General topology meets model theory, on $\mathfrak{p}$ and $\mathfrak{t}$

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Submitted to Proceedings of the National Academy of Sciences of the United States of America

Cantor proved in 1874 that the continuum is uncountable, and Hilbert's first problem asks whether it is the smallest uncountable cardinal. A program arose to study cardinal invariants of the continuum, which measure the size of the continuum in various ways. By Gödel 1939 and Cohen 1963, Hilbert's first problem is independent of ZFC. Much work both before and since has been done on inequalities between these cardinal invariants, but some basic questions have remained open despite Cohen's introduction of forcing. The oldest and perhaps most famous of these is whether "$\mathfrak{p} = \mathfrak{t}$," which was proved in a special case by Rothberger 1948, building on Hausdorff 1934. In this paper we explain how our work on the structure of Keisler's order, a large scale classification problem in model theory, led to the solution of this problem in ZFC as well as of an a priori unrelated open question in model theory.

Unstable model theory | Keisler's order | cardinal invariants of $\mathfrak{c}$ | $\mathfrak{p}$ and $\mathfrak{t}$

Introduction

In this paper we present our solution to two long standing, and a priori unrelated, questions: the question from set theory/general topology of whether $\mathfrak{p} = \mathfrak{t}$, the oldest problem on cardinal invariants of the continuum, and the question from model theory of whether SOP_2 is maximal in Keisler's order. Before motivating and precisely stating these questions, we note there were two big surprises in this work: first, the connection of the two questions, and second, that the question of whether "$\mathfrak{p} < \mathfrak{t}$" can be decided in ZFC.

There have been few connections between general topology and model theory, and these were exclusively in model theory: a key example is Morley's use of ideas from general topology such as the Cantor-Bendixson derivative in proving his 1965 categoricity theorem [1], the cornerstone of modern model theory. Here we have applications in the other direction: by our proofs below, model theoretic methods solve an old problem in general topology. It seems too early to tell what further interactions will arise from these methods, but it is a good sign that Claim 1 below will be generalized to $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, see Discussion 21. Full details of work presented here are available in our manuscript [2].

Cardinal invariants of the continuum

Cantor's 1874 proof of the uncountability of the continuum [3] focuses attention on the region between $\aleph_0$, the cardinality of the natural numbers, and the continuum $2^{\aleph_0}$, the cardinality of the reals. A productive way to map this region is to consider properties of families of subsets of $\mathbb{N}$ which hold for any countable family and fail for some family of size continuum. For any such property of interest, call the minimum size κ of a family for which the property may fail a *cardinal invariant* (or *characteristic*) *of the continuum.* Theorems about the relative sizes and interdependence of such cardinal invariants give fundamental structural information about families of subsets of $\mathbb{N}$, and give precise ways to test the strength of countability hypotheses. As a simply stated example of such a cardinal, the bounding number $\mathfrak{b}$ is the least size of a family $F \subseteq {}^{\omega}\omega$ which is not bounded, meaning that there is no single $g \in {}^{\omega}\omega$ which eventually dominates all $f \in F$, i.e. $f \leq^* g$ for all $f \in F$. As an example of results, the Cichoń diagram, [4] p. 424, gives implications between cardinal invariants relating primarily to measure and category.

The study of cardinal invariants of the continuum is a flourishing area, which lies at the intersection of set theory and general topology; some properties reflect ideas from measure theory, algebra, or combinatorics. The subject's growth and development from the 1930s and 40s, in part a response to Hilbert's problem, to the present can be seen in the surveys of van Douwen 1984 [5], Vaughan 1990 [6], and Blass 2009 [4].

By Gödel 1939 [7] and Cohen 1963 [8], Hilbert's first problem (whether $2^{\aleph_0}$ is the first uncountable cardinal) is independent of ZFC. In light of this, the study of cardinal invariants of the continuum becomes especially fertile. In most cases there are obvious ZFC results and usually independence proofs are deep and hard (using forcing, of Cohen). ZFC answers to problems which have remained open for some time are rare, and so, surprising.

The problem of "$\mathfrak{p}$ and $\mathfrak{t}$" appears throughout the literature. In the seminal survey paper mentioned above, van Douwen catalogued and consolidated much of the work on these cardinals prior to 1984. He focused on six primary invariants: $\mathfrak{b}$, $\mathfrak{p}$, $\mathfrak{t}$ (attributed to Rothberger 1939 [9] and 1948 [10]), $\mathfrak{d}$ (attributed to Katetov 1960), $\mathfrak{a}$ (attributed to Hechler 1972 and Solomon 1977) and $\mathfrak{s}$ (attributed to Booth 1974). As of Vaughan 1990 [6], only two inequalities about van Douwen's cardinals remained open: that of $\mathfrak{a}$ and $\mathfrak{d}$ and that of $\mathfrak{p}$ and $\mathfrak{t}$, and "we believe [whether $\mathfrak{p} < \mathfrak{t}$ is consistent with ZFC] to be the most interesting" ([6] 1.1). Following Shelah's [11] proof of the independence of $\mathfrak{a} < \mathfrak{d}$, attention focused on whether "$\mathfrak{p} = \mathfrak{t}$" as both the oldest and the only remaining open inequality about van Douwen's diagram.

We now define $\mathfrak{p}$ and $\mathfrak{t}$. Let $A \subseteq^* B$ mean $\{x : x \in A, x \notin B\}$ is finite. Let $D \subseteq [\mathbb{N}]^{\aleph_0}$ be a family of countable subsets of $\mathbb{N}$. Say that D has a pseudo-intersection if $\exists A \subseteq \mathbb{N}$, A infinite such that $(\forall B \in D)(A \subseteq^* B)$. Say that D has the strong finite intersection property, or s.f.i.p., if every nonempty finite subfamily of D has infinite intersection. Finally, say that D is a tower if it is linearly ordered by $\supseteq^*$ and has no infinite pseudo-intersection.

Definition 1. *The cardinal $\mathfrak{p}$ is the minimum size of a family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ such that $\mathcal{F}$ has the s.f.i.p. but no infinite pseudo-intersection. The cardinal $\mathfrak{t}$ is the minimum size of a tower.*

It is easy to see that $\mathfrak{p} \leq \mathfrak{t}$, since a tower has the s.f.i.p. Hausdorff proved in 1934 [12] that $\aleph_1 \leq \mathfrak{p}$, and Rothberger proved in 1948 (in our terminology) that $\mathfrak{p} = \aleph_1 \implies \mathfrak{p} = \mathfrak{t}$. This begs the question:

Question 2. *Does $\mathfrak{p} = \mathfrak{t}$?*

After Rothberger [10], there has been much work on $\mathfrak{p}$ and $\mathfrak{t}$, as noted in the surveys [5], [6], [4], the introduction to our paper [2]; see also the references [13], [14], [15], [16], [17] below. As noted, given the length of time the problem remained open, there was wide confidence of an independence result.

Reserved for Publication Footnotes

Model theory

From Morley's theorem to Keisler's order. Our solution to Question 2 arose in the context of our work on a model-theoretic classification program known as Keisler's order. As a frame for these investigations, we briefly discuss model theory, stability, and ultraproducts. Model theory begins with the study of *models*, which are given by the data of an underlying set along with an interpretation for all function, relation, and constant symbols in a fixed background language $\mathcal{L}$; (complete) *theories*, sets of $\mathcal{L}$-sentences of first order logic true in some $\mathcal{L}$-model; and *elementary classes*, classes of $\mathcal{L}$-models which share the same theory, written $Th(M) = Th(N)$ or $M \equiv N$.

It is well known that any two algebraically closed fields of the same characteristic and the same uncountable size are isomorphic. Los could not find any countable first order theory which had one model, up to isomorphism, in *some* but not *every* uncountable cardinality, and so conjectured there was none. The cornerstone of modern model theory is the affirmative answer:

Theorem A. (Morley 1965 [1]) *If T is a countable theory then either T is categorical [=has one model, up to isomorphism] in every uncountable cardinal or T is categorical in no uncountable cardinal.*

Proofs of Morley's theorem, due to Morley and later to Baldwin-Lachlan [18], show, in an abstract setting, that whenever a complete theory in a countable language is categorical in some uncountable cardinal then its models admit analogues of many properties familiar from algebraically closed fields: for instance, the existence of prime models, a well-defined notion of independence generalizing algebraic independence in algebraically closed fields, and existence of a maximal independent set whose size determines the model up to isomorphism. Often, model theoretic arguments extract local structural information from global constraints on a given elementary class (e.g. the property of having one model, up to isomorphism, in some uncountable size).

Model theory since Morley has developed in a number of ways; a central theme (and one crucial for our work) is *stability*, which we now describe. Let M be a model and $A \subseteq M$. The [1-]*types* over A are the maximal consistent sets of formulas $\varphi(x)$ in one free variable with parameters from A. Informally, types describe elements which may or may not exist in a given model M but which will always exist in some extension of M, e.g. a transcendental element which will exist in field extensions of $\mathbb{Q}^{\mathrm{alg}}$. A *realization* of a type p in a model M is an element a such that $\varphi(a)$ is true in M for all $\varphi \in p$. When λ is an infinite cardinal and the model M contains realizations of all types over all subsets of M of size $< \lambda$, we call M λ-saturated. For instance, an algebraically closed field of fixed characteristic is λ-saturated if and only if it has transcendence degree at least λ. Alternatively and informally, if a model M is considered as sitting inside a large universal domain $\mathcal{M}$, the 1-types over $A \subseteq M$ are the orbits under automorphisms of $\mathcal{M}$ fixing A pointwise, and M is λ-saturated if it contains representatives of all orbits under all automorphisms of $\mathcal{M}$ fixing some set $A \subseteq M$, $|A| < \lambda$ pointwise.

We now define "stability" in a given cardinal λ, which led to a fundamental structural dichotomy between theories which are stable in many λ and those which are never stable, Theorem B below.

Definition 3. *T is λ-stable if there are no more than λ types over any $M \models T$, $|M| = \lambda$.*

Example 4. *If M is an algebraically closed field, the type of an element over M is determined by the polynomials over M it does or does not satisfy; this gives $|M|$ types ["stable"]. If $M = (\mathbb{Q}, <)$, the types over M include all cuts; this gives $2^{|M|}$ types ["unstable"]. If M is the Rado graph, the types over M include all partitions of the vertex set of M; this gives gives $2^{|M|}$ types ["unstable"].*

The remarkable fact is that the stability or instability of models resolves into a very informative classification of theories (as in Morley's theorem above, much of the information comes from the structural analysis which goes into the proof). In Theorem B, $\kappa(T) \leq |T|^+$ is an invariant of the theory.

Theorem B. (Shelah 1978 [19]) *A complete theory T is either:*

- *stable, i.e. T is stable in all $\lambda = \lambda^{<\kappa(T)}$, or*
- *unstable, i.e. T is unstable in all λ.*

Under this analysis, the stable theories are "tame" or well-behaved, admitting a strong structure theory, while the unstable theories are "wild."

Discussion 5. *Moreover, stability is* local*: T is unstable if and only if it contains an unstable formula, i.e. a formula with the order property. Also, instability hides a structure/randomness phenomenon: any unstable theory either contains a configuration close to a bipartite random (Rado) graph, or one close to a definable linear order.*

Meanwhile, Keisler had proposed a very different way of classifying theories (and ultrafilters), from an asymptotic (ultrapower) point of view. It is of particular interest that Keisler's order gives an outside definition of the stable theories, see Discussion 10 below.

An asymptotic classification. Keisler's order, which we now explain, studies the relative complexity of theories according to the difficulty of ensuring regular ultrapowers of some $M \models T$ are saturated.

Definition 6. *The filter $\mathcal{D}$ on I, $|I| = \lambda$ is called* regular *if it contains a* regularizing family*, i.e. $\mathrm{X} = \{X_i : i < \lambda\} \subseteq \mathcal{D}$ such that any infinite subset of X has empty intersection.*

It is consistent that all nonprincipal ultrafilters are regular. Moreover, if $\mathcal{D}$ is regular and the background language $\mathcal{L}$ is countable then $M \equiv N \implies M^\lambda/\mathcal{D}$ is λ^+-saturated iff $N^\lambda/\mathcal{D}$ is, so the quantification over all models in Definition 7 below is justified.

An *ultraproduct* of models $\langle M_i : i \in I\rangle$ with respect to the ultrafilter $\mathcal{D}$ on I is the structure obtained by identifying elements of the product $\prod_i M_i$ which are "$\mathcal{D}$-almost everywhere" equivalent [20]. An ultrapower is an ultraproduct in which all the models M_i are the same. Ultrapowers are large (and, in some sense, generic) extensions of a given model M which remain in its elementary class. Their structure reflects and depends on the choice of ultrafilter $\mathcal{D}$ in subtle ways which are not yet well understood. However, precisely this dependence suggests it is informative to compare two theories by comparing ultrapowers of their respective models built using the *same* ultrafilter.

Definition 7. (Keisler's order, 1967 [21]) *Let T_1, T_2 be countable theories. We say that $T_1 \trianglelefteq T_2$ if: for all infinite λ, all $\mathcal{D}$ regular on λ, all $M_1 \models T_1$, all $M_2 \models T_2$, if $(M_2)^\lambda/\mathcal{D}$ is λ^+-saturated then $(M_1)^\lambda/\mathcal{D}$ is λ^+-saturated.*

Problem 1. *Determine the structure of Keisler's order.*

Keisler's order is a problem about constructing ultrafilters on the one hand, and understanding theories on the other hand. We now give some intuition for this problem and some of the main developments; for further details, see e.g. [22].

The power of Keisler's order comes from the fact that we may regard types as maps from $[\lambda]^{<\aleph_0}$ into $\mathcal{D}$. Los's theorem associates to each type p over $A \subseteq M^\lambda/\mathcal{D}$, $|A| \leq \lambda$ a monotonic [i.e. $u \subseteq v \implies f(v) \subseteq f(u)$] map $f : [p]^{<\aleph_0} \to \mathcal{D}$ given by: $\{\varphi_{i_0}(x; a_{i_0}), \dots, \varphi_{i_n}(x; a_{i_n})\} \mapsto$

$$\{t \in \lambda : M \models \exists x \bigwedge_{\ell \leq n} \varphi_{i_\ell}(x; a_{i_\ell}[t])\}$$

When $\mathcal{D}$ is regular, such a map can be chosen to be *multiplicative*, i.e. $f(u \cup v) = f(u) \cap f(v)$, precisely when the type p is realized, see e.g. [22] §1.2. Moreover, when $\mathcal{D}$ is regular, we may choose the map so that its image is a regularizing family.

Thus, in certain cases, the problem of ensuring saturation in ultrapowers is directly reflected in a property of the underlying ultrafilter.

Definition 8. (Keisler) *$\mathcal{D}$ is λ^+-good if every monotonic function $f : [\lambda]^{<\aleph_0} \to \mathcal{D}$ has a multiplicative refinement g, i.e. $u, v \in [\lambda]^{<\aleph_0}$ implies $f(u) \supseteq g(u) \in \mathcal{D}$ and $g(u) \cap g(v) = g(u \cup v)$.*

Good regular ultrafilters exist on all infinite λ by a theorem of Keisler under GCH [23] and of Kunen in ZFC [24]. If $\mathcal{D}$ is regular and λ^+-good then $M^\lambda/\mathcal{D}$ is therefore λ^+-saturated for any M in a countable language. This is half of proving that Keisler's order has a maximum class. In fact, in the paper defining the order $\trianglelefteq$, Keisler showed:

Theorem C. (Keisler 1967 [21])

$$\mathbf{T}_{min} \trianglelefteq \cdots ??? \cdots \trianglelefteq \mathbf{T}_{max}$$

i.e. Keisler's order has a minimum class and a maximum class.

Example 9. *It follows from Theorem A above that any uncountably categorical theory, e.g. the theory of algebraically closed fields of a fixed characteristic, is minimum in Keisler's order. On the other hand, one can show that any sufficiently rich theory, e.g. number theory, will code any failures of goodness into a non-realized type and will therefore be maximal.*

The only Keisler-equivalence classes characterized to date were given by:

Theorem D. (Shelah 1978 [19].VI)

$$\mathbf{T}_1 \triangleleft \mathbf{T}_2 \triangleleft \cdots ??? \cdots \trianglelefteq \mathbf{T}_{max}$$

where $\mathbf{T}_1 \cup \mathbf{T}_2$ is precisely the stable theories, and:

- $\mathbf{T}_1$*, the minimum class, consists of the theories without the finite cover property (f.c.p.).*
- $\mathbf{T}_2$*, the next largest class, consists of stable theories with f.c.p.*
- $\mathbf{T}_{max}$ *contains all linear orders, e.g. $Th(\mathbb{Q}, <)$, though its model-theoretic identity is not known. [Indeed, the weaker order property SOP_3 suffices for maximality, by Shelah 1996 [25].]*

Discussion 10. *In Theorem D the class of stable theories, characterized in terms of counting types in Theorem B, is independently characterized in terms of saturation of ultrapowers.*

Discussion 11. *From Theorems C and D we see that although Keisler's order has a maximum class, we have only weak upper and lower bounds on properties for membership in this class.*

In the intervening years, progress was slow, due in part to the perceived complexity of ultrafilters. Beginning in 2009, Malliaris advanced the problem of how ultrafilters and theories interact in a series of papers [26], [27], [28], [29], motivated by Discussion 10. That is, given that Keisler's order independently detects the jump in complexity between stable and unstable theories, the goal was to leverage Keisler's order to describe gradations in complexity among the unstable theories from a more uniform point of view. See Malliaris [29] for a further discussion. For the present work, we quote:

Theorem E. (Malliaris 2009 [26]) *For regular ultrapowers in a countable language, local saturation implies saturation. That is, Keisler's order reduces to the study of φ-types.*

Theorem F. (Malliaris 2010 [28]) *There is a property of filters, called flexibility, which is detected by any theory which is non-low (some formula k-divides with respect to arbitrarily large k).*

Theorem G. (Malliaris 2010 [28]-[29]) *There is a minimum Keisler class among the theories with TP_2, whose saturation is characterized by existence in $\mathcal{D}$ of internal maps between small sets, and by the ultrafilter $\mathcal{D}$ being "good for equality".*

Discussion 12. *Keisler's order imposes a hierarchy on the structure/randomness phenomenon from Discussion 5: the "structured" (in some sense, rich) theories, those with linear (strict) order, are Keisler-maximum whereas the "purest" random theory, that of the Rado graph, is Keisler-minimum among the unstable theories. See [29] or [22] §4. $TP2/SOP2$ is a randomness/structure phenomenon in non-simple theories with analogies to the independence/strict order phenomenon in non-stable theories. Theorem G shows that TP_2, the "random side" of this more complex example, also admits a Keisler-minimum class, of a particularly simple form. The very interesting idea that Keisler's order may further illuminate randomness/structure tradeoffs within instability is supported by our Theorem 3 below.*

Working together, Malliaris and Shelah have very recently made significant progress on the problem of Keisler's order in the 2012 papers [22], [30], [31], and [32]. These theorems are not prerequisites for the current proofs, so we refer the interested reader to the introduction of [31] or [32] for more details. However, this work provided strong motivation for addressing the long latent Question 16, which we now describe and solve in connection with "$\mathfrak{p} = \mathfrak{t}$".

The current approach: $\mathfrak{p}$, $\mathfrak{t}$, and SOP$_2$

In this section we motivate and announce the main results of our present work [2], with some comments on the proof.

Orders, trees, and the Keisler-maximal class. As described above, the problem:

Problem 13. *Give a model-theoretic characterization of the maximum class in Keisler's order.*

is an old important problem in model theory, for which we know lower bounds and upper bounds: instability on one hand, and the strict order property, or even SOP_3, on the other (Discussion 11). The main theorems of our present work arose as part of our solution to Question 16 below, which moves the conjectured boundary of this class onto what appears to be a major dividing line for which there are strong general indications of a theory.

Before explaining this comment, let us define:

Definition 14. *The formula φ has SOP_2 with respect to a theory T if in some sufficiently saturated $M \models T$ there are parameters $\{a_\eta : \eta \in {}^{\omega>}2\} \subseteq {}^{\ell(y)}M$ such that:*

for each $1 \leq n < \omega$ and and $\eta_1, \dots \eta_n \in {}^{\omega>}2$,

$$\{\varphi(x; a_{\eta_i}) : 1 \leq i \leq n\}$$

is consistent iff $\eta_1, \dots \eta_n$ lie along a single branch. We say the theory T has SOP_2 if some φ does, with respect to T.

Example 15. *In $(\mathbb{Q}, <)$ the formula $\varphi(x; y, z) = y < x < z$ has SOP_2; SOP_2 also arises in the generic K_n-free graph for any $n \geq 3$, a theory well into the "independent" region of instability.*

Recall from Theorem D that any theory which includes a definable linear order, e.g. $(\mathbb{Q}, <)$, or even just SOP_3 (which we shall leave as a black box; see [25] for details) , is maximal. Whereas SOP_3 retains many features of linear order, SOP_2 describes a kind of maximally inconsistent tree so did not appear amenable to the same kinds of arguments. We have the implications:

$$\text{not stable} \leftarrow SOP_2 \leftarrow SOP_3 \leftarrow \textit{strict order property}$$

Following Shelah 1996 [25] the key question, which we solve below, became:

Question 16. *Is any theory with SOP_2 maximal in Keisler's order?*

Why is the move from SOP_3 to SOP_2 so significant? First, there is the nontrivial matter of developing a framework to compare orders and trees. Second, recent evidence (in work in preparation of the authors) suggests that SOP_2 may characterize maximality, Conjecture 29 below.

In order to set the stage for the framework which led to the solution of both Questions 16 and 2, we present our strategy for solving Question 16. The first key idea was to describe the realization of SOP_2-types in terms of a property of an ultrafilter $\mathcal{D}$, which could then be compared against *goodness* for $\mathcal{D}$, as by Theorem D above $\mathcal{D}$ is good iff $(\mathbb{N}, <)^\lambda/\mathcal{D}$ has no (κ_1, κ_2)-cuts for $\kappa_1 + \kappa_2 \leq \lambda$.

Definition 17. *A subset Y of a partially ordered set $(X, <)$ is* cofinal *in X if $(\forall x \in X)(\exists y \in Y)(x \leq y)$. We use the word* coinitial *when the order is reversed. By the cofinality of a partially ordered set we mean the smallest size of a cofinal subset. Call an infinite cardinal κ* regular *if, considered as an ordinal, its cofinality is κ.*

Note: in this paper, we will have two different uses of the word "regular": as applied to ultrafilters, Definition 6 above, and as applied to cardinals, Definition 17.

Definition 18. *Let κ, θ be infinite regular cardinals and $(A, <)$ a total linear order. We say there is a (κ, θ)-*cut *or* gap *in A when we can write $A = A_1 \cup A_2$ such that (a) $A_1 \cap A_2 = \emptyset$, (b) $(\forall a \in A_1)(\forall b \in A_2)(a < b)$, (c) there is a cofinal κ-indexed sequence in $(A_1, <)$ and (d) there is a coinitial θ-indexed sequence in $(A_2, <)$. So in our terminology, cuts are not filled.*

Definition 19. *The regular ultrafilter $\mathcal{D}$ on λ has κ-*treetops *if: whenever $M = (\mathcal{T}, \trianglelefteq)$ is a tree and $N = M^\lambda/\mathcal{D}$ its ultrapower, any $\trianglelefteq$-increasing sequence in N of cofinality $< \kappa$ has an upper bound.*

We now have the following measure of the complexity of $\mathcal{D}$:

$$\mathcal{C}(\mathcal{D}) = \{(\kappa_1, \kappa_2) : \kappa_1, \kappa_2 \text{ regular}, \kappa_1 + \kappa_2 \leq |I|, \\ \text{and } (\mathbb{N}, <)^I/\mathcal{D} \text{ has a } (\kappa_1, \kappa_2)\text{-cut}\}$$

recalling that $\mathcal{C}(\mathcal{D}) = \emptyset$ iff $\mathcal{D}$ is good ("maximally complex," that is, able to saturate ultrapowers of linear order, which belong to the Keisler-maximum class).

We have therefore translated Question 16 to:

Question 20. *Suppose $\mathcal{D}$ is a regular ultrafilter on λ with λ^+-treetops. Is $\mathcal{C}(\mathcal{D}) = \emptyset$?*

To illustrate some simple properties of the cofinality spectrum $\mathcal{C}(\mathcal{D})$ under λ^+-treetops, and to motivate the later abstraction of "cofinality spectrum problems," we now sketch the proof of a uniqueness result in this context.

Claim 1. ([2] Claim 3.3, in the special case of ultrapowers) *Let $\mathcal{D}$ be a regular ultrafilter on λ, $\kappa = \mathrm{cf}(\kappa) \leq \lambda$, and suppose $\mathcal{D}$ has λ^+-treetops. If $(\kappa, \theta_0) \in \mathcal{C}(\mathcal{D})$ and $(\kappa, \theta_1) \in \mathcal{C}(\mathcal{D})$, then $\theta_0 = \theta_1$.*

Proof. Let $M = (\mathbb{N}, <)$ and let $N = M^\lambda/\mathcal{D}$. Suppose that in N, $(\langle a^0_\alpha : \alpha < \kappa\rangle, \langle b^0_\epsilon : \epsilon < \theta_0\rangle)$ represents a (κ, θ_0)-cut while $(\langle a^1_\alpha : \alpha < \kappa\rangle, \langle b^1_\epsilon : \epsilon < \theta_1\rangle)$ represents a (κ, θ_1)-cut.

As ultrapowers commute with reducts, we may suppose, in an expanded language [e.g. expanding M to $(\mathcal{H}(\aleph_1), \epsilon)$], that M includes a definable tree $\mathcal{T}$ of finite sequences of pairs of natural numbers, strictly increasing in each coordinate, with $\trianglelefteq$ denoting a definable partial order on $\mathcal{T}$ corresponding to initial segment. Identifying N with its induced expansion to the larger language, let $(\mathcal{T}^N, \trianglelefteq^N)$ denote the $\mathcal{D}$-ultrapower of this tree. For ease of reading, write $c(n, i)$ for the ith coordinate of $c(n)$.

By induction on $\alpha < \kappa$, we choose $c_\alpha \in \mathcal{T}^N$ and $n_\alpha = \max \mathrm{dom}(c_\alpha)$ such that $c_\alpha(n_\alpha, i) = a^i_\alpha$ for $i = 0, 1$. The base and successor case are easily done. In the limit case, we will have defined a sequence $\langle c_\beta : \beta < \alpha\rangle$ of elements lying along a single branch in $\mathcal{T}^N$. The assumption of treetops gives an upper bound $c_* \in \mathcal{T}^N$ for this sequence, i.e. $c_\beta \trianglelefteq c_*$ for $\beta < \alpha$. Then, in our expanded language, the set $\{n \leq n_* : c_*(n, 0) < a^0_\alpha \wedge c_*(n, 1) < a^1_\alpha\}$ is a nonempty, definable, bounded subset of $\mathbb{N}^N$, and so it has a maximal element m. To complete this step, set $c_\alpha := c_* \restriction_m {}^\frown (a^0_\alpha, a^1_\alpha)$.

Having completed the definition of the sequence $\langle c_\alpha : \alpha < \kappa\rangle$, we may again by treetops find an upper bound for this sequence, $c_\infty \in \mathcal{T}^N$. Let $n_\infty = \max \mathrm{dom}(c_\infty)$. For $i \in \{0, 1\}$ and $\epsilon < \theta_i$, let $n^i_\epsilon = \max\{n \leq n_\infty : c_\infty(n, t) < b^i_\epsilon\}$. This is again a nonempty, definable, bounded subset of the nonstandard natural numbers, so each $n^0_\epsilon, n^1_\epsilon$ are well defined. But now we have effectively "sewn together" the two original cuts. The sequences $\langle b^i_\epsilon : \epsilon < \theta\rangle$ and $\langle c(n^i_\epsilon, i) : i < \theta\rangle$ are mutually cofinal for $i = 0, 1$. Suppose for a contradiction that, in $\mathbb{N}^N$, the sequences $(\langle n_\alpha : \alpha < \kappa\rangle, \langle n^0_\epsilon : \epsilon < \theta_0\rangle)$ and $(\langle n_\alpha : \alpha < \kappa\rangle, \langle n^1_\epsilon : \epsilon < \theta_1\rangle)$ do not describe the same cut. Without loss of generality, $\langle n^0_\epsilon : \epsilon < \theta_0\rangle$ is not cofinal in $\langle n^1_\epsilon : \epsilon < \theta_1\rangle$. So there is some $\epsilon_* < \theta_1$ such that [recalling we have strict monotonicity in each coordinate] $c_\infty(n^1_{\epsilon_*}, 0)$ lies in what we assumed to be a cut described by $(\langle a^0_\alpha : \alpha < \kappa\rangle, \langle b^0_\epsilon : \epsilon < \theta_0\rangle)$, a contradiction. Since we assumed θ_0, θ_1 are regular, $\theta_0 = \theta_1$, as desired. •

Discussion 21. *In work in progress, we have an analogue of Claim 1 for $\mathcal{P}(\mathbb{N})/\,\mathrm{fin}$.*

Before describing our approach in full generality, we recall that the hypothesis $\mathfrak{p} < \mathfrak{t}$ also connects to questions about cuts.

Peculiar cuts in $\mathfrak{p}$ and $\mathfrak{t}$. Here we describe a further useful ingredient which connects $\mathfrak{p} < \mathfrak{t}$ to the appearance of a certain kind of cut (gap), Definition 18. The * in $\leq^*$ means "for all but finitely many".

Definition 22. (Peculiar cuts) *For κ_1, κ_2 infinite regular cardinals, a (κ_1, κ_2)-peculiar cut in ${}^\omega\omega$ is a pair $(\langle f_\alpha : \alpha < \kappa_1\rangle, \langle g_\beta : \beta < \kappa_2\rangle)$ of sequences of functions in ${}^\omega\omega$ such that:* (a) *$\alpha < \alpha' < \kappa_1$ and $\beta < \beta' < \kappa_2$ implies $f_\alpha \leq^* f_{\alpha'} \leq^* g_{\beta'} \leq^* g_\beta$;* (b) *$(\forall \alpha < \kappa_1)(f_\alpha \leq^* h)$ implies $(\exists \beta < \kappa_2)(g_\beta \leq^* h)$; and* (c) *$(\forall \beta < \kappa_1)(h \leq^* g_\beta)$ implies $(\exists \alpha < \kappa_1)(h \leq^* f_\alpha)$.*

Theorem H. (Shelah [33]) *Assume $\mathfrak{p} < \mathfrak{t}$. Then for some regular cardinal κ there exists a $(\kappa, \mathfrak{p})$-peculiar cut in ${}^\omega\omega$, where $\aleph_1 \leq \kappa < \mathfrak{p}$.*

Looking ahead, in the context of forcing, we will connect Theorem H to a general version of the "cut spectrum" $\mathcal{C}(\mathcal{D})$, as follows:

Example 23. *If $G \subseteq [\omega]^{\aleph_0}$ is a generic ultrafilter then writing $X = \prod_G g_0(n)$, we have that $\langle f_\alpha/G : \alpha < \kappa_1\rangle, \langle g_\beta/G : \beta < \kappa_2\rangle$ give a (κ_1, κ_2)-cut in X.*

A common context for both problems. The right framework, discovered in [2], is more general than regular ultrapowers; however, in defining it we draw on intuition from the proof of Claim 1 sketched above. The abstraction captures the fact that when studying regular ultrapowers of $(\mathbb{N}, <)$, since ultrapowers commute with reducts, we may assume that the nonstandard copy of $\mathbb{N}$ in the ultrapower N continues to behave in a "pseudofinite" way (e.g. all nonempty definable subsets of $\mathbb{N}^N$ have what N believes to be an $<^N$-least element, and all such bounded subsets have a $<^N$-greatest element). Moreover, we may expand the language in order to have available uniform definitions for relevant trees. Recall Definition 18 above.

Definition 24. (Informal; for full details, see [2] Definition 2.1 pps. 7–8) *Informally, we say that the triple M, M_1, Δ has "enough set theory for trees" when: $M \preceq M_1$; Δ is a nonempty set of formulas defining discrete linear orders in M_1, such that every nonempty M_1-definable subset of any such order has a first and last element; and for each instance of a formula φ in Δ (i.e. for each order) there is a definable tree of sequences, i.e. funcitons from the order to itself.*

We ask that for each such tree, length, projection, concatenation, the maximal element of the domain of a given function, and the partial ordering $\trianglelefteq$ are definable; and we ask that Δ be closed under Cartesian products.

Our true context is therefore the following:

Definition 25. ([2] 2.5–2.11) *We say that $\mathbf{s}$ is a CSP (cofinality spectrum problem) when:*
$\mathbf{s} = (M, M_1, M^+, M_1^+, \Delta)$, where Δ is a set of formulas of $\mathcal{L}(M)$, $M \preceq M_1$, and this pair can be expanded to $M^+ \preceq M_1^+$ such that (M^+, M_1^+, Δ) has enough set theory for trees. When $M = M^+$ or $M_1 = M_1^+$, we may omit it.

In this context we define:

1. $\mathrm{Or}(\mathbf{s}) = \mathrm{Or}(\Delta^{\mathbf{s}}, M_1^{\mathbf{s}})$ *is the set of orders defined in M_1 by instances of formulas from Δ*
2. $\mathcal{C}^{\mathrm{ct}}(\mathbf{s}) = \{(\kappa_1, \kappa_2) :$ *there is $\mathbf{a} \in \mathrm{Or}(\mathbf{s}, M_1)$ such that the linear order $\leq_{\mathbf{a}}$ on $X_{\mathbf{a}}$ has a (κ_1, κ_2)-cut* $\}$
   Note*: By definition of cut in Defn. 18 above, κ_1, κ_2 are regular.*
3. $\mathrm{Tr}(\mathbf{s}) = \{\mathcal{T}_{\mathbf{a}} : \mathbf{a} \in \mathrm{Or}(\mathbf{s})\}$ *is the set of definable trees associated to orders in* $\mathrm{Or}(\mathbf{s})$
4. $\mathcal{C}^{\mathrm{ttp}}(\mathbf{s}) = \{\kappa : \kappa \geq \aleph_0, \mathbf{a} \in \mathrm{Or}(\mathbf{s})$ *and there is in the tree $\mathcal{T}_{\mathbf{a}}$ a strictly increasing sequence of cofinality $\mathrm{cf}(\kappa)$ with no upper bound* $\}$
5. *Let $\mathfrak{t}_{\mathbf{s}}$ be $\min \mathcal{C}^{\mathrm{ttp}}(\mathbf{s})$ and let $\mathfrak{p}_{\mathbf{s}}$ be $\min\{\kappa : (\kappa_1, \kappa_2) \in \mathcal{C}^{\mathrm{ct}}(\mathbf{s})$ and $\kappa = \kappa_1 + \kappa_2\}$.*

Note that by definition of $\mathcal{C}^{\rm ct}(\mathbf{s})$ and $\mathcal{C}^{\rm ttp}(\mathbf{s})$, both $\mathfrak{t}_{\mathbf{s}}$ and $\mathfrak{p}_{\mathbf{s}}$ are regular.

We focus on $\mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}})$ where this means:

6. $\mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}}) = \{(\kappa_1,\kappa_2) : \kappa_1+\kappa_2 < \mathfrak{t}_{\mathbf{s}}, (\kappa_1,\kappa_2) \in \mathcal{C}^{\rm ct}(\mathbf{s})\}$.

First Reduction. If for every CSP $\mathbf{s}$, $\mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}}) = \emptyset$ then SOP_2 is maximal in Keisler's order. Why? This simply translates Question 20 to this more general context: the issue is whether knowing there are always paths through trees is sufficient to guarantee that there are no cuts of small cofinality.

However, our new context also encompasses other problems.

Second Reduction. To deal with $\mathfrak{p}$ and $\mathfrak{t}$ in this context we look for a cofinality spectrum problem, and for a cut, recalling Fact H above.

Suppose we are given $M = (\mathcal{H}(\aleph_1),\epsilon)$, a forcing notion $\mathbf{Q} = ([\omega]^{\aleph_0}, \supseteq^*)$, $\mathbf{V}$ a transitive model of ZFC, and G a fixed generic subset of $\mathbf{Q}$, forced to be an ultrafilter on the Boolean algebra $\mathcal{P}(\omega)^{\mathbf{V}}$.

Consider $N = M^\omega/G$ in the forcing extension $\mathbf{V}[G]$. Recall that first, the diagonal embedding is elementary, so $M \preceq N$; and second, $\mathbf{Q}$ is $\mathfrak{t}$-complete so forcing with $\mathbf{Q}$ adds no new bounded subsets of $\mathfrak{t}$, and $\mathfrak{p}^{\mathbf{V}} < \mathfrak{t}^{\mathbf{V}}$ iff $\mathfrak{p}^{\mathbf{V}[G]} < \mathfrak{t}^{\mathbf{V}[G]}$.

We focus attention on the cofinality spectrum problem $\mathbf{s}_* = (M, N, \Delta_{psf})$ where Δ_{psf} is the set of definable linear orders in N which are covered by G-ultraproducts of finite linear orders. We shall assume $\mathfrak{p} < \mathfrak{t}$ and prove several claims.

Claim 2. ([2] 5.8, assuming $\mathfrak{p} < \mathfrak{t}$) $\mathfrak{t} \leq \mathfrak{t}_{\mathbf{s}_*}$.

Proof Sketch. Without loss of generality, work in the tree $\mathcal{T}^N = ({}^{\omega >}\omega, \trianglelefteq)^N$. Let $\langle \tilde{f}_\alpha/G : \alpha < \theta\rangle$ be a proposed path in N, $\theta < \mathfrak{t}$. Let $B \Vdash$ "$\langle \tilde{f}_\alpha/G : \alpha < \theta\rangle$ is $\trianglelefteq^N$-increasing in $\mathcal{T}^N$", and without loss of generality $B \Vdash$ "$\tilde{f}_\alpha = f_\alpha$" as there are no new sequences of length $< \mathfrak{t}$. For each $n \in B$, look at the cone Y^n_α above $f_\alpha(n)$ in in the index model $\mathcal{T}^M$. Let $Y_\alpha = \bigcup_{n \in B} Y^n_\alpha$. Then $\alpha < \beta \implies Y_\beta \subseteq_* Y_\alpha$, as otherwise we contradict the choice of B. Thus $\{Y_\alpha : \alpha < \theta\}$ is a tower of length $\theta < \mathfrak{t}$, and by definition of $\mathfrak{t}$ as the tower number, there is an infinite Y with $Y \subseteq_* Y_\alpha$ for all $\alpha < \theta$. From any such Y it is easy to build an upper bound for our given path. •

Claim 3. ([2] 5.15, assuming $\mathfrak{p} < \mathfrak{t}$) $\mathfrak{p}_{\mathbf{s}_*} \leq \mathfrak{p}$.

The proof shows, using Fact H and the definition of G, that if $\mathfrak{p} < \mathfrak{t}$ then for some nonconstant $g_0 \in {}^\omega\omega$, $\prod_G g_0(n)$ contains a $(\kappa,\mathfrak{p})$-cut where κ is regular, $\aleph_1 \leq \kappa < \mathfrak{p}$.

Conclusion 26. ([2] 5.16) *If* $\mathfrak{p} < \mathfrak{t}$ *then* $\mathcal{C}(\mathbf{s}_*, \mathfrak{t}_{\mathbf{s}_*}) \neq \emptyset$.

We have shown: if for every CSP $\mathbf{s}$, $\mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}}) = \emptyset$, then $\mathfrak{p} = \mathfrak{t}$. This completes the second reduction.

Our main theorems. In Malliaris and Shelah 2012 [2] we prove the following. The first, main structure theorem is Theorem 2, proved by model-theoretic means; some key lemmas are described in the next section.

Theorem 2. ([2] Theorem 3.66) *For any CSP* $\mathbf{s}$, $\mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}}) = \emptyset$.

By the First Reduction described above, Theorem 2 gives:

Theorem 3. ([2] Theorem 4.48) *SOP_2 is maximal in Keisler's order.*

By some further arguments in the case of ultrafilters, our methods and Theorem 2 also give a new characterization of Keisler's notion of goodness:

Theorem 4. ([2] Theorem 4.49) *For a regular ultrafilter $\mathcal{D}$ on λ, (a) $\kappa \leq \lambda \implies (\kappa,\kappa) \notin \mathcal{C}(\mathcal{D})$ iff (b) $\mathcal{C}(\mathcal{D}) = \emptyset$ iff (c) $\mathcal{D}$ is λ^+-good.*

By Theorem 3 and Theorem G above, we obtain:

Theorem 5. ([2] Theorem 4.51)*The minimum TP_2 class is the minimum non-simple class in Keisler's order.*

Finally, by the Second Reduction described above, Theorem 2 gives:

Theorem 6. ([2] Theorem 5.17)

$$\mathfrak{p} = \mathfrak{t}$$

These results give insight into other structures, as well. For instance, it is a corollary of these methods that in any model of Peano arithmetic, if for all $\kappa < \lambda$ the underlying order has no (κ,κ)-cuts (gaps) then the model itself is λ-saturated.

Key steps in the proof. Here we state the intermediate claims in the proof that $\mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}}) = \emptyset$, Theorem 2 above. [For the connection to $\mathfrak{p}$, $\mathfrak{t}$ and SOP_2, see the sections "First Reduction" and "Second Reduction" above.]

Claim 4. (Uniqueness, [2] Claim 3.3) *Let* $\mathbf{s}$ *be a cofinality spectrum problem. Then for each regular $\kappa \leq \mathfrak{p}_{\mathbf{s}}$, $\kappa < \mathfrak{t}_{\mathbf{s}}$:*

1. $(\kappa,\lambda) \in \mathcal{C}^{\rm ct}(\mathbf{s})$ *for precisely one* λ.
2. $(\kappa,\lambda) \in \mathcal{C}^{\rm ct}(\mathbf{s})$ *iff* $(\lambda,\kappa) \in \mathcal{C}^{\rm ct}(\mathbf{s})$.

This proof amounts to a generalization of Claim 1 above to the general context of cofinality spectrum problems.

Claim 5. (Anti-symmetry, [2] Claim 3.45) *Let* $\mathbf{s}$ *be a cofinality spectrum problem. Then for all regular κ such that $\kappa \leq \mathfrak{p}_{\mathbf{s}}$, $\kappa < \mathfrak{t}_{\mathbf{s}}$, we have that $(\kappa,\kappa) \notin \mathcal{C}^{\rm ct}(\mathbf{s})$.*

The proof involves constructing a path through a tree of pairs of natural numbers, in the spirit of Claims 1 and 4.

Corollary 27. (c.f. [2] Cor. 3.9) *Let* $\mathbf{s}$ *be a cofinality spectrum problem. In light of the above we may, without loss of generality, study* $\mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}})$ *by looking at*

$$\{(\kappa_1,\kappa_2) : (\kappa_1,\kappa_2) \in \mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}}), \kappa_1 < \kappa_2\}$$

Corollary 27 summarizes the two previous claims.

Claim 6. (On $\mathrm{lcf}(\aleph_0)$, [2] Claim 3.58) *Let* $\mathbf{s}$ *be a cofinality spectrum problem and suppose* $\mathrm{lcf}(\aleph_1,\mathbf{s}) \geq \aleph_2$. *Then* $(\aleph_0,\aleph_1) \notin \mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}}) \implies (\aleph_0,\lambda) \notin \mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}})$ *for all regular* $\lambda \leq \mathfrak{p}_{\mathbf{s}}, \lambda < \mathfrak{t}_{\mathbf{s}}$.

The proof of Claim 6 is an order of magnitude more complicated, and requires using (or developing) more general trees; so as to always have these available, we prove from the definition that cofinality spectrum problems always have available a certain amount of Peano arithmetic. Claim 6 may be read as saying that the only case of concern for $\aleph_0$ is a "close" asymmetric cut, $(\aleph_0,\aleph_1)$. Indeed, in the general result, Lemma 1 below, the a priori most difficult case to rule out was a (κ,λ)-cut when $\kappa^+ = \lambda$ and $\lambda^+ = \mathfrak{t}_{\mathbf{s}}$.

Lemma 1. (Main lemma, [2] Lemma 3.65) *Let* $\mathbf{s}$ *be a cofinality spectrum problem. Suppose that κ, λ are regular and $\kappa < \lambda = \mathfrak{p}_{\mathbf{s}} < \mathfrak{t}_{\mathbf{s}}$. Then $(\kappa,\lambda) \notin \mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}})$.*

This lemma, which supercedes and substantially generalizes Claim 6, is the core of the argument. Its proof requires formalizing a robust picture of how sets of large size may be carried along one side of a cut to overspill into the other when the cofinality on either side is small relative to the size of "treetops". We also show that cofinality spectrum problems have available an internal notion of cardinality. So for suitably chosen representations of cuts such overspill is enough for a contradiction, which rules out the existence of cuts of the given cofinality. [2] 3.61 is an informal description of the proof.

With these results in hand we prove the paper's fundamental result:

Proof. (of Theorem 2) If $\mathfrak{t}_{\mathbf{s}} \leq \mathfrak{p}_{\mathbf{s}}$, then by definition of $\mathfrak{p}_{\mathbf{s}}$, $\mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}}) = \emptyset$. So we may assume $\mathfrak{p}_{\mathbf{s}} < \mathfrak{t}_{\mathbf{s}}$. Let κ,λ be such that $\kappa+\lambda = \mathfrak{p}_{\mathbf{s}}$ and $(\kappa,\lambda) \in \mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}})$. By Claims 4, 27 and 5, $\kappa \neq \lambda$ and we may assume $\kappa < \lambda = \mathfrak{p}_{\mathbf{s}}$. Then the hypotheses of Lemma 1 are satisfied, so $(\kappa,\lambda) \notin \mathcal{C}(\mathbf{s},\mathfrak{t}_{\mathbf{s}})$. This completes the proof. •

Discussion 28. *As regards Keisler's order, the work presented here is related to the non-structure side, giving sufficient conditions for maximality.*

Conjecture 29. *SOP_2 characterizes maximality in Keisler's order.*

ACKNOWLEDGMENTS. Malliaris was partially supported by NSF grant DMS-1001666, by Shelah's NSF grant DMS-1101597, and by a Gödel research fellowship. Shelah was partially supported by the Israel Science Foundation grants 710/07 and 1053/11. This is paper E74 in Shelah's list of publications.

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