G}_m) = 0$.

*Remark.* The theorem is well known. The essential part of the proof given below consists of references to works by Faltings, Beauville, Laszlo, and Sorger.

*Proof.* (i) and (iv) hold for $G = GL_n$. Indeed, there is an ind-proper ind-scheme $Gr(K^n)$ parametrizing c-lattices in $K^n$ (see 7.11.2(iii) for details). $GL_n(K)/GL_n(O)$ is identified with the closed sub-ind-scheme of $Gr(K^n)$ parametrizing $O$-invariant c-lattices. To prove (i) and (iv) for any $G$ we need the following lemma.

*Lemma.* Let $G_1 \subset G_2$ be affine algebraic groups such that the quotient $U := G_1 \setminus G_2$ is quasiaffine, i.e., $U$ is an open subscheme of an affine scheme

*)The definition of formal smoothness can be found in 7.11.1.

$Z$. Suppose that the fpqc quotient $G_2(K)/G_2(O)$ is an ind-scheme of ind-finite type. Then this also holds for $G_1(K)/G_1(O)$ and the morphism

$$G_1(K)/G_1(O) \to G_2(K)/G_2(O) \tag{221}$$

is a locally closed embedding. If $U$ is affine then (221) is a closed embedding.

The reader can easily prove the lemma using the global interpretation of $G(K)/G(O)$ from 4.5.2. We prefer to give a local proof.

*Proof.* Consider the morphism $f : G_1(K) \to Z(K)$. Clearly $Z(O)$ is a closed subscheme of $Z(K)$, and $U(O)$ is an open subscheme of $Z(O)$. So $Y := f^{-1}(U(O))$ is a locally closed sub-ind-scheme of $G_2(K)$; it is closed if $U$ is affine. Clearly $Y \cdot G_2(O) = Y$, so $Y$ is the preimage of a locally closed sub-ind-scheme $Y' \subset G_2(K)/G_2(O)$; if $U$ is affine then $Y'$ is closed. Since $G_1(K) \subset Y$ we have a natural morphism

$$G_1(K) \to Y'\,. \tag{222}$$

We claim that (222) is a $G_1(O)$-torsor ($G_1(O)$ acts on $G_1(K)$ by right translations) and therefore $G_1(K)/G_1(O) = Y'$. To see that (222) is a $G_1(O)$-torsor notice that the morphism $Y \to Y'$ is a $G_2(O)$-torsor, the morphism $\varphi : Y \to U(O) = G_1(O) \setminus G_2(O)$ is $G_2(O)$-equivariant, and $G_1(K) = \varphi^{-1}(\bar{e})$ where $\bar{e} \in G_1(O) \setminus G_2(O)$ is the image of $e \in G_2(O)$. $\square$

Let us prove (i) and (iv) for any $G$. Choose an embedding $G \hookrightarrow GL_n$. If $G$ is reductive then $GL_n/G$ is affine, so the lemma shows that $G(K)/G(O)$ is an ind-proper ind-scheme. For any $G$ we will construct an embedding $i : G \hookrightarrow G' := GL_n \times \mathbb{G}_m$ such that $G'/i(G)$ is quasiaffine; this will imply (i). To construct $i$ take a $GL_n$-module $V$ such that $G \subset GL_n$ is the stabilizer of some 1-dimensional subspace $l \subset V$. The action of $G$ in $l$ is defined by some $\chi : G \to \mathbb{G}_m$. Define $i : G \hookrightarrow G' := GL_n \times \mathbb{G}_m$ by $i(g) = (g, \chi(g)^{-1})$. To show that $G'/i(G)$ is quasiaffine consider $V$ as a $G'$-module ($\lambda \in \mathbb{G}_m$ acts as multiplication by $\lambda$) and notice that the stabilizer of a nonzero $v \in l$ in $G'$ equals $i(G)$. So $G'/i(G) \simeq G'v$ and $G'v$ is quasiaffine.

Let us finish the proof of (iv). If $G(K)/G(O)$ is ind-proper and $G'$ is a normal subgroup of $G$ then according to the lemma $G'(K)/G'(O)$ is also ind-proper. Clearly $\mathbb{G}_a(K)/\mathbb{G}_a(O)$ is not ind-proper. Therefore $G(K)/G(O)$ is ind-proper only if $G$ is reductive.

To prove (iii) it suffices to show that $p: G(K) \to G(K)/G(O)$ admits a section over a neighbourhood of any $\mathbb{C}$-point $x \in G(K)/G(O)$ (here we use that $\mathbb{C}$-points are dense in $G(K)/G(O)$ by virtue of (i)). Since $p$ is $G(K)$-equivariant we are reduced to the case where $x$ is the image of $e \in G(K)$. So one has to construct a sub-ind-scheme $\Gamma \subset G(K)$ containing $e$ such that the morphism

$$\Gamma \times G(O) \to G(K)\,, \quad (\gamma, g) \mapsto \gamma g \tag{223}$$

is an open immersion. According to Faltings [Fal94, p.350–351] the morphism (223) is an open immersion if the set of $R$-point of $\Gamma$ is defined by

$$\Gamma(R) = \operatorname{Ker}(G(R[t^{-1}]) \xrightarrow{f} G(R)) \subset G(R((t))) = G(R\widehat{\otimes}K)$$

where $f$ is evaluation at $t = \infty$. The proof of this statement is due to Beauville and Laszlo (Proposition 1.11 from [BLa94]). It is based on the global interpretation of $G(K)/G(O)$ in terms of $X = \mathbb{P}^1$ (see 4.5.2) and on the following property of $G$-bundles on $\mathbb{P}^1$: for a $G$-bundle $\mathcal{F}$ on $S \times \mathbb{P}^1$ the points $s \in S$ such that the restriction of $\mathcal{F}$ to $s \times \mathbb{P}^1$ is trivial form an *open* subset of $S$ (indeed, $H^1(\mathbb{P}^1, \mathcal{O} \otimes \mathfrak{g}) = 0$, $\mathfrak{g} := \operatorname{Lie} G$).

Let us deduce*) (ii) from (iii). Since $G(K)$ is formally smooth it follows from (iii) that each point of $G(K)/G(O)$ has a formally smooth neighbourhood. Since $G(K)/G(O)$ is of ind-finite type this implies (ii).

It remains to consider (v). $G(O)$ is reduced. So $G(K)$ is reduced if and only if $G(K)/G(O)$ is reduced. Laszlo and Sorger prove that if $\operatorname{Hom}(G, \mathbb{G}_m) = 0$ then $G(K)/G(O)$ is reduced (see the proof of Proposition 4.6 from [La-So]); their proof is based on a theorem of

*)In fact, one can prove (ii) without using (iii).

Shafarevich. If $\mathrm{Hom}(G, \mathbb{G}_m) \neq 0$ there exist morphisms $f : \mathbb{G}_m \to G$ and $\chi : G \to \mathbb{G}_m$ such that $\chi f = \varphi_n$, $n \neq 0$, where $\varphi_n(\lambda) := \lambda^n$. The image of the morphism $\mathbb{G}_m(K) \to \mathbb{G}_m(K)$ induced by $\varphi_n$ is not contained in $\mathbb{G}_m(K)_{\mathrm{red}}$, so $G(K)$ is not reduced. $\square$

**4.5.2.** Let $X$ be a connected smooth projective curve over $\mathbb{C}$, $x \in X(\mathbb{C})$, $O_x$ the completed local ring of $x$, and $K_x$ its field of fractions. Then according to Beauville – Laszlo (see 2.3.4) the fpqc quotient $G(K_x)/G(O_x)$ can be interpreted as the moduli space of pairs $(\mathcal{F}, \gamma)$ consisting of a principal $G$-bundle $\mathcal{F}$ on $X$ and its section (=trivialization) $\gamma : X \setminus \{x\} \to \mathcal{F}$: to $(\mathcal{F}, \gamma)$ one assigns the image of $\gamma/\gamma_x$ in $G(K_x)/G(O_x)$ where $\gamma_x$ is a section of $\mathcal{F}$ over $\operatorname{Spec} O_x$ and $\gamma/\gamma_x$ denotes the element $g \in G(K_x)$ such that $\gamma = g\gamma_x$ (we have identified $G(K_x)/G(O_x)$ with the moduli space of pairs $(\mathcal{F}, \gamma)$ at the level of $\mathbb{C}$-points; the readers can easily do it for $R$-points where $R$ is any $\mathbb{C}$-algebra).

**4.5.3.** Let us recall the algebraic definition of the topological fundamental group of $G$. Denote by $\pi_1^{\mathrm{et}}(G)$ the fundamental group of $G$ in Grothendieck's sense. A character $f : G \to \mathbb{G}_m$ induces a morphism $\pi_1^{\mathrm{et}}(G) \to \pi_1^{\mathrm{et}}(\mathbb{G}_m) = \widehat{\mathbb{Z}}(1)$ and therefore a morphism $f_* : (\pi_1^{\mathrm{et}}(G))(-1) \to \widehat{\mathbb{Z}}$. Denote by $\pi_1(G)$ the set of $\alpha \in (\pi_1^{\mathrm{et}}(G))(-1)$ such that $f_*(\alpha) \in \mathbb{Z}$ for all $f \in \mathrm{Hom}(G, \mathbb{G}_m)$. We consider $\pi_1(G)$ as a discrete group. In fact, $\pi_1(G)$ does not change if $G$ is replaced by its maximal reductive quotient. For reductive $G$ one identifies $\pi_1(G)$ with the quotient of the group of coweights of $G$ modulo the coroot lattice.

For any finite covering $p : \widetilde{G} \to G$ one has the coboundary map $G(K) \to H^1(K, A) = A(-1)$, $A := \operatorname{Ker} p$. These maps yield a homomorphism $G(K) \to (\pi_1^{\mathrm{et}}(G))(-1)$. Its image is contained in $\pi_1(G)$. So we have constructed a canonical homomorphism

$$\varphi : G(K) \to \pi_1(G) \tag{224}$$

where $G(K)$ is understood in the naive sense (i.e., as the group of $K$-points of $G$ or as the group of $\mathbb{C}$-points of the ind-scheme $G(K)$). The restriction of (224) to $G(O)$ is trivial, so (224) induces a map

$$G(K)/G(O) \to \pi_1(G) \tag{225}$$

where $G(K)/G(O)$ is also understood in the naive sense.

Now consider $G(K)$ and $G(K)/G(O)$ as ind-schemes. The set of $\mathbb{C}$-points of $G(K)/G(O)$ is dense in $G(K)/G(O)$, and the same is true for $G(K)$.

**4.5.4.** *Proposition.*

(i) The maps (224) and (225) are locally constant.

(ii) The corresponding maps

$$\pi_0(G(K)) \to \pi_1(G) \tag{226}$$

$$\pi_0(G(K)/G(O)) \to \pi_1(G) \tag{227}$$

are bijective.

*Proof.* We already proved (i) using a global argument (see the Remark at the end of 4.1.7). The same argument can be reformulated using the interpretation of $G(K_x)/G(O_x)$ from 4.5.2: the map (225) equals minus the composition of the natural map $G(K_x)/G(O_x) \to \mathrm{Bun}_G$ and the "first Chern class" map $c: \pi_0(\mathrm{Bun}_G) \to \pi_1(G)$. For a local proof of (i) see 4.5.5.

Now let us prove (ii). The map $\pi_0(G(K)) \to \pi_0(G(K)/G(O))$ is bijective (because $G$ is connected). So it suffices to consider (226). Since $G$ can be represented as a semi-direct product of a reductive group and a unipotent group we can assume that $G$ is reductive. Fix a Cartan subgroup $H \subset G$. We have $\pi_0(H(K)) = \pi_1(H)$ and the composition $\pi_0(H(K)) \to \pi_0(G(K)) \to \pi_1(G)$ is the natural map $\pi_1(H) \to \pi_1(G)$, which is surjective. So (226) is also surjective. The map $\pi_0(H(K)) \to \pi_0(G(K))$ is surjective (use the Bruhat decomposition for the abstract group $G(K)$). Therefore to prove the injectivity of (226) it suffices to show that the kernel of the natural morphism

$f : \pi_0(H(K)) \to \pi_1(G)$ is contained in the kernel of $\pi_0(H(K)) \to \pi_0(G(K))$. Since $\operatorname{Ker} f$ is the coroot lattice it is enough to prove that for any coroot $\gamma : \mathbb{G}_m \to H$ the image of $\mathbb{G}_m(K)$ in $G(K)$ belongs to the connected component of $e \in G(K)$. A coroot $\mathbb{G}_m \to H$ extends to a morphism $SL(2) \to G$, so it suffices to notice that $SL(2, K)$ is connected (because any matrix from $SL(2, K)$ can be represented as a product of unipotent matrices). $\square$

In the next subsection we give a local proof of 4.5.4(i).

**4.5.5.** *Lemma.* Let $M = \operatorname{Spec} R$ be a connected affine variety, $A$ a finite abelian group, $\alpha \in H^1_{\mathrm{et}}(\operatorname{Spec} R((t)), A)$. For $x \in M(\mathbb{C})$ denote by $\alpha(x)$ the restriction of $\alpha$ to the fiber of $\operatorname{Spec} R((t)) \to \operatorname{Spec} R$ over $x$, so $\alpha(x) \in H^1_{\mathrm{et}}(\operatorname{Spec} \mathbb{C}((t)), A) = A(-1)$. Then $\alpha(x) \in A(-1)$ does not depend on $x$.

*Proof.* It suffices to show that for any smooth connected $M'$ and any morphism $M' \to M$ the pullback of $\alpha$ to $M'(\mathbb{C})$ is constant*). So we can assume that $M$ is smooth. Set $V := \operatorname{Spec} R[[t]]$, $V' := \operatorname{Spec} R((t))$. We can assume that $A = \mu_n$. Then $\alpha$ corresponds to a $\mu_n$-torsor on $V'$, i.e., a line bundle $\mathcal{A}$ on $V'$ equipped with an isomorphism $\psi : \mathcal{A}^{\otimes n} \xrightarrow{\sim} \mathcal{O}_{V'}$. Since $V$ is regular $\mathcal{A}$ extends to a line bundle $\widetilde{\mathcal{A}}$ on $V$. Then $\psi$ induces an isomorphism $\widetilde{\mathcal{A}}^{\otimes n} \xrightarrow{\sim} t^k \mathcal{O}_V$ for some $k \in \mathbb{Z}$. Clearly $\alpha(x) \in \mathbb{Z}/n\mathbb{Z}$ is the image of $k$. $\square$

Here is a local proof of 4.5.4(i). Since $G(K)/G(O)$ is of ind-finite type it suffices to prove that for every connected affine variety $M = \operatorname{Spec} R$ and any morphism $f : M \to G(K)$ the composition $M(\mathbb{C}) \to G(K) \to \pi_1(G)$ is constant. For any finite abelian group $A$ an exact sequence $0 \to A \to \widetilde{G} \to G \to 0$ defines a map $\pi_1(G) \to A(-1)$ and it is enough to show that the composition $M(\mathbb{C}) \to G(K) \to \pi_1(G) \to A(-1)$ is constant. To prove this apply the lemma to $\alpha = \varphi^*\beta$ where $\varphi : \operatorname{Spec} R((t)) \to G$ corresponds

*)In fact, it is enough to consider only those $M'$ that are smooth curves.

to $f : \operatorname{Spec} R \to G(K)$ and $\beta \in H^1_{\text{et}}(G, A)$ is the class of $\widetilde{G}$ considered as an $A$-torsor on $G$.

*Remark.* In fact, one can prove that for every affine scheme $M = \operatorname{Spec} R$ over $\mathbb{C}$ the "Künneth morphism"

$$H^1_{\text{et}}(M, A) \oplus H^0(M, \mathbb{Z}) \otimes H^1_{\text{et}}(\operatorname{Spec} \mathbb{C}((t)), A) \to H^1_{\text{et}}(M((t)), A), \qquad M((t)) := \operatorname{Spec} R((t)), \tag{228}$$

is an isomorphism (clearly this implies the lemma). A similar statement holds for any ring $R$ such that the order of $A$ is invertible in $R$.

**4.5.6.** *Proposition.* Let $A \subset G$ be a finite central subgroup, $G' := G/A$.

(i) The morphism $G(K)/G(O) \to G'(K)/G'(O)$ induces an isomorphism between $G(K)/G(O)$ and the union of some connected components of $G'(K)/G'(O)$.

(ii) The morphism $G(K) \to G'(K)$ is an etale covering.

*Remark.* By 4.5.4 the components mentioned in (i) are labeled by elements of $\operatorname{Im}(\pi_1(G) \to \pi_1(G'))$. The same is true for the connected components of the image of $G(K)$ in $G'(K)$.

*Proof.* Clearly (i) and (ii) are equivalent.

Let us prove (i) under the assumption of semisimplicity of $G$ (which is equivalent to semisimplicity of $G'$). In this case the morphism $f : G(K)/G(O) \to G'(K)/G'(O)$ is ind-proper by 4.5.1(iv). By 4.5.4(i) the fibers of $f$ over geometric points*) of components $C \subset G'(K)/G'(O)$ such that $f^{-1}(C) \neq \emptyset$ contain exactly one point, and it is easy to see that these fibers are reduced. By 4.5.1(v) $G'(K)/G'(O)$ is reduced. So in the semisimple case (i) is clear.

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*) The statement for $\mathbb{C}$-points follows immediately from 4.5.4(i). Since 4.5.4 remains valid if $\mathbb{C}$ is replaced by an algebraically closed field $E \supset \mathbb{C}$ the statement is true for $E$-points as well.

Now let us reduce the proof of (ii) to the semisimple case. We can assume that $A$ is cyclic. It suffices to construct a morphism $\rho$ from $G$ to a semisimple group $G_1$ such that $\rho|_A$ is injective and $\rho(A) \subset G_1$ is central (then the morphism $G(K) \to G'(K)$ is obtained by base change from $G_1(K) \to G_1'(K)$, $G_1' := G_1/\rho(A)$). To construct $G_1$ and $\rho$ one can proceed as follows. Fix an isomorphism $\chi : A \xrightarrow{\sim} \mu_n$. Let $V$ be a finite-dimensional $G$-module such that $Z$ acts on $V$ via $\chi$. Denote by $W_{pq}$ the direct sum of $p$ copies of $V$ and $q$ copies of $\mathrm{Sym}^{n-1} V^*$. If $p \cdot \dim V = q(n-1) \cdot \dim \mathrm{Sym}^{n-1} V$ then one can set $G_1 := SL(W_{pq})$ (indeed, the image of $GL(V)$ in $GL(W_{pq})$ is contained in $SL(W_{pq})$. □

*Remarks*

(i) Proposition 4.5.6 is an immediate consequence of the bijectivity of (228).

(ii) It is easy to prove Proposition 4.5.6 using the global interpretation of $G(K)/G(O)$ from 4.5.2.

**4.5.7.** Suppose that $G$ is reductive. Denote by $G_{\mathrm{ad}}$ the quotient of $G$ by its center. Set $T := G/[G,G]$, $G' := G_{\mathrm{ad}} \times T$. Then $G' = G/A$ for some finite central subgroup $A \subset G$. So by 4.5.6 $G(K)/G(O)$ can be identified with the union of certain connected components of $G'(K)/G'(O) = G_{\mathrm{ad}}(K)/G_{ad}(O) \times T(K)/T(O)$.

The structure of $T(K)/T(O)$ is rather simple. For instance, the reduced part of $\mathbb{G}_m(K)/\mathbb{G}_m(O)$ is the discrete space $\mathbb{Z}$ and the connected component of $1 \in \mathbb{G}_m(K)/\mathbb{G}_m(O)$ is the formal group with Lie algebra $K/O$.

**4.5.8.** From now on we assume that $G$ is reductive and set $\mathcal{GR} := G(K)/G(O)$.

Recall that $G(O)$-orbits in $\mathcal{GR}$ are labeled by dominant coweights of $G$ or, which is the same, by $P_+(^LG) :=$ the set of dominant weights of $^LG$. More precisely, $\chi \in P_+(^LG)$ defines a conjugacy class of morphisms $\nu : \mathbb{G}_m \to G$ and, by definition, $\mathrm{Orb}_\chi$ is the $G(O)$-orbit of the image of $\nu(\pi)$ in $\mathcal{GR}$ where

$\pi$ is a prime element of $O$ (this image does not depend on the choice of $\pi$). Clearly $\mathrm{Orb}_\chi$ does not depend on the choice of $\nu$ inside the conjugacy class, so $\mathrm{Orb}_\chi$ is well-defined. According to [IM] the map $\chi \mapsto \mathrm{Orb}_\chi$ is a bijection between $P_+({}^L G)$ and the set of $G(O)$-orbits in $\mathcal{GR}$. It is easy to show that

$$\dim \mathrm{Orb}_\chi = (\chi, 2\rho) \tag{229}$$

where $2\rho$ is the sum of positive roots of $G$.

*Remark.* Clearly $\mathrm{Orb}_\chi$ is $\mathrm{Aut}^0 O$-invariant.

**4.5.9.** We have the bijection (227) between $\pi_0(\mathcal{GR})$ and $\pi_1(G)$. Let $Z$ be the center of the Langlands dual group ${}^L G$. We identify $\pi_1(G)$ with $Z^\vee := \mathrm{Hom}(Z, \mathbb{G}_m)$ using the duality between the Cartan tori of $G$ and ${}^L G$. So the connected components of $\mathcal{GR}$ are labeled by elements of $Z^\vee$.

*Remark.* The connected component of $\mathcal{GR}$ containing $\mathrm{Orb}_\chi$ corresponds to $\chi_Z \in Z^\vee$ where $\chi_Z$ is the restriction of $\chi \in P^+({}^L G)$ to $Z$.

**4.5.10.** There is a canonical morphism $\alpha : \mu_2 \to Z$. If $G$ is semisimple we have already defined it by (56). If $G$ is reductive this gives us a morphism $\mu_2 \to Z'$ where $Z'$ is the center of the commutant of ${}^L G$; then we define $\alpha$ to be the composition $\mu_2 \to Z' \hookrightarrow Z$.

According to 4.4.4 the dual morphism $\alpha^\vee : \pi_1(G) \to \mathbb{Z}/2\mathbb{Z}$ is the morphism of fundamental groups that comes from the adjoint representation $G \to SO(\mathfrak{g}_{ss})$, $\mathfrak{g}_{ss} := [\mathfrak{g}, \mathfrak{g}]$.

The composition of (227) and $\alpha^\vee$ defines a locally constant *parity function*

$$p : \mathcal{GR} \to \mathbb{Z}/2\mathbb{Z}\,. \tag{230}$$

We say that a connected component of $\mathcal{GR}$ is *even* (resp. *odd*) if (230) maps it to 0 (resp. 1).

**4.5.11.** *Proposition.* All the $G(O)$-orbits of an even (resp. odd) component of $\mathcal{GR}$ have even (resp. odd) dimension.

*Proof.* Let $x = gG(O) \in \mathcal{GR}$. Using the relation between $\alpha^\vee$ and the adjoint representation (see 4.5.10) as well as Remarks (ii) and (iii) from 4.3.4 we see that $x$ belongs to an even component of $\mathcal{GR}$ if and only if

$$\dim \mathfrak{g}_{ss} \otimes O / \big( (\mathfrak{g}_{ss} \otimes O) \cap \mathrm{Ad}_g(\mathfrak{g}_{ss} \otimes O) \big) \tag{231}$$

is even. But (231) is the dimension of the $G(O)$-orbit of $x$. □

Here is another proof. Using (229) and the Remark from 4.5.9 we see that the proposition is equivalent to the formula $\chi_Z(\alpha(-1)) = (-1)^{\langle \chi, 2\rho \rangle}$, which is obvious because according to (56) $\alpha : \mu_2 \to Z$ is the restriction of the morphism $\lambda^\# : \mathbb{G}_m \to H \subset G$ corresponding to $2\rho$.

**4.5.12.** The following properties of $G(O)$-orbits in $\mathcal{GR}$ will not be used in this work but still we think they are worth mentioning.

The closure of $\mathrm{Orb}_\chi$ is the union of $\mathrm{Orb}_{\chi'}$, $\chi' \leq \chi$. Indeed, if $\rho : G \to GL(V)$ is a representation with lowest weight $\lambda$ then for $g \in \mathrm{Orb}_\chi$ one has $\rho(g) \in t^{(\chi,\lambda)} \mathrm{End}(V \otimes O)$, $\rho(g) \notin t^{(\chi,\lambda)+1} \mathrm{End}(V \otimes O)$. So if $\mathrm{Orb}_{\chi'} \subset \overline{\mathrm{Orb}_\chi}$ then $(\chi - \chi', \lambda) \leq 0$ for every antidominant weight $\lambda$ of $G$ and therefore $\chi - \chi'$ is a linear combination of simple coroots of $G$ with non-negative coefficients; by 4.5.4(i) these coefficients are integer, so $\chi' \leq \chi$. On the other hand, a $GL(2)$ computation shows that the set of weights $\chi'$ of ${}^L G$ such that $\mathrm{Orb}_{\chi'} \subset \overline{\mathrm{Orb}_\chi}$ is saturated in the sense of [Bour75], Ch. VIII, §7, no. 2. So Proposition 5 from loc.cit shows that $\mathrm{Orb}_{\chi'} \subset \overline{\mathrm{Orb}_\chi}$ for every dominant $\chi'$ such that $\chi' \leq \chi$.

The above description of $\overline{\mathrm{Orb}_\chi}$ implies that $\mathrm{Orb}_\chi$ is closed if and only if $\chi$ is minimal. If $G$ is simple then $\chi$ is minimal if and only if $\chi = 0$ or $\chi$ is a microweight of ${}^L G$ (see [Bour68], Ch. VI, §2, Exercise 5). So on each connected component of $\mathcal{GR}$ there is exactly one closed $G(O)$-orbit (use 4.5.4 and the first part of the exercise from loc.cit). If $\mathrm{Orb}_\chi$ is closed it is projective, so in this case $G(O)$ acts on $\mathrm{Orb}_\chi$ via $G = G(O/tO)$ and $\mathrm{Orb}_\chi$ is the quotient of $G$ by a parabolic subgroup. In terms of 9.1.3 $\mathrm{Orb}_\chi = \mathrm{orb}_\chi = G/P_\chi^-$.

If $G$ is simple then there is exactly one $\chi$ such that $\overline{\mathrm{Orb}_\chi} \setminus \mathrm{Orb}_\chi$ consists of a single point[*)]; this $\chi$ is the coroot of $\mathfrak{g} := \operatorname{Lie} G$ corresponding to the maximal root $\alpha_{\max}$ of $\mathfrak{g}$ (see [Bour75], Ch. VIII, §7, Exercise 22). In this case $\overline{\mathrm{Orb}_\chi}$ can be described as follows. Set $V := \mathfrak{g} \otimes (\mathfrak{m}^{-1}/O)$ where $\mathfrak{m}$ is the maximal ideal of $O$. Denote by $\overline{V}$ the projective space containing $V$ as an affine subspace. So $\overline{V}$ is the space of lines in $V \oplus \mathbb{C}$; in particular $V^* = \mathfrak{g}^* \otimes (\mathfrak{m}/\mathfrak{m}^2)$ acts on $\overline{V}$ preserving $0 \in V$. Denote by $C$ the set of elements of $V$ that are $G$-conjugate to $\mathfrak{g}_{\alpha_{\max}} \otimes (\mathfrak{m}^{-1}/O)$. This is a closed subvariety of $V$. Its projective closure $\overline{C} \subset \overline{V}$ is $V^*$-invariant because $C$ is a cone. It is easy to show that the morphism $\exp : C \to G(K)/G(O)$ extends to an isomorphism $f : \overline{C} \xrightarrow{\sim} \overline{\mathrm{Orb}_\chi}$. Clearly $f$ is $\operatorname{Aut}^0 O$-equivariant and $G$-equivariant. The action of $\operatorname{Ker}(G(O) \to G(O/\mathfrak{m}))$ on $\overline{C}$ induced by its action on $\overline{\mathrm{Orb}_\chi}$ comes from the action of $V^*$ on $\overline{C}$ and the isomorphism

$$\operatorname{Ker}(G(O/\mathfrak{m}^2) \to G(O/\mathfrak{m})) \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} V^*$$

where the last arrow is induced by the invariant scalar product on $\mathfrak{g}$ such that $(\alpha_{\max}, \alpha_{\max}) = 2$.

**4.6. Local Pfaffian bundles.** Consider the affine Grassmannian $\mathcal{GR} := G(K)/G(O)$ where $O = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$. Set $Z := \operatorname{Hom}(\pi_1(G), \mathbb{G}_m)$ (by the Remark from 4.1.1 $Z$ is the center of ${}^L G$). In this subsection we will construct and investigate a functor $\mathcal{L} \mapsto \lambda_{\mathcal{L}} = \lambda_{\mathcal{L}}^{\mathrm{loc}}$ from the groupoid $Z\operatorname{tors}_\theta(O)$ (see 3.4.3) to the category of line bundles on $\mathcal{GR}$. We call $\lambda_{\mathcal{L}}$ the *local Pfaffian bundle* corresponding to $\mathcal{L}$.

We recommend the reader to skip this subsection for the moment.

**4.6.1.** In 4.4.9 we defined a functor $\mathcal{L} \mapsto \widetilde{G(K)}_{\mathcal{L}}$ from $Z\operatorname{tors}_\theta(O)$ to the category of central extensions of $G(K)$ by $\mathbb{G}_m$. For $\mathcal{L} \in Z\operatorname{tors}_\theta(O)$ we have the splitting $G(O) \to \widetilde{G(K)}_{\mathcal{L}}$ and therefore the principal $\mathbb{G}_m$-bundle

$$\widetilde{G(K)}_{\mathcal{L}}/G(O) \to G(K)/G(O) = \mathcal{GR}\ . \tag{232}$$

[*)]Of course, this point is the image of $e \in G(K)$.

**4.6.2.** *Definition.* $\lambda_{\mathcal{L}}$ is inverse to the line bundle on $\mathcal{GR}$ corresponding to the $\mathbb{G}_m$-bundle (232).

Clearly $\lambda_{\mathcal{L}}$ depends functorially on $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$.

**4.6.3.** *Remark.* $\widetilde{G(K)}_{\mathcal{L}}$ depends on the choice of a non-degenerate invariant bilinear form on $\mathfrak{g}$ (see 4.4.7). So this is also true for $\lambda_{\mathcal{L}}$.

**4.6.4.** Let $\bar{e} \in \mathcal{GR}$ denote the image of the unit $e \in G$. Our $\lambda_{\mathcal{L}}$ is the unique $\widetilde{G(K)}_{\mathcal{L}}$-equivariant line bundle on $\mathcal{GR}$ trivialized over $\bar{e}$ such that any $c \in \mathbb{G}_m \subset \widetilde{G(K)}_{\mathcal{L}}$ acts on $\lambda_{\mathcal{L}}$ as multiplication by $c^{-1}$. Uniqueness follows from the equality $\mathrm{Hom}(G(O), \mathbb{G}_m) = 0$.

**4.6.5.** By 4.4.11 the action of $\widetilde{G(K)}_{\mathcal{L}}$ on $\lambda_{\mathcal{L}}$ induces an action of $\widetilde{\mathfrak{g} \otimes K}$ on $\lambda_{\mathcal{L}}$ such that $\mathbf{1} \in \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K}$ acts as multiplication by $-1$. It is compatible with the action of $\mathfrak{g} \otimes K$ on $\mathcal{GR}$ by left infinitesimal translations.

**4.6.6.** The push-forward of (63) by the morphism (56) is an exact sequence

$$0 \to Z \to \mathrm{Aut}_Z O \to \mathrm{Aut} O \to 0\,. \tag{233}$$

For any $\mathcal{L} \in Z\,\mathrm{tors}(O)$ the exact sequence

$$0 \to Z \to \mathrm{Aut}(O, \mathcal{L}) \to \mathrm{Aut}\, O \to 0 \tag{234}$$

can be canonically identified with (233). Here $\mathrm{Aut}(O, \mathcal{L})$ is the group ind-scheme of pairs $(\sigma, \varphi)$, $\sigma \in \mathrm{Aut}\, O$, $\varphi : \mathcal{L} \xrightarrow{\sim} \sigma_* \mathcal{L}$ (the reader may prefer to consider $\mathcal{L}$ as an object of the category $\widetilde{Z}\,\mathrm{tors}_\omega(O)$ from 3.4.5). The isomorphism between (233) and (234) is induced by the obvious morphism $\mathrm{Aut}_2\, O := \mathrm{Aut}(O, \omega_O^{1/2}) \to \mathrm{Aut}(O, \mathcal{L})$.

$\mathrm{Aut}_Z\, O = \mathrm{Aut}(O, \mathcal{L})$ acts on the exact sequence (217) by transport of structure; the action of $\mathrm{Aut}_Z\, O$ on $\mathbb{G}_m$ is trivial and its action on $G(K)$ comes from the usual action of $\mathrm{Aut}\, O$ on $G(K)$. The subgroup $G(O) \subset \widetilde{G(K)}_{\mathcal{L}}$ is $\mathrm{Aut}_Z\, O$-invariant.

**4.6.7.** It follows from 4.6.6 that the action of $\operatorname{Aut} O$ on $\mathcal{GR}$ lifts canonically to an action of $\operatorname{Aut}_Z O$ on the principal bundle (232) and the line bundle $\lambda_{\mathcal{L}}$. The action of $\operatorname{Aut}_Z O$ on $\lambda_{\mathcal{L}}$ induces an action of $\operatorname{Der} O = \operatorname{Lie} \operatorname{Aut}_Z O$ on $\lambda_{\mathcal{L}}$.

**4.6.8.** The action of $Z = \operatorname{Aut} \mathcal{L}$ on the extension (217) comes from (215). So $Z$ acts on $\lambda_{\mathcal{L}}$ via the morphism

$$Z \to H^0(\mathcal{GR}, \mathcal{O}^*_{\mathcal{GR}}) \tag{235}$$

inverse to the composition of (215) and the natural embedding $\operatorname{Hom}(G(K), \mathbb{G}_m) \hookrightarrow H^0(\mathcal{GR}, \mathcal{O}^*_{\mathcal{GR}})$. Recall that $\pi_0(\mathcal{GR}) = Z^\vee$ (see 4.5.9), so $z \in Z$ defines $f_z : \pi_0(\mathcal{GR}) \to \mathbb{C}^*$ and (235) is the map $z \mapsto f_z^{-1}$.

**4.6.9.** *Remark.* (Do we need it ???). Consider the category of line bundles on $\mathcal{GR}$ as a $Z$-category in the sense of 3.4.4, the $Z$-structure being defined by (235). By 3.4.7 (i) we have a canonical Picard functor

$$Z\operatorname{tors}(O) = Z\operatorname{tors} \to \{\text{line bundles on } \mathcal{GR}\}\,. \tag{236}$$

Explicitly, (236) assigns to $\mathcal{E} \in Z\operatorname{tors}$ the $\mathcal{E}$-twist of $\mathcal{O}_{\mathcal{GR}}$ equipped with the $Z$-action (235). By 3.4.7 (iv) the functor $\mathcal{L} \mapsto \lambda_{\mathcal{L}}$, $\mathcal{L} \in Z\operatorname{tors}_\theta(O)$, is affine with respect to the Picard functor (236).

**4.6.10.** The morphism $\alpha : \mu_2 \to Z$ defined by (56) induces an action of $\mu_2$ on $\lambda_{\mathcal{L}}$, $\mathcal{L} \in Z\operatorname{tors}_\theta(O)$. It defines a $(\mathbb{Z}/2\mathbb{Z})$-grading on $\lambda_{\mathcal{L}}$. In 4.5.10 we introduced the notions of even and odd component of $\mathcal{GR}$. According to 4.6.8 the restriction of the $(\mathbb{Z}/2\mathbb{Z})$-graded bundle $\lambda_{\mathcal{L}}$ to an even (resp. odd) component of $\mathcal{GR}$ is even (resp. odd).

**4.6.11.** The functor

$$Z\operatorname{tors}_\theta(O) \to \{\text{line bundles on } \mathcal{GR}\}, \quad \mathcal{L} \mapsto \lambda_{\mathcal{L}} \tag{237}$$

is a $Z$-functor in the sense of 3.4.4 provided the $Z$-structure on the r.h.s. of (237) is defined by (235). Since $Z\operatorname{tors}_\theta(O)$ is equivalent to $\omega^{1/2}(O) \otimes_{\mu_2} Z$

(see 3.4.4) the functor (237) is reconstructed from the corresponding functor

$$\omega^{1/2}(O) \to \{\text{line bundles on}\, \mathcal{GR}\} \tag{238}$$

where $\omega^{1/2}(O)$ is the groupoid of square roots of $\omega(O)$. Since the extension (212) essentially comes from the "Clifford extension" (193) it is easy to give a Cliffordian description of (238). Here is the answer.

Let $\mathcal{L} \in \omega^{1/2}(O)$. We have fixed a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, so the Tate space $V = V_{\mathcal{L}} := \mathcal{L} \otimes_O (\mathfrak{g} \otimes K)$ carries a nondegenerate symmetric bilinear form (see 4.3.3) and $L := \mathcal{L} \otimes \mathfrak{g} \subset V$ is a Lagrangian c-lattice. Set $M = M_{\mathcal{L}} := \mathrm{Cl}(V)/\,\mathrm{Cl}(V)L$; this is an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded discrete module over $\mathrm{Cl}(V)$. We have the line bundle $\mathcal{P}_M$ on the ind-scheme $\mathrm{Lagr}(V)$ of Lagrangian c-lattices in $V$ (see 4.3.2). We claim that

$$\lambda_{\mathcal{L}} = \varphi^* \mathcal{P}_{M_{\mathcal{L}}} \tag{239}$$

where the morphism*) $\varphi : G(K)/G(O) \to \mathrm{Lagr}(V)$ is defined by $\varphi(g) := gLg^{-1}$; in other words

(240) the fiber of $\lambda_{\mathcal{L}}$ over $g \in G(K)/G(O)$ is $M^{gLg^{-1}} := \{m \in M_{\mathcal{L}} | (gLg^{-1}) \cdot m = 0\}$.

Indeed, the central extension (212) is opposite to the one induced from (193) and therefore the action of $\widetilde{O}(V)$ on $\mathcal{P}_{M_{\mathcal{L}}}$ (see 4.3.2) induces an action of $\widetilde{G(K)}_{\mathcal{L}}$ on $\varphi^*\mathcal{P}_{M_{\mathcal{L}}}$ such that $c \in \mathbb{G}_m \subset \widetilde{G(K)}_{\mathcal{L}}$ acts as multiplication by $c^{-1}$; besides, the fiber of $\varphi^*\mathcal{P}_{M_{\mathcal{L}}}$ over $\overline{e}$ is $\mathbb{C}$.

Clearly the isomorphism (239) is functorial in $\mathcal{L} \in \omega^{1/2}(O)$.

**4.6.12.** *Remarks*

(i) The line bundle $\mathcal{P}_M$ from 4.3.2 is $(\mathbb{Z}/2\mathbb{Z})$-graded. So both sides of (239) are $(\mathbb{Z}/2\mathbb{Z})$-graded. The gradings of both sides of (239) are induced by the action of $\mu_2 = \mathrm{Aut}\,\mathcal{L}$ (to prove this for the r.h.s.

*)It is easy to show that $\varphi$ is a closed embedding and its image is the ind-scheme of $\Lambda \in \mathrm{Lagr}(V)$ such that $O\Lambda = \Lambda$ and $\mathcal{L}^{-1} \otimes_O \Lambda$ is a Lie subalgebra of $\mathfrak{g} \otimes K$.

notice that the $(\mathbb{Z}/2\mathbb{Z})$-grading on $\mathrm{Cl}(V)$ is induced by the natural action of $\mu_2$ on $V$). Therefore (239) is a *graded* isomorphism.

(ii) According to 4.6.10 $-1 \in \mu_2 = \operatorname{Aut} \mathcal{L}$ acts on the r.h.s. of (239) as multiplication by $(-1)^p$ where $p$ is the parity function (230). This also follows from the equality $\chi = \theta$ (see the proof of Lemma 4.3.4) and Remark (ii) at the end of 4.3.4.

**4.6.13.** We should think about super-aspects, in particular: what is the inverse of a 1-dimensional superspace? (maybe this should be formulated in an arbitrary Picard category; there may be troubles if it is not STRICTLY commutative).

Consider a $G(O)$-orbit $\mathrm{Orb}_\chi \subset \mathcal{GR}$, $\chi \in P_+({}^L G)$ (see 4.5.8). We will compute $\lambda_{\mathcal{L},\chi} :=$ the restriction of $\lambda_{\mathcal{L}}$ to $\mathrm{Orb}_\chi$, $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$. By 4.6.4 $\lambda_{\mathcal{L},\chi}$ is $G(O)$-equivariant. The orbit $\mathrm{Orb}_\chi$ is $\operatorname{Aut}^0 O$-invariant and by 4.6.7 $\lambda_{\mathcal{L},\chi}$ is $\operatorname{Aut}^0_Z O$-equivariant where $\operatorname{Aut}^0_Z O$ is the preimage of $\operatorname{Aut}^0 O$ in $\operatorname{Aut}_Z O$ (see (233)). Finally $\lambda_{\mathcal{L},\chi}$ is $\mathbb{Z}/2\mathbb{Z}$-graded (but in fact $\lambda_{\mathcal{L},\chi}$ is even or odd depending on $\chi$; besides, the $\mathbb{Z}/2\mathbb{Z}$-grading can be reconstructed from the action of $Z \subset \operatorname{Aut}^0_Z O$.....). The groups $G(O)$ and $\operatorname{Aut}^0_Z O$ also act on the canonical sheaf $\omega_{\mathrm{Orb}_\chi}$ ($\operatorname{Aut}^0_Z O$ acts via $\operatorname{Aut}^0 O$). In 4.6.17-4.6.19 (???) we will construct a canonical isomorphism

$$\lambda_{\mathcal{L},\chi} \xrightarrow{\sim} \omega_{\mathrm{Orb}_\chi} \otimes (\mathfrak{d}_{\mathcal{L},\chi})^{-1} \tag{241}$$

for a certain 1-dimensional vector space $\mathfrak{d}_{\mathcal{L},\chi}$. This space is equipped with an action of $G(O)$ and $\operatorname{Aut}^0_Z O$ and (241) is equivariant with respect to these groups.

**4.6.14.** Let us define $\mathfrak{d}_{\mathcal{L},\chi}$. Of course the action of $G(O)$ on $\mathfrak{d}_{\mathcal{L},\chi}$ is defined to be trivial ($G(O)$ has no nontrivial characters). So we have to construct for each $\chi$ a functor

$$Z\,\mathrm{tors}_\theta(O) \to \{\operatorname{Aut}^0_Z O\text{-mod}\}, \quad \mathcal{L} \mapsto \mathfrak{d}_{\mathcal{L},\chi} \tag{242}$$

where $\{\operatorname{Aut}^0_Z O\text{-mod}\}$ denotes the category of $\operatorname{Aut}^0_Z O$-modules. First let us define a functor

$$\omega^{1/2}(O) \to \{\operatorname{Aut}^0_Z O\text{-mod}\}, \quad \mathcal{L} \mapsto \mathfrak{d}_{\mathcal{L},\chi} \tag{243}$$

For $\mathcal{L} \in \omega^{1/2}(O)$ set

$$\mathfrak{d}_{\mathcal{L},\chi} := (\mathcal{L}_0)^{\otimes d(\chi)} \tag{244}$$

where $\mathcal{L}_0$ is the fiber of $\mathcal{L}$ over the closed point $0 \in \operatorname{Spec} O$ and

$$d(\chi) := (\chi, 2\rho) = \dim \operatorname{Orb}_\chi \tag{245}$$

Define the representation of $\operatorname{Aut}^0_Z O$ in $\mathfrak{d}_{\mathcal{L},\chi}$ as follows: $\operatorname{Aut}^0_2 O = \operatorname{Aut}^0(O, \mathcal{L})$ acts in the obvious way and $Z \subset \operatorname{Aut}^0_Z O$ acts via

$$\chi_Z : Z \to \mathbb{G}_m \tag{246}$$

where $\chi_Z$ is the restriction of $\chi \in P^+({}^LG)$ to $Z \subset {}^LG$ (these two actions are compatible because the composition of $\chi_Z$ and the morphism (56) maps $-1 \in \mu_2$ to $(-1)^{(\chi, 2\rho)}$).

So we have constructed (243). $\omega^{1/2}(O)$ is a $\mu_2$-category in the sense of 3.4.4, $\{\operatorname{Aut}^0_Z O\text{-mod}\}$ is a $Z$-category, and (243) is a $\mu_2$-functor (the $\mu_2$-structure on $\{\operatorname{Aut}^0_Z O\}$ comes from the morphism (56) or, equivalently, from the canonical embedding $\mu_2 \to \operatorname{Aut}^0_2 O$). So (243) induces a $Z$-functor $Z \operatorname{tors}_\theta(O) = \omega^{1/2}(O) \otimes_{\mu_2} Z \to \{\operatorname{Aut}^0_Z O\text{-mod}\}$. This is the definition of (242).

**4.6.15.** Clearly $\operatorname{Lie} \operatorname{Aut}^0_Z O = \operatorname{Der}^0 O$ acts on the one-dimensional space $\mathfrak{d}_{\mathcal{L},\chi}$ as follows:

$$L_0 \mapsto (\chi, \rho) = -\frac{1}{2} \dim \operatorname{Orb}_\chi \ , \quad L_n \mapsto 0 \text{ for } n > 0 \tag{247}$$

As usual, $L_n := -t^{n+1} \frac{d}{dt} \in \operatorname{Der}^0 O$.

**4.6.16.** *Remark.* The definition of $\mathfrak{d}_{\mathcal{L},\chi}$ from 4.6.14 can be reformulated as follows. Using the equivalence $Z\operatorname{tors}_\theta(O) \xrightarrow{\sim} \tilde{Z}\operatorname{tors}_\omega(O)$ from 3.4.5 we interpret $\mathcal{L} \in Z\operatorname{tors}_\theta(O)$ in terms of (59) as a lifting of the $\mathbb{G}_m$-torsor $\omega_O$ to a $\tilde{Z}$-torsor. We have the canonical morphism $\tilde{Z} \to {}^LH$ from (62) where ${}^LH$ is *the* Cartan torus of ${}^LG$ or, which is the same, ${}^LH$ is a Cartan subgroup of ${}^LG$ with a fixed Borel subgroup containing it. Denote by $\chi_{\tilde{Z}}$ the composition of $\tilde{Z} \to {}^LH$ and $\chi : {}^LH \to \mathbb{G}_m$. The $\tilde{Z}$-torsor $\mathcal{L}$ on $\operatorname{Spec} O$ and the 1-dimensional representation $\chi_{\tilde{Z}} : \tilde{Z} \to \mathbb{G}_m$ define a line bundle $\mathfrak{d}^O_{\mathcal{L},\chi}$ on $\operatorname{Spec} O$. According to 4.6.6 $\operatorname{Aut}_Z O = \operatorname{Aut}(O, \mathcal{L})$, so the action of $\operatorname{Aut} O$ on $\operatorname{Spec} O$ lifts to a canonical action of $\operatorname{Aut}_Z O$ on $\mathfrak{d}^O_{\mathcal{L},\chi}$. Therefore $\operatorname{Aut}^0_Z O$ acts on the fiber of $\mathfrak{d}^O_{\mathcal{L},\chi}$ at $0 \in \operatorname{Spec} O$. The reader can easily identify this fiber with the $\mathfrak{d}_{\mathcal{L},\chi}$ from 4.6.14.

**4.6.17.** Let us construct the isomorphism (241) for $\mathcal{L} \in \omega^{1/2}(O)$. We use the Cliffordian description of $\lambda_{\mathcal{L}}$. Just as in 4.6.11 we set $V = V_{\mathcal{L}} := \mathcal{L} \otimes_O (\mathfrak{g} \otimes K)$, $L := \mathcal{L} \otimes \mathfrak{g} \subset V$, $M = M_{\mathcal{L}} := \operatorname{Cl}(V)/\operatorname{Cl}(V)L$. For $x \in \mathcal{GR} = G(K)/G(O)$ set $L_x := gLg^{-1}$ where $g$ is a preimage of $x$ in $G(K)$. By (240) the fiber of $\lambda_{\mathcal{L}}$ at $x$ equals

$$M^{L_x} := \{m \in M_{\mathcal{L}} | L_x \cdot m = 0\} \tag{248}$$

Suppose that $x \in \operatorname{Orb}_\chi$. Since $\operatorname{Orb}_\chi$ is the $G(O)$-orbit of $x$ the tangent space to $\operatorname{Orb}_\chi$ at $x$ is $(\mathfrak{g} \otimes O)/((\mathfrak{g} \otimes O) \cap g(\mathfrak{g} \otimes O)g^{-1}) = \mathcal{L}^{-1} \otimes_O (L/(L \cap L_x))$ where $g \in G(K)$ is a preimage of $x$. So the fiber of $\omega^{-1}_{\operatorname{Orb}_\chi}$ at $x$ equals $(\mathcal{L}_0)^{\otimes -d(\chi)} \otimes \det(L/(L \cap L_x))$ where $d(\chi) = \dim \operatorname{Orb}_\chi$. Taking (244) into account we see that the fiber of the r.h.s. of (241) at $x$ equals

$$(\det(L/(L \cap L_x)))^{-1} \tag{249}$$

So it remains to construct an isomorphism

$$\det(L/(L \cap L_x)) \otimes M^{L_x} \xrightarrow{\sim} \mathbb{C} \tag{250}$$

**4.6.18.** *Lemma.* Consider a Tate space $V$ equipped with a nondegenerate symmetric bilinear form. Let $L, \Lambda \subset V$ be Lagrangian c-lattices and $M$ an irreducible discrete module over the Clifford algebra $\mathrm{Cl}(V)$. Consider the operator

$$\bigwedge^d L \otimes M \to M \tag{251}$$

induced by the natural map $\bigwedge^d L \to \bigwedge^d V \to \mathrm{Cl}(V)$. If $d = \dim L/(L \cap \Lambda)$ then (251) induces an isomorphism

$$\bigwedge^d (L/(L \cap \Lambda)) \otimes M^\Lambda \xrightarrow{\sim} M^L \tag{252}$$

The proof is reduced to the case where $\dim V < \infty$ and $V = L \oplus \Lambda$.

**4.6.19.** We define (250) to be the isomorphism (252) for $\Lambda = L_x$ (in the situation of 4.6.17 $M^L = \mathbb{C}$). So for $\mathcal{L} \in \omega^{1/2}(O)$ we have constructed the isomorphism (241), which is equivariant with respect to $G(O)$ and $\mathrm{Aut}^0_2 O = \mathrm{Aut}^0(O, \mathcal{L})$.

Denote by $C_\chi$ the category of line bundles on $\mathrm{Orb}_\chi$. Both sides of (241) are $\mu_2$-functors $\omega^{1/2}(O) \to C_\chi$ extended to $Z$-functors

$$Z \operatorname{tors}_\theta(O) = \omega^{1/2}(O) \otimes_{\mu_2} Z \to C_\chi$$

(the $Z$-structure on $C_\chi$ is defined by the character of $Z$ inverse to (246)); for the l.h.s of (241) this follows from 4.6.8. Clearly (241) is an isomorphism of functors $\omega^{1/2}(O) \to C_\chi$. Therefore (241) is an isomorphism of functors $Z \operatorname{tors}_\theta(O) \to C_\chi$. The isomorphism (241) is $\mathrm{Aut}^0_Z O$-equivariant because it is $\mathrm{Aut}^0_2 O$-equivariant and $Z$-equivariant.

**4.6.20.** Recall that $\lambda_{\mathcal{L}}$ depends on the choice of a nondegenerate invariant bilinear form on $\mathfrak{g}$ (see 4.6.3 and 4.4.7). As explained in the footnote to 4.4.7 there is a more canonical version of $\lambda_{\mathcal{L}}$. In the case where $G$ is simple this version $\lambda_{\mathcal{L}}^{\mathrm{can}}$ depends on the choice of $\beta^{1/2}$ where $\beta$ is the line of invariant bilinear forms on $\mathfrak{g}$ (cf. 4.4.5); $\lambda_{\mathcal{L}}^{\mathrm{can}}$ comes from the version of (212) obtained by using $SO(\mathfrak{g}\otimes\beta^{1/2})$ instead of $SO(\mathfrak{g})$. It is easy to see that the $(\mathbb{Z}/2\mathbb{Z})$-grading on $\lambda_{\mathcal{L}}^{\mathrm{can}}$, corresponding to the action of $-1\in\operatorname{Aut}\beta^{1/2}$ coincides with the grading from 4.6.10. The "canonical" version of (241) is an isomorphism

$$\lambda_{\mathcal{L},\chi}^{\mathrm{can}} \xrightarrow{\sim} \omega_{\mathrm{Orb}_\chi}\otimes(\mathfrak{d}_{\mathcal{L},\chi})^{-1}\otimes(\beta^{1/2})^{\otimes -d(\chi)} \tag{253}$$

where $d(\chi)$ is defined by (245). Details are left to the reader.

# 5. Hecke eigen-$\mathcal{D}$-modules

## 5.1. Construction of $\mathcal{D}$-modules.

**5.1.1.** In this subsection we construct a family of $\mathcal{D}$-modules on $\mathrm{Bun}_G$ parametrized by $\mathcal{O}\mathfrak{p}_{^LG}(X)$, i.e., the stack of $^LG$-opers on $X$.

Denote by $Z$ the center of $^LG$. According to formula (57) from 3.4.3 we must associate to $\mathcal{L} \in Z\operatorname{tors}_\theta(X)$ a family of $\mathcal{D}$-modules on $\mathrm{Bun}_G$ parametrized by $\mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(X)$. In 4.4.3 we defined $\lambda_\mathcal{L} \in \mu_\infty \operatorname{tors}_\theta(\mathrm{Bun}_G)$. $\lambda_\mathcal{L}$ is a line bundle on $\mathrm{Bun}_G$ equipped with an isomorphism $\lambda_\mathcal{L}^{\otimes 2n} \xrightarrow{\sim} (\omega^\sharp_{\mathrm{Bun}_G})^{\otimes n}$ for some $n \neq 0$ (see 4.0.1). So $\lambda_\mathcal{L}$ is a $\mathcal{D}'$-module. Therefore $M_\mathcal{L} := \lambda_\mathcal{L}^{-1} \bigotimes_{\mathcal{O}_{\mathrm{Bun}_G}} \mathcal{D}'$ is a left $\mathcal{D}$-module on $\mathrm{Bun}_G$. According to 3.3.2 and 2.7.4 there is a canonical morphism of algebras $h_X\varphi_X : A_{^L\mathfrak{g}}(X) \to \Gamma(\mathrm{Bun}_G, D')$. So the right action of $\Gamma(\mathrm{Bun}_G, D')$ on $\mathcal{D}'$ yields an $A_{^L\mathfrak{g}}(X)$-module structure on $M_\mathcal{L}$. Therefore we may consider $M_\mathcal{L}$ as a family of left $\mathcal{D}$-modules on $\mathrm{Bun}_G$ parametrized by $\operatorname{Spec} A_{^L\mathfrak{g}}(X) = \mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(X)$.

So we have constructed a family of left $\mathcal{D}$-modules on $\mathrm{Bun}_G$ parametrized by $\mathcal{O}\mathfrak{p}_{^LG}(X)$. For an $^LG$-oper $\mathfrak{F}$ the corresponding $\mathcal{D}$-module $M_\mathfrak{F}$ is $M_\mathcal{L}/\mathfrak{m}_\mathfrak{F} M_\mathcal{L} = \lambda_\mathcal{L}^{-1} \otimes \mathcal{D}'/\mathcal{D}'\mathfrak{m}_\mathfrak{F}$ where $\mathcal{L}$ is the image of $\mathfrak{F}$ in $Z\operatorname{tors}_\theta(X)$ and $\mathfrak{m}_\mathfrak{F} \subset A_{^L\mathfrak{g}}(X)$ is the maximal ideal of the $^L\mathfrak{g}$-oper corresponding to $\mathfrak{F}$.

**5.1.2.** *Proposition.*

(i) For every $\mathcal{L} \in Z\operatorname{tors}_\theta(X)$ $M_\mathcal{L}$ is flat over $A_{^L\mathfrak{g}}(X)$.

(ii) For every $^LG$-oper $\mathfrak{F}$ the $\mathcal{D}$-module $M_\mathfrak{F}$ is holonomic. Its singular support coincides as a cycle with the zero fiber of Hitchin's fibration.

*Proof.* According to 2.2.4 (iii) $\operatorname{gr}\mathcal{D}'$ is flat$^{*)}$ over $\operatorname{gr} A_{^L\mathfrak{g}}(X)$. So $\mathcal{D}'$ is flat over $A_{^L\mathfrak{g}}(X)$. This implies i) and the equality $\operatorname{gr}(\mathcal{D}'/\mathcal{D}'I) = \operatorname{gr}\mathcal{D}'/(\operatorname{gr}\mathcal{D}' \cdot \operatorname{gr} I)$ for any ideal $I \subset A_{^L\mathfrak{g}}(X)$. If $I$ is maximal we obtain ii). $\square$

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$^{*)}$This means that if $f : S \to \mathrm{Bun}_G$ is smooth and $S$ is affine $\Gamma(S, f^*\operatorname{gr}\mathcal{D}')$ is a *free* module over $\operatorname{gr} A_{^L\mathfrak{g}}(X)$ (a flat $\mathbb{Z}_+$-graded module over a $\mathbb{Z}_+$-graded ring $A$ with $A_0 = \mathbb{C}$ is free).

## 5.2. Main theorems I: an introduction.

**5.2.1.** Our main global theorem 5.2.6 asserts that the $\mathcal{D}$-module $M_{\mathfrak{F}}$ is an eigenmodule of the *Hecke functors*. In order to define them we introduce the *big Hecke stack* $\mathcal{H}$ecke. The groupoid of $S$-points $\mathcal{H}\mathrm{ecke}(S)$ consists of quadruples $(\mathcal{F}_1, \mathcal{F}_2, x, \alpha)$ where $\mathcal{F}_1$, $\mathcal{F}_2$ are $G$-torsors on $X \times S$, $x \in X(S)$, and $\alpha : \mathcal{F}_1|_U \xrightarrow{\sim} \mathcal{F}_2|_U$ is an isomorphism over the complement $U$ to the graph of $x$. One has the obvious projection $p_{1,2,X} = (p_1, p_2, p_X) : \mathcal{H}\mathrm{ecke} \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times X$.

The stack $\mathcal{H}$ecke is ind-algebraic and the projections $p_i$, $p_{i,X}$ are ind-proper. Precisely, there is an increasing family of closed algebraic substacks $\mathcal{H}\mathrm{ecke}_1 \subset \mathcal{H}\mathrm{ecke}_2 \subset \cdots \subset \mathcal{H}\mathrm{ecke}$ such that $\mathcal{H}\mathrm{ecke} = \bigcup \mathcal{H}\mathrm{ecke}_a$ and $p_i : \mathcal{H}\mathrm{ecke}_a \to \mathrm{Bun}_G$, $p_{i,X} : \mathcal{H}\mathrm{ecke}_a \to \mathrm{Bun}_G \times X$ are proper morphisms.

**5.2.2.** *Remarks.* (i) The composition of $\alpha$'s makes $\mathcal{H}$ecke an $X$-family of groupoids on $\mathrm{Bun}_G$.

(ii) $\mathcal{H}$ecke is a family of twisted affine Grassmannians over $\mathrm{Bun}_G \times X$. Precisely, for $(\mathcal{F}_2, x) \in \mathrm{Bun}_G \times X$ the fiber $\mathcal{H}\mathrm{ecke}_{(\mathcal{F}_2,x)} := p_{2,X}^{-1}(\mathcal{F}_2, x)$ is canonically isomorphic to the affine Grassmannian $\mathcal{GR}_x := G(K_x)/G(O_x)$ twisted by the $G(O_x)$-torsor $\mathcal{F}_2(O_x)$ (with respect to the left $G(O_x)$-action). In the case where $\mathcal{F}_2$ is the trivial bundle we described this isomorphism in 4.5.2. In the general case the construction is similar: for fixed $\gamma_2 \in \mathcal{F}_2(O_x)$ we assign to $(\mathcal{F}_1, \mathcal{F}_2, x, \alpha)$ the image of $\gamma_2/\alpha(\gamma_1)$ in $G(K_x)/G(O_x)$ where $\gamma_1$ is any element of $\mathcal{F}_1(O_x)$ and $\gamma_2/\alpha(\gamma_1)$ denotes the element $g \in G(K_x)$ such that $g\alpha(\gamma_1) = \gamma_2$; by 2.3.4 the morphism $\mathcal{H}\mathrm{ecke}_{(\mathcal{F}_2,x)} \to G(K_x)/G(O_x)$ is an isomorphism.

**5.2.3.** The set of conjugacy classes of morphisms $\nu : \mathbb{G}_m \to G$ can be canonically identified with the set $P_+(^LG)$ of dominant weights of $^LG$. Recall that $G(O_x)$-orbits in $\mathcal{GR}_x = G(K_x)/G(O_x)$ are labeled by $\chi \in P_+(^LG)$; by definition, $\mathrm{Orb}_\chi$ is the orbit of the image of $\nu(t_x) \in G(K_x)$ in $\mathcal{GR}_x$ where $\nu : \mathbb{G}_m \to G$ is of class $\chi$ and $t_x \in O_x$ is a uniformizer.

According to 5.2.2 (ii) the stratification of $\mathcal{GR}_x$ by $\mathrm{Orb}_\chi$ yields a stratification of the stack $\mathcal{H}$ecke by substacks $\mathcal{H}\mathrm{ecke}_\chi$, $\chi \in P_+({}^LG)$. The $\mathbb{C}$-points of $\mathcal{H}\mathrm{ecke}_\chi$ are quadruples $(\mathcal{F}_1, \mathcal{F}_2, x, \alpha)$ such that for some $\gamma_i \in \mathcal{F}_i(O_x)$ and a formal parameter $t_x$ at $x$ one has $\gamma_2 = \nu(t_x)\alpha(\gamma_1)$ where $\nu : \mathbb{G}_m \to G$ is of class $\chi$. The involution $(\mathcal{F}_1, \mathcal{F}_2, x, \alpha) \mapsto (\mathcal{F}_2, \mathcal{F}_1, x, \alpha^{-1})$ identifies $\mathcal{H}\mathrm{ecke}_\chi$ with $\mathcal{H}\mathrm{ecke}_{\chi^\circ}$ where $\chi^\circ$ is the dual weight. So the fibers of $p_{2,X} : \mathcal{H}\mathrm{ecke}_\chi \to \mathrm{Bun}_G \times X$ are twisted forms of $\mathrm{Orb}_\chi$ while the fibers of $p_{1,X} : \mathcal{H}\mathrm{ecke}_\chi \to \mathrm{Bun}_G \times X$ are twisted forms of $\mathrm{Orb}_{\chi^\circ}$.

For every $\chi$ the stack $\mathcal{H}\mathrm{ecke}_\chi$ is smooth over $\mathrm{Bun}_G \times X$. Usually its closure $\overline{\mathcal{H}\mathrm{ecke}_\chi}$ is not smooth.

*Remarks.* (i) According to 4.5.12 $\overline{\mathcal{H}\mathrm{ecke}_\chi}$ is the union of the strata $\mathcal{H}\mathrm{ecke}_{\chi'}$, $\chi' \leq \chi$.

(ii) If $G = GL(n)$ then our labeling of strata coincides with the "natural" one. Namely, let $V_1, V_2$ be the vector bundles corresponding to $\mathfrak{F}_1, \mathfrak{F}_2$. Then $\mathcal{H}\mathrm{ecke}_\chi$ consists of all collections $(V_1, V_2, x, \alpha)$ such that for certain bases of $V_i$'s on the formal neighbourhood of $x$ the matrix of $\alpha$ equals $t_x^\chi$.

**5.2.4.** Let us define the Hecke functors $T_\chi^i : \mathcal{M}(\mathrm{Bun}_G) \to \mathcal{M}(\mathrm{Bun}_G \times X)$ where $\mathcal{M}$ denotes the category of $\mathcal{D}$-modules, $\chi \in P_+({}^LG)$, $i \in \mathbb{Z}$.

For $\chi \in P_+({}^LG)$, $M \in \mathcal{M}(\mathrm{Bun}_G)$ denote by $p_{1\chi}^\star M$ the minimal (= Goresky–MacPherson) extension to $\overline{\mathcal{H}\mathrm{ecke}_\chi}$ of the pullback of $M$ by the smooth projection $p_{1\chi} : \mathcal{H}\mathrm{ecke}_\chi \to \mathrm{Bun}_G$, $p_{1\chi} := p_1|_{\mathcal{H}\mathrm{ecke}_\chi}$. Notice that the fibration $p_{1X} : \overline{\mathcal{H}\mathrm{ecke}_\chi} \to \mathrm{Bun}_G \times X$ is locally trivial (see 5.2.2 (ii), 5.2.3), so the choice of a local trivialization identifies $p_{1\chi}^\star M$ (locally) with the external tensor product of $M$ and the "intersection cohomology" $\mathcal{D}$-module on the closure of the corresponding $G(O)$-orbit[*)] on the affine Grassmannian.

Define the Hecke functors $T_\chi^i : \mathcal{M}(\mathrm{Bun}_G) \to \mathcal{M}(\mathrm{Bun}_G \times X)$ by

$$T_\chi^i = H^i(p_{2,X})_* p_{1\chi}^\star \tag{254}$$

[*)]This orbit is $\mathrm{Orb}_{\chi^\circ}$ where $\chi^\circ$ is the dual weight, see 5.2.3.

where $H^i(p_{2,X})_*$ is the cohomological pushforward functor for the projection $p_{2,X} : \overline{\mathcal{H}\mathrm{ecke}_\chi} \to \mathrm{Bun}_G \times X$.

*Remark.* For a representable quasi-compact morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks of locally finite type the definition of $H^i f_* : \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{Y})$ is clear. Indeed, in the case of schemes one has a definition of $H^i f_*$ and one knows that $H^i f_*$ commutes with smooth base change.

**5.2.5.** For $\chi \in P_+({}^L G)$ we denote by $V^\chi$ the irreducible ${}^L G$-module of highest weight $\chi$ with marked highest vector. If $\mathfrak{F}$ is an ${}^L G$-oper on $X$ (or, more generally, an ${}^L G$-bundle with a connection) denote by $V^\chi_{\mathfrak{F}}$ the $\mathfrak{F}$-twist of $V^\chi$; this is a smooth $\mathcal{D}$-module on $X$.

**5.2.6.** *Main Global Theorem.* Let $\mathfrak{F}$ be an ${}^L G$-oper on $X$ and $M_{\mathfrak{F}}$ the $\mathcal{D}$-module on $\mathrm{Bun}_G$ defined in 5.1.1. Then $T^i_\chi M_{\mathfrak{F}} = 0$ for $i \neq 0$ and there is a canonical isomorphism of $\mathcal{D}$-modules on $\mathrm{Bun}_G \times X$

$$T^0_\chi M_{\mathfrak{F}} \xrightarrow{\sim} M_{\mathfrak{F}} \boxtimes V^\chi_{\mathfrak{F}} \,. \tag{255}$$

The isomorphisms (255) are compatible with composition of Hecke correspondences and tensor products of $V^\chi$. For the precise statement see 5.4.3. All this means that $M_{\mathfrak{F}}$ is a *Hecke eigen-$\mathcal{D}$-module of eigenvalue* $\mathfrak{F}$.

**5.2.7.** Laumon defined (see §§5.3 and 4.3.3 from [La87]) a conjectural "Langlands transform" $K_E$ of an irreducible local system $E$ on $X$ ($K_E$ does exist if rank $E \leq 2$). $K_E$ is a holonomic $\mathcal{D}$-module on $\mathrm{Bun}_{GL_n}$, $n = \mathrm{rank} E$, and at least for $n = 2$ its singular support is the zero fiber of Hitchin's fibration (see §5.5 from [La87]). Besides $K_E$ has regular singularities and its restriction to each connected component of $\mathrm{Bun}_{GL_n}$ is irreducible. If $E$ is an $SL_n$ local system then $K_E$ lives on $\mathrm{Bun}_{PGL_n}$.

Taking in account 5.1.2 and 5.2.6 it is natural to conjecture that for $G = PGL_n$ the $\mathcal{D}$-module $M_{\mathfrak{F}}$ from 5.1.1 equals $K_{\mathfrak{F}}$ (some results in this direction can be found in [Fr]). It would also be interesting to find out (for any $G$) whether $M_{\mathfrak{F}}$ has regular singularities and whether its restrictions to connected components of $\mathrm{Bun}_G$ are irreducible.

**5.2.8.** It is convenient and important to rewrite 5.2.6 in terms of the $\mathcal{D}$-modules $M_{\mathcal{L}}$ from 5.1.1, $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$. According to (57) $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ defines a family $\mathfrak{F}_{\mathcal{L}}$ of ${}^LG$-opers on $X$ parametrized by $\mathrm{Spec}\, A_{{}^L\mathfrak{g}}(X)$. Thus $\mathfrak{F}_{\mathcal{L}}$ is an ${}^LG$-torsor on $X \times \mathrm{Spec}\, A_{{}^L\mathfrak{g}}(X)$ equipped with a connection along $X$. For $\chi \in P_+({}^LG)$ the $\mathfrak{F}_{\mathcal{L}}$-twist of $V^\chi$ is a vector bundle on $X \times \mathrm{Spec}\, A_{{}^L\mathfrak{g}}(X)$ equipped with a connection along $X$. We consider it as a $\mathcal{D}$-module $V^\chi_{\mathcal{L}}$ on $X$ equipped with an action of $A_{{}^L\mathfrak{g}}(X)$.

Now consider the $\mathcal{D}$-module $M_{\mathcal{L}}$ on $\mathrm{Bun}_G$ (sec 5.1.1); $A_{{}^L\mathfrak{g}}(X)$ acts on it. It is easy to see (use 5.1.2 (i)) that 5.2.6 is a consequence of the following theorem.

**5.2.9.** *Theorem.* There is a canonical isomorphism of $\mathcal{D}$-modules on $\mathrm{Bun}_G \times X$

$$T^0_\chi M_{\mathcal{L}} \xrightarrow{\sim} M_{\mathcal{L}} \underset{A_{{}^L\mathfrak{g}}(X)}{\boxtimes} V^\chi_{\mathcal{L}} \tag{256}$$

compatible with the action of $A_{{}^L\mathfrak{g}}(X)$, and $T^i_\chi M_{\mathcal{L}} = 0$ for $i \neq 0$.

**5.2.10.** We will deduce the above global theorem from its local version which we are going to explain now. Consider the affine Grassmannian $\mathcal{GR} := G(K)/G(O)$ where $O := \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$. This is an ind-proper ind-scheme. Thus we have the "abstract" category $\mathcal{M}(\mathcal{GR})$ of $\mathcal{D}$-modules on $\mathcal{GR}$ defined as $\varinjlim \mathcal{M}(Y)$ where $Y$ runs over the set of all closed subschemes $Y \subset \mathcal{GR}$.

We are not able to represent $\mathcal{GR}$ as a union of an increasing sequence of smooth subschemes. However $\mathcal{GR}$ is a *formally smooth* ind-scheme. This permits to treat $\mathcal{D}$-modules on $\mathcal{GR}$ as "concrete" objects in the same way as if $\mathcal{GR}$ were a smooth finite dimensional variety, i.e., to identify them with certain sheaves of $\mathcal{O}$-modules equipped with some extra structure. Namely, assume we have an $\mathcal{O}$-module $P$ on $\mathcal{GR}$ such that each local section of $P$

is supported on some subscheme of $\mathcal{GR}$. Then one easily defines what is a continuous *right* action of $\operatorname{Der}\mathcal{O}_{\mathcal{GR}}$ on $P$. Such $P$ equipped with such an action is the same as a $\mathcal{D}$-module on $\mathcal{GR}$ (we also assume an appropriate quasi-coherency condition). Details can be found in ???.

**5.2.11.** *Remark.* We see that it is the *right* $\mathcal{D}$-modules that make sense as sheaves in this infinite dimensional setting. The reason for this is quite finite dimensional. Indeed, if $i : Y \hookrightarrow Z$ is a closed embedding of smooth manifolds and $M$ is a $\mathcal{D}$-module on $Y$ then in order to identify $M$ with a subsheaf of $i_*M$ one needs to consider *right* $\mathcal{D}$-modules.

**5.2.12.** According to 3.4.3 one has the groupoid $Z\operatorname{tors}_\theta(O)$, which is the local analog of $Z\operatorname{tors}_\theta(X)$. A choice of $\mathcal{L} \in Z\operatorname{tors}_\theta(O)$ (which essentially amounts to that of square root of $\omega_O$) defines the "local" Pfaffian line bundle $\lambda_{\mathcal{L}}^{\mathrm{loc}}$ on $\mathcal{GR}$ (see 4.6). The action of $\mathfrak{g}\otimes K$ on $\mathcal{GR}$ by left infinitesimal translations lifts to the action of the central extension $\widetilde{\mathfrak{g}\otimes K}$ from 2.5.1 on $\lambda_{\mathcal{L}}^{\mathrm{loc}}$ such that $\mathbf{1} \in \mathbb{C} \subset \widetilde{\mathfrak{g}\otimes K}$ acts as multiplication by $-1$ (see 4.6.5). This yields an antihomomorphism $\overline{U}' \to \Gamma(\mathcal{GR}, \mathcal{D}')$ where $\overline{U}' = \overline{U}'(\mathfrak{g}\otimes K)$ is the completed twisted universal enveloping algebra defined in 2.9.4 and $\Gamma(\mathcal{GR}, \mathcal{D}')$ is the ring of $\lambda_{\mathcal{L}}^{\mathrm{loc}}$-twisted differential operators on $\mathcal{GR}$. Hence for any $\mathcal{D}$-module $M$ on $\mathcal{GR}$ the algebra $\overline{U}'$ acts on $M\lambda_{\mathcal{L}}^{-1} := M \otimes_{\mathcal{O}_{\mathcal{GR}}} (\lambda_{\mathcal{L}}^{\mathrm{loc}})^{\otimes -1}$. So $\Gamma(\mathcal{GR}, M\lambda_{\mathcal{L}}^{-1})$ is a (left) $\overline{U}'$-module.

For example, consider the $\mathcal{D}$-module $I_1$ of $\delta$-functions at the distinguished point of $\mathcal{GR}$. The $\overline{U}'$-module $\Gamma(\mathcal{GR}, I_1\lambda_{\mathcal{L}}^{-1})$ is the vacuum module $\mathit{Vac}'$.

**5.2.13.** Recall (see 4.5.8) that $\mathcal{GR}$ is stratified by $G(O)$-orbits $\mathrm{Orb}_\chi$ labeled by $\chi \in P_+({}^L G)$. Denote by $I_\chi$ the irreducible "intersection cohomology" $\mathcal{D}$-module on $\mathcal{GR}$ that corresponds to $\overline{\mathrm{Orb}_\chi}$.

Here is the first part of our main local theorem.

**5.2.14.** *Theorem.* The $\overline{U}'$-module $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})$ is isomorphic to a sum of several copies of $\mathit{Vac}'$, and $H^i(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1}) = 0$ for $i > 0$.

*Remark.* This theorem means (see 5.4.8, 5.4.10) that the Harish-Chandra module $Vac'$ is an eigenmodule of the Harish-Chandra version of the Hecke functors from 7.8.2, 7.14.1.

**5.2.15.** The group $\operatorname{Aut} O$ acts on $\mathcal{GR}$, and the action of its Lie algebra $\operatorname{Der} O$ lifts to $\lambda_{\mathcal{L}}^{\mathrm{loc}}$ (see 4.6.7). The second part of our theorem describes the action of $\operatorname{Der} O$ on $\Gamma(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1})$.

Consider the scheme of local ${}^L\mathfrak{g}$-opers $\mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g}}(O) = \operatorname{Spec} A_{{}^L\mathfrak{g}}(O)$ from 3.2.1. Write $A$ instead of $A_{{}^L\mathfrak{g}}(O)$. Just as in 5.2.8 $\mathcal{L}$ defines a family of ${}^LG$-opers on $\operatorname{Spec} O$ parametrized by $\operatorname{Spec} A$. This family defines an ${}^LG$-torsor $\mathfrak{F}_A$ over $\operatorname{Spec} A$ equipped with an action of $\operatorname{Der} O$ compatible with its action on $A$; see 3.5.4*). The $\mathfrak{F}_A$-twist of the ${}^LG$-module $V^\chi$ is a vector bundle over $\operatorname{Spec} A$. Denote by $V^\chi_{\mathcal{L}A}$ the $A$-module of its sections; $\operatorname{Der} O$ acts on it.

**5.2.16.** *Theorem.* There is a canonical isomorphism of $\overline{U}'$-modules

$$\Gamma(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1}) \xrightarrow{\sim} Vac' \otimes_A V^\chi_{\mathcal{L}A} \tag{257}$$

compatible with the action of $\operatorname{Der} O$.

Here we use the $A$-module structure on $Vac'$ that comes from the Feigin–Frenkel isomorphism (80).

**5.2.17.** A few words about the proofs. The global theorem follows from the local one by an easy local-to-global argument similar to that used in 2.8. The proof of the local theorem is based on the interplay of the following two key structures:

(i) The *Satake equivalence* ([Gi95], [MV]) between the tensor category of representations of ${}^LG$ and the category of $\mathcal{D}$-modules on $\mathcal{GR}$ generated by $I_\chi$'s equipped with the "convolution" tensor structure.

(ii) The *"renormalized" enveloping algebra* $U^\natural$. The morphism of algebras $\overline{U}' \to \Gamma(\mathcal{GR}, \mathcal{D}')$ is neither injective (it kills the annihilator

*)In 3.5.4 we used the notation $\mathfrak{F}^0_G$ instead of $\mathfrak{F}_A$ and we considered the "particular" case where $\mathcal{L}$ is a square root of $\omega_O$.

$I$ of $Vac'$ in the center $\mathfrak{Z}$ of $\overline{U}'$) nor surjective (its image does not contain $\mathrm{Der}\, O$). We decompose it as $\overline{U}' \to U^\natural \to \Gamma(\mathcal{GR}, \mathcal{D}')$ where $U^\natural$ is obtained by "adding" to $\overline{U}'/I\overline{U}'$ the algebroid $I/I^2$ from 3.6.5 (the commutation relations between $\mathfrak{z}_\mathfrak{g}(O) = \mathfrak{Z}/I \subset \overline{U}'/I\overline{U}'$ and $I/I^2$ come from the algebroid structure on $I/I^2$, they are almost of Heisenberg type). The vacuum representation $Vac'$ is irreducible as an $U^\natural$-module; the same is true for $\Gamma(\mathcal{GR}, I_\chi\lambda^{-1})$, $\chi \in P_+({}^LG)$.

**5.2.18.** Here is the idea of the proof of 5.2.16 (we assume 5.2.14). Set $\mathfrak{z} := \mathfrak{z}_\mathfrak{g}(O)$. Consider the $\mathfrak{z}$-modules $V^\chi_{\mathcal{L}\mathfrak{z}} := \mathrm{Hom}_{\overline{U}'}(Vac', \Gamma(\mathcal{GR}, I_\chi\lambda_\mathcal{L}^{-1}))$, so $\Gamma(\mathcal{GR}, I_\chi\lambda_\mathcal{L}^{-1}) = Vac' \underset{\mathfrak{z}}{\otimes} V^\chi_{\mathcal{L}\mathfrak{z}}$. Some Tannakian formalism joint with Satake equivalence yields a canonical ${}^LG$-torsor $\mathfrak{F}_\mathfrak{z}$ over $\mathrm{Spec}\,\mathfrak{z}$ such that $V^\chi_{\mathcal{L}\mathfrak{z}}$ are $\mathfrak{F}$-twists of $V^\chi$. The $U^\natural$-module structure on $\Gamma(\mathcal{GR}, I_\chi\lambda_\mathcal{L}^{-1})$ defines the action of the Lie algebroid $I/I^2$ on $\mathfrak{F}_\mathfrak{z}$. Some extra geometric considerations define a canonical $B$-structure on $\mathfrak{F}_\mathfrak{z}$, which satisfies the "oper" property with respect to the action of $\mathrm{Der}\, O \subset I/I^2$. Now the results of 3.5, 3.6 yield a canonical identification $(\mathrm{Spec}\,\mathfrak{z}, \mathfrak{F}_\mathfrak{z}) \xrightarrow{\sim} (\mathrm{Spec}\, A, \mathfrak{F}_A)$ such that $A \xrightarrow{\sim} \mathfrak{z}$ is the Feigin–Frenkel isomorphism, and we are done.

**5.2.19.** DO WE NEED IT???

Here is a direct construction of $M$ that does not appeal to twisted $\mathcal{D}$-modules. For $x \in X$ consider the scheme $\mathrm{Bun}_{G,\bar{x}}$ (see 2.3.1). For $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ denote by $\lambda_{\mathcal{L},\bar{x}}$ the pull-back of the line bundle $\lambda_\mathcal{L}$ to $\mathrm{Bun}_{G,\bar{x}}$. Let $\widetilde{\mathfrak{g} \otimes K_x}$ be the central extension of $\mathfrak{g} \otimes K_x$ from 2.5.1, so the $\mathfrak{g} \otimes K_x$-action on $\mathrm{Bun}_{G,\bar{x}}$ lifts canonically to a $\widetilde{\mathfrak{g} \otimes K_x}$-action on $\lambda_{\mathcal{L},\bar{x}}$ such that $\mathbf{1} \in \mathbb{C}$ acts as identity (see 4.4.12). Denote by $\mathrm{Bun}_{G,\mathcal{L},\bar{x}}$ the space of the $\mathbb{G}_m$-torsor over $\mathrm{Bun}_{G,\bar{x}}$ that corresponds to $\lambda_{\mathcal{L},\bar{x}}$. We have a Harish-Chandra pair $(\widetilde{\mathfrak{g} \otimes K_x}, \mathbb{G}_m \times G(O_x)), \mathrm{Lie}\, \mathbb{G}_m = \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K_x}$. The $\widetilde{\mathfrak{g} \otimes K_x}$-action on $\mathrm{Bun}_{G,\mathcal{L},\bar{x}}$ extends to the action of this pair in the obvious way.

Note that $\mathrm{Bun}_G = \mathbb{G}_m \times G(O_x) \setminus \mathrm{Bun}_{G,\mathcal{L},\bar{x}}$. Therefore by 1.2.4 and 1.2.6 we have the functor $\Delta_\mathcal{L} : \left(\widetilde{\mathfrak{g} \otimes K_x}, \mathbb{G}_m \times G(O_x)\right)$ mod $\to \mathcal{M}^\ell(\mathrm{Bun}_G)$.

Consider the projection $\mathbb{G}_m \times G(O_x) \to \mathbb{G}_m$ as a character; let $Vac^{\sim}$ be the corresponding induced Harish-Chandra module. One has

$$M_{\mathcal{L}} = \Delta_{\mathcal{L}}(Vac^{\sim}). \tag{258}$$

Let us identify the $A_{^L\mathfrak{g}}(X)$-module structure on $M_{\mathcal{L}}$. The action of $\operatorname{End}(Vac^{\sim}) = \mathfrak{z}_{\mathfrak{g}}(O_x)$ on $\Delta_{\mathcal{L}}(Vac^{\sim})$ identifies, via Feigin-Frenkel's isomorphism $\varphi_{O_x}$ (see 3.2.2) with an $A_{^L\mathfrak{g}}(O_x)$-action. This action factors through the quotient $A_{^L\mathfrak{g}}(X)$.

**5.3. The Satake equivalence.** We recall the basic facts and constructions, and fix notation. For details and proofs see [MV]. The authors of [MV] use perverse sheaves; we use $\mathcal{D}$-modules.

**5.3.1.** Consider the affine (or loop) Grassmannian $\mathcal{GR} = G(K)/G(O)$ (as usual $K = \mathbb{C}((t))$, $O = \mathbb{C}[[t]]$); this is a formally smooth ind-projective ind-scheme (see 4.5.1). It carries the stratification by $G(O)$-orbits $\mathrm{Orb}_\chi$, $\chi \in P_+(^LG)$ (see 4.5.8). Each stratum is $\mathrm{Aut}^0 O$-invariant.

In 4.5.10 we introduced the notion of parity of a connected component of $\mathcal{GR}$. According to 4.5.11

(259) All the strata of an even (resp. odd) component of $\mathcal{GR}$ have even (resp. odd) dimension.

**5.3.2.** *Lemma.*

(i) Each stratum $\mathrm{Orb}_\chi$ is connected and simply connected.
(ii) Any smooth $\mathcal{D}$-module on $\mathrm{Orb}_\chi$ is constant.
(iii) $\mathrm{Orb}_\chi$ has cohomology only in even degrees.

*Proof.* Denote by $\mathrm{Stab}_x$ the stabilizer of $x \in \mathcal{GR}$ in $G(O)$. The image of $\mathrm{Stab}_x$ in $G(O/tO) = G$ is a parabolic subgroup $P_x$ and the morphism $G(O)/\mathrm{Stab}_x \to G/P_x$ is a locally trivial fibration whose fibers are isomorphic to an affine space. Now (i) and (iii) are clear. Notice that $\overline{\mathrm{Orb}_\chi}$ is projective and according to (259) $\overline{\mathrm{Orb}_\chi}\setminus\mathrm{Orb}_\chi$ has codimension $\geq 2$. So by

Deligne's theorem[*)] a smooth $\mathcal{D}$-module on $\mathrm{Orb}_\chi$ has regular singularities and therefore (ii) follows from (i).

$\square$

Denote by $\mathcal{P}$ the category of coherent (or, equivalently, holonomic) $\mathcal{D}$-modules on $\mathcal{GR}$ smooth along our stratification.

**5.3.3.** *Proposition.*

(i) The category $\mathcal{P}$ is semisimple.

(ii) If $M \in \mathcal{P}$ is supported on an even (resp. odd) component then $H^a_{DR}(\mathcal{GR}, M) = 0$ if $a$ is odd (resp. even).

*Proof.* Denote by $I_\chi$ the intersection cohomology perverse sheaf of $\mathbb{C}$-vector spaces on $\overline{\mathrm{Orb}_\chi}$. Denote by $\mathcal{GR}_{(\chi)}$ the connected component of $\mathcal{GR}$ containing $\mathrm{Orb}_\chi$ and by $p(\chi)$ the parity of $\mathcal{GR}_{(\chi)}$. According to Lusztig (Theorem 11c from [Lu83] ) $I_\chi$ has the following property: the cohomology sheaves $H^i(I_\chi)$ are zero unless $i \mod 2 = p(\chi)$. Denote by $C$ the category of all objects of $D^b(\mathcal{GR}_{(\chi)})$ having this property and smooth along our stratification. It follows from (259) and 5.3.2 (iii) that for any $M, N \in C$ one has $H^i(\mathcal{GR}_{(\chi)}, M) = 0$ unless $i \mod 2 = p(\chi)$ and $\mathrm{Ext}^i(M, N^*) = 0$ for odd $i$ (here $N^*$ is the Verdier dual of $N$). In particular $H^i(\mathcal{GR}, I_\chi) = 0$ unless $i \mod 2 = p(\chi)$ and $\mathrm{Ext}^1(I_{\chi_1}, I_{\chi_2}) = 0$. Using 5.3.2 (ii) one gets the Proposition. $\square$

**5.3.4.** According to 5.3.2 (ii) the simple objects of $\mathcal{P}$ are "intersection cohomology" $\mathcal{D}$-modules $I_\chi$ of the strata $\mathrm{Orb}_\chi$. Thus 5.3.3 (i) implies that any object of $\mathcal{P}$ has a structure of $G(O)$-equivariant or $\mathrm{Aut}^0 O \ltimes G(O)$-equivariant $\mathcal{D}$-module. Such structure is unique and any morphism is compatible with it (since our groups are connected). We see that

[*)] Instead of using Deligne's theorem one can notice that for any vector bundle on $\mathrm{Orb}_\chi$ its analytic sections are algebraic. Applying this to horizontal analytic sections of a vector bundle on $\mathrm{Orb}_\chi$ equipped with an integrable connection one sees that (ii) follows from (i).

$\mathcal{P}$ coincides with the category of $G(O)$-equivariant or $\mathrm{Aut}^0 O \ltimes G(O)$-equivariant coherent $\mathcal{D}$-modules on $\mathcal{GR}$.

*Remark.* The existence of $G(O)$-equivariant structure follows also directly from the facts that $G(O)$ is connected and $\mathrm{Hom}(G(O), \mathbb{G}_m) = 0$ (and 5.3.2 (ii)); one needs not to evoke 5.3.3 (i) and therefore Lusztig's theorem (which is a deep result).

**5.3.5.** The category $\mathcal{P}$ carries a canonical tensor structure. There are two ways to describe it: the "convolution" construction (see 5.3.5 - 5.3.9) and the "fusion" construction (presented, after certain preliminaries of 5.3.10 - 5.3.12, in 5.3.13 - 5.3.16); for the equivalence of these definitions see 5.3.17. We begin with the convolution picture [*)]. We have to define the convolution product functor $\circledast : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, the associativity constraint for $\circledast$, and the commutativity constraint.

According to [MV] the functor $\circledast$ is defined as follows. Denote by $G(K) \times_{G(O)} \mathcal{GR}$ the quotient of $G(K) \times \mathcal{GR}$ by $G(O)$ where $u \in G(O)$ acts on $G(K) \times \mathcal{GR}$ by $(g, x) \mapsto (gu^{-1}, ux)$. The morphism $p : G(K) \times_{G(O)} \mathcal{GR} \to G(K)/G(O) = \mathcal{GR}$ defined by $(g, x) \mapsto g \bmod G(O)$ is the locally tivial fibration with fiber $\mathcal{GR}$ associated to the principal $G(O)$-bundle $G(K) \to \mathcal{GR}$ and the action of $G(O)$ on $\mathcal{GR}$. So $G(K) \times_{G(O)} \mathcal{GR}$ is a twisted form of $\mathcal{GR} \times \mathcal{GR}$. Let $M, N \in \mathcal{P}$. Using the $G(O)$-equivariant structure on $M$ one defines a $\mathcal{D}$-module $M \boxtimes' N$ on $G(K) \times_{G(O)} \mathcal{GR}$, which is a "twisted form" of $M \boxtimes N$. Then

$$M \circledast N = m_*(M \boxtimes' N) \tag{260}$$

where $m : G(K) \times_{G(O)} \mathcal{GR} \to \mathcal{GR}$ comes from the action of $G(K)$ on $\mathcal{GR}$.

**5.3.6.** *Miraculous Theorem (Lusztig).* If $M, N \in \mathcal{P}$ then $M \circledast N \in \mathcal{P}$. $\square$

[*)] What follows is an algebraic version of Ginzburg's topological construction [Gi95]; we leave it to the interested reader to identify the two constructions.

*Remark.* The nontrivial statement is that $M \circledast N$ is a $\mathcal{D}$-module (not merely an object of the derived category). Since this $\mathcal{D}$-module is coherent and $G(O)$-equivariant it belongs to $\mathcal{P}$.

*Remark.* G. Lusztig's original proof [Lu83] of 5.3.6 is based on explicit estimates showing that the map $m$ restricted to the (twisted) product of two strata in $\mathcal{GR}$ is semi-small. Much later D. Gaitsgory observed [Ga2001] that 5.3.6 is immediate once we take in consideration geometric structures involved in the definition of the fusion tensor product.

So we have defined $\circledast : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$. The associativity constraint for $\circledast$ is defined in the obvious way. The commutativity constraint will be defined in 5.3.8.

**5.3.7.** *Remarks.* (i) Suppose that $G(K)$ is replaced by an ind-affine group ind-scheme $\mathcal{G}$ and $G(O)$ by its closed group subscheme $\mathcal{K}$; assume that $\mathcal{G}/\mathcal{K}$ is an ind-scheme of ind-finite type. The construction of $\circledast : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ from 5.3.5 is based on the miracle 5.3.6. In general there is no convolution on the category of $\mathcal{K}$-equivariant $\mathcal{D}$-modules on $\mathcal{G}/\mathcal{K}$ and one has to consider a certain derived category $\mathcal{H}$ (the *Hecke monoidal category;* see 7.6.1 and 7.11.17). This is a triangulated category with a t-structure whose core is the category of $\mathcal{K}$-equivariant $\mathcal{D}$-modules on $\mathcal{G}/\mathcal{K}$; in general $\circledast : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is not t-exact and there is no commutativity constraint for $\circledast$. In the case of $(G(K), G(O))$ the functor $\circledast$ is t-exact by 5.3.6 and the core of $\mathcal{H}$ is the category of ind-objects of $\mathcal{P}$.

(ii) The construction of $\mathcal{H}$ mentioned above is a part of the "Hecke pattern" developed in §7. Later we will see that this pattern is useful (or maybe indispensable) even in the miraculously good situation of $(G(K), G(O))$.

**5.3.8.** Let us define the commutativity constraint for $\circledast$. Let $\theta : G \to G$ be an automorphism that sends any dominant weight to its dual. The anti-automorphism $\theta'(g) := \theta(g)^{-1}$ of $G$ yields an anti-automorphism $\theta'_{\mathcal{H}}$ of the monoidal category $\mathcal{H}$, so for any $M, N \in \mathcal{H}$ one has a canonical isomorphism $l_{M,N} : \theta'_{\mathcal{H}}(M \circledast N) \stackrel{\sim}{\rightarrow} \theta'_{\mathcal{H}}(N) \circledast \theta'_{\mathcal{H}}(M)$.

For any $M \in \mathcal{P} \subset \mathcal{H}$ there is a canonical isomorphism $e_M : M \stackrel{\sim}{\rightarrow} \theta'_{\mathcal{H}}(M)$. To define $e_M$ it suffices, according to 5.3.3 (i), to consider the case $M = I_\chi$. The action of $\theta'$ on $G(K)$ preserves the stratification $G(K)_\chi$ by the double $G(O)$-classes (here $G(K)_\chi$ is the preimage of $\mathrm{Orb}_\chi \subset G(K)/G(O)$). So we have the induced automorphism $\theta'_\chi$ of $G(K)_\chi$. As an object of $\mathcal{H}$ our $I_\chi$ is the $\Omega$-complex $\Omega_{G(K)_\chi}[\dim \mathrm{Orb}_\chi]$ on $G(K)$. Now $e_{I_\chi}$ is the action of $\theta'_\chi$ on $\Omega_{G(K)_\chi}$.

For $M, N \in \mathcal{P}$ define

$$s : M \circledast N \stackrel{\sim}{\rightarrow} N \circledast M \tag{261}$$

as the composition

$$M \circledast N \stackrel{\sim}{\rightarrow} \theta'_{\mathcal{H}}(M \circledast N) \stackrel{\sim}{\rightarrow} \theta'_{\mathcal{H}}(N) \circledast \theta'_{\mathcal{H}}(M) \stackrel{\sim}{\rightarrow} N \circledast M$$

where the first arrow is the isomorphism $e$ corresponding to $M \circledast N$ and the other arrows are $l_{M,N}$ and $e_N^{-1} \circledast e_M^{-1}$.

**5.3.9.** *Proposition.* $s$ is a commutativity constraint for the convolution tensor product $\circledast$.

*Proof.* In 5.3.17 below we identify the convolution tensor product with the fusion tensor product in a way compatible with all the constraints. Since the latter data obviously define a tensor category structure on $\mathcal{P}$ we are done. □

So we have defined the promised convolution tensor structure on $\mathcal{P}$.

**5.3.10.** The fusion description of the tensor structure on $\mathcal{P}$ [*)] is based on the important *chiral semigroup* structure on the "space" GRAS $= \mathrm{GRAS}_G$ from 4.3.14. This structure may be described as follows.

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[*)]The construction apparently involves a curve $X$, but actually it is purely local.

(i) For a $\mathbb{C}$-algebra $R$ and $S \in \Sigma(R)$ (we use notation from 4.3.11, so $S$ is a subscheme of $X \otimes R$ finite and flat over $\operatorname{Spec} R$) one has a subset $\mathrm{GRAS}(R)_S \subset \mathrm{GRAS}(R)$ defined as the set of pairs $(\mathcal{F}, \gamma)$ where $\mathcal{F}$ is a $G$-torsor on $X \otimes R$, $\gamma$ is a section of $\mathcal{F}$ over the complement to $S$.

(ii) If $S$ is a disjoint union of subschemes $S_i$, $i \in I$, then one has a canonical identification

$$\mathrm{GRAS}(R)_S \xrightarrow{\sim} \prod_i \mathrm{GRAS}(R)_{S_i} \tag{262}$$

Namely, we identify $(\mathcal{F}, \gamma)$ with the collection $(\mathcal{F}_i, \gamma_i)$, $i \in I$, where $(\mathcal{F}_i, \gamma_i) \in \mathrm{GRAS}(R)_{S_i}$ coincides with $(\mathcal{F}, \gamma)$ over the complement to the union of $S_{i'}$, $i' \neq i$.

The data (i), (ii) enjoy the following properties:

a. If for $S_1, S_2 \in \Sigma(R)$ one has $S_{1red} \subset S_{2red}$ then $\mathrm{GRAS}(R)_{S_1} \subset \mathrm{GRAS}(R)_{S_2}$. The union of $\mathrm{GRAS}(R)_S$, $S \in \Sigma(R)$, coincides with $\mathrm{GRAS}(R)$. So $\mathrm{GRAS}(R)_S$ form a filtration on $\mathrm{GRAS}(R)$. This filtration is functorial (with respect to $R$).

b. The isomorphisms (ii) are also functorial and compatible with subdivisions of $I$ in the obvious manner.

c. The subfunctor $\mathcal{GR}_\Sigma \subset \Sigma \times \mathrm{GRAS}$ defined by

$$\mathcal{GR}_\Sigma(R) := \{(S, \mathcal{F}, \gamma) | S \in \Sigma(R), (\mathcal{F}, \gamma) \in \mathrm{GRAS}(R)_S\}$$

is an ind-scheme formally smooth over $\Sigma$.

*Remark.* Let us explain why $\mathcal{GR}_\Sigma = \mathcal{GR}_\Sigma^G$ is an ind-scheme for any affine algebraic group $G$. Moreover we will show that $\mathcal{GR}_\Sigma$ is of ind-finite type and if $G$ is reductive then $\mathcal{GR}_\Sigma$ is ind-proper. First consider the case $G = GL_n$. Then $\mathcal{GR}_\Sigma$ is the direct limit of $\mathcal{GR}_{\Sigma,k}$ where $\mathcal{GR}_{\Sigma,k}$ parametrizes pairs consisting of a finite subscheme $D \subset X$ and a subsheaf $\mathcal{E} \subset \mathcal{O}_X^n(kD)$ such that $\mathcal{E} \supset \mathcal{O}_X^n(-kD)$. The morphism $\mathcal{GR}_{\Sigma,k} \to \Sigma$ is proper, so $\mathcal{GR}_\Sigma$ is ind-proper. As explained in the proof of Theorem 4.5.1, to reduce the general case to the case of $GL_n$ it suffices to show that if $G \subset G'$ and $G'/G$ is affine (resp. quasiaffine) then the morphism $\mathcal{GR}_\Sigma^G \to \mathcal{GR}_\Sigma^{G'}$ is a closed (resp. locally closed) embedding. This is easy.

**5.3.11.** For a finite set $J$ we have the morphism $X^J \to \Sigma$ that assigns to $(x_j) \in X^J$ the subscheme $D \subset X$ corresponding to the divisor $\sum\limits_j x_j$. Denote by $\mathcal{GR}_{X^J}$ the fibered product of $\mathcal{GR}_\Sigma$ and $X^J$ over $\Sigma$. So an $R$-point of $\mathcal{GR}_{X^J}$ is a collection $((x_j), \mathcal{F}, \gamma)$ where $(x_j) \in X^J(R)$, $\mathcal{F}$ is a $G$-bundle on $X \otimes R$, and $\gamma$ is a section of $\mathcal{F}$ over the complement to the union of the graphs of the $x_j$'s. Our $\mathcal{GR}_{X^J}$ is a formally smooth ind-proper ind-scheme over $X^J$ (see the Remark at the end of 5.3.10).

According to 4.5.2 there is a canonical isomorphism between the fiber of $\mathcal{GR}_X$ over $x \in X(\mathbb{C})$ and the ind-scheme $\mathcal{GR}_x := G(K_x)/G(O_x)$. So according to 5.3.10 (ii) the fiber of $\mathcal{GR}_{X^J}$ over $(x_j) \in X^J(\mathbb{C})$ equals $\prod\limits_{x \in S} \mathcal{GR}_x$ where $S$ is the subset $\{x_j\} \subset X$.

The following description of $\mathcal{GR}_X$ will be of use. Consider the scheme $X^\wedge$ of "formal parameters" on $X$ (its points are smooth morphisms $\operatorname{Spec} O \to X$, see 2.6.5). This is an $\operatorname{Aut}^0 O$-torsor over $X$; a choice of coordinate,i.e., étale $\mathbb{A}^1$-valued map, on an open $U \subset X$ defines a trivialization of $X^\wedge$ over $U$. Now $\mathcal{GR}_X$ is the $X^\wedge$-twist of $\mathcal{GR}$ (with respect to the $\operatorname{Aut}^0 O$-action on $\mathcal{GR}$).

The stratification of $\mathcal{GR}$ defines a stratification of $\mathcal{GR}_X$ by strata $\operatorname{Orb}_{\chi X}$ smooth over $X$.

**5.3.12.** For the future references let us list some of the compatibilities between $\mathcal{GR}_{X^J}$'s that follow directly from 5.3.10.

a. For a surjective map $\pi : J \twoheadrightarrow J'$ there is an obvious Cartesian diagram

$$\begin{array}{ccc} \mathcal{GR}_{X^{J'}} & \overset{\widetilde{\Delta}^{(\pi)}}{\hookrightarrow} & \mathcal{GR}_{X^J} \\ \downarrow & & \downarrow \\ X^{J'} & \overset{\Delta^{(\pi)}}{\hookrightarrow} & X^J \end{array} \tag{263}$$

where $\Delta^{(\pi)}$ is the $\pi$-diagonal embedding. If $|J'|=1$ we have $\Delta^{(J)}: X\hookrightarrow X^J$ and $\widetilde{\Delta}^{(J)}: \mathcal{GR}_X\hookrightarrow \mathcal{GR}_{X^J}$.

b. Let $\nu^{(J)}: U^{(J)}\hookrightarrow X^J$ be the complement to the diagonal divisor. By 5.3.10 (ii) the restrictions to $U^{(J)}$ of the $X^J$-ind-schemes $\mathcal{GR}_{X^J}$ and $(\mathcal{GR}_X)^J$ are canonically identified. Therefore we have a Cartesian diagram

$$\begin{array}{ccc} (\mathcal{GR}_X)^J\big|_{U^{(J)}} & \overset{\widetilde{\nu}^{(J)}}{\hookrightarrow} & \mathcal{GR}_{X^J} \\ \downarrow & & \downarrow \\ U^{(J)} & \overset{\nu^{(J)}}{\hookrightarrow} & X^J \end{array} \tag{264}$$

**5.3.13.** Now we are ready to define the fusion tensor structure on $\mathcal{P}$. This amounts to a construction of tensor product functors *)

$$\underset{J}{\circledast}: \mathcal{P}^{\otimes J}\to\mathcal{P} \tag{265}$$

for any finite non-empty set $J$ together with identifications

$$\underset{J}{\circledast}=\underset{J'}{\circledast}\big(\underset{j'\in J'}{\otimes}\big(\underset{\pi^{-1}(j')}{\circledast}\big)\big) \tag{266}$$

for any surjective map $J\overset{\pi}{\twoheadrightarrow} J'$.

The construction goes as follows.

**5.3.14.** Since any $M\in\mathcal{P}$ is $\operatorname{Aut}^0 O$-equivariant it defines a $\mathcal{D}$-module on $\mathcal{GR}_X$ (see the description of $\mathcal{GR}_X$ at the end of 5.3.11). Denote by $M_X\in D(\mathcal{GR}_X)(:=D\mathcal{M}(\mathcal{GR}_X))$ its shift by 1 in the derived category. In other words for any open $U$ as above and a trivialization $\theta$ of $X^\wedge$ over $U$ one has $M_U=\pi_\theta^! M$, where $M_U:=M_X\big|_{\mathcal{GR}_U}$, $\pi_\theta:\mathcal{GR}_U\to\mathcal{GR}$ is the projection that corresponds to $\theta$, and we glue these objects together using the $\operatorname{Aut}^0 O$-action on $M$. The functor $\mathcal{P}\to D(\mathcal{GR}_X)$, $M\mapsto M_X$, is fully faithful. Its essential image consists of (shifted by 1) $\mathcal{D}$-modules isomorphic to a direct sum of (finitely many) copies of "intersection cohomology" $\mathcal{D}$-modules $I_{\chi X}$ that correspond to the trivial local system on $\mathrm{Orb}_{\chi X}$.

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*)Here $\mathcal{P}^{\otimes J}$ denotes the tensor product of $J$ copies of $\mathcal{P}$ (since $\mathcal{P}$ is semisimple the definition of tensor product is clear).

Let now $\{M_j\}_{j\in J}$ be a collection of objects of $\mathcal{P}$. Using (264) one interprets $\boxtimes M_{jX}\big|_{U^{(J)}}$ as a $\mathcal{D}$-module on $\mathcal{GR}_{X^J}\big|_{U^{(J)}}$ shifted by $|J|$. Denote by $\circledast\!\!\!\!\Box\, M_{jX}\in D(\mathcal{GR}_{X^J})$ its minimal (i.e., $\widetilde{\nu}^{(J)}_{!*}-$) extension to $\mathcal{GR}_{X^J}$. This is a $\mathcal{D}$-module on $\mathcal{GR}_{X^J}$ shifted by $|J|$. Therefore we have defined a functor

$$(267)\qquad \underset{J}{\boxast}\colon \mathcal{P}^{\otimes J}\to D(\mathcal{GR}_{X^J}),\quad \otimes M_j\mapsto \boxast M_{jX}$$

which is obviously fully faithful.

**5.3.15.** *Proposition.* ([MV])

For any $\pi: J\twoheadrightarrow J'$ the complex $\widetilde{\Delta}^{(\pi)!}(\boxast M_{jX})\in D(\mathcal{GR}_X^{(J')})$ belongs to the essential image of $\underset{J'}{\boxast}$. □

**5.3.16.** We get a functor

$$(268)\qquad \underset{\pi}{\circledast}\colon \mathcal{P}^{\otimes J}\to\mathcal{P}^{\otimes J'}$$

such that $\underset{J'}{\boxast}\underset{\pi}{\circledast}=\widetilde{\Delta}^{(\pi)!}\underset{J}{\boxast}$. In particular for $|J'|=1$ we have the functor $\underset{J}{\circledast}:\mathcal{P}^{\otimes J}\to\mathcal{P}$ which is our tensor product functor (265). The obvious identification $\underset{\pi}{\circledast}=\underset{j'\in J'}{\otimes}(\underset{\pi^{-1}(j')}{\circledast})$ (look at our $\mathcal{D}$-modules over $U^{(J')}$) and the standard isomorphism $\Delta^{(J)!}=(\Delta^{(\pi)}\Delta^{(J')})^!=\Delta^{(J')!}\Delta^{(\pi)!}$ yield the compatibility isomorphisms (266). So $\mathcal{P}$ is a tensor category. It is easy to see that $I_0$ is a unit object in $\mathcal{P}$.

**5.3.17.** Let us identify the convolution and fusion tensor structures on $\mathcal{P}$. Below in this subsection we denote by $\circledast^c$ the convolution tensor product, and by $\circledast^f$ the fusion tensor product on $\mathcal{P}$. We have to construct for $M,N\in\mathcal{P}$ a canonical isomorphism $M\circledast^c N\xrightarrow{\sim} M\circledast^f N$ compatible with the associativity and commutativity constraints.*)

Let $\mathcal{GR}'_{X^2}$ be the ind-scheme over $X^2$ such that $\mathcal{GR}'_{X^2}(R)$ is the set of collections $(x_1,x_2,\mathcal{F}_1,\mathcal{F}_2,\gamma_1,\gamma_2)$ where $x_1,x_2\in X(R)$, $\mathcal{F}_1,\mathcal{F}_2$ are $G$-torsors over $X\otimes R$, $\gamma_1$ is a section of $\mathcal{F}_1$ over the complement to the

*)The construction is borrowed from [MV] where it is written in more details; however the commutativity constraint 5.3.8 was not considered there.

graph of $x_1$, $\gamma_2$ is an isomorphism $\mathcal{F}_1 \to \mathcal{F}_2$ over the complement to the graph of $x_2$. We have the projection $q : \mathcal{GR}'_{X^2} \to \mathcal{GR}_{X^2}$ that sends the above data to $((x_1, x_2), \mathcal{F}_2, \gamma_2\gamma_1)$. This projection is ind-proper; over $U := X^2 \setminus \{\text{the diagonal}\}$ it is an isomorphism.*)

Denote by $M_X \circledast' N_X \in D(\mathcal{GR}'_{X^2})$ the minimal extension to $\mathcal{GR}'_{X^2}$ of $M_X \boxtimes N_X\big|_U$. This is a $\mathcal{D}$-module on $\mathcal{GR}'_{X^2}$ shifted by 2. According to [MV] the obvious identification over $U$ extends (uniquely) to a canonical isomorphism

$$q_*(M_X \circledast' N_X) \xrightarrow{\sim} M_X \circledast N_X \tag{269}$$

Now $\mathcal{GR}'_{X^2}$ is a twisted form of $(\mathcal{GR}_X)^2$. Indeed, a trivialization of $\mathcal{F}_1$ on the formal neighbourhood of $x_2$ yields an identification of the data $(\mathcal{F}_2, \gamma_2)$ above with $\mathcal{GR}_x$. These trivializations together with formal parameters at $x_2$ form an $\mathrm{Aut}^0 O \ltimes G(O)$-torsor over $\mathcal{GR}_X \times X$, and $\mathcal{GR}'_{X^2}$ identifies with the corresponding twist of $\mathcal{GR}$. So $M_X \circledast' N_X$ is the "twisted form" of $M_X \boxtimes N$. Restricting this picture to the diagonal $X \hookrightarrow X \times X$ we see that the pull-back of $q : \mathcal{GR}'_{X^2} \to \mathcal{GR}_{X^2}$ to $X$ coincides with the $X^\wedge$-twist of the morphism $m : G(K) \times_{G(O)} \mathcal{GR} \to \mathcal{GR}$ from (260) and the pull-back of $M_X \circledast' N_X$ to the preimage of $X$ in $\mathcal{GR}'_{X^2}$ equals $(M \boxtimes' N)_X$ where $M \boxtimes' N$ has the same meaning as in (260). Comparing (269) and (260) (and using the base change isomorphism) we get the desired canonical isomorphism $M \circledast^c N \xrightarrow{\sim} M \circledast^f N$.

Its compatibility with the associativity constraints comes from the similar picture over $X^3$. WRITE DOWN THE COMAT WITH COM CONSTRAINTS (use $Bun_G$ and $Hecke$)!

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*)Over the diagonal the fibers of $q$ are isomorphic to $\mathcal{GR}$; more precisely, the closed embedding $\mathcal{GR}'_{X^2} \to (\mathcal{GR}_X) \times_X (\mathcal{GR}_{X^2})$ defined by $(x_1, x_2, \mathcal{F}_1, \mathcal{F}_2, \gamma_1, \gamma_2) \mapsto (x_1, x_2, \mathcal{F}_1, \gamma_1, \mathcal{F}_2, \gamma_2\gamma_1)$ becomes an isomorphism when restricted to the diagonal $X \hookrightarrow X^2$. So the maximal open subset over which $q$ is an isomorphism has the form $\mathcal{GR}_{X^2} \setminus Z$ where $Z$ has codimension 1; this is an infinite-dimensional phenomenon.

**5.3.18.** For $M \in \mathcal{P}$ set $h^{\cdot}(M) := H^{\cdot}_{DR}(\mathcal{GR}, M)$. This is a $\mathbb{Z}$-graded vector space; denote by $h^{\varepsilon}(M)$ the corresponding $\mathbb{Z}/2\mathbb{Z}$-graded vector space.

Consider the projection $p : \mathcal{GR}_X \to X$. The $\mathcal{D}$-modules $H^a p_*(M_X)$ on $X$ are constant, i.e., isomorphic to a sum of copies of $\omega_X$ (recall that we play with right $\mathcal{D}$-modules). The corresponding fiber is $h^{\cdot}(M)$: for any $x \in X$ one has $H^{\cdot} i_x^! p_*(M_X) = h^{\cdot}(M)$ (here $i_x$ is the embedding $\{x\} \hookrightarrow X$).

**5.3.19.** *Proposition.* ([MV])

For any collection $\{M_j\}_{j\in J}$ of objects of $\mathcal{P}$ the $\mathcal{D}$-modules $H^a p_*^{(J)}(\boxtimes M_{jX})$ on $X^J$ are constant. □

For any $(x_j) \in X^J$ one has

$$H^{\cdot} i^!_{(x_j)} p_*^{(J)}(\boxtimes M_{jX}) = \otimes h^{\cdot}(M_j)\,. \tag{270}$$

This is clear from 5.3.18 for $(x_j) \in U^{(J)}$; then use 5.3.19.

**5.3.20.** For $(x_j) \in X \subset X^J$ (270) yields a canonical isomorphism $h^{\cdot}(\circledast M_j) = \otimes h^{\cdot}(M_j)$ which is obviously compatible with "constraints" (266). We see that

$$h^{\cdot} : \mathcal{P} \to \mathrm{Vect}^{\cdot}\,, \qquad h^{\varepsilon} : \mathcal{P} \to \mathrm{Vect}^{\varepsilon} \tag{271}$$

are *tensor functors.* Here $\mathrm{Vect}^{\cdot}$ is the tensor category of $\mathbb{Z}$-graded vector spaces with the *"super"* commutativity constraint, $\mathrm{Vect}^{\varepsilon}$ is the analogous tensor category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces.

**5.3.21.** One may twist the tensor structure on $\mathcal{P}$ to get rid of super vector spaces. To do this note that the objects of $\mathcal{P}$ carry a canonical $\mathbb{Z}/2\mathbb{Z}$-grading $\varepsilon$ by parity of the components of support (see 4.5.10). This grading is compatible with $\circledast$.

Denote by $\mathcal{P}^{\natural}$ the full subcategory of even objects in $\mathcal{P}^{\varepsilon} := \mathcal{P} \otimes \mathrm{Vect}^{\varepsilon}$ (with respect to tensor product of the $\mathbb{Z}/2\mathbb{Z}$-gradings). This is a tensor subcategory in $\mathcal{P}^{\varepsilon}$. The "forgetting of the grading" functor $o_{\varepsilon} : \mathrm{Vect}^{\varepsilon} \to \mathrm{Vect}$ yields an equivalence $\mathcal{P}^{\natural} \overset{\sim}{\to} \mathcal{P}$. This is an equivalence of monoidal categories (i.e., it is compatible with the tensor products and associativity constraints);

the commutativity constraints $A \otimes B \xrightarrow{\sim} B \otimes A$ for $\mathcal{P}$ and $\mathcal{P}^\natural$ differ by $(-1)^{p(A)p(B)}$.

The functor $h^\varepsilon$ is compatible with the $\mathbb{Z}/2\mathbb{Z}$-gradings by 5.3.3 (ii). Therefore it defines a tensor functor

$$h : \mathcal{P}^\natural \to \mathrm{Vect}\ . \tag{272}$$

Note that $h$ carries a canonical $\mathbb{Z}$-grading which we denote also by $h^{\cdot}$ by abuse of notation. So $h^{\cdot}$ is a tensor functor on $\mathcal{P}^\natural$ with values in the tensor category of graded vector spaces equipped with the plain (*not* super) commutativity constraint.

**5.3.22.** According to [MV] (WHAT ABOUT GINZBURG ??) the tensor category $\mathcal{P}^\natural$ is rigid, i.e., each object has a dual in the sense of §2.1.2 from [Del91] (the dual objects are explicitly constructed in [MV]). The tensor functor (272) is $\mathbb{C}$-linear and exact,*) so it is a fiber functor in the sense of [Del91]. Therefore by the general Tannakian formalism (272) induces an equivalence between the tensor categories $\mathcal{P}^\natural$ and $\mathrm{Rep}(\mathrm{Aut}^{\otimes} h)$ where $\mathrm{Aut}^{\otimes} h$ denotes the group scheme of tensor automorphisms of $h$ and Rep means the category of finite-dimensional representations. According to [MV] there is an isomorphism $\varkappa : {}^LG \xrightarrow{\sim} \mathrm{Aut}^{\otimes} h$, so we may rewrite the above equivalence as

$$h : \mathcal{P}^\natural \xrightarrow{\sim} \mathrm{Rep}\,{}^LG\,. \tag{273}$$

Here ${}^LG$ is the Langlands dual group, i.e., it is a semisimple group together with a fixed Cartan torus ${}^LH \subset {}^LG$, an identification of the corresponding root datum with the dual to the root datum of $G$, and a collection of fixed non-zero vectors $y_\alpha \in ({}^L\mathfrak{g})^\alpha$ for simple negative roots $\alpha$.

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*)Exactness is clear since $\mathcal{P}^\natural$ is semisimple. Mirković and Vilonen [MV] have to prove exactness because they want their proofs to work for perverse sheaves over arbitrary commutative rings.

**5.3.23.** We are going to define a *canonical* isomorphism

$$\varkappa : {}^L G \xrightarrow{\sim} \operatorname{Aut}^{\otimes} h \tag{274}$$

by listing some properties of the action of ${}^L G$ on $h$, which determine $\varkappa$ uniquely.

(i) Denote by

$$t \mapsto t^{2\rho} \tag{275}$$

the morphism $\mathbb{G}_m \to {}^L H$ corresponding to the weight $2\rho$ of $G$. Then $t^{2\rho}$ acts on $h^a$ as multiplication by $t^{-a}$ (so the action of the 1-parameter subgroup (275) corresponds to the grading $h^{\cdot}$ of $h$).

It follows from (i) that the action of ${}^L H$ on $h$ preserves the grading of $h$.

(ii) For any $\chi \in P_+({}^L G)$ the group ${}^L H$ acts on $h^{\min}(I_\chi) = h^{-\dim \mathrm{Orb}_\chi}(I_\chi)$ by the character $\chi$.

This means that the highest weight of the irreducible ${}^L G$-module $h(I_\chi)$ equals $\chi$.

*Remark.* Since $\dim \mathrm{Orb}_\chi = \langle \chi, 2\rho \rangle$ there is no contradiction between (i) and (ii).

The properties (i) and (ii) can be found in [MV]. They uniquely determine the restriction of (274) to ${}^L H$. So (274) is determined by (i) and (ii) up to ${}^L H$-conjugation. We normalize (274) by the following property.

(iii) Let $c \in (\mathrm{Sym}^2 \mathfrak{g}^*)^G$ be an invariant bilinear form on $\mathfrak{g}$ (or on $[\mathfrak{g}, \mathfrak{g}]$ in the reductive case???). Set

$$f_c := \frac{???}{2} \sum_{\alpha} c(\alpha, \alpha) y_\alpha \in {}^L \mathfrak{g} \tag{276}$$

(the expression $c(\alpha, \alpha)$ makes sense because $\alpha \in ({}^L \mathfrak{h})^* = \mathfrak{h} \subset \mathfrak{g}$). Then the Lie algebra element $f_c$ acts on $h(M) = H^{\cdot}_{DR}(\mathcal{GR}, M)$, $M \in \mathcal{P}^\natural = \mathcal{P}$, as multiplication by $\nu(c)$ where

$$\nu : (\mathrm{Sym}^2 \mathfrak{g}^*)^G \to H^2_{DR}(\mathcal{GR}) \tag{277}$$

is the standard morphism whose definition will be reminded in 5.3.24.

*Remark.* (iii) is formulated by V.Ginzburg [Gi95] in a slightly different form. In fact, he describes in a similar way the action on $h$ of the whole centralizer of $f_c$ in ${}^L\mathfrak{g}$.

**5.3.24.** In this subsection (which can be skipped by the reader) we define the canonical morphism (277). We use the folowing ad hoc definition: for any ind-scheme $Z$ one has $H^a_{DR}(Z) := \varprojlim H^a(Y, \Omega_Y)$ where $Y$ runs over the set of all closed subschemes of $Z$ and $\Omega_Y$ is the de Rham complex of $Y$ (in the most naive sense). To define $\nu$ let us assume for simplicity (simplicity twice?? BAD STYLE) that $G$ is semisimple *). Then the projection $G(K) \to \mathcal{GR}$ induces an isomorphism $H^2_{DR}(\mathcal{GR}) \simeq H^2_{DR}(G(K))$ (indeed, this projection is a $G(O)$-torsor, $G(O)$ is connected, and $H^1_{DR}(G(O)) = H^2_{DR}(G(O)) = 0$). Now our $c$ defines the Kac-Moody cocycle $u, v \mapsto \mathrm{Res}_{t=0}\, c(du, v)$ on $\mathfrak{g} \otimes K$. Let $\omega_c$ be the corresponding right invariant closed 2-form on $G(K)$. The image of its class by the inverse map to the above isomorphism is $\nu(c) \in H^2_{DR}(\mathcal{GR})$. WHAT ABOUT THE SIGN???

*Remark.* In 5.3.23(iii) we used the action of $H^{\cdot}_{DR}(\mathcal{GR})$ on $H^{\cdot}_{DR}(\mathcal{GR}, M)$ where $M$ is a $\mathcal{D}$-module on $\mathcal{GR}$. It is defined as follows. Consider the $\Omega^!$-complex $\Omega M$ (see 7.11.13). Then $H^{\cdot}_{DR}(\mathcal{GR}, M) = \varinjlim H^{\cdot}(Y, \Omega M_{(Y)})$ where $Y$ runs over the set of all subschemes of $\mathcal{GR}$. Now $\Omega M_{(Y)}$ is an $\Omega$-complex on $Y$, so $H^{\cdot}(Y, \Omega_Y)$ acts on $H^{\cdot}(Y, \Omega M_{(Y)})$. Therefore $H^{\cdot}_{DR}(\mathcal{GR})$ acts on $H^{\cdot}_{DR}(\mathcal{GR}, M)$.

**5.3.25.** The brief characterization of the canonical isomorphism (274) given in 5.3.23 is enough for our purposes. Those who want to understand (274) better may read ???-??? and [MV].

**5.3.26.**

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*) We leave it to the reader to define $\nu$ for arbitrary $G$.

*Remark.* Recall (see 4.5.9) that the connected components of $\mathcal{GR}$ are labeled by elements of $Z({}^LG)^\vee$ where $Z({}^LG)^\vee$ is the group of characters of the center $Z({}^LG) \subset {}^LG$. The connected component of $\mathcal{GR}$ corresponding to $\zeta \in Z({}^LG)^\vee$ will be denoted by $\mathcal{GR}_\zeta$. The support decomposition $D(\mathcal{GR}) = \prod D(\mathcal{GR}_\zeta)$, $\mathcal{P} = \oplus \mathcal{P}_\zeta$ defines a $Z({}^LG)^\vee$-grading, i.e., a $Z({}^LG)$-action, on $h$. This action coincides with the one induced by the ${}^LG$-action.

In the rest of the section we explain how the above constructions are compatible with passage to a Levi subgroup of ${}^LG$. When this subgroup is ${}^LH \subset {}^LG$ this amounts to an explicit description of the action of ${}^LH$ on the fiber functor $h$ due to Mirković – Vilonen.

**5.3.27.** Let $P \subset G$ be a parabolic subgroup, $N_P \subset P$ its unipotent radical, $F := P/N_P$ the Levi group. The Cartan tori of $F$ and $G$ are identified in the obvious way, and the root datum for $F$ is a subset of that for $G$. So ${}^LF$ is a Levi subgroup of ${}^LG$ for the standard torus ${}^LH \subset {}^LF \subset {}^LG$. Thus $Z({}^LG) \subset Z({}^LF)$.

We are going to define a canonical tensor functor

$$r_{\mathcal{P}}^\natural : \mathcal{P}_G^\natural \to \mathcal{P}_F^\natural \tag{278}$$

which corresponds, via the equivalences $h_G$, $h_F$, to the obvious restriction functor $r^{GF} : \operatorname{Rep} {}^LG \to \operatorname{Rep} {}^LF$.

**5.3.28.** The diagram $G \hookleftarrow P \twoheadrightarrow F$ yields the morphisms of the corresponding affine Grassmanians

$$\mathcal{GR}^G \xleftarrow{\ i\ } \mathcal{GR}^P \xrightarrow{\ \pi\ } \mathcal{GR}^F\ . \tag{279}$$

Here $\pi$ is a formally smooth ind-affine surjective projection. Its fibers are $N_P(K)$-orbits. Hence $\pi$ yields a bijection between the sets of connected components of $\mathcal{GR}^P$ and $\mathcal{GR}^F$. For any $\zeta \in Z({}^LF)^\vee$ let $\mathcal{GR}^P_\zeta$ be the corresponding component. Then the restriction $i_\zeta : \mathcal{GR}^P_\zeta \hookrightarrow \mathcal{GR}^G$ of $i$ is a locally closed embedding; its image lies in $\mathcal{GR}^G_{\overline{\zeta}}$ where $\overline{\zeta} := \zeta|_{Z({}^LG)}$. The ind-schemes $\mathcal{GR}^P_\zeta$ form a stratification of $\mathcal{GR}^G$ (i.e., for any closed subscheme $Y \subset \mathcal{GR}^G$ the intersections $Y_\zeta := Y \cap \mathcal{GR}^P_\zeta$ form a stratification of $Y$).

Set $\rho_{GF} := \rho_G - \rho_F \in \mathfrak{h}^*$. Since $2\rho_{GF}$ is a character of $F$ (the determinant of the adjoint action on $n_P$) we may consider it as a one-parameter subgroup of $Z({}^L F) \subset {}^L H$. So for any $\zeta$ as above one has an integer $\langle \zeta, 2\rho_{GF}\rangle$. Let $\mathcal{GR}^F_n$ be the union of components $\mathcal{GR}^F_\zeta$ with $\langle \zeta, 2\rho_{GF}\rangle = n$. We have the corresponding decomposition $D(\mathcal{GR}^F) = \prod D(\mathcal{GR}^F_n)$, $\mathcal{P}^F = \oplus \mathcal{P}^F_n$. Set $\mathcal{P}^{F'} = \oplus \mathcal{P}^F_n[-n] \subset D(\mathcal{GR}^F)$. As in 5.3.18 for $M \in \mathcal{P}^{F'}$ we set $h^\cdot_F(M) = H^\cdot(\mathcal{GR}^F, M) \in \mathrm{Vect}^\cdot$.

**5.3.29.** *Proposition.*

(i) The functor $r^{GF}_D := \pi_* i^! : D(\mathcal{GR}^G) \to D(\mathcal{GR}^F)$ sends $\mathcal{P}^G$ to $\mathcal{P}^{F'}$, so we have

$$r^{GF}_{\mathcal{P}} : \mathcal{P}^G \to \mathcal{P}^{F'} . \tag{280}$$

(ii) There is a canonical identification of functors

$$h^\cdot_G = h^\cdot_F r^{GF}_{\mathcal{P}} : \mathcal{P}^G \to \mathrm{Vect}^\cdot . \tag{281}$$

*Proof.* Assume first that $P = B$ is a Borel subgroup. Then $F = H$ and $\mathcal{GR}^H_{\mathrm{red}} = ({}^L H)^\vee$, so $\mathcal{D}$-modules on $\mathcal{GR}^H$ are the same as $({}^L H)^\vee$-graded vector spaces, i.e., ${}^L H$-modules. The strata $\mathcal{GR}^B_\zeta$ are just $N_B(K)$-orbits on $\mathcal{GR}^G$. Thus 5.3.29 is just the key theorem of [MV].

Recall that the identification (281) is constructed as follows (see [MV]). Let $\overline{\mathcal{GR}}^B_n \subset \mathcal{GR}^G$ be the closure of $\mathcal{GR}^B_n := \pi^{-1}(\mathcal{GR}^H_n)$ in $\mathcal{GR}^G$. Then $\overline{\mathcal{GR}}^B_\cdot$ is a decreasing filtration on $\mathcal{GR}^G$. For any $M \in \mathcal{P}^G$ the obvious morphisms $h^n_H r^{GH}_{\mathcal{P}}(M) = H^n(\mathcal{GR}^B_n, i^! M) \longleftarrow H^n_{\overline{\mathcal{GR}}^B_n}(\mathcal{GR}^G, M) \longrightarrow H^n(\mathcal{GR}^G, M) = h^n_G(M)$ are isomorphisms. Their composition is (281).

Now let $P$ be any parabolic subgroup. Choose a Borel subgroup $B \subset P$, so $B_F := B/N_P \cap B$ is a Borel subgroup of $F$. Consider the functors $r^{GH}_D : D(\mathcal{GR}^G) \to D(\mathcal{GR}^H)$, $r^{FH}_D : D(\mathcal{GR}^F) \to D(\mathcal{GR}^H)$. By base change one has a canonical identification of functors $r^{GH}_D = r^{FH}_D r^{GF}_D$. Let

$\mathcal{P}^{H'} \subset D(\mathcal{GR}^H)$ be the category defined by $B \subset G$, so we know that $r_D^{GH}(\mathcal{P}^G) \subset \mathcal{P}^{H'}$ and (since $\rho_{GF} = \rho_G - \rho_F$) one has $r_D^{FH}(\mathcal{P}^{F'}) \subset \mathcal{P}^{H'}$.

The functor $r_{\mathcal{P}}^{GH} : \mathcal{P}^{F'} \to \mathcal{P}^{H'}$ is faithful (since up to shift if coincides with $h_F^{\cdot}$). Hence an object $T \in D(\mathcal{GR}^F)$ such that all $H^iT$ are in $\mathcal{P}^F$ belongs to $\mathcal{P}^{F'}$ if and only if $r_D^{FH}(T) \subset \mathcal{P}^{H'}$. Applying this remark to $T = r_D^{GF}(M)$, $M \in \mathcal{P}^G$, we see that $r_D^{GF}(M) \in \mathcal{P}^{F'}$, which is 5.3.29 (i). We also know that $h_G^{\cdot}(M) = h_H^{\cdot}(r_{\mathcal{P}}^{GH}(M)) = h_H^{\cdot}(r_{\mathcal{P}}^{FH}(M)) = h_F^{\cdot} r^{GF}(M)$ which is the identification 5.3.29 (ii). We leave it to the reader check that it does not depend on the auxiliary choice of a Borel subgroup $B \subset P$. □

**5.3.30.** The category $\mathcal{P}^{F'}$ has a canonical tensor structure (defined by the same constructions that were used for $\mathcal{P}^F$). The functor $r_{\mathcal{P}}^{GF} : \mathcal{P}^G \to \mathcal{P}^{F'}$ is a tensor functor in a canonical manner. Indeed, (279) are morphisms of chiral semi-groups, so we may consider the corresponding functors $r_D^{GF} := \pi_* i^! : D(\mathcal{GR}_{X^J}^G) \to D(\mathcal{GR}_{X^J}^F)$ . We leave it to the reader to check (hint: use 5.3.19) that for $M_j \in \mathcal{P}^G$ this functor sends $\boxtimes M_j$ to $\boxtimes r_D^{GF}(M_j)$ (see 5.3.14 for notation). Since (by base change) it also commutes with the functors $\widetilde{\Delta}^{(J)!}$ we get the desired tensor product compatibilities. As in 5.3.19 we see that (281) is an isomorphism of tensor functors.

Finally let us replace, as in 5.3.21, the tensor category $\mathcal{P}^G$ by $\mathcal{P}^{G\natural}$. Since $\rho_{GF} = \rho_G - \rho_F$ we see that $r_{\mathcal{P}}^{GF}$ yields a tensor functor $r^{GF} : \mathcal{P}^{G\natural} \to \mathcal{P}^{F\natural}$ compatible with the fiber functors $h_G$, $h_F$. It defines a morphism $r : \mathrm{Aut}^{\otimes} h_F \to \mathrm{Aut}^{\otimes} h_G$.

**5.3.31.** *Lemma.* The morphism $\varkappa_G^{-1} r \varkappa_F : {}^L F \to {}^L G$ coincides with the canonical embedding from 5.3.27. □

**5.4. Main Theorems II: from local to global.** In this section we give the precise version of the main theorems from 5.2 and show that the local main theorem implies the global one. We use in essential way the "Hecke pattern" from Chapter 7. To understand what is going on it is necessary (and almost sufficient) to read 7.1.1 and 7.9.1.

**5.4.1.** We start with the definition of Hecke eigen-$\mathcal{D}$-module. Consider the pair $(G(K), G(O))$ equipped with the action of $\operatorname{Aut} O$. Let $\mathcal{H}$ be the corresponding $(\operatorname{Der} O, \operatorname{Aut}^0 O)$-equivariant Hecke category as defined in 7.9.2*). Since any object of $\mathcal{P}$ is an $\operatorname{Aut} O$-equivariant $\mathcal{D}$-module in a canonical way*) our $\mathcal{P}$ is a full subcategory of $\mathcal{H}$. It follows from the definitions that the embedding $\mathcal{P} \to \mathcal{H}$ is a monoidal functor.

Consider the canonical $\operatorname{Aut} O$-structure $X^\wedge$ on $X$ (see 2.6.5) and the scheme $M^\wedge$ over $X^\wedge$ defined in 2.8.3; it carries a canonical action of $\operatorname{Aut} O \ltimes G(K)$ (see 2.8.3 - 2.8.4). The quotient stack $(\operatorname{Aut}^0 O \ltimes G(O)) \setminus M^\wedge$ equals $\operatorname{Bun}_G \times X$. We arrive to the setting of 7.9.1, 7.9.4*). Thus $\mathcal{H}$ acts on $D(\operatorname{Bun}_G \times X)$. Therefore $D(\operatorname{Bun}_G \times X)$ is a $\mathcal{P}$-Module. Identifying the monoidal category $\mathcal{P}$ *) with $\operatorname{Rep} {}^L G$ via the Satake equivalence (273) one gets a canonical Action of $\operatorname{Rep} {}^L G$ on $D(\operatorname{Bun}_G \times X)$ called the Hecke Action. We denote it by $\circledast$.

Note that $D(\operatorname{Bun}_G \times X)$ also carries an obvious Action of the tensor category $\operatorname{Vect}^\nabla(X)$ of vector bundles with connection on $X$ (or, in fact, of the larger tensor category of torsion free left $\mathcal{D}$-modules on $X$) which we denote by $\otimes$. It commutes with the Hecke Action, so $D(\operatorname{Bun}_G \times X)$ is a $(\operatorname{Rep} {}^L G, \operatorname{Vect}^\nabla(X))$-biModule.

Let $\mathfrak{F}$ be an ${}^L G$-bundle with a connection on $X$. It yields a tensor functor $\operatorname{Rep} {}^L G \to \operatorname{Vect}^\nabla(X)$, $V \to V_{\mathfrak{F}}$, hence the corresponding Action of $\operatorname{Rep} {}^L G$ on $D(\operatorname{Bun}_G \times X)$.

**5.4.2.** Let $M$ be a $\mathcal{D}$-module on $\operatorname{Bun}_G$. Let $M_{(X)} \in \mathcal{M}(\operatorname{Bun}_G \times X)$ be the pull-back of $M$. Assume that for any $V \in \operatorname{Rep} {}^L G$ we are given a natural

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*)Our $(G(K), G(O))$, $(\operatorname{Der} O, \operatorname{Aut}^0 O)$ are $(G, K)$, $(\mathfrak{l}, P)$ of 7.9.2.

*)According to 5.3.4 any object of $\mathcal{P}$ carries a unique strong $\operatorname{Aut}^0 O$-action which is the same as a strong $\operatorname{Aut} O$-action.

*)Our $X^\wedge$ and $M^\wedge$ are $X^\wedge$ and $Y^\wedge$ of 7.9.4.

*)In this section (except Remarks 5.4.6) we use only the monoidal structure on $\mathcal{P}$ (the commutativity constraint plays no role). So we may identify $\mathcal{P}$ with $\mathcal{P}^\natural$.

isomorphism $\alpha_V : V \circledast M_{(X)} \xrightarrow{\sim} M_{(X)} \otimes V_{\mathfrak{F}}$ (so, in particular, $V \circledast M_{(X)}$ is a $\mathcal{D}$-module, and not merely an object of the derived category). We say that the $\alpha_V$'s define a *Hecke $\mathcal{F}$-eigenmodule structure* on $M$ if for any $V_1, V_2 \in \operatorname{Rep} {}^L G$ one has $\alpha_{V_1 \otimes V_2} = \alpha_{V_1} \circ (V_1 \circledast \alpha_{V_2})$. We call such $(M, \alpha_V)$, or simply $M$, a *Hecke $\mathfrak{F}$-eigenmodule.*

*Remark.* For any ${}^L G$-local system $\mathcal{F}$ on $X$ one would like to define the triangulated category of Hecke $\mathfrak{F}$-eigenmodules[*)].

The following theorem is the precise version of Theorem 5.2.6.

**5.4.3.** *Theorem.* For any ${}^L G$-oper $\mathfrak{F}$ the $\mathcal{D}$-module $M_{\mathfrak{F}}$ defined in 5.1.1 has a natural structure of Hecke $\mathfrak{F}$-eigenmodule.

We leave it to the reader to check that the functor $T^i_\chi$ coincides with $H^i V^\chi \circledast$ (see 5.2.4, 5.2.5 for notation). Thus Theorem 5.4.3 implies 5.2.6.

**5.4.4.** We need a version of 5.4.1-5.4.3 "with parameters". Let $A$ be a commutative ring. Denote by $\mathcal{M}(\operatorname{Bun}_G \times X, A)$ the category of $A$-modules in $\mathcal{M}(\operatorname{Bun}_G \times X)$ (i.e., $\mathcal{D}$-modules with $A$-action). It has a derived version $D(\operatorname{Bun}_G \times X, A)$, which is a t-category with core $\mathcal{M}(\operatorname{Bun}_G \times X, A)$ (see 7.3.13). The category $D(\operatorname{Bun}_G \times X, A)$ carries, as in 5.4.1, the Hecke Action of $\operatorname{Rep} {}^L G$.

We also have the obvious Action of the tensor category of $A \otimes \mathcal{O}_X$-flat $A \otimes \mathcal{D}_X$-modules on $D(\operatorname{Bun}_G \times X, A)$ which commutes with the Hecke Action. Therefore any flat $A$-family $\mathcal{F}_A$ of ${}^L G$-bundles with connection on $X$ yields an Action of $\operatorname{Rep} {}^L G$ on $D(\operatorname{Bun}_G \times X, A)$.

Now for $M \in \mathcal{M}(\operatorname{Bun}_G, A)$ one defines the notion of Hecke $\mathfrak{F}_A$-eigenmodule structure on $M$ as in 5.4.2. The following theorem is the precise version of 5.2.9; by 5.1.2(i) it implies 5.4.3.

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[*)] Certainly, in the above definition of Hecke eigenmodule you may take for $M$ any object of $D(\operatorname{Bun}_G)$ instead of just a $\mathcal{D}$-module. However in this generality the definition does not look reasonable (such objects do not form a triangulated category).

**5.4.5.** *Theorem.* The $\mathcal{D}$-module $M_{\mathcal{L}} \in \mathcal{M}(\mathrm{Bun}_G, A_{{}^L\mathfrak{g}}(X))$ defined in 5.1.1 has a canonical structure of Hecke $\mathfrak{F}_{\mathcal{L}}$-eigenmodule.

**5.4.6.** *Remarks.* (i) Sometimes (when you want to use the commutativity constraint, see, e.g., the next Remark or the next section) it is convenient to deal with the above notions in the setting of super $\mathcal{D}$-modules. Note that any $\mathcal{D}$-module $M$ on $\mathrm{Bun}_G$ has a canonical $\mathbb{Z}/2\mathbb{Z}$-grading such that $M$ is even or odd depending on whether $M$ is supported on even or odd components of $\mathrm{Bun}_G$. We denote this super $\mathcal{D}$-module by $M^\natural$. So $\natural$ identifies $\mathcal{M}(\mathrm{Bun}_G)$ with a full subcategory $\mathcal{M}(\mathrm{Bun}_G)^\natural$ of $\mathcal{M}(\mathrm{Bun}_G)^\varepsilon := \mathcal{M}(\mathrm{Bun}_G)\otimes\mathrm{Vect}^\varepsilon$. The same applies to $D(\mathrm{Bun}_G)$ and $D(\mathrm{Bun}_G \times X)$.

The Action of $\mathcal{P}$ on $D(\mathrm{Bun}_G\times X)$ yields an Action of $\mathcal{P}^\varepsilon$ on $D(\mathrm{Bun}_G\times X)^\varepsilon$. The Action of $\mathcal{P}^\natural \subset \mathcal{P}^\varepsilon$ preserves $D(\mathrm{Bun}_G \times X)^\natural$, as well as the $\mathrm{Vect}^\nabla(X)$-Action. Now one defines the notion of Hecke $\mathfrak{F}$-eigenobject of $\mathcal{M}(\mathrm{Bun}_G)^\natural$ exactly as in 5.4.2. This definition brings nothing new: a $\mathcal{D}$-module $M$ is a Hecke $\mathfrak{F}$-eigenmodule if and only if $M^\natural$ is.

(ii) In the above definition of the $\mathcal{F}$-eigenmodule structure on $M \in \mathcal{M}(\mathrm{Bun}_G)$ we used the convolution construction of the tensor structure on $\mathcal{P}$. One may rewrite it instead using the fusion construction of $\circledast$ as follows. DOPISAT'!!!

**5.4.7.** Let us turn to the main local theorems from 5.2. We are in the setting of 5.2.12, so we fix $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$, which defines the central extension $\widetilde{G(K)} = \widetilde{G(K)}_{\mathcal{L}}$ of $G(K)$ split over the group subscheme $G(O)$ (see 4.4.9). We have the corresponding category of twisted Harish-Chandra modules $\mathcal{M}(\mathfrak{g} \otimes K, G(O))'$ and the derived category $D(\mathfrak{g} \otimes K, G(O))'$ of Harish-Chandra complexes (see 7.8.1 and 7.14.1)[*)]. According to 7.8.2, 7.14.1, $D(\mathfrak{g}\otimes K, G(O))'$ carries a canonical Action $\circledast$ of the Hecke monoidal category $\mathcal{H}$ of the pair $(G(K), G(O))$. Since $\mathcal{P}$ is a monoidal subcategory of (the core of) $\mathcal{H}$ our $D(\mathfrak{g} \otimes K, G(O))'$ is a $\mathcal{P}$-Module.

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[*)] So $\mathbf{1} \in \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K}$ acts on the objects of these categories as identity.

Let $Vac' \in \mathcal{M}(\mathfrak{g}\otimes K, G(O))'$ be the twisted vacuum module.

**5.4.8.** *Theorem.* For any object $P \in \mathcal{P}$ the object $P \circledast Vac' \in D(\mathfrak{g}\otimes K, G(O))'$ is isomorphic to a direct sum of copies of $Vac'$ *).

This theorem is equivalent to 5.2.14. Indeed, according to (335) of 7.8.5 and 7.14.1, there is a canonical identification of $(\widetilde{\mathfrak{g}\otimes K}, G(O))$-modules

$$H^i(P \circledast Vac') = H^i(\mathcal{GR}, P\lambda_{\mathcal{L}}^{-1}). \tag{282}$$

Here $P\lambda_{\mathcal{L}}^{-1} := P\otimes\lambda_{\mathcal{L}}^{-1}$. The interested reader may pass directly to the proof of this theorem, which can be found in ???.

**5.4.9.** We need to incorporate the $\operatorname{Aut} O$ symmetry in the above setting. Recall (see 4.6.6) that the action of $\operatorname{Aut} O$ on $G(K)$ lifts to the action of $\operatorname{Aut}_Z O$ on $\widetilde{G(K)}$ that preserves $G(O)$. So we are in the setting of 7.9.5*). Let $D_{HC}$ be the derived category of Harish-Chandra complexes as defined in 7.9.5. This is a t-category with core $\mathcal{M}_{HC}$ equal to the category of Harish-Chandra modules for the pair $(\operatorname{Der} O \ltimes \widetilde{\mathfrak{g}\otimes K}, \operatorname{Aut}_Z^0 O \ltimes G(O))$ (we assume that the center $\mathbb{C}\subset\widetilde{\mathfrak{g}\otimes K}$ acts in the standard way).

The $(\operatorname{Der} O, \operatorname{Aut}_Z^0 O)$-equivariant Hecke category for $(G(K), G(O))$ (see 7.9.2) contains the corresponding $(\operatorname{Der} O, \operatorname{Aut}^0 O)$-equivariant categories $\mathcal{H}$ and $\mathcal{H}^c$ as full monoidal subcategory. So, by 7.9.5, $D_{HC}$ is an $\mathcal{H}$-Module. hence it is a $\mathcal{P}$-Module.

We will need to change slightly our setting. Let as usual $\mathfrak{Z}$ be the center of the completed twisted universal enveloping algebra of $\mathfrak{g}\otimes K$ and $\mathfrak{z}$ the endomorphism ring of the twisted vacuum module $Vac'$; we have the obvious morphism of algebras $e : \mathfrak{Z}\to\mathfrak{z}$. Let $D_{HC\mathfrak{z}}$ be the corresponding derived category of Harish-Chandra complexes as defined in 7.9.8 (see also

*)In particular it is a single Harish-Chandra module, not merely a complex of those.

*)Our $(\operatorname{Der} O, \operatorname{Aut}_Z^0 O)$ and $(\widetilde{G(K)}, G(O))$ are $(\mathfrak{l}, P)$ and $(G', K)$ of 7.9.5.

7.9.7(iii))*). This is a t-category with core $\mathcal{M}_{HC\mathfrak{z}}$ equal to the category of Harish-Chandra modules killed by $\operatorname{Ker} e$.

Let $\mathcal{H}_{\mathfrak{z}}$ be the $\mathfrak{z}$-linear version of the $(\operatorname{Der} O, \operatorname{Aut}^0_Z O)$-equivariant Hecke category for $(G(K), G(O))$ as defined in 7.9.7(i). According to 7.9.8 it acts on $D_{HC\mathfrak{z}}$. Due to the obvious monoidal functor $\mathcal{H} \to \mathcal{H}_{\mathfrak{z}}$ (see the Remark in 7.9.7) $\mathcal{H}_{\mathfrak{z}}$ contains $\mathcal{P}$, so $D_{HC\mathfrak{z}}$ is a $\mathcal{P}$-Module. As in 5.4.1 we will replace $\mathcal{P}$ by $\operatorname{Rep}{}^LG$ by means of the Satake equivalence and denote the corresponding Action of $\operatorname{Rep}{}^LG$ on $D_{HC\mathfrak{z}}$ by $\circledast$. On the other hand $\mathcal{H}_{\mathfrak{z}}$ contains in its center the tensor category $\mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}}$ of flat $\mathfrak{z}$-modules equipped with $\operatorname{Aut}_Z O$-action (see 7.9.7(i)). The corresponding Action of $\mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}}$ on $D_{HC\mathfrak{z}}$ is the obvious one: for $W \in \mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}}$, $V \in D_{HC\mathfrak{z}}$ one has $W \circledast V = W \otimes V := W \underset{\mathfrak{z}}{\otimes} V$. Therefore $D_{HC\mathfrak{z}}$ is a $(\operatorname{Rep}{}^LG, \mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}})$-biModule.

Let $\mathfrak{F}$ be an $\operatorname{Aut}_Z O$-equivariant ${}^LG$-torsor on $\operatorname{Spec}\mathfrak{z}$. It yields the tensor functor $\operatorname{Rep}{}^LG \to \mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}}$, $V \mapsto V_{\mathfrak{F}}$, hence the corresponding Action of $\operatorname{Rep}{}^LG$ on $D_{HC\mathfrak{z}}$.

**5.4.10.** Let us repeat the definition from 5.4.2 in the present Harish-Chandra setting. Namely, a *Hecke $\mathfrak{F}$-eigenmodule* is a Harish-Chandra module $M \in \mathcal{M}_{HC\mathfrak{z}}$ together with natural isomorphisms $\alpha_V : V \circledast M \risingdotseq M \otimes V_{\mathfrak{F}}$, $V \in \operatorname{Rep}{}^LG$, such that for any $V_1, V_2 \in \operatorname{Rep}{}^LG$ one has $\alpha_{V_1 \otimes V_2} = \alpha_{V_1} \circ (V_1 \circledast \alpha_{V_2})$.

Now we can formulate the precise version of 5.2.16. As in 5.2.15, our $\mathcal{L} \in Z\operatorname{tors}_{\theta}(O)$ (see 5.4.7) defines an $\operatorname{Aut}_Z O$-equivariant*) ${}^LG$-torsor over the moduli scheme of local ${}^L\mathfrak{g}$-opers. Identifying this scheme with $\operatorname{Spec}\mathfrak{z}$ via the Feigin-Frenkel isomorphism (80) we get the corresonding $\operatorname{Aut}_Z O$-equivariant torsor $\mathfrak{F}_{\mathcal{L}}$ over $\operatorname{Spec}\mathfrak{z}$.

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*)Our $D_{HC\mathfrak{z}}$ is $D^e_{HC\,A}$ of 7.9.8. In 7.9.8 $\mathfrak{Z}$ denotes the set of $G(K)$-invariant elements of the center, but according to 3.7.7(ii) all elements of the center are $G(K)$-invariant.

*)The action of $\operatorname{Aut}_Z O$ comes from the identification $\operatorname{Aut}_Z O = \operatorname{Aut}(O, \mathcal{L})$; see 4.6.6.

From now on we consider $Vac'$ as an object of $\mathcal{M}_{HC_{\mathfrak{z}}}$ (with respect to the $\mathrm{Aut}_Z O$-action that fixes the vacuum vector).

**5.4.11.** *Theorem.* $Vac'$ has a canonical structure of Hecke $\mathfrak{F}_{\mathcal{L}}$-eigenmodule.

This theorem implies 5.2.16. Indeed, the isomorphism (282) is $\mathrm{Aut}_Z O$-equivariant since $\mathrm{Aut}_Z O$ acts on both sides of (282) by transport of structure.

Where will it be proved???

Now we may turn to the main result of this section.

**5.4.12.** *Theorem.* Theorem 5.4.11 implies 5.4.5.

*Proof.* We will show that an appropriate "localization functor" $L\Delta$ transforms the local picture into the global one $^{*)}$.

We need to modify slightly the setting of 5.4.1 to be able to use the "equivariant Hecke pattern" from 7.9. Recall that in the formulation of the global theorem 5.4.5 we fixed $\mathcal{L}^{\mathrm{glob}} \in Z\,\mathrm{tors}_\theta(X)$ (see 5.2.8), while in the local theorem 5.4.11 we used $\mathcal{L}^{\mathrm{loc}} \in Z\,\mathrm{tors}_\theta(O)$. Consider the schemes $X_Z^\wedge$ and $M_Z^\wedge$ from 4.4.15 corresponding to $\mathcal{L}^{\mathrm{glob}}$ and $\mathcal{L}^{\mathrm{loc}}$ (they are etale $Z$-coverings of the schemes $X^\wedge$ and $M^\wedge$ used in 5.4.1). Recall that $\mathrm{Aut}_Z O$ acts on $X_Z^\wedge$ and $\mathrm{Aut}_Z O \ltimes G(K)$ acts on $M_Z^\wedge$ (see 4.4.15). One has $\mathrm{Aut}_Z^0 O \backslash X_Z^\wedge = X$, and the quotient stack $(\mathrm{Aut}_Z^0 O \ltimes G(O)) \setminus M_Z^\wedge$ equals $\mathrm{Bun}_G \times X$. It is clear that in the construction of the Hecke Action on $D(\mathrm{Bun}_G \times X)$ in 5.4.1 we may replace $(M^\wedge, \mathrm{Aut}\, O \ltimes G(K))$ by $(M_Z^\wedge, \mathrm{Aut}_Z O \ltimes G(K))$.

As in 5.1.1 let $\lambda_{\mathcal{L}^{\mathrm{glob}}}$ be the Pfaffian line bundle on $\mathrm{Bun}_G$ that corresponds to $\mathcal{L}^{\mathrm{glob}}$. Denote by $\widehat{\lambda} = \widehat{\lambda}_{\mathcal{L}^{\mathrm{glob}}}$ its pull-back to $M_Z^\wedge$. The action of $\mathrm{Aut}_Z O \ltimes G(K)$ on $M_Z^\wedge$ lifts in a canonical way to an action on $\widehat{\lambda}$ of the central extension $\mathrm{Aut}_Z O \ltimes \widetilde{G(K)}$ (see 4.4.16). So we are in the setting of

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$^{*)}$The reader may decide if there is a method in this madness.

7.9.6[*)], and therefore, one has the right t-exact localization functor

$$L\Delta \,:\, D_{HC} \to D(\mathrm{Bun}_G \times X)$$

One has also the corresponding picture in the setting of $\mathfrak{z}$-modules. Indeed, following 7.9.7(ii), consider the $\mathcal{D}_X$-algebra $\mathfrak{z}_X$ [*)] (which we already used in 2.7) and the corresponding category $D(\mathrm{Bun}_G \times X, \mathfrak{z}_X)$ which is the derived category of $\mathcal{D}$-modules on $\mathrm{Bun}_G \times X$ equipped with $\mathfrak{z}_X$-action (see 7.3.13). It carries a canonical Action of $\mathcal{H}_{\mathfrak{z}}$. One has a canonical localization functor

$$L\Delta_{\mathfrak{z}} \,:\, D_{HC_{\mathfrak{z}}} \to D(\mathrm{Bun}_G \times X, \mathfrak{z}_X)$$

which is a Morhism of $\mathcal{H}_{\mathfrak{z}}$-Modules. The above $L\Delta$'s are compatible (they commute with the forgetting of $\mathfrak{z}$-action).

Now our theorem is immediate consequence of the following facts:

(a) There is a natural identification

$$L\Delta(Vac') = \Delta(Vac') = M_{\mathcal{L}^{\mathrm{glob}}} \boxtimes \mathcal{O}_X \tag{283}$$

such that the $\mathfrak{z}_X$-action on $\Delta(Vac') = \Delta_{\mathfrak{z}}(Vac')$ coincides with the action of $\mathfrak{z}_X$ on $M_{\mathcal{L}^{\mathrm{glob}}} \boxtimes \mathcal{O}_X$ through the maximal constant quotient $\mathfrak{z}(X) \otimes \mathcal{O}_X = A_{{}^L\mathfrak{g}}(X) \otimes \mathcal{O}_X$ and the standard $A_{{}^L\mathfrak{g}}(X)$-module structure on $M_{\mathcal{L}^{\mathrm{glob}}}$. For a proof see 7.14.9 (and note that $\mathfrak{z}_X$ acts by transport of structure).

(b) The functor $L\Delta_{\mathfrak{z}}$ is a Morphism of $(\mathrm{Rep}\,{}^L G, \mathcal{M}(\mathrm{Aut}_Z\, O)^{fl}_{\mathfrak{z}})$-biModules. Indeed, this is a Morphism of $\mathcal{H}_{\mathfrak{z}}$-Modules.

(c) For any $W \in \mathcal{M}(\mathrm{Aut}_Z\, O)^{fl}_{\mathfrak{z}}$, $T \in D(\mathrm{Bun}_G \times X, \mathfrak{z}_X)$ one has $W \circledast T = W_X \underset{\mathfrak{z}_X}{\otimes} T$ where $W_X$ is the $\mathfrak{z}_X$-module that corresponds to $W$.

For a proof see 7.9.7(i).

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[*)] Sorry for a terrible discrepancy of notations: our $M_Z^{\wedge}$, $X^{\wedge}$, $\widehat{\lambda}$, $\mathrm{Der}\, O$, $\mathrm{Aut}^0_Z\, O$, $\widetilde{G(K)}$, $G(O)$ are $Y^{\wedge}$, $X^{\wedge}$, $\mathcal{L}^*$, $\mathfrak{l}$, $P$, $G'$, $K$ of 7.9.6.

[*)] Any $\mathrm{Aut}\, O$-module $V$ yields the $\mathcal{D}_X$-module $V_X$, see 2.6.6.

(d) For any $V \in \operatorname{Rep}{}^LG$ there is a canonical identification

$$(V_{\mathfrak{F}_{\mathcal{L}^{\rm loc}}})_X \underset{\mathfrak{z}_X}{\otimes} (\mathfrak{z}(X) \otimes \mathcal{O}_X) \rightleftarrows V_{\mathfrak{F}_{\mathcal{L}^{\rm glob}}}$$

compatible with tensor products of $V$'s (here $\mathfrak{F}_{\mathcal{L}^{\rm loc}}$ is $\mathfrak{F}_{\mathcal{L}}$ from 5.4.10). □

**5.5. The birth of opers.** *In this section we assume Theorem 5.4.8.* We first show that this theorem implies that $Vac'$ is a Hecke $\mathcal{F}$-eigenmodule for some $\operatorname{Aut}_Z O$-equivariant ${}^LG$-torsor $\mathcal{F}$ on $\operatorname{Spec}\mathfrak{z}$. The main point of this section is that $\mathcal{F}$ comes naturally from an $\operatorname{Aut}_Z O$-equivariant $\mathfrak{z}$-family of local opers. Later we will see that the corresponding map from $\operatorname{Spec}\mathfrak{z}$ to the moduli of local opers coincides with the Feigin-Frenkel isomorphism, which yields the main local theorem.

**5.5.1.** For any $V \in \operatorname{Rep}{}^LG$ set

$$F_{\mathcal{H}}(V) := \operatorname{Hom}_{\widehat{\mathfrak{g}\otimes K}}(Vac', V \circledast Vac') = (V \circledast Vac')^{G(O)}. \tag{284}$$

This is an $\operatorname{Aut}_Z O$-equivariant $\mathfrak{z}$-module[*)]. According to 5.4.8 it is a free $\mathfrak{z}$-module, so $F_{\mathcal{H}}(V) \in \mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}}$. One has a canonical isomorphism

$$V \circledast Vac' = Vac' \otimes F_{\mathcal{H}}(V). \tag{285}$$

Since the Action of $\mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}}$ commutes with the Hecke Action we get a canonical identification $F_{\mathcal{H}}(V_1 \otimes V_2) = F_{\mathcal{H}}(V_1) \otimes F_{\mathcal{H}}(V_2)$, which means that

$$F_{\mathcal{H}} : \operatorname{Rep}{}^LG \to \mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}} \tag{286}$$

is a monoidal functor.

**5.5.2.** *Lemma.* For any $V \in \operatorname{Rep}{}^LG$ the free $\mathfrak{z}$-module $F_{\mathcal{H}}(V)$ has finite rank.

*Proof.* Since $F_{\mathcal{H}}$ is a monoidal functor $F_{\mathcal{H}}(V^*)$ is dual to $F_{\mathcal{H}}(V)$ in the sense of monoidal categories (see 2.1.2 of [Del91]). If a free $\mathfrak{z}$-module has a dual in the sense of monoidal categories then its rank is finite. □

[*)] The two $\mathfrak{z}$-module structures on $V \circledast Vac'$ coincide because the Hecke functors are $\mathfrak{z}$-linear.

Let

$$F_{\mathfrak{F}_{\mathcal{L}}} : \operatorname{Rep}{}^L G \to \mathcal{M}(\operatorname{Aut}_Z O)^{fl}_{\mathfrak{z}} \tag{287}$$

be the tensor functor $F_{\mathfrak{F}_{\mathcal{L}}}(V) = V_{\mathfrak{F}_{\mathcal{L}}}$ (see 5.4.10).

Now our main local theorem 5.4.11 may be restated as follows.

**5.5.3.** *Theorem.* The monoidal functors $F_{\mathcal{H}}$ and $F_{\mathfrak{F}_{\mathcal{L}}}$ are canonically isomorphic.

We are going to show that $F_{\mathcal{H}}$ indeed comes from a *some* canonically defined family of local opers parametrized by $\operatorname{Spec}\mathfrak{z}$. First let us check that $F_{\mathcal{H}}$ indeed comes from an ${}^L G$-torsor on $\operatorname{Spec}\mathfrak{z}$.

**5.5.4.** *Proposition.* The monoidal functor $F_{\mathcal{H}}$ is a tensor functor, i.e., it is compatible with the commutativity constraints.

The proof has two steps. First we write down the compatibility isomorphism $F_{\mathcal{H}}(V_1) \otimes F_{\mathcal{H}}(V_2) \xrightarrow{\sim} F_{\mathcal{H}}(V_1 \otimes V_2)$ as convolution product of sections of (twisted) $\mathcal{D}$-modules (see 5.5.5, 5.5.6). Then, using the fusion picture for the convolution, we show that it is commutative (see **??**?).

**5.5.5.** Let us replace the tensor category of ${}^L G$-modules by that of $\mathcal{D}$-modules on the affine Grassmanian using the Satake equivalence $h$ (see (273)). For $P \in \mathcal{P}^{\natural}$ we set $F_{\mathcal{H}}(P) := F_{\mathcal{H}}(hP)$. Thus (see (282))

$$F_{\mathcal{H}}(P) = \Gamma(\mathcal{G}\mathcal{R}, P\lambda_{\mathcal{L}}^{-1})^{G(O)}. \tag{288}$$

*Remark.* Recall that $P$ is a "super" $\mathcal{D}$-module and $\lambda_{\mathcal{L}}$ is a "super" line bundle. However their parities coincide (being equal to the parity of components of $\mathcal{G}\mathcal{R}$), so $P\lambda_{\mathcal{L}}^{-1}$ is a plain even sheaf. These "super" subtleties will be relevant when we pass to the commutativity constraint.

To describe the compatibility isomorphism $F_{\mathcal{H}}(P_1) \otimes F_{\mathcal{H}}(P_2) \xrightarrow{\sim} F_{\mathcal{H}}(P_1 \circledast P_2)$ consider the integration morphism of $\mathcal{O}^!$-modules (we use notation of 5.3.5; for integration see 7.11.16 (**??**))

$$i_m : m.(P_1 \boxtimes' P_2) \to P_1 \circledast P_2. \tag{289}$$

The line bundle $\lambda_{\mathcal{L}}$ on $\mathcal{GR}$ is $G(O)$-equivariant and its pull-back by $m : G(K) \times_{G(O)} \mathcal{GR} \to \mathcal{GR}$ is identified canonically with the "twisted product" $\lambda_{\mathcal{L}} \boxtimes' \lambda_{\mathcal{L}}$ *). So, twisting $i_m$ by $\lambda_{\mathcal{L}}$, we get the morphism $m.((P_1\lambda_{\mathcal{L}}^{-1}) \boxtimes' (P_2\lambda_{\mathcal{L}}^{-1})) \to (P_1 \circledast P_2)\lambda_{\mathcal{L}}^{-1}$.

Passing to $G(O)$-invariant sections we get the *convolution map* (notice that $G(O)$-invariance permits to neglect the twist)

(290)

$$* : \Gamma(\mathcal{GR}, P_1\lambda_{\mathcal{L}}^{-1})^{G(O)} \otimes \Gamma(\mathcal{GR}, P_2\lambda_{\mathcal{L}}^{-1})^{G(O)} \to \Gamma(\mathcal{GR}, (P_1 \circledast P_2)\lambda_{\mathcal{L}}^{-1})^{G(O)}$$

**5.5.6.** *Lemma.* The convolution map coincides with the compatibility isomorphism $F_{\mathcal{H}}(P_1) \otimes F_{\mathcal{H}}(P_2) \xrightarrow{\sim} F_{\mathcal{H}}(P_1 \circledast P_2)$.

*Proof.* Consider the canonical isomorphism (the Action constraint) $a : P_1 \circledast (P_2 \circledast Vac') \xrightarrow{\sim} (P_1 \circledast P_2) \circledast Vac'$. For $f \in \mathrm{Hom}(Vac', P_1 \circledast Vac')$, $g \in \mathrm{Hom}(Vac', P_2 \circledast Vac')$ the compatibility isomorphism sends $f \otimes g$ to $(P_1 \circledast g) \circ f$.

$\square$

## 5.6. The renormalized universal enveloping algebra.

**5.6.1.** Let $A$ be the completed universal enveloping algebra of $\widetilde{\mathfrak{g} \otimes K}$. According to 3.6.2 $A$ is a flat algebra over $\mathbb{C}[h]$, $h := \mathbf{1} - 1$, and $A/hA = \overline{U}'$. The natural topology on $A$ induces a topology on $A[h^{-1}] := A \otimes_{\mathbb{C}[h]} \mathbb{C}[h, h^{-1}]$; in fact this is the inductive limit topology (represent $A[h^{-1}]$ as the inductive limit of $A \to A \to \dots$ where each arrow is multiplication by $h$).

Let $I \subset \mathfrak{Z}$ be the ideal from 3.6.5. Denote by $J$ the preimage of $I\overline{U}' \subset \overline{U}' = A/hA$ in $A$ ($I\overline{U}'$ is understood in the topological sense, i.e., $I\overline{U}'$ is the closed ideal of $\overline{U}'$ generated by $I$). $J$ is a closed ideal of $A$ containing $hA$. Denote by $A^\natural$ the union of the increasing sequence $A \subset h^{-1}J \subset h^{-2}J^2 \subset \dots$ where $J^k$ is understood in the topological sense. Finally set $U^\natural := A^\natural / hA^\natural$.

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*) This follows since, by definition, $\lambda_{\mathcal{L}}$ comes from a central extension of $G(K)$ equipped with a splitting over $G(O)$.

$A^\natural$ is a topological algebra over $\mathbb{C}[h]$ (the topology is induced from $A[h^{-1}]$). So $U^\natural$ is a topological $\mathbb{C}$-algebra ($U^\natural$ is equipped with the quotient topology).

**5.6.2.** Set $Vac_A = A/A(\mathfrak{g}\otimes O)$ where $A(\mathfrak{g}\otimes O)$ denotes the closed left ideal of $A$ generated by $\mathfrak{g}\otimes O$. $I$ acts trivially on $Vac' = Vac_A/h\, Vac_A$. Since $Vac_A$ is a flat $\mathbb{C}[h]$-module $A^\natural$ acts on $Vac_A$. Therefore $U^\natural$ acts on $Vac'$.

**5.6.3.** Denote by $U_0^\natural$ the image of $A$ in $U^\natural$. $U_0^\natural$ is a subalgebra of $U^\natural$. We equip $U_0^\natural$ with the induced topology. The map $A \to U_0^\natural$ factors through $A/hA = \overline{U}'$ and actually through $\overline{U}'/I\overline{U}'$. So we get a surjective continuous homomorphism $f : \overline{U}'/I\overline{U}' \to U_0^\natural$. Probably $f$ is a homeomorphism. Anyway $f$ induces a topological isomorphism $\mathfrak{z} = \mathfrak{Z}/I \xrightarrow{\sim} f(\mathfrak{z})$ (use the action of $U^\natural$ on $Vac'$). We will identify $\mathfrak{z}$ with $f(\mathfrak{z})$.

**5.6.4.** Let $J_I \subset A$ denote the preimage of $I \subset \overline{U}' = A/hA$. Denote by $U_1^\natural$ the image of $h^{-1}J_I$ in $U^\natural$. Equip $U_1^\natural$ with the topology induced from $U^\natural$. The topological algebra $U^\natural$ is generated by $U_1^\natural$.

**5.6.5.** *Lemma.*

(i) $U_1^\natural$ is a Lie subalgebra of $U^\natural$.
(ii) $U_0^\natural$ is an ideal of $U_1^\natural$.
(iii) $\mathfrak{z}U_1^\natural \subset U_1^\natural$, $U_1^\natural\mathfrak{z} \subset U_1^\natural$.
(iv) $[U_1^\natural, \mathfrak{z}] \subset \mathfrak{z}$.

*Proof.* We will use some properties of the Hayashi bracket $\{,\}$ defined in 3.6.2. (i) follows from the inclusion $[J_I, J_I] \subset hJ_I$, which is clear because $\{I, I\} \subset I$ (see 3.6.4 (i)). (ii) and (iii) are obvious. (iv) is clear because $\{I, \mathfrak{Z}\} \subset \{\mathfrak{Z}, \mathfrak{Z}\} \subset \mathfrak{Z}$. $\square$

**5.6.6.** It follows from 5.6.5 that $U_1^\natural/U_0^\natural$ is a topological Lie algebroid over $\mathfrak{z}$. Multiplication by $h^{-1}$ defines a map $J_I \to A^\natural$, which induces a Lie algebroid morphism

$$I/I^2 = J_I/(J_I^2 + hA) \to U_1^\natural/U_0^\natural \tag{291}$$

(see 3.6.5 for the definition of the algebroid structure on $I/I^2$ ). The morphism (291) is continuous and surjective. In fact it is a topological isomorphism (see ???).

**5.6.7.** Denote by $U_i^\flat$ the set of elements of $U_i^\natural$ annihilating the vacuum vector from $Vac'$, $i=0,1$. Lemma 5.6.5 remains valid if $U_i^\natural$ is replaced by $U_i^\flat$, $i=0,1$. So $U_1^\flat/U_0^\flat$ is a topological Lie algebroid over $\mathfrak{z}$. The natural map $U_1^\flat/U_0^\flat\to U_1^\natural/U_0^\natural$ is a topological isomorphism. So (291) induces a surjective continuous Lie algebroid morphism

$$I/I^2\to U_1^\flat/U_0^\flat\,. \tag{292}$$

**5.6.8.** Let $V$ be a topological $U^\natural$-module (in the applications we have in mind $V$ will be discrete). Then $V^{\mathfrak{g}\otimes O}$ is a (left) topological module over the Lie algebroid $I/I^2$. Indeed, first of all $V^{\mathfrak{g}\otimes O}$ is a $\mathfrak{z}$-module. Secondly, $V^{\mathfrak{g}\otimes O}=\{v\in V|\,U_0^\flat v=0\}$, so the Lie algebra $U_1^\flat/U_0^\flat$ acts on $V^{\mathfrak{g}\otimes O}$. If $v\in V^{\mathfrak{g}\otimes O}$, $z\in\mathfrak{z}$, $a\in U_1^\flat/U_0^\flat$, then $a(zv)-z(av)=\partial_a(z)v$ where $\partial_a\in\operatorname{Der}\mathfrak{z}$ corresponds to $a$ according to the algebroid structure on $U_1^\flat/U_0^\flat$. So $V^{\mathfrak{g}\otimes O}$ is a module over the algebroid $U_1^\flat/U_0^\flat$. Using (292) we see that $V^{\mathfrak{g}\otimes O}$ is a module over the Lie algebroid $I/I^2$.

**5.6.9.** According to (89) one has the continuous Lie algebra morphism $\operatorname{Der}O\to h^{-1}J_I\subset A[h^{-1}]$ such that $L_n\mapsto h^{-1}\widetilde{\mathfrak{L}}_n$, $n\geq -1$. It induces a continuous Lie algebra morphism $\operatorname{Der}O\to U_1^\flat\subset U^\natural$. On the other hand in 3.6.16 we defined a canonical morphism $\operatorname{Der}O\to I/I^2$. Clearly the diagram

$$\begin{array}{ccc}\operatorname{Der}O & \longrightarrow & U_1^\flat\\ \downarrow & & \downarrow\\ I/I^2 & \longrightarrow & U_1^\flat/U_0^\flat\end{array}$$

is commutative.

*Remark.* The morphism $\operatorname{Der}O\to U_1^\flat/U_0^\flat$ induces a homeomorphism of $\operatorname{Der}O$ onto its image. Since $U_1^\flat/U_0^\flat$ acts continuously on $\mathfrak{z}\subset U_0^\natural$ this follows from the analogous statement for the morphism $\operatorname{Der} O \to \operatorname{Der} \mathfrak{z}$, which is clear (look at the Sugawara elements of $\mathfrak{z}$).

**5.6.10.** Suppose we are in the situation of 5.6.8. According to 5.6.9 $\operatorname{Der} O$ acts on $V$ via the morphism $\operatorname{Der} O \to U^\natural$, the subspace $V^{\mathfrak{g}\otimes O}$ is $\operatorname{Der} O$-invariant and the action of $\operatorname{Der} O$ on $V^{\mathfrak{g}\otimes O}$ coincides with the one that comes from the morphism $\operatorname{Der} O \to I/I^2$.

**5.6.11.** *Remark.* The definition of $\widetilde{\mathfrak{g}\otimes K}$ from 2.5.1 involves the "critical" scalar product $c$ defined by (18). Suppose we consider the central extension $0 \to \mathbb{C} \to (\widetilde{\mathfrak{g}\otimes K})_\lambda \to \mathfrak{g}\otimes K \to 0$ corresponding to $\lambda c$, $\lambda \in \mathbb{C}^*$. Denote by $A_\lambda$ the completed universal enveloping algebra of $(\widetilde{\mathfrak{g}\otimes K})_\lambda$. The construction of $U^\natural$ and the map (291) remain valid if $A$ and $h = \mathbf{1} - 1$ are replaced by $A_\lambda$ and $h_\lambda := \mathbf{1}_\lambda - \lambda^{-1}$, where $\mathbf{1}_\lambda$ denotes $1 \in \mathbb{C} \subset (\widetilde{\mathfrak{g}\otimes K})_\lambda$. Denote by $U^\natural_\lambda$ and $f_\lambda$ the analogs of $U^\natural$ and (291) corresponding to $\lambda$. One can identify $A_\lambda$ and $U^\natural_\lambda$ with $A$ and $U^\natural$ using the canonical isomorphism $\widetilde{\mathfrak{g}\otimes K} \xrightarrow{\sim} (\widetilde{\mathfrak{g}\otimes K})_\lambda$ such that $\mathbf{1} \mapsto \lambda \cdot \mathbf{1}_\lambda$. Then $f_\lambda$ *does* depend on $\lambda$: indeed, $f_\lambda = \lambda f_1$.

# 6. The Hecke property II

## 6.1.

## 6.2. Proof of Theorem 8.1.6.

**6.2.1.** *Lemma.* Let $V$ be a non-zero $\overline{U}'$-module such that the representation of $\mathfrak{g}\otimes O$ on $V$ is integrable, and the ideal $I\subset\mathfrak{Z}$ annihilates $V$. Then $V$ has a non-zero $\mathfrak{g}\otimes O$-invariant vector.

*Proof.* Denote by $\mathfrak{m}$ the maximal ideal of $O$. The kernel of the morphism $G(O)\to G(O/\mathfrak{m})$ is pro-unipotent and its Lie algebra is $\mathfrak{g}\otimes\mathfrak{m}$. So $V^{\mathfrak{g}\otimes\mathfrak{m}}\neq 0$. Consider the Sugawara element $\mathcal{L}_0\in I$ (see 3.6.15, 3.6.16). A glance at (85) shows that $2\mathcal{L}_0$ acts on $V^{\mathfrak{g}\otimes\mathfrak{m}}$ as the Casimir of $\mathfrak{g}$. On the other hand, $\mathcal{L}_0 V=0$ because $\mathcal{L}_0\in I$. So the action of $\mathfrak{g}$ on $V^{\mathfrak{g}\otimes\mathfrak{m}}$ is trivial and $V^{\mathfrak{g}\otimes O}=V^{\mathfrak{g}\otimes\mathfrak{m}}\neq 0$. □

**6.2.2.** *Lemma.* Let $N$ be a $\mathfrak{z}_{\mathfrak{g}}(O)$-module equipped with an action of the Lie algebroid $I/I^2$. Suppose that the action of $L_0\in\operatorname{Der}O\subset I/I^2$ on $N$ is diagonalizable and the intersection of its spectrum with $c+\mathbb{Z}$ is bounded from below for every $c\in\mathbb{C}$. Then $N$ is a free $\mathfrak{z}_{\mathfrak{g}}(O)$-module.

*Proof.* Using (80), (81), and 3.6.17 we can replace $\mathfrak{z}_{\mathfrak{g}}(O)$ by $A_{{}^L\mathfrak{g}}(O)$ and $I/I^2$ by $\mathfrak{a}_{{}^L\mathfrak{g}}$. By definition, $\mathfrak{a}_{{}^L\mathfrak{g}}$ is the algebroid of infinitesimal symmetries of $\mathfrak{F}^0_G$. In 3.5.6 we described a trivialization of $\mathfrak{F}^0_G$. The corresponding splitting $\operatorname{Der}A_{{}^L\mathfrak{g}}(O)\to\mathfrak{a}_{{}^L\mathfrak{g}}$ is $\operatorname{Der}^0 O$-equivariant (see (69) and (70); the point is that the r.h.s. of these formulas are constant as functions on $\operatorname{Spec}A_{{}^L\mathfrak{g}}(O)$). So $N$ becomes a module over $\operatorname{Der}A_{{}^L\mathfrak{g}}(O)$ and the mapping $\operatorname{Der}A_{{}^L\mathfrak{g}}(O)\to\operatorname{End}N$ is $\operatorname{Der}^0 O$-equivariant. According to 3.5.6 $A_{{}^L\mathfrak{g}}(O)$ is the ring of polynomials in $u_{jk}$, $1\leq j\leq r$, $0\leq k<\infty$, and $L_0u_{jk}=(d_j+k)u_{jk}$ for some $d_j>0$. So $N$ is an $L_0$-graded module over the algebra generated by $u_{jk}$ and $\frac{\partial}{\partial u_{jk}}$, $\deg(\frac{\partial}{\partial u_{jk}})=-\deg u_{jk}=-(d_j+k)\to-\infty$ when $k\to\infty$. Therefore every element of $N$ is annihilated by almost all $\frac{\partial}{\partial u_{jk}}$ and by all monomials in the $\frac{\partial}{\partial u_{jk}}$ of sufficiently high degree. It is well known (see, e.g., Lemma 9.13 from [Kac90] or Theorem 3.5 from [Kac97]) that in this situation $N = A_{{}^L\mathfrak{g}}(O) \otimes N_0$ where $N_0$ is the space of $n \in N$ such that $\frac{\partial}{\partial u_{jk}} n = 0$ for all $j$ and $k$. □

**6.2.3.** Let us prove Theorem 8.1.6. According to 5.6.8 we can apply Lemma 6.2.2 to $N := V^{\mathfrak{g}\otimes O}$. So $N = \mathfrak{z}_{\mathfrak{g}}(O) \otimes W$ for some vector space $W$. We will show that the natural $\overline{U}'$-module morphism $f : \mathit{Vac}' \otimes W = \mathit{Vac}' \otimes_{\mathfrak{z}_{\mathfrak{g}}(O)} N \to V$ is an isomorphism. One has $(\operatorname{Ker} f)^{\mathfrak{g}\otimes O} = \operatorname{Ker} f \cap N = 0$, so by 6.2.1 $\operatorname{Ker} f = 0$. Suppose that $\operatorname{Coker} f \neq 0$. Then according to 6.2.1 there is a non-zero $\mathfrak{g} \otimes O$-invariant element of $\operatorname{Coker} f$, i.e., a non-zero $\overline{U}'$-module morphism $\mathit{Vac}' \to \operatorname{Coker} f$. It induces an extension $0 \to \mathit{Vac}' \otimes W \to P \to \mathit{Vac}' \to 0$ which does not split (the composition of a splitting $\mathit{Vac}' \to P$ and the natural morphism $P \to V$ would yield a $\mathfrak{g} \otimes O$-invariant vector of $V$ not contained in $N$). So it remains to prove the following statement.

**6.2.4.** *Proposition.* Any extension of discrete $\overline{U}'$-modules $0 \to \mathit{Vac}' \otimes W \to P \to \mathit{Vac}' \to 0$ such that $IP = 0$ splits (here $W$ is a vector space).

*Proof.* Let $p \in P$ belong to the preimage of the vacuum vector from $\mathit{Vac}'$. Then $(\mathfrak{g} \otimes O) \cdot p \subset \mathit{Vac}' \otimes W$. In fact $(\mathfrak{g} \otimes O) \cdot p \subset \mathit{Vac}' \otimes W_1$ for some finite-dimensional $W_1 \subset W$, so we can assume that $\dim W < \infty$. Moreover, since the functor Ext is additive we can assume that $W = \mathbb{C}$.

Let $p$ be as above. Define $\varphi : \mathfrak{g} \otimes O \to \mathit{Vac}'$ by $\varphi(a) = ap$, so $\varphi$ is a 1-cocycle and $\operatorname{Ker} \varphi$ is open. We must show that $\varphi$ is a coboundary. One has the standard filtration $\overline{U}'_k$ of $\overline{U}'$. The induced filtration $\mathit{Vac}'_k$ of $\mathit{Vac}'$ is $(\mathfrak{g} \otimes O)$-invariant because the vacuum vector is annihilated by $\mathfrak{g} \otimes O$. So $\mathfrak{g} \otimes O$ acts on $\operatorname{gr} \mathit{Vac}'$. There is a $k$ such that $\operatorname{Im} \varphi \subset \mathit{Vac}'_k$. Denote by $\psi$ the composition of $\varphi : \mathfrak{g} \otimes O \to \mathit{Vac}'_k$ and $\mathit{Vac}'_k \to \mathit{Vac}'_k / \mathit{Vac}'_{k-1} \subset \operatorname{gr} \mathit{Vac}'$. So $\psi : \mathfrak{g} \otimes O \to \operatorname{gr} \mathit{Vac}'$ is a 1-cocycle and it suffices to show that $\psi$ is a coboundary (then one can proceed by induction).

Denote by $Vac^{cl}$ the space of polynomials on $\mathfrak{g}^* \otimes \omega_O$ (by definition, a polynomial on $\mathfrak{g}^* \otimes \omega_O$ is a function $\mathfrak{g}^* \otimes \omega_O \to \mathbb{C}$ that comes from a polynomial on the vector space $\mathfrak{g}^* \otimes (\omega_O/\mathfrak{m}^n \omega_O)$ for some $n$). According to 2.4.1 one has a canonical $\mathfrak{g} \otimes O$-equivariant identification $\operatorname{gr} Vac' = \operatorname{Sym}(\mathfrak{g} \otimes K/\mathfrak{g} \otimes O) = Vac^{cl}$ (the action of $\mathfrak{g} \otimes O$ on $Vac^{cl}$ is induced by the natural action of $\mathfrak{g} \otimes O$ on $\mathfrak{g}^* \otimes \omega_O$). So we can consider $\psi$ as a 1-cocycle $\mathfrak{g} \otimes O \to Vac^{cl}$. Define $\beta_\psi : (\mathfrak{g} \otimes O) \times (\mathfrak{g}^* \otimes \omega_O) \to \mathbb{C}$ by

$$\beta_\psi(a, \eta) := (\psi(a))(\eta)\,. \tag{293}$$

We say that $\eta \in \mathfrak{g}^* \otimes \omega_O$ is *regular* if the image of $\eta$ in $\mathfrak{g}^* \otimes (\omega_O/\mathfrak{m}\omega_O)$ is regular.

*Lemma.* If $\eta \in \mathfrak{g}^* \otimes \omega_O$ is regular and $\mathfrak{c}(\eta)$ is the stabilizer of $\eta$ in $\mathfrak{g} \otimes O$ then

$$\beta_\psi(a, \eta) = 0 \quad \text{for} \quad a \in \mathfrak{c}(\eta)\,. \tag{294}$$

*Proof.* We will use that $IP = 0$. Let $F \in \operatorname{Ker}(\mathfrak{Z}^{cl} \to \mathfrak{z}_{\mathfrak{g}}^{cl}(O))$, i.e., $F$ is a $(\mathfrak{g} \otimes K)$-invariant polynomial function on $\mathfrak{g}^* \otimes \omega_K$ whose restriction to $\mathfrak{g}^* \otimes \omega_O$ is zero (see 2.9.8). Suppose that $F$ is homogeneous of degree $r$. By 3.7.8 $F$ is the symbol of some $z \in \mathfrak{Z}_r$. Since the image of $F$ in $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is zero the image of $z$ in $\mathfrak{z}_{\mathfrak{g}}(O)$ belongs to the $(r-1)$-th term of the filtration, so according to 2.9.5 it comes from some $z' \in \mathfrak{Z}_{r-1}$. Replacing $z$ by $z - z'$ we can assume that $z \in I \cap \mathfrak{Z}_r$.

Since $I \subset \overline{U}' \cdot (\mathfrak{g} \otimes O)$ we can write $z$ as

$$z = \sum_{i=1}^{\infty} u_i a_i\,, \quad a_i \in \mathfrak{g} \otimes O\,, \quad u_i \in \overline{U}'\,, \quad a_i \to 0 \quad \text{for} \quad i \to \infty\,. \tag{295}$$

It follows from the Poincaré – Birkhoff – Witt theorem that the decomposition (295) can be chosen so that $u_i \in \overline{U}'_{r-1}$ for all $i$. Rewrite the equality $zp = 0$ as

$$\sum_i u_i \varphi(a_i) = 0\,. \tag{296}$$

Denote by $\widetilde{u}_i$ the image of $u_i$ in $\overline{U}'_{r-1}/\overline{U}'_{r-2}$. (295) and (296) imply that

$$F = \sum_i \widetilde{u}_i a_i\,, \tag{297}$$

$$\sum_i \overline{u}_i \psi(a_i) = 0 \tag{298}$$

where $a_i \in \mathfrak{g} \otimes O$ is considered as a linear function on $\mathfrak{g}^* \otimes \omega_K$ and $\overline{u}_i$ is the restriction of $\widetilde{u}_i$ to $\mathfrak{g}^* \otimes \omega_O$. Denote by $\underline{d}F$ the restriction of the differential of $F$ to $\mathfrak{g}^* \otimes \omega_O$. Since $F$ vanishes on $\mathfrak{g}^* \otimes \omega_O$ we have $\underline{d}F \in Vac^{cl}\,\widehat{\otimes}(\mathfrak{g} \otimes O)$ where $\widehat{\otimes}$ is the completed tensor product. According to (297) $\underline{d}F = \sum\limits_i \overline{u}_i \otimes a_i$, so we can rewrite (298) as

$$\mu(\underline{d}F) = 0 \tag{299}$$

where $\mu$ is the composition of $\mathrm{id} \otimes \psi :\ Vac^{cl}\,\widehat{\otimes}(\mathfrak{g} \otimes O) \to Vac^{cl} \otimes Vac^{cl}$ and the multiplication map $Vac^{cl} \otimes Vac^{cl} \to Vac^{cl}$.

Now set

$$F(\eta) = \operatorname{Res} f(\eta)\nu\,, \quad \nu \in \omega_O^{\otimes(1-r)} \tag{300}$$

where $f$ is a homogeneous invariant polynomial on $\mathfrak{g}^*$ of degree $r$. In this case (299) can be rewritten as

$$\beta_\psi(A_f(\eta)\nu, \eta) = 0 \tag{301}$$

where $\beta_\psi$ is defined by (293) and $A_f$ is the differential of $f$ considered as a polynomial map $\mathfrak{g}^* \to \mathfrak{g}$ (so $A_f(\eta) \in \mathfrak{g} \otimes \omega_O^{\otimes(r-1)}$, $A_f(\eta)\nu \in \mathfrak{g} \otimes O$). Since $f$ is invariant $A_f(l)$ belongs to the stabilizer of $l \in \mathfrak{g}^*$ and if $l$ is regular the elements $A_f(l)$ for all invariant $f$ generate the stabilizer. So the lemma follows from (301) $\square$

To prove the Proposition it remains to show that any 1-cocycle $\psi$ : $\mathfrak{g} \otimes O \to Vac^{cl}$ with open kernel such that the function (293) satisfies (294) is a coboundary.

*Lemma.* Let $K$ be a connected affine algebraic group with $\mathrm{Hom}(K, \mathbb{G}_m) = 0$, $W$ a $K$-module, and $\psi$ a 1-cocycle $\mathrm{Lie}\, K \to W$. Then $\psi$ comes from a unique 1-cocycle $\Psi : K \to W$.

*Proof.* The uniqueness of $\Psi$ is clear. The proof of existence is reduced to the case where $K$ is unipotent (represent $K$ as a semidirect product of a semisimple subgroup $K_{ss}$ and a unipotent normal subgroup; then notice that the restriction of $\psi$ to $\mathrm{Lie}\, K_{ss}$ is a coboundary and reduce to the case where this restriction is zero). Let $\widetilde{K}$ denote the semidirect product of $K$ and $W$. A 1-cocycle $K \to W$ is the same as a morphism $K \to \widetilde{K}$ such that the composition $K \to \widetilde{K} \to K$ equals id. A 1-cocycle $\mathrm{Lie}\, K \to W$ has a similar interpretation. So we can use the fact that the functor $\mathrm{Lie} : \{\text{unipotent groups}\} \to \{\text{nilpotent Lie algebras}\}$ is an equivalence. □

So our 1-cocycle $\psi : \mathfrak{g} \otimes O \to Vac^{cl}$ comes from a 1-cocycle $\Psi : G(O) \to Vac^{cl}$ where $G(O)$ is considered as a group scheme. Define $B_\Psi : G(O) \times (\mathfrak{g}^* \otimes \omega_O) \to \mathbb{C}$ by $B_\Psi(g, \eta) = (\Psi(g))(\eta)$.

*Lemma.* If $\eta \in \mathfrak{g}^* \otimes \omega_O$ is regular and $C(\eta)$ is the stabilizer of $\eta$ in $G(O)$ then

$$B_\Psi(g, \eta) = 0 \quad \text{for} \quad g \in C(\eta)\,. \tag{302}$$

*Proof.* For fixed $\eta$ the map $g \mapsto B_\Psi(g, \eta)$ is a morphism of group schemes $f : C(\eta) \to \mathbb{G}_a$. According to (294) the differential of $f$ equals 0. So $f = 0$ (even if $C(\eta)$ is not connected $\mathrm{Hom}(\pi_0(C(\eta)), \mathbb{G}_a) = 0$ because $\pi_0(C(\eta))$ is finite; but in fact if $G$ is the adjoint group, which can be assumed without loss of generality, then $C(\eta)$ is connected). □

The fact that $\Psi$ is a cocycle means that

$$B_\Psi(g_1 g_2, \eta) = B_\Psi(g_1, \eta) + B_\Psi(g_2, g_1^{-1} \eta g_1)\,. \tag{303}$$

We have to prove that $B_\Psi$ is a coboundary, i.e.,

$$B_\Psi(g, \eta) = f(g^{-1} \eta g) - f(\eta) \tag{304}$$

for some polynomial function $f : \mathfrak{g}^* \otimes \omega_O \to \mathbb{C}$. Denote by $\mathfrak{g}^*_{\mathrm{reg}}$ the set of regular elements of $\mathfrak{g}^*$ and by $(\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}}$ the set of regular elements of $\mathfrak{g}^* \otimes \omega_O$ (i.e., the preimage of $\mathfrak{g}^*_{\mathrm{reg}}$ in $\mathfrak{g}^* \otimes \omega_O$). Since $\operatorname{codim}(\mathfrak{g}^* \setminus \mathfrak{g}^*_{\mathrm{reg}}) > 1$ it is enough to construct $f$ as a regular function on $(\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}}$.

Let $C$ have the same meaning as in 2.2.1. The morphism $\mathfrak{g}^*_{\mathrm{reg}} \to C$ is smooth and surjective, $G$ acts transitively on its fibers, and Kostant constructed in [Ko63] a subscheme Kos $\subset \mathfrak{g}^*_{\mathrm{reg}}$ such that Kos $\to C$ is an isomorphism. If $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ using an invariant scalar product on $\mathfrak{g}$ then $\mathrm{Kos} = i\left(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right) + V$ where $i$ and $V$ have the same meaning as in 3.1.9. Define $\mathrm{Kos}_O \subset \mathfrak{g}^* \otimes \omega_O$ by $\mathrm{Kos}_O := i\left(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right) \cdot dt + V \otimes \omega_O$.

The equation (304) has a unique solution $f$ that vanishes on $\mathrm{Kos}_O$. The restriction of $f$ to $(\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}}$ is defined by

$$f(g^{-1}\eta g) = B_\Psi(g, \eta) \quad \text{for} \quad \eta \in \mathrm{Kos}\,,\ g \in G(O)\,. \tag{305}$$

Here $f$ is well-defined since (as follows from (302) and (303)) one has $B_\Psi(g_1 g, \eta) = B_\Psi(g, \eta)$ for $\eta \in (\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}}$, $g_1 \in C(\eta)$. Now (303) implies that the function $f$ defined by (305) satisfies (304) $\square$

*Remark.* At the end of the proof we used Kostant's global section of the fibration $(\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}} \to \mathrm{Hitch}_{\mathfrak{g}}(O)$ (see 2.4.1 for the definition of $\mathrm{Hitch}_{\mathfrak{g}}(O)$). Instead one could use *local* sections and the equality $H^1(\mathrm{Hitch}_{\mathfrak{g}}(O), \mathcal{O}) = 0$, which is obvious because $\mathrm{Hitch}_{\mathfrak{g}}(O)$ is affine.

**6.2.5.** Proposition 6.2.4 seems to be related with [F91] (see, e.g., the Propositions in the lower parts of pages 97 and 98 of [F91]). Maybe a modification of the methods of [F91] would yield Proposition 6.2.4 and much more.

# 7. Appendix: $\mathcal{D}$-module theory on algebraic stacks and Hecke patterns

## 7.1. Introduction.

**7.1.1.** The principal goal of this section is to present a general Hecke format which is used in the proof of our main Theorem. Its (untwisted) finite-dimensional version looks as follows. Let $G$ be an algebraic group, $K \subset G$ an algebraic subgroup, $\mathfrak{g}$ the Lie algebra of $G$, and $Y$ a smooth variety with $G$-action. Denote by $\mathcal{H} := D(K \setminus G/K)$ the $\mathcal{D}$-module derived category of the stack $K \setminus G/K$. One has the similar derived category $D(K \setminus Y)$ and the derived category $D(\mathfrak{g}, K)$ of the category $\mathcal{M}(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$-modules. Then we have the following "Hecke pattern":

(a) $\mathcal{H}$ is a monoidal triangulated category,
(b) $D(K \setminus Y)$ is an $\mathcal{H}$-Module,
(c) $D(\mathfrak{g}, K)$ is an $\mathcal{H}$-Module,
(d) the standard functors

$$L\Delta : D(\mathfrak{g}, K) \longrightarrow D(K \setminus Y), \qquad R\Gamma : D(K \setminus Y) \longrightarrow D(\mathfrak{g}, K)$$

are Morphisms of $\mathcal{H}$-Modules.

Here $L\Delta$, $R\Gamma$ are derived versions of the functors $\Delta$, $\Gamma$ from 1.2.4. The tensor product on $\mathcal{H}$ and $\mathcal{H}$-Actions from (b) and (c) are appropriate "convolution" functors $\circledast$. For example, consider the case $K = \{1\}$. Denote by $\delta_g$ the $\mathcal{D}$-module of $\delta$-functions at $g \in G$. One has $\delta_{g_1} \circledast \delta_{g_2} = \delta_{g_1 g_2}$. For a $\mathcal{D}$-module $M$ on $Y$ $\delta_g \circledast M$ is the $g$-translation of $M$, and for a $\mathfrak{g}$-module $V$ $\delta_g \circledast V$ is $V$ equipped with the $\mathfrak{g}$-action turned by $\mathrm{Ad}_g$. The $\mathcal{D}$-module structure on $M$ identifies canonically $\delta_g \circledast M$ for infinitely close $g$'s; similarly, the $\mathfrak{g}$-action on $V$ identifies such $\delta_g \circledast V$'s. This allows to define the convolution functors for an arbitrary $\mathcal{D}$-module on $G$.

**7.1.2.** The accurate construction of Hecke functors requires some $\mathcal{D}$-module formalism for stacks. For example, one needs a definition of the $\mathcal{D}$-module derived category $D(\mathcal{Y})$ of a smooth stack $\mathcal{Y}$ (it might not coincide with the derived category of the category of $\mathcal{D}$-modules on $\mathcal{Y}$!). There seems to be no reference available (except in the specific case when $\mathcal{Y}$ is an orbit stack, i.e., the quotient of a smooth variety by an affine group action, that was treated in [BL], [Gi87] in a way not too convenient for the Hecke functor applications), so we have to supply some general nonsense to keep afloat.

We start in 7.2, following Kapranov [Kap91] and Saito [Sa89], with a canonical equivalence between the derived category of $\mathcal{D}$-modules and that of $\Omega$-modules (here $\Omega$ is the DG algebra of differential forms) which identifies a $\mathcal{D}$-module with its de Rham complex. When you deal with stacks, $\Omega$-modules are easier to handle: the reason is that $\Omega$ is a sheaf of rings on the smooth topology while $\mathcal{D}$ is not. In the important special case of a stack for which the diagonal morphism is affine this super*) format is especially convenient. Here one may define (see 7.3) the $\mathcal{D}$-module derived category directly using "global" $\Omega$-complexes. In 7.5, after a general homological algebra digression of 7.4, we give a "local" definition of the $\mathcal{D}$-module derived category that works for arbitrary smooth stacks. In 7.6 parts (a), (b) of the Hecke pattern are explained; we also show that for an orbit stack its $\mathcal{D}$-module derived category is equivalent to the equivariant derived category from [BL], [Gi87]. In 7.7 we describe a similar super format for Harish-Chandra modules; as a bonus we get in 7.7.12 a simple proof of the principal result of [BL]. The Harish-Chandra parts (c), (d) of the Hecke pattern are treated in 7.8. A version with extra symmetries and parameters needed in the main body of the article is presented in 7.9. Before passing to an infinite-dimensional setting we discuss in 7.10 a crystalline approach to $\mathcal{D}$-modules which is especially convenient when you deal with singular spaces (we owe this section to discussions with J.Bernstein back in 1980). Sections 7.11 and 7.12 contain some basic material about ind-schemes, Mittag-Leffler modules,

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*)A mathematician's abbreviation of Mary Poppins' coinage "supercalifragelistic-expialidocious".

and $\mathcal{D}$-modules on formally smooth ind-schemes. Section 7.13 is a review of BRST reduction. The infinite-dimensional rendering of parts (c), (d) of the Hecke pattern is in 7.14. Finally in 7.15 we show that positively twisted $\mathcal{D}$-modules on affine flag varieties are essentially the same as representations of affine Kac-Moody Lie algebras of less than critical level. In the particular case of $\mathcal{D}$-modules smooth along the Schubert stratification, similar result was found by Kashiwara and Tanisaki [KT95] (the authors of [KT95] do not use the language of $\mathcal{D}$-modules on ind-schemes). We also identify the corresponding de Rham and BRST cohomology groups.

Our exposition of $\mathcal{D}$-module theory is quite incomplete; basically we treat the subjects that are used in the main body of the paper. The exceptions are sections 7.4, 7.5 (the stack $\mathrm{Bun}_G$ fits into the formalism of 7.3), 7.10 (the singular spaces that we encounter are strata on affine Grassmannians, so one may use 7.11), and 7.15 (included for the mere fun of the reader).

Recall that $\mathcal{M}^\ell(X)$ (resp. $\mathcal{M}^r(X)$) denotes the category of left (resp. right) $\mathcal{D}$-modules on a smooth variety $X$; we often identify these categories and denote them by $\mathcal{M}(X)$. If $F$ is a complex then we denote by $F^{\cdot}$ the corresponding graded object (with the differential forgotten).

## 7.2. $\mathcal{D}$- and $\Omega$-modules.

**7.2.1.** Let $X$ be a smooth algebraic variety $^{*)}$. Denote by $\Omega_X$ the DG algebra of differential forms on $X$. Then $(X, \Omega_X)$ is a DG ringed space, so we have the category of $\Omega_X$-complexes (:= DG $\Omega_X$-modules). An $\Omega_X$-complex $F = (F^{\cdot}, d)$ is quasi-coherent if $F^i$ are quasi-coherent $\mathcal{O}_X$-modules; quasi-coherent $\Omega_X$-complexes will usually be called $\Omega$-*complexes on* $X$. Denote

$^{*)}$or, more generally, a smooth quasi-compact algebraic space over $\mathbb{C}$ such that the diagonal morphism $X \to X \times X$ is affine. The constructions and statements of this section (but 7.2.10) are local, so they make sense for any smooth algebraic space. The condition on $X$ is needed to ensure that the derived categories we define satisfy an appropriate local-to-global (descent) property. We discuss this in the more general setting of stacks in 7.5.

the DG category of $\Omega$-complexes on $X$ by $C(X,\Omega)$. This is a tensor DG category.

*Remark.* For an $\Omega_X$-complex $F$ the differential $d : F^{\cdot} \to F^{\cdot+1}$ is a differential operator of order $\leq 1$ with symbol equal to the product map $\Omega^1_X \otimes F^{\cdot} \to F^{\cdot+1}$. We see that the $\Omega^{\cdot}_X$-module structure on $F^{\cdot}$ can be reconstructed from the $\mathcal{O}_X$-module structure and $d$. In fact, forgetting the $\Omega^{\geq 1}_X$-action identifies $C(X,\Omega)$ with the category of complexes $(F^{\cdot}, d)$ where $F^{\cdot}$ are quasi-coherent $\mathcal{O}_X$-modules, $d$ are differential operators of order $\leq 1$.

**7.2.2.** Let $C(X,\mathcal{D}) := C(\mathcal{M}^r(X))$ be the DG category of complexes of right $\mathcal{D}$-modules on $X$ (*right $\mathcal{D}$-complexes*, or just *$\mathcal{D}$-complexes* for short), and $K(X,\mathcal{D})$ the corresponding homotopy category. We have a pair of adjoint DG functors

$$\mathcal{D} : C(X,\Omega) \longrightarrow C(X,\mathcal{D})\,, \quad \Omega : C(X,\mathcal{D}) \longrightarrow C(X,\Omega) \tag{306}$$

defined as follows. Denote by $DR_X$ the de Rham complex of $\mathcal{D}_X$ considered as a left $\mathcal{D}$-module, so $DR^{\cdot}_X = \Omega^{\cdot}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$. This is an $\Omega$-complex equipped with the right action of $\mathcal{D}_X$. Now for an $\Omega$-complex $F$ and a right $\mathcal{D}$-complex $M$ one has

$$\mathcal{D}F = F \underset{\Omega_X}{\otimes} DR_X\,, \quad \Omega M := Hom_{\mathcal{D}_X}(DR_X, M)\,. \tag{307}$$

The adjunction property is clear.

**7.2.3.** *Remarks.* (i) One has $\mathcal{D}F^{\cdot} = F^{\cdot} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X = \mathrm{Diff}(\mathcal{O}, F^{\cdot})$; the differential $d_{\mathcal{D}F} : \mathcal{D}F^{\cdot} \to \mathcal{D}F^{\cdot+1}$ sends a differential operator $a : \mathcal{O}_X \to F^{\cdot}$ to the composition $d{\cdot}a$. The $\Omega$-complex $\Omega M$, $(\Omega M)^i = \bigoplus\limits_{a-b=i} M^a \otimes \Lambda^b \Theta_X$ is the de Rham complex of $M$.

(ii) The category $\mathcal{M}^{\ell}(X)$ of left $\mathcal{D}$-modules on $X$ is a tensor category in the usual way (tensor product over $\mathcal{O}_X$), so the category of *left $\mathcal{D}$-complexes* $C(\mathcal{M}^{\ell}(X))$ is a tensor DG category. The DG functor $\Omega$ :

$C(\mathcal{M}^\ell(X)) \to C(X,\Omega)$ which assigns to a left $\mathcal{D}$-complex $N$ its de Rham complex, $(\Omega N)^{\cdot} = \Omega^{\cdot}_X \underset{\mathcal{O}_X}{\otimes} N$, is a tensor functor.

(iii) The DG categories $C(X,\Omega)$ and $C(X,\mathcal{D})$ are Modules over the tensor DG category $C(\mathcal{M}^\ell(X))$. The functors $\mathcal{D}$ and $\Omega$ are Morphisms of $C(\mathcal{M}^\ell(X))$-Modules.

**7.2.4.** *Lemma.* For any $\mathcal{D}$-complex $M$ the canonical morphism $\mathcal{D}\Omega M \to M$ is a quasi-isomorphism.

*Proof.* Set

$$V^i_j := \bigoplus_{\substack{a-b=i \\ b+c=j}} M^a \otimes \Lambda^b\Theta_X \otimes \mathcal{D}_X^{\leq c} \quad \subset \ (\mathcal{D}\Omega M)^i.$$

Then $V_*$ is a increasing filtration of $\mathcal{D}\Omega M$ by $\mathcal{O}$-subcomplexes such that $V_0 \stackrel{\sim}{\to} M$ and $V_i/V_{i-1}$ are acyclic for $i \geq 1$ (since $V_i/V_{i-1}$ is the tensor product of $M$ and the $i$-th Koszul complex for $\Theta_X$). $\square$

**7.2.5.** For an $\Omega$-complex $F$ set $H^{\cdot}_{\mathcal{D}}F = H^{\cdot}\mathcal{D}F$. Thus $H_{\mathcal{D}}$ is a cohomology functor on $K(X,\Omega)$ with values in the abelian category $\mathcal{M}^r(X)$. A morphism of $\Omega$-complexes $\phi : F_1 \to F_2$ is called *$\mathcal{D}$-quasi-isomorphism* if the morphism of $\mathcal{D}$-complexes $\mathcal{D}\phi : \mathcal{D}F_1 \to \mathcal{D}F_2$ is a quasi-isomorphism, i.e., $H^{\cdot}_{\mathcal{D}}F_1 \to H^{\cdot}_{\mathcal{D}}F_2$ is an isomorphism. We have the following simple properties (use 7.2.4 to prove (ii), (iii)):

(i) If $\phi$ is a $\mathcal{D}$-quasi-isomorphism, $N$ is a left $\mathcal{D}$-module flat as an $\mathcal{O}$-module then $\phi \otimes id_N : F_1 \otimes N \to F_2 \otimes N$ is a $\mathcal{D}$-quasi-isomorphism.

(ii) The canonical morphism $\alpha_F : F \to \Omega\mathcal{D}F$ is a $\mathcal{D}$-quasi-isomorphism.

(iii) $\Omega$ sends quasi-isomorphisms to $\mathcal{D}$-quasi-isomorphisms.

The following lemma will not be used in the sequel; the reader may skip it. We say that a morphism of $\Omega$-complexes $\phi : F_1 \to F_2$ is a *naive quasi-isomorphism* if it is a quasi-isomorphism of complexes of sheaves of vector spaces.

**7.2.6.** *Lemma.* (i) Any $\mathcal{D}$-quasi-isomorphism is a naive quasi-isomorphism.

(ii) A morphism $\phi$ as above is a $\mathcal{D}$-quasi-isomorphism if and only if for any bounded below complex $A$ of locally free $\Omega$-modules the morphism$\phi \otimes id_A : F_1 \otimes A \to F_2 \otimes A$ is a naive quasi-isomorphism.

(iii) Assume either that $\Omega^{\geq 1} F_i^{\cdot} = 0$ (i.e., the differential is $\mathcal{O}$-linear), or that $F_i^{\cdot}$ are bounded and $\mathcal{O}$-coherent. Then any naive quasi-isomorphism $\phi$ is a $\mathcal{D}$-quasi-isomorphism. For arbitrary $\Omega$-complexes this may be not true.

*Proof.* (i) For any $\Omega$-complex $F$ the canonical morphism $\alpha_F : F \to \Omega\mathcal{D}F$ is a naive quasi-isomorphism. Since $\Omega$ sends quasi-isomorphisms of $\mathcal{D}$-complexes to naive quasi-isomorphisms we see that $\Omega(\mathcal{D}\phi)$ is a naive quasi-isomorphism. Now our statement follows from the fact that $\alpha_{F_2}\phi = \Omega((\mathcal{D}\phi)\alpha_{F_1}$.

(ii) To prove the "if" statement just take $A = DR_X$. Conversely, assume that $\phi$ is a $\mathcal{D}$-quasi-isomorphism. There is a bounded below increasing filtration $A_i$ on $A$ such that $\cup A_i = A$ and each $gr_i A$ is a locally free $\Omega_X^{\cdot}$-module with generators in degree $i$ (set $A_i := \Omega_X \cdot A_{\leq i}$). So $\phi \otimes id_A$ is a naive quasi-isomorphism if all $\phi \otimes id_{gr_i A}$ are naive quasi-isomorphisms. Thus we may assume that $A$ is a locally free $\Omega_X^{\cdot}$-module with generators in fixed degree, say 0, i.e., $A = \Omega N$ where $N$ is a left $\mathcal{D}$-module locally free as an $\mathcal{O}$-module. Then $\phi \otimes id_A = \phi \otimes id_N$, and we are done by (i) from 7.2.5.

(iii) The $\mathcal{O}$-linear case is obvious (since in this situation $\mathcal{D}F = F \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$). The $\mathcal{O}$-coherent case follows from the Sublemma below applied to $\mathcal{D}\phi$ (notice that because of property (ii) from 7.2.5 the fiber of $\mathcal{D}F$ at $x$ coincides with $R\Gamma_x(X, F)$).

*Sublemma.* Let $\psi : M_1 \to M_2$ be a morphism of finite complexes of coherent $\mathcal{D}$-modules on $X$. Assume that for any $x \in X(\mathbb{C})$ the corresponding morphism of fibers[*)] $M_{1x} \to M_{2x}$ is a quasi-isomorphism. Then $\psi$ is a quasi-isomorphism.

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[*)]Certainly here we consider the $\mathcal{O}$-moduli fibers in the usual derived category sense.

*Proof of Sublemma.* Set $C = Cone(\psi)$; denote by $Y$ the support of $H^{\cdot}(C)$. Assume that $\psi$ is not a quasi-isomorphism, i.e., $Y$ is not empty. Restricting $X$ if necessary we may assume that $Y$ is a smooth subvariety of $X$ and the coherent $\mathcal{D}_Y$-modules $P^{\cdot} := i_Y^! H^{\cdot}(C) = H^{\cdot} i_Y^!(C)$ are free as $\mathcal{O}_Y$-modules. Since for $x \in Y$ one has $H^{\cdot}(C_x) = P_x^{\cdot+n}$ where $n$ is codimension of $Y$ in $X$ we see that $P^{\cdot} = 0$ which is a contradiction.

To get an example of a naive quasi-isomorphism which is not a $\mathcal{D}$-quasi-isomorphism it suffice to find a non-zero $\mathcal{D}$-module $M$ such that $\Omega M$ is an acyclic complex of sheaves. Take $M$ to be a constant sheaf of $\mathcal{D}_X$-modules equal to the field of fractions of the ring of differential operators (at the generic point of $X$). $\square$

**7.2.7.** Since $H_{\mathcal{D}}$ is a cohomology functor, $\mathcal{D}$-quasi-isomorphisms form a localizing family in the homotopy category of $C(X,\Omega)$. Therefore the corresponding localization $D(X,\Omega)$ is a triangulated category (see [Ve]); we call it $\mathcal{D}$-*derived category* of $\Omega$-complexes. The functors $\mathcal{D}$, $\Omega$ give rise to mutually inverse equivalences of triangulated categories

$$\mathcal{D}:\, D(X,\Omega) \longrightarrow D(X,\mathcal{D})\,, \qquad \Omega:\, D(X,\mathcal{D}) \longrightarrow D(X,\Omega)\,. \tag{308}$$

Here $D(X,\mathcal{D}) = D\mathcal{M}^r(X)$. We often denote these triangulated categories thus identified by $D(X)$. One may consider bounded derived categories as well.

*Remark.* For a bounded from below complex of injective $\mathcal{D}$-modules $M$ the corresponding $\Omega$-complex $\Omega M$ is injective. Thus the homotopy category $K^+(X,\Omega)$ has many injective objects.

**7.2.8.** Let $f: Y \to Z$ be a morphism of smooth varieties. It yields the morphism of DG ringed spaces $f_\Omega : (Y,\Omega_Y) \to (Z,\Omega_Z)$. Thus we have the corresponding DG functors $f_\Omega^{\cdot} : C(Z,\Omega) \to C(Y,\Omega)$, $f_{\cdot} = f_{\Omega\cdot} : C(Y,\Omega) \to C(Z,\Omega)$. Let us consider first the pull-back functor.

We have the usual pull-back functor for left $\mathcal{D}$-modules $f^\dagger : \mathcal{M}^\ell(Z) \to \mathcal{M}^\ell(Y)$, $f^\dagger(N) = \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_Z}{\otimes} f^{-1}N$. One has $\Omega f^\dagger(N) = f_\Omega^{\cdot}(\Omega N)$. One may replace left $\mathcal{D}$-modules by right ones*) and consider the corresponding functor $f^\dagger : \mathcal{M}^r(Z) \to \mathcal{M}^r(Y)$; then $f_\Omega^{\cdot}(\Omega M) = \Omega f^\dagger M[-\dim Y/Z]$.

If $f$ is smooth then for any $F \in C(Z,\Omega)$ one has $H_{\mathcal{D}}^{\cdot} f_\Omega^{\cdot} F = f^\dagger H_{\mathcal{D}}^{\cdot - \dim Y/Z} F$. So $f_\Omega^{\cdot}$ preserves $\mathcal{D}$-quasi-isomorphisms and we have the functor $f_\Omega^{\cdot} : D(Z,\Omega) \to D(Y,\Omega)$. The adjunction morphism $\mathcal{D} f_\Omega^{\cdot}(\Omega M) \to f^\dagger M[-\dim U/X]$ is a quasi-isomorphism.

**7.2.9.** *Lemma.* $\Omega$-complexes are local objects with respect to the smooth topology, i.e., the pull-back functors make $C(U,\Omega)$, $U \in X_{sm}$, a sheaf of DG categories on the smooth topology of $X$. The notion of $\mathcal{D}$-quasi-isomorphism is local on $X_{sm}$. □

**7.2.10.** Let us return to situation 7.2.8 and consider the DG functor $f_\cdot : C(Y,\Omega) \to C(Z,\Omega)$. The right derived functor $Rf_\cdot : D(Y,\Omega) \to D(Z,\Omega)$ is correctly defined. Indeed, let $U_\cdot$ be a (finite) affine covering (either étale or Zariski) of $Y$. For $F \in C(Y,\Omega)$ let $F \to \mathcal{C}(F)$ be the corresponding Čech resolution of $F$. Then*) $f_\cdot \mathcal{C}(F) \overset{\sim}{\to} Rf_\cdot F$.

We denote the corresponding functor $D(Y) \to D(Z)$ by $f_*$. It coincides with the usual $\mathcal{D}$-module push-forward functor. Indeed, for a $\mathcal{D}$-complex $M$ on $Y$ one has $\mathcal{D} f_\cdot \Omega M = f_\cdot(\Omega M \otimes f^\dagger \mathcal{D}_Z) = f_\cdot(\mathcal{D}(\Omega M) \underset{\mathcal{D}_Y}{\otimes} f^\dagger \mathcal{D}_Z)$. Since $f^\dagger \mathcal{D}_Z$ is a flat $\mathcal{O}_Y$-module and $\mathcal{D}(\Omega M)$ is a resolution of $M$ we see that $\mathcal{D}(\Omega M) \underset{\mathcal{D}_Y}{\otimes} f^\dagger \mathcal{D}_Z = M \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} f^\dagger \mathcal{D}_Z$. Thus $f_* M = Rf_\cdot(M \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} f^\dagger \mathcal{D}_Z)$, q.e.d.

We leave it to the reader to check that $Rf_\cdot$ is compatible with composition of $f$'s, i.e., that the canonical morphism $R(fg)_\cdot \to Rf_\cdot Rg_\cdot$ is an isomorphism*), and that this identification $(fg)_* = f_* g_*$ coincides with the standard identification from $\mathcal{D}$-module theory.

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*) using the standard equivalence $\mathcal{M}^\ell(Z) \overset{\sim}{\to} \mathcal{M}(Z)$, $N \mapsto N \otimes \omega_Z$.

*) this follows, e.g., from Remark after 7.3.9.

*) see 7.3.10(ii) for a proof of this statement in a more general situation.

**7.2.11.** For a $\mathcal{D}$-complex $M$ on $Y$ denote by $M_{\mathcal{O}} \in D(Y, \mathcal{O})$ same $M$ considered as a complex of $\mathcal{O}^!$-modules. One has a canonical *integration* morphism

$$i_f : Rf.(M_{\mathcal{O}}) \to (f_* M)_{\mathcal{O}} \tag{309}$$

in $D(Y, \mathcal{O})$ defined as follows. It suffice to define the morphism $i_f$ : $f.(M_{\mathcal{O}}) \to \mathcal{D}(f.\Omega M)$. Now $i_f$ is the composition

$$f.(M_{\mathcal{O}}) \to [\mathcal{D}(f.(M_{\mathcal{O}}))]_{\mathcal{O}} \to [\mathcal{D}(f.\Omega M)]_{\mathcal{O}}$$

where the arrows come from the canonical morphisms $N \to (\mathcal{D}N)_{\mathcal{O}}$ (for $N = f.(M_{\mathcal{O}})$) and $M_{\mathcal{O}} \to \Omega M$. In other words, $i_f$ comes by applying $Rf.$ to the obvious morphism $M_{\mathcal{O}} \to (M \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} f^\dagger \mathcal{D}_Z)_{\mathcal{O}}$.

We leave it to the reader to check that $i_f$ is compatible with composition of $f$'s.

**7.3. $\mathcal{D}$-module theory on smooth stacks I.** We establish the basic $\mathcal{D}$-module formalism for a smooth stack that satisfies condition (310) below. In 7.3.12 we modify the definitions so that one may drop the quasi-compactness assumption. The arbitrary smooth stacks will be treated in 7.5.

**7.3.1.** Let $\mathcal{Y}$ be a smooth quasi-compact algebraic stack. Assume that it satisfies the following condition$^{*)}$:

(310) The diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is affine.

Equivalently, this means that there exist a smooth affine surjective morphism $U \to \mathcal{Y}$ such that $U$ is an affine scheme. In other words, $\mathcal{Y}$ is a quotient of a smooth algebraic variety $X$ modulo the action of a smooth groupoid $Q^{*)}$ such that the structure morphism $Q \to X \times X$ is affine.

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$^{*)}$This condition is needed to ensure that the category $D(\mathcal{Y})$ we define has right local-to-global properties, see 7.5.3. The constructions 7.3.1-7.3.3 make sense for any smooth algebraic stack.

$^{*)}Q = X \underset{\mathcal{Y}}{\times} X$.

Note that $\Omega(U)$, $U \in \mathcal{Y}_{sm}$, form a sheaf of DG algebras $\Omega_{\mathcal{Y}}$ on $\mathcal{Y}_{sm}$. An *$\Omega$-complex on $\mathcal{Y}$* is a DG $\Omega_{\mathcal{Y}}$-module which is quasi-coherent as an $\mathcal{O}_{\mathcal{Y}}$-module. We denote the DG category of $\Omega$-complexes on $\mathcal{Y}$ by $C(\mathcal{Y}, \Omega)$.

*Remark.* The categories $C(U, \Omega)$, $U \in \mathcal{Y}_{sm}$, form a sheaf of DG categories $\mathcal{C}(\mathcal{Y}_{sm}, \Omega)$ on $\mathcal{Y}_{sm}$ (see 7.2.9), and an $\Omega$-complex on $\mathcal{Y}$ is the same as a Cartesian section of $\mathcal{C}(\mathcal{Y}_{sm}, \Omega)$. Equivalently, an $\Omega$-complex on $\mathcal{Y}$ is the same as a $Q$-equivariant $\Omega$-complex on $X$.

**7.3.2.** Recall that the categories of $\mathcal{D}$-modules $\mathcal{M}(U)$, $U \in \mathcal{Y}_{sm}$, form a sheaf of abelian categories on $\mathcal{Y}_{sm}$, and the category $\mathcal{M}(\mathcal{Y})$ of $\mathcal{D}$-modules on $\mathcal{Y}$ is the category of its Cartesian sections. By 7.2.8 there is a canonical cohomology functor $H^{\cdot}_{\mathcal{D}} : C(\mathcal{Y}, \Omega) \to \mathcal{M}(\mathcal{Y})$, $H^{\cdot}_{\mathcal{D}}(F)_U := H^{\cdot+\dim U/\mathcal{Y}}_{\mathcal{D}}(F_U)$. A morphism of $\Omega$-complexes is called a $\mathcal{D}$-quasi-isomorphism if it induces an isomorphism of $H^{\cdot}_{\mathcal{D}}$'s. Localizing the homotopy category of $\Omega$-complexes by $\mathcal{D}$-quasi-isomorphisms we get a triangulated category $D(\mathcal{Y}) = D(\mathcal{Y}, \Omega)$. One has the corresponding bounded derived categories as well.

There is a fully faithful embedding $\mathcal{M}(\mathcal{Y}) \hookrightarrow D(\mathcal{Y})$ which assigns to a $\mathcal{D}$-module $M$ on $\mathcal{Y}$ its de Rham complex $\Omega M$, $(\Omega M)_U := \Omega M_U[-\dim U/\mathcal{Y}]$. One has $H^0_{\mathcal{D}}\Omega M = M$ and $H^a_{\mathcal{D}}\Omega M = 0$ for $a \neq 0$. It is easy to see that $\Omega$ identifies $\mathcal{M}(\mathcal{Y})$ with the full subcategory of $D(\mathcal{Y})$ that consists of those $\Omega$-complexes $F$ that $H^a_{\mathcal{D}}(F) = 0$ for $a \neq 0$.

**7.3.3.** *Example.* Denote by $\Omega\mathcal{D}_{\mathcal{Y}}$ the $\Omega$-complex on $\mathcal{Y}$ defined by $\Omega\mathcal{D}_{\mathcal{Y}U} := \Omega_{U/\mathcal{Y}}[\dim \mathcal{Y}]$. Note that $H^a_{\mathcal{D}}(\Omega\mathcal{D}_{\mathcal{Y}}) = 0$ for $a > 0$. If $\mathcal{Y}$ is good then our $\Omega$-complex belongs to the essential image of $\mathcal{M}(\mathcal{Y})$; the corresponding $\mathcal{D}$-module $\mathcal{D}_{\mathcal{Y}} = H^0_{\mathcal{D}}(\Omega\mathcal{D}_{\mathcal{Y}})$ coincides with the left $\mathcal{D}$-module $\mathcal{D}_{\mathcal{Y}}$ from 1.1.3. More generally, for any $\mathcal{O}$-module $P$ on $\mathcal{Y}$ we have the $\Omega$-complex $\Omega(\mathcal{D}_{\mathcal{Y}} \otimes P)$ with $\Omega(\mathcal{D}_{\mathcal{Y}} \otimes P)_U := \Omega_{U/\mathcal{Y}} \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} P_U[\dim \mathcal{Y}]$. If $\mathcal{Y}$ is good and $P$ is locally free then our $\Omega$-complex sits in $\mathcal{M}(\mathcal{Y})$ and equals to the left $\mathcal{D}$-module $\mathcal{D}_{\mathcal{Y}} \otimes P = \mathcal{D}_{\mathcal{Y}} \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} P$.

Denote by $D(\mathcal{Y})^{\geq 0} \subset D(\mathcal{Y})$ the full subcategory of $\Omega$-complexes $F$ such that $H^a_{\mathcal{D}} F = 0$ for $a < 0$; define $D(\mathcal{Y})^{\leq 0}$ in the similar way.

**7.3.4.** *Proposition.* This is a $t$-structure on $D(\mathcal{Y})$ with core $\mathcal{M}(\mathcal{Y})$ and cohomology functor $H_{\mathcal{D}}$.

This proposition follows immediately from Lemma 7.5.3 below. A different proof in the particular case where $\mathcal{Y}$ is an orbit stack may be found in 7.6.11.

**7.3.5.** *Remark.* Consider the functor $\Omega : C(\mathcal{M}(\mathcal{Y})) \to C(\mathcal{Y}, \Omega)$. For $M \in C(\mathcal{M}(\mathcal{Y}))$ one has $H^{\cdot} M = H^{\cdot}_{\mathcal{D}}(\Omega M)$, so $\Omega$ yields the t-exact functor $\Omega : D(\mathcal{M}(\mathcal{Y})) \to D(\mathcal{Y})$ which extends the "identity" equivalence between the cores. This functor is an equivalence of categories if $\mathcal{Y}$ is a Deligne-Mumford stack*), but not in general.

**7.3.6.** Let $f : \mathcal{Y} \to \mathcal{Z}$ be a morphism of smooth stacks that satisfy (310). It yields a morphism of DG ringed topologies $(\mathcal{Y}_{sm}, \Omega_{\mathcal{Y}}) \to (\mathcal{Z}_{sm}, \Omega_{\mathcal{Z}})$ hence a pair of adjoint DG functors

$$f^{\cdot}_{\Omega} : C(\mathcal{Z}, \Omega) \to C(\mathcal{Y}, \Omega), \;\; f_{\cdot} : C(\mathcal{Y}, \Omega) \to C(\mathcal{Z}, \Omega) \tag{311}$$

and the corresponding adjoint triangulated functors between the homotopy categories (since $\mathcal{Y}$ is quasi-compact $f_{\cdot}$ preserves quasi-coherency).

If $f$ is smooth then $f^{\cdot}_{\Omega}$ preservres $\mathcal{D}$-quasi-isomorphisms, so it defines a t-exact functor $f^{\cdot} : D(\mathcal{Z}) \to D(\mathcal{Y})$. It is obviously compatible with composition of $f$'s.

Let $f$ be an arbitrary morphism. We define the push-forward functor $f_* : D^+(\mathcal{Y}) \to D^+(\mathcal{Z})$ as the right derived functor $Rf_{\cdot}$. We will show that $f_*$ is correctly defined in 7.3.10 below. One needs for this a sufficient supply of "flabby" objects.

*)which means that $\mathcal{Y}$ admits an etale covering by a variety. In this situation the functor $\mathcal{D} : C(\mathcal{Y}, \Omega) \to C(\mathcal{M}(\mathcal{Y}))$ makes obvious sense (which yields the inverse equivalence $D(\mathcal{M}(\mathcal{Y})) \to D(\mathcal{Y})$ as in 7.2.7.

**7.3.7.** *Definition.* We say that an $\mathcal{O}$-module $F$ on $\mathcal{Y}$ is *loose* if for any flat $\mathcal{O}$-module $P$ on $\mathcal{Y}$ one has $H^a(\mathcal{Y}, P \otimes F) = 0$ for $a > 0$. An $\mathcal{O}$- or $\Omega$-complex $F$ is loose if each $F^i$ is loose.

**7.3.8.** *Lemma.* (i) For any $\Omega$-complex $F'$ on $\mathcal{Y}$ there exists a $\mathcal{D}$-quasi-isomorphism $F' \to F$ such that $F$ is loose. If $F'$ is bounded from below then we may choose $F$ bounded from below.

(ii) Assume that $f$ (see 7.3.6) is smooth and affine. Then $f^{\boldsymbol{\cdot}}_{\Omega}, f_{\boldsymbol{\cdot}}$ send loose $\Omega$-complexes to loose ones.

(iii) If $F_1, F_2$ are loose $\Omega$-complexes on stacks $\mathcal{Y}_1, \mathcal{Y}_2$ then $F_1 \boxtimes F_2$ is a loose $\Omega$-complex on $\mathcal{Y}_1 \times \mathcal{Y}_2$.

*Proof.* (i) Since $\mathcal{Y}$ is quasi-compact, there exists a hypercovering $U_{\boldsymbol{\cdot}}$ of $\mathcal{Y}$ such that $U_a$ are affine schemes. Since the diagonal morphism for $\mathcal{Y}$ is affine, the projections $\pi_a : U_a \to \mathcal{Y}$ are affine. Take for $F$ the Čech complex of $F'$ for this hypercovering, so $F^i = \bigoplus\limits_{a \geq 0} \pi_{a\boldsymbol{\cdot}}(F'^{i-a}_{U_a})$.

(ii) Clear.

(iii) We may assume that $F_i$ are loose $\mathcal{O}_{\mathcal{Y}_i}$-modules. Let $P$ be a flat $\mathcal{O}$-module on $\mathcal{Y}_1 \times \mathcal{Y}_2$. Since $F_1$ is loose, one has $R^a p_{2\boldsymbol{\cdot}}(P \otimes p_1^* F_1) = 0$ for $a > 0$ and $p_{2\boldsymbol{\cdot}}(P \otimes p_1^* F_1)$ is a flat $\mathcal{O}$-module on $\mathcal{Y}_2$ (here $p_i : \mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_i$ are the projections). Thus $H^a(\mathcal{Y}_1 \times \mathcal{Y}_2, P \otimes (F_1 \boxtimes F_2)) = H^a(\mathcal{Y}_2, (p_{2\boldsymbol{\cdot}}(P \otimes p_1^* F_1)) \otimes F_2)$ which vanishes for $a > 0$ since $F_2$ is loose. $\square$

Let us return to the situation at the end of 7.3.6.

**7.3.9.** *Lemma.* If $F$ is a loose $\Omega$-complex on $\mathcal{Y}$ bounded from below then $f_{\boldsymbol{\cdot}} F = Rf_{\boldsymbol{\cdot}} F$.

*Proof.* It suffices to check that if our $F$ is in addition $\mathcal{D}$-acyclic (i.e., satisfies condition $H^{\boldsymbol{\cdot}}_{\mathcal{D}} F = 0$) then $f_{\boldsymbol{\cdot}} F$ is also $\mathcal{D}$-acyclic (use 7.3.8(i)).

a. We may assume that $\mathcal{Z}$ is a smooth affine scheme $Z$. Indeed, the statement we want to check is local with respect to $\mathcal{Z}$. Replace $\mathcal{Z}$ by an affine $Z \in \mathcal{Z}_{sm}$, $\mathcal{Y}$ by $\mathcal{Y} \underset{\mathcal{Z}}{\times} Z$, and $F$ by its pull-back to $\mathcal{Y} \underset{\mathcal{Z}}{\times} Z$. The new data satisfy all the conditions of the lemma.

b. We may assume that $\mathcal{Y}$ is a smooth affine scheme $Y$. Indeed, take $U$. as in (i), and denote by $A$ the Čech complex with terms $A^i = \bigoplus_{a \geq 0} (f\pi_a)\cdot(F^{i-a}_{U_a})$. This is an $\Omega$-complex on $Z$. Since $F$ is loose the obvious morphism $f.F \to A$ is a $\mathcal{D}$-quasi-isomorphism (use (310)). Note that $A$ carries an obvious filtration with successive quotients $(f\pi_a)\cdot(F_{U_a})[-a]$. If we know that these are $\mathcal{D}$-acyclic, then $A$ is $\mathcal{D}$-acyclic (use the fact that $F$ is bounded from below), hence $f.F$ is $\mathcal{D}$-acyclic.

c. Let $i : Y \to Y \times Z$ be the graph embedding for $f$. Then $G := i.F$ is $\mathcal{D}$-acyclic. Since $f.F = p.G$ (here $p$ is the projection $Y \times Z \to Z$) what we need to show is that $p.G$ is $\mathcal{D}$-acyclic. Let $T$ be the relative de Rham complex for $\mathcal{D}G$ along the fibers of $p$. We are in a direct product situation so $p.T$ is a $\mathcal{D}$-complex on $Z$. There is an obvious morphism of $\mathcal{D}$-complexes $\mathcal{D}p.G \to p.T$ which is a quasi-isomorphism. Since $p.T$ is acyclic ($T$ carries a filtration with successive quotients $\mathcal{D}G \otimes \Lambda\Theta_Y$, and $\mathcal{D}G$ is acyclic) we are done. □

*Remark.* If $f$ is an affine morphism then for any $F \in C(\mathcal{Y}, \Omega)$ one has $f.F = Rf.F$. Indeed, the statement is local with respect to $\mathfrak{z}$, so we may assume that $\mathfrak{z}$ is an affine scheme. Then $\mathcal{Y}$ is an affine scheme, hence any complex on $\mathcal{Y}$ is loose; now use 7.3.9.

**7.3.10.** *Corollary.* (i) The functor $f_* := Rf. : D^+(\mathcal{Y}) \to D^+(\mathcal{Z})$ is correctly defined.

(ii) $f_*$ is compatible with composition of $f$'s, i.e., the canonical morphism $(f_1 f_2)_* \to f_{1*} f_{2*}$ is an isomorphism.

*Proof.* (i) Use 7.3.8(i) and 7.3.9.

(ii) $f.$ sends loose $\Omega$-complexes to loose ones. □

**7.3.11.** *Remarks.* (i) The above lemmas are also true in the setting of $\mathcal{O}$-complexes.

(ii) Assume that the functor $f.$ on the category of $\mathcal{O}$-modules on $\mathcal{Y}$ has finite cohomology dimension (e.g., this happens when $f$ is representable). Then $f_* := Rf.$ is well-defined for the derived categories of $\Omega$-complexes with arbitrary boundary conditions. Indeed, 7.3.9 (together with its proof) remains valid for unbounded loose $\Omega$-complexes.

(iii) If our stacks are smooth varieties then the above functor $f_*$ is the standard push-forward functor of $\mathcal{D}$-module theory (see 7.2.10). In this situation lemma 7.3.9 (and its proof) remains valid if we assume only that the cohomology $H^a(U, F^i)$, $a > 0$, vanish for any Zariski open $U$ of $Y$ such that $U \to Y$ is an affine morphism.

**7.3.12.** Let now $\mathcal{Y}$ be any smooth stack such that the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is affine (i.e., we drop the quasi-compactness assumption). Then the category of $\Omega$-complexes on $\mathcal{Y}$ may be too small to define the right $\mathcal{D}$-module derived category. One extends the above formalism as follows.

To simplify the notations let us assume that $\mathcal{Y}$ admits a countable covering by quasi-compact opens. In other words $\mathcal{Y}$ is a union of an increasing sequence $\mathcal{Y}_1 \subset \mathcal{Y}_2 \subset \ldots$ of open quasi-comact substacks. An $\Omega$-*complex on* $\mathcal{Y}.$ is a collection $F = (F_i, a_i)$ where $F_i$ are $\Omega$-complexes on $\mathcal{Y}_i$ and $a_i : F_{i+1}|_{\mathcal{Y}_i} \to F_i$ are morphisms of $\Omega$-complexes which are $\mathcal{D}$-quasi-isomorphisms. Such $\Omega$-complexes form a DG category $C(\mathcal{Y}., \Omega)$, so we have the corresponding homotopy category $K(\mathcal{Y}., \Omega)$. It carries the cohomology functor $H_{\mathcal{D}}$ with values in the abelian category $\mathcal{M}(\mathcal{Y})$ of $\mathcal{D}$-modules on $\mathcal{Y}$, $H_{\mathcal{D}}(F)|_{\mathcal{Y}_i} = H_{\mathcal{D}}(F_i)$.

We define $D(\mathcal{Y}., \Omega)$ as the localization of $K(\mathcal{Y}., \Omega)$ with respect to $\mathcal{D}$-quasi-isomorphisms. The triangulated categories $D(\mathcal{Y}., \Omega)$ for different $\mathcal{Y}.$'s are canonically identified. Indeed, let $\mathcal{Y}'_j$ be another sequence of open substacks of $\mathcal{Y}$ as above. Choose an increasing function $j = j(i)$ such that $\mathcal{Y}_i \subset \mathcal{Y}'_{j(i)}$. Let us assign to an $\Omega$-complex $F'$ on $\mathcal{Y}'_.$ the $\Omega$-complex $F$ on $\mathcal{Y}.$, $F_i = F'_{j(i)}|_{\mathcal{Y}_i}$. This functor commutes with $H_{\mathcal{D}}$. The corresponding functor between the

$\mathcal{D}$-derived categories does not depend (in the obvious sense) on the auxiliary choice of $j(i)$, and it is an equivalence of categories.

We see that the category $D(\mathcal{Y}^{\cdot}, \Omega)$ depends only on $\mathcal{Y}$, so we denote it by $D(\mathcal{Y}, \Omega)$ or simply $D(\mathcal{Y})$. Our triangulated category carries the cohomology functor $H_{\mathcal{D}} : D(\mathcal{Y}) \to \mathcal{M}(\mathcal{Y})$ and there is a canonical fully faithful embedding $\Omega : \mathcal{M}(\mathcal{Y}) \hookrightarrow D(\mathcal{Y})$ (see 7.3.2). Proposition 7.3.4 remains true; the proof follows from 7.5.4.

Let $f : \mathcal{Y} \to \mathcal{Z}$ be a morphism of smooth stacks that satisfy our assumption. If $f$ is smooth then one defines the t-exact pull-back functor $f^{\cdot} : D(\mathcal{Z}) \to D(\mathcal{Y})$ in the obvious manner. If $f$ is an arbitrary quasi-compact morphism then one has a canonical push-forward functor $f_* : D(\mathcal{Y})^+ \to D(\mathcal{Z})^+$. We define it after a short digression about loose $\Omega$-complexes.

By definition, $F \in C(\mathcal{Y}^{\cdot}, \Omega)$ is loose if such are all $F_i \in C(\mathcal{Y}_i, \Omega)$. Lemma 7.3.8(i),(iii) remains true in our setting. This means that one may define the $\mathcal{D}$-derived category using only loose complexes. To prove 7.3.8(i) choose coverings $\pi_i : V_i \to \mathcal{Y}_i$ such that $V_i$ is an affine scheme. Denote by $U_i$ the disjoint union of $V_j$'s, $1 \leq j \leq i$, and by $U_i^{\cdot}$ the corresponding hypercovering of $\mathcal{Y}_i$, $U_{ia}$ is the $a$-multiple fibered product of $U_i$ over $\mathcal{Y}_i$. Now take any $F' \in C(\mathcal{Y}^{\cdot}, \Omega)$. Let $F_i$ be the Čech complex of $F_i'$ for the hypercovering $U_i^{\cdot}$ (see the proof of 7.3.8(i)). Then $F_i$ form an $\Omega$-complex $F$ on $\mathcal{Y}^{\cdot}$ in the obvious manner. This $F$ is loose, and the obvious morphism $F' \to F$ is a $\mathcal{D}$-quasi-isomorphism, q.e.d.

Now let us define $f_*$. Let $\mathcal{Z}_i$ be a sequence of open quasi-compact substacks of $\mathcal{Z}$ as above. Then $\mathcal{Y}_i := f^{-1}\mathcal{Z}_i$ is the corresponding sequence for $\mathcal{Y}$. Let $F$ be a bounded from below loose $\Omega$-complex on $\mathcal{Y}^{\cdot}$. Then $(f_{\cdot}F)_i := f_{\cdot}(F_i)$ form an $\Omega$-complex $f_{\cdot}F$ on $\mathcal{Z}^{\cdot}$ (use 7.3.9). The functor $f_{\cdot}$ preserves $\mathcal{D}$-quasi-isomorphisms (by 7.3.9). Our $f_*$ is the corresponding functor between the $\mathcal{D}$-derived categories. Corollary 7.3.10(ii) together with its proof remains true.

Assume that in addition all the functors $f_i. : \mathcal{M}(\mathcal{Y}_i, \mathcal{O}) \to \mathcal{M}(\mathcal{Z}_i, \mathcal{O})$ have finite cohomological dimension (e.g., this happens when $f$ is representable). Then the functor $f_*$ is correctly defined on the whole $D(\mathcal{Y})$. Indeed, let $F$ be any loose $\Omega$-complex on $\mathcal{Y}.$. Then $(f.F)_i := f.(F_i)$ form an $\Omega$-complex $f.F$ on $\mathcal{Z}.$ (use 7.3.11(ii)). The functor $f.$ preserves $\mathcal{D}$-quasi-isomorphisms, and we define $f_* : D(\mathcal{Y}) \to D(\mathcal{Z})$ as the corresponding functor between the $\mathcal{D}$-derived categories.

**7.3.13.** *Remark.* Let $A$ be a commutative algebra. Let $\mathcal{M}(\mathcal{Y}, A)$ be the abelian category of $\mathcal{D}$-modules on $\mathcal{Y}$ equipped with an action of $A$. One defines a t-category $D(\mathcal{Y}, A)$ with core $\mathcal{M}(\mathcal{Y}, A)$ as in 7.3.12 using $\Omega$-complexes with $A$-action. The standard functors render to the $A$-linear setting without problems. More generally, let $\mathcal{A}_{\mathcal{Y}}$ be a commutative $\mathcal{D}$-algebra on $\mathcal{Y}$ (:= a commutative algebra in the tensor category $\mathcal{M}^\ell(\mathcal{Y})$). We have the abelian category $\mathcal{M}(\mathcal{Y}, \mathcal{A}_{\mathcal{Y}})$ of $\mathcal{A}_{\mathcal{Y}}$-modules and its derived version $D(\mathcal{Y}, \mathcal{A}_{\mathcal{Y}})$ defined as in 7.3.12 using $\Omega$-complexes with $\mathcal{A}_{\mathcal{Y}}$-action.

**7.4. Descent for derived categories.** We explain a general homotopy inverse limit construction for derived categories. We need it to be able to formulate a "local" definition of the $\mathcal{D}$-module derived categories.

**7.4.1.** Denote by $(\Delta)$ the category of non-empty finite totally ordered sets $\Delta_n = [0, n]$ and increasing injections. Let $\mathcal{M}.$ be a family of abelian categories cofibered over $(\Delta)$ such that for any morphism $\alpha : \Delta_n \hookrightarrow \Delta_m$ the corresponding functor $\alpha. : \mathcal{M}_n \to \mathcal{M}_m$ is exact.

Denote by $\mathcal{M}_{\text{tot}}$ the category of cocartesian sections of $\mathcal{M}.$, so an object of $\mathcal{M}_{\text{tot}}$ is a collection $M = \{M_n, \alpha^*\}$, $M_n \in \mathcal{M}_n$, $\alpha^* = \alpha^*_M : \alpha.M_n \stackrel{\sim}{\to} M_m$ are isomorphisms such that $(\alpha\beta)^* = \alpha^* \alpha.(\beta^*)$ (here $\beta : \Delta_l \hookrightarrow \Delta_n$). This is an abelian category. Note that $\mathcal{M}_{\text{tot}}$ is compatible with duality: one has $(\mathcal{M}_{\text{tot}})^\circ = (\mathcal{M}^\circ)_{\text{tot}}$.

Our aim is to define a $t$-category $D_{\rm tot}(\mathcal{M}\cdot)$ with core $\mathcal{M}_{\rm tot}$ which satisfies the following key property:

(312) For any $M$, $N \in \mathcal{M}_{\rm tot}$ there is a canonical spectral sequence $E_r^{p,q}$ converging to $\mathrm{Ext}^{p+q}_{D_{\rm tot}(\mathcal{M}\cdot)}(N,M)$ with $E_1^{p,q}=\mathrm{Ext}^q_{\mathcal{M}_p}(N_p,M_p)$.

The construction of $D_{\rm tot}(\mathcal{M}\cdot)$ is compatible with duality.

**7.4.2.** Consider the category $\mathrm{sec}_+=\mathrm{sec}_+(\mathcal{M}\cdot)$ whose objects are collections $M=(M_n,\alpha^*)$ where $M_n\in\mathcal{M}_n$, $\alpha^*=\alpha^*_M:\alpha\cdot M_n\to M_m$ are morphisms such that $(\alpha\beta)^*=\alpha^*\alpha\cdot(\beta^*)$, $\mathrm{id}^*_{\Delta_n}=\mathrm{id}_{M_n}$. This is an abelian category which contains $\mathcal{M}_{\rm tot}$ as a full subcategory closed under extensions. Define $\mathrm{sec}_-=\mathrm{sec}_-(\mathcal{M}\cdot)$ by duality: $\mathrm{sec}_-(\mathcal{M}\cdot):=(\mathrm{sec}_+(\mathcal{M}^\circ_\cdot))^\circ$, so an object of $\mathrm{sec}_-$ is a collection $N=(N_n,\alpha_*)$, $N_n\in\mathcal{M}_n$, $\alpha_*=\alpha^N_*:N_m\to\alpha\cdot N_n$.

Consider the DG categories $C\,\mathrm{sec}_\pm$ of complexes in $\mathrm{sec}_\pm$ and the corresponding homotopy categories $K\,\mathrm{sec}_\pm$. There are adjoint DG functors

$$c_+: C\,\mathrm{sec}_-\longrightarrow C\,\mathrm{sec}_+\,,\qquad c_-: C\,\mathrm{sec}_+\longrightarrow C\,\mathrm{sec}_- \tag{313}$$

defined as follows. Take $M\in C\,\mathrm{sec}_+$. Then for any $m\geq 0$ we have a "cohomology type" coefficient system $\widetilde{M}_m$ on the simplex $\Delta_m$ with values in $C\mathcal{M}_m$. Namely, $\widetilde{M}_m$ assigns to a face $\alpha:\Delta_n\hookrightarrow\Delta_m$ the complex $\alpha\cdot M_n$, and if $\alpha':\Delta_l\hookrightarrow\Delta_m$ is a face of $\alpha$, i.e., $\alpha'=\alpha\beta$, then the corresponding connecting morphism $\alpha'_\cdot M_l\to\alpha\cdot M_n$ is $\alpha\cdot(\beta^*)$. Now $(c_-M)_m$ is the total cochain complex $C^\cdot(\Delta_m,\widetilde{M}_m)$ (so $c_-(M)^\cdot_m=\bigoplus\limits_{\alpha:\,\Delta_n\to\Delta_m}\alpha\cdot M_n^{\cdot-n}$), $\alpha_*^{c_-(M)}$ are the obvious projections. One defines $c_+$ by duality.

To see that $c_\pm$ are adjoint consider for $N$, $M$ as above the complex of abelian groups $Hom(N,M)$ with terms

$$Hom(N,M)^i=\prod_{a,n}\mathrm{Hom}(N_n^{a+n},M_n^{a+i})$$

and the differential wich sends $f=(f_{a,n})\in Hom(N,M)^i$ to $df$,

$$(df)_{a,n}=df_{a,n}-(-1)^{i+n}f_{a+1,n}d+\sum_{j=0,..,n}(-1)^j\alpha^*_j\alpha_j\cdot(f_{a+1,n-1})\alpha_{j*}.$$

Here $\alpha_j : \Delta_{n-1} \to \Delta_n$ is the $j^{th}$ face embedding. Now the adjunction property follows from the obvious identification of complexes of homomorphisms

$$Hom(c_+N, M) \overset{\sim}{\rightarrow} Hom(N, M) \overset{\sim}{\leftarrow} Hom(N, c_-M) \tag{314}$$

**7.4.3.** *Remark.* Fix some $m \geq 0$. For $i = 0, .., m$ let $\nu_i : c_-(M)_m \to M_m$ be the composition of the projector $c_-(M)_m \to \alpha_{i\cdot}M_0$ and $\alpha_i^* : \alpha_{i\cdot}M_0 \to M_m$; here $\alpha_i : \Delta_0 \to \Delta_m$ is the $i^{th}$ vertex. Now all the morphisms $\nu_i$'s are mutually homotopic (with canonical homotopies and "higher homotopies").

**7.4.4.** *Lemma.* The functors $c_\pm$ preserve quasi-isomorphisms. The adjunction morphisms $c_+c_-M \to M$, $N \to c_-c_+N$ are quasi-isomorphisms. □

We see that $c_\pm$ define mutually inverse equivalences between the derived categories $D\,\mathrm{sec}_\pm$. Let us denote these categories thus identified by $D\,\mathrm{sec}$. So $D\,\mathrm{sec}$ carries two $t$-structures with cores $\mathrm{sec}_\pm$ and cohomology functors $H_\pm :\, D\,\mathrm{sec} \to \mathrm{sec}_\pm$.

**7.4.5.** Let $C_{\mathrm{tot}\,+} \subset C\,\mathrm{sec}_+$ be the full subcategory of complexes $M$ such that $H^iM \in \mathcal{M}_{\mathrm{tot}} \subset \mathrm{sec}_+$ for any $i$. In other words $M \in C\,\mathrm{sec}_+$ belongs to $C_{\mathrm{tot}\,+}$ if all the morphisms $\alpha_M^*$ are quasi-isomorphisms. Define $C_{\mathrm{tot}\,-} \subset C\,\mathrm{sec}_-$ in the similar way. Let $K_{\mathrm{tot}\,\pm} \subset K\,\mathrm{sec}_\pm$, $D_{\mathrm{tot}\,\pm} \subset D\,\mathrm{sec}_\pm$ be the corresponding homotopy and derived categories; these are triangulated categories.

The derived categories $D(\mathcal{M}_n)$ form a cofibered category over $(\Delta)$. Denote by $D_{\mathrm{tot}}^{fake}$ the category of its cocartesian sections (this is *not* a triangulated category!). The cohomology functors for $\mathcal{M}.$ define a functor $H : D_{\mathrm{tot}}^{fake} \to \mathcal{M}_{\mathrm{tot}}$. One has an obvious functor $\epsilon_+ : D_{\mathrm{tot}\,+} \to D_{\mathrm{tot}}^{fake}$ which assigns to $M$ the data $(M_n, \alpha*)$ considered as an object of $D_{\mathrm{tot}}^{fake}$. There is a similar functor $\epsilon_- : D_{\mathrm{tot}} \to D_{\mathrm{tot}}^{fake}$.

**7.4.6.** *Lemma.* For any $M \in D_{\mathrm{tot}\,+}$ one has $c_-M \in D_{\mathrm{tot}\,-}$, and there is a unique isomorphism $\epsilon_-(c_-M) \overset{\sim}{\rightarrow} \epsilon_+(M)$ such that its $0^{th}$ component is $\mathrm{id}_{M_0}$. One also has the dual statement with + and - interchanged.

*Proof.* Use 7.4.3. □

**7.4.7.** We see that the functors $c_\pm$ identify the triangulated categories $D_{\mathrm{tot}\,\pm}$. In other words, the subcategories $D_{\mathrm{tot}\,\pm} \subset D\,\mathrm{sec}$ coincide; this is the category $D_{tot} = D_{\mathrm{tot}}(\mathcal{M}.)$ that was promised in 7.4.1. The functors $\epsilon_\pm$ are canonically identified, so we have the functor $\epsilon : D_{\mathrm{tot}} \to D_{\mathrm{tot}}^{fake}$. Note that $H_\pm = H\epsilon$, so we have a canonical cohomology functor $H : D_{\mathrm{tot}} \to \mathcal{M}_{\mathrm{tot}}$. This is a cohomology functor for a non-degenerate t-structure on $D_{\mathrm{tot}}$ with core $\mathcal{M}_{\mathrm{tot}}$. Note that the embedding $D_{\mathrm{tot}} \hookrightarrow D\,\mathrm{sec}$ is t-exact with respect to either of $\pm$ t-structures on $D\,\mathrm{sec}$; it identifies the core $\mathcal{M}_{\mathrm{tot}}$ with the intersection of cores $\mathrm{sec}_+$ and $\mathrm{sec}_-$.

**7.4.8.** Let us derive the spectral sequence (312) from 7.4.1. More generally, consider objects $N \in D^-\,\mathrm{sec}_- \subset D\,\mathrm{sec}$, $M \in D^+\,\mathrm{sec}_+ \subset D\,\mathrm{sec}$. Let us represent them by complexes $N \in K^-\,\mathrm{sec}_-$, $M \in K^+\,\mathrm{sec}_+$. Consider the complex $Hom(N,M)$ (see 7.4.2). It carries an obvious decreasing filtration $F^{\cdot}$ with $gr_F^n = Hom(N_n, M_n)[-n]$. Note that $Hom(N,M)$ is a bounded below complex and filtration $F^{\cdot}$ induces on each term $Hom(N,M)^i$ a finite filtration. We consider $Hom(N,M)$ as an object of the filtered derived category DF of such complexes. Let $RHom(N,\cdot)$ be the right derived functor of the functor $K^+\,\mathrm{sec}_+ \to \mathrm{DF}$, $M \to Hom(N,M)$. This functor is correctly defined, and the obvious morphism $gr_F^n RHom(N,M) \to RHom(N_n,M_n)[-n]$ is a quasi-isomorphism for any $n$. This follows from the fact that for any quasi-isomorphism $f : M_n \to I$ in $\mathcal{M}_n$ there exists a quasi-isomorphism $g : M \to J$ in $K^+\,\mathrm{sec}_+$ and a morphism $h : I \to J_n$ such that $g_n = hf$. Consider the spectral sequence $E_r^{p,q}$ of the filtered complex $RHom(N,M)$. It converges to $H^{\cdot}RHom(N,M)$, and $E_1^{p,q} = H^q RHom_{\mathcal{M}_p}(N_p, M_p)$.

**7.4.9.** *Remark.* Assume that the categories $\mathcal{M}_n$ have many injective objects. Then the category $K_{\mathrm{tot}-}^+$ has many injective objects (i.e., the functor $K_{\mathrm{tot}-}^+ \to D_{\mathrm{tot}}^+$ admits a right adjoint functor). Indeed, if $I \in K_{\mathrm{tot}+}^+$ is a complex such that eacb $I_n^a$ is an injective object of $\mathcal{M}_n$ then $c_- I$ is an injective object of $K^+_{\mathrm{tot}-}$, and any object in $K^+_{\mathrm{tot}-}$ is quasi-isomorphic to such $I$. Dually, if $\mathcal{M}_n$ have many projective objects then $K^-_{\mathrm{tot}+}$ has many projective objects.

**7.4.10.** This subsection will not be used in the sequel; the reader may skip it. One may define $D\sec$, hence $D_{\mathrm{tot}}$, in a slightly different way which is convenient in some applications[*)]. We define the category $hot_+ = hot_+(\mathcal{M}.)$ as follows. Its objects are families $A = (A_m)$, $A_m \in \mathcal{M}_m$. A morphism $f : A \to B$ is a collection $(f_\alpha)$ where for an arrow $\alpha : \Delta_n \to \Delta_m$ the corresponding $f_\alpha$ is a morphism $\alpha.A_n \to B_m$. The composition of morphisms is $(fg)_\alpha = \sum\limits_{\alpha=\beta\gamma} f_\beta \beta.(g_\gamma)$. This is an additive category. Set $hot_-(\mathcal{M}.) = (hot_+(\mathcal{M}^\circ_.))^\circ$. We have the corresponding DG categories of complexes $Chot_\pm$.

One has a DG functor $t_+ : C\sec_+ \to Chot_+$ which sends $M \in C\sec_+$ to a complex $t_+M \in Chot_+$ with components $(t_+M)^a_m = M^{a-m}_m$ and the differential $d = d_{t_+M}$ such that $d_{\mathrm{id}_{\Delta_m}} = (-1)^m d^{a-m}_{M_m} : M^{a-m}_m \to M^{a-m+1}_m$, and for the $i^{th}$ boundary map $\alpha_i : \Delta_m \hookrightarrow \Delta_{m+1}$ one has $d_{\alpha_i} = (-1)^i \alpha_i^* : \alpha_i.M^{a-m}_m \to M^{a-m}_{m+1}$, all other components of $d$ are zero. For $l \in Hom(M_1, M_2)$ one has $t_+(l)_{\mathrm{id}_{\Delta_m}} = l_m$, the other components are zero.

*Remark.* The functor $t_+$ is faithful. One may consider objects of $Chot_+$ as "generalized complexes" in $\sec_+$ with extra higher homotopies.

One also has a DG functor $s_- : Chot_+ \to C\sec_-$ defined as follows. For $A \in Chot_+$ the complex $s_-A$ has components $(s_-A)^a_m = \sum\limits_{\beta:\Delta_n\to\Delta_m} \beta.A^a_n$. The compatibility morphism $\alpha_* : (s_-A)^a_l \to \alpha.(s_-A)^a_m$ for $\alpha : \Delta_m \to \Delta_l$ has component $\gamma.A^a_k \to \alpha.\beta.A^a_n$ equal to $\mathrm{id}_{\gamma.A^a_k}$ if $k = n$, $\gamma = \alpha\beta$ and zero otherwise. A component $\gamma.A^a_k \to \alpha.\beta.A^a_n$ of the differential $d_{s_-A} : (s_-A)^a_m \to (s_-A)^{a+1}_m$ is equal to $\gamma.(d_{A\delta})$ if $\beta = \gamma\delta$ and zero otherwise.

*Remark.* The DG functor $s_-$ is fully faithful.

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[*)] This construction goes back to the works of Toledo and Tong.

We define DG functors $t_- : C\sec_- \to Chot_-$ and $s_+ : Chot_- \to C\sec_+$ by duality. Note that the composition $s_+t_- : C\sec_- \to C\sec_+$ coincides with the functor $c_+$ from 7.4.2; similarly, $s_-t_+ = c_-$. The functors $t_-s_- : Chot_+ \to Chot_-$ and $t_+s_+ : Chot_- \to Chot_+$ are adjoint (just as the functors $c_\pm$, see 7.4.2).

We say that a morphism $f : A \to B$ in the homotopy category $Khot_\pm$ of $Chot_\pm$ is a *quasi-isomorphism* if all the morphisms $f_m := f_{\mathrm{id}_{\Delta_m}} : A_m \to B_m$ are quasi-isomorphisms. Quasi-isomorphisms form a localizing family. Denote the corresponding localized triangulated categories by $Dhot_\pm$.

The functors $s_\pm, t_\pm$ preserve quasi-isomorphisms, so they define functors between the derived categories. The adjunction morphisms for compositions of these functors are quasi-isomorphisms. So our derived categories $D\sec_\pm, Dhot_\pm$ are canonicaly identified.

*Remarks.* (i) A complex $A \in Dhot_+$ belongs to $D_{\mathrm{tot}}$ if and only if for any $\alpha : \Delta_m \to \Delta_{m+1}$ the $\alpha$-component $d_{A\alpha} : \alpha.A_m \to A_{m+1}$ is a quasi-isomorphism of complexes (the differential on $A_m$ is $d_{A\,\mathrm{id}_{\Delta_m}}$, same for $A_{m+1}$).

(ii) If the categories $\mathcal{M}_n$ have many injective objects then $K^+hot_+$ has many injective objects. Dually, if $\mathcal{M}_n$ have many projective objects then $K^-hot_-$ has many projective objects (cf. 7.4.9).

**7.4.11.** Some of the above constructions make sense in the following slightly more general setting. Consider any family of DG categories $\mathcal{C}.$ cofibered over $(\Delta)$. One has the DG categories $\mathcal{C}\sec_\pm = \sec_\pm(\mathcal{C}.)$ (defined exactly as the categories $\sec_\pm(\mathcal{M}.)$ in 7.4.2), and the corresponding homotopy categories. One defines the adjoint functors $c_\pm$ between the $\pm$ categories as in 7.4.2.

Assume in addition that we have $\mathcal{M}$ as in 7.4.1 and a family of cohomology functors $H : \mathcal{C}. \to \mathcal{M}.$ compatible with the fibered category structures. We get the corresponding cohomology functors $H_\pm : \mathcal{C}\sec_\pm \to \sec_\pm$. Localising our homotopy categories by $H$-quasi-isomorphisms we get the derived categories $\mathcal{D}\sec_\pm$. As in Lemma 7.4.4 the functors $c_\pm$ identify the categories $\mathcal{D}\sec_\pm$, so we may denote them simply $\mathcal{D}\sec$. One defines the categories $\mathcal{C}_{\mathrm{tot}\pm}$, etc., as in 7.4.5. Lemma 7.4.6 remains true, so we have the full triangulated subcategory $\mathcal{D}_{\mathrm{tot}} \subset \mathcal{D}\,\mathrm{sec}$ and the cohomology functor $H : \mathcal{D}_{\mathrm{tot}} \to \mathcal{M}_{\mathrm{tot}}$.

### 7.5. $\mathcal{D}$-module theory on smooth stacks II.

**7.5.1.** Let $\mathcal{Y}$ be an arbitrary smooth algebraic stack. Let $U\cdot$ be a hypercovering of $\mathcal{Y}$ such that each $U_n$ is a disjoint union of (smooth) quasi-compact separated algebraic spaces (e.g., affine schemes). We call such $U\cdot$ an *admissible* hypercovering. Consider $U\cdot$ as a $(\Delta)^{\circ}$-algebraic space. The categories $\mathcal{M}(U\cdot)$ form a $(\Delta)$-family of abelian categories as in 7.4.1; the corresponding category $\mathcal{M}_{\mathrm{tot}}$ is $\mathcal{M}(\mathcal{Y})$. According to 7.4.7 we get the corresponding t-category $D_{\mathrm{tot}} = D_{\mathrm{tot}}(U\cdot, \mathcal{D})$ with core $\mathcal{M}(\mathcal{Y})$.

We may also consider DG categories $C(U\cdot, \Omega)$ together with the cohomology functors $H_{\mathcal{D}}\cdot : C(U\cdot, \Omega) \to \mathcal{M}(U\cdot)$, $H_{\mathcal{D}n} F_n = H_{\mathcal{D}} F_n[\dim U_n/\mathcal{Y}]$ for $F_n \in C(U_n, \Omega)$, and apply 7.4.11. We get a triangulated category $D_{\mathrm{tot}}(U\cdot, \Omega)$ together with a cohomology functor $H_{\mathcal{D}} : D_{\mathrm{tot}}(U\cdot, \Omega) \to \mathcal{M}(\mathcal{Y})$.

The categories $D_{\mathrm{tot}}(U\cdot, \mathcal{D})$ and $D_{\mathrm{tot}}(U\cdot, \Omega)$ are canonically identified. Namely, one has a functor $\Omega\cdot : C(U\cdot, \mathcal{D}) \to C(U\cdot, \Omega)$, $\Omega_n(M_n) := \Omega M_n[-\dim U_n/\mathcal{Y}]$. This functor is compatible with DG and fibered categories structures, and with the cohomology functors (i.e., $H = H_{\mathcal{D}}\cdot \Omega\cdot$). Therefore it yields an exact functor

$$\Omega : D_{\mathrm{tot}}(U\cdot, \mathcal{D}) \to D_{\mathrm{tot}}(U\cdot, \Omega) \tag{315}$$

This functor is an equivalence of categories. Indeed, though the functor $\mathcal{D}$ between $C(U\cdot, \Omega)$ and $C(U\cdot, \mathcal{D})$ is *not* compatible with the fibered category structures, it provides the functor $\mathcal{D} : C\,\mathrm{sec}_-(U\cdot, \Omega) \to C\,\mathrm{sec}_-(U\cdot, \mathcal{D})$, $(\mathcal{D}F)_n = \mathcal{D}F_n[\dim U_n/\mathcal{Y}]$ (use 7.2.8 to define $\alpha^*$'s). This $\mathcal{D}$ is left adjoint to the corresponding $\Omega$ functor, and is compatible with the cohomology functors. The $\mathcal{D}$-$\Omega$ adjunction morphisms are quasi-isomorphisms (see 7.2.4, 7.2.5), so $\mathcal{D}$ yields the functor inverse to (315).

We denote the categories $D_{\mathrm{tot}}(U\cdot,\mathcal{D})$ and $D_{\mathrm{tot}}(U\cdot,\Omega)$ thus identified simply by $D_{\mathrm{tot}}(U\cdot)$.

**7.5.2.** *Propositon.* There exists a canonical identification of t-categories $D_{\mathrm{tot}}(U\cdot)$ for different admissible coverings of $\mathcal{Y}$.

For a proof see 7.5.5 below. We denote these categories thus identified by $D(\mathcal{Y})$; this is a t-category with core $\mathcal{M}(\mathcal{Y})$.

Before proving 7.5.2 let us show that if $\mathcal{Y}$ satisfies condition (310) then, indeed, we get the same category $D(\mathcal{Y})$ as in 7.3.2. By the way, this implies 7.3.4.

Choose a hypercovering $U\cdot$ of $\mathcal{Y}$ such that $U_n$ are affine schemes. There is an obvious exact functor (restriction to $U\cdot$)

$$r: D(\mathcal{Y},\Omega)\to D_{\mathrm{tot}}(U\cdot,\Omega) \tag{316}$$

**7.5.3.** *Lemma.* The functor $r$ is an equivalence of categories.

*Proof.* Let us construct the inverse functor. For $F\in K_{\mathrm{tot}\,+}(\Omega)$ define the $\Omega$-complex $\pi\cdot F$ on $\mathcal{Y}$ as the total complex of Čech bicomplex with terms $\pi\cdot F^{ab}:=\pi_b(F^a)$, so $(\pi\cdot F)^n=\bigoplus\limits_{a+b=n}F^{ab}$; here $\pi_b$ are projections $U_b\to\mathcal{Y}$. Thus we have the exact functor $\pi\cdot: K_{\mathrm{tot}\,+}(\Omega)\to K(\mathcal{Y},\Omega)$. This functor preserves $\mathcal{D}$-quasi-isomorphisms (since, by (310), the projections $\pi_b$ are affine), so it defines a functor $D_{\mathrm{tot}}(U\cdot,\Omega)\to D(\mathcal{Y},\Omega)$.

We leave it to the reader to check that this functor is inverse to $r$ (hint: for $F$ as above the adjunction quasi-isomorphism $\pi^{\cdot}_{\Omega}\pi\cdot F\to F$ comes from a canonical morphism $\pi^{\cdot}_{\Omega}\pi\cdot F\to c_-F$ in $C\,\mathrm{sec}_-(U\cdot,\Omega)$).

$\square$

**7.5.4.** *Remark.* The above lemma renders to the setting of 7.3.12 as follows. Let $\mathcal{Y}$ be any smooth stack such that the diagonal morphism $\mathcal{Y}\to\mathcal{Y}\times\mathcal{Y}$ is affine. Then the categories $D(\mathcal{Y})$ as defined in 7.3.12 and 7.5.1 are canonically equivalent. Indeed, let $\mathcal{Y}_i$ be a sequence of open substacks of $\mathcal{Y}$ as in 7.3.12, and $V_i\to\mathcal{Y}$ be a covering such that $V_i$ are affine schemes.

Then the $V_i$'s form a covering of $\mathcal{Y}$. Let $U.$ be the corresponding Čech hypercovering. Therefore $U_a$ is disjoint union of components $U_\alpha$ labeled by sequences $\alpha = (\alpha_1, \alpha_2, ...)$, $\alpha_i \geq 0$, $\Sigma\alpha_i = a + 1$, where $U_\alpha$ is fibered product over $\mathcal{Y}$ of $\alpha_1$ copies of $V_1$, $\alpha_2$ copies of $V_2$,... For $F \in C(\mathcal{Y}., \Omega)$ set $F_{U_\alpha} := F_{i_\alpha U_\alpha}$ where $i_\alpha$ is the minimal $i$ such that $\alpha_i$ is non-zero (note that $U_\alpha \in \mathcal{Y}_{sm}$. These $F_{U_\alpha}$ form an $\Omega$-complex $F$ on $U.$ in the obvious manner which lies in $C_{\mathrm{tot}}(U., \Omega)$. The functor $C(\mathcal{Y}., \Omega) \to C_{\mathrm{tot}}(U., \Omega)$ commutes with the functor $H_{\mathcal{D}}$ so it defines a triangulated functor

$$r : D(\mathcal{Y}., \Omega) \to D_{\mathrm{tot}}(U., \Omega) \tag{317}$$

We leave it to the reader to check that this functor is an equivalence of categories, and that the corresponding identification of $D(\mathcal{Y})$'s in the sense of 7.3.12 and 7.5.2 does not depend on the auxiliary data of $\mathcal{Y}.$ and $V.$.

**7.5.5.** *Proof of 7.5.2.* We need to identify canonically the t-categories $D_{\mathrm{tot}}(U.)$ for different $U.$'s. Let $U'_.$ be another admissible hypercovering. First we define a t-exact functor $\Phi = \Phi_V : D_{\mathrm{tot}}(U.) \to D_{\mathrm{tot}}(U'_.)$ in terms of some auxiliary data $V$. Then we show that $\Phi$ actually does not depend $V$, and it is an equivalence of categories.

Our $V$ is a $(\Delta)^\circ \times (\Delta)^\circ$ -algebraic space $V..$ over $\mathcal{Y}$ together with smooth morphisms $\pi : V_{mn} \to U_m$, $\pi' : V_{mn} \to U'_n$. We assume that $\pi$, $\pi'$ are compatible with $(\Delta)$ projections in the obvious manner, $\pi'_{\cdot n} : V_{\cdot n} \to U'_n$ are hypercoverings, and $\pi'_{mn} : V_{mn} \to U'_n$ are affine morphisms. For $F \in K_{\mathrm{tot}\,+}(U., \Omega)$ we have $\Omega$-complexes $F_{Vn} \in K_{\mathrm{tot}\,+}(V_{\cdot n}, \Omega)$ - the pull-back of $F$ to $V_{\cdot n}$. Set $\Phi_{Vn}F := \pi'_. F_{Vn}$ (see the proof of 7.5.3 for the notation). This is an $\Omega$-complex on $U'_n$. The $\Omega$-complexes $\Phi_{Vn}$ form an $\Omega$-complex $\Phi_V F \in K_{\mathrm{tot}\,+}(U'_., \Omega)$ in the obvious way such that $H_{\mathcal{D}}F = H_{\mathcal{D}}\Phi_V F$. Therefore we have a t-exact functor $\Phi_V : D_{\mathrm{tot}}(U., \Omega) \to D_{\mathrm{tot}}(U'_., \Omega)$ which induces the identity functor between the cores $\mathcal{M}(\mathcal{Y})$.

Assume that we have $V_1$ and $V_2$ as above. To identify the functors $\Phi_{V_i}$ choose another $V$ as above, together with embeddings $V_1, V_2 \subset V$

compatible with all the projections which identify $(V_1)_{mn}$, $(V_2)_{mn}$ with a union of connected components of $V_{mn}$. The embeddings induce projections $\Phi_V F \to \Phi_{V_1}$, $\Phi_V F \to \Phi_{V_2} F$ which are obviously quasi-isomorphisms. Therefore we have identified the functors $\Phi_{V_i}$ between the derived categories. We leave it to the reader to check that this identification does not depend on the auxiliary data of $V$.

Thus we have a canonical functor $\Phi = \Phi_{UU'} : D_{\mathrm{tot}}(U^{\cdot}, \Omega) \to D_{\mathrm{tot}}(U'^{\cdot}, \Omega)$. If $U''^{\cdot}$ is the third hypercovering then there is a canonical isomorphism of functors $\Phi_{UU''} = \Phi_{U'U''}\Phi_{UU'}$; we leave its definition to the reader, as well as verification of the usual compatibilities. Since $\Phi_{UU}$ is the identity functor we see that $\Phi$'s identify simultaneously all the categories $D_{\mathrm{tot}}(U^{\cdot})$. $\square$

**7.5.6.** Let $f : \mathcal{Y} \to \mathcal{Z}$ be a quasi-compact morphism of smooth stacks. Let us define the push-forward functor $f_* : D(\mathcal{Y})^+ \to D(\mathcal{Z})^+$. To do this consider any admissible hypercoverings $U^{\cdot}$ of $\mathcal{Y}$ and $W^{\cdot}$ of $\mathcal{Z}$. We get the $(\Delta)^\circ \times (\Delta)^\circ$-algebraic space $U^{\cdot} \underset{\mathcal{Z}}{\times} W^{\cdot}$. One may find a $(\Delta)^\circ \times (\Delta)^\circ$-algebraic space $V^{\cdot\cdot}$ together with morphism $\phi = (\phi_1, \phi_2) : V^{\cdot\cdot} \to U^{\cdot} \underset{\mathcal{Z}}{\times} W^{\cdot}$ such that the projections $V_{mn} \to U_m$ are smooth, $V_{mn} \to W_n$ are affine, and $V^{\cdot}{}_n \to \mathcal{Y} \underset{\mathcal{Z}}{\times} W_n$ are hypercoverings. Now for $F \in K^+_{\mathrm{tot}\,+}(U^{\cdot}, \Omega)$ let $F_{Vn} \in K^+_{\mathrm{tot}\,+}(V^{\cdot}{}_n, \Omega)$ be its pull-back to $V^{\cdot}{}_n$. Define the $\Omega$ -complex $f_{\cdot} F_n$ on $W_n$ as the total complex of the Čech bicomplex with terms $\phi_{2\cdot} F_{Vn}$. These $\Omega$-complexes form an object $f_{\cdot} F$ of $K^+_{\mathrm{tot}\,+}(W^{\cdot}, \Omega)$. The functor $f_{\cdot} : K^+_{\mathrm{tot}\,+}(U^{\cdot}, \Omega) \to K^+_{\mathrm{tot}}(W^{\cdot}, \Omega)$ preserves $\mathcal{D}$-quasi-isomorphisms hence it yields a functor $f_* : D(\mathcal{Y})^+ \to D(\mathcal{Z})^+$. We leave it to the reader to check that the construction of $f_*$ does not depend on the auxiliary choices of $U, W, V$, and is compatible with composition of $f$'s.

A smooth morphism of smooth stacks $f : \mathcal{Y} \to \mathcal{Z}$ yields a t-exact functor $f^\dagger = f^{\cdot}_{\Omega} : D(\mathcal{Z}) \to D(\mathcal{Y})$. Namely, choose admissible hypercoverings $U^{\cdot}$ of $\mathcal{Y}$, $W^{\cdot}$ of $\mathcal{Z}$ and a morphism $f_{\cdot} : U^{\cdot} \to W^{\cdot}$ compatible with f. The functor $f^{\cdot}_{\cdot\Omega} : K_{\mathrm{tot}\,\pm}(W^{\cdot}, \Omega) \to K_{\mathrm{tot}\,\pm}(U^{\cdot}, \Omega)$ preserves $\mathcal{D}$-quasi-isomorphisms, so it defines a functor $f^{\cdot}_{\Omega}$ between the derived categories. We leave it to the reader to check that this definition does not depend on the auxiliary choices, that our pull-back functor is compatible with composition of $f$'s, and that in case when $f$ is quasi-compact the functor $f^{\cdot}_{\Omega}$ is left adjoint to $f_*$.

**7.5.7.**

**7.5.8.** *Remarks.* (i) One may also try to define $D(\mathcal{Y})$ using appropriate non-quasi-coherent $\Omega$-complexes in a way similar to the definition of derived category of $\mathcal{O}$-modules from [LMB93]6.3. Probably such a definition yields the same category $D^+(\mathcal{Y})$.

(ii) The "local" construction of derived categories is also convenient in the setting of $\mathcal{O}$-modules. For example, it helps to define the cotangent complex of an algebraic stack as a true object of the derived category (and not just the projective limit of its truncations as in [LMB93]9.2), and also to deal with Grothendieck-Serre duality.

(iii) Replacing $\mathcal{D}$-modules by perverse sheaves we get a convenient definition of the derived category of constructible sheaves on any algebraic stack locally of finite type.

## 7.6. Equivariant setting.

**7.6.1.** Let us explain parts 7.1.1 (a), (b) of the (finite dimensional) Hecke pattern. So let $G$ be an algebraic group and $K \subset G$ an algebraic subgroup. Assume for simplicity that $K$ is affine; then the stacks below satisfy condition (310) of 7.3.1. Set*) $\mathcal{H}^c := C(K\backslash G/K, \Omega)$, $\mathcal{H} := D(K\backslash G/K)$. We call these categories *pre Hecke* and *Hecke* category respectively. They carry canonical monoidal structures defined as follows.

Consider the morphisms of stacks

$$(K \backslash G/K) \times (K \backslash G/K) \xleftarrow{p} K \backslash G \underset{K}{\times} G/K \xrightarrow{\bar{m}} K \backslash G/K \tag{318}$$

*)Here the superscript "c" means that we deal with the true DG category of complexes, not the derived category.

Here $G \underset{K}{\times} G$ is the quotient of $G \times G$ modulo the $K$-action $k(g_1, g_2) = (g_1 k^{-1}, kg_2)$, $p$ is the obvious projection, and $\bar{m}$ is the product map. For $F_1$, $F_2 \in \mathcal{H}^c$ set $F_1 \overset{c}{\circledast} F_2 := \bar{m}_{\cdot} p^{\cdot}_{\Omega}(F_1 \boxtimes F_2)$ . The *convolution tensor product* $\overset{c}{\circledast}$ satisfies the obvious associativity constraint, so we have a monoidal structure on $\mathcal{H}^c$. We define the convolution tensor product $\circledast : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ as the right derived functor of $\overset{c}{\circledast}$. One has $F_1 \circledast F_2 = \bar{m}_* p^{\cdot}_{\Omega}(F_1 \boxtimes F_2)$ ; if $\Omega$-complexes $F_1$, $F_2$ are loose (see 7.3.7) then $F_1 \circledast F_2 = F_1 \overset{c}{\circledast} F_2$. Thus the associativity constraint for $\circledast$ follows from the one of $\overset{c}{\circledast}$, so $\mathcal{H}$ is a monoidal triangulated category. $\mathcal{H}^c$ and $\mathcal{H}$ have a unit object $E$: one has $E_G = i_{K\cdot} \Omega_K$ (here $i_K : K \hookrightarrow G$ is the embedding).

Let $Y$ be a smooth variety with $G$-action. Consider the stack $\mathcal{B} := K \setminus Y$. The Hecke Action on $D(\mathcal{B})$ arises from the diagram

$$(K \setminus G/K) \times \mathcal{B} \xleftarrow{p_Y} K \setminus (G \underset{K}{\times} Y) \xrightarrow{\bar{m}_Y} \mathcal{B}\,. \tag{319}$$

Namely, for $F \in \mathcal{H}^c$, $T \in C(\mathcal{B}, \Omega)$ set $F \overset{c}{\circledast} T := \bar{m}_{Y \cdot} p^{\cdot}_{Y\Omega}(F \boxtimes T)$. As above $\overset{c}{\circledast}$ satisfies the obvious associativity constraint, so $C(\mathcal{B}, \Omega)$ is a unital $\mathcal{H}^c$-Module. Define $\circledast : \mathcal{H} \times D(\mathcal{B}) \longrightarrow D(\mathcal{B})$ as the right derived functor of $\overset{c}{\circledast}$. One has $F \circledast T = \bar{m}_{Y*} p^{\cdot}_{Y\Omega}(F \boxtimes T)$, and if $F$,$T$ are loose (see 7.3.7) then $F \circledast T = F \overset{c}{\circledast} T$. Thus $D(\mathcal{B})$ is a $\mathcal{H}$-Module.

**7.6.2.** *Remarks.* (i) In the above definitions we were able to consider the unbounded derived categories since the projections $\bar{m}, \bar{m}_Y$ are representable (see7.3.11(ii)).

(ii) If $f : Z \to Y$ is a morphism of smooth varieties with $G$-action then $f_* : D(K \setminus Z) \to D(K \setminus Y)$ is a Morphism of $\mathcal{H}$-Modules.

**7.6.3.** Let $Y$ be a smooth variety equipped with an action of an affine algebraic group $K$. Consider the stack $\mathcal{B} := K \setminus Y$. In the rest of 7.6 we are going to describe $D(\mathcal{B})$ in terms of appropriate equivariant complexes on $Y$. We will also introduce certain derived category $D(K \overset{\cdot}{\setminus} Y)$ intermediate between $D(K \setminus Y)$ and $D(Y)$ that will be of use in 7.7.

Set $K_\Omega = (K, \Omega_K)$, $K^{\cdot}_\Omega = (K, \Omega^{\cdot}_K)$ (so $K^{\cdot}_\Omega$ is $K_\Omega$ with its de Rham differential skipped). These are group objects in the category of DG ringed spaced and graded ringed spaces respectively. Denote by $\mathfrak{k}$, $\mathfrak{k}_\Omega$, $\mathfrak{k}^{\cdot}_\Omega$ the Lie algebras of $K$, $K_\Omega$, $K^{\cdot}_\Omega$ respectively. As a plain complex, $\mathfrak{k}_\Omega$ is equal to the cone of $id_{\mathfrak{k}}$ so $\mathfrak{k}^0_\Omega = \mathfrak{k} = \mathfrak{k}^{-1}_\Omega$. Since $K$ is a subgroup of $K_\Omega$ and $K^{\cdot}_\Omega$ we have the corresponding Harish-Chandra pairs $(\mathfrak{k}_\Omega, K)$, $(\mathfrak{k}^{\cdot}_\Omega, K)$. Note that $K_\Omega$ modules are the same as DG $(\mathfrak{k}_\Omega, K)$-modules, and $K^{\cdot}_\Omega$-modules are the same as graded $(\mathfrak{k}^{\cdot}_\Omega, K)$-modules.

The $K$-action on $Y$ yields the action of $K_\Omega$ on $Y_\Omega = (Y, \Omega)$ hence the action of $K^{\cdot}_\Omega$ on $Y^{\cdot}_\Omega = (Y, \Omega^{\cdot})$. For a graded $\Omega^{\cdot}_Y$-module $F^{\cdot}_Y$ a $K^{\cdot}_\Omega$-action on $F^{\cdot}_Y$ is the same as a $(\mathfrak{k}^{\cdot}_\Omega, K)$-action. Explicitly, this is a $K$-action on $F^{\cdot}_Y$ together with a $K$-equivariant morphism $\mathfrak{k} \otimes F^{\cdot}_Y \to F^{\cdot -1}_Y$, $\xi \otimes f \mapsto i_\xi(f)$ (we assume that $K$ acts on $\mathfrak{k}$ in the adjoint way) such that $i_\xi(\nu f) =< \xi, \nu > f + \nu i_\xi(f)$, $i^2_\xi = 0$ for any $\xi \in \mathfrak{k}$ and $\nu \in \Omega^1_Y$.

**7.6.4.** Let $F_Y$ be an $\Omega$-complex on $Y$. A $K$-*action* on $F_Y$ is a $K$-action on the graded $\mathcal{O}_Y$-module $F^{\cdot}_Y$ such that for any $k \in K$ the translation $k^* F^{\cdot}_Y \approx F^{\cdot}_Y$ is a morphism of $\Omega$-complexes (i.e., it commutes with the differential). A $K_\Omega$-*action* on $F_Y$ is an action of $K_\Omega$ on $F_Y$ considered as a DG module on $Y_\Omega$. In other words, this is a $K^{\cdot}_\Omega$-action on the graded $\Omega^{\cdot}_Y$-module $F^{\cdot}_Y$ such that $K$ acts on $F_Y$ as on an $\Omega$-complex and $\mathfrak{k}_\Omega$ acts on $F_Y$ as a DG Lie algebra. The latter condition means that for any $\xi \in \mathfrak{k}$ one has $di_\xi + i_\xi d = Lie_\xi$ (here $Lie$ is the $\mathfrak{k}$-action on $F^{\cdot}_Y$ that comes from the $K$-action). An $\Omega$-complex equipped with a $K$-action is called a *weakly $K$-equivariant* $\Omega$-complex, and that with $K_\Omega$-action is called $K_\Omega$-*equivariant* $\Omega$-complex.

It is clear that for any $\Omega$-complex $F$ on the stack $\mathcal{B} := K\backslash Y$ the $\Omega$-complex $F_Y$ carries automatically a $K_\Omega$-action.

**7.6.5.** *Lemma.* The functor $C(K \setminus Y, \Omega) \longrightarrow$ ($K_\Omega$-equivariant $\Omega$-complexes on $Y$) is an equivalence of DG categories. $\square$

**7.6.6.** *Remark.* Assume we are in situation 7.6.1. Let $m : K \setminus G \times G/K \to K \setminus G/K$ be the product map. Set $F_1 \widetilde{\circledast} F_2 = m_\cdot(F_{1K\setminus G} \boxtimes F_{2G/K})$; this is an $\Omega$-complex on $K \setminus G/K$. The $K$-action along the fibers of the projection $G \times G \to G \underset{K}{\times} G$ yields a $K_\Omega$-action on $F_1 \widetilde{\circledast} F_2$ (with respect to the trivial $K$-action on $K \setminus G/K$). Its invariants coincide with $F_1 \overset{c}{\circledast} F_2$. Similarly, consider the map $m_Y : (K \setminus G) \times Y \to \mathcal{B}$ ; set $F \widetilde{\circledast} T := m_{Y\cdot}(F_{K\setminus G} \boxtimes T)$. The obvious $K$-action on $(K \setminus G) \times Y$ yields a $K_\Omega$-action on this $\Omega$-complex whose invariants coinside with $F \overset{c}{\circledast} T$.

**7.6.7.** We denote the category of weakly $K$-equivariant $\Omega$-complexes on $Y$ by $C(K \overset{\setminus}{\setminus} Y, \Omega)$ and the corresponding homotopy and $\mathcal{D}$-derived categories by $K(K \overset{\setminus}{\setminus} Y, \Omega)$, $D(K \overset{\setminus}{\setminus} Y, \Omega)$ (a morphism of weakly equivariant $\Omega$-complexes is called a $\mathcal{D}$-quasi-isomorphism if it is a $\mathcal{D}$-quasi-isomorphism of plain $\Omega$-complexes).

**7.6.8.** *Remarks.* (i) The forgetful functor $C(\mathcal{B}, \Omega) \to C(K \overset{\setminus}{\setminus} Y, \Omega)$ admits left and right adjoint functors $c^l, c^r : C(K \overset{\setminus}{\setminus} Y, \Omega) \to C(\mathcal{B}, \Omega)$, $c^l(F_Y) = U(\mathfrak{k}_\Omega) \underset{U(\mathfrak{k})}{\otimes} F_Y$, $c^r(F_Y) = \mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{k}_\Omega), F_Y)$. These functors preserve quasi-isomorphisms, so they define adjoint functors between the derived categories.

(ii) The forgetful functor $C(K \overset{\setminus}{\setminus} Y, \Omega) \to C(Y, \Omega)$ admits a right adjoint functor $\mathrm{Ind} : C(Y, \Omega) \to C(K \overset{\setminus}{\setminus} Y, \Omega)$, $\mathrm{Ind}(T_Y)^\cdot = p_* m^*(T_Y^\cdot)$ where $m, p : K \times Y \rightrightarrows Y$ are the action and projection maps. These functors preserve quasi-isomorphisms so they yield the adjoint functors between the derived categories. The composition $c^r \,\mathrm{Ind}$ is the push-forward functor for the projection $Y \to \mathcal{B}$.

(iii) Remark 7.6.6 (ii) remains valid for weakly equivariant $\Omega$-complexes.

(iv) Let $f : Z \to Y$ be a morphism of smooth varieties equipped with $K$-actions. The construction of the direct image functor from 7.3.6 passes to the weakly equivariant setting without changes, so we have the functor $f_* = Rf_\cdot : D(K \overset{\setminus}{\setminus} Z, \Omega) \to D(K \overset{\setminus}{\setminus} Y, \Omega)$. The functors $f_*$ commute with the functors from (i), (ii) above. The same holds for the pull-back functors $f^{\cdot}_{\Omega}$ from 7.2.8, 7.3.6.

(v) Here is a weakly equivariant version of 7.6.1. Assume that $Y$ from 7.6.1 carries in addition an action of an affine algebraic group $G'$ that commutes with the $G$-action (we will write it as a right action). Consider the category $C(K \setminus Y \,_{/}{}^{/}\, G', \Omega) = C(\mathcal{B}\,_{/}{}^{/}\, G', \Omega)$ of $\Omega$-complexes on $Y$ equipped with commuting $K_{\Omega}$- and $G$-actions. Then the corresponding derived category $D(\mathcal{B}\,_{/}{}^{/}\, G', \Omega)$ is an $\mathcal{H}$-Module. The $\mathcal{H}$-action is defined in the same way as in 7.6.1. Remark 7.6.6 remains valid.

**7.6.9.** Let us describe the $\mathcal{D}$-module counterpart of the above equivariant categories (see [BL] for details). For a $\mathcal{D}$-module $M$ on $Y$ a *weak $K$-action* on $M$ is a $K$-action on $M$ as on an $\mathcal{O}_Y$-module such that for any $k \in K$ the translation $k^* M \xrightarrow{\sim} M$ is a morphism of $\mathcal{D}$-mosules. A $\mathcal{D}$-module equipped with a weak $K$-action is called a *weakly $K$-equivariant* $\mathcal{D}$-module; the category of those is denoted by $\mathcal{M}(K \,{}^{\setminus}{}_{\setminus}\, Y)$ (as usual we write $\mathcal{M}^{\ell}$ or $\mathcal{M}^{r}$ to specify left and right $\mathcal{D}$-modules). The notations $C(K \,{}^{\setminus}{}_{\setminus}\, Y, \mathcal{D})$, $K(K \,{}^{\setminus}{}_{\setminus}\, Y, \mathcal{D})$, $D(K \,{}^{\setminus}{}_{\setminus}\, Y, \mathcal{D}) = D(K \,{}^{\setminus}{}_{\setminus}\, Y)$ are clear (cf. 7.2).

The functors $\mathcal{D}$ and $\Omega$ from 7.2.2 send weakly equivariant complexes to weakly equivariant ones, thus we have the adjoint DG functors

$$\mathcal{D}:\, C(K \,{}^{\setminus}{}_{\setminus}\, Y, \Omega) \to C(K \,{}^{\setminus}{}_{\setminus}\, Y, \mathcal{D}),\; \Omega:\, C(K \,{}^{\setminus}{}_{\setminus}\, Y, \mathcal{D}) \to C(K \,{}^{\setminus}{}_{\setminus}\, Y, \Omega) \tag{320}$$

and the mutually inverse equivalences of triangulated categories

$$D(K \,{}^{\setminus}{}_{\setminus}\, Y, \mathcal{D}) \Longleftrightarrow D(K \,{}^{\setminus}{}_{\setminus}\, Y, \Omega)\,. \tag{321}$$

As usual we denote these categories thus identified by $D(K \,{}^{\setminus}{}_{\setminus}\, Y)$.

**7.6.10.** *Remark.* For a weakly $K$-equivariant $\mathcal{D}$-module $M$ the $\mathfrak{k}$-action on $Y$ lifts to the $\mathcal{O}$-module $M$ in two ways: either as the infinitesimal action defined by the $K$-action on $M$ or via the $\mathfrak{k}$-action on $Y$ $\sigma : \mathfrak{k} \to \Theta_Y$ and the $\mathcal{D}$-module structure on $M$. Denote these actions by $\xi, m \mapsto \mathrm{Lie}_{\xi}\, m$, $\sigma_{\xi} m$ respectively. Set $\xi^{\natural} m ::= \mathrm{Lie}_{\xi}\, m - \sigma_{\xi} m$. Then $\xi^{\natural} \in \mathrm{End}_{\mathcal{D}}\, M$ and

$\natural : \mathfrak{k} \to \operatorname{End}_{\mathcal{D}} M$ is a $\mathfrak{k}$-action on $M$. Note that $\natural$ is trivial if and only if $M$ is a $K$-equivariant $\mathcal{D}$-module, i.e., $M \in \mathcal{M}(\mathcal{B})$.

**7.6.11.** A *$K$-equivariant $\mathcal{D}$-complex on $Y$* is a complex $N$ of weakly $K$-equivariant $\mathcal{D}$-modules together with morphisms $\mathfrak{k}\otimes N^{\cdot} \to N^{\cdot-1}$, $\xi\otimes n \mapsto i_\xi n$, such that for any $\xi \in \mathfrak{k}$ our has $i_\xi^2 = 0$, $di_\xi + i_\xi d = \xi^\natural$. By abuse of notation we denote the DG category of such complexes by $C(\mathcal{B}, \mathcal{D})$. Note that any $K$-equivariant $\mathcal{D}$-module is a $K$-equivariant $\mathcal{D}$-complex in the obvious way, and for any $K$-equivariant $\mathcal{D}$-complex its cohomology sheaves are $K$-equivariant $\mathcal{D}$-modules. So we have the cohomology functor $H : C(\mathcal{B}, \mathcal{D}) \to \mathcal{M}(\mathcal{B})$. Localizing the homotopy category of $C(\mathcal{B}, \mathcal{D})$ by $H$-quasi-isomorphisms we get a triangulated category $D(\mathcal{B}, \mathcal{D})$. It is easy to see that it is a $t$-category with core $\mathcal{M}(\mathcal{B})$.

For any $F \in C(\mathcal{B}, \Omega)$ the $\mathcal{D}$-complex $\mathcal{D}F$ equipped with operators $i_\xi^{\mathcal{D}F} = i_\xi^F \otimes \mathrm{id}_{\mathcal{D}_Y}$ is $K$-equivariant. For any $N \in C(\mathcal{B}, \mathcal{D})$ the $\Omega$-complex $\Omega N$ equipped with the operators $i_\xi^{\Omega N}$ which act on $N^i \otimes \Lambda^{-j}\Theta_Y$ as $n \otimes \tau \mapsto i_\xi n \otimes \tau + (-1)^i n \otimes \sigma(\xi) \wedge \tau$ is a $K_\Omega$-equivariant $\Omega$-complex. Thus we have the adjoint functors $\mathcal{D}$, $\Omega$

$$C(\mathcal{B}, \Omega) \rightleftarrows C(\mathcal{B}, \mathcal{D}) \tag{322}$$

and the mutually inverse equivalences of triangulated categories

$$D(\mathcal{B}, \Omega) \rightleftarrows D(\mathcal{B}, \mathcal{D})\,. \tag{323}$$

The latter equivalence identifies the above $t$-structure on $D(\mathcal{B}, \mathcal{D})$ with that on $D(\mathcal{B}, \Omega)$ defined in 7.3.2. This provides another proof of 7.3.4 in the particular case when our stack is a quotient of a smooth variety by a group action.

### 7.7. Harish-Chandra modules and their derived category.

**7.7.1.** Let $G$ be an affine algebraic group, $K \subset G$ an algebraic subgroup, so we have the Harish-Chandra pair $(\mathfrak{g}, K)$. Consider the category $\mathcal{M}(K \setminus G {}_{/}{}^{/} G) = \mathcal{M}((K \setminus G) {}_{/}{}^{/} G)$ of $\mathcal{D}$-modules on $G$ equipped with commuting $K$- and weak $G$-actions (where $K$ and $G$ act on $G$ by left and right translations respectively). For $M \in \mathcal{M}(K \setminus G {}_{/}{}^{/} G)$ set $\gamma(M) = \gamma^r(M) := \Gamma(G, M_G)^G$; here we consider $M_G$ as a right $\mathcal{D}$-module on $G$. This is a $(\mathfrak{g}, K)$-module: $\mathfrak{g}$ acts on $\gamma(M)$ by vector fields invariant by right $G$-translations (according to $\mathcal{D}$-module structure on $M$), and $K$ acts by left $K$-translations.

**7.7.2.** *Lemma.* The functor $\gamma : \mathcal{M}(K \setminus G {}_{/}{}^{/} G) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is an equivalence of categories.

*Proof.* Left to the reader (or see [Kas]). □

**7.7.3.** *Remarks.* (i) Set $\gamma^l(M) := \Gamma(G, M_G^l)^G$ where $M_G^l$ is the left $\mathcal{D}$-module realization of $M$. This is a $(\mathfrak{g}, K)$-module by the same reason as above; one has the obvious identification $\gamma^l(M) = \gamma^r(M) \otimes \det \mathfrak{g}$.

(ii) There is a canonical isomorphism of vector spaces $\gamma^l(M) \xrightarrow{\sim} M^l_{G,1} = M^l_{K\setminus G,1}$ which assigns to a $G$-invariant section its value at $1 \in G$. The $(\mathfrak{g}, K)$-module structure on $M^l_{K\setminus G,1}$ may be described as follows. The $K$-action comes from the (weak) action of right $K$-translations on $K \setminus G$ (note that $K$ is the stabilizer of $1 \in K \setminus G$), and the $\mathfrak{g}$-action comes from $\natural$-action of $\mathfrak{g}$ that corresponds to the weak $G$-action (see 7.6.10).

(iii) Let $P$ be a $K$-module, and $\mathcal{P}$ the corresponding $G$-equivariant vector bundle on $K \setminus G$ with fiber $\mathcal{P}_1 = P$. We have $\mathcal{D}\mathcal{P} = \mathcal{P} \otimes \mathcal{D}_{K\setminus G} \in \mathcal{M}((K \setminus G) {}_{/}{}^{/} G)$, and $\gamma(\mathcal{D}\mathcal{P}) = U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} (P \otimes \det \mathfrak{k}^*)$.

**7.7.4.** The above lemma provides, as was promised in 7.1.1(c), a canonical $\mathcal{H}$-Action on the derived category $D(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$-modules. Indeed, by 7.6.8(v) (and 7.6.9) we know that $D(K \setminus G {}_{/}{}^{/} G)$ is an $\mathcal{H}$-Module. And 7.7.2 identifies $D(\mathfrak{g}, K)$ with this category.

We give a different description of this Action in 7.8.2 below. Its equivalence with the present definition is established in 7.8.9, 7.8.10(i).

The rest of the Section (7.7.5-7.7.11) is a digression about $\mathcal{D}$-$\Omega$ equivalences in the Harish-Chandra setting; as a bonus we get in 7.7.12 a simple proof of Bernstein-Lunts theorem [BL]1.3. The reader may skip it and go directly to 7.8.

**7.7.5.** Here is a version of 7.7.2 for $\Omega$-complexes.

Let $\Omega_{\mathfrak{g}}$ be the Chevalley DG-algebra of cochains of $\mathfrak{g}$, so $\Omega_{\mathfrak{g}}^{\cdot} = \Lambda^{\cdot}\mathfrak{g}^*$. It carries a canonical "adjoint" action of $K_\Omega$ (see 7.6.3 for notations). Namely, $K$ acts on $\Omega_{\mathfrak{g}}^{\cdot}$ in coadjoint way, and $\xi \in \mathfrak{k} = \mathfrak{k}_\Omega^{-1}$ acts as the derivation $i_\xi$ of $\Omega_{\mathfrak{g}}^{\cdot}$ which sends $\nu \in \mathfrak{g}^* = \Omega_{\mathfrak{g}}^1$ to $\langle \nu, \xi \rangle$.

A $\Omega_{(\mathfrak{g},K)}$-*complex* is a DG $(\Omega_{\mathfrak{g}}, K_\Omega)$-module, i.e., it is a complex equipped with $\Omega_{\mathfrak{g}}$- and $K_\Omega$-actions which are compatible with respect to the $K_\Omega$-action on $\Omega_{\mathfrak{g}}$. For an $\Omega_{(\mathfrak{g},K)}$-complex $T$ we denote the action of $\nu \in \mathfrak{g}^* = \Omega_{\mathfrak{g}}^1$, $\xi \in \mathfrak{k} = \mathfrak{k}_\Omega^{-1}$ on $T^{\cdot}$ by $a_\nu$, $i_\xi$. Denote the DG category of $\Omega_{(\mathfrak{g},K)}$-complexes by $C\Omega_{(\mathfrak{g},K)}$ and its homotopy category by $K\Omega_{(\mathfrak{g},K)}$.

For $F \in C(K \setminus G\,_{/}{}^{/}\,G, \Omega)$ set $\gamma(F) := \Gamma(G, F_G)^G$. This is an $\Omega_{(\mathfrak{g},K)}$-complex. Indeed, $\Omega_{\mathfrak{g}}$ acts on it via the usual identification with DG algebra of differential forms on $G$ that are invariant with respect to right $G$-translations, and $K_\Omega$ acts on $\gamma(F)$ since it acts on $F_G$ (see 7.6.4, 7.6.5).

**7.7.6.** *Lemma.* The functor $\gamma : C(K \setminus G\,_{/}{}^{/}\,G, \Omega) \longrightarrow C\Omega_{(\mathfrak{g},K)}$ is an equivalences of DG categories.

*Proof.* Left to the reader. $\square$

**7.7.7.** We identified $(\mathfrak{g}, K)$- and $\Omega_{(\mathfrak{g},K)}$-complexes with weakly $G$-equivariant complexes on $K \setminus G$. Let us write down the standard functors $\mathcal{D}$ and $\Omega$ in Harish-Chandra's setting. It is convenient to introduce a DG Harish-Chandra pair $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$ (the structure embedding $LieK \hookrightarrow \mathfrak{k}_\Omega \times \mathfrak{g}$ is the diagonal map).

Let $DR_{\mathfrak{g}}$ be the Chevalley complex of cochains of $\mathfrak{g}$ with coefficients in $U\mathfrak{g}$ (considered as a left $U\mathfrak{g}$-module), so $DR_{\mathfrak{g}}^i = \Lambda^i\mathfrak{g}^* \otimes U\mathfrak{g}$. Now $DR_{\mathfrak{g}}$ is an $\Omega_{\mathfrak{g}}$-complex, and an $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-complex; those actions are compatible (here

$(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$ acts on $\Omega_\mathfrak{g}$ via the projection $(\mathfrak{k}_\Omega \times \mathfrak{g}, K) \to (\mathfrak{k}_\Omega, K)$ , see 7.7.5). Namely, for $\nu \in \Omega_\mathfrak{g}$, $\epsilon = (\epsilon_l, \epsilon_r) \in \mathfrak{k} \times \mathfrak{g} = \mathfrak{k}^0_\Omega \times \mathfrak{g}$, $\xi \in \mathfrak{k} = \mathfrak{k}^{-1}_\Omega$, $k \in K$, and $a = \alpha \otimes \upsilon \in DR_\mathfrak{g}$ one has $\nu a = \nu\alpha \otimes \upsilon$, $\epsilon a = \mathrm{Ad}_{\epsilon_l}(\alpha) \otimes \upsilon + \alpha \otimes (\epsilon_l \upsilon - \upsilon \epsilon_r)$, $\xi a = i_\xi(\alpha) \otimes \upsilon$, $ka = \mathrm{Ad}_k(\alpha) \otimes \mathrm{Ad}_k(\upsilon)$.

For a complex of $(\mathfrak{g}, K)$-modules ($(\mathfrak{g}, K)$-complex for short) $V$, set $\Omega V := \mathrm{Hom}_\mathfrak{g}(DR_\mathfrak{g}, V)$; this is an $\Omega_{(\mathfrak{g},K)}$-complex in the obvious way. For an $\Omega_{(\mathfrak{g},K)}$-complex $T$ set $\mathcal{D}T = \mathcal{D}_{(\mathfrak{g},K)}T := T \underset{\Omega_\mathfrak{g}, \mathfrak{k}_\Omega}{\otimes} DR_\mathfrak{g} = (T \underset{\Omega_\mathfrak{g}}{\otimes} DR_\mathfrak{g})_{\mathfrak{k}_\Omega}$; this a $(\mathfrak{g}, K)$-complex. Thus we have the adjoint DG functors

$$\mathcal{D} = \mathcal{D}_{(\mathfrak{g},K)} : C\Omega_{(\mathfrak{g},K)} \longrightarrow C(\mathfrak{g}, K)\,, \qquad \Omega : C(\mathfrak{g}, K) \longrightarrow C\Omega_{(\mathfrak{g},K)}\,. \tag{324}$$

*Remark.* For $T$ as above let $\overline{T}^{\cdot} \subset T^{\cdot}$ be the kernel of all operators $i_\xi$, $\xi \in \mathfrak{k}$. This is a $K$- and $\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k})^*$-submodule of $T^{\cdot}$ (here $\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k})^* \subset \Lambda^{\cdot}\mathfrak{g}^* = \Omega^{\cdot}_\mathfrak{g}$), and the obvious morphisms

$$\Omega^{\cdot}_\mathfrak{g} \underset{\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k})^*}{\otimes} \overline{T}^{\cdot} \longrightarrow T^{\cdot}\,, \qquad \overline{T}^{\cdot} \underset{U\mathfrak{k}}{\otimes} U\mathfrak{g} \longrightarrow \mathcal{D}T^{\cdot} \tag{325}$$

are isomorphisms.

**7.7.8.** Let us return to the geometric situation. One has the obvious identification $\Gamma(G, DR_G)^G = DR_\mathfrak{g}$ (see 7.2.2 for notation; $G$ acts on itself by right translations). For $M \in C((K \setminus G)\,/^{\prime}\, G, \mathcal{D})$ there is a canonical isomorphism $\gamma(\Omega M) \mathrel{\tilde{\to}} \Omega(\gamma M)$ of $\Omega_{(\mathfrak{g},K)}$-complexes defined as composition $\Gamma(G, Hom_{\mathcal{D}_G}(DR_G, M_G))^G = \mathrm{Hom}_{\mathcal{D}_G}(DR_G, M_G)^G = \mathrm{Hom}_{U\mathfrak{g}}(DR_\mathfrak{g}, \gamma M)$. For $F \in C(K \setminus G\,/^{\prime}\, G, \Omega)$ there is a similar canonical isomorphism $\gamma\mathcal{D}F \mathrel{\tilde{\to}} \mathcal{D}\gamma F$ whose definition is left to the reader.

**7.7.9.** For an $\Omega_{(\mathfrak{g},K)}$-complex $T$ set $H^{\cdot}_\mathfrak{g}T = H^{\cdot}\mathcal{D}T \in \mathcal{M}(\mathfrak{g}, K)$. Then $H^{\cdot}_\mathfrak{g} : K\Omega_{(\mathfrak{g},K)} \to \mathcal{M}(\mathfrak{g}, K)$ is a cohomological functor. Define a $\mathfrak{g}$-*quasi-isomorphism* as a morphism in $K\Omega_{(\mathfrak{g},K)}$ that induces isomorphism between $H^{\cdot}_\mathfrak{g}$'s. The $\mathfrak{g}$-quasi-isomorphisms form a localizing family; define $D\Omega_{(\mathfrak{g},K)}$ as the corresponding localization of $K\Omega_{(\mathfrak{g},K)}$. The functors $\mathcal{D}$, $\Omega$ yield mutually inverse equivalences of derived categories

$$D\Omega_{(\mathfrak{g},K)} \rightleftarrows D(\mathfrak{g},K) \tag{326}$$

where $D(\mathfrak{g},K) := D\mathcal{M}(\mathfrak{g},K)$. The equivalences $\gamma$ yield equivalences of derived categories

$$D(K\setminus G\,/^{\prime}\,G,\Omega) \overset{\sim}{\to} D\Omega_{(\mathfrak{g},K)}\,,\quad D((K\setminus G)\,/^{\prime}\,G,\mathcal{D}) \overset{\sim}{\to} D(\mathfrak{g},K)\,. \tag{327}$$

**7.7.10.** *Remarks.* (i) Any $\mathfrak{g}$-quasi-isomorphism is a quasi-isomorphism; the converse might be not true.

(ii) Any $\Omega_{(\mathfrak{g},K)}$-complex $T$ may be considered as an $\Omega_{\mathfrak{g}} = \Omega_{(\mathfrak{g},1)}$-complex (forget the $K_\Omega$-action), so we have the corresponding complex of $\mathfrak{g}$-modules $\mathcal{D}_{\mathfrak{g}}T := T\otimes_{\Omega_{\mathfrak{g}}} DR_{\mathfrak{g}}$. The obvious projection $\mathcal{D}_{\mathfrak{g}}T \to \mathcal{D}_{(\mathfrak{g},K)}T$ is a quasi-isomorphism. This implies that a morphism of $\Omega_{(\mathfrak{g},K)}$-complexes is a $\mathfrak{g}$-quasi-isomorphism if and only if it is a $\mathfrak{g}$-quasi-isomorphism of $\Omega_{\mathfrak{g}}$-complexes.

**7.7.11.** The format of 7.7.7, 7.7.9 admits the folowing version. Recall that $DR_{\mathfrak{g}}$ is a $(\mathfrak{k}_\Omega\times\mathfrak{g},K)$-complex. Thus the above $\mathcal{D}_{\mathfrak{g}}T$ is a $(\mathfrak{k}_\Omega\times\mathfrak{g},K)$-complex, and for a $(\mathfrak{k}_\Omega\times\mathfrak{g},K)$-complex $V$ the complex $\Omega V := Hom_{\mathfrak{g}}(DR_{\mathfrak{g}},V)$ is a $\Omega_{(\mathfrak{g},K)}$-complex. The functors

$$\mathcal{D}_{\mathfrak{g}} : C\Omega_{(\mathfrak{g},K)} \longrightarrow C(\mathfrak{k}_\Omega\times\mathfrak{g},K),\quad \Omega : C(\mathfrak{k}_\Omega\times\mathfrak{g},K)\longrightarrow C\Omega_{(\mathfrak{g},K)} \tag{328}$$

are adjoint, as well as the corresponding functors between the homotopy categories. Passing to derived categories they become (use 7.7.10(ii)) mutually inverse equivalences

$$D\Omega_{(\mathfrak{g},K)} \rightleftarrows D(\mathfrak{k}_\Omega\times\mathfrak{g},K). \tag{329}$$

The projection $(\mathfrak{k}_\Omega\times\mathfrak{g},K)\to(\mathfrak{g},K)$ yields a fully faithful embedding $C(\mathfrak{g},K)\longrightarrow C(\mathfrak{k}_\Omega\times\mathfrak{g},K)$ hence the exact functor

$$D(\mathfrak{g},K)\longrightarrow D(\mathfrak{k}_\Omega\times\mathfrak{g},K). \tag{330}$$

The following theorem is due to Bernstein and Lunts [BL] 1.3[*)]:

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[*)]The authors of [BL] consider only bounded derived categories.

**7.7.12.** *Theorem.* The functor (330) is equivalence of categories.

*Proof.* The functor $\Omega$ from (328) restricted to $C(\mathfrak{g}, K)$ coincides with $\Omega$ from (324). Now 7.7.12 follows from (326) and (329). The inverse functor $D(\mathfrak{k}_\Omega \times \mathfrak{g}, K) \longrightarrow D(\mathfrak{g}, K)$ sends $V$ to $\mathcal{D}_{(\mathfrak{g},K)}\Omega V$. □

**7.8. The Hecke Action and localization functor.**

**7.8.1.** We are going to describe a canonical Hecke Action on the derived category of Harish-Chandra modules. We consider a twisted situation, i.e., representations of a central extension of $\mathfrak{g}$. Here is the list of characters.

Let $G'$ be a central extension of $G$ by $\mathbb{G}_m$ equipped with a splitting $K \to G'$. Therefore the preimage $K' \subset G'$ of $K$ is identified with $K \times \mathbb{G}_m$. Set $\mathfrak{g}' := \operatorname{Lie} G'$, $\mathfrak{k}' := \operatorname{Lie} K' = \mathfrak{k} \times \mathbb{C}$. We have a Harish-Chandra pair $(\mathfrak{g}', K')$ and the companion DG pair $(\mathfrak{k}_\Omega \times \mathfrak{g}', K')$ (here the first component of the structure embedding $\mathfrak{k}' \hookrightarrow \mathfrak{k}_\Omega \times \mathfrak{g}'$ is the projection $\mathfrak{k}' \to \mathfrak{k}$).

Let $\mathcal{M}(\mathfrak{g}, K)'$ be the category of $(\mathfrak{g}', K')$-modules on which $\mathbb{G}_m \subset K'$ acts by the standard character; we call its objects $(\mathfrak{g}, K)'$-modules or, simply, Harish-Chandra modules. This is an abelian category. Similarly, let $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$ be the category of those $(\mathfrak{k}_\Omega \times \mathfrak{g}', K')$-complexes on which $\mathbb{G}_m$ acts by the standard character; its objects are called $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-complexes or, simply, *Harish-Chandra complexes*. This is a DG category which carries an obvious cohomology functor with values in $\mathcal{M}(\mathfrak{g}, K)'$. Denote the corresponding derived category by $D(\mathfrak{g}, K)'$; this is a t-category with core $\mathcal{M}(\mathfrak{g}, K)'$.

*Remark.* By a twisted version of the Bernstein-Lunts theorem $D(\mathfrak{g}, K)'$ is equivalent to the derived category of $\mathcal{M}(\mathfrak{g}, K)'$ [*)]. We will not use this fact in the sequel since the Hecke Action is naturally defined in terms of $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-complexes.

[*)] The twisted Bernstein-Lunts follows from the straight one (see 7.7.12) applied to the Harish-Chandra pair $(\mathfrak{g}', K')$.

**7.8.2.** Now let us define a canonical $\mathcal{H}$-Action on $D(\mathfrak{g},K)'$. First we define an Action of the pre Hecke monoidal DG category $\mathcal{H}^c := C(K\setminus G/K,\Omega)$ on $C(\mathfrak{k}_\Omega\times\mathfrak{g},K)'$; the Hecke Action comes after passing to derived categories.

Denote by $\mathcal{L}_G$ the line bundle over $G$ that corresponds to the $\mathbb{G}_m$-torsor $G'\to G$. The left and right translation actions of $G$ on itself lift canonically to $G'$-actions on $\mathcal{L}_G$. So a section of $\mathcal{L}_G$ is the same as a function $\phi$ on $G'$ such that for $c\in\mathbb{G}_m$, $g'\in G'$ one has $\phi(cg')=c^{-1}\phi(g')$. Therefore the right translation action of $\mathbb{G}_m\subset G'$ on sections of $\mathcal{L}_G$ is multiplication by the character inverse to the standard one.

Take a Harish-Chandra complex $V\in C(\mathfrak{k}_\Omega\times\mathfrak{g},K)'$. Set $\mathcal{V}_G:=\mathcal{L}_G\otimes V$. Then $\mathcal{V}_G$ is a complex of left $\mathcal{D}$-modules on $G$. Indeed, the tensor product of the infinitesimal right translation action of $\mathfrak{g}'$ on $\mathcal{L}_G$ and the $\mathfrak{g}'$-action on $V$ is a $\mathfrak{g}$-action on $\mathcal{V}_G$. The left $\mathcal{D}$-module structure on $\mathcal{V}_G$ is such that the left invariant vector fields act on $\mathcal{V}_G$ via the above $\mathfrak{g}$-action. The $\mathcal{D}$-complex $\mathcal{V}_G$ is weakly equivariant with respect to left $G'$-translations: they act as tensor product of the corresponding action on $\mathcal{L}_G$ and the trivial action on $\mathcal{V}$. Therefore, by 7.6.10, it carries a canonical $\mathfrak{g}'$-action $\natural$.

*Remark.* For $\theta\in\mathfrak{g}'$ consider a function $\theta^\natural : G\to\mathfrak{g}'$, $\theta^\natural(g):=Ad_g(\theta)$. Then for $v\in V$, $l\in\mathcal{L}_G$ one has $\theta^\natural(l\otimes v)=l\otimes\theta^\natural(v)$.

Take $F\in\mathcal{H}^c$. Then $F_G\otimes\mathcal{V}_G$ is an $\Omega$-complex on $G$ (see 7.2.3(ii)). It is $K_\Omega$-equivariant with respect to the right $K$-translations. Namely, $K$ acts as tensor product of the corresponding actions on $F$, $\mathcal{L}_G$, and the structure action on $V$; the operators $i_\xi$ act as the sum of the corresponding operators for the right translation action on $F$ and the structure ones for $V$. Denote by $(F\otimes\mathcal{V})_{G/K}$ the corresponding $\Omega$-complex on $G/K$. The action of $\mathfrak{g}'$ on $F_G\otimes\mathcal{V}_G$ that comes from the action $\natural$ on $\mathcal{V}_G$ commutes with this $K_\Omega$-action, so it defines $\mathfrak{g}'$-action on $(F\otimes\mathcal{V})_{G/K}$. We also denote it as $\natural$.

*Remark.* If $V$ is a complex of $(\mathfrak{g},K)'$-modules then $\mathcal{V}_G$ is a complex of left $\mathcal{D}_G$-modules strongly equivariant with respect to right $K$-translations.

Let $\mathcal{V}_{G/K}$ be the corresponding complex of left $\mathcal{D}$-modules on $G/K$. One has $(F \otimes \mathcal{V})_{G/K} = F_{G/K} \otimes \mathcal{V}_{G/K}$.

Set $F \widetilde{\circledast} V := \Gamma(G, F_G \otimes \mathcal{V}_G)$ and

$$F \overset{c}{\circledast} V = \Gamma(G/K, (F \otimes \mathcal{V})_{G/K}) = (F \widetilde{\circledast} V)^{K_\Omega}. \tag{331}$$

These are $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-complexes. Indeed, $\mathfrak{g}'$ acts according to $\natural$ action, $K$ acts by tensor product of the left translation actions for $F$ and $\mathcal{V}$, and the operators $i_\xi$ are the corresponding operators for $F$. We leave it to the reader to check the Harish-Chandra compatibilities.

Now $\overset{c}{\circledast}$ defines an $\mathcal{H}^c$-Module structure on $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$. Indeed, the associativity constraint $(F_1 \overset{c}{\circledast} F_2) \overset{c}{\circledast} V = F_1 \overset{c}{\circledast} (F_2 \overset{c}{\circledast} V)$ follows from the obvious identification

$$\Gamma(G, (F_1 \overset{c}{\circledast} F_2)^{\cdot} \otimes \mathcal{L}_G) = [\Gamma(G, F_1^{\cdot} \otimes \mathcal{L}_G) \otimes \Gamma(G, F_2^{\cdot} \otimes \mathcal{L}_G)]^{K_\Omega^{\cdot}}$$

where $K_\Omega^{\cdot}$ acts by tensor product of the right and left translation actions (see 7.6.5). We define the Hecke Action $\circledast : \mathcal{H} \times D(\mathfrak{g}, K)' \to D(\mathfrak{g}, K)'$ as the right derived functor of $\overset{c}{\circledast}$. If $F$ is loose then $F \circledast V = F \overset{c}{\circledast} V$ so the associativity constraint for $\circledast$ follows from that of $\overset{c}{\circledast}$.

*Remark.* As folows from the previous Remark, for $M \in \mathcal{M}(K \setminus G/K) \subset \mathcal{H}$, $V \in \mathcal{M}(\mathfrak{g}, K)'$ one has

$$H^{\cdot} M \circledast V = H_{DR}^{\cdot}(G/K, M \otimes \mathcal{V}_{G/K}). \tag{332}$$

**7.8.3.** *Remark.* Assume that our twist is trivial, so $G' = G \times \mathbb{G}_m$. One has obvious equivalences $\mathcal{M}(\mathfrak{g}, K)' = \mathcal{M}(\mathfrak{g}, K)$ and $D(\mathfrak{g}, K) = D(\mathfrak{g}, K)'$ (see 7.7.11). So we defined a Hecke Action on $D(\mathfrak{g}, K)$. We will see in 7.8.9 that this Action indeed coincides with the one from 7.7.4.

Let us return to the general situation. Let $U'$ be the twisted enveloping algebra of $\mathfrak{g}$; denote by $\mathfrak{Z}$ its subalgebra of $\operatorname{Ad} G$-invariant elements. The commutative algebra $\mathfrak{Z}$ acts on any Harish-Chandra complex in the obvious manner, so $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$, hence $D(\mathfrak{g}, K)$, is a $\mathfrak{Z}$-category.

**7.8.4.** *Lemma.* The Hecke Actions on $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$, $D(\mathfrak{g}, K)'$ are $\mathfrak{Z}$-linear.

*Proof.* Use the first Remark in 7.8.2. □

**7.8.5.** *Example.* (to be used in 5). Let $Vac' := U'/U' \cdot \mathfrak{k}$ be the twisted vacuum module. Let us compute $F \circledast Vac'$ explicitely. We use notation of 7.8.2. So, according to the second Remark in 7.8.2, we have the left $\mathcal{D}$-module $\mathcal{V}_{G/K}$ on $G/K$, weakly equivariant with respect to left $G$-translations, such that $\mathcal{V}_G = \mathcal{L}_G \otimes Vac'$. The embedding $\mathbb{C} \subset Vac'$ yields an embedding $\mathcal{L}_{G/K} \subset \mathcal{V}_{G/K}$. It is easy to see that the corresponding morphism of left $\mathcal{D}_{G/K}$-modules $\mathcal{D}_{G/K} \underset{\mathcal{O}_{G/K}}{\otimes} \mathcal{L}_{G/K} \to \mathcal{V}_{G/K}$ is an isomorphism of weakly $G$-equivariant $\mathcal{D}$-modules.

*Remark.* The $\mathfrak{g}'$-action on $\mathcal{D}_{G/K} \otimes \mathcal{L}_{G/K}$ that corresponds to $\natural$ is given by formula $\alpha'(\psi \otimes l) = \psi \otimes \alpha'(l) - \psi \cdot \alpha \otimes l$ where $\alpha' \in \mathfrak{g}'$, $\alpha$ is the corresponding left translation vector field on $G/K$, and $\alpha'(l)$ is the infinitesimal left translation of $l \in \mathcal{L}_{G/K}$.

So for $F \in \mathcal{H}^c$ one has $(F \otimes \mathcal{V})_{G/K} = F_{G/K} \otimes \mathcal{D}_{G/K} \otimes \mathcal{L}_{G/K} = \mathcal{D}(F_{G/K}) \underset{\mathcal{O}_{G/K}}{\otimes} \mathcal{L}_{G/K}$. Therefore

$$F \overset{c}{\circledast} Vac' = \Gamma(G/K, \mathcal{D}(F_{G/K}) \otimes \mathcal{L}_{G/K}). \tag{333}$$

Here the $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-action on $\Gamma(G/K, \mathcal{D}(F_{G/K}) \otimes \mathcal{L}_{G/K})$ is defined as follows. The $\mathfrak{g}'$-action comes from the $\mathfrak{g}'$-action on $\mathcal{D}(F_{G/K}) \otimes \mathcal{L}_{G/K}$ described in the Remark above, the $K$-action is the action by left translations, and the operators $i_\xi$ come from the corresponding operators on $F_{G/K}$.

Passing to the derived functors (which amounts to considering loose $F$ in the above formula) we get

$$F \circledast Vac' = R\Gamma(G/K, \mathcal{D}(F_{G/K}) \otimes \mathcal{L}_{G/K}). \tag{334}$$

In particular, for $M \in \mathcal{M}(K \setminus G/K)$ one has

$$M \circledast Vac' = R\Gamma(G/K, M_{G/K} \otimes \mathcal{L}_{G/K}). \tag{335}$$

Here the $\mathfrak{g}'$-action on the r.h.s. comes from the $\mathfrak{g}'$-action on $M_{G/K} \otimes \mathcal{L}_{G/K}$ given by formula $\alpha'(m \otimes l) = m \otimes \alpha'(l) - m\alpha \otimes l$.

**7.8.6.** Let us explain part (d) of the "Hecke pattern" from 7.1.1. Let us first define the localization functor $\Delta$. We use the notation of 7.8.1. Let $Y$ be a smooth variety on which $G$ acts, $\mathcal{L} = \mathcal{L}_Y$ a line bundle on $Y$. Assume that $\mathcal{L}$ carries a $G'$-action which lifts the $G$-action on $Y$ in a way that $\mathbb{G}_m \subset G'$ acts by the character opposite to the standard one. The line bundle $\omega_Y \otimes \mathcal{L}$ carries the similar action.

We define a DG functor

$$\Delta_\Omega = \Delta_{\Omega\mathcal{L}} : C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)' \to C(K \setminus Y, \Omega) \tag{336}$$

as follows. Note that $(\mathfrak{g}', K')$, hence $(\mathfrak{k}_\Omega \times \mathfrak{g}', K')$, acts on $\omega_Y \otimes \mathcal{L}$ (since $G'$ does). For a Harish-Chandra complex $V$ consider the complex of $\mathcal{O}$-modules $\omega_Y \otimes \mathcal{L} \otimes V$. The tensor product of $(\mathfrak{k}_\Omega \times \mathfrak{g}', K')$-actions on $\omega_Y \otimes \mathcal{L}$ and $V$ yields a $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-action on $\omega_Y \otimes \mathcal{L} \otimes V$. Set

$$\Delta_\Omega(V) := Hom_{\mathfrak{g}}(DR_{\mathfrak{g}}, \omega_Y \otimes \mathcal{L} \otimes V)[-\dim K]$$

(see 7.7.7 for notation). In other words $\Delta_\Omega(V)$ is the shifted Chevalley chain complex of $\mathfrak{g}$ with coefficients in $\omega_Y \otimes \mathcal{L} \otimes V$. This is an $\Omega$-complex on $Y$. Since $DR_{\mathfrak{g}}$ and $\omega_Y \otimes \mathcal{L} \otimes V$ are $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-complexes our $\Delta_\Omega(V)$ is $K_\Omega$-equivariant, i.e., $\Delta_\Omega(V) \in C(K \setminus Y, \Omega)$.

Note that $\Delta_\Omega(V)$ carries a canonical increasing finite filtration with successive quotients equal to $\Lambda^i \mathfrak{g} \otimes \omega_Y \otimes \mathcal{L} \otimes V[i - \dim K]$. Therefore $\Delta_\Omega$ sends quasi-isomorphisms to $\mathcal{D}$-quasi-isomorphisms. So it yields a triangulated functor

$$L\Delta = L\Delta_{\mathcal{L}} : D(\mathfrak{g}, K)' \to D(K \setminus Y) \tag{337}$$

The above remark also shows that $L\Delta$ is a right t-exact functor. The corresponding right exact functor between the cores $\Delta_{\mathcal{L}} : \mathcal{M}(\mathfrak{g}, K)' \to$

$\mathcal{M}^\ell(K\setminus Y)$ sends a $(\mathfrak{g},K)'$-module $V$ to a $K$-equivariant left $\mathcal{D}_Y$-module $(\mathcal{D}_Y\otimes\mathcal{L})\underset{U(\mathfrak{g}')}{\otimes} V$. More generally, $H^i_{\mathcal{D}}L\Delta_{\mathcal{L}}(V)=H_{-i}(\mathfrak{g},\mathcal{D}_Y\otimes\mathcal{L}\otimes V)$.

**7.8.7.** *Remarks.* (i) The above construcion used only the action of $(\mathfrak{g}',K')$ on $(Y,\mathcal{L})$ (we do not need the whole $G'$-action).

(ii) One may show that $L\Delta_{\mathcal{L}}$ is a left derived functor of $\Delta_{\mathcal{L}}$ (see Remark in 7.8.1).

(iii) Assume that $(\mathfrak{g}',K')$ is the trivial extension of $(\mathfrak{g},K)$, so $(\mathfrak{g},K)'$-modules are the same as $(\mathfrak{g},K)$-modules, and $\mathcal{L}$ is $\mathcal{O}_Y$ with the obvious action of $(\mathfrak{g}',K')$. Then $\Delta_{\mathcal{L}}(V)=\mathcal{D}_Y\underset{U(\mathfrak{g})}{\otimes}V$, i.e., $\Delta_{\mathcal{L}}$ coincides with the functor $\Delta$ from 1.2.4.

**7.8.8.** *Proposition.* The functor $L\Delta_{\mathcal{L}}:D(\mathfrak{g},K)'\to D(K\backslash Y)$ is a Morphism of $\mathcal{H}$-Modules.

*Proof.* It suffices to show that the functor $\Delta_{\Omega\mathcal{L}}:C(\mathfrak{k}_\Omega\times\mathfrak{g},K)'\to C(K\backslash Y,\Omega)$ is a Morphism of $\mathcal{H}^c$-Modules.

Take $F$, $V$ as in 7.8.2. We have to define a canonical identification of $\Omega$-complexes $\alpha:\Delta_\Omega(F\overset{c}{\circledast}V)\overset{\sim}{\rightarrow}F\overset{c}{\circledast}\Delta_\Omega(V)$ compatible with the associativity constraints. We will establish a canonical isomorphism $\widetilde{\alpha}$ : $\Delta_\Omega(F\widetilde{\circledast}V)\overset{\sim}{\rightarrow}F\widetilde{\circledast}\Delta_\Omega(V)$ compatible with the $K_\Omega$-actions (see 7.6.6, 7.8.2 for notation). One gets $\alpha$ by passing to $K_\Omega$-invariants.

Let $m,p:G\times Y\to Y$ be the action and projection maps, $i:G\times Y\to G\times Y$ the symmetry $i(g,x)=(g,gx)$; one has $pi=m$. The $G'$-action on $\mathcal{L}_Y$ provides an $i$-isomorphism of line bundles $\widetilde{i}:\mathcal{O}_G\boxtimes\mathcal{L}_Y\overset{\sim}{\rightarrow}\mathcal{L}_G\boxtimes\mathcal{L}_Y$.

Below for a $\mathfrak{g}$-complex $P$ we denote by $C(P)$ the Chevalley complex of Lie algebra chains with coefficients in $P$ shifted by $\dim K$. So $C(P)^\cdot=C^\cdot\otimes P^\cdot$ where $C^a:=\Lambda^{\dim K-a}\mathfrak{g}$. Consider the $\Omega$-complexes $F_G\boxtimes\Delta_\Omega(V)=F_G\boxtimes C(\omega_Y\otimes\mathcal{L}_Y\otimes V)$ and $C((F_G\otimes\mathcal{V}_G)\boxtimes(\omega_Y\otimes\mathcal{L}_Y))=C((F_G\otimes(\mathcal{L}_G\otimes V))\boxtimes(\omega_Y\otimes\mathcal{L}_Y))$; here the $\mathfrak{g}$-action on $(F_G\otimes\mathcal{V}_G)\boxtimes(\omega_Y\otimes\mathcal{L}_Y)$ is the tensor product of the $\mathfrak{g}'$-action $\natural$ and the standard $\mathfrak{g}'$-action on $\omega_Y\otimes\mathcal{L}_Y$ (see 7.8.2).

There is a canonical $i$-isomorphism of $\Omega$-complexes

$$\widetilde{\alpha}' : F_G \boxtimes \Delta_\Omega(V) \stackrel{\sim}{\rightarrow} C((F_G \otimes \mathcal{V}_G) \boxtimes (\omega_Y \otimes \mathcal{L}_Y))$$

defined as follows. For $f \in F_G$, $\lambda \in C^{\cdot}$, $l \in \omega_Y \otimes \mathcal{L}_Y$, $v \in V$ one has $\widetilde{\alpha}'(f \otimes \lambda \otimes l \otimes v) = a(\lambda) \otimes f \otimes \widetilde{i}(l) \otimes v$; here $a(\lambda) \in \mathcal{O}_{G\times Y} \otimes C^{\cdot}$ is a function $a(\lambda)(g, y) = Ad_g(\lambda)$. We leave it to the reader to check that $\alpha$ commutes with the differentials (use Remark in 7.8.2).

Now one has the obvious identifications $m_\cdot(F_G \boxtimes \Delta_\Omega(V)) = F\widetilde{\circledast}\Delta_\Omega(V)$ and $p_\cdot C((F_G \otimes \mathcal{V}_G) \boxtimes (\omega_Y \otimes \mathcal{L}_Y)) = \Delta_\Omega(F\widetilde{\circledast}V)$. Thus $\widetilde{\alpha}'$ defines the desired canonical isomorphism $\widetilde{\alpha}$. We leave it to the reader to check its compatibility with the $K_\Omega$-actions and associativity constraints. □

**7.8.9.** Consider the case when $Y = G$ with the left translation $G$-action, and $\mathcal{L} = \mathcal{L}_Y$ is the line bundle dual to $\mathcal{L}_G$ (see 7.8.2) equipped with the obvious $G'$-action by left translations. The right $G'$-translations act on our data. Therefore the $\Omega$-complexes $\Delta_\Omega(V)$ are weakly $G'$-equivariant with respect to the right translation action of $G'$.

Let $C(K \setminus G\,{}_/{}^/\,G, \Omega)' \subset C(K \setminus G\,{}_/{}^/\,G', \Omega)$ be the subcategory of those weakly $G'$-equivariant $\Omega$-complexes $T$ that $\mathbb{G}_m \subset G'$ acts on $T$ by the standard character. Let $D(K \setminus G\,{}_/{}^/\,G)'$ be the corresponding $\mathcal{D}$-derived category. The complexes $\Delta_\Omega(V)$ lie in this subcategory, so we have a triangulated functor $L\Delta : D(\mathfrak{g}, K)' \to D(K \setminus G\,{}_/{}^/\,G)'$. This categories are $\mathcal{H}$-Modules (for the latter one see 7.6.8(v), 7.6.9). By 7.8.8, $L\Delta$ is a Morphism of $\mathcal{H}$-modules. A variant of 7.7.6 and 7.7.11 shows that $L\Delta$ is an equivalence of t-categories.

**7.8.10.** *Remarks.* i) If $G'$ is the trivial extension of $G$ then $D(\mathfrak{g}, K)' = D(\mathfrak{g}, K)$ and $L\Delta$ coincides with the equivalence defined by the functor $\gamma^{-1}$ from 7.7.2. This shows that the Hecke Actions from 7.7.4 and in 7.8.3 do coincide.

(ii) Assume that our extension is arbitrary. Then the pull-back functor $r : D(K \setminus G/K) \to D(K' \setminus G'/K')$ is a Morphism of monoidal categories, and the fully faithful embedding $D(\mathfrak{g},K)' \hookrightarrow D(\mathfrak{g}',K')$ is $r$-Morphism of Hecke Modules. So the twisted picture is essentially equivalent to untwisted one for $(\mathfrak{g}',K')$. However in applications it is convenient to keep the twist (alias level, alias central charge) separately.

**7.8.11.** Let us explain the $\Gamma$ part of the "Hecke pattern" (d) from 7.1.1. This subject is not needed for the main part of this paper, so the reader may skip the rest of the section. We treat a twisted version, so we are in situation 7.8.6. For $T \in C(K \setminus Y, \Omega)$ the $\mathcal{D}$-complex $\mathcal{D}T_Y$ on $Y$ is $K$-equivariant (see 7.6.11). Let us consider $\mathcal{D}T_Y$ as an $\mathcal{O}$-complex equipped with a $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-action. Set $\Gamma_{\mathcal{L}}(T) := \Gamma(Y, \mathcal{D}T_Y \otimes (\omega_Y \otimes \mathcal{L}_Y)^*)$. This is a Harish-Chandra complex (recall that $(\mathfrak{g}',K)$ acts on $\omega_Y \otimes \mathcal{L}_Y$), so we have a DG functor $\Gamma_{\mathcal{L}} : C(K \setminus Y, \Omega) \to C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$. Let

$$R\Gamma_{\mathcal{L}} : D(K \setminus Y) \to D(\mathfrak{g},K)'$$

be its right derived functor. If $T$ is loose then $\Gamma_{\mathcal{L}}(T) = R\Gamma_{\mathcal{L}}(T)$, so $R\Gamma_{\mathcal{L}}$ is correctly defined.

Note that $R\Gamma_{\mathcal{L}}$ is a left t-exact functor; let $\Gamma_{\mathcal{L}} : \mathcal{M}(K \setminus Y) \to \mathcal{M}(\mathfrak{g},K)'$ be the corresponding left exact functor. One has $\Gamma_{\mathcal{L}}(M) = \Gamma(Y, M \otimes (\omega_Y \otimes \mathcal{L}_Y)^*)$. If we are in situation 7.8.7(iii) then this functor coincides, after the standard identification of right and left $\mathcal{D}$-modules, with the functor $\Gamma$ from 1.2.4.

**7.8.12.** *Lemma.* The functor $R\Gamma_{\mathcal{L}}$ is a Morphism of $\mathcal{H}$-Modules.

*Proof.* It suffices to show that $\Gamma_{\mathcal{L}}$ is a Morphism of $\mathcal{H}^c$-Modules, i.e., to define for $F \in \mathcal{H}^c$, $T$ as above a canonical isomorphism $\beta : \Gamma_{\mathcal{L}}(F \overset{c}{\circledast} T) \mathrel{\widetilde{\to}} F \overset{c}{\circledast} \Gamma_{\mathcal{L}}(T)$ compatible with the associativity constraints. Let us write down a canonical isomorphism $\widetilde{\beta} : \Gamma_{\mathcal{L}}(F \widetilde{\circledast} T) \mathrel{\widetilde{\to}} F \widetilde{\circledast} \Gamma_{\mathcal{L}}(T)$ compatible with the $K_\Omega$-actions; one gets $\beta$ by passing to $K_\Omega$-invariants.

The $G'$-action on $\mathcal{L}$ yields an isomorphism $m_Y^*((\omega_Y \otimes \mathcal{L}_Y)^*) = \mathcal{L}_G \boxtimes (\omega_Y \otimes \mathcal{L}_Y)^*$, and the $G$-action on $\mathcal{D}_Y$ (as on a left $\mathcal{O}_Y$-module yields an isomorphism $m_Y^*(\mathcal{D}_Y) = \mathcal{O}_G \boxtimes \mathcal{D}_Y$. These isomorphisms identify $\Gamma_{\mathcal{L}}(F \widetilde{\circledast} T)^{\cdot}$ with $\Gamma(G \times Y, (F^{\cdot} \otimes \mathcal{L}_G) \boxtimes (\mathcal{D}T_Y^{\cdot} \otimes (\omega_Y \otimes \mathcal{L}_Y)^*))$. This vector space coincides with $\Gamma(G, F^{\cdot} \otimes \mathcal{L}_G) \otimes \Gamma(Y, \mathcal{D}T_Y^{\cdot} \otimes (\omega_Y \otimes \mathcal{L}_Y)^*)$ which is $(F \widetilde{\circledast} \Gamma_{\mathcal{L}}(T))^{\cdot}$. Our $\widetilde{\beta}$ is composition of these identifications. We leave it to the reader to check that this is an isomorphism of Harish-Chandra complexes compatible with the $K_\Omega$-actions. $\square$

### 7.9. Extra symmetries and parameters.

**7.9.1.** In the main body of this paper (namely, in 5.4) we use an equivariant version of the Hecke pattern from 7.1.1. Namely, we are given an extra Harish-Chandra pair $(\mathfrak{l}, P)$ that acts on $(G, K)$, and we are looking for an $(\mathfrak{l}, P)$-equivariant version of 7.1.1(a)-(d). Let us explain very briefly the setting; for all the details see the rest of this section. The Hecke category $\mathcal{H}$ is a derived version of the category of *weakly* $(\mathfrak{l}, P)$-equivariant $\mathcal{D}$-modules on $K \backslash G / K$. This is a monoidal triangulated category (which is the analog of 7.1.1(a) in the present setting). $\mathcal{H}$ acts on the appropriate derived category $D_{HC}$ of $(\mathfrak{l} \ltimes \mathfrak{g}, P \ltimes K)$-modules; this is the Harish-Chandra counterpart similar to 7.1.1(c). The geometric counterpart looks as follows. Let $X$ be a "parameter" space equipped with an $(\mathfrak{l}, P)$-structure $X^\wedge$ (see 2.6.4). We consider a family $Y^\wedge$ of smooth varieties with $G$-action parametrized by $X^\wedge$. We assume that the $(\mathfrak{l}, P)$-action on $X^\wedge$ is lifted to $Y^\wedge$ in a way compatible with the $G$-action. Then $\mathcal{H}$ acts on the $\mathcal{D}$-module derived category $D(\mathcal{B})$ of the $X$-stack $\mathcal{B} = (P \ltimes K) \setminus Y^\wedge$ (which is the version of 7.1.1(b)). We have an appropriate localization functor $L\Delta : D_{HC} \to D(\mathcal{B})$ which commutes with the Hecke Actions (this is 7.1.1(d)). For an algebra $A$ with an $(\mathfrak{l}, P)$-action one has an $A$-linear version of the above constructions: one looks at Harish-Chandra modules with $A$-action and $\mathcal{D}$-modules with $A_X$-action (see 2.6.6 for the definition of $A_X$). The corresponding triangulated categories are denoted by $\mathcal{H}_A$, $D_{HC\,A}$, and $D(\mathcal{B}, A_X)$.

The constructions are essentially straightforward modifications of constructions from the previous sections; we write them down for the sake of direct reference in 5.4.

*Remark.* The equivariant Hecke pattern does not reduce to the plain one with $G$ replaced by the group ind-scheme that corresponds to the Harish-Chandra pair $(\mathfrak{l} \ltimes \mathfrak{g}, P \ltimes G)$. Indeed, our $\mathcal{H}$ is much larger then the corresponding "plain" Hecke category: the latter is formed by *strongly* $P$-equivariant $\mathcal{D}$-modules on $K \backslash G/K$. In particular, $\mathcal{H}$ contains as a tensor subcategory the tensor category of $(\mathfrak{l}, P)$-modules. The above structure of fibration $Y/X$ is needed to make the whole $\mathcal{H}$ act on $D(\mathcal{B})$.

**7.9.2.** So we consider a Harish-Chandra pair $(\mathfrak{l}, P)$ that acts on $(G, K)$. Here $P$ could be any affine group scheme (it need not be of finite type), but we assume that $\operatorname{Lie} P$ has finite codimension in $\mathfrak{l}$. Consider the DG category $\mathcal{H}^c$ of $\Omega$-complexes $F$ on $K \backslash G/K$ equipped with an $(\mathfrak{l}, P)$-action on $F$ that lifts the $(\mathfrak{l}, P)$-action on $G/K$. Such $F$ is the same as an $(\mathfrak{l}, P) \ltimes (K_\Omega \times K_\Omega)$-equivariant $\Omega$-complex on $G$. We call $\mathcal{H}^c$ the $(\mathfrak{l}, P)$-*equivariant pre Hecke category.* The morphisms in the homotopy category of $\mathcal{H}^c$ which are $\mathcal{D}$-quasi-isomorphisms of plain $\Omega$-complexes form a localizing family. The $(\mathfrak{l}, P)$-*equivariant Hecke category* $\mathcal{H}$ is the corresponding localization. So $\mathcal{H}$ is a t-category with core equal to the category of $\mathcal{D}$-modules on $G/K$ equipped with a weak $(\mathfrak{l} \ltimes \mathfrak{k}, P \ltimes K)$-action (here $K$ acts on $G/K$ by left translations) such that the action of $K$ is actually a strong one.

Now $\mathcal{H}^c$ is a DG monoidal category, and $\mathcal{H}$ is a monoidal triangulated category. Indeed, all the definitions from 7.6.1 work in the present situation.

*Remark.* Take a Harish-Chandra module $V \in \mathcal{M}(\mathfrak{l}, P)$. Assign to it the corresponding skyscraper sheaf at the distinguished point of $G/K$ considered as an $\Omega$-complex sitting in degree zero and equipped with the trivial $K_\Omega$-action. This is an object of $\mathcal{H}^c$. The functors $\mathcal{M}(\mathfrak{l}, P) \to \mathcal{H}^c, \mathcal{H}$ are fully faithful monoidal functors. Note that $\mathcal{M}(\mathfrak{l}, P)$ belongs in a canonical way to the center of the (pre)Hecke monoidal category, i.e., for any $V$ as above,

$F \in \mathcal{H}$ there is a canonical isomorphism $V \circledast F \rightleftarrows F \circledast V$ compatible with tensor products of $F$'s and $V$'s. Indeed, both objects coincide with $V \otimes F$.

**7.9.3.** To define the Hecke Action on $\mathcal{D}$-modules we need to fix some preliminaries.

Let $X$ be a smooth variety, $Y$ be a $\mathcal{D}_X$-scheme. A $\mathcal{D}_X\Omega_{/X}$*-complex on* $Y$ is a DG $\Omega_{Y/X}$-module equipped with a $\mathcal{D}_X$-structure (:= flat connection along the leaves of the structure connection on $Y/X$). Precisely, the $\mathcal{D}_X$-structure on $Y$ defines on $\Omega_{Y/X}(\mathcal{D}_X) := \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \Omega_{Y/X}$ the structure of an associative DG algebra. Now a $\mathcal{D}_X\Omega_{/X}$-complex on $Y$ is a left DG $\Omega_{Y/X}(\mathcal{D}_X)$-module which is quasi-coherent as an $\mathcal{O}_Y$-module.

The DG category $C(Y, \mathcal{D}_X\Omega_{/X})$ of $\mathcal{D}_X\Omega_{/X}$-complexes on $Y$ is a tensor category (the tensor product is taken over $\Omega_{Y/X}$). The pull-back functor $C(\mathcal{M}^\ell(X)) \longrightarrow C(Y, \mathcal{D}_X\Omega_{/X})$, $M \to \Omega_{Y/X} \underset{\mathcal{O}_X}{\otimes} \Omega_{Y/X}$, is a tensor functor. In particular $C(Y, \mathcal{D}_X\Omega_{/X})$ is an $\mathcal{M}^\ell(X)$-Module (one has $M \circledast F = M \underset{\mathcal{O}_X}{\otimes} F$).

Note that for a $\mathcal{D}_X\Omega_{/X}$-complex $F$ on $Y$ we have an absolute $\Omega$-complex $\Omega_X F$ defined as de Rham complex along $X$ with coefficient in $F$*). So if $Y$ is a smooth variety then we have a notion of $\mathcal{D}$-quasi-isomorphism of $\mathcal{D}_X\Omega_{/X}$-complexes. The corresponding localization of the homotopy category of $C(Y, \mathcal{D}_X\Omega_{/X})$ is denoted $D(Y, \mathcal{D}_X\Omega_{/X})$. The functor $\Omega_X$ : $D(Y, \mathcal{D}_X\Omega_{/X}) \longrightarrow D(Y, \Omega)$ is an equivalence of triangulated categories.

**7.9.4.** Now let $X$ be a smooth variety equipped with a $(\mathfrak{l}, P)$-structure $X^\wedge$ (see 2.6.4). Let $Y^\wedge$ be a scheme equipped with an action of $(\mathfrak{l}, P) \ltimes G$ and a smooth morphism $p^\wedge : Y^\wedge \to X^\wedge$ compatible with the actions (so $G$ acts along the fibers and $p^\wedge$ commutes with the actions of $(\mathfrak{l}, P)$). Set $Y := P \setminus Y^\wedge$. This is a smooth variety equipped with a smooth projection $p : Y \to X$. The $(\mathfrak{l}, P)$-action on $Y^\wedge$ defines a structure of $\mathcal{D}_X$-scheme on $Y$. The $G$-action on $Y^\wedge$ yields a horisontal $G_X$-action on $Y$ (the group $\mathcal{D}_X$-scheme $G_X$ was defined in 2.6.6).

*)As in 7.2 the functor $\Omega_X$ admits left adjoint functor $\mathcal{D}_X$.

Consider the stack $\mathcal{B} := K_X \backslash Y = (P \ltimes K)\backslash Y^\wedge$ fibered over $X$ so we have the corresponding category of left $\mathcal{D}$-modules $\mathcal{M}^\ell(\mathcal{B})$ and the t-category $D(\mathcal{B})$ of $\Omega$-complexes on $\mathcal{B}$. This t-category has a different realization in terms of $\mathcal{D}_X\Omega_{/X}$-complexes that we are going to describe.

Consider the DG group $\mathcal{D}_X$-schemes $G_{\Omega X} := (G_X, \Omega_{G_X/X})$, $K_{\Omega X}$. One defines a $K_{\Omega X}$-action on a $\mathcal{D}_X\Omega_{/X}$-complex on $Y$ as in 7.6.4. Now we have the DG category $C(K_X\backslash Y, \mathcal{D}_X\Omega_{/X})$ of $K_{\Omega X}$-equivariant $\mathcal{D}_X\Omega_{/X}$-complexes on $Y$. Localizing its homotopy category by $\mathcal{D}$-quasi-isomorphisms we get the triangulated category $D(K_X \backslash Y, \mathcal{D}_X\Omega_{/X})$. The de Rham functor $\Omega_X$ identifies it with $D(\mathcal{B})$.

Now we can define the Hecke Action on $D(\mathcal{B})$. First let us construct the Action $\overset{c}{\circledast}$ of $\mathcal{H}^c$ on $C(K_X \backslash Y, \mathcal{D}_X\Omega_{/X})$. Indeed, for $F \in \mathcal{H}^c$ we have a $\mathcal{D}_X\Omega_{/X}$-complex $F_X$ on $G_X$ which is $K_{\Omega X}$-equivariant with respect to the left and right translations. So for $T \in C(K_X\backslash Y, \mathcal{D}_X\Omega_{/X})$ we have a $\mathcal{D}_X\Omega_{/X}$-complex $F \boxtimes T$ on the $\mathcal{D}_X$-scheme $G_X \times Y$ (the fiber product of $G_X$ and $Y$ over $X$). It is $K_{\Omega X}$-equivariant with respect to all the $K_X$-actions on $G_X\times Y$. So $F \boxtimes T$ descends to $G_X \underset{K_X}{\times} Y$. We define $F \overset{c}{\circledast} T \in C(K_X \backslash Y, \mathcal{D}_X\Omega_{/X})$ as the push-forward of the above complex by the action map $G_X \underset{K_X}{\times} Y \to Y$. The Hecke Action $\circledast : \mathcal{H} \times D(\mathcal{B}) \longrightarrow D(\mathcal{B})$ is the right derived functor of $\overset{c}{\circledast}$; as usually you may compute it using loose $\mathcal{D}_X\Omega_{/X}$-complexes.

*Remark.* For $W \in \mathcal{M}(\mathfrak{l}, P) \subset \mathcal{H}^c$ and $T$ as above one has $W \overset{c}{\circledast} T = W \circledast T = W_X \otimes T$ (the $\mathcal{D}_X$-module $W_X$ was defined in 2.6.6).

**7.9.5.** Let us define the Harish-Chandra categories. Let $G'$ be as in 7.8.1 and assume that we are given a lifting of the $(\mathfrak{l}, P)$-action on $G$ to that on $G'$ which preserves $K \subset G'$ and fixes $\mathbb{G}_m \subset G'$. So we have the Harish-Chandra pair $(\mathfrak{l}, P) \ltimes (\mathfrak{g}', K')$. Let $C_{HC}$ be the category of $(\mathfrak{l}, P) \ltimes (\mathfrak{k}_\Omega \ltimes \mathfrak{g}, K)'$-complexes, i.e., $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-complexes equipped with a compatible $(\mathfrak{l}, P)$-action (see 7.8.1 for notation). Let $D_{HC}$ be the corresponding derived category. This is a t-category with core $\mathcal{M}_{HC} = \mathcal{M}(\mathfrak{l} \ltimes \mathfrak{g}, P \ltimes K)'$. Below we call the objects of $C_{HC}$ and $D_{HC}$ simply *Harish-Chandra complexes* and those of $\mathcal{M}_{HC}$ *Harish-Chandra modules.*

The pre Hecke category $\mathcal{H}^c$ acts on $C_{HC}$. Indeed, the constructions of 7.8.2 make perfect sense in our situation (($\mathfrak{l}, P$) acts on $F \overset{c}{\circledast} V$ by transport of structure). The $\mathcal{H}$-Action $\circledast$ on $D_{HC}$ is the right derived functor of $\overset{c}{\circledast}$. The results of 7.8.4-7.8.5 render to the present setting without changes.

*Remark.* For $W \in \mathcal{M}(\mathfrak{l}, P) \subset \mathcal{H}^c$ and a Harich-Chandra complex $V$ one has a canonical isomorphism of Harish-Chandra complexes $W \overset{c}{\circledast} V = W \circledast V = W \otimes V$.

**7.9.6.** Let us pass to the localization functor. The construction of 7.8.6 renders to our setting as follows. We start with $Y^\wedge$ as in 7.9.4. Assume that it carries a line bundle $\mathcal{L}_{Y^\wedge}$ and the $(\mathfrak{l}, P) \ltimes G$-action on $Y^\wedge$ is lifted to an action of $(\mathfrak{l}, P) \ltimes G'$ on $\mathcal{L}_{Y^\wedge}$ such that $\mathbb{G}_m \subset G'$ acts by the character opposite to the standard one. Let $\mathcal{L}_Y$ be the descent of $\mathcal{L}_{Y^\wedge}$ to $Y$ defined by the action of $P$. This line bundle carries a canonical $\mathcal{D}_X$-structure that comes from the $\mathfrak{l}$-action on $\mathcal{L}_{Y^\wedge}$. It also carries a horisontal action of $G'_X$.

We have a DG functor

$$\Delta_\Omega = \Delta_{\Omega\mathcal{L}} : \ C_{HC} \longrightarrow C(K_X \setminus Y, \mathcal{D}_X \Omega_{/X}), \tag{338}$$

$\Delta_\Omega(V) = \mathrm{Hom}_{\mathfrak{g}_X}(DR_{\mathfrak{g}_X}, \omega_{Y/X} \times \mathcal{L}_Y \times V)[-\dim K]$ (cf. (336)). As in 7.8.6 this functor sends quasi-isomorphisms to $\mathcal{D}$-quasi-isomorphisms, so it yields a triangulated functor

$$L\Delta = L\Delta_{\mathcal{L}} : \ D_{HC} \longrightarrow D(\mathcal{B}) \tag{339}$$

which is right t-exact. The corresponding right exact functor between the cores $\Delta_{\mathcal{L}} : \mathcal{M}_{HC} \longrightarrow \mathcal{M}^\ell(\mathcal{B})$ sends $V$ to the $K_X$-equivariant left $\mathcal{D}_Y$-module $(\mathcal{D}_{Y/X} \otimes \mathcal{L}_Y) \underset{U(\mathfrak{g}'_X)}{\otimes} V_X$.

The functors $\Delta_\Omega, L\Delta$ commute with the Hecke Action. Indeed, the proof of 7.8.8 renders to our setting word-by-word. In particular for any $W \in \mathcal{M}(\mathfrak{l}, P), V \in D_{HC}$ one has $L\Delta(W \otimes V) = W_X \otimes L\Delta(V)$.

**7.9.7.** *$A$-linear version.* Assume that in addition we are given a commutative algebra $A$ equipped with an $(\mathfrak{l},P)$-action. One attaches it to the above pattern as follows.

(i) Denote by $\mathcal{H}^c_A$ the DG category of objects $F\in\mathcal{H}^c$ equipped with an action of $A$ such that the actions of $A$ and $(\mathfrak{l},P)$ are compatible and $F$ is $A$-flat. Let $\mathcal{H}_A$ be the corresponding $\mathcal{D}$-derived category. One defines the convolution product as in 7.9.2 (the tensor product is taken over $A$) so $\mathcal{H}^c_A$ and $\mathcal{H}_A$ are monoidal categories. Let $\mathcal{M}(\mathfrak{l},P)^{fl}_A$ be the tensor category of flat $A$-modules equipped with an action of $(\mathfrak{l},P)$. As in the Remark in 7.9.2 one has canonical fully faithful monoidal functors $\mathcal{M}(\mathfrak{l},P)^{fl}_A\longrightarrow\mathcal{H}^c_A,\mathcal{H}_A$ which send $\mathcal{M}(\mathfrak{l},P)^{fl}_A$ to the center of Hecke categories.

(ii) Assume we are in situation 7.9.4. Consider the category $\mathcal{M}^\ell(\mathcal{B},A_X)$ of left $\mathcal{D}$-modules on $\mathcal{B}$ equipped with $A_X$-action (the $\mathcal{D}_X$-algebra $A_X$ was defined in 2.6.6). Let $C(\mathcal{B},A_X\otimes\Omega)$ be the DG category of $\Omega$-complexes on $\mathcal{B}$ equipped with an $A_X$-action and $D(\mathcal{B},A_X)$ be the localization of the corresponding homotopy category with respect to $\mathcal{D}$-quasi-isomorphisms. This is a t-category with core $\mathcal{M}^\ell(\mathcal{B},A_X)$. As in 7.9.4 one may also define this t-category in terms of $\mathcal{D}_X\Omega_{/X}$-complexes. Namely, let $C(K_X\setminus Y,A_X\mathcal{D}_X\Omega_{/X})$ be the DG category of objects of $C(K_X\setminus Y,\mathcal{D}_X\Omega_{/X})$ equipped with an $A_X$-action (commuting with the $K_{\Omega X}$-action). Localizing it by $\mathcal{D}$-quasi-isomorphisms we get the triangulated category $D(K_X\setminus Y,A_X\mathcal{D}_X\Omega_{/X})$. The de Rham functor $\Omega_X$ identifies it with $D(\mathcal{B},A_X)$.

The Hecke Action in the $A$-linear setting is defined exactly as in 7.9.4. The statement of the Remark in 7.9.4 remains true (you take the tensor product over $A_X$).

(iii) Assume we are in situation 7.9.5. One defines $C_{HC\,A}$ as the category of Harish-Chandra complexes equipped with a compatible $A$-action (so the actions of $A$ and $(\mathfrak{k}_\Omega\times\mathfrak{g},K)'$ commute). Let $D_{HC\,A}$ be the corresponding derived category. This is a t-category with core $\mathcal{M}_{HC\,A}$ equal to the category of $(\mathfrak{l}\ltimes\mathfrak{g},P\ltimes K)'$-modules equipped with a compatible $A$-action. All the constructions and results about the Hecke Action remain valid without changes. In the Remark in 7.9.5 you take $W \in \mathcal{M}(\mathfrak{l}, P)^{fl}_A$; the tensor product $W \otimes V$ is taken over $A$. The $A$-linear setting for the localization functors requires no changes.

*Remark.* There are obvious functors (tensoring by $A$) which send the plain categories as above to those with $A$ attached. These functors are compatible with all the structures we considered. The forgetting of the $A$-action functors $D(\mathcal{B}, A_X) \to D(\mathcal{B})$, $D_{HC\,A} \to D_{HC}$ are Morphisms of $\mathcal{H}$-Modules. They commute with the localization functors.

**7.9.8.** *Variant.* Assume that in addition to $A$ we are given a morphism of commutative algebras $e : \mathfrak{Z} \to A$ compatible with the $(\mathfrak{l}, P)$-actions. Here $\mathfrak{Z} := U(\mathfrak{g})'^{\,\mathrm{Ad}\,G}$ (so if $G$ is connected then $\mathfrak{Z}$ is the center of $U(\mathfrak{g})'$). Then $\mathfrak{Z}$ acts on any object of $\mathcal{M}_{HC\,A}$ or $C_{HC\,A}$ in two ways. Denote by $\mathcal{M}^e_{HC\,A}, C^e_{HC\,A}$ the categories of those objects on which the two actions of $\mathfrak{Z}$ coincide; let $D^e_{HC}$ be the corresponding derived category. The Action of $\mathcal{H}^c_A$ on $C_{HC\,A}$ is $\mathfrak{Z}$-linear (see 7.8.4) so it preserves $C^e_{HC\,A}$. Thus we have an Action of $\mathcal{H}_A$ on $D^e_{HC\,A}$. The obvious functor $D^e_{HC\,A} \to D_{HC\,A}$ is a Morphism of $\mathcal{H}_A$-Modules.

*Remark.* If $e$ is surjective then $\mathcal{M}^e_{HC\,A}$ is the full subcategory of $\mathcal{M}_{HC}$ that consists of Harish-Chandra modules killed by $\operatorname{Ker} e$. Same for $C^e_{HC\,A}$.

**7.10.** $\mathcal{D}$**-crystals.** Below we sketch a crystalline approach to $\mathcal{D}$-module theory. As opposed to the conventional formalism it makes no distinction between smooth and non-smooth schemes.

In this section "scheme" means "$\mathbb{C}$-scheme locally of finite type". Same for algebraic spaces and stacks. The formal schemes or algebraic spaces are assumed to be locally of ind-finite type*).

**7.10.1.** Let $f : Y \to X$ be a quasi-finite morphism of schemes. Then Grothendieck's functor $Rf^! : D^b(X, \mathcal{O}) \to D^b(Y, \mathcal{O})$ is left t-exact. Set

*):= any closed subscheme is of finite type.

$f^! := H^0 Rf^! : \mathcal{M}(X,\mathcal{O}) \to \mathcal{M}(Y,\mathcal{O})$; this is a left exact functor. Therefore the categories $\mathcal{M}(X,\mathcal{O})$ together with functors $f^!$ form a fibered category over the category of schemes and quasi-finite morphisms.

Here is an explicit description of $f^!$. According to Zariski's Main Theorem any quasi-finite morphism is composition of a finite morphism and an open embedding; let us describe $f^!$ in these two cases. If $f$ is an open embedding (or, more generally, if $f$ is étale) then $f^! = f^*$. If $f$ is finite then $f^!$ is the functor right adjoint to the functor $f_* : \mathcal{M}(Y,\mathcal{O}) \to \mathcal{M}(X,\mathcal{O})$. Explicitely, $f_*\mathcal{O}_Y$ is a finite $\mathcal{O}_X$-algebra, and the functor $f_*$ identifies $\mathcal{M}(Y,\mathcal{O})$ with the category of $f_*\mathcal{O}_Y$-modules which are quasi-coherent as $\mathcal{O}_X$-modules. Now for an $\mathcal{O}$-module $M$ on $X$ the corresponding $f_*\mathcal{O}_Y$-module $f_*f^!M$ is $\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, M)$. In particular, if $f$ is a closed embedding then $f^!M \subset M$ is the submodule of sections supported (scheme-theoretically) on $Y$.

The above picture extends to the setting of formal schemes (or algebraic spaces) as follows. For a formal scheme $\hat{X}$ we denote by $\mathcal{M}(\hat{X},\mathcal{O})$ the category of discrete quasi-coherent $\mathcal{O}_{\hat{X}}$-modules*). For example, if $\hat{X}$ is the formal completion of a scheme $V$ along its closed subscheme $X$ then $\mathcal{M}(\hat{X},\mathcal{O})$ coinsides with the category of $\mathcal{O}$-modules on $V$ supported set-theoretically on $X$. If $\hat{X}$ is affine then for any $M \in \mathcal{M}(\hat{X},\mathcal{O})$ one has $M = \bigcup M_{X'}$ where $X'$ runs the (directed) set of closed subschemes of $\hat{X}$ and $M_{X'} \in \mathcal{M}(X',\mathcal{O})$ is the submodule of sections supported scheme-theoretically on $X'$. The pull-back functors $f^!$ extend in a unique manner*) to the setting of quasi-finite morphisms of formal algebraic spaces. Indeed, if $\hat{f} : \hat{Y} \to \hat{X}$ is such a morphism then to define $\hat{f}^! : \mathcal{M}(\hat{X},\mathcal{O}) \to \mathcal{M}(\hat{Y},\mathcal{O})$ we may assume that $\hat{X}$, $\hat{Y}$ are affine; now $\hat{f}^!M = \bigcup \hat{f}|^!_{Y'} M_{X'}$ where $Y'$ is a closed subscheme of $\hat{Y}$ and $\hat{f}(Y') \subset X'$. We leave it to the reader to describe $\hat{f}^!$ explicitely if $\hat{f}$ is ind-finite*).

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*)This category is abelian. For a more general setting see 7.11.4.

*)We assume that they are compatible with composition of $f$'s.

*)$:= Y_{\mathrm{red}} \to X_{\mathrm{red}}$ is finite.

**7.10.2.** For a scheme or an algebraic space $X$ denote by $X_{cr}$ the category of diagrams $X \xleftarrow{j} S \overset{i}{\hookrightarrow} \hat{S}$ where $j$ is a quasi-finite morphism and $i$ a closed embedding of affine schemes such that the corresponding ideal $\mathcal{I} \subset \mathcal{O}_{\hat{S}}$ is nilpotent. We usually write this object of $X_{cr}$ as $(S, \hat{S})$ or simply $\hat{S}$. A morphism $(S, \hat{S}) \to (S', \hat{S}')$ in $X_{cr}$ is a morphism of schemes $\phi : \hat{S} \to \hat{S}'$ such that $\phi(S) \subset S'$ and $\phi|_S : S \to S'$ is a morphism of $X$-schemes.

Note that for any $\phi$ as above the morphism $\phi : \hat{S} \to \hat{S}'$ is quasi-finite. Therefore the categories $\mathcal{M}(\hat{S}, \mathcal{O})$ together with the pull-back functors $\phi^!$ form a fibered category $\mathcal{M}^!(X_{cr}, \mathcal{O})$ over $X_{cr}$.

Sometimes it is convenient to consider a larger category $X_{\hat{cr}}$ which consists of similar diagrams as above but we permit $\hat{S}$ to be a formal scheme (so $\mathcal{I}$ is a pronilpotent ideal, i.e., $\hat{S}_{\mathrm{red}} = S_{\mathrm{red}}$). As above we have the fibered category $\mathcal{M}^!(X_{\hat{cr}}, \mathcal{O})$ over $X_{\hat{cr}}$.

**7.10.3.** *Definition.* A *$\mathcal{D}$-crystal on* $X$ is a Cartesian section of $\mathcal{M}^!(X_{cr}, \mathcal{O})$. $\mathcal{D}$-crystals on $X$ form a $\mathbb{C}$-category $\mathcal{M}_{\mathcal{D}}(X)$.

Explicitely, a $\mathcal{D}$-crystal $M$ is a rule that assigns to any $(S, \hat{S}) \in X_{cr}$ an $\mathcal{O}$-module $M_{\hat{S}} = M_{(S,\hat{S})}$ on $\hat{S}$ and to a morphism $\phi : (S, \hat{S}) \to (S', \hat{S}')$ an identification $\alpha_\phi : M_{\hat{S}} \overset{\sim}{\rightarrow} \phi^! M_{\hat{S}'}$ compatible with composition of $\phi$'s.

In particular, if $\phi$ is a closed embedding defined by an ideal $\mathcal{I} \subset \mathcal{O}_{\hat{S}'}$ then $M_{\hat{S}}$ is the submodule of $M_{\hat{S}'}$ that consists of sections killed by $\mathcal{I}$.

In definition 7.10.3 one may replace $X_{cr}$ by $X_{\hat{cr}}$: we get the same category of $\mathcal{D}$-crystals. Indeed, for $(S, \hat{S}) \in X_{\hat{cr}}$ one has $M_{\hat{S}} = \bigcup M_{(S,\hat{S}')}$ where $\hat{S}'$ runs the set of all subschemes $S \subset \hat{S}' \subset \hat{S}$.

**7.10.4.** *Variants.* Let $X_{cr}^{(i)}, .., X_{cr}^{(iv)}$ be the full subcategories of $X_{cr}$ that consist of objects $(S, \hat{S})$ which satisfy, respectively, one of the following conditions (in (ii)-(iv) we assume that $X$ is a scheme):

(i) $S \to X$ is étale.

(ii) $S \to X$ is an open embedding.

(iii) (assuming that $X$ is affine) $S \overset{\sim}{\rightarrow} X$.

(iv) $S \to X$ is a locally closed embedding.

Denote by $\mathcal{M}^{(i)}_{\mathcal{D}}(X), .., \mathcal{M}^{(iv)}_{\mathcal{D}}(X)$ the categories of Cartesian sections of $\mathcal{M}^!(X_{cr}, \mathcal{O})$ over the corresponding subcategories $X^{(a)}_{cr}$. One has the obvious restriction functors $\mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}^{(a)}_{\mathcal{D}}(X)$. We leave it to the reader to check that these functors are equivalences of categories$^{*)}$.

*Remark.* The category $X^{(ii)}_{cr}$ is (the underlying category of) Grothendieck's crystalline site of $X$, so $\mathcal{D}$-crystals are the same as crystals for the fibered category $\mathcal{M}^!(X^{(ii)}_{cr}, \mathcal{O})$ in Grothendieck's terminology. We consider $X_{cr}$ as the basic set-up since it directly generalizes to the setting of ind-schemes (see 7.11.6).

**7.10.5.** Let $f : Y \to X$ be a quasi-finite morphism. It yields a faithful functor $Y_{cr} \to X_{cr}$ which sends $Y \xleftarrow{j} S \hookrightarrow \hat{S}$ to $Y \xleftarrow{fj} S \hookrightarrow \hat{S}$. We get the corresponding "restriction" functor $f^! : \mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}_{\mathcal{D}}(Y)$. It is compatible with composition of $f$'s.

In particular, categories $\mathcal{M}_{\mathcal{D}}(U)$, where $U$ is étale over $X$, form a fibered category over the small étale site $X_{ét}$ which we denote by $\mathcal{M}_{\mathcal{D}}(X_{ét})$.

**7.10.6.** *Lemma.* $\mathcal{D}$-crystals are local objects for the étale topology, i.e., $\mathcal{M}_{\mathcal{D}}(X_{ét})$ is a sheaf of categories. □

**7.10.7.** Below we give a convenient "concrete" description of $\mathcal{D}$-crystals.

Assume we have a closed embedding $X \hookrightarrow V$ where $V$ is a formally smooth$^{*)}$ formal algebraic space such that $X_{\mathrm{red}} = V_{\mathrm{red}}{}^{*)}$. Such thing always exists if $X$ is affine: one may embed $X$ into a smooth scheme $W$ and take for $V$ the formal completion of $W$ along $X$.

For $n \geq 1$ let $V^{<n>}$ denotes the formal completion of $V^n$ along the diagonal $V \subset V^n$ (or, equivalently, along $X \subset V^n$). The projections $p_1, p_2$ :

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$^{*)}$It suffices to notice that 7.10.6, 7.10.7, 7.10.8 together with the proofs remain literally valid if we replace $\mathcal{M}_{\mathcal{D}}(X)$ by $\mathcal{M}^{(a)}_{\mathcal{D}}(X)$.

$^{*)}$see 7.11.1.

$^{*)}$i.e., the ideal of $X$ in $\mathcal{O}_V$ is pronilpotent.

$V^{<2>} \to V$, $p_{12}, p_{23}, p_{13} : V^{<3>} \to V^{<2>}$ are ind-finite, so we have the functors $p_i^! : \mathcal{M}(V,\mathcal{O}) \to \mathcal{M}(V^{<2>},\mathcal{O})$, $p_{ij}^! : \mathcal{M}(V^{<2>},\mathcal{O}) \to \mathcal{M}(V^{<3>},\mathcal{O})$. Since $V$ is formally smooth these functors are exact.

Denote by $\mathcal{M}_{\mathcal{D}V}(X)$ the category of pairs $(M_V, \tau)$ where $M_V \in \mathcal{M}(V,\mathcal{O})$ and $\tau : p_1^! M_V \overset{\sim}{\rightarrow} p_2^! M_V$ is an isomorphism such that

$$p_{23}^!(\tau) p_{12}^!(\tau) = p_{13}^!(\tau). \tag{340}$$

**7.10.8.** *Proposition.* The categories $\mathcal{M}_{\mathcal{D}}(X)$ and $\mathcal{M}_{\mathcal{D}V}(X)$ are canonically equivalent.

*Proof.* We deal with local objects, so we may assume that $X$ is affine. For $M \in \mathcal{M}_{\mathcal{D}}(X)$ we have $M_V = M_{(X,V)} \in \mathcal{M}(V,\mathcal{O})$. Since $p_i^! M_V = M_{V^{<2>}}$ we have $\tau$ that obviously satisfies (340). Conversely, assume we have $(M_V, \tau) \in \mathcal{M}_{\mathcal{D}V}(X)$; let us define the corresponding $\mathcal{D}$-crystal $M$. For $(S, \hat{S}) \in X_{cr}$ choose $j' : \hat{S} \to V$ that extends the structure morphism $j : S \to X$ (such $j'$ exists since $V$ is formally smooth). Consider the $\mathcal{O}_{\hat{S}}$-module $j'^! M_V$. If $j'' : \hat{S} \to V$ is another extension of $j$ then there is a canonical isomorphism $\nu_{j'j''} : j'^! M_V \overset{\sim}{\rightarrow} j''^! M_V$. Namely, $(j', j'')$ maps $\hat{S}$ to $V^{<2>}$, hence $j'^! M_V = (j', j'')^! p_1^! M_V$; now use the similar description of $j''^! M_V$ and set $\nu_{j'j''} := (j', j'')^!(\tau)$. By (340) these identifications are transitive, so $j'^! M_V$ does not depend on the choice of $j'$. This is $M_{(S,\hat{S})}$. The definition of structure isomorphisms $\alpha_\phi$ for $M$ is clear. $\square$

**7.10.9.** *Corollary.* (i) For any $X$ the category $\mathcal{M}_{\mathcal{D}}(X)$ is abelian.

(ii) For $\hat{S} \in X_{cr}$ the functor $\mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}(\hat{S},\mathcal{O})$, $M \mapsto M_{\hat{S}}$ is left exact.

(iii) For a quasi-finite $j : Y \to X$ the functor $j^! : \mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}_{\mathcal{D}}(Y)$ is left exact. If $j$ is étale then $j^!$ is exact.

*Proof.* The statement (i) is true if $X$ is affine. Indeed, choose $X \hookrightarrow V$ as in 7.10.7. The category $\mathcal{M}_{\mathcal{D}V}(X)$ is abelian since the functors $p_i^!$, $p_{ij}^!$ are exact, so we are done by 7.10.8.

If $j : U \to X$ is an étale morphism of affine schemes then the functor $j^! : \mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}_{\mathcal{D}}(U)$ is exact. Indeed, let $U \hookrightarrow V_U$ be the $U$-localization of $X \hookrightarrow V$ (so $V_U$ is étale over $V$); then $j^!$ coincides with the étale localization functor $\mathcal{M}_{\mathcal{D}V}(X) \to \mathcal{M}_{\mathcal{D}V_U}(U)$ which is obviously exact.

Now (i) follows from 7.10.6. The rest is left to the reader. $\square$

**7.10.10.** *Lemma.* For an étale morphism $p : U \to X$ the functor $p^!$ admits a right adjoint functor $p_* : \mathcal{M}_{\mathcal{D}}(U) \to \mathcal{M}_{\mathcal{D}}(X)$. If $p$ is an open embedding then $p^! p_*$ is identity functor.

*Proof.* Here is an explicit construction of $p_*$. For $(S, \hat{S}) \in X_{cr}$ set $S_U := S \underset{X}{\times} U$; let $\hat{p}_S : \hat{S}_U \to \hat{S}$ be the étale morphism whose pull-back to $S \hookrightarrow \hat{S}$ is the projection $S_U \to S$. So $(S_U, \hat{S}_U) \in U_{cr}$, and we have the functor $X_{cr} \to U_{cr}$, $(S, \hat{S}) \mapsto (S_U, \hat{S}_U)$.

Now for $N \in \mathcal{M}_{\mathcal{D}}(U)$ set $(p_* N)_{\hat{S}} := (\hat{p}_S)_\cdot N_{\hat{S}_U}$. The identifications $\alpha_\phi$ come from the base change isomorphism $\phi^! \hat{p}_{S'\cdot} = \hat{p}_{S\cdot} \phi_U^!$. $\square$

Now let $i : Y \hookrightarrow X$ be a closed embedding and $j : U := X \setminus Y \hookrightarrow X$ the complementary open embedding. Denote by $\mathcal{M}_{\mathcal{D}}(X)_Y$ the full subcategory of $\mathcal{M}_{\mathcal{D}}(X)$ that consists of those $\mathcal{D}$-crystals $M$ that $j^! M = 0$.

**7.10.11.** *Lemma.* (i) The functor $i^!$ admits a left adjoint functor $i_* : \mathcal{M}_{\mathcal{D}}(Y) \to \mathcal{M}_{\mathcal{D}}(X)$.

(ii) $i_*$ sends $\mathcal{M}_{\mathcal{D}}(Y)$ to $\mathcal{M}_{\mathcal{D}}(X)_Y$ and

$$i_* : \mathcal{M}_{\mathcal{D}}(Y) \to \mathcal{M}_{\mathcal{D}}(X)_Y, \quad i^! : \mathcal{M}_{\mathcal{D}}(X)_Y \to \mathcal{M}_{\mathcal{D}}(Y)$$

are mutually inverse equivalences of categories.

(iii) Let $p : Z \to X$ be a quasi-finite morphism; set $Y_Z := Y \underset{Z}{\times} X$, so we have $i_Z : Y_Z \hookrightarrow Z$ and $p_Y : Y_Z \to Y$. Then one has a canonical identification of functors $p^! i_* = i_{Y*} p_Y^! : \mathcal{M}_{\mathcal{D}}(Y) \to \mathcal{M}_{\mathcal{D}}(Z)$.

*Proof.* Here is an explicit construction of $i_*$. Take a $\mathcal{D}$-crystal $N$ on $Y$. For $(S, \hat{S}) \in X_{cr}$ set $S_Y := S \underset{X}{\times} Y$, so $S_Y$ is a closed subscheme of $S$, hence of $\hat{S}$. The projection $S_Y \to Y$ is quasi-finite, so $N$ yields a $\mathcal{D}$-crystal on $S_Y$. We define $(i_*N)_{(S,\hat{S})}$ as the corresponding $\mathcal{O}$-module on $\hat{S}$ (see 7.10.3). The structure isomorphisms $\alpha_\phi$ for $i_*N$ come from the corresponding isomorphisms for $N$ in the obvious manner.

The adjunction property of $i_*$, as well as properties (ii), (iii), are clear. $\square$

**7.10.12.** *Proposition.* If $X$ is smooth then $\mathcal{M}_{\mathcal{D}}(X)$ is canonically equivalent to the category $\mathcal{M}(X)$ of $\mathcal{D}$-modules on $X$.

*Proof.* We use description 7.10.7 of $\mathcal{M}_{\mathcal{D}}(X)$ for $V = X$. So a $\mathcal{D}$-crystal $M$ amounts to a pair $(M_X, \tau)$ where $M_X \in \mathcal{M}(X, \mathcal{O})$ and $\tau : p_1^! M_X \xrightarrow{\sim} p_2^! M_X$ is an isomorphism of $\mathcal{O}$-modules on $X^{<2>}$ which satisfies (340). Let us show that such $\tau$ is the same as a right $\mathcal{D}$-module structure on $M_X$.

Consider $\mathcal{D}_X$ as an object of $\mathcal{M}(X^{<2>}.\mathcal{O})$ (via the $\mathcal{O}_X$-bimodule structure). There is a canonical isomorphism $\mathcal{D}_X \xrightarrow{\sim} p_1^! \mathcal{O}_X$ which identifies $\partial \in \mathcal{D}_X$ with the section $(f \otimes g \mapsto f\partial(g)) \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X^{<2>}}, \mathcal{O}_X) = p_1^! \mathcal{O}_X$. Therefore we have $M_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \xrightarrow{\sim} M_X \underset{\mathcal{O}_X}{\otimes} p_1^! \mathcal{O}_X \xrightarrow{\sim} p_1^! M_X$. Hence, by adjunction,

$$\text{(341)}\quad \operatorname{Hom}(p_1^! M_X, p_2^! M_X) = \operatorname{Hom}(p_{2\cdot} p_1^! M_X, M_X) = \operatorname{Hom}(M_X \otimes \mathcal{D}_X, M_X).$$

Here we consider $M_X \otimes \mathcal{D}_X$ as an $\mathcal{O}_X$-module via the right $\mathcal{O}$-module structure on $\mathcal{D}_X$. So $\tau : p_1^! M_X \to p_2^! M_X$ is the same as a morphism $M_X \otimes \mathcal{D}_X \to M_X$. One checks that the conditions on $\tau$ just mean that this arrow is a right unital action of $\mathcal{D}_X$ on $M_X$. See the next Remark for a comment and some details. $\square$

**7.10.13.** *Remark.* Let us discuss certain points of 7.10.12 in a more general setting. Since $\mathcal{O}_{X^{<2>}}$ is a completion of $\mathcal{O}_X \underset{\mathbb{C}}{\otimes} \mathcal{O}_X$ one may consider objects of $\mathcal{M}(X^{<2>}, \mathcal{O})$ as certain sheaves of $\mathcal{O}_X$-bimodules called Diff-*bimodules* on $X$[*)]. If $A, B$ are Diff-bimodules then such is $A \underset{\mathcal{O}_X}{\otimes} B$ (so $\mathcal{M}(X^{<2>}, \mathcal{O})$ is a monoidal category). Notice that $A \underset{\mathcal{O}_X}{\otimes} B$ is actually an object of

[*)] In [BB93] the term "differential bimodule" was used; we refer there for the details.

$\mathcal{M}(X^{<3>},\mathcal{O})$ in the obvious way. By adjunction, for any $C\in\mathcal{M}(X^{<2>},\mathcal{O})$ a morphism of Diff-bimodules $A\underset{\mathcal{O}_X}{\otimes}B\to C$ is the same as a morphism $A\underset{\mathcal{O}_X}{\otimes}B\to p_{13}^!C$ in $\mathcal{M}(X^{<3>},\mathcal{O})$. Thus for a Diff-algebra*) $A$ its product amounts to a morphism $m:A\underset{\mathcal{O}_X}{\otimes}A\to p_{13}^!A$ in $\mathcal{M}(X^{<3>},\mathcal{O})$ (we leave it to the reader to write associativity property in these terms). Similarly, for a (right) $A$-module $M_X$ we may write the $A$-action as a morphism $a:M_X\underset{\mathcal{O}_X}{\otimes}A\to p_2^!M_X$ in $\mathcal{M}(X^{<2>},\mathcal{O})$; the action (associativity) property just says that the two morphisms $M_X\underset{\mathcal{O}_X}{\otimes}A\underset{\mathcal{O}_X}{\otimes}A\to p_3^!M$ in $\mathcal{M}(X^{<3>},\mathcal{O})$ obtained from $m$ and $a$ coincide. Assume now that $A=\mathcal{D}_X$ or, more generally, $A$ is a tdo. Then $m:A\underset{\mathcal{O}_X}{\otimes}A\to p_{13}^!A$ is an isomorphism*). If $M_X$ is a (possibly, non-unital) $A$-module then $a:M_X\underset{\mathcal{O}_X}{\otimes}A\to p_2^!M_X$ is an isomorphism if and only if our module is unital.

**7.10.14.** We leave it to the reader to identify (in the smooth setting) the functors $f^!$, $p_*$, $i_*$ from, respectively, 7.10.5, 7.10.10, and 7.10.11(i), with the standard $\mathcal{D}$-module functors.

Combining 7.10.12 and 7.10.11(ii) we see that if $X$ is any algebraic space then $\mathcal{D}$-crystals on $X$ are the same as $\mathcal{D}$-modules on $X$ in the sense of [Sa91]*).

**7.10.15.** The rest of the section is a sketch of crystalline setting for tdo and twisted $\mathcal{D}$-modules. First we discuss crystalline $\mathcal{O}^*$-gerbes. In case of a smooth scheme such gerbe amounts to an étale localized version of the notion "tdo up to a twist by a line bundle". Then we define for a crystalline $\mathcal{O}^*$-gerbe $\mathcal{C}$ the corresponding abelian category of twisted $\mathcal{D}$-crystals $\mathcal{M}_{\mathcal{C}}(X)$.

**7.10.16.** As before, $X$ is any algebraic space. The category $X_{cr}$ carries a structure of site (étale crystalline topology): a covering is a family of

*)i.e., an algebra in the monoidal category of Diff-bimodules.

*)Probably this property characterizes tdo's.

*)Saito prefers to deal with analytic setting, but his definitions have obvious algebraic version (and the above definitions have obvious analytic version).

morphisms $\{(S_i, \hat{S}_i) \to (S, \hat{S})\}$ such that $\{\hat{S}_i \to \hat{S}\}$ is an étale covering of $\hat{S}$. It carries a sheaf of rings $\mathcal{O}_{cr}$ where $\mathcal{O}_{cr}(S, \hat{S}) = \mathcal{O}(\hat{S})$. So we have the corresponding sheaf $\mathcal{O}^*_{cr}$ of invertible elements.

**7.10.17.** *Definition.* A *crystalline $\mathcal{O}^*$-gerbe* on $X$ is an $\mathcal{O}^*_{cr}$-gerbe on $X_{cr}$ *).

Explicitely, this means the following. Consider the sheaf of Picard groupoids $\mathcal{P}ic_{cr}$ on $X_{cr}$ where $\mathcal{P}ic_{cr}(S, \hat{S}) := \mathcal{P}ic(\hat{S})$ (= the Picard groupoid of line bundles on $\hat{S}$). Now a crystalline $\mathcal{O}^*$-gerbe on $X$ is a $\mathcal{P}ic_{cr}$-Torsor $\mathcal{C}$ over $X_{cr}$ (i.e., $\mathcal{C}$ is a fibered category over $X_{cr}$ equipped with an Action of $\mathcal{P}ic_{cr}$ which makes each fiber $\mathcal{C}(\hat{S}) = \mathcal{C}(S, \hat{S})$ a $\mathcal{P}ic(\hat{S})$-Torsor) such that locally on $X_{cr}$ our $\mathcal{C}(S, \hat{S})$ is non-empty.

Crystalline $\mathcal{O}^*$-gerbes form a Picard 2-groupoid $\mathcal{G}_{cr}(X)$. The group of equivalence classes of gerbes is $H^2(X_{cr}, \mathcal{O}^*_{cr})$. For a pair of gerbes $\mathcal{C}$, $\mathcal{C}'$ Morphisms $\phi : \mathcal{C} \to \mathcal{C}'$ form a $\mathcal{P}ic(X_{cr})$-Torsor. Here $\mathcal{P}ic(X_{cr})$ is the Picard groupoid of $\mathcal{O}^*_{cr}$-torsors on $X_{cr}$ *).

**7.10.18.** *Remarks.* (i) Let $X_{\acute{e}t\,cr}$ be the small étale crystalline site of $X$ (as a category it equals $X^{(i)}_{cr}$ from 7.10.4, the topology is induced from $X_{cr}$). A crystalline $\mathcal{O}^*$-gerbe on $X$ yields by restriction an $\mathcal{O}^*_{cr}$-gerbe on $X_{\acute{e}t\,cr}$. We leave it to the reader to check that we get an equivalence of the Picard 2-groupoids of gerbes *).

(ii) Our $\mathcal{G}_{cr}(X)$ is the Picard 2-groupoid associated to the complex $\tau_{\leq 2} R\Gamma(X_{cr}, \mathcal{O}^*_{cr}) = \tau_{\leq 2} R\Gamma(X_{\acute{e}t\,cr}, \mathcal{O}^*_{cr})$. To compute $R\Gamma$ look at the canonical ideal $\mathcal{I}_{cr} \subset \mathcal{O}_{cr}$ defined by $(\mathcal{O}_{cr}/\mathcal{I}_{cr})(S, \hat{S}) = \mathcal{O}(S)$. There is a canonical morphism of ringed topologies $i : X_{\acute{e}t} \to X_{\acute{e}t\,cr}$, $i^{-1}(S, \hat{S}) = S$, and $\mathcal{I}_{cr}$ fits into short exact sequence $0 \to \mathcal{I}_{cr} \to \mathcal{O}_{cr} \to i_{\cdot}\mathcal{O}_X \to 0$. Passing

*) i.e., a gerbe over $X_{cr}$ with band $\mathcal{O}^*_{cr}$ in terminology of [De-Mi].

*) If $X$ is smooth then such torsor is the same as a line bundle with flat connection on $X$.

*) We consider $X_{cr}$ as the basic setting since it directly generalizes to the case of ind-schemes, see 7.11.6).

to sheaves of invertible elements we get the short exact sequence

$$0 \longrightarrow \mathcal{I}_{cr} \xrightarrow{\exp} \mathcal{O}^*_{cr} \longrightarrow i.\mathcal{O}^*_X \longrightarrow 0 \tag{342}$$

where exp is the exponential map (since each $\mathcal{I}_{cr}(S,\hat{S})$ is a nilpotent ideal our exp is correctly defined). Since $R\Gamma(X_{\acute{e}t\,cr}, i.\mathcal{O}^*_X) = R\Gamma(X_{\acute{e}t}, \mathcal{O}^*)$. one may use (343) to compute $R\Gamma(X_{cr}, \mathcal{O}^*_{cr})$. For example, since $H^0(X_{cr}, \mathcal{I}_{cr}) = 0$ the group $H^0(X_{cr}, \mathcal{O}^*_{cr})$ is the group $\mathcal{O}^*(X)_{con}$ of locally constant invertible functions on $X$.

(iii) Assume that $X$ is smooth. Set $\Omega_X^{\geq 1} := (0 \to \Omega^1_X \to \Omega^2_X ..)$. According to Grothendieck, one has $R\Gamma(X_{cr}, \mathcal{O}_{cr}) = R\Gamma(X, \Omega_X)$ and $R\Gamma(X_{cr}, \mathcal{I}_{cr}) = R\Gamma(X_{cr}, \mathcal{C}one(\mathcal{O}_{cr} \to i.\mathcal{O}_X)[-1]) = R\Gamma(X, \Omega_X^{\geq 1})$. Thus (342) yields the long cohomology sequence

$$0 \longrightarrow \mathcal{O}^*(X)_{con} \longrightarrow \mathcal{O}^*(X) \xrightarrow{d\,log} \Omega^{1cl}(X) \longrightarrow H^1(X_{cr}, \mathcal{O}^*_{cr}) \longrightarrow$$

$$\longrightarrow Pic(X) \xrightarrow{c_1} H^2(X, \Omega_X^{\geq 1}) \longrightarrow H^2(X_{cr}, \mathcal{O}^*_{cr}) \longrightarrow Br(X) \longrightarrow 0.$$

Here $H^1(X_{cr}, \mathcal{O}^*_{cr})$ is the group of isomorphism classes of line bundles with flat connection on $X$. One has 0 at the right since $H^2(X_{\acute{e}t}, \mathcal{O}^*) = Br(X)$ is a torsion group and $H^3(X_{cr}, \mathcal{I}_{cr})$ is a $\mathbb{C}$-vector space.

(iv) If $X$ is a scheme then one may consider a weaker topology $X_{Zar\,cr}$ (as a category it equals $X^{(ii)}_{cr}$ from 7.10.4). We get the corresponding Picard 2-groupoid $\mathcal{G}_{Zar\,cr}(X)$ of $\mathcal{O}^*_{cr}$-gerbes on $X_{Zar\,cr}$. By étale descent the pull-back functor $\mathcal{G}_{Zar\,cr}(X) \to \mathcal{G}_{cr}(X)$ is a fully faithful Morphism of Picard 2-groupoids, i.e., $\mathcal{G}_{Zar\,cr}(X)$ is the 2-groupoid of Zariski locally trivial crystalline $\mathcal{O}^*$-gerbes. It is easy to see*) that $\mathcal{C} \in \mathcal{G}_{cr}(X)$ belongs to $\mathcal{G}_{Zar\,cr}(X)$ if (and only if) the $\mathcal{O}^*$-gerbe $i^{\cdot}\mathcal{C}$ on $X_{\acute{e}t}$ is Zariski locally trivial. For example, if $X$ is smooth then $H^2(X_{Zar}, \mathcal{O}^*) = 0$, so $\mathcal{G}_{cr}(X)/\mathcal{G}_{Zar\,cr}(X) = Br(X)$.

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*)cf. 7.10.22.

**7.10.19.** Below we give a convenient "concrete" description of (appropriately rigidified) crystalline $\mathcal{O}^*$-gerbes.

Assume we have $X \hookrightarrow V$ as in 7.10.7. For $\mathcal{C} \in \mathcal{G}_{cr}(X)$ and an infinitesimal neighbourhood $X' \subset V$ of $X$ we have the $\mathcal{P}ic(X')$-Torsor $\mathcal{C}(X')$. Set $\mathcal{C}(V) := \varprojlim \mathcal{C}(X')$ (:= the groupoid of Cartesian sections of $\mathcal{C}$ over the directed set of $X'$'s); this is a $\mathcal{P}ic(V)$-Torsor.

Consider pairs $(\mathcal{C}, \mathcal{E}_V)$ where $\mathcal{C} \in \mathcal{G}_{cr}(X)$ and $\mathcal{E}_V \in \mathcal{C}(V)$. Such objects form a Picard groupoid $\mathcal{G}^V_{cr}(X)$. Namely, a morphism $(\mathcal{C}, \mathcal{E}_V) \to (\mathcal{C}', \mathcal{E}'_V)$ is a pair $(F, \nu)$ where $F$ is a Morphism $\mathcal{C} \to \mathcal{C}'$ and $\nu : F(\mathcal{E}_V) \stackrel{\sim}{\rightarrow} \mathcal{E}'_V$ *). We are going to describe $\mathcal{G}^V_{cr}(X)$.

We use notation from 7.10.7. Let $\mathcal{R}$ be a line bundle on $V^{<2>}$ and $\beta : p_{12}^*\mathcal{R} \otimes p_{23}^*\mathcal{R} \stackrel{\sim}{\rightarrow} p_{13}^*\mathcal{R}$ an isomorphism of line bundles on $V^{<3>}$ such that the following diagram of isomorphisms of line bundles on $V^{<4>}$ commutes (associativity condition):

$$
\begin{array}{ccc}
\mathcal{R}_{12} \otimes \mathcal{R}_{23} \otimes \mathcal{R}_{34} & \longrightarrow & \mathcal{R}_{13} \otimes \mathcal{R}_{34} \\
\big\downarrow & & \big\downarrow \\
\mathcal{R}_{12} \otimes \mathcal{R}_{24} & \longrightarrow & \mathcal{R}_{14}
\end{array}
\tag{343}
$$

Here $\mathcal{R}_{ij}$ is the pull-back of $\mathcal{R}$ by projection $p_{ij} : V^{<4>} \to V^{<2>}$ and the arrows come from $\beta$.

Such pairs $(\mathcal{R}, \beta)$ form a Picard groupoid $G(V)$ (with respect to tensor product).

**7.10.20.** *Proposition.* The Picard groupoids $\mathcal{G}^V_{cr}(X)$ and $G(V)$ are canonically equivalent.

*Proof.* For $(\mathcal{C}, \mathcal{E}_V) \in \mathcal{G}^V_{cr}(X)$ set $\mathcal{R} := \mathcal{H}om(p_1^*\mathcal{E}_V, p_2^*\mathcal{E}_V) \in \mathcal{P}ic(V)$ and define $\beta$ as the composition isomorphism; it is clear that $(\mathcal{R}, \beta) \in G(V)$. So we have the Morphism of Picard groupoids $\mathcal{G}^V_{cr}(X) \to G(V)$.

*) Notice that such pairs have no symmetries, so $\mathcal{G}^V_{cr}(X)$ is a plain groupoid (while $\mathcal{G}_{cr}(X)$ is a 2-groupoid).

The inverse Morphism assigns to $(\mathcal{R},\beta)$ the pair $(\mathcal{C},\mathcal{E}_V)$ glued from trivial gerbes by means of $(\mathcal{R},\beta)$. Namely, one defines $(\mathcal{C},\mathcal{E}_V)$ as follows. Since $V$ is formally smooth the structure morphism $j: S\to X$ extends to $j': \hat{S}\to V$. Now $\mathcal{C}(\hat{S})$ is a $\mathcal{P}ic(\hat{S})$-Torsor together with the following extra structure:

(i) For any $j'$ as above we are given an object of $\mathcal{C}(\hat{S})$ denoted by $j'^*\mathcal{E}_V$.

(ii) If $j'': \hat{S}\to V$ is another extension of $j$ then we have an identification of line bundles $\theta_{j''j'}: \mathcal{H}om(j'^*\mathcal{E}_V, j''^*\mathcal{E}_V)\xrightarrow{\sim}(j'',j')^*\mathcal{R}$.

We demand that (ii) identifies composition of $\mathcal{H}om$'s with the isomorphism defined by $\beta$. It is easy to see that such $\mathcal{C}(\hat{S})$ exists and unique (up to a unique equivalence). The fibers $\mathcal{C}(\hat{S})$ glue together to form a crystalline $\mathcal{O}^*$-gerbe in the obvious way. We have $\mathcal{E}_V\in\mathcal{C}(V)$ by construction. $\square$

**7.10.21.** *Remark.* Let $\mathcal{E}'_V$ be another object of $\mathcal{C}(V)$ and $(\mathcal{R}',\beta')\in G(V)$ the pair that corresponds to $(\mathcal{C},\mathcal{E}'_V)$. Set $\mathcal{L}:=\mathcal{H}om(\mathcal{E}_V,\mathcal{E}'_V)\in\mathcal{P}ic(V)$. Then $\mathcal{R}'=\mathrm{Ad}_{\mathcal{L}}\,\mathcal{R}:=(p_2^*\mathcal{L})\otimes\mathcal{R}\otimes(p_1^*\mathcal{L})^{\otimes-1}$ and $\beta'=\mathrm{Ad}_{\mathcal{L}}\,\beta$.

Now let $\mathcal{C}$ be any crystalline $\mathcal{O}^*$-gerbe on $X$, and assume that we have $X\hookrightarrow V$ as above. To use 7.10.20 for description of $\mathcal{C}$ one has to assure that $\mathcal{C}(V)$ is non-empty.

**7.10.22.** *Lemma.* Assume that $X$ is affine and $V$ is a union of countably many subschemes. Then $\mathcal{C}(V)$ is non-empty if*) $\mathcal{C}(X,X)$ is non-empty.

*Proof.* Let $X'\subset V$ be an infinitesimal neighbourhood of $X$. Then any $\mathcal{E}_X\in\mathcal{C}(X,X)$ admits an extension $\mathcal{E}_{X'}\in\mathcal{C}(X,X')$, and all such extensions are isomorphic. Now we have a sequence $X\subset X^{(1)}\subset X^{(2)}$... of infinitesimal neighbourhoods of $X$ such that $V=\varinjlim X^{(n)}$. One defines by induction a sequence $\mathcal{E}_{X^{(n)}}\in\mathcal{C}(X,X^{(n)})$ together with identifications $\mathcal{E}_{X^{(n+1)}}|_{X^{(n)}}=\mathcal{E}_{X^{(n)}}$. This is $\mathcal{E}_V\in\mathcal{C}(V)$. $\square$

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*) and, certainly, only if

**7.10.23.** *Remarks.* (i) Consider the $\mathcal{O}^*$-gerbe $i^{\cdot}\mathcal{C}$ on $X_{\acute{e}t}$ (so $i^{\cdot}\mathcal{C}(U) = \mathcal{C}(U,U)$). Then $\mathcal{C}(X,X) \neq \emptyset$ if and only if $i^{\cdot}\mathcal{C}$ is a trivial gerbe, i.e., its class in $H^2(X_{\acute{e}t}, \mathcal{O}^*) = Br(X)$ vanishes*).

(ii) For any algebraic space $X$ and $\mathcal{C} \in \mathcal{G}_{cr}(X)$ one may use 7.10.20 to describe $\mathcal{C}$ locally on $X_{\acute{e}t}$. Namely, there exists an étale covering $U_i$ of $X$ such that $U_i$ are affine and $\mathcal{C}(U_i, U_i) \neq \emptyset$. Embed $U_i$ into a smooth scheme and take for $V_i$ the corresponding formal completion. Now, by 7.10.22, we may use 7.10.20, 7.10.21 to describe $\mathcal{C}_{U_i}$.

**7.10.24.** *Definition.* For $\mathcal{C} \in \mathcal{G}_{cr}(X)$ a *$\mathcal{C}$-twisted $\mathcal{D}$-crystal on* $X$ is a Cartesian functor $M : \mathcal{C} \to \mathcal{M}^!(X_{cr}, \mathcal{O})$ such that for any $\mathcal{E} \in \mathcal{C}(\hat{S})$ and $f \in \mathcal{O}^*(\hat{S})$ one has $M(f_{\mathcal{E}}) = f \cdot \mathrm{id}_{M(\mathcal{E})}$.

The $\mathcal{C}$-twisted $\mathcal{D}$-crystals form a $\mathbb{C}$-category $\mathcal{M}_{\mathcal{C}}(X)$. It depends on $\mathcal{C}$ in a functorial way (to a Morphism $\mathcal{C} \to \mathcal{C}'$ there corresponds an equivalence of categories $\mathcal{M}_{\mathcal{C}}(X) \rightleftarrows \mathcal{M}_{\mathcal{C}'}(X)$, etc.).

The categories $\mathcal{M}_{\mathcal{C}}(U) = \mathcal{M}_{\mathcal{C}_U}(U)$, $U \in X_{\acute{e}t}$, form a sheaf of categories $\mathcal{M}_{\mathcal{C}}(X_{\acute{e}t})$ over $X_{\acute{e}t}$ in the obvious way.

Let $\mathcal{C}^{triv}$ be the trivialized gerbe, so $\mathcal{C}^{triv}(\hat{S}) = \mathcal{P}ic(\hat{S})$. The $\mathcal{C}^{triv}$-twisted $\mathcal{D}$-crystals are the same as plain $\mathcal{D}$-crystals. Namely, one identifies $M \in \mathcal{M}_{\mathcal{C}^{triv}}(X)$ with the $\mathcal{D}$-crystal $M_{\hat{S}} := M(\mathcal{O}_{\hat{S}})$.

*Remark.* In the above definition we may replace $X_{cr}$ by $X_{\acute{e}t\,cr}$. If $X$ is a scheme and $\mathcal{C} \in \mathcal{G}_{Zar\,cr}(X)$ then we may replace $X_{cr}$ by $X_{Zar\,cr}$. One gets the same category $\mathcal{M}_{\mathcal{C}}(X)$.

**7.10.25.** Here is a twisted version of 7.10.7, 7.10.8. Assume we are in situation 7.10.19, so we have $(\mathcal{C}, \mathcal{E}_V) \in \mathcal{G}_{cr}^V(X)$ and the corresponding $(\mathcal{R}, \beta) \in G(V)$ (see 7.10.20). The category $\mathcal{M}_{\mathcal{C}}(X)$ may be described as follows. Let $\mathcal{M}_{\mathcal{R}}(X) = \mathcal{M}_{\mathcal{R}\beta}(X)$ be the category of pairs $(M_V, \tau)$ where $M_V \in \mathcal{M}(V, \mathcal{O})$ and $\tau : (p_1^! M_V) \otimes \mathcal{R} \rightleftarrows p_2^! M_V$ is an isomorphism

*)This class is the image of the class of $\mathcal{C}$ by the map $H^2(X_{cr}, \mathcal{O}_{cr}^*) \to H^2(X_{cr}, i.\mathcal{O}^*) = Br(X)$.

in $\mathcal{M}(V^{<2>}, \mathcal{O})$ such that*)

$$p_{23}^!(\tau)p_{12}^!(\tau) = p_{13}^!(\tau). \tag{344}$$

**7.10.26.** *Lemma.* The categories $\mathcal{M}_{\mathcal{C}}(X)$ and $\mathcal{M}_{\mathcal{R}}(X)$ are canonically equivalent.

*Proof.* For $M \in \mathcal{M}_{\mathcal{C}}(X)$ set $M_V = M(\mathcal{E}_V) := \bigcup M(\mathcal{E}_{(X,X')})$, and define $\tau$ as composition of the isomorphisms $(p_1^! M_V) \otimes \mathcal{R} = M(p_1^* \mathcal{E}_V) \otimes \mathcal{R} = M((p_1^*\mathcal{E}_V) \otimes \mathcal{R}) = M(p_2^*\mathcal{E}_V) = p_2^! M_V$. The rest is an immediate modification of the proof of 7.10.8. $\square$

**7.10.27.** *Lemma.* For any $X$ and $\mathcal{C} \in \mathcal{G}_{cr}(X)$ the category $\mathcal{M}_{\mathcal{C}}(X)$ is abelian.

*Proof.* An obvious modification of the proof of 7.10.9. Use 7.10.23(ii), 7.10.22, 7.10.26. $\square$

**7.10.28.** From now on we assume that $X$ is a smooth algebraic space. We want to compare the above picture with the usual setting of tdo and twisted $\mathcal{D}$-modules. First let us relate crystalline $\mathcal{O}^*$-gerbes and tdo*).

Look at 7.10.19 for $V = X$. Consider the Picard groupoid $\mathcal{G}_{cr}^{\sim}(X) := \mathcal{G}_{cr}^V(X)$ of pairs $(\mathcal{C}, \mathcal{E}_X)$ where $\mathcal{C}$ is a crystalline $\mathcal{O}^*$-gerbe on $X$ and $\mathcal{E}_X \in \mathcal{C}(X)$.

Here is a convenient interpretation of $\mathcal{G}_{cr}^{\sim}(X)$. Consider $\mathcal{I}_{cr}$-gerbes on $X$ (i.e., $\mathcal{I}_{cr}$-gerbes on $X_{cr}$). Since $H^0(X_{cr}, \mathcal{I}_{cr}) = 0$ these gerbes form a (shifted) Picard groupoid $\mathcal{G}\mathcal{I}_{cr}(X)$. The exponential morphism $\mathcal{I}_{cr} \hookrightarrow \mathcal{O}_{cr}^*$ yields the functor $\exp : \mathcal{G}\mathcal{I}_{cr}(X) \to \mathcal{G}_{cr}(X)$. Since $\mathcal{I}_{(X,X)} = 0$, for any $\mathcal{I}_{cr}$-gerbe $\mathcal{B}$ the groupoid $\mathcal{B}_X$ is trivial, so the groupoid $(\exp \mathcal{B})_X$ has a distinguished object $\mathcal{E}_{\mathcal{B}X}$ (defined up to a canonical isomorphism). Thus we defined a Morphism of Picard groupoids

$$\exp : \mathcal{G}\mathcal{I}_{cr}(X) \longrightarrow \mathcal{G}_{cr}^{\sim}(X), \tag{345}$$

*)We use $\beta$ to identify the modules where the l.h.s. and r.h.s. of the equality lie.

*)see, e.g., [BB93] 2.1 for basic facts about tdo.

$\mathcal{B} \mapsto (\exp \mathcal{B}, \mathcal{E}_{\mathcal{B}X})$. This is an equivalence of Picard groupoids (as follows from (342)).

*Example.* The "boundary map" for (342) yields the morphism of Picard groupoids $c : \mathcal{P}ic(X) \to \mathcal{GI}_{cr}(X)$ (the crystalline Chern class). In terms of (345) it assigns to $\mathcal{L} \in \mathcal{P}ic(X)$ the pair $(\mathcal{C}^{triv}, \mathcal{L})$.

**7.10.29.** *Proposition.* $\mathcal{G}_{cr}^{\sim}(X)$ is canonically equivalent to the Picard groupoid $\mathcal{TDO}(X)$ of tdo's on $X$.

*Proof.* Let us identify, according to 7.10.20 for $V = X$, our $\mathcal{G}_{cr}^{\sim}(X)$ with $G(X)$. Now for $(\mathcal{R}, \beta) \in G(X)$ the corresponding tdo $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{(\mathcal{R},\beta)}$ is defined as follows. We use notation from 7.10.13. Consider $\mathcal{D}_X$ as a Diff-bimodule (an object of $\mathcal{M}(X^{<2>}, \mathcal{O})$). Set $\mathcal{D}_{\mathcal{R}} := \mathcal{D}_X \underset{\mathcal{O}_{X^{<2>}}}{\otimes} \mathcal{R}$. The multiplication morphism $m_{\mathcal{R}} : \mathcal{D}_{\mathcal{R}} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{\mathcal{R}} \to p_{13}^! \mathcal{D}_{\mathcal{R}}$ is the tensor product of the corresponding morphism for $\mathcal{D}_X$ and $\beta$. One checks easily that $\mathcal{D}_{\mathcal{R}}$ is a tdo and $G(X) \to \mathcal{TDO}(X)$, $(\mathcal{R}, \beta) \mapsto \mathcal{D}_{\mathcal{R}}$ is a Morphism of Picard groupoids.

The inverse Morphism assigns to a tdo $A$ on $X$ the object $(\mathcal{R}, \beta)$ where $\mathcal{R} := \mathcal{H}om_{\mathcal{O}_{X^{<2>}}}(\mathcal{D}_X, A)$ and $\beta$ is defined by comparison of the multiplication morphisms $m$ for $\mathcal{D}_X$ and $A$. We leave the details for the reader. $\square$

**7.10.30.** *Remark.* Here is another (equivalent) way to spell out the above equivalence. By (345) $\mathcal{G}_{cr}^{\sim}(X)$ is equivalent to $\mathcal{GI}_{crys}(X)$, i.e., to the Picard groupoid associated with complex $\tau_{\leq 1}(R\Gamma(X_{crys}, \mathcal{I}_{X_{crys}})[1])$. According to [BB93] 2.1.6, 2.1.4, $\mathcal{TDO}(X)$ is the Picard groupoid associated with the complex $\tau_{\leq 1}(R\Gamma(X, \Omega_X^{\geq 1})[1])$. Now the above complexes are canonically quasi-isomorphic (see 7.10.18(iii)).

**7.10.31.** Here is a twisted version of 7.10.12. For $(\mathcal{C}, \mathcal{E}_X) \in \mathcal{G}_{cr}^{\sim}(X)$ consider the corresponding $(\mathcal{R}, \beta) \in G(X)$ and the tdo $\mathcal{D}_{\mathcal{R}}$. Take $M \in \mathcal{M}_{\mathcal{C}}(X)$. According to 7.10.26 we may consider $M$ as pair $(M_X, \tau) \in \mathcal{M}_{\mathcal{R}}(X)$.

Since[*)] $p_1^! M_X = M_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ and $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_X \underset{\mathcal{O}_{X^{<2>}}}{\otimes} \mathcal{R}$ we may rewrite $\tau$ as an isomorphism

$$M_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{\mathcal{R}} \xrightarrow{\sim} p_2^! M_X \tag{346}$$

in $\mathcal{M}(X^{<2>}, \mathcal{O})$. By adjunction, one may consider (346) as a morphism of $\mathcal{O}_X$-modules

$$M_X \otimes \mathcal{D}_{\mathcal{R}} \to M_X. \tag{347}$$

Denote by $\mathcal{M}^r(X, \mathcal{D}_{\mathcal{R}})$ the category of right $\mathcal{D}_{\mathcal{R}}$-modules on $X$.

**7.10.32.** *Proposition.* The morphism (347) is a right unital action of $\mathcal{D}_{\mathcal{R}}$ on $M_X$. The functor $\mathcal{M}_{\mathcal{C}}(X) \to \mathcal{M}^r(X, \mathcal{D}_{\mathcal{R}})$, $M \mapsto M_X$, is an equivalence of categories.

*Proof.* Left to the reader (see 7.10.12, 7.10.13). □

**7.11. $\mathcal{D}$-modules on ind-schemes.** In this section we discuss $\mathcal{D}$-module theory on formally smooth ind-schemes. Notice that the $\mathcal{D}$-crystal picture (see 7.10) makes immediate sense in the ind-scheme setting, and it is the conventional approach (differential operators, etc.) that takes some space to be written down.

**7.11.1.** An *ind-scheme* (in the strict sense) $X$ is a "space" (i.e., a set valued functor on the category of commutative $\mathbb{C}$-algebras $A \mapsto X(A) = X(\operatorname{Spec} A)$) which may be represented as $\varinjlim X_\alpha$ where $\{X_\alpha\}$ is a directed family of quasi-compact schemes such that all the maps $i_{\alpha\beta} : X_\alpha \to X_\beta$, $\alpha \leq \beta$, are closed embeddings. If $X$ can be represented as above so that the set of indices $\alpha$ is countable then $X$ is said to be an $\aleph_0$-*ind-scheme.*[*)] If $P$

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[*)]See the proofs of 7.10.12 and 7.10.29.

[*)]Not all natural examples of ind-schemes are $\aleph_0$-ind-schemes; e.g., for every infinite-dimensional vector space $V$ the functor $A \mapsto \operatorname{End}_A(V \otimes A)$ is an ind-scheme but not an $\aleph_0$-ind-scheme.

is a property of schemes stable under passage to closed subschemes then we say that $X$ satisfies the *ind-$P$* property if each $X_\alpha$ satisfies $P$.

Set $X_{\text{red}} := \varinjlim X_{\alpha\,\text{red}}$; an ind-scheme $X$ is said to be *reduced* if $X_{\text{red}} = X$.

A *formal scheme* is an ind-scheme $X$ such that $X_{\text{red}}$ is a scheme (see 7.12.17 for a discussion of the relation between this definition of formal scheme and the one from EGA). An $\aleph_0$-*formal scheme* is a formal scheme which is an $\aleph_0$-ind-scheme. The *completion* of an ind-scheme $Z$ along a closed subscheme $Y \subset Z$ is the direct limit of closed subschemes $Y' \subset Z$ such that $Y'_{\text{red}} = Y_{\text{red}}$. In the case of formal schemes we write "affine" instead of "ind-affine". A formal scheme $X$ is affine if and only if $X_{\text{red}}$ is affine.

Following Grothendieck ([Gr64], [Gr67]), we say that $X$ is *formally smooth* if for every $A$ and every nilpotent ideal $I \subset A$ the map $X(A) \to X(A/I)$ is surjective. It is easy to see that for ind-schemes of ind-finite type formal smoothness is a local property (cf. the proof of Proposition 17.1.6 from [Gr67]).*) A morphism $X \to Y$ is said to be *formally smooth* if for any $A$, $I$ as above the map from $X(A)$ to the fiber product of $Y(A)$ and $X(A/I)$ over $Y(A/I)$ is surjective.

Let $X$ be an ind-scheme. A closed quasi-compact subscheme $Y \subset X$ is called *reasonable* if for any closed subscheme $Z \subset X$ such that $Y \subset Z$ the ideal of $Y$ in $\mathcal{O}_Z$ is finitely generated. We say that $X$ is *reasonable* if $X$ is a union of its reasonable subschemes, i.e., it may be represented as $\varinjlim X_\alpha$ where all $X_\alpha$ are reasonable. A closed subspace of a reasonable ind-scheme is a reasonable ind-scheme; a product of two reasonable ind-schemes is reasonable.

*)We do not know whether this is true for ind-schemes that are not of ind-finite type. For schemes the answer is "yes". This follows from Remark 9.5.8 in [Gr68a] and the following surprising result ([RG], p.82, 3.1.4): the property of being a projective module is local for the Zariski topology and even for the fpqc topology (without any finiteness assumptions!).

*Remark.* Replacing the word "schemes" in the above definition by "algebraic spaces" we get the notion of an ind-algebraic space. All the discussion passes automatically to the setting of ind-algebraic spaces.

**7.11.2.** *Examples.* (i) An ind-affine ind-scheme $X$ is the same as a pro-algebra, i.e., a pro-object $R$ of the category of commutative algebras that can be represented as $\varprojlim R_\alpha$ so that the maps $R_\beta \to R_\alpha$, $\beta \geq \alpha$, are surjective. We write $X = \operatorname{Spf} R := \varinjlim \operatorname{Spec} R_\alpha$. A complete topological commutative algebra $R$ whose topology is defined by open ideals $I_\alpha \subset R$ can be considered as a pro-algebra (set $R_\alpha := R/I_\alpha$). Not all pro-algebras are of this type because if the set of indices $\alpha$ is uncountable then the map from the set-theoretical projective limit of the $R_\alpha$ to $R_{\alpha_0}$ is not necessarily surjective*). Of course, an ind-affine $\aleph_0$-ind-scheme is the same as a complete topological algebra whose topology is defined by a countable or finite system of open ideals of $R$.

(ii) Let $V$ be a Tate vector space (see 4.2.13). Then $V$ (or, more precisely, the functor $A \mapsto V \widehat{\otimes} A$) is a reasonable ind-affine ind-scheme. Indeed, every c-lattice in $V$ is an affine scheme. One has $V = \operatorname{Spf} R$ where $R = \varprojlim \operatorname{Sym}(U_\alpha^*)$, $U_\alpha$ runs over the set of c-lattices in $V$.

If $X$ is a reasonable ind-scheme then for $x \in X(\mathbb{C})$ the tangent space $\Theta_x$ of $X$ at $x$ is a Tate vector space: the topology of $\Theta_x$ is defined by tangent spaces at $x$ of reasonable subschemes of $X$ that contain $x$. So if $H$ is a reasonable group ind-scheme then its Lie algebra $\operatorname{Lie} H$ is a Lie algebra in the category of Tate vector spaces.

(iii) For $V$ as above denote by $Gr(V)$ the "space" of c-lattices in $V$. More precisely, $Gr(V)$ is the functor that assigns to a commutative algebra $A$ the set of c-lattices in $V \widehat{\otimes} A$ (in the sense of 4.2.14). Clearly $Gr(V)$ is an ind-proper formally smooth ind-scheme (indeed, it is a union of the Grassmannians of $U_2/U_1$'s for all pairs of c-lattices $U_1 \subset U_2 \subset V$).

*)even if the maps $R_\beta \to R_\alpha$, $\beta \geq \alpha$, are surjective (as we assume).

(iv) Let $K$ be a local field, $O \subset K$ the corresponding local ring (so $K \simeq \mathbb{C}((t))$, $O \simeq \mathbb{C}[[t]]$). For any "space" $Y$ we have "spaces" $Y(O) \subset Y(K)$ defined as $Y(O)(A) := Y(A \widehat{\otimes} O)$, $Y(K)(A) = Y(A \widehat{\otimes} K)$ (here $A \widehat{\otimes} O = A[[t]]$, $A \widehat{\otimes} K = A((t))$). Assume that $Y$ is an affine scheme. Then $Y(O)$ is also an affine scheme, and $Y(K)$ is an ind-affine $\aleph_0$-ind-scheme. If $Y$ is of finite type then $Y(K)$ is reasonable. If $Y$ is smooth then $Y(O)$ and $Y(K)$ are formally smooth.

Let $G$ be an affine algebraic group, $\mathfrak{g}$ its Lie algebra. Consider the group ind-scheme $G(K)$. One has $\mathrm{Lie}(G(K)) = \mathfrak{g}(K) = \mathfrak{g} \otimes K$, $\mathrm{Lie}(G(O)) = \mathfrak{g}(O) = \mathfrak{g} \otimes O$.

(v) Let $G$ be a reasonable group ind-scheme such that $G_{\mathrm{red}}$ is an affine group scheme. The functor $G \mapsto (\mathrm{Lie}\, G, G_{\mathrm{red}})$ is an equivalence between the category of $G$'s as above and the category of Harish-Chandra pairs. For an ind-scheme $X$ an action of $G$ on $X$ is the same as a $(\mathrm{Lie}\, G, G_{\mathrm{red}})$-action on $X$. Similarly, a $G$-module is the same as a $(\mathrm{Lie}\, G, G_{\mathrm{red}})$-module, etc.

**7.11.3.** There are two different ways to define $\mathcal{O}$-modules in the setting of ind-schemes; the corresponding objects are called $\mathcal{O}^p$-modules and $\mathcal{O}^!$-modules. We start with the more immediate (though less important) notion of $\mathcal{O}^p$-module*) which makes sense for any "space" $X$ (see 7.11.1).

An *$\mathcal{O}^p$-module $P$ on $X$* is a rule that assigns to a commutative algebra $A$ and a point $\phi \in X(A)$ an $A$-module $P_\phi$, and to any morphism of algebras $f : A \to B$ an identification of $B$-modules $f_P : B \underset{f}{\otimes} P_\phi \stackrel{\sim}{\rightarrow} P_{f\phi}$ in a way compatible with composition of $f$'s. If $X = \underset{\longrightarrow}{\lim} X_\alpha$ is an ind-scheme then such $P$ is the same as a collection of (quasi-coherent) $\mathcal{O}$-modules $P_{X_\alpha}$ on $X_\alpha$ together with identifications $i^*_{\alpha\beta} P_{X_\beta} = P_{X_\alpha}$ for $\alpha \leq \beta$ that satisfy the obvious transitivity property. We say that $P$ is flat if each $P_\phi$ (or each $P_{X_\alpha}$) is flat. One defines invertible $\mathcal{O}^p$-modules on $X$ (alias line bundles) in the similar way.

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*) Here "p" stands for "projective limit".

We denote the category of $\mathcal{O}^p$-modules on $X$ by $\mathcal{M}^p(X,\mathcal{O})$. This is a tensor $\mathbb{C}$-category. The unit object in $\mathcal{M}^p(X,\mathcal{O})$ is the "sheaf" of functions $\mathcal{O}_X$. Note that $\mathcal{M}^p(X,\mathcal{O})$ need not be an abelian category. The category $\mathcal{M}^{p\,fl}(X,\mathcal{O})$ of flat $\mathcal{O}^p$-modules is an exact category (in Quillen's sense).

For any $P,P' \in \mathcal{M}^p(X,\mathcal{O})$ the vector space $\mathrm{Hom}(P,P')$ carries the obvious topology; the composition of morphisms is continuous. In particular $\Gamma(X,P) := \mathrm{Hom}(\mathcal{O}_X,P)$ is a topological vector space which is a module over the topological ring $\Gamma(X,\mathcal{O}_X)$.

*Remarks.* (i) The above definitions makes sense if we replace $\mathcal{O}$-modules by any category fibered over the category of affine schemes. For example, one can consider left $\mathcal{D}$-modules (alias $\mathcal{O}$-modules with integrable connection); the corresponding objects over ind-schemes called *(left) $\mathcal{D}^p$-modules*.

(ii) If $X$ is an ind-affine $\aleph_0$-ind-scheme, $X = \mathrm{Spf}\, R = \varinjlim \mathrm{Spec}\, R/I_\alpha$ (see 7.11.2(i)), then an $\mathcal{O}^p$-module on $X$ is the same as a complete and separated topological $R$-module $P$ such that the closures of $I_\alpha P \subset P$ form a basis of the topology.

**7.11.4.** Now let us pass to $\mathcal{O}^!$-modules. Here we must assume that our $X$ is a reasonable ind-scheme. An *$\mathcal{O}^!$-module $M$ on $X$* is a rule that assigns to a reasonable subscheme $Y \subset X$ a quasi-coherent $\mathcal{O}_Y$-module $M_{(Y)}$ together with morphisms $M_{(Y)} \to M_{(Y')}$ for $Y \subset Y'$ which identify $M_{(Y)}$ with $i^!_{YY'}M_{(Y')} := \mathrm{Hom}_{\mathcal{O}_{Y'}}(\mathcal{O}_Y, M_{(Y')})$ and satisfy the obvious transitivity condition*). If we write $X = \varinjlim X_\alpha$ where $X_\alpha$'s are reasonable then it suffices to consider only $X_\alpha$'s instead of all reasonable subschemes. $\mathcal{O}^!$-modules on $X$ form an abelian category $\mathcal{M}(X,\mathcal{O})$. Note that for any closed subscheme $Y \subset X$, the category $\mathcal{M}(Y,\mathcal{O})$ is a full subcategory of $\mathcal{M}(X,\mathcal{O})$ closed under subquotients, and that for any $\mathcal{O}^!$-module $M$ one has $M = \varinjlim M_{(X_\alpha)}$.

The category $\mathcal{M}(X,\mathcal{O})$ is a Module over the tensor category $\mathcal{M}^p(X,\mathcal{O})$. Namely, for $M \in \mathcal{M}(X,\mathcal{O})$, $P \in \mathcal{M}^p(X,\mathcal{O})$ their tensor product $M \otimes P \in$

*) We need to consider reasonable subschemes to assure that $i^!$ preserves quasi-coherency.

$\mathcal{M}(X,\mathcal{O})$ is $\varinjlim M_{(X_\alpha)} \underset{\mathcal{O}_{X_\alpha}}{\otimes} P_{X_\alpha}$. The functor $\otimes : \mathcal{M}(X,\mathcal{O}) \times \mathcal{M}^{pfl}(X,\mathcal{O}) \to \mathcal{M}(X,\mathcal{O})$ is biexact.

For an $\mathcal{O}^!$-module $M$ we define the space of its global sections $\Gamma(X,M)$ as $\varinjlim \Gamma(X_\alpha, M_{(X_\alpha)})$. The functor $\Gamma(X,\cdot)$ is left exact.

*Remarks.* (i) The categories $\mathcal{M}(Y,\mathcal{O})$ together with the functors $i^!_{YY'}$ form a fibered category over the category (ordered set) of reasonable subschemes of $X$, and $\mathcal{M}(X,\mathcal{O})$ is the category of its Cartesian sections.

(ii) If $X = \operatorname{Spf} R$ and the pro-algebra $R$ is a topological algebra (see 7.11.2) then an $\mathcal{O}^!$-module on $X$ is the same as a discrete $R$-module (where "discrete" means that the $R$-action is continuous with respect to the discrete topology on $M$).

(iii) If $P$ is flat then $(M \otimes P)_{(X_\alpha)} = M_{(X_\alpha)} \otimes P_{X_\alpha}$.

**7.11.5.** Assume that we have a group ind-scheme (or any group "space") $K$ that acts on $X$. Then for any commutative algebra $A$ the group $K(A)$ acts on $\operatorname{Spec} A \times X$. For $M \in \mathcal{M}(X,\mathcal{O})$ an *action of* $K$ *on* $M$ is defined by $K(A)$-actions on $\mathcal{O}_{\operatorname{Spec} A} \boxtimes M \in \mathcal{M}(\operatorname{Spec} A \times X, \mathcal{O})$ such that for any morphism $A \to A'$ the corresponding actions are compatible. We denote the category of $K$-equivariant $\mathcal{O}^!$-modules on $X$ by $\mathcal{M}(K \setminus X, \mathcal{O})$. We leave it to the reader to define $K$-equivariant $\mathcal{O}^p$-modules.

**7.11.6.** All the basic definitions and results of 7.10 (the definitions of topology $X_{cr}$, $\mathcal{D}$-crystals, crystalline $\mathcal{O}^*$-torsors, twisted $\mathcal{D}$-crystals, basic functoriality) make obvious sense for any ind-scheme $X$ of ind-finite type. So, from the $\mathcal{D}$-crystalline point of view, $\mathcal{D}$-module theory generalizes automatically to the setting of ind-schemes.

What we will discuss in the rest of this section is the conventional approach to $\mathcal{D}$-modules (rings of differential operators, etc.) which works when our ind-scheme is formally smooth. The results 7.10.12, 7.10.29, 7.10.32 comparing the $\mathcal{D}$-crystalline and $\mathcal{D}$-module setting remain literally true for formally smooth ind-schemes.

Below we will no more mention $\mathcal{D}$-crystals. In the main body of this book we employ conventional $\mathcal{D}$-modules (the ind-schemes we meet are affine Grassmannians, they are formally smooth). Notice, however, that $\mathcal{D}$-crystal approach is needed to make obvious the following fact (we use it for $Y$ equal to a Schubert cell): Let $i : Y \hookrightarrow X$ be a closed embedding of a scheme $Y$ of finite type into formally smooth $X$ as above. Then the category of $\mathcal{D}$-modules on $X$ supported (set-theoretically) on $Y$ depends only on $Y$ (and not on $i$ and $X$). Indeed, this category identifies canonically with the category of $\mathcal{D}$-crystals on $X$.

**7.11.7.** Let us explain what are differential operators in the setting of ind-schemes. Assume that our $X$ is an ind-scheme of ind-finite type. For an $\mathcal{O}^!$-module $M$ on $X$ set

$$\mathrm{Der}(\mathcal{O}_X, M) := \varinjlim \mathrm{Der}(\mathcal{O}_Y, M_{(Y)}) = \varinjlim \mathrm{Hom}(\Omega_Y, M_{(Y)}). \tag{348}$$

Here $Y$ is a closed subscheme of $X$. We consider $\mathrm{Der}(\mathcal{O}_X, M)$ as an $\mathcal{O}^!$-module on $X$. Similarly, set

$$\mathcal{D}(M) = \mathrm{Diff}(\mathcal{O}_X, M) := \varinjlim \mathrm{Diff}(\mathcal{O}_Y, M_{(Y)}). \tag{349}$$

We consider the sheaf of differential operators $\mathrm{Diff}(\mathcal{O}_Y, M_{(Y)})$ as a "differential $\mathcal{O}_Y$-bimodule" in the sense of [BB93], i.e., an $\mathcal{O}$-module on $Y \times Y$ supported set-theoretically on the diagonal. So $\mathcal{D}(M)$ is an $\mathcal{O}^!$-module on $X \times X$ supported set-theoretically on the diagonal. We may consider it as an $\mathcal{O}^!$-module on $X$ with respect to either of the two $\mathcal{O}_X$-module structures. Note that $\mathcal{D}(M)$ carries a canonical increasing filtration $\mathcal{D}_\cdot(M)$ where $\mathcal{D}_i(M)$ is the submodule of sections supported on the $i^{th}$ infinitesimal neighbourhood of the diagonal; equivalently, $\mathcal{D}_i(M) = \varinjlim \mathrm{Diff}_i(\mathcal{O}_Y, M_{(Y)})$ is the submodule of differential operators of order $\leq i$. One has $\mathcal{D}_0(M) = M$, $\bigcup \mathcal{D}_i(M) = \mathcal{D}(M)$, and the two $\mathcal{O}^!$-module structures on $\mathrm{gr}_i\, \mathcal{D}(M)$ coincide. There is an obvious embedding $\mathrm{Der}(\mathcal{O}_X, M) \subset \mathcal{D}_1(M)$.

Assume now that $X$ is formally smooth. In the next proposition we consider $\mathcal{D}(M)$ as an $\mathcal{O}^!$-module on $X$ with respect to the left $\mathcal{O}$-module structure.

**7.11.8.** *Proposition.* (i) The functors $\mathrm{Der}(\mathcal{O}_X, \cdot)$, $\mathcal{D}$, $\mathcal{D}_i$ are exact and commute with direct limits. So there are flat $\mathcal{O}^p$-modules $\Theta_X$, $\mathcal{D}_X$ and a filtration of $\mathcal{D}_X$ by flat submodules $\mathcal{D}_{iX}$ such that

$$\mathrm{Der}(\mathcal{O}_X, M) = M \otimes \Theta_X,\ \mathcal{D}(M) = M \otimes \mathcal{D}_X,\ \mathcal{D}_i(M) = M \otimes \mathcal{D}_{iX}.$$

(ii) There is a canonical identification $\mathrm{gr}.\, \mathcal{D}_X = \mathrm{Sym}^{\cdot}\, \Theta_X$.

*Remark.* In 7.12.12 we will show that the $\mathcal{O}^p$-modules $\Theta_X$, $\mathcal{D}_X$, and $\mathcal{D}_{iX}$ are Mittag-Leffler modules in the sense of Raynaud-Gruson (see 7.12.1, 7.12.2, 7.12.9). If $X$ is an $\aleph_0$-ind-scheme the restrictions of these $\mathcal{O}^p$-modules to subschemes of $X$ are locally free (see 7.12.13 for a more precise statement).

*Proof.* (i) Our functors are obviously left exact and commute with direct limits. The right exactness of $\mathrm{Der}(\mathcal{O}_X, \cdot)$ follows from formal smoothness of $X$ (use the standard interpretation of derivations $\mathcal{O}_X \to M$ as morphisms $\mathrm{Spec}(\mathrm{Sym}^{\cdot} M / \mathrm{Sym}^{\geq 2} M) \to X$). So we have our $\Theta_X \in \mathcal{M}^{p\,fl}(X, \mathcal{O})$.

(ii) We define a canonical isomorphism*)

$$\sigma.: \mathrm{gr}.\, \mathcal{D}(M) \xrightarrow{\sim} M \otimes \mathrm{Sym}^{\cdot}\, \Theta_X. \tag{350}$$

This clearly implies the proposition.

Notice that for any $n \geq 0$ the obvious morphism $M \otimes \Theta_X^{\otimes n} \to \varinjlim \mathrm{Hom}(\Omega_Y^{\otimes n}, M_{(Y)})$ is an isomorphism (use the fact that $\Omega_Y$ are coherent). Therefore (350) is equivalent to identifications

$$\sigma_n : \mathrm{gr}_n\, \mathcal{D}(M) \xrightarrow{\sim} \varinjlim \mathrm{Hom}(\mathrm{Sym}^n\, \Omega_Y, M_{(Y)}). \tag{351}$$

*) In the general case (when the base field may have non-zero characteristic) one has to replace $\mathrm{Sym}^{\cdot}$ by $\Gamma^{\cdot}$ where for any flat $A$-module $P$ we define $\Gamma^n(P)$ as $S_n$-invariants in $P^{\otimes n}$. Notice that (since $P$ is inductive limit of projective modules) $\Gamma^n(P)$ is flat and for any $A$-module $M$ one has $(M \otimes P^{\otimes n})^{S_n} = M \otimes \Gamma^n(P)$.

Our $\sigma_n$ is the inductive limit of the maps

$$\sigma_{nY} : \operatorname{gr}_n \operatorname{Diff}(\mathcal{O}_Y, M_{(Y)}) \to \operatorname{Hom}(\operatorname{Sym}^n \Omega_Y, M_{(Y)})$$

defined as follows. One has $\operatorname{Diff}_n(\mathcal{O}_Y, M_{(Y)}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}, M_{(Y)})$ where $\mathcal{I} \subset \mathcal{O}_{Y\times Y}$ is the ideal of the diagonal (and we consider the source as an $\mathcal{O}_Y$-module via one of the projection maps). Now $\mathcal{I}/\mathcal{I}^2 = \Omega_Y$ hence $\mathcal{I}^n/\mathcal{I}^{n+1}$ is a quotient of $\operatorname{Sym}^n \Omega_Y$, and our $\sigma_{nY}$ comes from the map $\operatorname{Sym}^n \Omega_Y \to \mathcal{I}^n/\mathcal{I}^{n+1} \subset \mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}$.

It remains to show that $\sigma_n$ is an isomorphism; we may assume that $n \geq 1$. It is clear that $\sigma_{nY}$ are injective, hence such is $\sigma_n$. To see that $\sigma_n$ is surjective look at the scheme $Z := \operatorname{Spec}(\operatorname{Sym}^{\cdot} \Omega_Y / \operatorname{Sym}^{\geq n+1} \Omega_Y)$. The embedding of its subscheme $\operatorname{Spec}(\operatorname{Sym}^{\cdot} \Omega_Y / \operatorname{Sym}^{\geq 2} \Omega_Y) = \operatorname{Spec}(\mathcal{O}_{Y\times Y}/\mathcal{I}^2) \subset Y \times Y \subset Y \times X$ extends, by formal smoothness of $X$, to a morphism $i : Z \to Y \times X$ over $Y$. It is easy to see that $i$ is a closed embedding. There is a closed subscheme $Y' \subset X$ such that $Y \subset Y'$ and $Z \subset Y \times Y'$. Thus $Z$ is a subscheme of the $n^{th}$ infinitesimal neighbourhood of the diagonal in $Y' \times Y'$. Therefore we get embeddings $\operatorname{Hom}(\operatorname{Sym}^n \Omega_Y, M_{(Y)}) \subset \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Z, M_{(Y)}) \subset \operatorname{Diff}_n(\mathcal{O}_{Y'}, M_{(Y')})$. The composition of them with $\sigma_{nY'}$ coincides with the embedding $\operatorname{Hom}(\operatorname{Sym}^n \Omega_Y, M_{(Y)}) \subset \operatorname{Hom}(\operatorname{Sym}^n \Omega_{Y'}, M_{(Y')})$. This implies surjectivity of $\sigma_n$. $\square$

**7.11.9.** To explain what are $\mathcal{D}$-modules on ind-schemes it is convenient to use the language of differential bimodules.

Let $X$ be any reasonable ind-scheme. A Diff*-bimodule $D$ on $X$* (cf. [BB93]) is a rule that assigns to any reasonable subscheme $Y \subset X$ an $\mathcal{O}^!$-module $D_Y$ on $Y \times X$ supported set-theoretically on the diagonal $Y \subset Y \times X$; for $Y \subset Y'$ one has identifications $D_{Y'} \otimes \mathcal{O}_Y \xrightarrow{\sim} D_Y$ which are transitive in the obvious sense.

The category $\mathcal{M}^{di}(X, \mathcal{O})$ of Diff-bimodules is a monoidal $\mathbb{C}$-category. Namely, for $D, D' \in \mathcal{M}^{di}(X, \mathcal{O})$ their tensor product $D \otimes D'$ is defined by $(D \otimes D')_Y := \varinjlim (D_Y)_{(Y\times Y')} \underset{\mathcal{O}_{Y'}}{\otimes} D'_{Y'}$. Our $\mathcal{O}_X$ is the unit object in

$\mathcal{M}^{di}(X,\mathcal{O})$ (see Remark (i) below). The category $\mathcal{M}(X,\mathcal{O})$ is a *right* $\mathcal{M}^{di}(X,\mathcal{O})$-Module: for an $\mathcal{O}^!$-module $M$ one has $M \otimes D = \varinjlim M_{(Y)} \otimes D_Y$ where we consider $M_{(Y)} \otimes D_Y$ as an $\mathcal{O}^!$-module on $X$ with respect to the right $\mathcal{O}^!$-module structure on $D_Y$.

*Remarks.* (i) An $\mathcal{O}^p$-module on $X$ is the same as a differential $\mathcal{O}_X$-bimodule supported scheme-theoretically on the diagonal. So we have a fully faithful embedding of monoidal categories $\mathcal{M}^p(X,\mathcal{O}) \subset \mathcal{M}^{di}(X,\mathcal{O})$. It is compatible with the Actions on $\mathcal{M}(X,\mathcal{O})$ from 7.11.4, 7.11.9.

(ii) The forgetful*) functor $\mathcal{M}^{di}(X,\mathcal{O}) \to \mathcal{M}^p(X,\mathcal{O})$ is faithful, so one may consider Diff-bimodules as $\mathcal{O}^p$-modules on $X$ equipped with certain extra structure. We say that a Diff-bimodule is flat if it is flat as an $\mathcal{O}^p$-module. The category of flat Diff-bimodules is an exact category (cf. 7.11.3).

A Diff*-algebra on* $X$ is a unital associative algebra $D$ in the monoidal category $\mathcal{M}^{di}(X,\mathcal{O})$. A $D^!$*-module on* $X$ is a (necessarily right) $D$-module $M$ in $\mathcal{M}(X,\mathcal{O})$. Often we call such $M$ simply a $D$-module. We denote the category of $D$-modules by $\mathcal{M}(X,D)$; this is an abelian category.

*Remarks.* (i) The forgetful functor $\mathcal{M}(X,D) \to \mathcal{M}(X,\mathcal{O})$ admits a left adjoint functor, namely $M \mapsto M \otimes D$.

(ii) The category $\mathcal{M}^p(X,\mathcal{O})$ is a *left* $\mathcal{M}^{di}(X,\mathcal{O})$-module in the obvious way. So one may consider $D^p$-modules := left $D$-modules in $\mathcal{M}^p(X,\mathcal{O})$.

For $D \in \mathcal{M}^{di}(X,\mathcal{O})$ set $\Gamma(X,D) := \varprojlim \Gamma(Y \times X, D_Y)$; this is a topological vector space. One has an obvious continuous map $\Gamma(X,D) \otimes \Gamma(X,D') \to \Gamma(X, D \otimes D')$. For $M \in \mathcal{M}(X,\mathcal{O})$ there is a similar map $\Gamma(X,M) \otimes \Gamma(X,D) \to \Gamma(X, M \otimes D)$. Therefore for a Diff-algebra $D$ our $\Gamma(X,\mathcal{D})$ is a topological ring and for any $D$-module $M$ the vector space $\Gamma(X,M)$ is a discrete $\Gamma(X,D)$-module.

Assume that we have a group ind-scheme (or any group "space") $K$ that acts on $X$. One defines a *weak**) *action of* $K$ on a Diff-algebra $D$ as follows.

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*)forgetting the right $\mathcal{O}$-module structure

*)For strong actions see [BB93].

For any commutative algebra $A$ we have the action of the group $K(A)$ on $\operatorname{Spec} A \times X$. Now a weak action of $K$ on $D$ is a rule that assigns to $A$ a lifting of this action to the Diff-algebra $\mathcal{O}_{\operatorname{Spec} A} \boxtimes D$ on $\operatorname{Spec} A \times X$. For any morphism $A \to A'$ the correspondings actions must be compatible in the obvious way. If $M$ is a $D$-module then a *weak action of $K$ on $M$* is an action of $K$ on $M$ as on $\mathcal{O}^!$-module (see 7.11.4) such that the $D$-action morphism $M \otimes D \to M$ is compatible with the $K$-actions. We denote the category of weakly $K$-equivariant $D$-modules by $\mathcal{M}(K \setminus X, D)$.

**7.11.10.** Here is a more concrete "sheaf-theoretic" way to look at differential bimodules and algebras on a reasonable $\aleph_0$-ind-scheme $X$.[*)] We explain it in two steps.

(i) Assume that $X_{\text{red}}$ is a scheme, so $X$ is a formal scheme[*)]. Then the underlying topological space of $X$ is well-defined, and $\mathcal{O}_X$ is a sheaf of topological algebras. Any Diff-bimodule $D$ yields a sheaf of topological $\mathcal{O}_X$-bimodules $\varprojlim D_{X_\alpha}$ which we denote also by $D$ by abuse of notation. It satisfies the following properties:

- The basis of the topology on $D$ is formed by closures of $\mathcal{I}{\cdot}D$, where $\mathcal{I} \subset \mathcal{O}_X$ is an open ideal; the topology is complete and separated.

- The quotients $D/\mathcal{I}{\cdot}D$ are $\mathcal{O}^!$-modules on $X \times X$ supported set-theoretically at the diagonal.

It is clear that $\mathcal{M}^{di}(X, \mathcal{O})$ is equivalent to the category of such sheaves of topological $\mathcal{O}_X$-bimodules. Notice that $D \otimes D' = D \underset{\mathcal{O}_X}{\widehat{\otimes}} D'$. Therefore a Diff-algebra on $X$ is the same as a sheaf $D$ of topological algebras on $X$ equipped with a continuous morphism of sheaves of algebras $\epsilon : \mathcal{O}_X \to D$ such that the $\mathcal{O}_X$-bimodule structure on $D$ satisfies the above conditions.

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[*)] The $\aleph_0$ assumption enables us to work with topological algebras instead of pro-algebras; see 7.11.2(i).

[*)] See 7.12.22 and 7.12.23 for a description of formally smooth affine $\aleph_0$-formal schemes of ind-finite type.

A $D$-module on $X$ is the same as a sheaf of discrete right $D$-modules which is quasi-coherent as an $\mathcal{O}_X$-module (i.e., it is an $\mathcal{O}^!$-module on $X$).

(ii) Let $X$ be any reasonable $\aleph_0$-ind-scheme. For a reasonable subscheme $Y \subset X$ denote by $Y^\wedge$ the completion of $X$ along $Y$. This is a formal scheme as in (i) above. For a Diff-bimodule $D$ on $X$ let $D_{Y^\wedge}$ be the ($\mathcal{O}^p$-module) pull-back of $D$ to $Y^\wedge$. This is a Diff-bimodule on $Y^\wedge$, so it may be viewed as a sheaf of $\mathcal{O}_{Y^\wedge}$-bimodules as in (i) above. If $Y' \subset X$ is another reasonable subscheme that contains $Y$ then we have a continuous morphism of sheaves of $\mathcal{O}_{Y'^\wedge}$-bimodules $D_{Y'^\wedge} \to D_{Y^\wedge}$ which identifies $D_{Y^\wedge}$ with the completion of $D_{Y'^\wedge}$ with respect to the topology generated by closures of $\mathcal{I}{\cdot}D_{Y'^\wedge}$ where $\mathcal{I} \subset \mathcal{O}_{Y'^\wedge}$ is an open ideal such that $\operatorname{Spec}(\mathcal{O}/\mathcal{I})_{\mathrm{red}} = Y_{\mathrm{red}}$. These morphisms satisfy the obvious transitivity property. It is clear that Diff-bimodules on $X$ are the same as such data.

Therefore a Diff-algebra $D$ on $X$ may be viewed as the following data:

- a collection of sheaves of topological algebras $D_{Y^\wedge}$ equipped with morphisms $\epsilon_{Y^\wedge} : \mathcal{O}_{Y^\wedge} \to D_{Y^\wedge}$ defined for any reasonable subscheme $Y \subset X$ that satisfy the conditions of (i) above.

- for $Y \subset Y'$ we have a continuous morphism $r_{YY'} : D_{Y'^\wedge} \to D_{Y^\wedge}$ which identifies $D_{Y^\wedge}$ with the completion of $D_{Y'^\wedge}$ as above. We demand the compatibilities $r_{YY'}\epsilon_{Y'^\wedge} = \epsilon_{Y^\wedge}$, $r_{YY''} = r_{YY'}r_{Y'Y''}$.

We leave it to the reader to describe $D$-modules in this language.

*Remark.* For a Diff-algebra $D$ the topological algebra $\Gamma(X, D)$ is the projective limit of topological algebras $\Gamma(Y, D_{Y^\wedge})$.

**7.11.11.** *The key example.* Assume that our $X$ is a formally smooth ind-scheme of ind-finite type. Consider the $\mathcal{O}^p$-module $\mathcal{D}_X$ as defined in 7.11.8(i). So for a subscheme $Y \subset X$ the $\mathcal{O}_Y$-module $(\mathcal{D}_X)_Y$ is $\mathcal{D}(\mathcal{O}_Y) := \varinjlim \operatorname{Diff}(\mathcal{O}_{Y'}, \mathcal{O}_Y)$ with its left $\mathcal{O}_Y$-module structure. Our $\mathcal{D}_X$ carries an obvious structure of Diff-bimodule. The composition of differential operators makes $\mathcal{D}_X$ a Diff-algebra on $X$. According to 7.11.8 our $\mathcal{D}_X$ carries a canonical ring filtration $\mathcal{D}_{iX}$ such that $\operatorname{gr}.\mathcal{D}_X = \operatorname{Sym}^{\cdot}\Theta_X$. The topological algebra $\Gamma(X, \mathcal{D}_X)$ is called *the ring of global differential operators on* $X$. We denote the category of $\mathcal{D}_X$-modules by $\mathcal{M}(X, \mathcal{D})$ or simply $\mathcal{M}(X)$.

If a group "space" $K$ acts on $X$ then $\mathcal{D}_X$ carries a canonical weak $K$-action (defined by transport of structure). Thus we have the category $\mathcal{M}(K \backslash X, \mathcal{D}_X) = \mathcal{M}(K \backslash X)$ of weakly $K$-equivariant $\mathcal{D}$-modules.

*A twisted version.* In the main body of the paper we also need to consider the rings of twisted differential operators (alias tdo), families of such rings and modules over them. The corresponding definitions are immediate modifications of the usual ones in the finite-dimensional setting (see e.g. [BB93]). Below we describe explicitely particular examples of tdo we need.

Let $X$ be as above, $\mathcal{L}$ a line bundle on $X$ (see 7.11.3).

a. The Diff-algebra $\mathcal{D}_{\mathcal{L}}$ of differential operators acting on $\mathcal{L}$ is defined exactly as $\mathcal{D}_X$ replacing in (349) $\mathcal{D}(M)$ by $\mathcal{D}_{\mathcal{L}}(M) = \mathrm{Diff}(\mathcal{L}, M \otimes \mathcal{L}) := \varinjlim \mathrm{Diff}(\mathcal{L}_Y, M_{(Y)} \otimes \mathcal{L}_Y)$; proposition 7.11.8 (as well as its proof) remains true without any changes. Equivalently, $\mathcal{D}_{\mathcal{L}} = \mathcal{L} \otimes \mathcal{D}_X \otimes \mathcal{L}^{\otimes -1}$.

b. We define a Diff-algebra $\mathcal{D}_{\mathcal{L}^h}$ on $X$ as follows. Let $\pi : X^{\sim} \to X$ be the $\mathbb{G}_m$-torsor over $X$ that corresponds to $\mathcal{L}$ (so $X^{\sim} = \mathcal{L} \backslash$(zero section)). Consider the Diff-algebra $\mathcal{D}^{\sim} := \pi_* \mathcal{D}_{X^{\sim}}$ on $X$ (so for a subscheme $Y \subset X$ one has $(\mathcal{D}^{\sim})_Y := \pi_*((\mathcal{D}_{X^{\sim}})_{\pi^{-1}Y}))$. The weak $\mathbb{G}_m$-action on $\mathcal{D}_{X^{\sim}}$ yields a weak $\mathbb{G}_m$-action on $\mathcal{D}^{\sim}$ (with respect to the trivial $\mathbb{G}_m$-action on $X$). Our $\mathcal{D}_{\mathcal{L}^h}$ is the subalgebra of $\mathbb{G}_m$-invariants in $\mathcal{D}^{\sim}$.

Denote by $h$ the global section of $\mathcal{D}_{\mathcal{L}^h}$ that corresponds to the action of $-t\frac{d}{dt} \in \mathrm{Lie}\, \mathbb{G}_m$. Then $\mathcal{D}_{\mathcal{L}^h}$ is the centralizer of $h$ in $\mathcal{D}^{\sim}$. Notice that for any subscheme $Y \subset X$ a trivialization of $\mathcal{L}_{Y^\wedge}$ (which exists locally on $Y$) yields an identification $\mathcal{D}_{\mathcal{L}^h Y^\wedge} \rightleftarrows \mathcal{D}_{Y^\wedge} \widehat{\otimes} \mathbb{C}[h]$.

*Remarks.* (i) Consider the $\mathcal{O}^p$-module $\pi_*(\mathcal{O}_{X^{\sim}}) = \oplus \mathcal{L}^{\otimes n}$. It carries the action of $\mathcal{D}_{\mathcal{L}^h}$ which preserves the grading. The action of $\mathcal{D}_{\mathcal{L}^h}$ on $\mathcal{L}^{\otimes n}$ identifies $\mathcal{D}_{\mathcal{L}^h}/(h-n)\mathcal{D}_{\mathcal{L}^h}$ with $\mathcal{D}_{\mathcal{L}^{\otimes n}}$.

(ii) Let $M^{\sim}$ be a weakly $\mathbb{G}_m$-equivariant $\mathcal{D}$-module on $X^{\sim}$. Set $M := (\pi_* M^{\sim})^{\mathbb{G}_m}$; this is a $\mathcal{D}_{\mathcal{L}^h}$-module. The functor $\mathcal{M}(\mathbb{G}_m \backslash X^{\sim}) \to \mathcal{M}(X, \mathcal{D}_{\mathcal{L}^h})$, $M^{\sim} \mapsto M$, is an equivalence of categories.

**7.11.12.** Let us explain the $\mathcal{D}$-$\Omega$ complexes interplay in the setting of ind-schemes. First let us define $\Omega$-complexes. Here we assume that $X$ is any reasonable ind-scheme.

For any reasonable subschemes $Y \subset Y'$ one has a surjective morphism of commutative DG algebras $\Omega_{Y'} \to \Omega_Y$. An $\Omega^!$-*complex* $F$ *on* $X$ (or simply an $\Omega$-complex) is a rule that assigns to a reasonable subscheme $Y \subset X$ a DG $\Omega_Y$-module $F_{[Y]}$ together with morphisms of $\Omega_{Y'}$-modules $F_{[Y]} \to F_{[Y']}$ for $Y \subset Y'$ which identify $F_{[Y]}$ with $i^!_{\Omega YY'} F_{[Y']} := \mathrm{Hom}_{\Omega_{Y'}}(\Omega_Y, F_{[Y']})$ and satisfy the obvious transitivity condition. We assume that $F^i_{[Y]}$ is quasi-coherent as an $\mathcal{O}_Y$-module. As in 7.11.4 it suffice to consider only $X_\alpha$'s instead of all reasonable $Y$'s. As in Remark in 7.2.1 such an $F$ is the same as a complex of $\mathcal{O}^!$-modules whose differential is a differential operator of order $\leq 1$. We denote by $C(X, \Omega)$ the DG category of $\Omega^!$-complexes.

If $f : Y \to X$ is a representable quasi-compact morphism of ind-schemes (so $Y = \varinjlim Y_\alpha$ where $Y_\alpha := f^{-1}(X_\alpha)$) then one has a pull-back functor $f^{\cdot}_\Omega : C(X, \Omega) \to C(Y, \Omega)$, $f^{\cdot}_\Omega(F) := \varinjlim \Omega_{Y_\alpha} \underset{f^{-1}\Omega_{X_\alpha}}{\otimes} F_\alpha$. If $f$ is surjective and formally smooth then $f^{\cdot}_\Omega$ satisfies the descent property.

Assume that a group "space" $K$ acts on $X$. One defines a $K$-action on an $\Omega$-complex $F$ on $X$ as a rule that assigns to any $g \in K(A)$ a lifting of the action of $g$ on $\operatorname{Spec} A \times X$ to $\mathcal{O}_{\operatorname{Spec} A} \otimes F \in C(\operatorname{Spec} A \times X, \Omega)$; the obvious compatibilities should hold. We denote the corresponding category by $C(K \backslash X, \Omega)$.

*Remarks.* (i) Assume that $K$ is a group ind-scheme, so we have the Lie algebra $\operatorname{Lie} K$. The definition of $K_\Omega$-action on $F$ in terms of operators $i_\xi$ from 7.6.4 renders immediately to the present setting. The category of $K_\Omega$-equivariant $\Omega$-complexes is denoted by $C(K \setminus X, \Omega)$.

(ii) If our $K$ is an affine group scheme then a $K_\Omega$-equivariant $\Omega$-complex is the same as an $\Omega$-complex $F$ equipped with an isomorphism $m_\Omega^{\cdot}F = p_X^{\cdot}F$ of $\Omega$-complexes on $K \times X$ that satisfy the usual condition (see 7.6.5).

**7.11.13.** Assume that $X$ is a formally smooth ind-scheme of ind-finite type. Denote by $C(X,\mathcal{D})$ the DG category of complexes of $\mathcal{D}$-modules ($\mathcal{D}$-complexes for short) on $X$. We have the DG functor

$$\mathcal{D} :\, C(X,\Omega) \to C(X,\mathcal{D}) \tag{352}$$

which sends an $\Omega$-complex $F$ to the $\mathcal{D}$-complex $\mathcal{D}F$ with components $(\mathcal{D}F)^n := \mathcal{D}(F^n) = F^n \otimes \mathcal{D}_X$ (see 7.11.8) and the differential defined by formula $d(a) := d_F \circ a$ (here $a \in \mathcal{D}(F^n) = \mathrm{Diff}(\mathcal{O}_X, F^n)$). This functor admits a right adjoint functor

$$\Omega :\, C(X,\mathcal{D}) \to C(X,\Omega) \tag{353}$$

which may be described explicitely as follows. For a subscheme $Y \subset X$ we have the $\mathcal{D}$-complex $DR_Y := \mathcal{D}(\Omega_Y)$. It is also a left DG $\Omega_Y$-module. Now for a $\mathcal{D}$-complex $M$ one has $\Omega M = \varinjlim \mathrm{Hom}(DR_Y, M) = \bigcup \mathrm{Hom}(DR_Y, M)$.

Lemma 7.2.4 remains true as well as its proof. As in 7.2.5 we have the cohomology functor $H_{\mathcal{D}}^{\cdot} :\, C(X,\Omega) \to \mathcal{M}(X)$, $H_{\mathcal{D}}^{\cdot}(F) = H^{\cdot}(\mathcal{D}F)$, and the corresponding notion of $\mathcal{D}$-quasi-isomorphism. The adjunction morphisms for $\mathcal{D}$, $\Omega$ are quasi-isomorphism and $\mathcal{D}$-quasi-isomorphism*).

**7.11.14.** We say that an $\mathcal{O}^!$-complex or $\mathcal{O}^!$-module has *quasi-compact support* if it vanishes on the complement to some closed subscheme. Same definition applies to $\mathcal{D}$- and $\Omega$-complexes. We mark the corresponding categories by lower "c" index. The functors $\mathcal{D}$ and $\Omega$ preserve the corresponding full DG subcategories $C_c(X,\Omega) \subset C(X,\Omega)$, $C_c(X,\mathcal{D}) \subset C(X,\mathcal{D})$.

*)The fact that de Rham complexes of $\mathcal{D}$-modules are not bounded from below does not spoil the picture.

In order to ensure that our derived categories are the right ones (i.e., that they have nice functorial properties) we assume in addition that *the diagonal morphism* $X \to X \times X$ *is affine* (cf. 7.3.1). For example, it suffices to assume that $X$ is separated.

Denote by $D(X, \mathcal{O})$ the homotopy category of $C_c(X, \mathcal{O})$ localized with respect to quasi-isomorphisms; this is a t-category with core $\mathcal{M}_c(X, \mathcal{O})$. We define $D(X, \mathcal{D})$ (assuming that $X$ is formally smooth of ind-finite type) in the similar way; this is a t-category with core $\mathcal{M}_c(X)$. Let $D(X, \Omega)$ be localization of the homotopy category of $C_c(X, \Omega)$ by $\mathcal{D}$-quasi-isomorphisms. The functors $\mathcal{D}$ and $\Omega$ yield canonical identification of $D(X, \mathcal{D})$ and $D(X, \Omega)$, so, as usual, we denote these categories thus identified simply $D(X)^{*)}$.

We say that an $\mathcal{O}^!$-module $F$ with quasi-compact support is *loose* if for any closed subscheme $Y \subset X$ such that $F$ is supported on $Y^\wedge$ and a flat $\mathcal{O}^p$-module $P$ on $Y^\wedge$ one has $H^a(X, P \otimes F) = 0$ for $a > 0$. An $\mathcal{O}^!$- $\mathcal{D}$- or $\Omega$-complex $F$ is loose if each $\mathcal{O}^!$-module $F^i$ is loose. One has the following lemma parallel to 7.3.8:

**7.11.15.** *Lemma.* i) For any $F' \in C_c(X, \Omega)$ there exists a $\mathcal{D}$-quasi-isomorphism $F' \to F$ such that $F$ is loose and the supports of $F, F'$ coincide.

(ii) If $f : X \to X'$ is a formally smooth affine morphism of ind-schemes then the functors

$$f_\Omega^\cdot : C_c(X', \Omega) \to C_c(X, \Omega),\ f_\cdot : C_c(X, \Omega) \to C_c(X', \Omega)$$

send loose complexes to loose ones.

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*) To get a t-category with core $\mathcal{M}(X)$ one may consider complexes which are unions of subcomplexes with quasi-compact support; however to ensure the good functorial properties of this category one has to assume that $X$ satisfies certain extra condition (e.g., that there exists a formally smooth surjective morphism $Y \to X$ such that $Y$ is ind-affine). The category formed by all complexes has unpleasant homological and functorial properties. Notice that the usual remedy - to consider only $\Omega$-complexes bounded from below - does not work here (the de Rham complexes of $\mathcal{D}$-modules do not satisfy this condition).

(iii) If $F_1, F_2$ are loose complexes on $X_1, X_2$ then $F_1 \boxtimes F_2$ is a loose $\Omega$-complex on $X_1 \times X_2$.

*Proof.* Modify the proof of 7.3.8 in the obvious way. □

We see that one can define the derived category $D(X)$ using loose complexes.

**7.11.16.** Any morphism $f : X \to Y$ of ind-schemes yields the push-forward functor $f. : C(X, \Omega) \to C(Y, \Omega)$ which preserves the subcategories $C_c$. We leave it to the reader to check that $f.$ preserves $\mathcal{D}$-quasi-isomorphisms between loose complexes with quasi-compact support (cf. 7.3.9, 7.3.11(ii)). Thus the right derived functor $Rf. = f_* : D(X) \to D(Y)$ is well-defined: one has $f_*F = f.F$ if $F$ is a loose complex with quasi-compact support. Since $f.$ sends loose complexes to loose ones we see that $f_*$ is compatible with composition of $f$'s.

For $M \in D(X, \mathcal{D})$ denote by $M_{\mathcal{O}} \in D(X, \mathcal{O})$ same $M$ considered as a complex of $\mathcal{O}^!$-modules. One has a canonical integration morphism

$$i_f : Rf.(M_{\mathcal{O}}) \to (f_*M)_{\mathcal{O}}$$

in $D(Y, \mathcal{O})$ defined as in 7.2.11. It is compatible with composition of $f$'s.

**7.11.17.** Let us define the Hecke monoidal category $\mathcal{H}$ as in 7.6.1. We start with an ind-affine group ind-scheme $G$ and its affine group subscheme $K \subset G$. We assume that $G/K$ (the quotient of sheaves with respect to fpqc topology) is a ind-scheme of ind-finite type; it is automatically formally smooth and its diagonal morphism is affine. Clearly $G$ is a reasonable ind-scheme, and $K$ is its reasonable subscheme. Consider the DG category $\mathcal{H}^c$ of $(K \times K)_\Omega$-equivariant $\Omega^!$-complexes on $G$ with quasi-compact support (see Remark (i) in 7.11.12). By descent such a complex is the same as a $K_\Omega$-equivariant admissible $\Omega^!$-complex either on $G/K$ or on $K \setminus G$. The corresponding notions of $\mathcal{D}$-quasi-isomorphism are equivalent. Our $\mathcal{H}$ is the corresponding $\mathcal{D}$-derived category.

The constructions of 7.6.1 make perfect sense in our setting. Thus $\mathcal{H}^c$ is a DG monoidal category, and $\mathcal{H}$ is a triangulated monoidal category.

**7.11.18.** Assume that we have a scheme $Y$ equipped with a $G$-action such that there exists an increasing family $U_0 \subset U_1 \subset ...$ of open quasi-compact subschemes of $Y = \bigcup U_i$ with property that for some reasonable group subscheme $K_i \subset G$ the action of $K_i$ on $U_i$ is free and $K_i \setminus U_i$ is a smooth scheme (in particular, of finite type). Then the stack $\mathcal{B} = K \setminus Y$ is smooth (it has a covering by schemes $(K_i \cap K) \setminus U_i$). The diagonal morphism for $\mathcal{B}$ is affine, so we may use the definition of $D(\mathcal{B})$ from 7.3.12.

To define the $\mathcal{H}$-Action on $D(\mathcal{B})$ you proceed as in 7.6.1 with the following modifications that arise due to possible non-quasi-compactness of $Y$ and $G$. We may assume that the above $U_i$'s are $K$-invariant; set $\mathcal{B}_i = K \setminus U_i \subset \mathcal{B}$. Take loose $\Omega$-complexes $F = \cup F_n \in C_a(K \setminus G/K, \Omega)$ (so the supports $S_n$ of $F_n$ are quasi-compact) and $T \in C(\mathcal{B}., \Omega)$. Let $j(n,i)$ be an increasing (with respect to both $n$ and $i$) sequence such that $S_n^{-1} \cdot U_i \subset U_{j(n,i)}$. Consider the $\Omega$-complexes $(F_n \circledast T)_i := \bar{m}_{U_i} \cdot p^{\cdot}_{U_i \Omega}(F_n \boxtimes T_{j(n,i)})|_{\mathcal{B}_i}$ and $(F_n \circledast T)'_i := \bar{m}_{U_i} \cdot p^{\cdot}_{U_i \Omega}(F_n \boxtimes T_{j(n+1,i)})|_{\mathcal{B}_i}$ on $\mathcal{B}_i$. There are the obvious morphisms $(F_n \circledast T)'_i \to (F_{n+1} \circledast T)_i$, $(F_n \circledast T)'_i \to (F_n \circledast T)_i$; the latter is a quasi-isomorphism. Set $(F \circledast T)_i := Cone(\oplus (F_n \circledast T)_i \to \oplus (F_n \circledast T)_i)$ where the arrow is the (componentwise) difference of the above morphisms. These $(F \circledast T)_i$ form in the obvious manner an object $F \circledast T \in C(\mathcal{B}, \Omega)$. We leave it to the reader to check that $F \circledast T$ as an object of $D(\mathcal{B})$ does not depend on the choice of the auxiliary data (of $U_i$ and $j(n,i)$), and that $\circledast$ is an $\mathcal{H}$-Action on $D(\mathcal{B})$.

**7.12. Ind-schemes and Mittag-Leffler modules.** Raynaud and Gruson [RG] introduced a remarkable notion of Mittag-Leffler module. In this section we show that the notion of flat Mittag-Leffler module is, in some sense, a linearized version of the notion of formally smooth ind-scheme of ind-finite type (see 7.12.12, 7.12.14, 7.12.15). Using the fact that countably generated flat Mittag-Leffler modules are projective we describe formally smooth affine $\aleph_0$-formal schemes of ind-finite type (see 7.12.22, 7.12.23).

The reader can skip this section because its results are not used in the rest of this work (we include them only to clarify the notion of formally smooth ind-scheme).

In 7.11 we assumed that "ind-scheme" means "ind-scheme over $\mathbb{C}$" (this did not really matter). In this section we prefer to drop this assumption.

**7.12.1.** Let $A$ be a ring*). Denote by $\mathcal{C}$ the category of $A$-modules of finite presentation. According to [RG], p.69 an $A$-module $M$ is said to be a *Mittag-Leffler module* if every morphism $f : F \to M$, $F \in \mathcal{C}$, can be decomposed as $F \xrightarrow{u} G \to M$, $G \in \mathcal{C}$, so that for every decomposition of $f$ as $F \xrightarrow{u'} G' \to M$, $G' \in \mathcal{C}$, there is a morphism $\varphi : G' \to G$ such that $u = \varphi u'$.

**7.12.2.** Suppose that $M = \varinjlim M_i$, $i \in I$, where $I$ is a directed ordered set and $M_i \in \mathcal{C}$. According to loc.cit, $M$ is a Mittag-Leffler module if and only if for every $i \in I$ there exists $j \geq i$ such that for every $k \geq i$ the morphism $u_{ij} : M_i \to M_j$ can be decomposed as $\varphi_{ijk} u_{ik}$ for some $\varphi_{ijk} : F_k \to F_j$. A similar statement holds if $I$ is a filtered category; if $I$ is the category of all morphisms from objects of $\mathcal{C}$ to $M$ and $F_i \in \mathcal{C}$ is the source of the morphism $i$ then the above statement is tautological.

**7.12.3.** The above property of inductive systems $(M_i)$, $M_i \in \mathcal{C}$, makes sense if $\mathcal{C}$ is replaced by any category $\mathcal{C}'$. If $\mathcal{C}'$ is dual to the category of sets, i.e., if we have a projective system of sets $(E_i, u_{ij} : E_j \to E_i)$ one gets the *Mittag-Leffler condition* from EGA $0_{\mathrm{III}}$ 13.1.2: for every $i \in I$ there exists $j \geq i$ such that $u_{ij}(E_j) = u_{ik}(E_k)$ for all $k \geq j$.

This condition is satisfied if and only if the projective system $(E_i, u_{ij})$ is equivalent to a projective system $(\widetilde{E}_\alpha, \widetilde{u}_{\alpha\beta})$ where the maps $\widetilde{u}_{\alpha\beta}$ are

*) We assume that $A$ is commutative but in 7.12.1–7.12.8 this is not essential (one only has to insert in the obvious way the words "left" and "right" before the word "module").

surjective. To prove the "only if" statement it suffices to set $\widetilde{E}_i := u_{ij}(E_j)$ for $j$ big enough.

**7.12.4.** Suppose that $M = \underrightarrow{\lim} M_i$, $M_i \in \mathcal{C}$. According to [RG] $M$ is a Mittag-Leffler module if and only if for any contravariant functor $\Phi$ from $\mathcal{C}$ to the category of sets the projective system $(\Phi(M_i))$ satisfies the Mittag-Leffler condition (to prove the "if" statement consider the functor $\Phi(N) = \mathrm{Hom}(N, \prod_i M_i)$ or $\widetilde{\Phi}(N) = \bigsqcup_i \mathrm{Hom}(N, M_i)$ ).

Assume that $M$ is flat. Set $M_i^* = \mathrm{Hom}(M_i, A)$. According to [RG] $M$ is a Mittag-Leffler module if and only if the projective system $(M_i^*)$ satisfies the Mittag-Leffler condition. This is clear if the modules $M_i$ are projective. The general case follows by Lazard's lemma (there is an inductive system equivalent to $(M_i)$ consisting of finitely generated projective modules).

**7.12.5.** Consider the following two classes of functors from the category of $A$-modules to the category of abelian groups:

1) For an $A$-module $M$ one has the functor

$$L \mapsto L \otimes_A M; \tag{354}$$

2) For a projective system of $A$-modules $N_i$ (where $i$ belong to a directed ordered set) one has the functor

$$L \mapsto \underset{\underset{i}{\longrightarrow}}{\lim} \mathrm{Hom}(N_i, L) \tag{355}$$

**7.12.6.** *Proposition.* (i) The functor (354) is isomorphic to a functor of the form (355) if and only if $M$ is flat.

(ii) The functor (354) is isomorphic to the functor (355) corresponding to a projective system $(N_i)$ with surjective transition maps $N_j \to N_i$, $i \leq j$, if and only if $M$ is a flat Mittag-Leffler module.

(iii) The functor (355) corresponding to a projective system $(N_i)$ with surjective transition maps $N_j \to N_i$, $i \leq j$, is isomorphic to a functor of the form (354) if and only if the functor (355) is exact and the modules $N_i$ are finitely generated.

*Proof.* If (354) and (355) are isomorphic then (354) is left exact, so $M$ is flat. If $M$ is flat then by Lazard's lemma $M = \varinjlim P_i$ where the modules $P_i$ are projective and finitely generated, so the functor (355) corresponding to $N_i = P_i^*$ is isomorphic to (354).

We have proved (i). To deduce (ii) from (i) notice that for $P_i$ as above the projective system $(P_i^*)$ is equivalent to a projective system $(N_i)$ with surjective transition maps $N_j \to N_i$ if and only if $(P_i^*)$ satisfies the Mittag-Leffler condition (see 7.12.3).

To prove (iii) notice that functors of the form (354) are those additive functors which are right exact and commute with infinite direct sums (then they commute with inductive limits). A functor of the form (355) is right exact if and only if it is exact. If the modules $N_i$ are finitely generated then (355) commutes with infinite direct sums. If the transition maps $N_j \to N_i$ are surjective and (355) commutes with inductive limits then the modules $N_i$ are finitely generated. $\square$

**7.12.7.** According to 7.12.6 a flat Mittag-Leffler module is "the same as" an equivalence class of projective systems $(N_i)$ of finitely generated modules with surjective transition maps $N_j \to N_i$, $i \leq j$, such that the functor (355) is exact. More precisely, $M = \varinjlim_i \operatorname{Hom}(N_i, A)$ (then the functors (354) and (355) are isomorphic).

**7.12.8.** *Theorem.* (Raynaud–Gruson). (What about D.Lazard? according to [RG], p.73 the idea goes back to Theorems 3.1 and 3.2 from chapter I of D.Lazard's thesis in Bull.Soc.Math.France, vol.97 (1969), 81–128; see also D.Lazard's work in Bull.Soc.Math.France, vol.95 (1967), 95–108)

The following conditions are equivalent:

(i) $M$ is a flat Mittag-Leffler module;

(ii) every finite or countable subset of $M$ is contained in a countably generated projective submodule $P \subset M$ such that $M/P$ is flat;

(iii) every finite subset of $M$ is contained in a projective submodule $P \subset M$ such that $M/P$ is flat.

In particular, a projective module is Mittag-Leffler and a countably generated*) flat Mittag-Leffler module is projective.

The implication (iii)$\Rightarrow$(i) is easy. (It suffices to show that if $F$ and $F'$ are modules of finite presentation and $\varphi : F \to F'$, $\psi : F' \to M$ are morphisms such that $\psi\varphi(F) \subset P$ then there exists $\widetilde{\psi} : F' \to M$ such that $\widetilde{\psi}(F') \subset P$ and $\widetilde{\psi}\varphi = \psi\varphi$; use the fact that $\mathrm{Hom}(L, M) \to \mathrm{Hom}(L, M/P)$ is surjective for every $L$ of finite presentation, in particular for $L = \mathrm{Coker}\,\varphi$).

The implication (i)$\Rightarrow$(ii) is proved in [RG], p.73–74. The key argument is as follows. Suppose we have a sequence $P_1 \to P_2 \to \dots$ where $P_1, P_2, \dots$ are finitely generated projective modules and the projective system $(P_i^*)$ satisfies the Mittag-Leffler property. To prove that $P := \varinjlim P_i$ is projective one has to show that for every exact sequence $0 \to N' \to N \to N'' \to 0$ the map $\mathrm{Hom}(P, N) \to \mathrm{Hom}(P, N'')$ is surjective. For each $i$ the sequence

$$0 \to P_i^* \otimes N' \to P_i^* \otimes N \to P_i^* \otimes N'' \to 0$$

is exact and the problem is to show that the projective limit of these sequences is exact. According to EGA $0_{\mathrm{III}}$ 13.2.2 this follows from the Mittag-Leffler property of the projective system $(P_i^* \otimes N')$.

*Remark.* If the set of indices $i$ were uncountable we would not be able*) to apply EGA $0_{\mathrm{III}}$ 13.2.2.

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*)The countable generatedness assumption is essential; see 7.12.24.

*)The argument from EGA $0_{\mathrm{III}}$ 13.2.2 is based on the following fact: if a projective system of non-empty sets $(Y_i)_{i \in I}$ parametrized by a countable set $I$ satisfies the Mittag-Leffler condition then its projective limit is non-empty. This is wrong in the uncountable case. For instance, consider an uncountable set $S$, for every finite $F \subset A$ denote by $Y_F$ the set of injections $F \to \mathbb{N}$; the maps $Y_{F'} \to Y_F$, $F' \supset F$, are surjective but $\varprojlim_F Y_F = \emptyset$.

Here is another proof of the projectivity of $P$ (in fact, another version of the same proof). Denote by $f_i$ the map $P_i \to P_{i+1}$. The Mittag-Leffler property means that after replacing the sequence $\{P_i\}$ by its subsequence there exist $g_i : P_{i+1} \to P_i$ such that $g_{i+1}f_{i+1}f_i = f_i$. Set $\mathcal{P} := \bigoplus\limits_i P_i$. Denote by $f : \mathcal{P} \to \mathcal{P}$ and $g : \mathcal{P} \to \mathcal{P}$ the operators induced by the $f_i$ and $g_i$. Then $gf^2 = f$. We have the exact sequence

$$0 \to \mathcal{P} \xrightarrow{1-f} \mathcal{P} \to P \to 0$$

Since $\mathcal{P}$ is projective it suffices to show that this sequence splits, i.e., there is an $h : \mathcal{P} \to \mathcal{P}$ such that $h(1-f) = 1$. Indeed, set $h = 1 - (1-g)^{-1}gf$ and use the equality $gf^2 = f$.[*)]

**7.12.9.** *Proposition.* Let $B$ be an $A$-algebra. If $M$ is a Mittag-Leffler $A$-module then $B \otimes_A M$ is a Mittag-Leffler $B$-module. If $B$ is faithfully flat over $A$ then the converse is true.

This is proved in [RG]. The proof is easy: represent $M$ as an inductive limit of modules of finite presentation and use 7.12.2.

So the notion of a Mittag-Leffler $\mathcal{O}$-module on a scheme is clear as well as the notion of Mittag-Leffler $\mathcal{O}^p$-module on an ind-scheme.

**7.12.10.** *Proposition.* A flat Mittag-Leffler $\mathcal{O}$-module $\mathcal{F}$ of countable type on a noetherian scheme $S$ is locally free. If $S$ is affine and connected, and $\mathcal{F}$ is of infinite type then $\mathcal{F}$ is free.

This is an immediate consequence of 7.12.8 and the following result.

**7.12.11.** *Theorem.* If $R$ is noetherian and $\operatorname{Spec} R$ is connected then every nonfinitely generated projective $R$-module is free.

This theorem was proved by Bass (see Corollary 4.5 from [Ba63]).

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[*)] D.Arinkin noticed that it is clear a priori that if $f$ and $g$ are elements of a (noncommutative) ring $R$ such that $gf^2 = f$ and $1-g$ has a left inverse then $1-f$ has a left inverse. Indeed, denote by $\mathbf{1}$ the image of 1 in $R/R(1-f)$. Then $f\mathbf{1} = \mathbf{1}$, $gf^2\mathbf{1} = g\mathbf{1}$, so $g\mathbf{1} = \mathbf{1}$ and therefore $\mathbf{1} = 0$.

**7.12.12.** *Proposition.* Let $X$ be a formally smooth ind-scheme of ind-finite type over a field. Then the $\mathcal{O}^p$-modules $\Theta_X$, $\mathcal{D}_X$, $\mathcal{D}_{iX}$ (see 7.11.8) are flat Mittag-Leffler modules.

*Proof.* Let us prove that the restriction of $\mathcal{D}_X$ to a closed subscheme $Y \subset X$ is a flat Mittag-Leffler $\mathcal{O}_Y$-module (the same argument works for $\Theta_X$ and $\mathcal{D}_{iX}$). We can assume that $Y$ is affine (otherwise replace $X$ by $X \setminus F$ for a suitable closed $F \subset Y$). According to 7.12.6 it suffices to prove that

(i) The functor $L \mapsto L \otimes \mathcal{D}_X$ defined on the category of $\mathcal{O}_Y$-modules is exact,

(ii) it has the form (355) where the $\mathcal{O}_Y$-modules $N_i$ are coherent.

By definition, $L \otimes \mathcal{D}_X$ is the sheaf $\mathcal{D}(L)$ defined by (349). So (ii) is clear. We have proved (i) in 7.11.8. $\square$

**7.12.13.** *Proposition.* Let $X$ be a formally smooth $\aleph_0$-ind-scheme of ind-finite type over a field, $Y \subset X$ a locally closed subscheme. Then the restriction of $\Theta_X$ to $Y$ is locally free. If $Y$ is affine and connected, and the restriction of $\Theta_X$ to $Y$ is of infinite type then it is free.

This follows from 7.12.12 and 7.12.10.

**7.12.14.** *Proposition.* Let $A$ be a ring, $M$ an $A$-module. Define an "$A$-space" $F_M$ (i.e., a functor from the category of $A$-algebras to that of sets) by $F_M(R) = M \otimes R$. Then $F_M$ is an ind-scheme if and only if $M$ is a flat Mittag-Leffler module. In this case $F_M$ is formally smooth over $A$ and of ind-finite type over $A$.

*Proof.* If $M$ is a flat Mittag-Leffler module then by 7.12.6(ii) $F_M$ is an ind-scheme and by 7.12.6(iii) it is of ind-finite type over $A$. Formal smoothness follows from the definition. Now suppose that $F_M$ is an ind-scheme. Represent $F_M$ as $\varinjlim S_i$ where the $S_i$ are closed subshemes of $F_M$ containing the zero section $0 \in F_M(A)$. Denote by $N_i$ the restriction of the cotangent sheaf of $S_i$ to $0 : \operatorname{Spec} A \hookrightarrow S_i$. Then the functor (355) is isomorphic to (354), so by 7.12.6(ii) $M$ is a flat Mittag-Leffler module. $\square$

*Remark.* If $M$ is an arbitrary flat $A$-module then $M$ is an inductive limit of a directed family of finitely generated projective $A$-modules $M_i$, so $F_M = \varinjlim F_{M_i}$ is an ind-scheme in the broad sense (the morphisms $F_{M_i} \to F_{M_j}$ are not necessarily closed embeddings). It is easy to see that if $F_M$ is an ind-scheme in the broad sense then $M$ is flat.

**7.12.15.** *Proposition.* Let $(N_i)_{i\in I}$ be a projective system of finitely generated $A$-modules parametrized by a directed set $I$ such that all the transition maps $N_j \to N_i$, $j \geq i$, are surjective. Set $\mathbb{A}(N_i) := \operatorname{Spec} \operatorname{Sym}(N_i)$, $S := \varinjlim\limits_{i} \mathbb{A}(N_i)$. The ind-scheme $S$ is formally smooth over $A$ if and only if $S$ is isomorphic to the ind-scheme $F_M$ from 7.12.14 corresponding to a flat Mittag-Leffler module $M$.

*Proof.* $S$ is formally smooth if and only if the functor (355) is exact (apply the definition of formal smoothness to $A$-algebras of the form $A\oplus J$, $A\cdot J \subset J$, $J^2 = 0$). Now use 7.12.6(iii). $\square$

**7.12.16.** *Proposition.* Let $M$ be a flat Mittag-Leffler module, $F_M$ the ind-scheme from 7.12.14. The following conditions are equivalent:

(i) the pro-algebra corresponding to $F_M$ (see 7.11.2(i) ) is a topological algebra;

(ii) $M$ is a strictly Mittag-Leffler module in the sense of [RG].

According to [RG], p.74 a module $M$ is *strictly Mittag-Leffler* if for every $f : F \to M$, $F \in \mathcal{C}$, there exists $u : F \to G$, $G \in \mathcal{C}$, such that $f = gu$ and $u = hf$ for some $g : G \to M$, $h : M \to G$ (recall that $\mathcal{C}$ is the category of modules of finite presentation). If $M = \varinjlim M_i$, $M_i \in \mathcal{C}$, and $u_{ij} : M_i \to M_j$, $u_i : M_i \to M$ are the canonical maps then $M$ is strictly Mittag-Leffler if and only if for every $i$ there exists $j \geq i$ such that $u_{ij} = \varphi_{ij} u_j$ for some $\varphi_{ij} : M \to M_j$. Clearly a projective module is stritly Mittag-Leffler and a strictly Mittag-Leffler module is Mittag-Leffler. The converse statements are not true in general (see 7.12.24).

*Proof.* Represent $M$ as $\varinjlim P_i$ where the modules $P_i$ are finitely generated and projective. Set $N_i := \operatorname{Im}(P_j^* \to P_i^*)$ where $j$ is big enough. Consider the following conditions:

(a) the maps $\varphi_i : \varprojlim_r \operatorname{Sym}(N_r) \to \operatorname{Sym}(N_i)$ are surjective;

(b) $\operatorname{Im}\varphi_i \supset N_i$ for every $i$;

(c) the map $\varprojlim_r N_r \to N_i$ is surjective for every $i$;

(d) for every $i$ there exists $j \geq i$ such that the images of $\operatorname{Hom}(M, A)$ and $\operatorname{Hom}(P_j, A)$ in $\operatorname{Hom}(P_i, A)$ are equal.

Clearly (i)⇔(a)⇔(b)⇔(c)⇔(d). For $i \leq j$ consider the maps $u_{ij} : P_i \to P_j$ and $u_i : P_i \to M$. To show that (d)⇔(ii) it suffices to prove that the images of $\operatorname{Hom}(M, A)$ and $\operatorname{Hom}(P_j, A)$ in $\operatorname{Hom}(P_i, A)$ are equal if and only if $u_{ij} = \varphi u_j$ for some $\varphi : M \to P_j$. To prove the "only if" statement notice that the images of $\operatorname{Hom}(M, P_j)$ and $\operatorname{Hom}(P_j, P_j)$ in $\operatorname{Hom}(P_i, P_j)$ are equal and therefore the image of $\mathrm{id} \in \operatorname{Hom}(P_j, P_j)$ in $\operatorname{Hom}(P_i, P_j)$ is the image of some $\varphi \in \operatorname{Hom}(M, P_j)$. □

**7.12.17.** Before passing to the structure of formally smooth affine $\aleph_0$-ind-schemes let us discuss the relation between the definition of formal scheme from 7.11.1 and Grothendieck's definition (see EGA I). They are not equivalent even in the affine case. A formal affine scheme in our sense is an ind-scheme $X$ that can be represented as $\varinjlim \operatorname{Spec} R_\alpha$ where $(R_\alpha)$ is a projective system of rings such that the maps $u_{\alpha\beta} : R_\beta \to R_\alpha$, $\beta \geq \alpha$, are surjective and the elements of $\operatorname{Ker} u_{\alpha\beta}$ are nilpotent. Grothendieck requires the possibility to represent $X$ as $\varinjlim \operatorname{Spec} R_\alpha$ so that the maps

$$\varprojlim_\beta R_\beta \to R_\alpha \tag{356}$$

are surjective[*)] and the ideals $\operatorname{Ker} u_{\alpha\beta}$ are nilpotent. A reasonable $\aleph_0$-formal scheme in our sense is a formal scheme in the sense of EGA I. A quasi-compact formal scheme in Grothendieck's sense having a fundamental system of "defining ideals (English?)" ("Idéaux de définition"; see EGA I 10.5.1) is a formal scheme in our sense; in particular, this is true for noetherian formal schemes in the sense of EGA I.

Since we are mostly interested in affine $\aleph_0$-formal schemes of ind-finite type over a field the difference between our definition and that of EGA I is not essential.

**7.12.18.** *Proposition.* Let $X$ be a formally smooth $\aleph_0$-ind-scheme of ind-finite type over $A$, $S \subset X$ a closed subscheme such that $S \to \operatorname{Spec} A$ is an isomorphism. Suppose that $X_{\mathrm{red}} = S_{\mathrm{red}}$ (in particular, $X$ is a formal scheme). Let $M$ denote the $A$-module of global sections of the restriction of the relative tangent sheaf $\Theta_{X/A}$ to $S$. Then $M$ is a countably generated projective module and $(X, S)$ is isomorphic to the completion $\widehat{F}_M$ of the ind-scheme $F_M$ (see 7.12.14) along the zero section.

*Remark.* The $\mathcal{O}^p$-module $\Theta_{X/A}$ on a formally smooth ind-scheme $X$ of ind-finite type over $A$ is defined just as in the case $A = \mathbb{C}$ (see 7.11.8, 7.11.7).

*Proof.* Just as in 7.12.12 one shows that $M$ is a flat Mittag-Leffler module. The $\aleph_0$ assumption implies that $M$ is countably generated. By 7.12.8 $M$ is projective.

Represent $X$ as $\varinjlim X_n$, $n \in \mathbb{N}$, where the $X_n$ are closed subschemes of $X$ containing $S$ such that $X_n \subset X_{n+1}$. Let $X^{(1)}$ be the first infinitesimal neighbourhood of $S$ in $X$, i.e., $X^{(1)}$ is the union of the first infinitesimal

[*)] This is stronger than surjectivity of $u_{\alpha\beta}$; e.g., if $M$ is a flat Mittag-Leffler $A$-module that is not strictly Mittag-Leffler then the arguments from 7.12.6 show that the completion of $F_M$ along the zero section cannot be represented as $\varinjlim \operatorname{Spec} R_\alpha$ so that the maps (356) are surjective.

neighbourhoods of $S$ in $X_n$, $n \in \mathbb{N}$. Clearly $X^{(1)} = F_M^{(1)}$ :=the first infinitesimal neighbourhood of $0 \in F_M$. The embedding $X^{(1)} \to \widehat{F}_M$ can be extended to a morphism $\varphi : X \to \widehat{F}_M$ (to construct $\varphi$ define $\varphi_n : X_n \to \widehat{F}_M$ so that $\varphi_n|_{X_{n-1}} = \varphi_{n-1}$ and the restriction of $\varphi_n$ to $X_n \cap X^{(1)}$ is the canonical embedding $X_n \cap X^{(1)} \hookrightarrow F_M^{(1)}$; this is possible because $\widehat{F}_M$ is formally smooth over $A$). Quite similarly one extends the embedding $F_M^{(1)} = X^{(1)} \hookrightarrow X$ to a morphism $\psi : \widehat{F}_M \to X$. Since $\varphi$ and $\psi$ induce isomorphisms between $F_M^{(1)}$ and $X^{(1)}$ we see that $\varphi$ and $\psi$ are ind-closed embeddings and $\varphi\psi$ is an isomorphism. So $\varphi$ and $\psi$ are isomorphisms. $\square$

**7.12.19.** *Example.* We will construct a pair $(X, S)$ satisfying the conditions of 7.12.18 except the $\aleph_0$ assumption such that $(X, S)$ is not $A$-isomorphic to a formal scheme of the form $\widehat{F}_M$.

Suppose we have a nontrivial extension of flat Mittag-Leffler modules

$$0 \to N' \to N \to L \to 0. \tag{357}$$

Such extensions do exist for "most" rings $A$; see 7.12.24(b, a$''$, d). After tensoring (357) by $A[t]$ we get the extension $0 \to N'[t] \to N[t] \to L[t] \to 0$. Multiplying this extension by $t$ we get $0 \to N'[t] \to Q \to L[t] \to 0$. The ind-scheme $F_Q$ is formally smooth over $A[t]$ and therefore over $A$. Let $S \subset F_Q$ be the image of the composition of the zero sections $\operatorname{Spec} A \to \operatorname{Spec} A[t] \to F_Q$. Denote by $X$ the completion of $F_Q$ along $S$.

Before proving the desired property of $(X, S)$ let us describe $X$ more explicitly. For an $A$-algebra $R$ an $R$-point of $F_Q$ is a pair consisting of an $A$-morphism $A[t] \to R$ and an element of $Q \otimes_{A[t]} R$. In other words, an $R$-point of $F_Q$ is defined by a triple $(n, l, t)$, $n \in N \otimes_A R$, $l \in L \otimes_A R$, $t \in R$, such that

$$\pi(n) = tl \tag{358}$$

where $\pi$ is the projection $N \otimes_A R \to L \otimes_A R$.

So $F_Q$ is a closed ind-subscheme of $F_N \times F_L \times \mathbb{A}^1$ defined by the equation (358). Therefore $X \subset \widehat{F}_N \times \widehat{F}_L \times \widehat{\mathbb{A}}^1$ is defined by the same equation (358) (here $\widehat{\mathbb{A}}^1$ is the completion of $\mathbb{A}^1$ at $0 \in \mathbb{A}^1$).

Now suppose that $(X, S)$ is $A$-isomorphic to $\widehat{F}_M$. Then $M$ is the module of global sections of the restriction of $\Theta_{X/A}$ to $S$. Linearizing (358) we see that

$$M = N' \oplus L \oplus A \subset N \oplus L \oplus A. \tag{359}$$

The composition

$$\widehat{F}_M \xrightarrow{\sim} X \hookrightarrow \widehat{F}_N \times \widehat{F}_L \times \mathbb{A}^1 \tag{360}$$

is defined by a "Taylor series" $\sum_{n=1}^{\infty} \varphi_n$ where $\varphi_n$ is a homogeneous polynomial map $M \to N \oplus L \oplus A$ of degree $n$; clearly $\varphi_1$ is the embedding (359). Set $f = \mathrm{pr}_N \circ \varphi_2$ where $\mathrm{pr}_N$ is the projection $N \oplus L \oplus A \to N$. Since $M = N' \oplus L \oplus A$ the module of quadratic maps $M \to N$ contains as a direct summand the module of bilinear maps $L \times A \to N$, i.e., $\mathrm{Hom}(L, N)$. The image of $f$ in $\mathrm{Hom}(L, N)$ defines a splitting of (357) (use the fact that the morphism (360) factors through the ind-subscheme $X \subset \widehat{F}_N \times \widehat{F}_L \times \mathbb{A}^1$ defined by the equation (358)). So we get a contradiction.

**7.12.20.** *Proposition.* Let $X$ be a formally smooth ind-scheme over a ring $A$. Suppose that one of the following two assumptions holds:

(i) $X$ is ind-affine;

(ii) $A$ is noetherian and $X$ is of ind-finite type over $A$.

Then $X$ is the union of a directed family of ind-closed $\aleph_0$-ind-schemes formally smooth over $A$.

*Proof.* It suffices to show that for every increasing sequence of closed subschemes $Y_n \subset X$ there is an ind-closed $\aleph_0$-ind-scheme $Y \subset X$ formally smooth over $A$ such that $Y \supset Y_n$ for all $n$.

Suppose that $X$ is ind-affine. Then each $Y_n$ is affine. Represent $Y_n$ as a closed subscheme of a formally smooth scheme $V_n$ over $A$ (e.g., represent the coordinate ring of $Y_n$ as a quotient of a polynomial algebra over $A$). Let $Y'_n \subset V_n$ be the first infinitesimal neighbourhood of $Y_n$ in $V_n$. Since $X$ is formally smooth the morphism $Y_n \hookrightarrow X$ extends to a morphism $Y'_n \to Z_n \subset X$ for some closed subscheme $Z_n \subset X$. Set $Y_n^{(2)} := Z_1 \cup \ldots \cup Z_n$. Now apply the above construction to $(Y_n^{(2)})$ and get a new sequence $(Y_n^{(3)})$, etc. The union of all $Y_n^{(k)}$ is formally smooth over $A$.

If $X$ is ind-quasicompact but not ind-affine an obvious modification of the above construction yields an ind-closed $\aleph_0$-ind-scheme $Y \subset X$ containing all the $Y_n$ such that for any affine scheme $S$ over $A$ and any closed subscheme $S_0 \subset S$ defined by an Ideal $\mathcal{I} \subset \mathcal{O}_S$ with $\mathcal{I}^2 = 0$ every $A$-morphism $S_0 \to Y$ extends *locally* to a morphism $S \to Y$. If assumption (ii) holds then this implies the existence of a global extension. $\square$

**7.12.21.** We are going to describe formally smooth affine $\aleph_0$-formal schemes of ind-finite type over a field $C$ (according to 7.12.20 the general case can, in some sense, be reduced to the $\aleph_0$ case). First of all we have the following examples.

(0) Set $R_{mn} := C[x_1, \ldots, x_m][[x_{m+n}, \ldots, x_{m+n}]]$. Then $\operatorname{Spf} R_{mn}$ is a formally smooth affine $\aleph_0$-formal scheme over $C$.

(i) Let $I \subset R_{mn}$ be an ideal, $A := R_{mn}/I$. Denote by $\mathcal{I}$ the sheaf of ideals on $\operatorname{Spf} R_{mn}$ corresponding to $I$. Of course, $\operatorname{Spf} A$ is an affine $\aleph_0$-formal scheme of ind-finite type over $C$. It is formally smooth if and only if for every $u \in \operatorname{Spf} A$ the stalk of $\mathcal{I}$ at $u$ is generated by some $f_1, \ldots, f_r \in I$ such that the Jacobi matrix $(\frac{\partial f_i}{\partial x_j}(u))$ has rank $r$.

(ii) Suppose that $A$ is as in (i) and $\operatorname{Spf} A$ is formally smooth. Then $\operatorname{Spf} A[[y_1, y_2, \ldots]]$ is a formally smooth affine $\aleph_0$-formal scheme of ind-finite type over $C$.

In 7.12.22 and 7.12.23 we will show that every connected formally smooth affine $\aleph_0$-formal scheme of ind-finite type over a field is isomorphic to a formal scheme from Example (i) or (ii).

**7.12.22.** *Proposition.* Let $X$ be a formally smooth affine formal scheme of ind-finite type over a field $C$ such that $\Theta_X$ is coherent (i.e., the restriction of $\Theta_X$ to every closed subscheme of $X$ is finitely generated). Then $X$ is isomorphic to a formal scheme from Example 7.12.21(i).

*Proof.* Represent $X$ as $\varinjlim \operatorname{Spec} A_i$ so that for $i \leq j$ the morphism $A_j \to A_i$ is surjective with nilpotent kernel. The algebras $A_i$ are of finite type. We can assume that the set of indices $i$ has a smallest element 0. Put $I_i := \operatorname{Ker}(A_i \to A_0)$.

*Lemma.* For every $k \in \mathbb{N}$ there exists $i_1$ such that the morphisms $A_i/I_i^k \to A_{i_1}/I_{i_1}^k$ are bijective for all $i \geq i_1$.

Assuming the lemma set $A_{(k)} := A_i/I_i^k$ for $i$ big enough, $I_{(k)} := \operatorname{Ker}(A_{(k)} \to A_0)$. Clearly $A_{(1)} = A_0$, $A_{(k)} = A_{(k+1)}/I_{(k+1)}^k$, $I_{(k)} = I_{(k+1)}/I_{(k+1)}^k$. One has $X = \operatorname{Spf} A$, $A := \varprojlim A_{(k)}$. Choose generators $\bar{x}_1, \ldots, \bar{x}_m$ of the algebra $A_{(1)} = A_0$ and generators $\bar{x}_{m+1}, \ldots, \bar{x}_{m+n}$ of the $A_0$-module $I_{(2)}$. Lift $\bar{x}_1, \ldots, \bar{x}_{m+n}$ to $\widetilde{x}_1, \ldots, \widetilde{x}_{m+n} \in A$. Set $R_{mn} := C[x_1, \ldots, x_m][[x_{m+1}, \ldots, x_{m+n}]]$. There is a unique continuous homomorphism $f : R_{mn} \to A$ such that $x_i \mapsto \widetilde{x}_i$. Clearly $f$ is surjective. Moreover, $f$ induces surjections $\mathfrak{a}^k \to \operatorname{Ker}(A \to A_{(k)})$, where $\mathfrak{a} \subset R_{mn}$ is the ideal generated by $x_{m+1}, \ldots, x_{m+n}$. So $f$ is an open map. Therefore $f$ induces a topological isomorphism between $A$ and a quotient of $R_{mn}$. The proposition follows.

It remains to prove the lemma. There exists $i_0$ such that for every $i \geq i_0$ the morphism $\operatorname{Spec} A_{i_0} \to \operatorname{Spec} A_i$ induces isomorphisms between tangent spaces (indeed, since the restriction of $\Theta_X$ to $\operatorname{Spec} A_0$ is finitely generated the functor (355) corresponding to the $A_0$-modules $N_i := \Omega_i \otimes_{A_i} A_0$ is isomorphic to the functor $L \mapsto \operatorname{Hom}(Q, L)$ for some $A_0$-module $Q$, so there exists $i_0$ such that $N_i = N_{i_0}$ for $i \geq i_0$). We can assume that $i_0 = 0$. Set $Y_i := \operatorname{Spec} A_i/I_i^k$ (in particular, $Y_0 = \operatorname{Spec} A_0$). The morphisms $Y_0 \to Y_i$ induce isomorphisms between tangent spaces.

Represent $A_0$ as $C[x_1,\dots,x_n]/J$ and set $\widetilde{Y}_0 := \operatorname{Spec} C[x_1,\dots,x_n]/J^k$. Since $X$ is formally smooth the morphism $Y_0 \hookrightarrow X$ extends to a morphism $\widetilde{Y}_0 \to X$. Its image is contained in $Y_{i_1}$ for some $i_1$. Let us show that for $i \geq i_1$ the embedding $\nu : Y_{i_1} \hookrightarrow Y_i$ is an isomorphism. We have the morphism $f : \widetilde{Y}_0 \to Y_{i_1}$. On the other hand, the morphism $Y_0 \hookrightarrow \widetilde{Y}_0$ extends to $g : Y_i \to \widetilde{Y}_0$. The composition $\nu f g : Y_i \to Y_i$ induces the identity on $Y_0$. So $\nu f g$ is finite and induces isomorphisms between tangent spaces. Therefore $\nu f g$ is a closed embedding. Since $Y_i$ is noetherian a closed embedding $Y_i \to Y_i$ is an isomorphism. So $\nu f g$ is an isomorphism and therefore $\nu$ is an isomorphism. □

**7.12.23.** *Proposition.* Let $X$ be a connected formally smooth affine $\aleph_0$-formal scheme of ind-finite type over a field $C$ such that $\Theta_X$ is not coherent (i.e., the restriction of $\Theta_X$ to $X_{\mathrm{red}}$ is of infinite type). Then $X$ is isomorphic to a formal scheme from Example 7.12.21(ii).

*Proof.* We will construct a formally smooth morphism

$$X \to \operatorname{Spf} C[[y_1, y_2, \dots]]$$

whose fiber over $0 \in \operatorname{Spf} C[[y_1, y_2, \dots]]$ is a formal scheme from 7.12.21(i). Represent $X$ as $\varinjlim \operatorname{Spec} A_n$, $n \in \mathbb{N}$, so that for every $n$ the morphism $A_{n+1} \to A_n$ is surjective with nilpotent kernel. The algebras $A_n$ are of finite type. By 7.12.13 the restriction of $\Theta_X$ to $\operatorname{Spec} A_n$ is free; it has countable rank. This means that for every $n$ the projective system $(\Omega_{A_i} \otimes_{A_i} A_n)$, $i \geq n$, is equivalent to the projective system

$$\dots \to A_n^3 \to A_n^2 \to A_n$$

(here the map $A_n^{k+1} \to A_n^k$ is the projection to the first $k$ coordinates). So after replacing the sequence $(A_n)$ by its subsequence one gets the diagram

$$\dots \twoheadrightarrow \Omega_{A_3} \twoheadrightarrow F_2 \twoheadrightarrow \Omega_{A_2} \twoheadrightarrow F_1 \twoheadrightarrow \Omega_{A_1}$$

where the $F_n$ are finitely generated free $A_n$-modules and the $A_n$-modules $G_n := \mathrm{Ker}(F_{n+1} \otimes_{A_{n+1}} A_n \to F_n)$ are also free. For each $n$ choose a base $e_{n1}, \ldots, e_{nk_n} \in G_n$. Lift $e_{ni}$ to $\widetilde{e}_{ni} \in \mathrm{Ker}(\Omega_{A_{n+2}} \otimes_{A_{n+2}} A_n \to F_n) \subset \mathrm{Ker}(\Omega_{A_{n+2}} \otimes_{A_{n+2}} A_n \to \Omega_{A_n})$ and represent $\widetilde{e}_{ni}$ as $df_{ni}$, $f_{ni} \in \mathrm{Ker}(A_{n+2} \to A_2)$. Finally lift $f_{ni}$ to $\widetilde{f}_{ni} \in A := \varprojlim\limits_m A_m$ and organize the $f_{ni}$, $n \in \mathbb{N}$, $i \leq k_n$, into a sequence $\varphi_1, \varphi_2, \ldots$ . This sequence converges to 0, so one has a continuous morphism $C[[y_1, y_2, \ldots]] \to A$ such that $y_i \mapsto \varphi_i$. It induces a morphism

$$f : X \to Y := \operatorname{Spf} C[[y_1, y_2, \ldots]] \tag{361}$$

It follows from the construction that the differential

$$df : \Theta_X \to f^* \Theta_Y \tag{362}$$

is surjective and its kernel is coherent (indeed, it is clear that these properties hold for the restriction of (362) to $\operatorname{Spec} A_1 \subset X$, so they hold for the restriction to $\operatorname{Spec} A_n$, $n \in \mathbb{N}$).

*Lemma.* A morphism $f : X \to Y$ of formally smooth ind-schemes of ind-finite type is formally smooth if and only if its differential (362) is surjective. In this case $\Theta_{X/Y}$ is the kernel of (362).

Assuming the lemma we see that (361) is formally smooth and $\Theta_{X/Y}$ is coherent. So the fiber $X_0$ of (361) over $0 \in Y$ satisfies the conditions of Proposition 7.12.22. Therefore $X_0$ is isomorphic to a formal scheme from Example 7.12.21(i). Let us show that $X$ is isomorphic to $\widetilde{X} := X_0 \times Y$. Indeed, since $X$ is formally smooth over $Y$ the embedding $X_0 \hookrightarrow X$ extends to a $Y$-morphism $\alpha : \widetilde{X} \to X$. Since $\widetilde{X}$ is formally smooth over $Y$ the embedding $X_0 \hookrightarrow \widetilde{X}$ extends to a $Y$-morphism $\beta : X \to \widetilde{X}$. Both $\alpha$ and $\beta$ are ind-closed embeddings (if a morphism $\nu : Y \to Z$ of schemes of finite type induces an isomorphism $Y_{\mathrm{red}} \to Z_{\mathrm{red}}$ and each geometric fiber of $\nu$ is reduced then $\nu$ is a closed embedding). The $Y$-morphism $\beta\alpha : X_0 \times Y \to X_0 \times Y$

induces the identity over $0 \in Y$, so $\beta\alpha$ is an isomorphism. Therefore $\alpha$ and $\beta$ are isomorphisms, so we have proved the proposition.

The proof of the lemma is standard. The statement concerning $\Theta_{X/Y}$ follows from the definitions. To prove the first statement take an affine scheme $S$ with an Ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I}^2 = 0$ and let $S_0 \subset S$ be the subscheme corresponding to $\mathcal{I}$. For a morphism $\psi : S_0 \to X$ denote by $E_X(S, \mathcal{I}, \psi)$ (resp. $E_Y(S, \mathcal{I}, \psi)$) the set of extensions of $\psi$ (resp. of $f\psi$) to a morphism $S \to X$ (resp. $S \to Y$). Formal smoothness of $f$ means that $f_* : E_X(S, \mathcal{I}, \psi) \to E_Y(S, \mathcal{I}, \psi)$ is surjective for all $S$, $\mathcal{I}$, $\psi$ as above. Since $X$ and $Y$ are formally smooth $E_X(S, \mathcal{I}, \psi)$ and $E_Y(S, \mathcal{I}, \psi)$ are non-empty. According to 16.5.14 from [Gr67] they are torsors (i.e., non-empty affine spaces) over $V_X(S, \mathcal{I}, \psi) := \mathrm{Hom}(\psi^*\Omega_X, \mathcal{I}) = \Gamma(S_0, \psi^*\Theta_X \otimes \mathcal{I})$ and $V_Y(S, \mathcal{I}, \psi) = \Gamma(S_0, \psi^* f^*\Theta_Y \otimes \mathcal{I})$. The map $f_*$ is affine and the corresponding linear map $\Gamma(S_0, \psi^*\Theta_X \otimes \mathcal{I}) \to \Gamma(S_0, \psi^* f^*\Theta_Y \otimes \mathcal{I})$ is induced by (362). So the first statement of the lemma is clear. $\square$

**7.12.24.** *Examples of Mittag-Leffler modules.*

(a) According to [RG], p.77, 2.4.1 for every noetherian $A$ and projective $A$-module $P$ the $A$-module $P^* := \mathrm{Hom}_A(P, A)$ is strictly Mittag-Leffler and flat. To prove that $P^*$ is strictly Mittag-Leffler one can argue as follows: for any $f : F \to P^*$ with $F$ of finite type the image of $f^* : P \to F^*$ is generated by some $l_1, \ldots, l_n \in F^*$; the $l_i$ define $u : F \to A^n$ such that $f = gu$ and $u = hf$ for some $g : A^n \to P^*$, $h : P^* \to A^n$.

In particular, if $A$ is noetherian then for every set $I$ the $A$-module $A^I$ is strictly Mittag-Leffler and flat.

(a′) It is well known that if $A$ is a Dedekind ring and not a field then $A^I$ is not projective for infinite $I$. Indeed, we can assume that $I$ is countable. Fix a non-zero prime ideal $\mathfrak{p} \subset A$ and consider the submodule $M$ of elements $a = (a_i) \in A^I$ such that $a_i \to 0$ in the

$\mathfrak{p}$-adic topology. If $A^I$ were projective the localization $M_\mathfrak{p}$ would be free. Since $M/\mathfrak{p}M$ has countable dimension $M_\mathfrak{p}$ would have countable rank. But $M$ contains a submodule isomorphic to $A^I$, so $(A^I)_\mathfrak{p}$ would have countable rank. This is impossible because the dimension of $(A^I)_\mathfrak{p}/\mathfrak{p}\cdot(A^I)_\mathfrak{p} = (A/\mathfrak{p})^I$ is uncountable.

(a″) Suppose that $A$ is finitely generated over $\mathbb{Z}$ or over a field*). If $A$ is not Artinian and $I$ is infinite then $A^I$ is not projective: use (a′) and the existence of a Dedekind ring $B$ finite over $A$.

(b) If $L$ is a non-projective flat Mittag-Leffler module then there exists a non-split exact sequence $0 \to N' \to N \to L \to 0$ where $N$ and $N'$ are flat Mittag-Leffler modules. Indeed, if $N$ is a projective module and $N \to L$ is an epimorphism then it does not split and $\operatorname{Ker}(N \to L)$ is Mittag-Leffler ([RG], p.71, 2.1.6).

(c) It is noticed in [RG] that if

$$0 \to A \xrightarrow{f} M' \to M \to 0 \tag{363}$$

is a non-split exact sequence of $A$-modules and $M$ is flat and Mittag-Leffler then $M'$ is Mittag-Leffler but not strictly Mittag-Leffler. Indeed, if $M'$ were strictly Mittag-Leffler then there would exist a module $G$ of finite presentation and a morphism $u : A \to G$ such that $f = gu$ and $u = hf$ for some $g : G \to M'$, $h : M' \to G$. Since $M$ is a direct limit of finitely generated projective modules one can assume that $\operatorname{Im} g \subset \operatorname{Im} f$. Then $gh$ would define a splitting of (363), i.e., one gets a contradiction.

Here is another argument. The fiber of $F_{M'}$ over $0 \in F_M$ is a closed subscheme of $F_{M'}$ canonically isomorphic to $\operatorname{Spec} A \times \mathbb{A}^1$; if (363) is non-split then the projection $\operatorname{Spec} A \times \mathbb{A}^1 \to \mathbb{A}^1$ cannot be extended to a function $F_{M'} \to \mathbb{A}^1$, so by 7.12.16 $M'$ is not strictly Mittag-Leffler.

*) We do not know whether it suffices to assume $A$ noetherian.

(d) Let $A$ be a Dedekind ring which is neither a field nor a complete local ring. Then according to [RG], p.76 there is a non-split exact sequence (363) such that $M$ is a flat strictly Mittag-Leffler $A$-module. Here is a construction. Let $K$ denote the field of fractions of $A$. Fix a non-zero prime ideal $\mathfrak{p} \subset A$ and consider the completions $\widehat{A}_{\mathfrak{p}}$, $\widehat{K}_{\mathfrak{p}}$; then $\widehat{A}_{\mathfrak{p}} \neq A$, $\widehat{K}_{\mathfrak{p}} \neq K$. Denote by $M$ the module of sequences $(a_n)$ such that $a_n \in \mathfrak{p}^{-n}$ and $(a_n)$ converges in $\widehat{K}_{\mathfrak{p}}$; we have the morphism $\lim : M \to \widehat{K}_{\mathfrak{p}}$. Notice that $M$ is a strictly Mittag-Leffler module*). Indeed, according to (a) above $\prod_{n=1}^{\infty} \mathfrak{p}^{-n}$ is strictly Mittag-Leffler and $(\prod_{n=1}^{\infty} \mathfrak{p}^{-n})/M$ is flat, so $M$ is strictly Mittag-Leffler. We claim that $\mathrm{Ext}(M, A) \neq 0$, i.e., the morphism $\varphi : \mathrm{Hom}(M, K) \to \mathrm{Hom}(M, K/A)$ is not surjective. More precisely, let $l : M \to K/A$ be the composition of $\lim : M \to \widehat{K}_{\mathfrak{p}}$ and the morphisms $\widehat{K}_{\mathfrak{p}} \to \widehat{K}_{\mathfrak{p}}/\widehat{A}_{\mathfrak{p}} \hookrightarrow K/A$. We will show that $l \notin \mathrm{Im}\, \varphi$.

Suppose that $l$ comes from $\widetilde{l} : M \to K$. The restriction of $\widetilde{l}$ to $\mathfrak{p}^{-n} \subset M$ defines $c_n \in \mathrm{Hom}(\mathfrak{p}^{-n}, A) = \mathfrak{p}^n$. Then $\widetilde{l} = \widetilde{l}'$ where $\widetilde{l}' : M \to K_{\mathfrak{p}}$ maps $(a_n) \in M$ to

$$\sum_{n=1}^{\infty} c_n a_n + \lim_{n\to\infty} a_n \,. \tag{364}$$

Indeed, $\widetilde{l}' - \widetilde{l}$ is a morphism $M/M_0 \to \widehat{A}_{\mathfrak{p}}$ where $M_0$ is the set of $(a_n) \in M$ such that $a_n = 0$ for $n$ big enough; on the other hand, $\mathrm{Hom}(M/M_0, \widehat{A}_{\mathfrak{p}}) = 0$ because $M/M_0$ is $\mathfrak{p}$-divisible (i.e., $\mathfrak{p}M + M_0 = M$). Since $\widetilde{l}' = \widetilde{l}$ the expression (364) belongs to $K \subset \widehat{K}_{\mathfrak{p}}$ for every sequence $(a_n) \in M$. This is impossible (consider separately the case where the number of nonzero $c_n$'s is finite and the case where it is infinite).

*)The fact that $M$ is a Mittag-Leffler module is clear: $A$ is a Dedekind ring, $M$ is flat, and for every finite-dimensional subspace $V \subset M \otimes K$ the module $V \cap M$ is finitely generated

*Remark.* In (d) we had to exclude the case where $A$ is a complete local ring. The true reason for this is explained by the following results:

1) according to [J] if $A$ is a complete local noetherian ring, $M$ is a flat $A$-module, and $N$ is a finitely generated $A$-module then $\mathrm{Ext}(M, N) = 0$;
2) according to [RG] (p.76, Remark 4 from 2.3.3) if $A$ is a projective limit of Artinian rings (is this the meaning of the words "linearly compact" from [RG]?) then every (flat?) Mittag-Leffler $A$-module is strictly Mittag-Leffler. (In [RG] there is no flatness assumption, but is their argument correct without this assumption? e.g., why the $F_i$ from [RG] are linearly compact?)

**7.13. BRST basics.** The BRST construction is a refined version of Hamiltonian reduction; it is especially relevant in the infinite-dimensional setting. In the main body of this article we invoke BRST twice: first to define the Feigin-Frenkel isomorphism and then to construct the localization functor $L\Delta$ used in the proof of the Hecke property. In this section we give a brief account of the general BRST construction; the functor $L\Delta$ is studied in the next section.

The usual mathematical references for BRST are [F84], [FGZ86], [KS], and [Ak]. We tried to write down an exposition free from redundand structures (such as $\mathbb{Z}$-grading, normal ordering, etc.).

We start with the finite-dimensional setting. Then, after a digression about the Tate central extension, we explain the infinite-dimensional version.

**7.13.1.** Let $F$ be a finite-dimensional vector space. Denote by $\mathrm{Cl}^{\cdot} = \mathrm{Cl}^{\cdot}_F$ the Clifford algebra of $F \oplus F^*$ equipped with the grading such that $F$ has degree -1 and $F^*$ has degree 1. We consider $\mathrm{Cl}^{\cdot}$ as an algebra in the tensor category of graded vector spaces[*)]. Set $\mathrm{Cl}^{\cdot}_i := \Lambda^{\leq i} F {\cdot} \Lambda F^* \subset \mathrm{Cl}^{\cdot}$. Then $\mathrm{Cl}^{\cdot}_0 = \Lambda^{\cdot} F^* \subset \mathrm{Cl}^{\cdot}_1 \subset ...$ is a ring filtration on $\mathrm{Cl}^{\cdot}$. The *classical Clifford algebra* $\mathcal{C}l^{\cdot} = \mathcal{C}l^{\cdot}_F := \mathrm{gr}\,\mathrm{Cl}^{\cdot}$ is commutative (as a graded algebra), so it is

[*)] with the "super" commutativity constraint.

a Poisson algebra in the usual way. Set $\mathcal{C}l_i^{\cdot} := \mathrm{gr}_i\, \mathrm{Cl}^{\cdot}$. The commutative graded algebra $\mathcal{C}l^{\cdot}$ is freely generated by $F = \mathcal{C}l_1^{-1}$ and $F^* = \mathcal{C}l_0^1$. The Poisson bracket $\{,\}$ vanishes on $F$ and $F^*$, and for $f \in F$, $f^* \in F^*$ one has $\{f, f^*\} = f^*(f)$.

The subspace $\mathcal{C}l_1^0$ is a Lie subalgebra of $\mathcal{C}l$; it normalizes $F$ and $F^*$ and the corresponding adjoint action identifies it with $\mathrm{End}_F$ and $\mathrm{End}_{F^*}$. Let $E^{\mathrm{Lie}} = \mathrm{End}_F^{\mathrm{Lie}}$ be $\mathrm{End}_F$ considered as a Lie algebra. Then $E^\flat = \mathrm{End}_F^\flat := \mathrm{Cl}_1^0$ is a central extension of $E^{\mathrm{Lie}}$ by $\mathbb{C}$.

*Remarks.* (i) The action of Cl on $\Lambda F^* \overset{\sim}{\rightarrow} \mathrm{Cl} / \mathrm{Cl} \cdot F$ identifies it with the algebra of differential operators on the "odd" vector space $F^{odd}$. The filtration on Cl is the usual filtration by degree of the differential operator, so $\mathcal{C}l$ is the Poisson algebra of functions on the cotangent bundle to $F^{odd}$.

(ii) (valid only in the finite-dimensional setting) The extension $\mathrm{End}_F^\flat$ splits (in a non-unique way). Indeed, we have splittings $s', s" : E^{\mathrm{Lie}} \to E^\flat$ which identify $E^{\mathrm{Lie}}$ with, respectively, $F^* \cdot F$ and $F \cdot F^*$. Any other splitting equals $s_\lambda = \lambda s' + (1-\lambda)s"$ for certain $\lambda \in \mathbb{C}$. For example $s_{1/2}$ is the "unitary" splitting which may also be defined as follows. Notice that Cl carries a canonical anti-automorphism (as a *graded* algebra) which is identity on $F$ and $F^*$. It preserves $\mathrm{Cl}_1^0$, and the "unitary" splitting is the -1 eigenspace.

**7.13.2.** Here is the "classical" version of the BRST construction. Let $\mathfrak{n}$ be a finite-dimensional Lie algebra, $\mathcal{R}$ a Poisson algebra, $l^c : \mathfrak{n} \to \mathcal{R}$ a morphism of Lie algebras[*)]. Set $\mathcal{C}l^{\cdot} := \mathcal{C}l_{\mathfrak{n}}^{\cdot}$. The adjoint action of $\mathfrak{n}$ yields a morphism of Lie algebras $a^c : \mathfrak{n} \to \mathcal{C}l_1^0$. Set $\mathcal{A}^{\cdot} := \mathcal{C}l^{\cdot} \otimes \mathcal{R}$; this is a Poisson graded algebra. It also carries an additional grading $\mathcal{A}_{(i)}^{\cdot} := \mathcal{C}l_i^{\cdot} \otimes \mathcal{R}$ compatible with the product (but not with the Poisson bracket). We have the morphism of Lie algebras $\mathcal{L}ie : \mathfrak{n} \to \mathcal{A}^0$, $n \mapsto \mathcal{L}ie_n := a^c(n) \otimes 1 + 1 \otimes l^c(n)$. Below for $n \in \mathfrak{n}$ we denote by $i_n^c$ the corresponding element of $\mathcal{C}l_1^{-1} \subset \mathcal{A}_{(1)}^{-1}$. One has $\{\mathcal{L}ie_{n_1}, i_{n_2}^c\} = i_{[n_1,n_2]}^c$.

*) "c" for "classical".

The following key lemma, as well as its "quantum" version 7.13.7, is due essentially to Akman [Ak].

**7.13.3.** *Lemma.* There is a unique element $Q^c = Q^c_{\mathcal{A}} \in \mathcal{A}^1$ such that for any $n \in \mathfrak{n}$ one has $\{Q^c, i^c_n\} = \mathcal{L}ie_n$. In fact, $Q^c \in \mathcal{A}^1_{(\leq 1)}$. One has $\{Q^c, Q^c\} = 0$.

*Proof.* Let us consider $\mathcal{A}$ as a $\Lambda\mathfrak{n}$-module where $n \in \mathfrak{n} = \Lambda^1\mathfrak{n}$ acts as $Ad_{i^c_n} = \{i^c_n, \cdot\}$. The subspace of elements killed by all $Ad_{i^c_n}$'s (i.e., the centralizer of $\mathfrak{n} \subset \mathcal{A}^{-1}_{(1)}$) equals $\Lambda\mathfrak{n} \otimes \mathcal{R}$. This is a subspace of $\mathcal{A}^{\leq 0}$, so the unicity of $Q^c$ is clear. Our $\Lambda\mathfrak{n}$-module is free, so the existence of $Q^c$ follows from the fact that the map $n_1, n_2 \mapsto \{\mathcal{L}ie_{n_1}, i^c_{n_2}\}$ is skew-symmetric. Our $Q^c$ belongs to $\mathcal{A}^1_{(\leq 1)}$ since $\mathcal{L}ie_n \in \mathcal{A}^0_{(\leq 1)}$. Finally, since $\{Q^c, Q^c\} \in \mathcal{A}^2$, to check that it vanishes it suffices to show that $Ad_{i^c_n} Ad_{i^c_{n'}}(\{Q^c, Q^c\}) = 0$ for any $n, n' \in \mathfrak{n}$. Indeed, $Ad_{i^c_n} Ad_{i^c_{n'}}(\{Q^c, Q^c\}) = 2Ad_{i^c_n}(\{\mathcal{L}ie_{n'}, Q^c\}) = 2\{i^c_{[n,n']}, Q^c\} + 2\{\mathcal{L}ie_{n'}, \mathcal{L}ie_n\} = 0$. □

*Remark.* Denote by $\mathfrak{n}^{\cdot}_{\heartsuit}$ the Lie graded algebra whose non-zero components are $\mathfrak{n}^{-1}_{\heartsuit} = \mathfrak{n}$, $\mathfrak{n}^0_{\heartsuit} = \mathfrak{n}$, $\mathfrak{n}^1_{\heartsuit} = \mathbb{C} = \mathbb{C}\cdot Q$, the Lie bracket on $\mathfrak{n}^0_{\heartsuit}$ coincides with that of $\mathfrak{n}$, the adjoint action of $\mathfrak{n}^0_{\heartsuit}$ on $\mathfrak{n}^{-1}_{\heartsuit}$ is the adjoint action of $\mathfrak{n}$, and the operator $Ad_Q : \mathfrak{n}^{-1}_{\heartsuit} \to \mathfrak{n}^0_{\heartsuit}$ is $\mathrm{id}_{\mathfrak{n}}$. So $\mathfrak{n}_{\heartsuit}$ equipped with the differential $Ad_Q$ is a Lie DG algebra[*)]. Then 7.13.3 says that there is a canonical morphism of Lie graded algebras $\mathcal{L}ie : \mathfrak{n}^{\cdot}_{\heartsuit} \to \mathcal{A}^{\cdot}$ whose components are, respectively, $n \mapsto i^c_n$, $n \mapsto \mathcal{L}ie_n$, $Q \mapsto Q^c$.

**7.13.4.** Set $d := Ad_{Q^c} = \{Q^c, \cdot\}$. This is a derivation of $\mathcal{A}^{\cdot}$ of degree 1 and square 0. Thus $\mathcal{A}$ is a Poisson DG algebra; it is called the *BRST reduction* of $\mathcal{R}$. The morphism $\mathcal{L}ie : \mathfrak{n}_{\heartsuit} \to \mathcal{A}$ is a morphism of Lie DG algebras.

One says that the BRST reduction is *regular* if $H^i\mathcal{A} = 0$ for $i \neq 0$.

It is easy to see that $Q^c = Q_1 + Q_0$ where $Q_1 \in \mathcal{A}^1_{(1)} = \mathfrak{n} \otimes \Lambda^2\mathfrak{n}^* \otimes \mathcal{R}$ and $Q_0 \in \mathcal{A}^1_{(0)} = \mathfrak{n}^* \otimes \mathcal{R}$ are, respectively, the image of $\frac{1}{2}a^c \in \mathrm{Hom}(\mathfrak{n}, \mathcal{C}l^0_1) = \mathfrak{n}^* \otimes \mathcal{C}l^0_1 \subset \mathcal{A}^1 \otimes \mathcal{A}^0$ by the product map, and $l \in \mathrm{Hom}(\mathfrak{n}, \mathcal{R}) = \mathcal{A}^1_{(1)}$. Decomposing

*)Notice that $\mathfrak{n}_{\heartsuit}/\mathfrak{n}^1_{\heartsuit}$ is the Lie DG algebra $\mathfrak{n}_{\Omega}$ from 7.6.3.

the differential by the bigrading we see that $\mathcal{A}$ is the total complex of the bicomplex with bidifferentials $d' : \mathcal{A}^i_{(j)} \to \mathcal{A}^{i+1}_{(j)}$, $d'' : \mathcal{A}^i_{(j)} \to \mathcal{A}^{i+1}_{(j-1)}$.

The BRST differential preserves the filtration $\mathcal{A}_{(\leq i)}$. In particular $\mathcal{A}_{(0)} = C(\mathfrak{n}, \mathcal{R})$ is a DG subalgebra of $\mathcal{A}$, hence one has a canonical morphism of graded algebras

$$H^{\cdot}(\mathfrak{n}, \mathcal{R}) \to H^{\cdot}\mathcal{A}. \tag{365}$$

Notice that $(\mathcal{A}^{\cdot}_{(-\cdot)}, d'')$ is the Koszul complex $P := \Lambda^{-\cdot}\mathfrak{n} \otimes \mathcal{R}$ for $l^c : \mathfrak{n} \to \mathcal{R}$. So $\mathcal{A}$ is the Chevalley complex $C^{\cdot}(\mathfrak{n}, P)$ of Lie algebra cochains of $\mathfrak{n}$ with coefficients in $P$. The obvious projection $P \to \mathcal{R}/\mathcal{R}l^c(\mathfrak{n})$ yields an isomorphism of DG algebras $\mathcal{A}/\mathcal{I} \overset{\sim}{\rightarrow} C(\mathfrak{n}, \mathcal{R}/\mathcal{R}l^c(\mathfrak{n}))$ where $\mathcal{I} \subset \mathcal{A}$ is the DG ideal generated by elements $i^c_n$, $n \in \mathfrak{n}$. Passing to cohomology we get a canonical morphism of graded algebras

$$H^{\cdot}\mathcal{A} \to H^{\cdot}(\mathfrak{n}, \mathcal{R}/\mathcal{R}l^c(\mathfrak{n})). \tag{366}$$

We say that $l^c$ is *regular* if $H_i(P) = 0$ for $i \neq 0$.

**7.13.5.** *Lemma.* If $l^c$ is regular then (366) is an isomorphism.

*Proof.* Regularity means that the projection $P \to \mathcal{R}/\mathcal{R}l^c(\mathfrak{n})$ is a quasi-isomorphism. Hence $\mathcal{A} \to C^{\cdot}(\mathfrak{n}, \mathcal{R}/\mathcal{R}l^c(\mathfrak{n}))$ is also a quasi-isomorphism. $\square$

Thus $H^i\mathcal{A}$ vanish for negative $i$ and $H^0\mathcal{A} \overset{\sim}{\rightarrow} [\mathcal{R}/\mathcal{R}l^c(\mathfrak{n})]^{\mathfrak{n}}$ which is the usual Hamiltonian reduction of $\mathcal{R}$ with respect to the Hamiltonian action $l^c$.

**7.13.6.** Now let us pass to the "quantum" version of BRST. Let $\mathfrak{n}$ be a finite-dimensional Lie algebra. Set $\mathrm{Cl}^{\cdot} := \mathrm{Cl}^{\cdot}_{\mathfrak{n}}$. Denote by $\mathfrak{n}^{\flat}$ the central extension of $\mathfrak{n}$ by $\mathbb{C}$ defined as the pull-back of $\mathrm{End}^{\flat}_{\mathfrak{n}}$ by the adjoint action morphism $\mathfrak{n} \to \mathrm{End}_{\mathfrak{n}}$ (see the end of 7.13.1 for the notation). In other words, $\mathfrak{n}^{\flat}$ is a central extension of $\mathfrak{n}$ by $\mathbb{C}$ equipped with a Lie algebra map $a : \mathfrak{n}^{\flat} \to \mathrm{Cl}^0$ such that $a(1_{\mathfrak{n}^{\flat}}) = 1$*) and the action of $\mathfrak{n}$ on Cl induced by the adjoint action on $\mathfrak{n} \oplus \mathfrak{n}^*$ coincides with the adjoint action by $a$.

*)Here $1_{\mathfrak{n}^{\flat}}$ is the generator of $\mathbb{C} \subset \mathfrak{n}^{\flat}$.

Let $R$ be an associative algebra, $l : \mathfrak{n}^\flat \to R$ a morphism of Lie algebras such that $l(1_{\mathfrak{n}^\flat}) = -1$. Set $A^{\cdot} := \mathrm{Cl}^{\cdot} \otimes R$; this is an associative graded algebra. We have the morphism of Lie algebras $\mathrm{Lie} := a + l : \mathfrak{n} \to A^0$, $n \mapsto \mathrm{Lie}_n := a(n^\flat) + l(n^\flat)$ where $n^\flat$ is any lifting of $n$ to $\mathfrak{n}^\flat$. Below for $n \in \mathfrak{n}$ we denote by $i_n$ the corresponding element of $\mathrm{Cl}_1^{-1} \subset A^{-1}$. One has $[\mathrm{Lie}_{n_1}, i_{n_2}] = i_{[n_1,n_2]}$.

**7.13.7.** *Lemma.* There is a unique element $Q = Q_A \in A^1$ such that for any $n \in \mathfrak{n}$ one has $[Q, i_n] = \mathrm{Lie}_n$. In fact, $Q \in \mathrm{Cl}_1^1 \otimes R$. One has $Q^2 = 0$.

*Proof.* Coincides with that of the "classical" version 7.13.3. □

Set $d := Ad_Q$ *); this is a derivation of $A$ of degree 1 and square 0. Thus $A$ is an associative DG algebra called the *BRST reduction* of $R$. As in Remark after 7.13.3 and 7.13.4 we have a canonical morphism of Lie DG algebras $\mathrm{Lie} : \mathfrak{n}_\heartsuit \to A$ with components $n \mapsto i_n$, $n \mapsto \mathrm{Lie}_n$, $Q \mapsto Q_A$.

One says that the BRST reduction is *regular* if $H^i A = 0$ for $i \neq 0$.

Denote by $C(\mathfrak{n}, R)$ the Chevalley DG algebra of Lie algebra cochains of $\mathfrak{n}$ with coefficients in $R$ (with respect to the action $Ad_l$). As a graded algebra it equals $\Lambda^{\cdot}\mathfrak{n}^* \otimes R$, so it is a subalgebra of $A^{\cdot}$.

**7.13.8.** *Lemma.* The embedding $C(\mathfrak{n}, R) \subset A$ is compatible with the differentials.

*Proof.* It suuffices to show that on $R, \mathfrak{n}^* \subset A$ our differential equals, respectively, the dual to $\mathfrak{n}$-action map $R \to \mathfrak{n}^* \otimes R$ and the dual to bracket map $\mathfrak{n}^* \to \Lambda^2\mathfrak{n}^*$. As in the proof of unicity of $Q$ it suffices to check that $[i_n, [Q, r]] = [l(n), r]$ and $[i_{n_1}, [i_{n_2}, [Q, n^*]]] = n^*([n_1, n_2])$ for any $n, n_1, n_2 \in \mathfrak{n}, n^* \in \mathfrak{n}^*, r \in R$; this is an immediate computation. □

*Remark.* We see that $d$ preserves the ring filtration $\mathrm{Cl}_. \otimes R$. On $\mathrm{Cl}_i \otimes R / \mathrm{Cl}_{i-1} \otimes R = \Lambda^{\cdot+i}\mathfrak{n}^* \otimes \Lambda^i \mathfrak{n} \otimes R = C^{\cdot+i}(\mathfrak{n}, \Lambda^i\mathfrak{n} \otimes R)$ it coincides with the Chevalley differential.

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*) Of course, we take $Ad$ in the "super" sense, so for $v \in A^{odd}$ one has $dv = Qv + vQ$.

The embedding of DG algebras $C(\mathfrak{n}, R) \subset A$ yields the morphism of graded algebras

$$H^{\cdot}(\mathfrak{n}, R) \to H^{\cdot} A. \tag{367}$$

In particular, since the center $\mathfrak{z}$ of $R$ lies in $R^{\mathfrak{n}}$, we get the morphism

$$\mathfrak{z} \to H^0 A. \tag{368}$$

**7.13.9.** *Remark.* (valid only in the finite-dimensional setting) Let $I$ be the left DG ideal of $A$ generated by elements $i_n$, $n \in \mathfrak{n}$. The quotient complex $A/I$ may be computed as follows. Let $\mathfrak{n} \hookrightarrow \mathfrak{n}^{\flat}$ be the splitting defined by the splitting $s'$ from Remark (ii) in 7.13.1. Then $I$ is generated as a plain ideal by elements $i_n$ and $l(n)$, $n \in \mathfrak{n}$. Restricting the projection $A \to A/I$ to $C(\mathfrak{n}, R)$, we get the isomorphism of complexes $A/I \xrightarrow{\sim} C(\mathfrak{n}, R/Rl(\mathfrak{n}))$ which yields a morphism

$$H^{\cdot} A \to H^{\cdot}(\mathfrak{n}, R/Rl(\mathfrak{n})). \tag{369}$$

**7.13.10.** *Remark.* Let $C^{\cdot}$ be an irreducible graded $\mathrm{Cl}^{\cdot}$-module (such $C^{\cdot}$ is unique up to isomorphism and shift of the grading). If $M = (M^{\cdot}, d_M)$ is an $R$-complex (:= complex of $R$-modules) then $M \otimes C := (M^{\cdot} \otimes C^{\cdot}, d)$, where $d := d_M \otimes \mathrm{id}_C + Q\cdot$, is an $A$-complex (i.e., a DG $A$-module). The functor $\cdot \otimes C : (R\text{-complexes}) \to (A\text{-complexes})$ is an equivalence of categories.

**7.13.11.** Let us compare the "quantum" and "classical" settings. Assume that we are in situation 7.13.6. Let $R_0 \subset R_1 \subset \ldots$ be an increasing ring filtration on $R$ such that $\cup R_i = R$ and $\mathcal{R} := \mathrm{gr}\, R$ is commutative. Then $\mathcal{R}$ is a Poisson algebra in the usual way. We endow $A$ with the filtration $A.$ equal to the tensor product of filtrations $\mathrm{Cl}.$ and $R..$ Then $\mathcal{A} := \mathrm{gr}\, A$ equals $\mathcal{C}l \otimes \mathcal{R}$ as a Poisson graded algebra. Set $\mathcal{A}_i := \mathrm{gr}_i\, A$.

Assume that $l(\mathfrak{n}^{\flat}) \subset R_1$; let $l^c$ be the corresponding morphism $\mathfrak{n} \to \mathcal{R}_1$. Then $(\mathcal{R}, l^c)$ are data to define the "classical" BRST construction from 7.13.2. By 7.13.3 we have the corresponding "classical" BRST element $Q^c$. It is easy to see that $Q \in A_1$ and $Q^c$ equals to the image of $Q$ in $\mathcal{A}_1$.

Therefore the filtration $A.$ is stable with respect to the differential, and $\operatorname{gr} A$ coincides with the corresponding "classical" $\mathcal{A}$ as a Poisson DG algebra. Hence we have the spectral sequence converging to $H^{\cdot}A$ with the first term $E_1^{p,q} = H^{p+q}\mathcal{A}_{-p}$.

**7.13.12.** *Lemma.* (i) Assume that $l^c$ is regular. Then $H^i A = 0$ for $i < 0$ and $\operatorname{gr} H^0 A \subset [\mathcal{R}/\mathcal{R}l^c(\mathfrak{n})]^{\mathfrak{n}}$.

(ii) If, in addition, $H^i(\mathfrak{n}, \mathcal{R}/\mathcal{R}l^c(\mathfrak{n})) = 0$ for $i > 0$ then $H^i A = 0$ for $i \neq 0$ and $\operatorname{gr} H^0 A \overset{\sim}{\rightarrow} [\mathcal{R}/\mathcal{R}l^c(\mathfrak{n})]^{\mathfrak{n}}$.

*Proof.* Look at the spectral sequence and 7.13.5. □

**7.13.13.** One may compute the algebra $H^0 A$ explicitely in the following situation. Assume we are in situation 7.13.11 and $l : \mathfrak{n}^\flat \to R_1$ is injective. Denote by $\mathfrak{b}'$ the normalizer of $l(\mathfrak{n}^\flat)$ in $R_1$. So $\mathfrak{b}'$ is a Lie algebra which contains $\mathfrak{n}^\flat$, and we have the embedding of Lie algebras $l^{\mathfrak{b}} : \mathfrak{b}' \to R_1$ which extends $l$. Set $\mathfrak{b} := \mathfrak{b}'/\mathbb{C}$, so $\mathfrak{b}'$ is a central extension of $\mathfrak{b}$ by $\mathbb{C}$. The adjoint action of $\mathfrak{b}$ yields a morphism of Lie algebras $\mathfrak{b} \to \mathrm{End}_{\mathfrak{n}}$; denote by $\mathfrak{b}^\flat$ the pull-back of the central extension $\mathrm{End}^\flat$ (see 7.13.1). Then $\mathfrak{n}^\flat$ is a Lie subalgebra of $\mathfrak{b}^\flat$, and we have the morphism of Lie algebras $a^{\mathfrak{b}} : \mathfrak{b}^\flat \to \mathrm{Cl}_1^0$ which extends $a$.

Let $\mathfrak{b}^\natural$ be the Baer sum of extensions $\mathfrak{b}'$ and $\mathfrak{b}^\flat$. By construction we have a canonical splitting $s : \mathfrak{n} \to \mathfrak{b}^\natural$. It is invariant with respect to the adjoint action of $\mathfrak{b}$, so $s(\mathfrak{n})$ is an ideal in $\mathfrak{b}^\natural$. Set $\mathfrak{h}^\natural := \mathfrak{b}^\natural/s(\mathfrak{n})$; this is a central extension of $\mathfrak{h} := \mathfrak{b}/\mathfrak{n}$ by $\mathbb{C}$.

Set $\mathrm{Lie}^{\mathfrak{b}} := a^{\mathfrak{b}} \otimes 1 + 1 \otimes l^{\mathfrak{b}} : \mathfrak{b}^\natural \to A_1^0$. This is a morphism of Lie algebras which equals $\mathrm{id}_{\mathbb{C}}$ on $\mathbb{C} \subset \mathfrak{b}^\natural$. Its image commutes with $Q$ (since all our constructions were natural), i.e., it belongs to $\operatorname{Ker} d$. One has $\mathrm{Lie}^{\mathfrak{b}} \circ s = \mathrm{Lie} = d \circ i : \mathfrak{n} \to A^0$, so $\mathrm{Lie}^{\mathfrak{b}}$ yields a canonical morphism $\mathrm{Lie}^{\mathfrak{h}} : \mathfrak{h}^\natural \to H^0 A$. Let $U^\natural\mathfrak{h}$ be the twisted enveloping algebra of $\mathfrak{h}$ that corresponds to $\mathfrak{h}^\natural$. Our $\mathrm{Lie}^{\mathfrak{h}}$ yields a canonical morphism of associative algebras

$$h : U^{\natural}\mathfrak{h} \to H^0 A. \tag{370}$$

This morphism has the obvious "classical" version $h^c : \operatorname{Sym}\mathfrak{h} \to H^0\mathcal{A}$. Its composition with the projection $H^0\mathcal{A} \to [\mathcal{R}/l^c(\mathfrak{n})\mathcal{R}]^{\mathfrak{n}}$ (see (366)) is the obvious morphism $\operatorname{Sym}\mathfrak{h} \to [\mathcal{R}/l^c(\mathfrak{n})\mathcal{R}]^{\mathfrak{n}}$ whose restriction to $\mathfrak{h}$ is the composition of $l^{\mathfrak{b}}$ with the projection $R_1 \to R_1/R_0$.

**7.13.14.** *Lemma.* Assume that $l^c$ is regular and the morphism $\operatorname{Sym}\mathfrak{h} \to [\mathcal{R}/l^c(\mathfrak{n})\mathcal{R}]^{\mathfrak{n}}$ is an isomorphism. Then (370) is an isomorphism.

*Proof.* Use 7.13.12(i). □

**7.13.15.** *Examples.* (cf. [Ko78]) (i) We use notation of 7.13.13. Let $\mathfrak{g}$ be a (finite-dimensional) semi-simple Lie algebra, $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalgebra, $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$. Set $R := U\mathfrak{g}$ and let $R.$ be the standard filtration on $R$, so $\mathcal{R} = \operatorname{Sym}\mathfrak{g}$. The extension $\mathfrak{n}^{\flat}$ trivializes canonically since the adjoint action of $\mathfrak{n}$ is nilpotent. Let $l : \mathfrak{n} \to \mathfrak{g} \subset R$ be the obvious embedding. Then $\mathfrak{b}'$ is equal to $\mathfrak{b} \oplus \mathbb{C}$, so this extension is trivialized. Let us trivialize the extension $\mathfrak{b}^{\flat}$ by means of the splitting $s'$ from Remark (ii) from 7.13.1. Therefore we split the extension $\mathfrak{b}^{\natural}$, hence $U^{\natural}\mathfrak{h} = \operatorname{Sym}\mathfrak{h}$.

The conditions of 7.13.14 are valid. Indeed, $l^c$ is clearly regular, and the obvious embedding $i^c : \operatorname{Sym}\mathfrak{h} \hookrightarrow [\operatorname{Sym}(\mathfrak{g}/\mathfrak{n})]^{\mathfrak{n}}$ is an isomorphism since $\mathfrak{n}$ acts simply transitively along the generic fiber of the projection $(\mathfrak{g}/\mathfrak{n})^* \to \mathfrak{h}^*$. Therefore $h : \operatorname{Sym}\mathfrak{h} \xrightarrow{\sim} H^0 A$.

Let us show that the canonical morphism (368) $\mathfrak{z} \to H^0 A = \operatorname{Sym}\mathfrak{h}$ is the usual Harish-Chandra morphism. The obvious embedding $i : \operatorname{Sym}\mathfrak{h} \xrightarrow{\sim} [R/Rl(\mathfrak{n})]^{\mathfrak{n}}$ is an isomorphism, and, by definition, the Harish-Chandra morphism is composition of the embedding $\mathfrak{z} \hookrightarrow R^{\mathfrak{n}}$ and the inverse to this isomorphism. Consider the map $p : H^0 A \to [R/Rl(\mathfrak{n})]^{\mathfrak{n}}$ from (369). As follows from the definition of $p$ one has $ph = i$ which implies our assertion.

(ii) Let now $\psi : \mathfrak{n} \to \mathbb{C}$ be a non-degenerate character of $\mathfrak{n}$ (we use notation of 7.13.15 (i)). Set $R_t := R[t]$, $l_t := l + t\psi : \mathfrak{n} \to R_t$.

**7.13.16.** Let us pass to the infinite-dimensional setting. We need to fix some Clifford algebra notation. Let $F$ be a Tate vector space, so we have the ind-scheme $Gr(F)$ (see 7.11.2(iii)). The ind-scheme $Gr(F) \times Gr(F)$ carries a canonical line bundle $\lambda$ of "relative determinants". This is a graded line bundle equipped with canonical isomorphisms

$$\lambda_{(P,P'')} = \lambda_{(P,P')} \otimes \lambda_{(P',P'')} \tag{371}$$

and identifications $\lambda_{(P,P')} = \det(P/P')$ for $P' \subset P$ that satisfy the obvious compatibilities; here we assume that $\det(P/P')$ sits in degree $-\dim(P/P')$.

Consider the Tate vector space $F \oplus F^*$ equipped with the standard symmetric form and the Clifford algebra $\mathrm{Cl} = \mathrm{Cl}_F := \mathrm{Cl}(F \oplus F^*)$. Let $C$ be an irreducible discrete Cl-module*). Since $C$ is unique up to tensoring by a one-dimensional vector space*), the corresponding projective space $\mathbb{P}$ is canonically defined (this is an ind-scheme). For any c-lattice $P \subset F\widehat{\otimes}A$ denote by $\lambda^C_P$ the set of elements of $C \otimes A$ annihilated by Clifford operators from $P$ and $P^\perp \subset F^*\widehat{\otimes}A$. The $A$-submodule $\lambda^C_P \subset C \otimes A$ is a "line" (i.e., a direct summand of rank 1), so $\lambda^C$ is a line subbundle of $C \otimes \mathcal{O}_{Gr(F)}$. It defines a canonical embedding $Gr(F) \hookrightarrow \mathbb{P}$. There is a canonical identification

$$\lambda_{(P,P')} = \lambda^C_P \otimes (\lambda^C_{P'})^* \tag{372}$$

compatible with (371): if $P' \subset P$ the isomorphism $\lambda_{(P,P')} \otimes \lambda^C_{P'} \overset{\sim}{\rightarrow} \lambda^C_P$ is induced by the obvious map $\lambda_{(P,P')} = \det(P/P') \to \mathrm{Cl}_F / \mathrm{Cl}_F \cdot P'$.

The algebra Cl carries a canonical grading such that $F \subset \mathrm{Cl}^{-1}$, $F^* \subset \mathrm{Cl}^1$. Let $C^{\cdot}$ be a grading on $C$ compatible with the grading on $\mathrm{Cl}_F$; it is unique

*) Here "discrete" means that annihilator of any element of $C$ is an open subspace of $F \oplus F^*$.

*) $C$ is isomorphic to the fermionic Fock space $\varinjlim_U \bigwedge(F/U) \otimes \det(P/U)^*$ (cf. (182)), where $P$ is a c-lattice in $F$ and $U$ belongs to the set of all c-sublattices of $P$.

up to a shift. Then $\lambda^C$ is a homogenuous line, and (372) is an isomorphism of graded line bundles.

**7.13.17.** Denote by $\overline{\mathrm{Cl}}^{\cdot} = \overline{\mathrm{Cl}}^{\cdot}_F$ the completion of $\mathrm{Cl}^{\cdot}$ (as a graded algebra) with respect to the topology generated by left ideals $\mathrm{Cl}\cdot U$ where $U \subset F \oplus F^*$ is an open subspace. Thus $C$ is a discrete $\overline{\mathrm{Cl}}$-module. The action of $\overline{\mathrm{Cl}}$ yields an isomorphism of topological graded algebras $\overline{\mathrm{Cl}}^{\cdot} \stackrel{\sim}{\rightarrow} \mathrm{End}^{\cdot}_{\mathbb{C}}\, C$.

The graded algebra $\mathrm{Cl}^{\cdot}$ has a canonical filtration $\mathrm{Cl}^{\cdot}_0 = \Lambda^{\cdot} F^* \subset \mathrm{Cl}^{\cdot}_1 \subset ...$ (see 7.13.1). We define the filtration $\overline{\mathrm{Cl}}^{\cdot}_i$ on $\overline{\mathrm{Cl}}^{\cdot}$ as the closure of $\mathrm{Cl}^{\cdot}_i$. As in 7.13.1 the classical Clifford algebra $\overline{\mathcal{C}l}^{\cdot} := \mathrm{gr}\,\overline{\mathrm{Cl}}^{\cdot}$ is a Poisson graded topological algebra. It carries an additional grading $\overline{\mathcal{C}l}^{\cdot}_i := \mathrm{gr}_i\,\overline{\mathrm{Cl}}^{\cdot}$; one has $\overline{\mathcal{C}l}^a_i = \lim\limits_{\longleftarrow U,V} \Lambda^i(F/U) \otimes \Lambda^{a+i}(F^*/V)$ where $U, V$ are, respectively, c-lattices in $F, F^*$.

Denote by $E = E_F$ the associative algebra of endomorphisms of $F$. Let $E^{\mathrm{Lie}}$ be $E$ considered as a Lie algebra. Notice that $\overline{\mathcal{C}l}^0_1$ is a Lie subalgebra of $\overline{\mathcal{C}l}$ which normalizes $\overline{\mathcal{C}l}^{-1}_1$. The adjoint action of $\overline{\mathcal{C}l}^0_1$ on $\overline{\mathcal{C}l}^{-1}_1 = F$ identifies $\overline{\mathcal{C}l}^0_1$ with $E^{\mathrm{Lie}}$*). Set $E^\flat := \overline{\mathrm{Cl}}^0_1$; this is a Lie subalgebra of Cl which is a central extension of $\overline{\mathcal{C}l}^0_1 = E^{\mathrm{Lie}}$ by $\mathbb{C}$.

We see that $E^\flat$ acts on $C$ in a way compatible with the Clifford action; this action preserves the grading on $C$.

The next few sections 7.13.18 - 7.13.22 provide a convenient description of $E^\flat$ and some of its subalgebras. The reader may skip them and pass directly to 7.13.23.

**7.13.18.** Here is an explicit description of the central extension $E^\flat$ of $E^{\mathrm{Lie}}$ due essentially to Tate [T].

Let $E_+ \subset E$ be the (two-sided) ideal of bounded operators (:= operators with bounded image), $E_- \subset E$ that of discrete operators (:= operators with open kernel). One has $E_+ + E_- = E$; set $E_{tr} := E_+ \cap E_-$. For any $A \in E_{tr}$ its trace $trA$ is well-defined (if $U' \subset U \subset F$ are c-lattices such that

*)Use the above explicit description of $\overline{\mathcal{C}l}^0_1$.

$A(F) \subset U$, $A(U') = 0$ then we have $A^\sim : U/U' \to U/U'$ and $trA := trA^\sim$). The functional $tr : E_{tr} \to \mathbb{C}$ is invariant with respect to the adjoint action of $E^{\mathrm{Lie}}$; it also vanishes on $[E_+, E_-] \subset E_{tr}$.

Our extension $E^\flat$ is equipped with canonical splittings $s_+ : E_+ \to E^\flat$, $s_- : E_- \to E^\flat$. Namely, for $A \in E_+$ its lifting $s_+(A)$ is characterised by the property that $s_+(A)$ kills any element in $C$ annihilated by all Clifford operators from $\mathrm{Im}\, A \subset \mathfrak{g}$. Similarly, $s_-(A)$ is the unique lifting of $A \in E_-$ that kills any element in $C$ annihilated by all Clifford operators from $(\mathrm{Ker}\, A)^\perp \subset F^*$. The sections $s_\pm$ commute with the adjoint action of $E$, and for $A \in E_{tr}$ one has $s_-(A) - s_+(A) = trA \in \mathbb{C} \subset E^\flat$. It is easy to see that the data $(E^\flat, s_\pm)$ with these properties are uniquely defined. Indeed, consider the exact sequence of $E$-bimodules

$$0 \longrightarrow E_{tr} \xrightarrow{(-,+)} E_+ \oplus E_- \xrightarrow{(+,+)} E \longrightarrow 0. \tag{373}$$

Now $s = (s_+, s_-)$ identifies $E^\flat$ with the push-forward of the extension (373) by $tr : E_{tr} \to \mathbb{C}$. The adjoint action of $E^{\mathrm{Lie}}$ on $E^\flat$ comes from the adjoint action on the $E$-bimodule $E_+ \oplus E_-$.

*Remarks.* (i) The vector space $F \otimes F^*$ carries 4 natural topologies with bases of open subspaces formed, respectively, by $U \otimes V$, $U \otimes F^*$, $F \otimes V$, and $U \otimes F^* + F \otimes V$, where $U \subset F$, $V \subset F^*$ are open subspaces. The corresponding completions are equal, respectively, to $E_{tr}$, $E_+$, $E_-$, and $E$. The trace functional is the continuous extension of the canonical pairing $F \otimes F^* \to \mathbb{C}$.

(ii) Set $(E_-/E_{tr})^\flat := E_-/\,\mathrm{Ker}\, tr$; this is a central extension of $(E_-/E_{tr})^{\mathrm{Lie}}$ by $\mathbb{C}$. Note that $E_-/E_{tr} \overset{\sim}{\rightarrow} E/E_+$, so we have the projection $\pi_- : E^{\mathrm{Lie}} \to (E_-/E_{tr})^{\mathrm{Lie}}$. It lifts canonically to a morphism of extensions $\pi_-^\flat : E^\flat \to (E_-/E_{tr})^\flat$ with kernel $s_+(E_+)$. In other words, $E^\flat$ is the pull-back of $(E_-/E_{tr})^\flat$ by $\pi_-$. Same for $\pm$ interchanged.

(iii) Let $F^i$ be a finite filtration of $F$ by closed subspaces; denote by $B \subset E_F$ the subalgebra of endomorphisms that preserve the filtration. We have the induced central extension $B^\flat$ of $B^{\mathrm{Lie}}$. On the other hand, we have the obvious projections $gr^i : B \to E_{\mathrm{gr}^i F}$; let $B^{\flat i}$ be the pull-back of the extension $E^\flat_{\mathrm{gr}^i F}$ of $E^{\mathrm{Lie}}_{\mathrm{gr}^i F}$. Denote by $B^{\flat\prime}$ the Baer sum of the extensions $B^{\flat i}$. Then there is a canonical (and unique) isomorphism of extensions $B^{\flat\prime} \stackrel{\sim}{\to} B^\flat$. Indeed, $B^{\flat\prime}$ coincides with the extension defined by the exact subsequence

$$0 \to B \cap E_{tr} \to (B \cap E_+) \oplus (B \cap E_-) \to B \to 0$$

of (373) (notice that for $e \in B \cap E_{tr}$ one has $tr(e) = \Sigma tr(gr^i e)$). In particular we see that $B^\flat$ splits canonically over the Lie subalgebra $\mathrm{Ker}\, gr^\cdot$.

**7.13.19.** Set $K = \mathbb{C}((t))$, $O := \mathbb{C}[[t]]$. Let $F$ be a finite-dimensional $K$-vector space equipped with the usual topology; this is a Tate $\mathbb{C}$-vector space. Let $i : D \hookrightarrow E$ be the agebra of $K$-differential operators acting on $F$, so we have the induced central extension $D^\flat$ of the Lie algebra $D^{\mathrm{Lie}}$. Let us rephrase (following [BS]2.4) the Tate description of $D^\flat$ in geometric terms.

Set $F' := \mathrm{Hom}_K(F, K)$, $F^\circ := F' \underset{K}{\otimes} \omega_K$. Clearly $F^\circ$ coincides with the Tate dual $F^*$ (use the pairing $f^\circ, f \mapsto < f^\circ, f > := \mathrm{Res}(f^\circ, f)$). Our $F$ is a left $D$-module, and $F^\circ$ carries a unique structure of right $D$-module such that $<,>$ is a $D$-invariant pairing; notice that $D$ acts on $F^\circ$ by differential operators, and this is the usual geometric "adjoint" action. Let $K \widehat{\otimes} K$ be the completion of $K \otimes K$ with respect to the topology with basis $(t^n O) \otimes (t^n O)$, i.e. $K \widehat{\otimes} K := \mathbb{C}[[t_1, t_2]][t_1^{-1}][t_2^{-1}]$. Let $F \widehat{\otimes} F^\circ$ be the similar completion of $F \otimes F^\circ$; this is a finite-dimensional $K \widehat{\otimes} K$-module. Denote by $F \widehat{\otimes} F^\circ(\infty\Delta)$ the localization of $F \widehat{\otimes} F^\circ$ by $(t_1 - t_2)^{-1}$, i.e., by the equation of the diagonal.

Consider the standard exact sequence

$$0 \longrightarrow F \widehat{\otimes} F^\circ \longrightarrow F \widehat{\otimes} F^\circ(\infty\Delta) \xrightarrow{r} D \longrightarrow 0 \tag{374}$$

where the projection $r$ sends a "kernel" $k = k(t_1, t_2) dt_2 \in F \widehat{\otimes} F^\circ(\infty\Delta)$ to the differential operator $r(k) : F \to F$, $f(t) \mapsto \mathrm{Res}_{t_2 = t}(k(t, t_2), f(t_2)) dt_2$. Note that $F \widehat{\otimes} F^\circ$ is a $D$-bimodule in the obvious way. This biaction extends in a unique way to the $D$-biaction on $F\widehat{\otimes}F^\circ(\infty\Delta)$ compatible with the $K$-bimodule structure. It is easy to see that (374) is an exact sequence of $D$-bimodules. Let $tr : F^\circ\widehat{\otimes}F \to \mathbb{C}$ be the morphism $f\otimes f^\circ \mapsto < f^\circ, f >$ (i.e., it is the residue of the restriction to the diagonal). It is invariant with respect to the adjoint action of $D^{\mathrm{Lie}}$. Denote by $D^{\flat\prime}$ the push-forward of (374) by $tr$. The adjoint action of on $F\widehat{\otimes}F^\circ(\infty\Delta)$ yields a $D^{\mathrm{Lie}}$-module structure on $D^{\flat\prime}$. For $l_1^\flat, l_2^\flat \in D^{\flat\prime}$ set $[l_1^\flat, l_2^\flat] := l_1(l_2^\flat)$ where $l_1$ is the image of $l_1^\flat$ in $D^{\mathrm{Lie}}$.

**7.13.20.** *Lemma.* The bracket $[,]$ is skew-symmetric, so it makes $D^{\flat\prime}$ a central extension of $D^{\mathrm{Lie}}$ by $\mathbb{C}$. There is a unique isomorphism of central extensions

$$D^{\flat\prime} \xrightarrow{\sim} D^\flat.$$

*Proof.* It suffices to establish an isomorphism of $D^{\mathrm{Lie}}$-module extensions $D^{\flat\prime} \xrightarrow{\sim} D^\flat$. It comes from a canonical embedding $i^\sim : (374) \hookrightarrow (373)$ of exact sequences of $D$-bimodules defined as follows. The morphism $D \hookrightarrow E$ is our standard embedding $i$, and $i^\sim : F\widehat{\otimes}F^\circ = F\widehat{\otimes}F^* \xrightarrow{\sim} E_{tr}$ is the obvious isomorphism (see Remark (i) in 7.13.18). The map $i^\sim = (i_+^\sim, i_-^\sim) : F\widehat{\otimes}F^\circ(\infty\Delta) \to E_+ \oplus E_-$ sends the "kernel" $k$ to the operators $i_-^\sim(k)$ equal to $f \mapsto -\mathrm{Res}_{t_2=0}(k(t,t_2), f(t_2))dt_2$ and $i_+^\sim(k)$ equal to $f \mapsto (\mathrm{Res}_{t_2=t} + \mathrm{Res}_{t_2=0})(k(t,t_2), f(t_2))dt_2$. Here $f\in F$ and $(k(t,t_2), f(t_2))dt_2 \in F((t_2))dt_2$. We leave it to the reader to check that the operators $i_\pm^\sim(k)$ belong to $E_\pm$ *). Since $i^\sim$ identifies the trace functionals it yields the desired isomorphism of $D^{\mathrm{Lie}}$-modules $D^{\flat\prime} \xrightarrow{\sim} D^\flat$. □

*Remark.* Let $D_i \subset D$ be the subspace of differential operators of degree $\leq i$. The extension $D_i^\flat$ carries a natural topology induced by the embedding

*) This is clear for $i_-^\sim(k)$. To check that $i_+^\sim(k) \in E_+$ one may use Parshin's residue formula ([Pa76], §1, Proposition 7) applied to 2-forms $(k(t_1,t_2), g(t_1)f(t_2))dt_1\wedge dt_2$ where $g$ belongs to a sufficiently small c-lattice in $F^*$.

$D_i^\flat \subset \overline{\mathrm{Cl}}_F$. This is a Tate topology; the quotient topology on $D_i$ coincides with its natural topology of a finite-dimensional $K$-vector space.

**7.13.21.** *Example.* Set $\mathcal{E} := \mathrm{End}_K F = D_0 \subset D$, so we have the central extension $\mathcal{E}^\flat$ of $\mathcal{E}^{\mathrm{Lie}}$. Let $\mathcal{L} \subset D^{\mathrm{Lie}}$ be the normaliser of $\mathcal{E}$; it acts on $\mathcal{E}^\flat$ by the adjoint action. We will describe the extension $\mathcal{E}^\flat$ as an $\mathcal{L}$-module*).

It is easy to see that $\mathcal{L}$ coincides with the Lie algebra of differential operators of order $\leq 1$ whose symbol belongs to $\mathrm{Der}_K \cdot \mathrm{id}_F$. In other words, $\mathcal{L}$ consists of pairs $(\tau, \tau^\sim)$ where $\tau \in \mathrm{Der}\, K$ and $\tau^\sim$ is an action of $\tau$ on $F$, i.e., $\mathcal{L}$ is the Lie algebra of infinitesimal symmetries of $(K, F)$.

As above, set $\mathcal{E}^\circ := \mathcal{E} \underset{K}{\otimes} \omega_K$. We identify $\mathcal{E}^\circ$ with the Tate vector space dual $\mathcal{E}^*$ using the pairing $<,>$: $\mathcal{E}^\circ \times \mathcal{E} \to \mathbb{C}$, $< a, b >:= \mathrm{Res}\, tr_K(ab)$. The adjoint action of $\mathcal{L}$ on $\mathcal{E}^\circ$ is $(\tau,\tau^\sim)(e \otimes \nu) = [\tau^\sim, e] \otimes \nu + e \otimes Lie_\tau \nu$. Let $\omega_K^{\otimes 1/2}$ be a sheaf of half-forms on $\mathrm{Spec}\, K$. It carries an $\mathcal{L}$-action ($(\tau,\tau^\sim)$ acts by $\mathrm{Lie}_\tau$), so $\mathcal{L}$ acts on $\otimes\omega_K^{\otimes 1/2}$. Consider the set $\mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2})$ of connections on $F \otimes \omega_K^{\otimes 1/2}$*). Since $\mathrm{End}_K F = \mathrm{End}_K(F \otimes \omega_K^{\otimes 1/2})$ our $\mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2})$ is an $\mathcal{E}^\circ$-torsor; $\mathcal{L}$ acts on it in the obvious way.

**7.13.22.** *Lemma.* There is a unique $\mathcal{L}$- and $\mathcal{E}^\circ$-invariant pairing

$$<,>:\ \mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2}) \times \mathcal{E}^\flat \to \mathbb{C}$$

such that $< \nabla, 1_{\mathcal{E}^\flat} >= 1$ for any $\nabla \in \mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2})$.

*Remarks.*(i) An element $\lambda \in \mathcal{E}^\circ$ acts on $\mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2})$ and $\mathcal{E}^\flat$ according to formulas $\nabla \mapsto \nabla + \lambda$ and $e^\flat \mapsto e^\flat + < \lambda, e >$ (here $e := e^\flat \mod \mathbb{C}_{\mathcal{E}^\circ} = \mathcal{E}$). So $\mathcal{E}^\circ$-invariance of $<,>$ means that $< \nabla + \lambda, e^\flat >=< \nabla, e^\flat > - < \lambda, e >$.

(ii) Clearly $<,>$ identifies $\mathcal{E}^\flat$ with the $\mathcal{L}$-module of continuous affine functionals on $\mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2})$. This is the promised description of $\mathcal{E}^\flat$.

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*)Since $\mathcal{E} \subset \mathcal{L}$ we describe in particular the adjoint action of $\mathcal{E}$ which amounts to the Lie bracket on $\mathcal{E}^\flat$.

*)It does not depend on the choice of $\omega_K^{\otimes 1/2}$.

*Proof.* The unicity of $<,>$ follows since $\mathrm{Conn}(F\otimes\omega_K^{\otimes 1/2})$ has no $\mathcal{L}$-invariant elements.

To define $<\nabla, e^\flat>$ let us choose connections $\nabla_F$ on $F$ and $\nabla_\omega$ on $\omega_K$ such that $\nabla=\nabla_F+\frac{1}{2}\nabla_\omega$.

a. The connection $\nabla_F$ identifies the restrictions of $F\otimes K$ and $K\otimes F$ to the formal neighbourhood of the diagonal, i.e., it yields an isomorphism of $K\widehat{\otimes}K$-modules $\epsilon(\nabla_F):F\widehat{\otimes}K\overset{\sim}{\rightarrow}K\widehat{\otimes}F$. Let $\varepsilon(\nabla_F):F\widehat{\otimes}F^\circ\to K\widehat{\otimes}\omega_K$ be the composition of $\epsilon(\nabla_F)\otimes\mathrm{id}_{F^\circ}$ and the obvious morphism $K\widehat{\otimes}(F\otimes F^\circ)\to K\widehat{\otimes}\omega_K$ defined by the pairing $F\otimes F^\circ\to\omega_K$. Localizing $\varepsilon(\nabla_F)$ by the equation of the diagonal we get the morphism $F\widehat{\otimes}F^\circ(\infty\Delta)\to K\widehat{\otimes}\omega_K(\infty\Delta)$. Applying it to $e^\flat$ we get a 1-form $\varepsilon(\nabla_F,e^\flat)\in K\widehat{\otimes}\omega_K(\Delta)$ well-defined up to the subspace of those forms $\phi(t_1,t_2)dt_2\in K\widehat{\otimes}\omega_K$ that $\mathrm{Res}_0\,\phi(t,t)dt=0$. Notice that for $\lambda\in\mathcal{E}^\circ$ one has $\varepsilon(\nabla_F+\lambda,e^\flat)=\varepsilon(\nabla_F,e^\flat)-tr_K(\lambda\cdot e)$ (here $tr_K(\lambda\cdot e)\in\omega_K=K\widehat{\otimes}\omega_K/(t_1-t_2)K\widehat{\otimes}\omega_K$).

b. Let $\nu\in\omega_K\widehat{\otimes}K(\Delta)$ be a form with residue 1 at the diagonal (i.e., $\nu$ equals $\frac{dt_1}{t_1-t_2}$ modulo $\omega_K\widehat{\otimes}K$). Let $\psi(\nabla_\omega)$ be a similar form such that $\psi(\nabla_\omega)^{\otimes 2}=-\nabla_\omega^{(1)}\nu$*). Notice that $\psi(\nabla_\omega)$ is well-defined modulo $(t_1-t_2)\omega_K\widehat{\otimes}K$. For $l\in\omega_K$ one has $\psi(\nabla_\omega+l)=\psi(\nabla_\omega)-l$ (here we consider $l$ as an element in $\omega_K\widehat{\otimes}K/(t_1-t_2)\omega_K\widehat{\otimes}K$).

c. Consider the 2 form $\varepsilon(\nabla_F,e^\flat)\wedge\nu$. Set

$$<\nabla,e^\flat>:=\mathrm{Res}_0\,\mathrm{Res}_\Delta(\varepsilon_\nabla(e^\flat)\wedge\nu)$$

Then $<\nabla,e^\flat>$ is well-defined (i.e., it does not depend on the auxiliary choices) and $<,>$ is $\mathcal{E}^\circ$-invariant. Since all the constructions where natural it is also $\mathcal{L}$-invariant. $\square$

*Remarks.* (i) Let $e_\alpha$ be an $F$-basis of $F$, $e'_\alpha$ the dual basis of $F'$, and $\nabla$ the connection such that $e'_\alpha\cdot(dt)^{-1/2}$ are horisontal sections. Denote by $(e_\alpha\cdot e'_\beta)\flat\in\mathcal{E}^\flat$ the image of $e_\alpha\otimes e'_\beta\frac{dt_2}{t_2-t_1}$. Then $<\nabla,(e_\alpha\cdot e_\beta)^\flat>=\delta_{\alpha,\beta}$.

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*) here $\nabla_\omega^{(1)}$ is the covariant derivative along the first variable.

(ii) The above lemma is a particular case of the local Riemann-Roch formula; see, e.g., Appendix in [BS].

**7.13.23.** Now let $\mathfrak{n}$ be a Lie algebra in the Tate setting, i.e., a Tate vector space equipped with a continuous Lie bracket $[\ ,\ ]$. The following lemma may help the reader to feel more comfortable.

*Lemma.* $\mathfrak{n}$ admits a base of neighbourhoods of 0 that consists of Lie subalgebras of $\mathfrak{n}$.

*Proof.* Take any c-lattice $P \subset \mathfrak{n}$. We want to find an open Lie algebra $\mathfrak{k} \subset P$. Note that

$$\mathfrak{n}_P := \{\alpha \in \mathfrak{n} :\ [\alpha, P] \subset P\} \tag{375}$$

is an open Lie subalgebra. Set $\mathfrak{k} := P \cap \mathfrak{n}_P$. □

**7.13.24.** We use the notation of 7.13.17 for $F = \mathfrak{n}$. So we have the Clifford graded topological algebra $\overline{\mathrm{Cl}}^{\cdot} = \overline{\mathrm{Cl}}^{\cdot}_{\mathfrak{n}}$, the corresponding classical Clifford algebra $\overline{\mathcal{C}l}^{\cdot} = \mathrm{gr}\,\overline{\mathrm{Cl}}^{\cdot}$ (which is a Poisson graded topological algebra), the central extension $E^\flat$ of the Lie algebra $E^{\mathrm{Lie}}$ of endomorphisms of the Tate vector space $\mathfrak{n}$ and the embedding $E^\flat \hookrightarrow \overline{\mathrm{Cl}}^0$. The adjoint action defines a morphism $\mathfrak{n} \to E^{\mathrm{Lie}}$; denote by $\mathfrak{n}^\flat$ the pull-back of the extension $E^\flat$ to $\mathfrak{n}$. So $\mathfrak{n}^\flat$ is a central extension of $\mathfrak{n}$ by $\mathbb{C}$. We equip $\mathfrak{n}^\flat$ with the weakest topology such that the projection $\mathfrak{n}^\flat \to \mathfrak{n}$ and the morphism $\mathfrak{n}^\flat \to \overline{\mathrm{Cl}}^0$ are continuous. Then $\mathfrak{n}^\flat$ is a Tate space and the map $\mathfrak{n}^\flat/\mathbb{C} \to \mathfrak{n}$ is a homeomorphism*).

**7.13.25.** Now we are ready to render the BRST construction to the infinite-dimensional setting. Let us start with the "classical" version. Let $\mathcal{R}$ be a topological Poisson algebra. We assume that $\mathcal{R}$ is complete and separated and topology.

*)Indeed, the extension $\mathfrak{n}^\flat$ has a canonical continuous splitting over any subalgebra of the form (375) (its image consists of operators annihilating $\lambda_P$).

**7.13.26.** Denote by $\mathcal{M}(\mathfrak{g})^\flat$ the category of discrete $\mathfrak{g}^\flat$-modules $V$ such that $1 \in \mathbb{C} \subset \mathfrak{g}^\flat$ acts as $-\mathrm{id}_V$. For such $V$, the $\mathfrak{g}^\flat$-actions on $C^\cdot$ and $V$ yield a $\mathfrak{g}$-module structure on $C^\cdot \otimes V$. It is also a $\mathrm{Cl}_\mathfrak{g}$-module in the obvious manner, and the $\mathfrak{g}$ -action is compatible with the Clifford action. For $\alpha \in \mathfrak{g}$ we denote its action on $C^\cdot \otimes V$ by $Lie_\alpha$, and the Clifford operator $C^\cdot \otimes V \to C^{\cdot -1} \otimes V$ by $i_\alpha$.

It is convenient to rewrite the operators acting on $C^\cdot \otimes V$ as follows (cf. 7.7.5). Let $\Omega_\mathfrak{g}$ be the DG algebra of continuous Lie algebra cochains of $\mathfrak{g}$. The corresponding plane graded algebra $\Omega^\cdot_\mathfrak{g}$ is the completed exterior algebra of $\mathfrak{g}^*$. We identify it with the closed subalgebra of the completed Clifford algebra $\overline{\mathrm{Cl}_\mathfrak{g}}$ generated by $\mathfrak{g}^* \subset \mathrm{Cl}_\mathfrak{g}$, so $\Omega^\cdot_\mathfrak{g}$ acts on $C^\cdot \otimes V$ by Clifford operators. Now let $\mathfrak{g}_\Omega$ be a DG Lie algebra defined as follows. The only non-zero components are $\mathfrak{g}^0_\Omega = \mathfrak{g}^{-1}_\Omega = \mathfrak{g}$, the differential $\mathfrak{g}^{-1}_\Omega \to \mathfrak{g}^0_\Omega$ is $\mathrm{id}_\mathfrak{g}$, the bracket on $\mathfrak{g}^0_\Omega$ is the bracket of $\mathfrak{g}$. Recall that $\mathfrak{g}_\Omega$ acts on $\Omega_\mathfrak{g}$ (namely, $\mathfrak{g}^0_\Omega$ acts in coadjoint way, and $\mathfrak{g}^{-1}_\Omega$ acts by "constant" derivations). The graded Lie algebra $\mathfrak{g}^\cdot_\Omega$ acts on $C^\cdot \otimes V$ via the operators $Lie_\alpha$ and $i_\alpha$. So $C^\cdot \otimes V$ is a graded $(\Omega^\cdot_\mathfrak{g}, \mathfrak{g}^\cdot_\Omega)$ -module.

**7.13.27.** *Proposition.* There is a unique linear map $d : C^\cdot \otimes V \to C^{\cdot+1} \otimes V$ such that for any $\alpha \in \mathfrak{g}$ one has $Lie_\alpha = di_\alpha + i_\alpha d$. One has $d^2 = 0$, and $C_\mathfrak{g}(V) := (C^\cdot \otimes V, d)$ is a DG $(\Omega_\mathfrak{g}, \mathfrak{g}_\Omega)$ -module.

*Proof.* Uniqueness. The difference of two such $d$'s is an operator that commutes with any $i_\alpha$. It is easy to see that the algebra of all such operators coincides with the closed subalgebra generated by $\mathfrak{g}^{-1}_\Omega$ and $\mathrm{End}\, V$. Since it has no operators of positive degree we are done.

A similar argument shows that the action of $(\Omega_\mathfrak{g}, \mathfrak{g}_\Omega)$ is compatible with the differentials and that $d^2 = 0$ (first you prove that $[d, Lie_\alpha] = 0$, then the rest of properties).

Existence. We write $d$ explicitely. Let $e_i$, $i \in I$, be a topological basis of $\mathfrak{g}$ (see 4.2.13), $e_i^*$ the dual basis of $\mathfrak{g}^*$. For a semi-infinite (with respect to $\mathfrak{g}$)

subset $A \subset I$ denote by $\lambda_A \subset C^{\cdot}$ the homogenuous line $\lambda^C$ that corresponds to the c-lattice generated by $e_a$, $a \in A$ (see 7.13.16). In other words $\lambda_A$ is the subspace of vectors killed by the Clifford operators $e_a$, $e_b^*$ for $a \in A$, $b \in I \setminus A$. Our $C^{\cdot}$ is the direct sum of $\lambda_A$'s. Note that for $a, b$ as above one has $e_a^*(\lambda_A) = \lambda_{A\setminus a}$, $e_b(\lambda_A) = \lambda_{A\cup b}$.

Set $V_A := \lambda_A \otimes V$; then $C^{\cdot} \otimes V$ is direct sum of $V_A$'s. For $c \in I$ set $L_c := Lie_{e_c}, i_c := i_{e_c}$; for semi-infinite $A, A'$, we denote by $L_c^{A,A'}$, $i_c^{A,A'}$ the $A, A'$-components $V_A \to V_{A'}$ of these operators.

Let $A, B$ be semi-infinite subsets such that $|A| - |B| = 1$ (here $|A| - |B| := |A \setminus (A \cap B)| - |B \setminus (A \cap B)|$). Choose any $a = a_{A,B} \in A \setminus (A \cap B)$ (this set is not empty). Denote by $d^{A,B}$ the composition $V_A \to V_{B\cup a} \to V_B$ where the first arrow is $L_a^{A,B\cup a}$, the second one is the Clifford operator $e_a^*$. It is easy to see that the operator $d : C^{\cdot} \otimes V \to C^{\cdot+1} \otimes V$ with components $d^{A,B}$ is correctly defined (use the fact that for any $v \in V$ and there is only finitely many $a \in A$ such that $L_a(\lambda_A \otimes v)$ is non-zero).

It remains to show that our $d$ satisfies the condition of the Proposition, i.e., that for any $c \in I$ one has $[d, i_c] = L_c$. One checks this fact by a direct computation; the key point is the skew-symmetry of $[L_a, i_b]$ with respect to $a, b$. We leave the details for the reader. $\square$

**7.13.28.** If V is a complex in $\mathcal{M}(\mathfrak{g})^\flat$ then we denote by $C_{\mathfrak{g}}(V)$ the total complex for the bicomplex $C(V^{\cdot})$. This is a discrete DG $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$-module (an $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$-complex for short). The functor $C_{\mathfrak{g}}$ is an equivalence between the DG category $C(\mathfrak{g})^\flat$ of complexes in $\mathcal{M}(\mathfrak{g})^\flat$ (we call them $\mathfrak{g}^\flat$-complexes) and the DG category $C(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$ of $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$-complexes. The inverse functor assigns to $F \in C(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$ the complex $\mathrm{Hom}_{\mathrm{Cl}_{\mathfrak{g}}}(C^{\cdot}, F)$.

**7.13.29.** Let $\mathfrak{k} \subset \mathfrak{g}$ be an open bounded Lie subalgebra. For $a \geq 0$ denote by $C_a^{\cdot} \subset C^{\cdot}$ the subspace of elements killed by product of any $a+1$ Clifford operators from $\mathfrak{k}^\perp \subset \mathfrak{g}^*$. Then $0 = C_{-1}^{\cdot} \subset C_0^{\cdot} \subset C_1^{\cdot} \subset ...$ is an increasing filtration on $C^{\cdot} = \cup C_a^{\cdot}$. Any Clifford operator $\nu \in \mathfrak{g}^*$ preserves our filtration; if $\nu$ belongs to $\mathfrak{k}^\perp$ then it sends $C^\cdot_a$ to $C^{\cdot+1}_{a-1}$. Any Clifford operator from $\mathfrak{g}$ sends $C^\cdot_a$ to $C^{\cdot-1}_{a+1}$; if it belongs to $\mathfrak{k}$ then it preserves the filtration. Thus $gr_*C^\cdot$ is a module over the Clifford algebra $\mathrm{Cl}_{\mathfrak{g}:\mathfrak{k}}$ of the vector space $(\mathfrak{g}/\mathfrak{k})\oplus(\mathfrak{g}/\mathfrak{k})^*\oplus\mathfrak{k}\oplus\mathfrak{k}^*$ (equipped with the standard "hyperbolic" form).

This is an irreducible $\mathrm{Cl}_{\mathfrak{g}:\mathfrak{k}}$-module; and $C^\cdot_0$ is an irreducible module over the subalgebra $\mathrm{Cl}_{\mathfrak{k}} \subset \mathrm{Cl}_{\mathfrak{g},\mathfrak{k}}$. The homogenuous line $\lambda_{\mathfrak{k}} = \lambda_{\mathfrak{k}}^{(C)}$ (see 7.13.16) sits in $C^\cdot_0$, and $gr_*C^\cdot$ is a free module over the subalgebra $\Lambda(\mathfrak{g}/\mathfrak{k})\otimes\Lambda\mathfrak{k}^* \subset \mathrm{Cl}_{\mathfrak{g}:\mathfrak{k}}$ generated by this line. If $\lambda_{\mathfrak{k}} \subset C^0$ (we may assume this shifting the $\cdot$ filtration if necessary) then $gr_aC^b = \Lambda^a(\mathfrak{g}/\mathfrak{k})\otimes\Lambda^{b+a}\mathfrak{k}^*\otimes\lambda_{\mathfrak{k}}$.

Let $\mathfrak{k}^\flat \subset \mathfrak{g}^\flat$ be the preimage of $\mathfrak{k}$. This is a central extension of $\mathfrak{k}$ by $\mathbb{C}$ which splits canonically: the image of the splitting $\mathfrak{k}\to\mathfrak{k}^\flat$ consists of those elements that kill $\lambda_{\mathfrak{k}}$ (we consider the Lie algebra action of $\mathfrak{k}^\flat$ on $C^\cdot$).

For $V\in C(\mathfrak{g})^\flat$ the subspaces $C^\cdot_a\otimes V$ are subcomplexes of $C_{\mathfrak{g}}(V)$; denote them by $C_{\mathfrak{g}}(V)_a$. We get a filtration on $C_{\mathfrak{g}}(V)$ preserved by the Clifford operators from $\mathfrak{g}^*$ and $\mathfrak{k}$; the successive quotients $\mathrm{gr}_a C_{\mathfrak{g}}(V)$ are $(\Omega_{\mathfrak{k}},\mathfrak{k}_\Omega)$-complexes. For a $\mathfrak{k}$-complex $P$ denote by $C_{\mathfrak{k}}(P)$ the Chevalley complex of Lie algebra cochains of $\mathfrak{k}$ with coefficients in $P$; this is an $(\Omega_{\mathfrak{k}},\mathfrak{k}_\Omega)$-complex. The identification $\mathrm{gr}_a C_{\mathfrak{g}}(V)^\cdot = \Lambda^{\cdot+a}\mathfrak{k}^*\otimes(V^\cdot\otimes\Lambda^a(\mathfrak{g}/\mathfrak{k})\otimes\lambda_{\mathfrak{k}})$ is an isomorphism of $(\Omega_{\mathfrak{k}},\mathfrak{k}_\Omega)$-complexes

$$\mathrm{gr}_a C_{\mathfrak{g}}(V) \overset{\sim}{\to} C_{\mathfrak{k}}(V\otimes\Lambda^a(\mathfrak{g}/\mathfrak{k})\otimes\lambda_{\mathfrak{k}})[a] \tag{376}$$

Here $\mathfrak{k}$ acts on $\Lambda^a(\mathfrak{g}/\mathfrak{k})$ according to the adjoint action. The corresponding spectral sequence converges to $H^\cdot C_{\mathfrak{g}}(V)$; its first term is $E_1^{p,q} = H^{p+q}\,\mathrm{gr}_{-p} C_{\mathfrak{g}}(V) = H^q(\mathfrak{k},\Lambda^{-p}(\mathfrak{g}/\mathfrak{k})\otimes V\otimes\lambda_{\mathfrak{k}})$.

**7.13.30.** *Remark.* Assume that we have a $\mathfrak{k}^\flat$-subcomplex $T\subset V$ such that $V$ is induced from $T$, i.e., $V = U(\mathfrak{g}^\flat)\underset{U(\mathfrak{k}^\flat)}{\otimes} T$. Then the composition of embeddings $C_{\mathfrak{k}}(T\otimes\lambda_{\mathfrak{k}})\subset C_{\mathfrak{g}}(V)_0\subset C_{\mathfrak{g}}(V)$ is a quasi-isomorphism.

**7.14. Localization functor in the infinite-dimensional setting.** Now we may explain the parts (c), (d) of the "Hecke pattern" from 7.1.1 in the present infinite-dimensional setting.

**7.14.1.** Let $G$, $K$ be as in 7.11.17 and $G'$ be a central extension of $G$ by $\mathbb{G}_m$ equipped with a splitting $K \to G'$ (cf. 7.8.1). Then $\mathfrak{g}$, $\mathfrak{g}'$ are Lie algebras in Tate's setting, and $\mathfrak{k} = LieK$ is an open bounded Lie subalgebra of $\mathfrak{g}$, $\mathfrak{g}'$. All the categories from 7.8.1 make obvious sense in the present setting.

One defines the Hecke Action on the category $D(\mathfrak{g}, K)'$ as in 7.8.2. Now the line bundle $\mathcal{L}_G$ is an $\mathcal{O}^p$-module on $G$, and $\mathcal{V}_G$ is a complex of left $\mathcal{D}^p$-modules (see 7.11.3). All the constructions of 7.8.2 pass to our situation word-by-word, as well as 7.8.4-7.8.5 (in 7.8.4 we should take for $U'$, as usual, the completed twisted enveloping algebra).

**7.14.2.** To define the localization functor $L\Delta$ we need some preliminaries. Let $Y$ be a scheme, $F$ a Tate vector space. A $\mathrm{Cl}_F$*-module on* $Y$ is a $\mathbb{Z}$-graded $\mathcal{O}$-module $C^{\cdot}$ on $Y$ equipped with a continuous action of the graded Clifford algebra $\mathrm{Cl}^{\cdot}_F$ (see 7.13.16). For any c-lattice $P \subset F$ denote by $\lambda_P(C^{\cdot})$ the graded $\mathcal{O}$-submodule of $\mathcal{C}^{\cdot}$ that consists of local sections killed by Clifford operators from $P \subset F$ and $P^{\perp} \subset F^*$. The functor $\lambda_P : \mathcal{C}(Y) \to \{$ the category of graded $\mathcal{O}$-modules on $Y\}$ is an equivalence of categories*). For two c-lattices $P_1, P_2$ there is a canonical isomorphism

$$\lambda_{P_1}(C^{\cdot}) \overset{\sim}{\to} \lambda_{(P_1,P_2)} \otimes \lambda_{P_2}(C^{\cdot}) \tag{377}$$

that satisfies the obvious transitivity property (see 7.13.16). Same is true for $Y$-families of c-lattices (see loc. cit.).

**7.14.3.** Now assume we are in situation 7.11.18. Then $Y$ carries a *canonical* $\mathrm{Cl}_{\mathfrak{g}}$-module $C^{\cdot}_Y$ defined as follows. Let $K \subset G$ be a reasonable group subscheme, $\mathfrak{k} := \mathrm{Lie}\, K$. Denote by $\omega_{(K\backslash Y)}$ the pull-back of the canonical

*)The inverse functor is tensoring by an appropriate irreducible graded Clifford module over $\mathbb{C}$.

bundle $\omega_{K\backslash Y} = \det \Omega_{K\backslash Y}$ by the projection $Y \to K \backslash Y$ (recall that $K \backslash Y$ is a smooth stack). This is a graded line bundle that sits in degree $\dim K \backslash Y$. If $K_1, K_2 \subset G$ are two reasonable group subschemes as above, then there is a canonical isomorphism

$$\omega_{(K_1\backslash Y)} = \lambda_{(\mathfrak{k}_1,\mathfrak{k}_2)} \otimes \omega_{(K_2\backslash Y)} \tag{378}$$

which satisfies the obvious transitivity property. Indeed, to define (378) it suffices to consider the case $K_2 \subset K_1$. The pull-back to $Y$ of the relative tangent bundle for the smooth projection $K_2 \backslash Y \to K_1 \backslash Y$ equals $(\mathfrak{k}_1/\mathfrak{k}_2) \otimes \mathcal{O}_Y$, which yields (378). The transitivity property is clear.

Now our $C^{\cdot}_Y \in \mathcal{C}(Y)$ is a Clifford module together with data of isomorphisms $\lambda_{\mathfrak{k}}(C^{\cdot}_Y) \overset{\sim}{\to} \omega_{(K\backslash Y)}$ for any reasonable subgroup $K \subset G$ that are compatible with (377) and (378). Such $C^{\cdot}_Y$ exists and unique (up to a unique isomorphism).

The action of $G$ on $Y$ lifts canonically to a $G$-action on $C^{\cdot}_Y$ compatible with adjoint action of $G$ on the Clifford operators $\mathfrak{g} \oplus \mathfrak{g}^*$. Indeed, $G(\mathbb{C})$ acts on all the objects our $C^{\cdot}_Y$ is cooked up with, so it acts on $C^{\cdot}_Y$. To define the action of $A$-points $G(A)$ on $C^{\cdot}_Y \otimes A$ one has to spell out the characteristic property of the Clifford module $C^{\cdot}_Y \otimes A$ on $Y \times \operatorname{Spec} A$ using $A$-families of reasonable group subschemes of $G$. We leave it to the reader.

*Remark.* Take any $y \in Y$. The fiber $C^{\cdot}_y$ of $C^{\cdot}_y$ at $y$ is an irreducible graded $\mathrm{Cl}^{\cdot}_{\mathfrak{g}}$-module which may be described as follows. Consider the "action" map $\mathfrak{g} \to \Theta_y$. Its kernel $\mathfrak{g}_y$ (the stabilizer of $y$) is a d-lattice in $\mathfrak{g}$. The cokernel $T$ is a finite-dimensional vector space. Let $C^{\cdot}_{y\mathfrak{g}_y}$ be the graded vector space of $\mathfrak{g}_y$-coinvariants in $C^{\cdot}_y$ (with respect to the Clifford action of $\mathfrak{g}_y$). Now there is a canonical identification $C^{\dim T}_{y\mathfrak{g}_y} \overset{\sim}{\to} \det(T^*)$, and $C^{\cdot}_y$ is uniquely determined by this normalization.

**7.14.4.** Let $\mathcal{L} = \mathcal{L}_Y$ be a line bundle on $Y$ equipped with a $G'$-action that lifts the $G$-action on $Y$; we assume that $\mathbb{G}_m \subset G$ acts on $\mathcal{L}$ by the character opposite to the standard.

Take $V \in \mathcal{M}(\mathfrak{g})'$, so $V$ is a discrete $\mathfrak{g}'$-module on which $\mathbb{C} \subset \mathfrak{g}'$ acts by the standard character. Then the tensor product $\mathcal{L} \otimes V$ is a $\mathfrak{g}$-module, as well as $C_Y^{\cdot} \otimes \mathcal{L} \otimes V$ (i.e., the $\mathfrak{g}$-action on $Y$ lifts to a continuous $\mathfrak{g}$-action on these $\mathcal{O}$-modules). We denote the action of $\alpha \in \mathfrak{g}$ on $C_Y^{\cdot} \otimes \mathcal{L} \otimes V$ by $Lie_\alpha$. Note that $C_Y^{\cdot} \otimes \mathcal{L} \otimes V$ is also a Clifford module, and the above $\mathfrak{g}$-action is compatible with the Clifford operators. As usual we denote the Clifford action of $\alpha \in \mathfrak{g}$ by $i_\alpha$. So, as in 7.13.26, our $C_Y^{\cdot} \otimes \mathcal{L} \otimes V$ is a graded $(\Omega_{\mathfrak{g}}^{\cdot}, \mathfrak{g}_\Omega^{\cdot})$-module.

The following proposition is similar to 7.13.27, as well as its proof which we leave to the reader.

**7.14.5.** *Proposition.* There is a unique morphism of sheaves

$$d : C_Y^{\cdot} \otimes \mathcal{L} \otimes V \to C_Y^{\cdot+1} \otimes \mathcal{L} \otimes V$$

such that for any $\alpha \in \mathfrak{g}$ one has $Lie_\alpha = d i_\alpha + i_\alpha d$. This $d$ is a differential operator of first order, $d^2 = 0$, and $C_{\mathcal{L}}(V) := (C_Y^{\cdot} \otimes \mathcal{L} \otimes V, d)$ is a DG $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$-module.

*Remark.* One may deduce 7.14.5 directly from 7.13.27. Namely, pick any $K$ as in 7.14.3. Then $C_Y^{\cdot} \otimes \omega_{(K\backslash Y)}^{*}$ is a "constant" Clifford module: it is canonically isomorphic to $C^{\cdot} \otimes \mathcal{O}_Y$ for some irreducible Clifford module $C^{\cdot}$. The $\mathfrak{g}^\flat$-action on $C^{\cdot}$ and the $\mathfrak{g}$-action on $C_Y^{\cdot}$ yield a $\mathfrak{g}^\flat$-action on $\omega_{(K\backslash Y)} = \mathrm{Hom}(C_Y^{\cdot}, C^{\cdot} \otimes \mathcal{O}_Y)$ which lifts the $\mathfrak{g}$-action on $Y$. Thus $\mathfrak{g}^\flat$-acts on $\omega_{(K\backslash Y)} \otimes \mathcal{L} \otimes V$, and $d$ from 7.14.5 coincides with $d$ from 7.13.27 for $C^{\cdot} \otimes (\omega_{(K\backslash Y)} \otimes \mathcal{L} \otimes V)$.

**7.14.6.** So we defined an $\Omega$-complex $C_{\mathcal{L}}(V)$ on $Y$. One extends this definition to the case when $V$ is a complex in $\mathcal{M}(\mathfrak{g})'$ in the obvious manner.

Now assume we have $K$ as in 7.14.1. For a Harish-Chandra complex $V \in C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$ the $\Omega$-complex $C_{\mathcal{L}}(V)$ is $K_\Omega$-equivariant. Indeed, $K$ acts on $C_{\mathcal{L}}(V)$ according to the $K$-actions on $C_Y^{\cdot}$, $\mathcal{L}$, and $V$, and the operators $i_\xi$, $\xi \in \mathfrak{k}$, are sums of the corresponding Clifford operators for $C_Y^{\cdot}$ and the operators for the $\mathfrak{k}_\Omega^{-1}$-action on $V$.

Set $\Delta_{\Omega\mathcal{L}}(V) := C_{\mathcal{L}}(V)[\dim(K\setminus Y)]$. We have defined a DG functor

$$\Delta_\Omega = \Delta_{\Omega\mathcal{L}} :\ C(\mathfrak{k}_\Omega\times\mathfrak{g},K)'\longrightarrow C(K\setminus Y,\Omega) \tag{379}$$

**7.14.7.** *Remark.* The $\Omega$-complex $\Delta_\Omega(V)$ carries a canonical filtrartion $\Delta_\Omega(V)$. where $\Delta_\Omega(V)_a$ consists of sections killed by product of any $a+1$ Clifford operators from $\mathfrak{k}^\perp\subset\mathfrak{g}^*$ (see 7.13.29). By (376) one has a canonical isomorphism of $K_\Omega$-equivariant $\Omega$-complexes

$$gr_a\Delta_\Omega(V)\xrightarrow{\sim} C_{\mathfrak{k}}(\omega_{(K\setminus Y)}\otimes\mathcal{L}\otimes V\otimes\Lambda^a(\mathfrak{g}/\mathfrak{k}))[a] \tag{380}$$

**7.14.8.** *Lemma.* (i) The functor $\Delta_\Omega$ sends quasi-isomorphisms to $\mathcal{D}$-quasi-isomorphisms, so it yields a triangulated functor

$$L\Delta = L\Delta_{\mathcal{L}} :\ D(\mathfrak{g},K)'\to D(K\setminus Y) \tag{381}$$

(ii) The functor $L\Delta$ is right t-exact, and the corresponding right exact functor $\Delta=\Delta_{\mathcal{L}} :\ \mathcal{M}((\mathfrak{g},K)'\to\mathcal{M}^\ell(K\setminus Y)$ is

$$\Delta_{\mathcal{L}}(V)_Y = (\mathcal{D}_Y\otimes\mathcal{L})\underset{U(\mathfrak{g})}{\widehat{\otimes}}V = \mathcal{L}^*\otimes\mathcal{D}_{Y,\mathcal{L}}\underset{U(\mathfrak{g}')}{\widehat{\otimes}}V \tag{382}$$

Here $\mathcal{D}_Y$ is the topological algebra of differential operators on $Y$ (see 1.2.6), $\mathcal{D}_{Y,\mathcal{L}} := \mathcal{L}\otimes\mathcal{D}_Y\otimes\mathcal{L}^*$ is the corresponding $\mathcal{L}$-twisted algebra.

*Proof.* (i) Our statement is local, so, shrinking $K$ if necessary, we may assume that the $K$-action on $Y$ is free. Let us consider $\Delta_\Omega(V)$ as a filtered $\Omega$-complex on $K\setminus Y$. For a $K$-module $P$ denote by $P^\sim$ the $Y$-twist of $P$ which is an $\mathcal{O}$-module on $K\setminus Y$. The projection $C_{\mathfrak{k}}\to C_{\mathfrak{k}}/C_{\mathfrak{k}}^{\geq 1}$ yields, according to (380), a canonical isomorphism

$$gr_a\Delta_\Omega(V)_{K\setminus Y} = \omega_{K\setminus Y}\otimes\mathcal{L}_{K\setminus Y}\otimes V^\sim\otimes\Lambda^a(\mathfrak{g}/\mathfrak{k})^\sim[a] \tag{383}$$

The r.h.s. is an $\mathcal{O}$-complex, so a quasi-isomorphism between $V$'s defines a (filtered) $\mathcal{D}$-quasi-isomorphism of $\Delta_\Omega(V)$'s.

(ii) As above we may assume that the $K$-action is free. For $V\in\mathcal{M}(\mathfrak{g},K)'$ we can rewrite (383) as an isomorphism $\Delta_\Omega(V)^a_{K\setminus Y} = \omega_{K\setminus Y}\otimes\mathcal{L}_{K\setminus Y}\otimes V^\sim\otimes\Lambda^{-a}(\mathfrak{g}/\mathfrak{k})^\sim$. This shows that $\Delta_\Omega$ is right t-exact. One describes the differential in $\Delta_\Omega(V)_{K\backslash Y}$ as follows. The $\mathfrak{g}$-action on $Y$ defines on $(\mathfrak{g}/\mathfrak{k})^\sim$ the structure of Lie algebroid on $K\setminus Y$. The $\mathfrak{g}$-action on $\mathcal{L}_Y\otimes V$ defines on $\mathcal{L}_{K\backslash Y}\otimes V^\sim$ the structure of a left $(\mathfrak{g}/\mathfrak{k})^\sim$-module, hence $\omega_{K\backslash Y}\otimes\mathcal{L}_{K\backslash Y}\otimes V^\sim$ is a right $(\mathfrak{g}/\mathfrak{k})^\sim$-module. Now $\Delta_\Omega(V)_{K\backslash Y}$ is the Chevalley homology complex of $(\mathfrak{g}/\mathfrak{k})^\sim$ with coefficients in $\omega_{K\backslash Y}\otimes\mathcal{L}_{K\backslash Y}\otimes V^\sim$. The right $\mathcal{D}$-module $H^0_{\mathcal{D}}(L\Delta(V))$ on $K\backslash Y$ is $(\omega_{K\backslash Y}\otimes\mathcal{L}_{K\backslash Y}\otimes V^\sim)\underset{(\mathfrak{g}/\mathfrak{k})^\sim}{\otimes}\mathcal{D}_{K\backslash Y}$; the corresponding left $\mathcal{D}$-module is $\mathcal{D}_{K\backslash Y}\underset{(\mathfrak{g}/\mathfrak{k})^\sim}{\otimes}(\mathcal{L}_{K\backslash Y}\otimes V^\sim)$. Lifting this isomorphism to $Y$ we get (382). □

**7.14.9.** *Example.* Let us compute $L\Delta(Vac')$. The embedding $\mathbb{C}\to Vac'$ yields an embedding of $\Omega$-complexes on $Y$ $C_{\mathfrak{k}}(\omega_{(K\backslash Y)}\otimes\mathcal{L}_Y)\to\Delta_{\Omega\mathcal{L}}(Vac')_0$. We leave it to the reader to check that the corresponding morphism

$$C_{\mathfrak{k}}(\omega_{(K\backslash Y)}\otimes\mathcal{L}_Y)\to\Delta_{\Omega\mathcal{L}}(Vac')$$

of $K_\Omega$-equivariant $\Omega$-complexes is a $\mathcal{D}$-quasi-isomorphism. Now the l.h.s. is the $\Omega$-complex $\Omega(\mathcal{D}_{K\backslash Y}\otimes\mathcal{L}_{K\backslash Y}$ on $K\setminus Y$ (see 7.3.3). Therefore if $K\setminus Y$ is a good stack then

$$L\Delta(Vac')=\Delta(Vac')=\mathcal{D}_{K\backslash Y}\otimes\mathcal{L}_{K\backslash Y}.$$

*Remark.* Since $\operatorname{End}Vac'$ is anti-isomorphic to the algebra $D'_{(\mathfrak{g},K)}$ from 1.2.5 (cf. also 1.2.2) we have a right action of $D'_{(\mathfrak{g},K)}$ on $\Delta(Vac')=\mathcal{D}_{K\backslash Y}\otimes\mathcal{L}_{K\backslash Y}$, i.e., a homomorphism from $D'_{(\mathfrak{g},K)}$ to the twisted differential operator ring $\Gamma(K\setminus Y,D'_{K\backslash Y})$. This is the homomorphism $h$ from 1.2.5 (cf. also 1.2.3 and 1.2.4).

**7.14.10.** *Proposition.* The functor $L\Delta : D(\mathfrak{g},K)'\to D(K\setminus Y)$ is a Morphism of $\mathcal{H}$-Modules.

*Proof.* The constructions and arguments of 7.8.8 render to our infinite-dimensional setting in the obvious manner. □

The infinite-dimensional versions of 7.9 are straightforward.

**7.15. Affine flag spaces are $\mathcal{D}$-affine.** In this section we show that representations of affine Lie algebras of less than critical level are related to $\mathcal{D}$-modules on affine flag spaces just as they do in the usual finite-dimensional situation.

**7.15.1.** Below as usual $K = \mathbb{C}((t))$, $O = \mathbb{C}[[t]]$. Let $\mathfrak{g}$ be a simple (finite-dimensional) Lie algebra$^{*)}$, $G$ the corresponding simply connected simple group. We have the group ind-scheme $G(K)$ and its group subscheme $G(O)$ (see 7.11.2(iv)). The adjoint action of $G(K)$ on the Tate vector space $\operatorname{Lie} G(K) = \mathfrak{g}(K)$ yields the central extension $G(K)^\flat$ of $G(K)$ by $\mathbb{G}_m$ (see **??**). Its Lie algebra is the central extension $\mathfrak{g}(K)^\flat$ of $\mathfrak{g}(K)$ defined by cocycle $\phi, \psi \mapsto \operatorname{Res}(d\phi, \psi)$ where $(a, b) := \operatorname{Tr}(\operatorname{ad}_a \cdot \operatorname{ad}_b)$ (see **??**). Let $G(O)^\flat \subset G(K)^\flat$ be the preimage of $G(O)$. The adjoint action of $G(O)$ preserves the c-lattice $\mathfrak{g}(O) \subset \mathfrak{g}(K)$, so we have a canonical identification $s : G(O)^\flat \mathrel{\widetilde{\rightarrow}} G(O) \times \mathbb{G}_m{}^{*)}$.

Let $N \subset B \subset G$ be a Borel subgroup and its radical, so $H = B/N$ is the Cartan group of $G$. Let $N^+, B^+$ be the preimages of $N, B$ by the obvious projection $G(O) \to G$, so $B^+/N^+ = H$, $G(O)/B^+ = G/B$. Let $B^\dagger \subset G(K)^\flat$ be the preimage of $B^+$. There is a unique section $N^+ \to G(K)$; set $H^\flat := B^\dagger/N^+$, $\mathfrak{h}^\flat = \operatorname{Lie} H^\flat$. The section $s$ yields an isomorphism $B^+ \times \mathbb{G}_m \mathrel{\widetilde{\rightarrow}} B^\dagger$, hence isomorphisms $H \times \mathbb{G}_m \mathrel{\widetilde{\rightarrow}} H^\flat$, $\mathfrak{h} \times \mathbb{C} \mathrel{\widetilde{\rightarrow}} \mathfrak{h}^\flat$.

Set $X := G(K)/B^+ = G(K)^\flat/B^\dagger$ (the quotient of sheaves with respect to either flat or Zariski topology - the result is the same, as follows from 4.5.1). One calls $X$ the *affine flag space*. This is a reduced connected ind-projective formally smooth ind-scheme$^{*)}$. Set $X^\dagger := G(K)^\flat/N^+$: this is a left $H^\flat$-torsor over $X$ (the action is $h^\flat \cdot x^\dagger = x^\dagger h^{\flat -1}$). It carries the obvious action of $G(K)^\flat$. Denote the projection $X^\dagger \to X$ by $p$.

$^{*)}$A generalization to the case when $\mathfrak{g}$ is any reductive Lie algebra is immediate.

$^{*)}$Since $G$ is simple the splitting $G(O) \to G(O)^\flat$ is unique.

$^{*)}X$ is smoothly fibered over the affine Grassmannian $G(K)/G(O)$, see 4.5.1.

**7.15.2.** Let $\mathcal{M}^\dagger(X)$ be the category of weakly $H^\flat$-equivariant $\mathcal{D}$-modules on $X^\dagger$ (see 7.11.11). This is an abelian category. For $M \in \mathcal{M}^\dagger(X)$ set $M_X := (p_\cdot M)^{H^\flat} \in \mathcal{M}(X, \mathcal{O})$. The functor $\mathcal{M}^\dagger(X) \to \mathcal{M}(X, \mathcal{O})$, $M \mapsto M_X$, is exact and faithful.

Set $\mathcal{D}^\dagger := (p_\cdot \mathcal{D}_{X^\dagger})^{H^\flat}$. This is a Diff-algebra on $X$. The map

$$\mathfrak{h}^\flat \to \Gamma(X, \mathcal{D}^\dagger) = \Gamma(X^\dagger, \mathcal{D}_{X^\dagger})^{H^\flat} \tag{384}$$

equal to *minus* the left action along the fibers of $p$ takes values in the center of $\mathcal{D}^\dagger$. In fact, $\mathcal{D}^\dagger$ is a $\mathrm{Sym}(\mathfrak{h}^\flat)$-family of tdo (see 7.11.11(b)).

Notice that $\mathcal{D}^\dagger$ acts (from the right) on any $M_X$ as above in the obvious manner, so we have a functor

$$\mathcal{M}^\dagger(X) \to \mathcal{M}(X, \mathcal{D}^\dagger). \tag{385}$$

One has (see Remark (ii) in 7.11.11):

**7.15.3.** *Lemma.* The functor (385) is an equivalence of categories. □

**7.15.4.** For $\chi = (\chi_0, c) \in \mathfrak{h}^{\flat *} = \mathfrak{h}^* \times \mathbb{C}$ we denote by $\mathcal{D}^\chi$ the corresponding tdo from our family $\mathcal{D}^\dagger$. Thus $\mathcal{D}^{(0,0)} = \mathcal{D}_X$. Set $\mathcal{M}^\chi(X) := \mathcal{M}(X, \mathcal{D}^\chi) \subset \mathcal{M}(X, \mathcal{D}^\dagger)$. Consider the topological algebra $\Gamma\mathcal{D}^\chi = \Gamma(X, \mathcal{D}^\chi)$ (see 7.11.9, 7.11.10). We have the functor

$$\Gamma : \mathcal{M}^\chi(X) \to \mathcal{M}^r(\Gamma\mathcal{D}^\chi) \tag{386}$$

where $\mathcal{M}^r(\Gamma\mathcal{D}^\chi)$ is the category of discrete right $\Gamma\mathcal{D}^\chi$-modules and $\Gamma M := \Gamma(X, M)$.

The action of $\mathfrak{g}(K)^\flat$ on $X^\dagger$ yields a continuous morphism $\mathfrak{g}(K)^\flat \to \Gamma(X, \mathcal{D}^\dagger)$. The corresponding morphism $\mathfrak{g}(K)^\flat \to \Gamma\mathcal{D}^\chi$ sends $1^\flat \in \mathfrak{g}(K)^\flat$ to $-c$.

**7.15.5.** We say that $\chi$ is *anti-dominant* if the Verma $\mathfrak{g}(K)^\flat$-module $M(\chi)$ is irreducible. As follows from [KK] 3.1 this amounts to the following three conditions:

(i) One has $c \neq -1/2$.

(ii) For any positive coroot $h_\alpha \in \mathfrak{h}$ of $\mathfrak{g}$ one has $(\chi_0+\rho_0)(h_\alpha) \neq 1,2,..$

(iii) For any $h_\alpha$ as above and any integer $n>0$ one has

$$\pm(\chi_0+\rho_0)(h_\alpha)+2n\frac{c+1/2}{(\alpha,\alpha)} \neq 1,2,..$$

Here $\rho_0 \in \mathfrak{h}^*$ is the half sum of the positive roots of $\mathfrak{g}$ and $(,)$ is the scalar product on $\mathfrak{h}^*$ that corresponds to $(,)$ on $\mathfrak{h}$ (see 7.15.1).

*Remark.* To deduce the above statement from [KK] 3.1 it suffices to notice that the "real" positive coroots of $\mathfrak{g}(K)^\flat$ are $h_\alpha$ and $\pm h_\alpha + 2n(\alpha,\alpha)^{-1}\cdot 1^\flat$ for $h_\alpha$, $n$ as above, and that the weight $\rho$ from [KK] is given by the next formula.

Set $\rho := (\rho_0, 1/2) \in \mathfrak{h}^{\flat *}$. We say that $\chi$ is *regular* if the stabilizer of $\chi+\rho$ in the affine Weyl group $W_{\mathrm{aff}}$ is trivial$^{*)}$.

**7.15.6.** *Theorem.* Assume that $\chi$ is anti-dominant and regular. Then (386) is an equivalence of categories.

We prove 7.15.6 in 7.15.8-**??** below.

**7.15.7.** *Remarks.* (i) Let $\mathcal{M}^c(\mathfrak{g}(K))$ be the category of discrete $\mathfrak{g}(K)^\flat$-modules on which $1^\flat$ acts as multiplication by $c$. Let

$$\Gamma : \mathcal{M}^\chi(X) \to \mathcal{M}^c(\mathfrak{g}(K)) \tag{387}$$

be the composition of (386) and the obvious "restriction" functor $\mathcal{M}^r(\Gamma\mathcal{D}^\chi) \to \mathcal{M}^c(\mathfrak{g}(K))$. According to 7.15.6 this functor is exact and faithful.

(ii) One may hope that $\mathfrak{g}(K)^\flat$ generates a dense subalgebra in $\Gamma\mathcal{D}^{\chi *)}$. In other words, $\Gamma\mathcal{D}^{\chi\circ}$ is a completion of the enveloping algebra $\bar{U}^c = \bar{U}^c\mathfrak{g}(K)$ of level c by certain topology. Can one determine this topology explicitely?

Notice that in the finite-dimensional setting (see [BB81] or [Kas]) one usually deduces the corresponding statement from its "classical" version

$^{*)}$Remind that the action of $W_{\mathrm{aff}}$ on $\mathfrak{h}^{\flat *}$ comes from the adjoint action of $G(K)$ on $\mathfrak{g}(K)^\flat$.

$^{*)}$This amounts to the property that for $M \in \mathcal{M}^\chi(X)$ any $\mathfrak{g}(K)^\flat$-submodule of $\Gamma M$ comes from a $\mathcal{D}^\chi$-submodule of $M$.

(using Kostant's normality theorem). This "classical" statement (which says that $\mathfrak{g}(K) \hookrightarrow \Gamma(X, \Theta_X)$ generates a dense subalgebra in $\underset{n\geq 0}{\oplus} \Gamma(X, \Theta_X^{\otimes n}))$ is *false* for the affine flags (e.g., the map $\mathfrak{g}(K) \hookrightarrow \Gamma(X, \Theta_X)$ is not surjective).

As in [BB81] or [Kas] it is easy to see that 7.15.6 follows from the next statement:

**7.15.8.** *Theorem.* (i) If $\chi$ is anti-dominant then for any $M \in \mathcal{M}^\chi(X)$ one has $H^r(X, M) = 0$ for any $r > 0^{*)}$.

(ii) If, in addition, $\chi$ is regular and $M \neq 0$ then $\Gamma M \neq 0$.

*Remark.* The proof of 7.15.8(i) is very similar to the proof of the corresponding finite-dimensional statement (see [BB81] or [Kas]). It would be nice to find a proof of 7.15.8(ii) similar to that in [BB81] (using translation functors) for it could be of use for understanding 7.15.7(ii).

**7.15.9.** Let us begin the proof of 7.15.8(i). Let $\psi = (\psi_0, b)$ be a character of $H^\flat$ and $\mathcal{L} = \mathcal{L}^\psi$ the corresponding $G(K)^\flat$-equivariant line bundle on $X$ (defined by $X^\dagger$). Assume that $\mathcal{L}$ is ample. This amounts$^{*)}$ to the following property of $\psi$: for any positive coroot $h_\alpha$ of $\mathfrak{g}$ one has $\frac{2b}{(\alpha,\alpha)} < \psi_0(h_\alpha) < 0$.

Denote by $V$ be the dual to the pro-finite dimensional vector space $\Gamma(X, \mathcal{L})$. This is a $G(K)^\flat$-module in the obvious way, hence an integrable $\mathfrak{g}(K)^\flat$-module$^{*)}$ of level $-b$. Consider the canonical section of $V \widehat{\otimes} \mathcal{L}$; this is a $G(K)^\flat$-equivariant morphism $\mathcal{O}_X \to V \widehat{\otimes} \mathcal{L}$ of $\mathcal{O}^p$-modules. Tensoring it by $M$ we get a morphism of $\mathcal{O}^!$-modules

$$i : M \to V \otimes \mathcal{L} \otimes M \tag{388}$$

that commutes with the action of $\mathfrak{g}(K)^\flat$.

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$^{*)}$Here $H^r(X, M) := \varinjlim H^r(Y, M_{(Y)})$; we use notation of 7.11.4.

$^{*)}$See Remark in 7.15.5.

$^{*)}$According to a variant of Borel-Weil theorem (see, e.g., [?]) $V$ is an irreducible $\mathfrak{g}(K)^\flat$-module.

**7.15.10.** Below we will consider *!-sheaves* of vector spaces on $X$. Such object $F$ is a rule that assigns to a closed subscheme $Y\subset X$ a sheaf $F_{(Y)}$ on the Zariski topology of $Y$ together with identifications $i^!_{YY'}F_{(Y')}=F_{(Y)}$ *) for $Y\subset Y'$ that satisfy the obvious transitivity property (cf. Remark (i) in 7.11.4). Notice that !-sheaves form an abelian category. It contains the categories of sheaves on $Y$'s as full subcategories closed under subquotients and extensions. Any $\mathcal{O}^!$-module $M$ on $X$ yields a !-sheaf $\varinjlim M_{(Y)}$ on $X$ (so the corresponding sheaf on $Y$ is $M_{(Y^\wedge)}$ *)); we denote it by $M$ by abuse of notation. We will also consider !-sheaves of $\mathfrak{g}(K)^\flat$-modules which are !-sheaves of vector spaces equipped with $\mathfrak{g}(K)^\flat$-action such that the action on each $F_{(Y)}$ is discrete in the obvious sense. Any $\mathcal{O}^!$-module equipped with $\mathfrak{g}(K)^\flat$-action may be considered as a !-sheaf of $\mathfrak{g}(K)^\flat$-modules.

**7.15.11.** *Proposition.* Considered as a morphism of !-sheaves of $\mathfrak{g}(K)^\flat$-modules, (388) is a direct summand embedding.

**7.15.12.** *Proof of 7.15.8(i).* Take any $\alpha\in H^r(X,M)=\varinjlim H^r(X_{(Y)},M_{(Y)})$. It comes from certain closed subscheme $Y\subset X$ and an $\mathcal{O}$-coherent submodule $F\subset M_{(Y)}$. Choose an ample $\mathcal{L}$ as above such that $H^r(Y,\mathcal{L}\otimes F)=0$. Since $i(\alpha)$ belongs to the image of $H^r(Y,V\otimes\mathcal{L}\otimes F)$ it vanishes. We are done by 7.15.11. □

**7.15.13.** *Proof of 7.15.11.* We are going to define an endomorphism $A$ of $V\otimes\mathcal{L}\otimes M$ such that

$$\operatorname{Ker} A = M,\ \ V\otimes\mathcal{L}\otimes M=\operatorname{Ker} A\oplus\operatorname{Im} A. \tag{389}$$

This settles 7.15.11.

Let $\bar U:=\bar U\mathfrak{g}(K)^\flat$ be the usual completed enveloping algebra of $\mathfrak{g}(K)^\flat$. Consider the Sugawara element $\widetilde{\mathfrak{L}}_0\in\bar U$ defined by formula (85). For any $ft^r\in\mathfrak{g}((t))\subset\bar U$ we have $[\widetilde{\mathfrak{L}}_0,ft^r]=(1^\flat+1/2)rft^r$ (see (87)). For any

*)Here $i^!_{YY'}F_{(Y')}:=$ the subsheaf of sections supported (set-theoretically) on $Y$.

*)See 7.11.4 for notation.

$N \in \mathcal{M}^e(\mathfrak{g}(K))$ where $e \neq -1/2$ consider the operator $\Delta_N := (e+1/2)^{-1}\widetilde{\mathfrak{L}}_0$ acting on $N$. If also $e - b \neq -1/2$ we set

$$A_{V,N} := \Delta_{V\otimes N} - \Delta_V \otimes \mathrm{id}_N - \mathrm{id}_V \otimes \Delta_N \in \mathrm{End}(V \otimes N). \tag{390}$$

This operator commutes with the action of $\mathfrak{g}(K)^\flat$.

Let us apply this construction to the !-sheaf of $\mathfrak{g}(K)^\flat$-modules $N := \mathcal{L}\otimes M$ (so $e = b + c$ and the condition on levels is satisfied). Set

$$A := A_{V,\mathcal{L}\otimes M} \in \mathrm{End}(V \otimes \mathcal{L} \otimes M). \tag{391}$$

Let us show that $A$ satisfies (389). $\square$

**7.15.14.** Now let us turn to 7.15.8(ii). It is an immediate consequence of the following proposition which shows, in particular, how to compute fibers of $M$ in terms of $\Gamma M$. We start with notation.

Consider the stratification of $X$ by $N^+$-orbits (Schubert cells). The cells are labeled by elements of the affine Weyl group $W_{\mathrm{aff}}$. For $w \in W_{\mathrm{aff}}$ the corresponding cell is $i_w : Y_w \hookrightarrow X$; it has dimension $l(w)$. The restriction to $Y_w$ of the $H^\flat$-torsor $X^\dagger$ is trivial*). Since any invertible function on $Y_w$ is constant, the trivialization is unique up to a constant shift. Therefore the pull-back of the tdo $\mathcal{D}^\chi$ to $Y_w$ is canonically trivialized.

Let $M$ be any object of the derived category $D(X, \mathcal{D}^\chi)$*). For any $w \in W_{\mathrm{aff}}$ we have (untwisted, as we just explained) $\mathcal{D}$-complexes $i_w^! M \in D(Y_w)$.

We want to compute Lie algebra (continuous) cohomology $H^a(\mathfrak{n}^+, \Gamma M)$ (notice that, because of 7.15.8(i), $\Gamma = R\Gamma$). Since $\mathfrak{h}^\flat = \mathfrak{b}^\dagger/\mathfrak{n}^+$ these are $\mathfrak{h}^\flat$-modules. We assume that $\chi$ is regular.

**7.15.15.** *Proposition.* There is a canonical isomorphism

$$H^a(\mathfrak{n}^+, \Gamma M) \;\tilde{\rightarrow}\; \bigoplus_{w\in W_{\mathrm{aff}}} H_{DR}^{a-l(w)}(Y_w, i_w^! M).$$

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*)A section is provided by any $N^+$-orbit in $X^\dagger$ over $Y_w$.

*)Its definition is similar to one given in 7.11.14 in the untwisted situation.

such that $\mathfrak{h}^\flat$ acts on the $w$-summand as multiplication by $w(\chi)$ $^{*)}$.

**7.15.16.** *Proof of 7.15.8(ii).* Since $\Gamma$ is exact we may assume that $M$ is compactly supported and finitely generated. Let $Y \subset X$ be a smooth Zariski open subset of the (reduced) support of $M$. Then $M_{(Y)}$ is a coherent $\mathcal{D}$-module on a smooth scheme $Y$. So, shrinking $Y$ farther, we may assume that $M_{(Y)}$ is a free $\mathcal{O}_Y$-module. Now for any $x \in Y$ one has $H^{\cdot} i_x^! M \neq 0$. Translating $M$ we may assume that $x = Y_1$. By 7.15.15 $H^{\cdot}(\mathfrak{n}^+, \Gamma M) \neq 0$, hence $\Gamma M \neq 0$. □

**7.15.17.** *Proof of 7.15.15.* We may assume that $M = i_{w*}N$ for certain $N \in D(Y_w)$. Indeed, any $M \in D(X, \mathcal{D}^\chi)$ carries a canonical filtration with $\operatorname{gr}_i M = \underset{l(w)=i}{\oplus} i_{w*} i_w^! M$. Now the isomorphism 7.15.15 for $M$ comes from the corresponding isomorphisms for $i_{w*} i_w^! M$'s together with the spectral decomposition for the action of $\mathfrak{h}^\flat$. Here we use the assumption of regularity of $\chi$; for the rest of the argument one needs only anti-dominance of $\chi$.

Consider first the case $M = \delta$, so $\Gamma\delta$ is the Verma module from 7.15.5 (see 7.15.7(iii)). This Verma module is cofree $N^+$-module of rank 1 (it is cofreely generated by any functional $\nu$ which does not kill the vacuum vector) $^{*)}$. Thus $H^{\cdot}(\mathfrak{n}_x^+, \Gamma\delta) = H^0(\mathfrak{n}_x^+, \Gamma\delta)^\chi = \mathbb{C}\cdot vac$. Since also $H^{\cdot} i_x^! \delta = H^0 i_x^! \delta = \mathbb{C}\cdot vac$, we get the desired isomorphism.

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$^{*)}$Remind that the adjoint action of $G(K)$ on $\mathfrak{g}(K)^\flat$ yields the $W_{\mathrm{aff}}$-action on $\mathfrak{h}^\flat$.

$^{*)}$The kernel of $\nu$ contains no non-trivial $\mathfrak{n}^+$-submodule (otherwise, since $\mathfrak{n}^+$ is nilpotent, it would contain $\mathfrak{n}^+$-invariant vectors which contradicts 7.15.5(i)). So the morphism defined by $\nu$ from $\Gamma\delta$ to the cofree $N^+$-module is injective. Then it is an isomorphism by dimensional reasons.

# 8. To be inserted into 5.x

## 8.1.

**8.1.1.** Choose $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$. Recall that $\lambda_{\mathcal{L}}$ denotes the corresponding local Pfaffian bundle on $\mathcal{GR} = G(K)/G(O)$ (see 4.6.2). We are going to prove the following statement, which is weaker than 5.2.14 and will be used in the proof of Theorem 5.2.14 itself.

**8.1.2.** *Proposition.* For any $\chi \in P_+({}^LG)$ and $i \in \mathbb{Z}$ the $\overline{U}'$-module $H^i(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1})$ is isomorphic to a direct sum of copies of $Vac'$.

At this stage we do not claim that the number of copies is finite.

Proposition 8.1.2 is an immediate consequence of Theorems 8.1.4 and 8.1.6 formulated below (the first theorem is geometric while the second one is representation-theoretic).

**8.1.3.** For any $\mathcal{D}$-module $M$ on $\mathcal{GR}$ the renormalized universal enveloping algebra $U^\natural$ acts on the sheaf $M\lambda_{\mathcal{L}}^{-1}$ (see ???). So the canonical morphism $\mathrm{Der}\, O \to U^\natural$ from 5.6.9 yields an action of $\mathrm{Der}\, O$ on $M\lambda_{\mathcal{L}}^{-1}$. According to ??? this action is induced by the action of $\mathrm{Der}\, O$ on the sheaf $M$ ($\mathrm{Der}\, O$ is mapped to the algebra of vector fields on $\mathcal{GR}$, which acts on $M$) and the action of $\mathrm{Der}\, O$ on $\lambda_{\mathcal{L}}$ (see 4.6.7). The action of $\mathrm{Der}\, O$ on the sheaf $I_\chi$ integrates to the action of $\mathrm{Aut}\, O$. The action of $\mathrm{Der}\, O$ on $\lambda_{\mathcal{L}}$ comes from the action of $\mathrm{Aut}_Z\, O$ on $\lambda_{\mathcal{L}}$ (see 4.6.7). Therefore the action of $\mathrm{Der}\, O$ on $I_\chi\lambda_{\mathcal{L}}^{-1}$ integrates to the action of $\mathrm{Aut}_2\, O$. So the action of $L_0 \in \mathrm{Der}\, O$ on $H^i(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1})$ is diagonalizable and its spectrum is contained in $\frac{1}{2}\mathbb{Z}$ (in fact, it is contained in $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$ depending on the parity of $\mathrm{Orb}_\chi$).

**8.1.4.** *Theorem.* The eigenvalues of $L_0$ on $H^i(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1})$ are $\geq -d(\chi)/2$ where $d(\chi) = \dim \mathrm{Orb}_\chi$.

The proof will be given in 9.1; we will also obtain the following description of the eigenspace corresponding to $-d(\chi)/2$. Set $F_\chi := \overline{\mathrm{Orb}_\chi} \setminus \mathrm{Orb}_\chi$, $U_\chi := \mathcal{GR} \setminus F_\chi$. The restriction of $I_\chi$ to $U_\chi$ is the direct image of the (right)

$\mathcal{D}$-module $\omega_{\mathrm{Orb}_\chi}$. It contains the sheaf-theoretic direct image of $\omega_{\mathrm{Orb}_\chi}$, so $H^0(U_\chi, I_\chi\lambda_{\mathcal{L}}^{-1}) \supset H^0(\mathrm{Orb}_\chi, \omega_{\mathrm{Orb}_\chi} \otimes \lambda_{\mathcal{L},\chi}^{-1})$ where $\lambda_{\mathcal{L},\chi}$ is the restriction of $\lambda_{\mathcal{L}}$ to $\mathrm{Orb}_\chi$. Therefore (241) yields an embedding

$$\mathfrak{d}_{\mathcal{L},\chi} \hookrightarrow H^0(U_\chi, I_\chi\lambda_{\mathcal{L}}^{-1}) \tag{392}$$

where $\mathfrak{d}_{\mathcal{L},\chi}$ is the 1-dimensional representation of $\mathrm{Aut}_Z^0 O$ constructed in 4.6.14. According to 4.6.15 $L_0$ acts on $\mathfrak{d}_{\mathcal{L},\chi}$ as multiplication by $-d(\chi)/2$.

**8.1.5.** *Proposition.* The image of (392) is contained in $H^0(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})$. It equals the eigenspace of $L_0$ on $H^0(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})$ corresponding to the eigenvalue $-d(\chi)/2$.

The proof is contained in 9.1.

*Remark.* The natural map $\varphi : H^0(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1}) \to H^0(U_\chi, I_\chi\lambda_{\mathcal{L}}^{-1})$ is injective because $I_\chi$ is irreducible and therefore the morphism $f : I_\chi \to R^0 j_* j^* I_\chi$ is injective, where $j$ denotes the immersion $U_\chi \hookrightarrow \mathcal{GR}$. In fact, the semisimplicity theorem 5.3.3(i) implies that $f$ is an isomorphism and therefore $\varphi$ is an isomorphism. So the first statement of Proposition 8.1.5 is obvious modulo the highly nontrivial theorem by Lusztig used in the proof of 5.3.3.

Proposition 8.1.2 is a consequence of Theorem 8.1.4 and the following statement, which will be proved in 6.2.

**8.1.6.** *Theorem.* Let $V$ be a discrete $U^\natural$-module such that

1) the representation of $\mathfrak{g} \otimes O \subset U^\natural$ in $V$ is integrable (i.e., it comes from a representation of $G(O)$),
2) the action of $L_0 \in \mathrm{Der}\, O \subset U^\natural$ on $V$ is diagonalizable and the intersection of its spectrum with $c + \mathbb{Z}$ is bounded from below for every $c \in \mathbb{C}$.

Then $V$ considered as a $\overline{U}'$-module is isomorphic to a direct sum of copies of $Vac'$ (i.e., to $Vac' \otimes W$ for some vector space $W$).

*Remark.* Suppose that $V$ is a discrete $U^{\natural}$-module such that $V$ is isomorphic to $Vac' \otimes W$ as a $\overline{U}'$-module. Write $V$ more intrinsically as $Vac' \otimes_{\mathfrak{z}} N$, $\mathfrak{z} := \mathfrak{z}_{\mathfrak{g}}(O)$, $N := \mathrm{Hom}_{\overline{U}'}(Vac', V) = V^{\mathfrak{g}\otimes O}$. According to 5.6.8 $N$ is a module over the Lie algebroid $I/I^2$. The $U^{\natural}$-module $V$ can be reconstructed from the $(I/I^2)$-module $N$ as follows: $V$ is the quotient of $U^{\natural} \otimes_{\mathfrak{z}} N$ by the closed $U^{\natural}$-submodule generated by $u \otimes n - 1 \otimes an$ where $n \in N$, $u \in U_1^{\flat}$, $a \in I/I^2$, and the images of $u$ and $a$ in $U_1^{\flat}/U_0^{\flat}$ coincide (see 5.6.7).

# 9. To be inserted into Section 6

**9.1. Proof of Theorem 8.1.4 and Proposition 8.1.5.** We keep the notation of 5.2.13, 8.1.1, and 8.1.4. Theorem 8.1.4 and Proposition 8.1.5 can be easily deduced from the following statement.

**9.1.1.** *Theorem.* The eigenvalues of $L_0$ on $H^i(U_\chi, I_\chi\lambda_{\mathcal{L}}^{-1})$ are $\geq -d(\chi)/2$. If $i>0$ they are $>-d(\chi)/2$. If $i=0$ the eigenvalue $-d(\chi)/2$ occurs with multiplicity 1 and the corresponding eigenspace is the image of (392).

Let us start to prove the theorem. Denote by $I_\chi^U$ the restriction of $I_\chi$ to $U_\chi$, i.e., $I_\chi^U$ is the direct image of the right $\mathcal{D}$-module $\omega_{\mathrm{Orb}_\chi}$ with respect to the closed embedding $\mathrm{Orb}_\chi \hookrightarrow U_\chi$. Consider the $\mathcal{O}$-module filtration on $I_\chi^U\lambda_{\mathcal{L}}^{-1}$ whose $k$-th term is formed by sections supported on the $k$-th infinitesimal neighbourhood of $\mathrm{Orb}_\chi$. The filtration is $\mathrm{Aut}_2^0\,O$-invariant and $\mathrm{gr}_j(I_\chi^U\lambda_{\mathcal{L}}^{-1}) = \omega_{\mathrm{Orb}_\chi}\otimes\lambda_{\mathcal{L}}^{-1}\otimes\mathrm{Sym}^j\,\mathcal{N}_\chi$ where $\mathcal{N}_\chi$ is the normal sheaf of $\mathrm{Orb}_\chi\subset U_\chi$. Using (241) we get an $\mathrm{Aut}_2^0\,O$-equivariant isomorphism $\mathrm{gr}_j(I_\chi^U\lambda_{\mathcal{L}}^{-1}) = \mathfrak{d}_{\mathcal{L},\chi}\otimes\mathrm{Sym}^j\,\mathcal{N}_\chi$. By 4.6.15 $L_0$ acts on $\mathfrak{d}_{\mathcal{L},\chi}$ as multiplication by $-d(\chi)/2$. So it remains to prove the following.

**9.1.2.** *Proposition.* i) The eigenvalues of $L_0$ on $H^i(\mathrm{Orb}_\chi, \mathrm{Sym}^j\,\mathcal{N}_\chi)$ are non-negative.

ii) They are positive if $i>0$ or $j>0$. There are no $L_0$-invariant regular functions on $\mathrm{Orb}_\chi$ except constants.

*Remark.* The eigenvalues of $L_0$ on $H^i(\mathrm{Orb}_\chi, \mathrm{Sym}^j\,\mathcal{N}_\chi)$ are integer because $\mathcal{N}_\chi$ is an $\mathrm{Aut}^0\,O$-equivariant sheaf.

Before proving the proposition we need some lemmas.

**9.1.3.** Let us introduce some notation. Recall that $\chi$ is a dominant coweight of $G$. Fix a Cartan subgroup $H\subset G$ and a Borel subgroup $B\subset G$ containing $H$. We will understand "coweight" as "coweight of $H$" and "dominant" as "dominant with respect to $B$". Let $t^\chi\in H(K)$ denote the image of $t\in\mathbb{C}((t))^* = K^*$ by $\chi:\mathbb{G}_m\to H$. Recall that $\mathrm{Orb}_\chi$ is the $G(O)$-orbit of $[\chi]$, where $[\chi]$ is the image of $t^\chi$ in $\mathcal{GR} = G(K)/G(O)$. Denote by $\mathrm{orb}_\chi$ the $G$-orbit of $[\chi]$ and by $P_\chi^-$ the stabilizer of $[\chi]$ in $G$, i.e., $P_\chi^- = \{g \in G | t^{-\chi} g t^\chi \in G(O)\}$. $P_\chi^-$ is the parabolic subgroup of $G$ such that $\mathrm{Lie}\, P_\chi^-$ is the sum of $\mathrm{Lie}\, H$ and the root spaces corresponding to roots $\alpha$ with $(\alpha, \chi) \leq 0$ (in particular $P_\chi^-$ contains the Borel subgroup $B^- \supset H$ opposite to $B$). So $\mathrm{orb}_\chi = G/P_\chi^-$ is a projective variety. Clearly the action of $\mathrm{Aut}^0 O$ on $\mathrm{orb}_\chi$ is trivial.

**9.1.4.** Endomorphisms of $O$ form an affine semigroup scheme $\mathrm{End}^0 O$ (for a $\mathbb{C}$-algebra $R$ an $R$-point of $\mathrm{End}^0 O$ is an $R$-morphism $f : R[[t]] \to R[[t]]$ such that $f(t) \in tR[[t]]$ ). $\mathrm{Aut}^0 O$ is dense in $\mathrm{End}^0 O$. Let $\mathbf{0} \in \mathrm{End}^0 O$ denote the endomorphism of $O = \mathbb{C}[[t]]$ such that $t \mapsto 0$.

**9.1.5.** *Lemma.* i) The action of $\mathrm{Aut}^0 O$ on $\mathrm{Orb}_\chi$ extends to an action of $\mathrm{End}^0 O$ on $\mathrm{Orb}_\chi$.

ii) Let $\varphi$ be the endomorphism of $\mathrm{Orb}_\chi$ corresponding to $\mathbf{0} \in \mathrm{End}^0 O$. Then $\varphi^2 = \varphi$ and the scheme of fixed points of $\varphi$ equals $\mathrm{orb}_\chi$.

iii) The morphism $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ induced by $\varphi$ is affine. Its fibers are isomorphic to an affine space.

*Proof.* i)$\mathrm{Orb}_\chi = G(O)/S$ where $S$ is the stabilizer of $[\chi]$ in $G(O)$. The action of $\mathrm{Aut}^0 O$ on $G(O)$ extends to an action of $\mathrm{End}^0 O$. Since $S$ is $\mathrm{Aut}^0 O$-invariant it is $\mathrm{End}^0 O$-invariant.

ii) The morphism $f : G(O) \to G(O)$ corresponding to $\mathbf{0} \in \mathrm{End}^0 O$ is the composition $G(O) \to G \hookrightarrow G(O)$. So $\varphi(\mathrm{Orb}_\chi) \subset \mathrm{orb}_\chi$. Clearly the restriction of $\varphi$ to $\mathrm{orb}_\chi$ equals id.

iii) $G(O) = G \cdot U$ where $U := \mathrm{Ker}(G(O) \to G)$. One has $f(S) \subset S$, so $S = S_G \cdot S_U$, $S_G := S \cap G$, $S_U := S \cap U$. $p$ is the natural morphism $G(O)/S \to G(O)/(S_G \cdot U) = G/S_G = \mathrm{orb}_\chi$. Since $U$ is prounipotent $(S_G \cdot U)/S = U/S_U$ is isomorphic to an affine space. $\square$

**9.1.6.** *Remark.* It follows from 9.1.5(ii) that the scheme of fixed points of $L_0$ on $\mathrm{Orb}_\chi$ equals $\mathrm{orb}_\chi$.

**9.1.7.** Since $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ is affine

$$H^i(\mathrm{Orb}_\chi, \mathrm{Sym}^j \mathcal{N}_\chi) = H^i(\mathrm{orb}_\chi, p_* \mathrm{Sym}^j \mathcal{N}_\chi).$$

$p$ is $\mathrm{Aut}^0 O$-equivariant, so $\mathrm{Aut}^0 O$ and therefore $L_0$ acts on $p_* \mathrm{Sym}^j \mathcal{N}_\chi$. To prove Proposition 9.1.2 it suffices to show the following.

**9.1.8.** *Lemma.* The eigenvalues of $L_0$ on $p_* \mathrm{Sym}^j \mathcal{N}_\chi$ are non-negative. If $j > 0$ they are positive. If $j = 0$ the zero eigensheaf of $L_0$ equals the structure sheaf of $\mathrm{orb}_\chi$.

*Proof.* Denote by $\mathcal{O}_{\mathrm{Orb}}$ and $\mathcal{O}_{\mathrm{orb}}$ the structure sheaves of $\mathrm{Orb}_\chi$ and $\mathrm{orb}_\chi$. It follows from 9.1.5(i) that the eigenvalues of $L_0$ on $p_*\mathcal{O}_{\mathrm{Orb}}$ are non-negative. 9.1.5(ii) or 9.1.6 implies that the cokernel of $L_0 : p_*\mathcal{O}_{\mathrm{Orb}} \to p_*\mathcal{O}_{\mathrm{Orb}}$ equals $\mathcal{O}_{\mathrm{orb}}$.

The obvious morphism $\mathcal{O}_{\mathrm{Orb}} \otimes (\mathfrak{g} \otimes K/\mathfrak{g} \otimes O) \to \mathcal{N}_\chi$ is surjective and $\mathrm{Aut}^0 O$-equivariant. It induces an $\mathrm{Aut}^0 O$-equivariant epimorphism $p_*\mathcal{O}_{\mathrm{Orb}} \otimes \mathrm{Sym}^j(\mathfrak{g} \otimes (K/O)) \to p_* \mathrm{Sym}^j \mathcal{N}_\chi$. Since the eigenvalues of $L_0$ on $K/O$ are positive we are done. $\square$

**9.1.9.** So we have proved 9.1.2 and therefore 8.1.4, 8.1.5. Now we are going to compute the canonical bundle of $\mathrm{Orb}_\chi$ in terms of the morphism $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$. The answer (see 9.1.12, 9.1.13) will be used in 10.1.7.

**9.1.10.** $\mathrm{Orb}_\chi$ is a homogeneous space of $G(O)$, while $\mathrm{orb}_\chi$ is a homogeneous space of $G$. Using the projection $G(O) \to G(O/tO) = G$ we get an action of $G(O)$ on $\mathrm{orb}_\chi$. The morphism $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ is $G(O)$-equivariant.[*)]

**9.1.11.** *Proposition.* The functor $p^*$ induces an equivalence between the groupoid of $G$-equivariant line bundles on $\mathrm{orb}_\chi$ and the groupoid of $G(O)$-equivariant line bundles on $\mathrm{Orb}_\chi$.

[*)] Of course the embedding $\mathrm{orb}_\chi \hookrightarrow \mathrm{Orb}_\chi$ is not $G(O)$-equivariant. DO WE NEED THIS FOOTNOTE?

*Proof.* One has $\mathrm{Orb}_\chi = G(O)/S$, $\mathrm{orb}_\chi = G/S_G$ where $S$ is the stabilizer of $[\chi]$ in $G(O)$ and $S_G = S \cap G$. In fact, $S_G$ is the image of $S$ in $G$ and $p : G(O)/S \to G/S_G$ is induced by the projection $G(O) \to G$ (see the proof of 9.1.5(iii) ). We have to show that the morphism $\pi : S \to S_G$ induces an isomorphism $\mathrm{Hom}(S_G, \mathbb{G}_m) \to \mathrm{Hom}(S, \mathbb{G}_m)$. This is clear because $\operatorname{Ker} \pi \subset \operatorname{Ker}(G(O) \to G)$ is prounipotent. $\square$

*Remark.* We formulated the proposition for equivariant bundles because we will use it in this form. Of course the statement still holds if one drops the word "equivariant" (indeed, $p$ is a locally trivial fibration whose fibers are isomorphic to an affine space). Besides, if $G$ is simply connected then a line bundle on $\mathrm{orb}_\chi$ has a unique $G$-equivariant structure (because by 9.1.3 $\mathrm{orb}_\chi = G/P_\chi^-$ and $P_\chi^-$ is parabolic).

**9.1.12.** The canonical sheaf $\omega_{\mathrm{Orb}_\chi}$ is a $G(O)$-equivariant line bundle on $\mathrm{Orb}_\chi$. By 9.1.11 it comes from a unique $G$-equivariant line bundle $\mathcal{M}_\chi$ on $\mathrm{orb}_\chi$. Since $\mathrm{orb}_\chi = G/P_\chi^-$ (see 9.1.3) isomorphism classes of $G$-equivariant line bundles on $\mathrm{orb}_\chi$ are parametrized by $\mathrm{Hom}(P_\chi^-, \mathbb{G}_m)$. The embedding $H \hookrightarrow P_\chi^-$ induces an embedding $\mathrm{Hom}(P_\chi^-, \mathbb{G}_m) \hookrightarrow \mathrm{Hom}(H, \mathbb{G}_m)$. So $\mathcal{M}_\chi$ defines a weight of $H$, which can be considered as an element $l_\chi \in \mathfrak{h}^*$.

**9.1.13.** *Proposition.* $l_\chi = B\chi$ where $\chi \in \mathrm{Hom}(\mathbb{G}_m, H)$ is identified in the usual way with an element of $\mathfrak{h}$ and $B : \mathfrak{h} \to \mathfrak{h}^*$ is the linear operator corresponding to the scalar product (18).

*Proof.* The tangent space to $\mathrm{Orb}_\chi$ at $[\chi]$ equals

$$(\mathfrak{g} \otimes O)/((\mathfrak{g} \otimes O) \cap t^\chi (\mathfrak{g} \otimes O) t^{-\chi}). \tag{393}$$

The action of $H$ on (393) comes from the adjoint action of $H$ on $\mathfrak{g} \otimes O$. So the weights of $H$ occuring in (393) are positive roots, and for a positive root $\alpha$ its multiplicity in (393) equals $(\chi, \alpha)$. Therefore the weight of $\mathfrak{h}$

corresponding to the determinant of the vector space dual to (393) equals

$$-\sum_{\alpha>0}(\chi,\alpha)\cdot\alpha=-\frac{1}{2}\sum_{\alpha}(\chi,\alpha)\cdot\alpha=B\chi.$$

$\square$

Note for the authors: the notation $U := \mathrm{Ker}(G(O)\to G)$ is not quite compatible with the notation $U_\chi$. Is this OK ???

# 10. To be inserted into Section 6, too

**10.1. Delta-functions.** Is the title of the section OK ???

**10.1.1.** According to 8.1.5 we have the canonical embedding $\mathfrak{d}_{\mathcal{L},\chi} \hookrightarrow \Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})$. Its image is contained in $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$. The Lie algebroid $I/I^2$ acts on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ (see ??? and 5.6.8). Using (81) we identify $I/I^2$ with the Lie algenroid $\mathfrak{a}_{^L\mathfrak{g}}$ from 3.5.11, where ${}^L\mathfrak{g} := \operatorname{Lie}{}^LG$ and ${}^LG$ is understood in the sense of 5.3.22 (in particular, ${}^L\mathfrak{g}$ has a distinguished[*)] Borel subalgebra ${}^L\mathfrak{b}$ and a distinguished Cartan subalgebra ${}^L\mathfrak{h} \subset {}^L\mathfrak{b}$; we set ${}^L\mathfrak{n} := [{}^L\mathfrak{b}, {}^L\mathfrak{b}]$). By 3.5.16 we have the Lie subalgebroids $\mathfrak{a}_{^L\mathfrak{n}} \subset \mathfrak{a}_{^L\mathfrak{b}} \subset \mathfrak{a}_{^L\mathfrak{g}}$ and a canonical isomorphism of $A_{^L\mathfrak{g}}(O)$-modules $\mathfrak{a}_{^L\mathfrak{b}}/\mathfrak{a}_{^L\mathfrak{n}} = A_{^L\mathfrak{g}}(O) \otimes {}^L\mathfrak{h}$. In particular ${}^L\mathfrak{h} \subset \mathfrak{a}_{^L\mathfrak{b}}/\mathfrak{a}_{^L\mathfrak{n}}$.

**10.1.2.** *Theorem.* i) $\mathfrak{a}_{^L\mathfrak{n}}$ annihilates $\mathfrak{d}_{\mathcal{L},\chi}$, so $a\delta$ makes sense for $a \in {}^L\mathfrak{h}$, $\delta \in \mathfrak{d}_{\mathcal{L},\chi}$ .

ii) $a\delta = \chi(a)\delta$ for $a \in {}^L\mathfrak{h}$, $\delta \in \mathfrak{d}_{\mathcal{L},\chi}$.

*Remark.* We identify $\chi \in P_+({}^LG)$ with a linear functional on ${}^L\mathfrak{h}$, so $\chi(a)$ makes sense.

Statement (i) is easy. Indeed, $\operatorname{Der} O$ acts on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ (see 5.6.10) and the action of $\mathfrak{a}_{^L\mathfrak{g}}$ on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ is compatible with the actions of $\operatorname{Der} O$ on $\mathfrak{a}_{^L\mathfrak{g}}$ and $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ (use the $\operatorname{Der} O$-equivariance of (81) and the Remark at the end of 3.6.16).[*)] So statement (i) follows from Theorem 8.1.4, Proposition 8.1.5, and (77). In a similar way one proves using (78) that $a\mathfrak{d}_{\mathcal{L},\chi} \subset \mathfrak{d}_{\mathcal{L},\chi}$ for $a \in {}^L\mathfrak{h}$, which is weaker than (ii). We will prove (ii)

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[*)] In §3 (where we worked with $G$-opers rather than ${}^LG$-opers) we assumed that a Borel subgroup $B \subset G$ is fixed (see 3.1.1), so we are pleased to have a distinguished ${}^L\mathfrak{b} \subset {}^L\mathfrak{g}$. But in fact this is not essential here: one could rewrite §3 without fixing $B$; in this case we would have the Lie algebroids $\mathfrak{a}_\mathfrak{b}$ and $\mathfrak{a}_\mathfrak{n}$ without having concrete $\mathfrak{b}, \mathfrak{n} \subset \mathfrak{g}$.

[*)] In fact, a stronger statement is true: the action of $\operatorname{Der} O$ on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ coincides with the one coming from the morphism $\operatorname{Der} O \to \mathfrak{a}_{^L\mathfrak{g}}$ defined in 3.5.11 and the action of $\mathfrak{a}_{^L\mathfrak{g}}$ on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ (this follows from 3.6.17).

in 10.1.3 – 10.1.7. In this proof we fix*) $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$ and write $\lambda$ instead of $\lambda_{\mathcal{L}}$, $\mathfrak{d}_\chi$ instead of $\mathfrak{d}_{\mathcal{L},\chi}$, etc.

**10.1.3.** By 3.6.11 we can reformulate 10.1.2(ii) as follows:

$$a\delta = -(d(a), B\chi)\cdot\delta \text{ for } a \in I^{\leq 0},\ \delta \in \mathfrak{d}_\chi \tag{394}$$

where $d : I^{\leq 0} \to \mathfrak{h}$ is the map (83), $\chi$ is considered as an element of $\mathfrak{h}$ (see the Remark from 10.1.2) and $B : \mathfrak{h} \to \mathfrak{h}^*$ corresponds to the scalar product (18).

*Remark.* The "critical" scalar product (18) appears in the r.h.s. of (394) because the definition of the l.h.s. involves the map (291), which depends on the choice of the scalar product on $\mathfrak{g}$ (see 5.6.11).

**10.1.4.** The method of the proof of (394) will be described in 10.1.5. Let us explain the difficulty we have to overcome. The action of $I/I^2$ on $\Gamma(\mathcal{GR}, I_\chi\lambda^{-1})^{G(O)}$ comes from the action of the renormalized universal enveloping algebra $U^\natural$ on $\Gamma(\mathcal{GR}, I_\chi\lambda^{-1})$, which is defined by deforming the critical level (see ???). So the naive idea would be to deform $I_\chi$, i.e., to try to construct a family of $\lambda^h$-twisted $\mathcal{D}$-modules $M_h^?$, $h \in \mathbb{C}$, such that $M_0^? = I_\chi$. But this turns out to be impossible (at least globally) because $\lambda^h$-twisted $\mathcal{D}$-modules on $\mathrm{Orb}_\chi$ that are invertible $\mathcal{O}$-modules exist only for a discrete set of values of $h$. Therefore we have to modify the naive idea (see 10.1.5 and 10.1.7).

**10.1.5.** We are going to use the notion of $\mathcal{D}_{\lambda^h}$-module from 7.11.11 (so $h \in \mathbb{C}[h]$ is a parameter). In 10.1.7 we will construct a $\mathcal{D}_{\lambda^h}$-module $M$ on $U_\chi$ and an embedding

$$\mathfrak{d}_\chi \hookrightarrow \Gamma(U_\chi, M\lambda^{-1}) \tag{395}$$

such that

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*) By the way, all objects of $Z\,\mathrm{tors}_\theta(O)$ are isomorphic.

(i) $M$ is a flat $\mathbb{C}[h]$-module$^{*)}$;

(ii) There is a $\mathcal{D}$-module morphism $M_0 := M/hM \to I_\chi^U := I_\chi|_{U_\chi}$ such that the composition

$$\mathfrak{d}_\chi \hookrightarrow \Gamma(U_\chi, M\lambda^{-1}) \to \Gamma(U_\chi, M_0\lambda^{-1}) \to \Gamma(U_\chi, I_\chi^U\lambda^{-1})$$

equals (392);

(iii) The image of (395) is annihilated by $\mathfrak{g}\otimes\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of $O$;

(iv) for $c \in C :=$ the center of $U\mathfrak{g}$ and $\delta \in \mathfrak{d}_\chi$ one has

$$c\delta_h = \varphi(c)\delta_h \tag{396}$$

where $\delta_h \in \Gamma(U_\chi, M\lambda^{-1})$ is the image of $\delta$ under (395), $\varphi : C \to \mathbb{C}[h]$ is the character corresponding to the Verma module with highest weight $-hB\chi$, and $B : \mathfrak{h} \to \mathfrak{h}^*$ is the scalar product (18).

*Remarks.* 1) $M\lambda^{-1}$ is a $\mathcal{D}_{\lambda^{h+1}}$-module.

2) Of course, $\mathcal{D}_{\lambda^{h+1}} := \mathcal{D}_{\lambda^s} \otimes_{\mathbb{C}[s]} \mathbb{C}[h]$ where the morphism $\mathbb{C}[s] \to \mathbb{C}[h]$ is defined by $s \mapsto h+1$. Quite simialrly one defines, e.g., $\mathcal{D}_{\lambda^{-h}}$ (this notation will be used in 10.1.7).

**10.1.6.** Let us deduce (394) from (i) – (iv). By 5.6.7 – 5.6.8 the l.h.s. of (394) equals $a^\flat\delta$ where $a^\flat \in U_1^\flat$ and $a \in I^{\le 0}$ have the same image in $U_1^\flat/U_0^\flat$. To construct $a^\flat$ we can lift $a$ to an element $\tilde{a} \in A :=$ the completed universal enveloping algebra of $\widetilde{\mathfrak{g}\otimes K}$ so that $\tilde{a}$ belongs to the ideal of $A$ topologically generated by $\mathfrak{g}\otimes O$; then $h^{-1}\tilde{a}$ belongs to the algebra $A^\natural$ from 5.6.1 and we can set $a^\flat :=$ the image of $h^{-1}\tilde{a}$ in $U^\natural$.

We will show that for a suitable choice$^{*)}$ of $\tilde{a}$

$$a^\flat\delta_0 = -(d(a), B\chi)\cdot\delta_0 \tag{397}$$

$^{*)}$So for each $a \in \mathbb{C}$ we have the module $M_a := M/(h-a)M$ over $\mathcal{D}_{\lambda^a} := \mathcal{D}_{\lambda^h}/(h-a)$, and $M$ is, so to say, a flat family formed by $M_a$, $a \in \mathbb{C}$.

$^{*)}a^\flat\delta$ does not depend on the choice of $\tilde{a}$ while $a^\flat\delta_0$ does (because $\delta_0$ is annihilated by $\mathfrak{g}\otimes\mathfrak{m}$, but not by $\mathfrak{g}\otimes\mathcal{O}$).

where $\delta_0$ is the image of $\delta_h$ in $\Gamma(U_\chi, M_0\lambda^{-1})$ and $d$, $B$ have the same meaning as in (394). By 10.1.5(ii) the equality (397) implies (394).

Let us describe our choice of $\tilde{a}$. We can write $a \in I^{\leq 0}$ as $c + a'$ where $c \in C$ and $a'$ belongs to the left ideal of $\overline{U}'$ topologically generated by $\mathfrak{g} \otimes \mathfrak{m}$ (in terms of 3.6.8 – 3.6.9 $c = \pi(a)$). We choose $\tilde{a} \in A$ so that $\tilde{a} \mapsto a$ and $\tilde{a} - c$ belongs to the left ideal of $A$ topologically generated by $\mathfrak{g} \otimes \mathfrak{m}$. Then (397) holds.

Indeed, $M\lambda^{-1}$ is a $\mathcal{D}_{\lambda^{h+1}}$-module. Therefore by ??? $A^\natural$ acts on $\Gamma(U_\chi, M\lambda^{-1})$ (can we write simply $M\lambda^{-1}$ ???) so that $h := \mathbf{1} - 1 \in \widetilde{\mathfrak{g} \otimes K} \subset A^\natural$ acts as multiplication by $h$ (is this expression OK ???). We can rewrite (397) as

$$h^{-1}\tilde{a} \cdot \delta_h \equiv -(d(a), B\chi) \cdot \delta_h \mod h\,. \tag{398}$$

By 10.1.5(iii) and 10.1.5(iv) we have $\tilde{a}\delta_h = c\delta_h = \varphi(c)\delta_h$. On the other hand, $\varphi(c) \in \mathbb{C}[h]$ is congruent to $-(d(a), B\chi)h$ modulo $h^2$ (see the definition of $\varphi$ from 10.1.5 and the definition of $d$ from 3.6.10). So we get (398).

**10.1.7.** Let us construct the $\mathcal{D}_{\lambda^h}$-module $M$ and the morphism (395) satisfying 10.1.5(i) – 10.1.5(iv).

We have the $G(O)$-equivariant line bundle $\lambda = \lambda_{\mathcal{L}}$ on $\mathcal{GR}$. Denote by $\lambda_\chi$ its restriction to $\mathrm{Orb}_\chi$. Let $\mathrm{orb}_\chi$ and $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ have the same meaning as in 9.1.3 and 9.1.5. Recall that $G(O)$ acts on $\mathrm{orb}_\chi$ via $G(O/tO) = G$ and $p$ is $G(O)$-equivariant. By 9.1.11 there is a unique $G$-equivariant line bundle $\underline{\lambda}_\chi$ on $\mathrm{orb}_\chi$ such that $\lambda_\chi = p^*\underline{\lambda}_\chi$.

On $\mathrm{orb}_\chi$ we have the sheaf of twisted differential operators $\mathcal{D}_{\underline{\lambda}_\chi^h}$. Set $N := p^\dagger \mathcal{D}_{\underline{\lambda}_\chi^{-h}}$ where $\mathcal{D}_{\underline{\lambda}_\chi^{-h}}$ is considered as a left $\mathcal{D}_{\underline{\lambda}_\chi^{-h}}$-module and $p^\dagger$ is the usual pullback functor. $N$ is a left $\mathcal{D}_{\lambda_\chi^{-h}}$-module on $\mathrm{Orb}_\chi$ equipped with a canonical section $\mathbb{I} := p^\dagger(1) \in \Gamma(\mathrm{Orb}_\chi, N)$. Clearly $\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N$ is a right

$\mathcal{D}_{\lambda_\chi^h}$-module*) on $\mathrm{Orb}_\chi$. The section $1\!\!1$ induces an $\mathcal{O}$-module morphism

$$\omega_{\mathrm{Orb}_\chi} \to \omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N\,. \tag{399}$$

We define $M$ to be the direct image of $\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N$ under the closed embedding $\mathrm{Orb}_\chi \hookrightarrow U_\chi$. The morphism (395) is defined to be the composition

$$\mathfrak{d}_\chi \hookrightarrow \Gamma(\mathrm{Orb}_\chi, \omega_{\mathrm{Orb}_\chi} \otimes \lambda_\chi^{-1}) \hookrightarrow \Gamma(\mathrm{Orb}_\chi, (\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N)\lambda_\chi^{-1}) \hookrightarrow \Gamma(U_\chi, M\lambda^{-1})$$

where the first morphism is induced by (241) and the second one is induced by (399).

The property 10.1.5(i) is clear. The property 10.1.5(ii) is also clear: the morphism $M_0 \to I_\chi^U$ comes from the $\mathcal{D}$-module morphism $N_0 = p^\dagger \mathcal{D}_{\mathrm{orb}_\chi} \to \mathcal{O}_{\mathrm{Orb}_\chi}$ such that $1\!\!1 \mapsto 1$ (is it OK to write $1\!\!1$ instead of $1\!\!1$ mod $h$, or $1\!\!1_0$, etc. ???). Notice that 10.1.5(iii) and 10.1.5(iv) are properties of the action of $\mathfrak{g} \otimes O$ on the image of (395). This image is contained in the $\mathfrak{g} \otimes O$-invariant subspace (or $\mathbb{C}[h]$-submodule ???)

$$\Gamma(\mathrm{Orb}_\chi, (\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N)\lambda_\chi^{-1}) = \Gamma(\mathrm{Orb}_\chi, \lambda_\chi^{-1}\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N)\,. \tag{400}$$

So to prove 10.1.5(iii) and 10.1.5(iv) it suffices to work on $\mathrm{Orb}_\chi$. Using (241) we identify (400) with

$$\mathfrak{d}_\chi \otimes \Gamma(\mathrm{Orb}_\chi, N)\,. \tag{401}$$

The isomorphism between (400) and (401) is $\mathfrak{g} \otimes O$-equivariant (the action of $\mathfrak{g} \otimes O$ on $\mathfrak{d}_\chi$ is trivial), because the isomorphism (241) is $\mathfrak{g} \otimes O$-equivariant. So 10.1.5(iii) and 10.1.5(iv) are equivalent to the following properties of

*) By the way, $\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N$ is canonically isomorphic to the pullback of the right $\mathcal{D}_{\underline{\lambda}_\chi^h}$-module $\omega_{\mathrm{orb}_\chi} \otimes_{\mathcal{O}} \mathcal{D}_{\underline{\lambda}_\chi^h}$. Indeed, the image of $\omega_{\mathrm{orb}_\chi} \otimes_{\mathcal{O}} \mathcal{D}_{\underline{\lambda}_\chi^h}$ under the usual functor $M \mapsto M \otimes_{\mathcal{O}} \omega_{\mathrm{orb}_\chi}^{-1}$ transforming right $\mathcal{D}_{\underline{\lambda}_\chi^h}$-modules into left $\mathcal{D}_{\underline{\lambda}_\chi^{-h}}$-modules is freely generated by $1 \in \Gamma(\mathrm{orb}_\chi, \omega_{\mathrm{orb}_\chi} \otimes_{\mathcal{O}} \mathcal{D}_{\underline{\lambda}_\chi^h} \otimes_{\mathcal{O}} \omega_{\mathrm{orb}_\chi}^{-1})$ and therefore is canonically isomorphic to $\mathcal{D}_{\underline{\lambda}_\chi^{-h}}$.

$\mathbb{1} \in \Gamma(\mathrm{Orb}_\chi, N)$:

$$(\mathfrak{g} \otimes \mathfrak{m})\mathbb{1} = 0\,, \tag{402}$$

$$c\mathbb{1} = \varphi(c)\mathbb{1} \quad \text{for} \quad c \in C\,. \tag{403}$$

Recall that $C$ :=the center of $U\mathfrak{g}$, $\varphi : C \to \mathbb{C}[h]$ denotes the character corresponding to the Verma module with highest weight $-hB\chi$, and $B : \mathfrak{h} \to \mathfrak{h}^*$ is the scalar product (18).

So it remains to prove (402) and (403). Recall that $N := p^\dagger \mathcal{D}_{\underline{\lambda}_\chi^{-h}}$, $\mathbb{1} := p^\dagger(1)$, and $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ is $G(O)$-equivariant. Therefore (402) is clear (because the action of $\mathfrak{g} \otimes \mathfrak{m}$ on $(\mathrm{orb}_\chi, \underline{\lambda}_\chi)$ is trivial) and (403) is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} C & \hookrightarrow & U\mathfrak{g} \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ \mathbb{C}[h] & \hookrightarrow & \Gamma(\mathrm{orb}_\chi, \mathcal{D}_{\underline{\lambda}_\chi^{-h}}) \end{array} \tag{404}$$

Recall that $\underline{\lambda}_\chi$ is the $G$-equivariant line bundle on $\mathrm{orb}_\chi$ such that $\lambda_\chi = p^*\underline{\lambda}_\chi$. Since $\mathrm{orb}_\chi = G/P_\chi^-$ (see 9.1.3) the isomorphism class of $\underline{\lambda}_\chi$ is defined by some $l \in \mathrm{Hom}(P_\chi^-, \mathbb{G}_m) \subset \mathrm{Hom}(H, \mathbb{G}_m) \subset \mathfrak{h}^*$. In fact,

$$l = B\chi\,. \tag{405}$$

Indeed, there is a $G(O)$-equivariant isomorphism $\lambda_\chi = \omega_{\mathrm{Orb}_\chi}$ (see (241)), so $\underline{\lambda}_\chi$ is $G$-isomorphic to the line bundle $\mathcal{M}_\chi$ from 9.1.12 and (405) is equivalent to Proposition 9.1.13. The commutativity of (404) follows from (405) (see ???). So we are done.

## References


[Ak] F. Akman.A characterization of the differential in semi-infinite cohomology. Journal of Algebra, vol. 162 (1993), 194–209.

[Ar] M. Artin. Algebraization of formal moduli: II. Existence of modifications. Annals of Math., vol. 91 (1970), No. 1, 88–135.

[At] M. F. Atiyah. Riemann surfaces and spin structures. *Ann. Sci. École Norm. Sup.*, ser. 4, vol. 4 (1971), 47–62.

[Ba63] H. Bass. Big projective modules are free. Illinois J. of Math., vol. 7 (1963), 24–31.

[Ba68] H. Bass. Algebraic K-theory. Benjamin, New York, 1968.

[BB81] A. Beilinson and J. Bernstein. Localisation de $\mathfrak{g}$-modules. Comptes Rendus Acad. Sci. Paris, vol. 292, série I (1981), 15–18.

[BB93] A. Beilinson and J. Bernstein. A proof of Jantzen conjectures. In: I. M. Gelfand Seminar, *Advances in Soviet Math.*, vol. 16, part 1, 1–50. Amer. Math. Soc., Providence, 1993.

[BD94] A. Beilinson and V. Drinfeld. Affine Kac-Moody algebras and polydifferentials. International Mathematics Research Notices, 1994, No. 1, 1–11.

[BD96] A. Beilinson and V. Drinfeld. Quantization of Hitchin's fibration and Langlands program. In: *Algebraic and Geometric Methods in Mathematical Physics* (A. Boutet de Monvel and V. A. Marchenko, editors), *Mathematical Physics Studies*, vol. 19, 3–7. Kluwer Academic Publishers, Netherlands, 1996.

[BD] A. Beilinson and V. Drinfeld. Chiral algebras. To appear.

[BS] A. Beilinson and V. Schechtman. Determinant bundles and Virasoro algebras. Commun. Math. Phys., vol. 118 (1988), 651–701.

[BL] J. Bernstein and V. Lunts. Localization for derived categories of $(\mathfrak{g}, K)$-modules. Journal of Amer. Math. Soc., vol. 8 (1995), No. 4, 819–856.

[BLa94] A. Beauville and Y. Laszlo. Conformal blocks and generalized theta functions. Commun. Math. Phys., vol. 164 (1994), No. 2, 385–419.

[BLa95] A. Beauville and Y. Laszlo. Un lemme de descente. Comptes Rendus Acad. Sci. Paris, vol. 320, série I (1995), 335–340.

[BLaSo] A. Beauville, Y. Laszlo, and C. Sorger. The Picard group of the moduli of $G$-bundles on a curve. Preprint alg-geom/9608002.

[BR] I.N. Bernstein and B.I. Rozenfel'd. Homogeneous spaces of infinite-dimensional Lie algebras and characteristic classes of foliations. Russian Math. Surveys, vol. 28 (1973), no. 4, 107–142 (Russian original: Uspekhi Mat. Nauk, vol. 28 (1973), no. 4, 103–138)

[Bourb59] N. Bourbaki. Algèbre, chap. IX. Hermann, Paris, 1959.

[Bour68] N. Bourbaki. Groupes et algèbres de Lie, chap. IV–VI. Hermann, Paris, 1968.

[Bour75] N. Bourbaki. Groupes et algèbres de Lie, chap. VII, VIII. Hermann, Paris, 1975.

[De-Mi] P. Deligne and J. Milne. Tannakian categories. In: *Hodge Cycles, Motives, and Shimura Varieties*, *Lecture Notes in Mathematics*, vol. 900, 101–228. Springer-Verlag, 1982.

[Del70] P. Deligne. Équations différentielles à points singuliers réguliers. *Lecture Notes in Math*, vol. 163. Springer-Verlag, 1970.

[Del73] P. Deligne. La formule de dualité globale. In: *SGA 4*, tome 3, *Lecture Notes in Mathematics*, vol. 305, 481–587. Springer-Verlag, 1973.

[Del87] P. Deligne. Le déterminant de la cohomologie. In: *Current Trends in arithmetical algebraic geometry, Contemporary Mathematics*, vol. 67, 93–177. Amer. Math. Soc., Providence, 1987.

[Del91] P. Deligne. Catégories tannakiennes. In: Grothendieck Festschrift, vol. II (P. Cartier et al., editors), 111–195, Birkhauser, Boston, 1991.

[D71] J. Dieudonné. La géometrie des groupes classiques. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 5. Springer-Verlag, 1971.

[DS85] V.G. Drinfeld and V.V. Sokolov. Lie algebras and equations of Korteweg-deVries type. *Journal of Soviet Math*, vol. 30 (1985), 1975–2036. Reprinted in [W], pp. 25–86.

[DSim] V. Drinfeld and C. Simpson. $B$-structures on $G$-bundles and local triviality. Mathematical Research Letters, vol. 2 (1995), No. 6, 823–829.

[Fal93] G. Faltings. Stable $G$-bundles and projective connections. *Journal of Algebraic Geometry*, vol. 2 (1993), No. 3, 507–568.

[Fal94] G. Faltings. A proof for the Verlinde formula. *Journal of Algebraic Geometry*, vol. 3 (1994), No. 2, 347–374.

[F84] B.L. Feigin. The semi-infinite homology of Kac–Moody and Virasoro Lie algebras. Russian Math. Surveys, vol. 39 (1984), no. 2, 155–156 (Russian original: Uspekhi Mat. Nauk, vol. 39 (1984), no. 2, 195–196)

[F91] B. Feigin. Differential operators on the moduli space of $G$-bundles on algebraic curve and Lie algebra cohomologies. In: *Special Functions*, editors M. Kashiwara, T. Miwa, 90–103. Springer-Verlag, Tokyo, 1991.

[FF92] B. Feigin and E. Frenkel. Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras. In: *Infinite Analysis*, editors A. Tsuchiya, T. Eguchi, M.

Jimbo, *Adv. Ser. in Math. Phys.*, vol. 16, 197–215. World Scientific, Singapore, 1992.

[Fr] E. Frenkel. Affine algebras, Langlands duality, and Bethe ansatz. In: Proceedings of the International Congress on Mathematical Physics, Paris, 1994 (D. Iagolnitzer editor), 606–642. International Press, 1995.

[Fr81] I. B. Frenkel. Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory. *Journal of functional analysis*, vol. 44 (1981), 259–327.

[FGZ86] I. B. Frenkel, H. Garland, and G. J. Zuckerman. Semi-infinite cohomology and string theory. Proc. Natl. Acad. Sci. USA 83 (1986), pp. 8442–8446.

[Ga2001] D. Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. Invent. Math., vol. 144 (2001), no. 2, 253–280.

[GD76] I.M. Gelfand and L.A. Dikii. Fractional powers of operators and Hamiltonian systems. Functional Analysis and Applications, vol. 10 (1976), no. 4, 259–273 (Russian original: pp. 13–29).

[GD78] I.M. Gelfand and L.A. Dikii. A family of Hamiltonian structures connected with integrable nonlinear differential equations. Preprint Inst. Appl. Math. no. 136 (Moscow, 1978). Reprinted in [W], pp. 3-24.

[Gr64] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et de morphismes de schémas (première partie). Publications Mathématiques IHES, No. 20, 1964.

[Gr67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et de morphismes de schémas (quatrième partie). Publications Mathématiques IHES, No. 32, 1967.

[Gr68] A. Grothendieck. Crystals and the De Rham cohomology of schemes. In: *Dix Exposés sur la Cohomologie des Schémas*, 306–358. North-Holland, Amsterdam, 1968.

[Gr68a] A. Grothendieck. Catégories cofibrées additives et complexe cotangent relatif. Lecture Notes in Math., vol. 79. Springer-Verlag, 1968.

[Gi87] V. Ginzburg. Equivariant cohomology and Kähler geometry. Functional Anal. Appl., vol. 21 (1987), no. 4, 271–283 (Russian original: pp. 19–34).

[Gi95] V. Ginzburg. Perverse sheaves on a loop group and Langlands' duality. Preprint alg-geom/9511007.

[Gi97] V. Ginzburg. Global nilpotent variety is Lagrangian. Preprint alg-geom/9704005.

[GK] I.M. Gel'fand and D.A. Kazhdan. Some problems of differential geometry and the calculation of cohomologies of Lie algebras of vector fields. Soviet Math. Doklady, vol. 12 (1971), no. 5, 1367–1370 (Russian original: Doklady Akademii Nauk SSSR, vol. 200 (1971), no. 2, 269–272).

[GKF] I.M. Gel'fand, D.A. Kazhdan, and D.B. Fuks. The actions of infinite-dimensional Lie algebras. Functional Anal. Appl., vol. 6 (1972), no. 1, 9–13 (Russian original: pp. 10–15).

[Ha88] T. Hayashi. Sugawara operators and Kac-Kazhdan conjecture. Inventiones Math., vol. 94 (1988), 13–52.

[Hit87] N. Hitchin. Stable bundles and integrable systems. *Duke Math. Jour.*, vol. 54 (1987), 91–114.

[IM] N. Iwahori and H. Matsumoto. On some Bruhat decompositiopns and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups. Publications Mathématiques IHES, No. 25, 1965.

[J] C.U. Jensen. On the vanishing of $\varprojlim^{(i)}$. Jour. of Algebra, vol. 15 (1970), 151–166.

[Kac90] V.G. Kac. Infinite dimensional Lie algebras. Cambridge University Press, 1990.

[Kac97] V.G. Kac. Vertex algebras for beginners. University lecture series, vol. 10. Amer.Math.Society, Providence, USA, 1997.

[KK] V.G. Kac and D.A. Kazhdan. Structure of representations with highest weight of infinite-dimensional Lie algebras. Adv. in Math., vol. 34 (1979), No. 1, 97–108.

[KP] V.G. Kac and D.H. Peterson. Spin and wedge representations of infinite-dimensional Lie algebras and groups. Proc. Nat. Acad. Sci. USA, vol. 78 (1981), No. 6, 3308–3312.

[KR] V.G. Kac and A.K. Raina. Bombay lectures on highest weight representations of infinite dimensional Lie algebras. World Scientific, Singapore, 1987.

[Kap91] M. Kapranov. On DG-modules over the de Rham complex and the vanishing cycles functor. In: Lecture Notes in Mathematics, vol. 1479, 57–86. Springer-Verlag, Berlin, 1991.

[Kap97] M. Kapranov. Injective resolutions of $BG$ and derived moduli spaces of local systems Preprint alg-geom/9710027.

[Kn-Mu] F. Knudsen and D. Mumford. The projectivity of the moduli space of stable curves I: preliminaries on "det" and "div". *Math. Scand.*, vol. 39 (1976), 19–55.

[Kas] M. Kashiwara. Representation theory and $D$-modules on flag varieties. Astérisque, vol. 173–174 (1989), 55–109.

[KT95] M. Kashiwara and T. Tanisaki. Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. *Duke Math. Jour.*, vol. 77 (1995), No. 1, 21–62.

[Ko61] B. Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. Annals of Math., vol. 74 (1961), No. 2, 329–387.

[Ko63] B. Kostant. Lie group representations on polynomial rings. *Amer. Jour. of Math.*, vol. 85 (1963), 327–404.

[Ko78] B. Kostant. On Whittaker vectors and representation theory. Inventiones Math., vol. 48 (1978), 101–184.

[KS] B. Kostant and S. Sternberg. Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. Annals of Physics, vol 176 (1987), 49–113.

[La-So] Y. Laszlo and C. Sorger. The line bundles on the moduli of parabolic $G$-bundles over curves and their sections. *Ann. Sci. École Norm. Sup.*, ser. 4, vol. 30 (1997), 499–525.

[La87] G. Laumon. Correspondance de Langlands géométrique pour les corps de fonctions. Duke Math. Jour., vol. 54 (1987), 309–359.

[La88] G. Laumon. Un analogue global du cône nilpotent. Duke Math. Jour., vol. 57 (1988), 647–671.

[LMB93] G. Laumon and L. Moret-Bailly. Champs Algébriques. Preprint 92-42, Université Paris-Sud, 1992.

[Lu83] G. Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. Astérisque, vol. 101-102 (1983), 208-229.

[Ma87] K. Mackenzie. Lie Groupoids and Lie Algebroids in Differential Geometry. London Math. Soc. Lecture Note Ser., vol. 124. Cambridge Univ. Press, 1987.

[Ma96] K. Mackenzie. Lie algebroids and Lie pseudoalgebras. Bull. London Math. Soc., vol. 27 (1995), part 2, 97–147.

[Mu] D. Mumford. Theta characteristics on an algebraic curve. *Ann. Sci. École Norm. Sup.*, ser. 4, vol. 4 (1971), 181–192.

[MV] I.Mirković and K.Vilonen. Perverse sheaves on Loop Grassmannians and Langlands duality. Preprint alg-geom/9703010.

[Pa] B. Pareigis. Non-additive ring and module theory. II. $C$-categories, $C$-functors and $C$-morphisms. *Publ. Math. Debrecen*, vol. 24 (1977), No. 3–4, 351–361.

[Pa76] A. Parshin. On the arithmetic of two-dimensional schemes, I. Distributions and residues. Mathematics of USSR – Izvestiya, vol. 10 (1976), 695–729 (Russian original: vol. 40, 736–773).

[Phys] L. Fehér, L. O'Raifertaigh, and I. Tsutsui. The vacuum preserving Lie algebra of a classical $W$-algebra. Phys. Lett. B 316 (1993), 275–281.

[RG] M. Raynaud and L. Gruson. Critères de platitude et de projectivité. Inventiones Math., vol. 13 (1971), 1–89.

[R] G. S. Rinehart. Differential forms on general Lie algbras. Trans.Amer.Math.Soc., vol. 108 (1963), 195–222.

[Sa89] M. Saito. Induced $\mathcal{D}$-modules and differential complexes. Bull. Soc. Math. France, tome 117 (1989), fasc. 3, 361–387.

[Sa91] M. Saito. $\mathcal{D}$-modules on analytic spaces. Publ. Res. Inst. Math. Sci., vol. 27 (1991), No. 2, 291–332.

[Se] J.-P. Serre. Modules projectifs et espaces fibrés à fibre vectorielle. In: Collected papers by J.-P. Serre, vol. I, 531-543, Springer-Verlag, Berlin, 1986.

[T] J. Tate. Residues of differentials on curves. *Ann. Sci. École Norm. Sup.*, ser. 4, vol. 1 (1968), 149–159.

[Ve] J.L.Verdier. Catégories dérivées. In: SGA $4\frac{1}{2}$, Lecture Notes in Mathematics, vol. 569, 262–311. Springer-Verlag, Berlin, 1977.

[W] W-symmetry (P. Bouwknegt and K. Schoutens editors). Adv. Ser. in Math. Phys., vol. 22. World Scientific, Singapore, 1995.

[We83] A. Weinstein. Sophus Lie and symplectic geometry. Expositiones Matematicae, vol. 1 (1983), 95–96.

[We88] A. Weinstein. Coisotropic calculus and Poisson groupoids. *Jour. Math. Soc. Japan*, vol. 40 (1988), No. 4, 705–727.

[We] A. Weinstein. Groupoids: unifying internal and external symmetry. Notices of AMS, vol. 43 (1996), No. 7, 744–752.

[Yet] D. N. Yetter. Tangles in prisms, tangles in cobordisms. In: *Topology '90* (B. Apanasov et al., editors), 399–443. Walter de Gruyter, Berlin, New York, 1992.

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