Gamma Swap

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A gamma swap on an underlying Y is a *weighted variance swap* on $\log Y$, with weight function

$$w(y) := y/Y_0. \tag{1}$$

In practice, the gamma swap monitors Y discretely, typically daily, for some number of periods N, annualizes by a factor such as $252/N$, and multiplies by notional, for a total payoff

$$\text{Notional} \times \text{Annualization} \times \sum_{n=1}^{N} \frac{Y_n}{Y_0} \left(\log \frac{Y_n}{Y_{n-1}} \right)^2. \tag{2}$$

If the contract makes dividend adjustments (as typical for single-stock gamma swaps but not index gamma swaps), then the term inside the parentheses becomes $\log((Y_n + D_n)/Y_{n-1})$, where D_n denotes the dividend payment, if any, of the nth period.

Gamma swaps allow investors to acquire variance exposures proportional to the underlying level. One application is dispersion trading of a basket's volatility against its components' single-name volatilities; as a component's value increases, so does its proportion of the total basket value, and hence so does the desired volatility exposure of the single-name contract; this variable exposure to volatility is provided by gamma swaps, according to point 1 below. A second application is to trade the volatility skew; for example, to express a view that the skew slopes too steeply downward, the investor can go long a gamma swap and short a variance swap, to create a weighting $y/Y_0 - 1$, which is short downside variance and long upside variance. A third application is to trade single-stock variance without the caps often embedded in variance swaps to protect the seller from crash risk; in a gamma swap, the weighting inherently dampens the downside variance, so caps are typically regarded as unnecessary.

Model-free replication and valuation

The continuously-monitored gamma swap admits model-free replication by a static position in options and dynamic trading of shares, under conditions specified in the *weighted variance swap* article, which include all positive continuous semimartingale share prices Y under deterministic interest rates and proportional dividends.

Explicitly, one replicates by using that article's (7), with payoff function

$$\lambda(y) = \frac{2}{Y_0}\Big[y\log(y/\kappa) - y + \kappa\Big] = \int_0^\infty \frac{2}{Y_0 K}\mathrm{Van}(y,K)\mathrm{d}K, \tag{3}$$

where $\mathrm{Van}(y,K) := (K-y)^+\mathbb{I}_{K<\kappa} + (y-K)^+\mathbb{I}_{K>\kappa}$ for an arbitrary put/call separator κ. Forms of this payoff were derived in, for instance, [2] and [3].

Therefore, in the case that the interest rate equals the dividend yield (otherwise, see the weighted variance swap article), a replicating portfolio statically holds $2/(Y_0K)\mathrm{d}K$ out-of-the-money vanilla calls or puts at each strike K. The gamma swap model-independently has the same initial value as this portfolio of Europeans. Additionally, the replication strategy trades shares dynamically according to a "zero-vol" delta-hedge, meaning that its share holding equals the negative of what would be the European portfolio's delta under zero volatility.

Further properties

Points 2–5 follow from (3). Point 1 uses only the definition (1).

1. For an index $Y_t := \sum_{j=1}^J \theta_j Y_{j,t}$, let $\alpha_{j,t} := \theta_j Y_{j,t}/Y_t$ be the fraction of total index value due to the quantity θ_j of the jth component $Y_{j,t}$. Define the cumulative *dispersion* D_t by

$$\mathrm{d}D_t = \sum_{j=1}^J \alpha_{j,t}\mathrm{d}[\log Y_j]_t - \mathrm{d}[\log Y]_t.$$

   Then going long $\alpha_{j,0}$ gamma swaps (non-dividend-adjusted) on each Y_j and short a gamma swap on Y creates the payoff

$$\sum_{j=1}^J \alpha_{j,0}\int_0^T \frac{Y_{j,t}}{Y_{j,0}}\mathrm{d}[\log Y_j]_t - \int_0^T \frac{Y_t}{Y_0}\mathrm{d}[\log Y]_t = \int_0^T \frac{Y_t}{Y_0}\mathrm{d}D_t,$$

   as noted in [2]. Hence a static combination of gamma swaps produces cumulative index-weighted dispersion.

2. By Corollary 2.7 in [1], if the implied volatility smile is symmetric in log-moneyness, and the dividend yield equals the interest rate ($q_t = r_t$), and there are no discrete dividends, then a gamma swap has the same value as a variance swap.

3. Assuming that $Y_T = Y_t R_{t,T}$ for all t, where the time-t conditional distribution of each $R_{t,T}$ does not depend on Y_t, the gamma swap has time-t gamma equal to a discounting/dividend-dependent factor times the risk-neutral expectation

$$\frac{2}{Y_0}\mathbb{E}_t\bigg(\frac{\partial^2}{\partial y^2}\bigg|_{y=Y_t} yR_{t,T}\log(yR_{t,T})\bigg) = \frac{2\mathbb{E}_t R_{t,T}}{Y_0 Y_t}.$$

   Therefore its *share gamma*, defined to be Y_t times the gamma, does not depend on Y_t. This property motivates the term *gamma swap*.

4. Within the family of weight functions proportional to $w(y) = y^n$, the gamma swap takes $n = 1$. In that sense, the gamma swap is intermediate between the usual logarithmic variance swap (which takes $n = 0$) and an arithmetic variance swap (which, in effect, takes $n = 2$).

   Expressed in terms of put and call holdings, the replicating portfolios in these three cases hold, at each strike K, a quantity proportional to K^{n-2}. The gamma swap $O(1/K)$ is intermediate between logarithmic variance $O(1/K^2)$ and arithmetic variance $O(1)$.

5. Let F_{X_t} be the characteristic function of $X_t := \log Y_t$. Then

$$\mathbb{E} Y_t \log Y_t = -i F'_{X_t}(-i).$$

   Gamma swap valuations are therefore directly computable in continuous models for which F_{X_t} is known, such as the Heston model.

References

[1] Peter Carr and Roger Lee. Put-call symmetry: Extensions and applications. *Mathematical Finance*. Forthcoming.

[2] Nicolas Mougeot. Variance swaps and beyond. BNP Paribas, 2005.

[3] Marcus Overhaus, Ana Bermúdez, Hans Buehler, Andrew Ferraris, Christopher Jordinson, and Aziz Lamnouar. *Equity Hybrid Derivatives*. John Wiley & Sons, 2007.