CONSTRUCTING REGULAR ULTRAFILTERS
FROM A MODEL-THEORETIC POINT OF VIEW

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ABSTRACT. This paper contributes to the set-theoretic side of understanding Keisler's order. We consider properties of ultrafilters which affect saturation of unstable theories: the lower cofinality $\mathrm{lcf}(\aleph_0, \mathcal{D})$ of $\aleph_0$ modulo $\mathcal{D}$, saturation of the minimum unstable theory (the random graph), flexibility, goodness, goodness for equality, and realization of symmetric cuts. We work in ZFC except when noted, as several constructions appeal to complete ultrafilters thus assume a measurable cardinal. The main results are as follows. First, we investigate the strength of flexibility, known to be detected by non-low theories. Assuming $\kappa > \aleph_0$ is measurable, we construct a regular ultrafilter on $\lambda \geq 2^\kappa$ which is flexible but not good, and which moreover has large $\mathrm{lcf}(\aleph_0)$ but does not even saturate models of the random graph. This implies (a) that flexibility alone cannot characterize saturation of any theory, however (b) by separating flexibility from goodness, we remove a main obstacle to proving non-low does not imply maximal. Since flexible is precisely OK, this also shows that (c) from a set-theoretic point of view, consistently, ok need not imply good, addressing a problem from Dow 1985. Second, under no additional assumptions, we prove that there is a loss of saturation in regular ultrapowers of unstable theories, and also give a new proof that there is a loss of saturation in ultrapowers of non-simple theories. More precisely, for $\mathcal{D}$ regular on κ and M a model of an unstable theory, $M^\kappa/\mathcal{D}$ is not $(2^\kappa)^+$-saturated; and for M a model of a non-simple theory and $\lambda = \lambda^{<\lambda}$, $M^\lambda/\mathcal{D}$ is not λ^{++}-saturated. In the third part of the paper, we investigate realization and omission of symmetric cuts, significant both because of the maximality of the strict order property in Keisler's order, and by recent work of the authors on SOP_2. We prove that if $\mathcal{D}$ is a κ-complete ultrafilter on κ, any ultrapower of a sufficiently saturated model of linear order will have no (κ, κ)-cuts, and that if $\mathcal{D}$ is also normal, it will have a (κ^+, κ^+)-cut. We apply this to prove that for any $n < \omega$, assuming the existence of n measurable cardinals below λ, there is a regular ultrafilter D on λ such that any D-ultrapower of a model of linear order will have n alternations of cuts, as defined below. Moreover, D will λ^+-saturate all stable theories but will not $(2^\kappa)^+$-saturate any unstable theory, where κ is the smallest measurable cardinal used in the construction.

INTRODUCTION

The motivation for our work is a longstanding, and far-reaching, problem in model theory: namely, determining the structure of Keisler's order on countable first-order theories. Introduced by Keisler in 1967, this order suggests a way of comparing the complexity of first-order theories in terms of the difficulty of producing saturated regular ultrapowers. Much of the power of this order comes from the interplay of model-theoretic structure and set-theoretic constraints. However, this interplay also contributes to its difficulty: progress requires advances in model-theoretic analysis on the one hand, and advances in ultrapower construction on the other. Our work in this paper is of the second kind and is primarily combinatorial set theory, though the model-theoretic point of view is fundamental.

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As might be expected from a problem of this scope, surprising early results were followed by many years of little progress. Results of Shelah in [19], Chapter VI (1978) had settled Keisler's order for stable theories, as described in §1.2 below. Apart from this work, and the result on maximality of SOP_3 in [22], the problem of understanding Keisler's order on unstable theories was dormant for many years and seemed difficult.

Very recently, work of Malliaris and Shelah has led to considerable advances in the understanding of how ultrafilters and theories interact (Malliaris [12]-[14], Malliaris and Shelah [16]-[17]). In particular, we now have much more information about properties of ultrafilters which have model-theoretic significance. However, the model-theoretic analysis gave little information about the relative strength of the ultrafilter properties described. In the current paper, we substantially clarify the picture. We establish various implications and non-implications between model-theoretic properties of ultrafilters, and we develop a series of tools and constraints which help the general problem of constructing ultrafilters with a precise degree of saturation. Though we have framed this as a model-theoretically motivated project, it naturally relates to questions in combinatorial set theory, and our results answer some questions there. Moreover, an interesting and unexpected phenomenon in this paper is the relevance of measurable cardinals in the construction of regular ultrafilters, see 2.8 below.

This paper begins with several introductory sections which frame our investigations and collect the implications of the current work. We give two extended examples in §1, the first historical, the second involving results from the current paper. Following this, we give definitions and fix notation in §2. §3 gives an overview of our main results in this paper. §4 includes some summary theorems for our constructions. Sections §5-§11 contain the main proofs.

In this paper, we focus on product constructions and cardinality constraints. In a related paper in preparation [15] we will focus on constructions via families of independent functions.

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CONTENTS

1. Background and examples

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In this section we give two extended examples. The first is historical; we motivate the problem of Keisler's order, i.e. of classifying first-order theories in terms of saturation of ultrapowers, by explaining the classification for the stable case. The second involves a proof from the current paper: we motivate the idea that model-theoretic properties can give a useful way of calibrating the "strength" of ultrafilters by applying saturation arguments to prove that consistently flexible (=OK) does not imply good.

Some definitions will be given informally; formal versions can be found in §2 below.

1.1. **Infinite and pseudofinite sets: Theories through the lens of ultrafilters.** This first example is meant to communicate some intuition for the kinds of model-theoretic "complexity" to which saturation of ultrapowers is sensitive.

First, recall that questions of saturation and expressive power already arise in the two fundamental theorems of ultrapowers.

Theorem A. (Łos' theorem for first-order logic) *Let $\mathcal{D}$ be an ultrafilter on $\lambda \geq \aleph_0$, M an $\mathcal{L}$-structure, $\varphi(\overline{x})$ an $\mathcal{L}$-formula, and $\overline{a} \subseteq N = M^\lambda/\mathcal{D}$, $\ell(\overline{a}) = \ell(\overline{x})$. Fixing a canonical representative of each $\mathcal{D}$-equivalence class, write $\overline{a}[i]$ for the value of $\overline{a}$ at index i. Then*

$$N \models \varphi(\overline{a}) \iff \{i \in \lambda : M \models \varphi(\overline{a}[i])\} \in \mathcal{D}$$

Theorem B. (Ultrapowers commute with reducts) *Let M be an $\mathcal{L}'$-structure, $\mathcal{L} \subseteq \mathcal{L}'$, $\mathcal{D}$ an ultrafilter on $\lambda \geq \aleph_0$, $N = M^\lambda/\mathcal{D}$. Then*

$$\left(M^\lambda/\mathcal{D}\right)|_{\mathcal{L}} = (M|_{\mathcal{L}})^\lambda/\mathcal{D}$$

That is: By itself, Theorem A may appear only to guarantee that $M \equiv M^\lambda/\mathcal{D}$. Yet combined with Theorem B, it has consequences for saturation of ultrapowers, as we now explain.

Consider the following three countable models in the language $\mathcal{L} = \{E, =\}$, for E a binary relation symbol, interpreted as an equivalence relation.

- In M_1, E is an equivalence relation with two countable classes.
- In M_2, E is an equivalence relation with countably many countable classes.
- In M_3, E is an equivalence relation with exactly one class of size n for each $n \in \mathbb{N}$.

What variations are possible in ultrapowers of these models? That is, for $N_i = M_i^\lambda/\mathcal{D}$, what can we say about: (a) the number of E^{N_i}-classes, (b) the possible sizes of E^{N_i}-classes, (b)′ if two E^{N_i}-classes can have unequal sizes?

Observation 1.1. *For any index set I and ultrafilter $\mathcal{D}$ on I,*

(1) $N_1 = (M_1)^I/\mathcal{D}$ *will have two E-classes each of size $|N_1|$*
(2) $N_2 = (M_2)^I/\mathcal{D}$ *will have $|N_2|$ E-classes each of size $|N_2|$*

Proof. (1) Two classes follows by Łos' theorem, so we prove the fact about size. By Theorem B $(M_{11}^\lambda/\mathcal{D})|_{\mathcal{L}} = M_1^\lambda/\mathcal{D}$, where M_{11} is the expansion of M_1 to $\mathcal{L}' = \mathcal{L} \cup \{f\}$ and f is interpreted as a bijection between the equivalence classes. By Łos' theorem, f will remain a bijection in N_1, but Theorem B means that whether we forget the existence of f before or after taking the ultrapower, the result is the same.

(2) Similarly, M_2 admits an expansion to a language with a bijection f_1 between M_2 and a set of representatives of E-classes; a bijection f_2 between M_2 and a fixed E-class; and a parametrized

family $f_3(x, y, z)$ where for each a, b, $f_3(x, a, b)$ is a bijection between the equivalence class of b and that of b. So once more, by Theorems A and B, the ultrapowers of M_2 are in a sense one-dimensional: if $N_2 = M_2^\lambda/\mathcal{D}$ is an ultrapower, it will be an equivalence relation with $|N_2|$ classes each of which has size $|N_2|$. □

Now for M_3, the situation is a priori less clear. Any nonprincipal ultrapower will contain infinite (pseudofinite) sets by Łos' theorem, but it is a priori not obvious whether induced bijections between these sets exist. It is easy to choose infinitely many distinct pseudofinite sets (let the nth set project a.e. to a class whose size is a power of the nth prime) which do not clearly admit bijections to each other in the index model M, nor to M itself.

We have reached the frontier of what Theorem B can control, and a property of ultrafilters comes to the surface:

Definition 1.2. ([20] Definition III.3.5) *Let $\mathcal{D}$ be an ultrafilter on λ.*

$$\mu(\mathcal{D}) := \min\left\{\prod_{t<\lambda} n_t/\mathcal{D} \ : \ n_t < \aleph_0, \ \prod_{t<\lambda} n_t/\mathcal{D} \geq \aleph_0\right\}$$

be the minimum value of the product of an unbounded sequence of cardinals modulo $\mathcal{D}$.

Observation 1.3. *Let $\mathcal{D}$ be an ultrafilter on λ, let M_3 be the model defined above, and $N_3 = (M_3)^\lambda/\mathcal{D}$. Then:*

(1) *N_3 will have $|N_3|$ E-classes.*
(2) *E^{N_3} will contain only classes of size $\geq \mu(\mathcal{D})$, and will contain at least one class of size $\mu(\mathcal{D})$.*

Proof. (1) As for the number of classes, Theorem B still applies.

(2) Choose a sequence of cardinals n_t witnessing $\mu(\mathcal{D})$, and consider the class whose projection to the tth index model has cardinality n_t. □

Definition 1.2 isolates a well-defined set-theoretic property of ultrafilters, and indeed, an early theorem of the second author proved that one could vary the size of $\mu(\mathcal{D})$:

Theorem C. (Shelah, [20].VI.3.12) *Let $\mu(\mathcal{D})$ be as in Definition 1.2. Then for any infinite λ and $\nu = \nu^{\aleph_0} \leq 2^\lambda$ there exists a regular ultrafilter $\mathcal{D}$ on λ with $\mu(\mathcal{D}) = \nu$.*

Whereas the saturation of $(M_1)^\lambda/\mathcal{D}$ and of $(M_2)^\lambda/\mathcal{D}$ will not depend on $\mu(\mathcal{D})$, $N_3 = (M_3)^\lambda/\mathcal{D}$ will omit a type of size $\leq \kappa$ of the form $\{E(x, a)\} \cup \{\neg x = a' \ : \ N_3 \models E(a', a)\}$ if and only if $\mu(\mathcal{D}) \leq \kappa$. Restricting to regular ultrafilters, so that saturation of the ultrapower does not depend on saturation of the index model but only on its theory, the same holds if we replace each M_i by some elementarily equivalent model, and is thus a statement about their respective theories.

This separation of theories by means of their sensitivity to $\mu(\mathcal{D})$ is, in fact, characteristic within stability. Recall that a formula $\varphi(x; y)$ has the finite cover property with respect to a theory T if for all $n < \omega$, there are $a_0, \dots a_n$ in some model $M \models T$ such that the set $\Sigma_n = \{\varphi(x; a_0), \dots \varphi(x; a_n)\}$ is inconsistent but every n-element subset of Σ_n is consistent.

Theorem D. (Shelah [20] VI.5) *Let T be a countable stable theory, $M \models T$, and $\mathcal{D}$ a regular ultrafilter on $\lambda \geq \aleph_0$. Then:*

(1) *If T does not have the finite cover property, then $M^\lambda/\mathcal{D}$ is always λ^+-saturated.*
(2) *If T has the finite cover property, then $M^\lambda/\mathcal{D}$ is λ^+-saturated if and only if $\mu(\mathcal{D}) \geq \lambda^+$.*

Thus Keisler's order on stable theories has exactly two classes, linearly ordered.

Proof. (Sketch) This relies on a characterization of saturated models of stable theories: N is λ^+-saturated if and only if it is $\kappa(T)$-saturated and every maximal indiscernible set has size $\geq \lambda^+$. [This relies heavily on uniqueness of nonforking extensions: given a type p one hopes to realize over some A, $|A| \leq \lambda$, restrict p to a small set over which it does not fork, and use $\kappa(T)$-saturation to find a countable indiscernible sequence of realizations of the restricted type. By hypothesis, we may assume this indiscernible sequence extends to one of size λ^+, and by uniqueness of nonforking extensions, any element of this sequence which does not fork with A will realize the type.]

Returning to ultrapowers: for countable theories, $\kappa(T) \leq \aleph_1$ and any nonprincipal ultrapower is $\aleph_1$-saturated. So it suffices to show that any maximal indiscernible set is large, and the theorem proves, by a coding argument, that this is true whenever the size of every pseudofinite set is large. $\quad \square$

Discussion. As mentioned above, in this paper we construct ultrafilters with "model-theoretically significant properties." The intent of this example was to motivate our work by showing what "model-theoretically significant" might mean. However, the example also illustrates what kinds of properties may fit the bill. We make two general remarks.

(1) "Only formulas matter": The fact that $\mu(\mathcal{D})$ was detected by a property of a single formula, the finite cover property, is not an accident. For $\mathcal{D}$ a regular ultrafilter and $M \models T$ any countable theory, $M^\lambda/\mathcal{D}$ is λ^+-saturated if and only if it is λ^+-saturated for φ-types, for all formulas φ, Malliaris [11] Theorem 12. Thus, from the point of view of Keisler's order, it suffices to understand properties of regular ultrafilters which are detected by formulas.

(2) The role of pseudofinite structure is fundamental, reflecting the nature of the objects involved (regular ultrapowers, first-order theories). On one hand, pseudofinite phenomena can often be captured by a first-order theory. On the other, saturation of regular ultrapowers depends on finitely many conditions in each index model, since by definition regular ultrafilters $\mathcal{D}$ on I, $|I| = \lambda$ contain regularizing families, i.e. $\{X_i : i < \lambda\}$ such that for each $t \in I$, $|\{i < \lambda : t \in X_i\}| < \aleph_0$.

1.2. **Flexibility without goodness: Ultrafilters through the lens of theories.** Our second example takes the complementary point of view. The following is a rich and important class of ultrafilters introduced by Keisler:

Definition 1.4. (Good ultrafilters, Keisler [4]) *The filter $\mathcal{D}$ on I is said to be μ^+-good if every $f : \mathcal{P}_{\aleph_0}(\mu) \to \mathcal{D}$ has a multiplicative refinement, where this means that for some $f' : \mathcal{P}_{\aleph_0}(\mu) \to \mathcal{D}$, $u \in \mathcal{P}_{\aleph_0}(\mu) \implies f'(u) \subseteq f(u)$, and $u, v \in \mathcal{P}_{\aleph_0}(\mu) \implies f'(u) \cap f'(v) = f'(u \cup v)$.*

We may assume the functions f are monotonic (11.6).

$\mathcal{D}$ is said to be good *if it is $|I|^+$-good.*

It is natural to ask for meaningful weakenings of this notion, e.g. by requiring only that certain classes of functions have multiplicative refinements. An important example is the notion of OK, which appeared without a name in Keisler [4], was named and studied by Kunen [9] and investigated generally by Dow [3] and by Baker and Kunen [1]. We follow the definition from [3] 1.1.

Definition 1.5. (OK ultrafilters) *The filter $\mathcal{D}$ on I is said to be λ-OK if each monotone function $g : \mathcal{P}_{\aleph_0}(\lambda) \to \mathcal{D}$ with $g(u) = g(v)$ whenever $|u| = |v|$ has a multiplicative refinement $f : \mathcal{P}_{\aleph_0}(\lambda) \to \mathcal{D}$.*

It is immediate that λ^+-good implies λ-OK. Though OK is an a priori weaker notion, the relative strength of OK and good was not clear. For instance, in [3] 3.10 and 4.7, Dow raises the problem of constructing ultrafilters which are λ^+-OK but not λ^+-good; to our knowledge, even the question of

constructing λ-OK not λ^+-good ultrafilters on λ was open. Before discussing how a model-theoretic perspective can help with such questions, we define the main objects of interest in this paper:

Definition 1.6. (Regular filters) *A filter $\mathcal{D}$ on an index set I of cardinality λ is said to be λ-regular, or simply regular, if there exists a λ-regularizing family $\langle X_i : i < \lambda \rangle$, which means that:*

- *for each $i < \lambda$, $X_i \in \mathcal{D}$, and*
- *for any infinite $\sigma \subset \lambda$, we have $\bigcap_{i \in \sigma} X_i = \emptyset$*

Equivalently, for any element $t \in I$, t belongs to only finitely many of the sets X_i.

Now we make a translation. As Keisler observed, good regular ultrafilters can be characterized as those regular ultrafilters able to saturate any countable theory. (By "$\mathcal{D}$ saturates T" we will always mean: $\mathcal{D}$ is a regular ultrafilter on the infinite index set I, T is a countable complete first-order theory and for any $M \models T$, we have that $M^I/\mathcal{D}$ is λ^+-saturated, where $\lambda = |I|$.) We state this as a definition and an observation, which together say simply that the distance between *consistency* of a type (i.e. finite consistency, reflected by Łos' theorem) and *realization* of a type in a regular ultrapower can be explained by whether or not certain monotonic functions have multiplicative refinements.

Definition 1.7. *Let T be a countable complete first-order theory, $M \models T$, $\mathcal{D}$ a regular ultrafilter on I, $|I| = \lambda$, $N = M^\lambda/\mathcal{D}$. Let $p(x) = \{\varphi_i(x; a_i) : i < \lambda\}$ be a consistent partial type in the ultrapower N. Then a* distribution *of p is a map $d : \mathcal{P}_{\aleph_0}(\lambda) \to \mathcal{D}$ which satisfies:*

(1) *For each $\sigma \in [\lambda]^{<\aleph_0}$, $d(\sigma) \subseteq \{t \in I : M \models \exists x \bigwedge \{\varphi_i(x; a_i[t]) : i \in \sigma\}\}$. Informally speaking, d refines the Łos map.*

(2) *d is monotonic, meaning that $\sigma, \tau \in [\lambda]^{<\aleph_0}$, $\sigma \subseteq \tau$ implies $d(\sigma) \supseteq d(\tau)$*

(3) *The set $\{d(\sigma) : \sigma \in [\lambda]^{<\aleph_0}\}$ is a regularizing family, i.e. each $t \in I$ belongs to only finitely many elements of this set.*

Observation 1.8. *Let T be a countable complete first-order theory, $M \models T$, $\mathcal{D}$ a regular ultrafilter on λ, $N = M^\lambda/\mathcal{D}$. Then the following are equivalent:*

(1) *For every consistent partial type p in N of size $\leq \lambda$, some distribution d of p has a multiplicative refinement.*

(2) *N is λ^+-saturated.*

Proof. The obstacle to realizing the type is simply that, while Łos' theorem guarantees each finite subset of p is almost-everywhere consistent, there is no a priori reason why, at an index $t \in I$ at which $M \models \exists x \bigwedge \{\varphi_i(x; a_i[t]) : i \in \sigma\}$, $M \models \exists x \bigwedge \{\varphi_i(x; a_j[t]) : j \in \tau\}$, these two sets should have a common witness. The statement that d has a multiplicative refinement is precisely the statement that there is, in fact, a common witness almost everywhere, in other words $t \in d(\sigma) \cap d(\tau) \implies t \in d(\sigma \cup \tau)$. When this happens, we may choose at each index t an element c_t such that $\sigma \in [\lambda]^{<\aleph_0} \wedge t \in d(\sigma) \implies M \models \bigwedge \{\varphi_i(c_t : a_i[t]) : i \in \sigma\}$, by 1.7(1). Then by Łos' theorem and 1.7(1), $\prod_{t < \lambda} c_t$ will realize p in N.

The other direction is clear (choose a realization a and use Łos' theorem to send each finite subset of the type to the set on which it is realized by a). $\square$

Keisler's characterization of good ultrafilters then follows from showing that there are first order theories which can "code" enough possible patterns to detect whether any $f : \mathcal{P}_{\aleph_0}(\lambda) \to \mathcal{D}$ fails to have a multiplicative refinement.

Note that first-order theories correspond naturally to monotonic functions of a certain kind (depending, very informally speaking, on some notion of the pattern-complexity inherent in the

theory) and thus, were one to succeed in building ultrafilters which were able to saturate certain theories and not saturate others, this would likewise show a meaningful weakening of goodness.

In this context we mention a property which arose in the study of certain unstable simple theories, called non-low. The original definition is due to Buechler.

Definition 1.9. *The formula $\varphi(x; y)$ is called* non-low *with respect to the theory T if in some sufficiently saturated model $M \models T$, for arbitrarily large $k < \omega$, there exists an infinite indiscernible sequence $\{a_i : i < \omega\}$, with $i < \omega \implies \ell(a_i) = \ell(y)$, such that every k-element subset of*

$$\{\varphi(x; a_i) : i < \omega\}$$

is consistent, but every $k + 1$-element subset is inconsistent.

Here we make a second translation. Recall from Definition 1.6 above that the characteristic objects of regular filters $\mathcal{D}$ on λ are (λ)-*regularizing families*, i.e. sets of the form $\{X_i : i < \lambda\}$ with $t \in I \implies |\{i < \lambda : t \in X_i\}| = n_t < \aleph_0$. Malliaris had noticed in [12] that non-low formulas could detect the *size* (i.e. the nonstandard integer whose tth coordinate is n_t) of the regularizing families in $\mathcal{D}$, and thus had defined and studied the "flexibility" of a filter, Definition 1.10.

Definition 1.10. (Flexible ultrafilters, Malliaris [12], [13]) *We say that the filter $\mathcal{D}$ is λ-flexible if for any $f \in {}^{I}\mathbb{N}$ with $n \in \mathbb{N} \implies n <_{\mathcal{D}} f$, we can find $X_\alpha \in \mathcal{D}$ for $\alpha < \lambda$ such that for all $t \in I$*

$$f(t) \geq |\{\alpha : t \in X_\alpha\}|$$

Informally, given any nonstandard integer, we can find a λ-regularizing family below it.

Specifically, Malliaris had shown that if $\mathcal{D}$ is not λ-flexible then it fails to λ^+-saturate any theory containing a non-low formula. (Note that by Keisler's observation about good ultrafilters, *any* property of ultrafilters which can be shown to be detected by formulas must necessarily hold of good ultrafilters.) Moreover, there is a useful convergence. Kunen had brought the definition of "OK" filters to Malliaris' attention in 2010; "λ-flexible" and "λ-OK" are easily seen to be equivalent, Claim 6.1 below.

We now sketch the proof from §7 below that consistently, flexible need not imply good. (This paper and its sequel [15] contain at least three distinct proofs of that fact, of independent interest.) The numbering of results follows that in §7.

To begin, we use a diagonalization argument to show that saturation decays in ultrapowers of the random graph, i.e. the Rado graph, Definition 2.15 below. ("The random graph" means, from the set-theoretic point of view, that the function which fails to have a multiplicative refinement will code the fact that there are two sets A, B in the final ultrapower N, $|A| = |B| = \lambda$, which are disjoint in N but whose projections to the index models cannot be taken to be a.e. disjoint.)

Claim 5.2. *Assume $\lambda \geq \kappa \geq \aleph_0$, $T = T_{rg}$, M a λ^+-saturated model of T, E a uniform ultrafilter on κ such that $|\kappa^\kappa/E| = 2^\kappa$ Then M^κ/E is not $(2^\kappa)^+$-saturated.*

Note that the hypothesis of the claim will be satisfied when E is regular, and also when E is complete. Our strategy will be to take a product of ultrafilters $D \times E$, where D is a regular ultrafilter on λ and E is an ultrafilter on κ. (The idea of taking such a product goes back at least as far as Keisler's work on ultraproducts of finite sets.) Then $D \times E$ will be regular, and if $\lambda \geq 2^\kappa$, it will fail to saturate the random graph, thus fail to be good. What remains is to ensure flexibility, and for this we will need E to be $\aleph_1$-complete. In the following Corollary, $\mathrm{lcf}(\aleph_0, \mathcal{D})$ is the coinitiality of $\mathbb{N}$ in $(\mathbb{N}, <)^I/\mathcal{D}$, i.e. the cofinality of the set of $\mathcal{D}$-nonstandard integers.

Corollary 7.3 *Let $\lambda, \kappa \geq \aleph_0$ and let $\mathcal{D}_1, E$ be ultrafilters on λ, κ respectively where $\kappa > \aleph_0$ is measurable. Let $\mathcal{D} = \mathcal{D}_1 \times E$ be the product ultrafilter on $\lambda \times \kappa$. Then:*

(1) *If $\mathcal{D}_1$ is λ-flexible and E is $\aleph_1$-complete, then $\mathcal{D}$ is λ-flexible.*
(2) *If $\lambda \geq \kappa$ and $\mathrm{lcf}(\aleph_0, \mathcal{D}_1) \geq \lambda^+$, then $\mathrm{lcf}(\aleph_0, \mathcal{D}) \geq \lambda^+$, so in particular, $\mathcal{D} = D \times E$ will λ^+-saturate any countable stable theory.*

Proof. (Sketch) (1) We first show that the following are equivalent: (i) any $\mathcal{D}$-nonstandard integer projects E-a.e. to a $\mathcal{D}_1$-nonstandard integer, (ii) E is $\aleph_1$-complete. Then, since we have assumed (ii) holds, let some $\mathcal{D}$-nonstandard integer n_* be given. By (ii), for E-almost all $t \in \kappa$, $n_*[t]$ is $\mathcal{D}_1$-nonstandard and by the flexibility of $\mathcal{D}_1$ there is a regularizing family $\{X_i^t : i < \lambda\} \subseteq \mathcal{D}_1$ below any such $n_*[t]$. Let $X_i = \{(s,t) : s \in X_i^t\} \subseteq \mathcal{D}$. It follows that $\{X_i : i < \lambda\}$ is a regularizing family in $\mathcal{D}$ below m_* and thus below n_*.

(2) From the first sentence of (1), we show that if E is $\aleph_1$-complete and in addition $\mathrm{lcf}(\aleph_0, \mathcal{D}) \geq \lambda^+$, then the $\mathcal{D}_1$-nonstandard integers (under the diagonal embedding) are cofinal in the $\mathcal{D}$-nonstandard integers. This suffices. For the second clause, see Theorem F, §4 below.   □

Thus we obtain:

Theorem 7.4. *Assume $\aleph_0 < \kappa < \lambda = \lambda^\kappa$, $2^\kappa \leq \lambda$, κ measurable. Then there exists a regular ultrafilter $\mathcal{D}$ on λ such that $\mathcal{D}$ is λ-flexible, yet for any model M of the theory of the random graph, $M^\lambda/\mathcal{D}$ is not $(2^\kappa)^+$-saturated. Thus $\mathcal{D}$ is not good, and will fail to $(2^\kappa)^+$-saturate any unstable theory. However, $\mathcal{D}$ will λ^+-saturate any countable stable theory.*

Note that the model-theoretic failure of saturation is quite strong, more so than simply "not good." The random graph is known to be minimum among unstable theories in Keisler's order (meaning that any regular $\mathcal{D}$ which fails to saturate the random graph will fail to saturate any other unstable theory). This is the strongest failure of saturation one could hope for given that $\mathrm{lcf}(\aleph_0, \mathcal{D})$ is large, see Section 4 for details.

Theorem 7.4 has the following immediate corollary in the language of OK and good:

Corollary 1.11. *Assume $\aleph_0 < \kappa < 2^\kappa \leq \mu_1 \leq \mu_2 < \lambda = \lambda^\kappa$ and κ is measurable. Then there exists a regular ultrafilter $\mathcal{D}$ on λ such that $\mathcal{D}$ is λ-flexible, thus λ-OK, but not $(2^\kappa)^+$-good. In particular, $\mathcal{D}$ is $(\mu_2)^+$-OK but not $(\mu_1)^+$-good.*

In particular, this addresses the problem raised by Dow in [3] 3.10 and 4.7, namely, the problem of constructing ultrafilters which are α^+-OK and not α^+-good.

Discussion. The intent of this example was to show that model theory can contribute to calibrating ultrafilters. Note that in terms of determining the strength of a priori weakenings of goodness, the model-theoretic perspective has given both positive and negative results:

(1) On one hand, Theorem 7.4 applies model-theoretic arguments to show that multiplicative refinements for size-uniform functions $f : \mathcal{P}_{\aleph_0}(\lambda) \to \mathcal{D}$ are not enough to guarantee multiplicative refinements for all such functions.
(2) On the other, the second author's proof of the maximality of strict order (see Theorem F(6) below) *does* isolate an a priori weaker class of functions which have such a guarantee – namely, those corresponding to distributions of types in linear order. The set-theoretic question of why these functions suffice appears to be deep. The model-theoretic formulation of "determine a minimum such set of functions" is: determine a necessary condition for maximality in Keisler's order.

This concludes our two examples. We now fix definitions and notation, before giving a summary of our results in §3.

2. Definitions and conventions

This section contains background, most definitions, and conventions. Note that the definition of $\mu(\mathcal{D})$ was given in Definition 1.2, and the definitions of good, regular and flexible filters were Definitions 1.4, 1.6 and 1.10 above. (Recall that a filter is said to be λ-*regular* if it contains a family of λ sets any countable number of which have empty intersection, 1.6 above.)

Let $I = \lambda \geq \aleph_0$ and fix $f : \mathcal{P}_{\aleph_0}(\lambda) \to I$. Then $\{\{s \in I : \eta \in f^{-1}(s)\} : \eta < \lambda\}$ can be extended to a regular filter on I, so regular ultrafilters on $\lambda \geq \aleph_0$ always exist, see [2].

Keisler proposed in 1967 [5] that saturation properties of regular ultrapowers might be used to classify countable first-order theories. His preorder $\trianglelefteq$ on theories is often thought of as a partial order on the $\trianglelefteq$-equivalence classes, and so known as "Keisler's order."

Definition 2.1. (Keisler [5]) *Given countable theories T_1, T_2, say that:*

  (1) $T_1 \trianglelefteq_\lambda T_2$ *if for any $M_1 \models T_1, M_2 \models T_2$, and $\mathcal{D}$ a regular ultrafilter on λ,*
      if $M_2^\lambda/\mathcal{D}$ is λ^+-saturated then $M_1^\lambda/\mathcal{D}$ must be λ^+-saturated.
  (2) (Keisler's order) $T_1 \trianglelefteq T_2$ *if for all infinite λ, $T_1 \trianglelefteq_\lambda T_2$.*

Question 2.2. *Determine the structure of Keisler's order.*

The hypothesis *regular* justifies the quantification over all models: when T is countable and $\mathcal{D}$ is regular, saturation of the ultrapower does not depend on the choice of index model.

Theorem E. (Keisler [5] Corollary 2.1 p. 30; see also Shelah [20].VI.1) *Suppose that $M_0 \equiv M_1$, the ambient language is countable, and $\mathcal{D}$ is a regular ultrafilter on λ. Then $M_0{}^\lambda/\mathcal{D}$ is λ^+-saturated iff $M_1{}^\lambda/\mathcal{D}$ is λ^+-saturated.*

More information on Keisler's order, including many examples and a summary of all known results through early 2010, may be found in the introduction to the first author's paper [13].

Definition 2.3. (Complete ultrafilters) *The ultrafilter $\mathcal{E}$ on κ is said to be μ-complete if for any $\{X_i : i < \mu' < \mu\} \subseteq \mathcal{E}$, $\bigcap\{X_i : i < \mu'\} \in \mathcal{E}$.*

Working with complete ultrafilters, we are obliged to make large cardinal hypotheses. We will use measurable, normal and to a lesser extent, weakly compact cardinals. Their utility for our arguments will be clear from the choice of definitions:

Definition 2.4. (Measurable, weakly compact)

  (1) *The uncountable cardinal κ is said to be* measurable *if there is a κ-complete nonprincipal ultrafilter on κ.*
  (2) *The uncountable cardinal κ is said to be* weakly compact *if $\kappa \to (\kappa)_2^2$.*

Fact 2.5. *If $\kappa > \aleph_0$ is weakly compact, $n < \aleph_0$ and $\rho < \kappa$, then for any $\alpha : [\kappa]^n \to \rho$ there exists $\mathcal{U} \subseteq \kappa$, $|\mathcal{U}| = \kappa$ such that $\langle \alpha(\epsilon_1, \ldots \epsilon_n) : \epsilon_1 < \cdots < \epsilon_n$ from $\mathcal{U}\rangle$ is constant.*

Definition 2.6. (Normal ultrafilters) *A filter D on κ is* normal *when, for any sequence $\langle A_i : i < \kappa \rangle$ with $i < \kappa \implies A_i \in D$,*

$$\{\alpha < \kappa : (\forall j < 1 + \alpha)(\alpha \in A_j)\} \in D$$

Fact 2.7. *Let κ be a measurable cardinal. Then*

  (1) *there exists a normal, κ-complete, uniform ultrafilter D on κ.*
  (2) *for any $f : \kappa \to \kappa$ which is regressive on $X \in D$, there is a set $Y \in D$, $Y \subseteq X$ on which f is constant.*

Discussion 2.8. *An interesting and unexpected phenomenon visible in this work is the relevance of measurable cardinals, and in particular κ-complete nonprincipal ultrafilters, in the construction of regular ultrafilters. In the 1960s, model theorists pointed out regularity as a central property of ultrafilters, and generally concentrated on this case. Regularity ensures that saturation [of ultrapowers of models of complete countable theories] does not depend on the saturation of the index model, and that the cardinality of ultrapowers can be settled ($M^I/\mathcal{D} = M^{|I|}$). Meanwhile, the construction of various non-regular ultrafilters was investigated by set theorists. However, many questions about regular ultrafilters remained opaque from the model-theoretic point of view. For example, from the point of view of Theorem D, p. 4 above, the regular ultrafilters with large $\mathrm{lcf}(\aleph_0)$ – a condition which implies that these ultrafilters saturate ultrapowers of stable theories – appeared to look alike. Moreover, it was not clear whether various a priori weakenings of goodness (e.g. flexible/ok) were indeed weaker. Here, in several different constructions, we combine both lines of work, using κ-complete ultrafilters to construct regular ultrafilters on $\lambda > \kappa$ with model-theoretically meaningful properties, i.e. presence or absence of some specific kind of saturation.*

Definition 2.9. (Good for equality, Malliaris [14]) *Let $\mathcal{D}$ be a regular ultrafilter. Say that $\mathcal{D}$ is* good for equality *if for any set $X \subseteq N = M^I/\mathcal{D}$, $|X| \leq |I|$, there is a distribution $d : X \to \mathcal{D}$ such that $t \in \lambda, t \in d(a) \cap d(b)$ implies that $(M \models a[t] = b[t]) \iff (N \models a = b)$.*

Definition 2.10. (Lower cofinality, $\mathrm{lcf}(\kappa, D)$) *Let D be an ultrafilter on I and κ a cardinal. Let $N = (\kappa, <)^I/\mathcal{D}$. Let $X \subset N$ be the set of elements above the diagonal embedding of κ. We define $\mathrm{lcf}(\kappa, D)$ to be the cofinality of X considered with the reverse order.*

Definition 2.11. (Product ultrafilters) *Let I_1, I_2 be infinite sets and let D_1, D_2 be ultrafilters on I_1, I_2 respectively. Then the product ultrafilter $D = D_1 \times D_2$ on $I_1 \times I_2$ is defined by:*

$$X \in D \iff \{t \in I_2 \ : \ \{s \in I_1 \ : \ (s,t) \in X\} \in D_1\} \in D_2$$

for any $X \subseteq I_1 \times I_2$.

Finally, it will be useful to have a name for functions, or relations, to which Łos' theorem applies since they are visible in an expanded language:

Definition 2.12. (Induced structure) *Let $N = M^\lambda/\mathcal{D}$ be an ultrapower and $X \subseteq N^m$. Say that X is an* induced *function, or relation, if there exists a new function, or relation, symbol P of the correct arity, and an expansion M_t' of each index model M_t to $\mathcal{L} \cup \{P\}$, so that $P^N \equiv X \mod \mathcal{D}$.*

Equivalently, X is the ultraproduct modulo $\mathcal{D}$ of its projections to the index models.

Definition 2.13. (Cuts in regular ultrapowers of linear orders)

(1) *For a model M expanding the theory of linear order, a (κ_1, κ_2)-cut in M is given by sequences $\langle a_i : i < \kappa_1 \rangle, \langle b_j : j < \kappa_2 \rangle$ of elements of M such that*
  - $i_1 < i_2 < \kappa_1 \implies a_{i_1} < a_{i_2}$
  - $j_1 < j_2 < \kappa_2 \implies b_{j_2} < b_{j_1}$
  - $i < \kappa_1, \ j < \kappa_2$ *implies $a_i < b_j$ and*
  - *the type $\{a_i < x < b_j : i < \kappa_1, j < \kappa_2\}$ is omitted in M.*

(2) *For $\mathcal{D}$ a (regular) ultrafilter on I we define:*

$$\mathcal{C}(\mathcal{D}) = \left\{ (\kappa_1, \kappa_2) \in (\mathrm{Reg} \cap |I|^+) \times (\mathrm{Reg} \cap |I|^+) : \ (\mathbb{N}, <)^I/\mathcal{D} \text{ has a } (\kappa_1, \kappa_2)\text{-cut} \right\}$$

Here we list the main model-theoretic properties of formulas used in this paper. For TP_1/SOP_2 and TP_2, see §8. The finite cover property is from Keisler [5] and the order property, independence property and strict order property are from Shelah [20].II.4.

Definition 2.14. (Properties of formulas) *Let $\varphi = \varphi(x; y)$ be a formula of T and $M \models T$ be any sufficiently saturated model. Note that $\ell(x), \ell(y)$ are not necessarily 1. Say that the formula $\varphi(x; y)$ has:*

(1) not the finite cover property, *written* nfcp, *if there exists $k < \omega$ such that: for any $A \subseteq M$ and any set $X = \{\varphi(x; a) : a \in A\}$ of instances of φ, k-consistency implies consistency. (This does not depend on the model chosen.)*
(2) *the* finite cover property, *written* fcp, *if it does not have nfcp.*
(3) *the* order property *if there exist elements a_i $(i < \omega)$ such that for each $n < \omega$, the following partial type is consistent:*

$$\{\neg\varphi(x; a_i) : i < n\} \cup \{\varphi(x; a_j) : j \geq n\}$$

Formulas with the order property are called unstable.
(4) *the* independence property *if there exist elements a_i $(i < \omega)$ such that for each $\sigma, \tau \in [\omega]^{<\aleph_0}$ with $\sigma \cap \tau = \emptyset$, the following partial type is consistent:*

$$\{\neg\varphi(x; a_i) : i \in \sigma\} \cup \{\varphi(x; a_j) : j \in \tau\}$$

Note that the independence property implies the order property.
(5) *the* strict order property *if there exist elements a_i $(i < \omega)$ such that for all $j \neq i < \omega$,*

$$(\exists x(\neg\varphi(x; a_j) \wedge \varphi(x; a_i))) \iff j < i$$

Note that (4), (5) each imply (3).

A theory T is said to have the finite cover property, the order property, the independence property or the strict order property iff one of its formulas does, and to have $nfcp$ if all of its formulas do.

The "random graph" is known to be minimum in Keisler's order among the unstable theories, and so will feature in our proofs with some regularity.

Definition 2.15. *The* random graph, *i.e. the Rado graph, is (the unique countable model of) the complete theory in the language with equality and a binary relation R axiomatized by saying that there are infinitely many elements, and that for each n, and any two disjoint subsets of size n, there is an element which R-connects to all elements in the first set and to none in the second set.*

We conclude this section with some conventions which hold throughout the paper.

Convention 2.16. (Conventions)

- *The letters $D, E, \mathcal{D}, \mathcal{E}$ are used for filters. Generally, we reserve $\mathcal{D}$ for a regular filter or ultrafilter, and $\mathcal{E}$ for a κ-complete ultrafilter where $\kappa \geq \aleph_0$, though this is always stated in the relevant proof.*
- *Throughout, tuples of variables may be written without overlines, that is: when we write $\varphi = \varphi(x; y)$, neither x nor y are necessarily assumed to have length 1.*
- *For transparency, all languages are assumed to be countable.*
- *As mentioned in §1.2, by "$\mathcal{D}$ saturates T" we will always mean: $\mathcal{D}$ is a regular ultrafilter on the infinite index set I, T is a countable complete first-order theory and for any $M \models T$, we have that $M^I/\mathcal{D}$ is λ^+-saturated, where $\lambda = |I|$.*
- *We will also say that the ultrafilter D is "good" (or: "not good") for the theory T to mean that D saturates (or: does not saturate) the theory T.*

- *We reserve the word* cut *in models of linear order for omitted types. A partial type in a model M given by some pair of sequences $(\langle a_\alpha : \alpha < \kappa_1 \rangle, \langle b_\beta : \beta < \kappa_2 \rangle)$ with $\alpha < \alpha' < \kappa_1, \beta < \beta' < \kappa_2 \implies M \models a_\alpha < a_{\alpha'} < b_{\beta'} < b_\beta$, which may or may not have a realization in M, is called a* pre-cut. *See also Definition 2.13.*

3. Description of results

In this section, we describe the main results of the paper.

Some notes: For relevant definitions and conventions ("$\mathcal{D}$ saturates T," "good for," "pre-cut") see §2 above, in particular 2.16. Lists of the properties mentioned from the point of view of Keisler's order can be found in Theorems F-G, §4. The reader unused to phrases of the form "not good for the random graph therefore not good" is referred to §1.2, especially following 1.6.

The first main result, Theorem 7.4, was discussed in §1.2 above. We first show that for E an ultrafilter on κ, if E is κ-regular or E is κ-complete then M^κ/E will not be $(2^\kappa)^+$-saturated for any κ^+-saturated model of T_{rg}, the theory of the random graph. Note that this gives a useful way of producing regular ultrafilters which are not good: for any λ-regular $\mathcal{D}_1$ on λ, the product ultrafilter $\mathcal{D} = \mathcal{D}_1 \times E$ will remain regular but will not be good for T_{rg} when $2^\kappa \le \lambda$. We then show, as sketched above, that when E is $\aleph_1$-complete and $\mathrm{lcf}(\aleph_0, \mathcal{D}_1)$ is large the nonstandard integers of $\mathcal{D}_1$ will be cofinal in those of $\mathcal{D}$ (under the diagonal embedding) and thus that $\mathcal{D}$ will inherit both the large $\mathrm{lcf}(\aleph_0)$ and the flexibility of $\mathcal{D}_1$.

This yields Theorem 7.4: for any $\lambda \ge 2^\kappa$, κ measurable there is a regular ultrafilter on λ which is flexible but not good, which has large $\mathrm{lcf}(\aleph_0)$ and thus saturates all stable theories, but does not saturate any unstable theory. In at least one sense, this is a surprising reversal. From the model-theoretic point of view, flexible ultrafilters had appeared "close" in power to ultrafilters capable of saturating any first-order theory. By Theorem 7.4, consistently flexibility cannot guarantee the saturation of any unstable theory, since the random graph is minimum among the unstable theories in Keisler's order. Thus the space between flexible and good is potentially quite large.

Discussion. In fact we obtain several different flexible-not-good theorems, including Conclusion 10.4 and a related theorem in [15]. These results have other advantages, and less dramatic failures of saturation. In fact, if the stated failures of saturation can be shown to be sharp, this could be quite useful for obtaining further dividing lines within the unstable theories.

Second, we give a new proof that there is a loss of saturation in ultrapowers of non-simple theories, Conclusion 8.6. Specifically, we show that if M is a model of a non simple theory and $\mathcal{D}$ a regular ultrafilter on $\lambda = \lambda^{<\lambda}$, then for some formula φ $M^\lambda/\mathcal{D}$ is not λ^{++}-saturated for φ-types. (The second author's book [20].VI contains a first proof of this result.) The argument uses a combinatorial principle from [7] against the definition of the tree property.

The remainder of the paper focuses on realization and omission of symmetric cuts, and here complete ultrafilters are very useful; see Discussion 2.8 above. Relatively little was known about the possible spectrum of cofinalities of cuts in ultrapowers of linear order, and our theorems here contribute to these investigations. The specific model-theoretic importance of symmetric cuts in ultrapowers has come to light in subsequent results of the authors [17], [18].

We prove in Claim 9.1 that if the ultrafilter $\mathcal{E}$ is κ-complete not κ^+-complete, any $\mathcal{E}$-ultrapower of a sufficiently saturated model of linear order will have no (κ, κ)-cuts. This is a fairly direct proof, and we also show there that if we further assume that $\mathcal{E}$ is normal then it follows that $\mathcal{E}$ is good (though not regular: see the Appendix e.g. 11.7). We then prove, in Claim 10.1, that if $\mathcal{E}$ is κ-complete and normal on κ, then any ultrapower of a sufficiently saturated model of linear order will contain a (κ^+, κ^+)-cut.

Finally, we leverage these proofs for a last existence result, Theorem 11.3. There, assuming n measurable cardinals below λ, we show how to take products of ultrafilters to produce any number of finite alternations of cuts in an ultrafilter on λ with clearly described saturation properties. The proof is by downward induction. Note that if the first ultrafilter in our $(n+1)$-fold product is regular, the final ultrafilter will be regular. By a result mentioned at the beginning of this section, if the first ultrafilter is flexible with large $\mathrm{lcf}(\aleph_0)$ and all remaining ultrafilters are at least $\aleph_1$-complete, the final ultrafilter will inherit flexibility and large $\mathrm{lcf}(\aleph_0)$, thus be able to saturate any stable theory. By the result about loss of saturation for the random graph, it will fail to $(2^\kappa)^+$-saturate any unstable theory, where κ is the smallest measurable cardinal used in the construction.

This completes the summary of our main results. In the appendix, we collect some easy observations and extensions of previous results relevant to these theorems.

4. Summary theorems

In this section, we state and prove several comprehensive theorems which give the picture of Keisler's order, Definition 2.1, in light of our current work. The results justify the phrase "properties of ultrafilters with model-theoretic significance." Recall our conventions in 2.16, especially with respect to "saturates" and "good". Minimum, maximum, etc. refer to Keisler's order.

The first theorem collects the currently known correspondences between properties of regular ultrafilters and properties of first-order theories.

Theorem F. *In the following table, for each of the rows* (1),(3),(5),(6) *the regular ultrafilter* $\mathcal{D}$ *on* λ *fails to have the property in the left column if and only if it omits a type in every formula with the property in the right column. For rows* (2) *and* (4), *left to right holds: if* $\mathcal{D}$ *fails to have the property on the left then it omits a type in every formula with the property on the right.*

Set theory: properties of filters		*Model theory: properties of formulas*
(1) $\mu(\mathcal{D}) \geq \lambda^+$		A. *finite cover property*
(2) $\mathrm{lcf}(\aleph_0, \mathcal{D}) \geq \lambda^+$	**	B. *order property*
(3) *good for* T_{rg}		C. *independence property*
(4) *flexible, i.e.* λ-*flexible*	**	D. *non-low*
(5) *good for equality*		E. TP_2
(6) *good, i.e.* λ^+-*good*		F. *strict order property*

Proof. (Discussion - Sketch)

　(1) $\leftrightarrow$ (A) Shelah [20].VI.5, see §1.1 above.

　(2) $\leftarrow$ (B) Shelah [20].VI.4.8, see also Theorem 11.8 below which generalizes that result.

　(3) $\leftrightarrow$ (C) Straightforward by quantifier elimination, see [13]. More generally, Malliaris [14] shows that the random graph, as the minimum non-simple theory, and T_{feq}, as the minimum TP_2 theory, are in a natural sense characteristic of "independence properties" seen by ultrafilters.

　(4) $\leftarrow$ (D) Malliaris [12], see §1.2 above, or [15].

　(5) $\leftrightarrow$ (E) Malliaris [13] §6, which proves the existence of a minimum TP_2-theory, the theory T^*_{feq} of a parametrized family of independent (crosscutting) equivalence relations.

　(6) $\leftrightarrow$ (F) Keisler observed that good ultrafilters can saturate any countable theory, and proved that goodness is equivalent to the saturation of certain ("versatile") formulas [5], thus establishing the existence of a maximum class in Keisler's order; see §1.2 above. The result (6) $\leftrightarrow$ (F) follows from Shelah's proof in [20].VI.3 that any theory with the strict order property is maximum in Keisler's order. Thus any ultrafilter able to saturate SOP-types must be good, and by Keisler's

observation the reverse holds. In fact, SOP_3 is known to be sufficient for maximality by [22]-[23], but this formulation is more suggestive here given our focus on order-types and cuts. A model-theoretic characterization of the maximum class is not known.   □

Remark 4.1. *Moreover, by work of the authors in [17] if $\mathcal{D}$ on λ has "treetops," i.e. it realizes a certain set of SOP_2-types then it must realize all symmetric pre-cuts, that is, there can be no (κ, κ)-cuts in ultrapowers of linear order for $\kappa \leq \lambda$. So we will also be interested in the property of realizing symmetric cuts.*

As rows (2) and (4) of Theorem F suggest, there are subtleties to the correspondence. If T is not Keisler-maximal then any formula φ of T with the order property has the independence property, as does any non-low formula. Yet consistently neither (4) nor (2) imply (3), as the rest of this section explains. So while we have model-theoretic sensitivity to properties (2) and (4), this is not enough for a characterization: in fact it follows from the theorems below that there is consistently no theory (and no formula φ) which is saturated by $\mathcal{D}$ if and only if (2), or (4).

Theorem G. *Using the enumeration of properties of ultrafilters from Theorem F, we have that:*

 (1) *is necessary and sufficient for saturating stable theories,*
 (2) *is necessary for saturating unstable theories,*
 (3) *is necessary and sufficient for saturating the minimum unstable theory,*
 (4) *is necessary for saturating non-low theories,*
 (5) *is necessary and sufficient for saturating the minimum TP_2 theory,*
 (6) *is necessary and sufficient for saturating any Keisler-maximum theory, e.g. $Th(\mathbb{Q}, <)$; note that the identity of the maximum class is not known.*

Discussion. The sources follow those of Theorem F, but we make some additional remarks.

(1) Note that Shelah's proof of (1) in [20].VI.5, quoted and sketched as Theorem D, §1.1 above, gives the only two known equivalence classes in Keisler's order.

(2) By Shelah [20].VI.4.8 (or Theorem 11.8 below) if $\mathcal{D}$ is regular and $\mathrm{lcf}(\mathcal{D}, \aleph_0) \leq \lambda^+$ then any $\mathcal{D}$-ultrapower will omit a λ-type in some unstable formula, i.e., a formula with the order property. From the set-theoretic point of view, (2) $\not\rightarrow$ (3) of Theorem 4.2 shows that $\mathrm{lcf}(\aleph_0, \mathcal{D}) \geq \lambda^+$ is weaker than ensuring λ^+-saturation for the random graph (or equivalently, for some formula with the independence property). From the model-theoretic point of view, since any unstable theory has either the strict order property or the independence property, this gap is not visible.

(3), (5) In fact what the characterization in Malliaris [13] shows is that a necessary and sufficient condition for an ultrafilter $\mathcal{D}$ on λ to saturate the minimum TP_2 theory is that it be "good for equality," meaning that for any set $X \subseteq N = M^\lambda/\mathcal{D}$, $|X| = \lambda$, there is a distribution $d : X \rightarrow \mathcal{D}$ such that $t \in \lambda, t \in d(a) \cap d(b)$ implies that $(M \models a[t] = b[t]) \iff (N \models a = b)$. By contrast, saturation of the minimum unstable theory asks only that for any two disjoint sets $X, Y \subseteq M^\lambda/\mathcal{D}$, $|X| = |Y| = \lambda$, there is a distribution $d : X \cup Y \rightarrow \mathcal{D}$ such that for any $t \in \lambda$, $a \in X, b \in Y$, $t \in d(a) \cap d(b)$ implies $M \models a[t] \neq b[t]$.

(4) This was discussed in §1.2 above. Note that by work of Shelah [21] non-simple theories have an inherent structure/randomness "dichotomy" of TP_1 versus TP_2, analogous to the structure/randomness dichotomy for unstable theories of SOP versus IP; see §8 below. We know from [13] that any ultrafilter which saturates the minimum TP_2-theory must be flexible, however we do not know whether an ultrafilter which saturates some SOP_2 theory must be flexible.

(6) See the proof of Theorem F(6).   □

With the progress in this paper, and in other recent work of the authors, we may summarize the current picture of implications as follows:

Theorem 4.2. *Let properties* $(1) - (6)$ *be as in Theorem F. Assume that* $\mathcal{D}$ *is a regular ultrafilter on* λ (*note that not all of these properties imply regularity*). *Then:*

$(1) \leftarrow (2) \leftarrow (3) \leftarrow (5) \leftarrow (6)$, *with* $(1) \not\rightarrow (2)$, *consistently* $(2) \not\rightarrow (3)$, $(3) \not\rightarrow (5)$, *and whether* (5) *implies* (6) *is open. Moreover* $(1) \leftarrow (4) \leftarrow (5) \leftarrow (6)$, *where* $(3) \not\rightarrow (4)$ *thus* $(2) \not\rightarrow (4)$, $(4) \not\rightarrow (3)$, *consistently* $(4) \not\rightarrow (5)$, *consistently* $(4) \not\rightarrow (6)$; *and* (4) *implies* (2) *is open.*

Proof. $(6) \rightarrow (x)$ Since good ultrafilters saturate any countable theory and properties (1)-(5) are all detected by formulas via Theorem F, property (6) implies all the others.

$(5) \rightarrow (3)$ By Theorem G and the fact that the random graph is minimum among unstable theories.

$(3) \rightarrow (2)$ By Theorem F lines (2)-(3), i.e. Shelah [20].VI.4.8.

$(2) \rightarrow (1)$ Clearly the failure of (1) implies the failure of (2).

$(1) \not\rightarrow (2)$ Shelah [20].VI.5, see Theorem D, §1.2 above.

$(2) \not\rightarrow (3)$ Consistently (assuming an $\aleph_1$-complete ultrafilter) by Theorem 7.4 below.

$(5) \rightarrow (4)$ Malliaris [13] §6.

$(4) \rightarrow (1)$ Proved in [15].

$(3) \not\rightarrow (4)$ Proved in a paper of the authors on excellent ultrafilters [16].

$(4) \not\rightarrow (3)$ Consistently (assuming an $\aleph_1$-complete ultrafilter) by Theorem 7.4 below.

$(4) \not\rightarrow (5), (6)$ We give several proofs of independent interest (each assuming the existence of an $\aleph_1$-complete ultrafilter): Theorem 7.4 proves $(4) \not\rightarrow (3)$ thus a fortiori $(4) \not\rightarrow (5)$, and in [15], we give a different proof that $(4) \not\rightarrow (6)$. See also Conclusion 10.4. $\square$

5. $M^\lambda/\mathcal{D}$ IS NOT $(2^\lambda)^+$-SATURATED FOR $Th(M)$ UNSTABLE

In this section and the next we prove that flexibility does not imply saturation of the random graph, and thus a fortiori that flexibility does not imply goodness for equality. This gives a proof (assuming the existence of an $\aleph_1$-compact ultrafilter, equivalent to the existence of a measurable cardinal) that flexible need not mean good.

Fact 5.1. ([20] Conclusion 1.13 p. 332) *If* κ *is an infinite cardinal and* $\mathcal{D}$ *is a regular ultrafilter on* I *then* $\kappa^I/\mathcal{D} = \kappa^{|I|}$.

Claim 5.2. *Assume* $\kappa \geq \aleph_0$, $T = T_{rg}$, M *a* κ^+-*saturated model of* T, E *a uniform ultrafilter on* κ *such that* $|\kappa^\kappa/E| = 2^\kappa$ (*i.e. we can find a sequence* $\langle g_\alpha : \alpha < 2^\kappa \rangle$ *of members of* $^\kappa\kappa$ *such that* $\alpha < \beta \implies g_\alpha \neq g_\beta \mod E$). *Then* M^κ/E *is not* $(2^\kappa)^+$-*saturated.*

Remark 5.3. *The claim is interesting when* $\mathcal{D}$ *is* $\aleph_1$-*complete.*

Proof. (of 5.2) Let $\mathcal{F} = \{f : f : \kappa \times \kappa \rightarrow \{0, 1\}\}$, so $|\mathcal{F}| = 2^\kappa$, and let $\langle f_\alpha : \alpha < 2^\kappa \rangle$ list $\mathcal{F}$. Let $\langle g_\alpha : \alpha < 2^\kappa \rangle$ be the distinct sequence given by hypothesis. First, for each $\alpha < 2^\kappa$, we define $\mathbf{t}_\alpha \in \{0, 1\}$ by:

$$\mathbf{t}_\alpha = 1 \iff \{i : f_\alpha(i, g_\alpha(i)) = 1\} \notin E$$

Second, since $|M| \geq \kappa$, we may fix some distinguished sequence $\langle a_i : i < \kappa \rangle$ of elements of M. Let $\hat{g}_\alpha \in {}^\kappa M$ be give by $\hat{g}_\alpha(i) = a_{g_\alpha(i)}$. Together these give a set

$$p(x) = \{(xR\hat{g}_\alpha/E)^{\text{if }\mathbf{t}_\alpha} : \alpha < 2^\kappa\}$$

We check that $p(x)$ is a consistent partial type in M^κ/E. It is clearly a set of formulas with parameters in M^κ/E, as $\hat{g}_\alpha/E \in M^\kappa/E$. Since $\alpha < \beta \implies g_\alpha/E \neq g_\beta/E \implies \hat{g}_\alpha/E \neq \hat{g}_\beta/E$, the parameters are distinct and so the type is consistent (note that for the given sequence of parameters, any choice of exponent sequence $\langle \mathbf{t}_\alpha : \alpha < 2^\kappa \rangle$ would produce a consistent partial type). Moreover, $|p| = 2^\kappa$ again by the choice of $\langle g_\alpha : \alpha < 2^\kappa \rangle$.

We now show that $p(x)$ is omitted in M^κ/E. Towards a contradiction, suppose that $h \in {}^\kappa M$ were such that h/E realized p. Let $f : \kappa \times \kappa \to \{0, 1\}$ be defined by $f(i, j) = 1 \iff M \models h(i)Ra_j$. Then $f \in \mathcal{F}$, hence for some $\alpha_* < 2^\kappa$ we have that $f_{\alpha_*} = f$. Thus:

$$\begin{aligned}
\mathbf{t}_{\alpha_*} = 1 \quad &\text{iff} \quad M^\kappa/E \models (h/E)R(\hat{g}_{\alpha_*}/E) && \text{(by choice of p, since $h/E \models p$)} \\
&\text{iff} \quad \{i < \kappa : M \models h(i)Ra_{g_{\alpha_*}(i)}\} \in E && \text{(by Łoś' theorem)} \\
&\text{iff} \quad \{i : f(i, g_{\alpha_*}(i)) = 1\} \in E && \text{(by the choice of f)} \\
&\text{iff} \quad \{i : f_{\alpha_*}(i, g_{\alpha_*}(i)) = 1\} \in E && \text{(as $f_{\alpha_*} = f$)}
\end{aligned}$$

But by definition of the truth values $\mathbf{t}$,

$$\mathbf{t}_{\alpha_*} = 1 \iff \{i : f_{\alpha_*}(i, g_\alpha(i)) = 1\} \notin E$$

This contradiction completes the proof. $\qquad\square$

We state the following corollary, which will be improved in certain cases by Theorem 7.4 below. Recall the independence and strict order properties from 2.14 above.

Claim 5.4. *If $\mathcal{D}$ is a λ-regular ultrafilter on λ, then $M^\lambda/\mathcal{D}$ is not $(2^\lambda)^+$-saturated for M a model of any unstable theory.*

Proof. As T is unstable, it has the finite cover property. By Theorem F, $M^\lambda/\mathcal{D}$ is not μ^+-saturated where $\mu = \mu(\mathcal{D})$, 1.2 above. ("$\mathcal{D}$ is not $\aleph_1$-complete" suffices.) But $\mu(\mathcal{D}) \leq 2^\lambda$, so we are done. $\qquad\square$

Observation 5.5. *The hypothesis of Claim 5.2 is satisfied when E is a regular ultrafilter on κ and when E is a κ-complete ultrafilter on κ.*

Remark 5.6. *5.5 is also true when E is a uniform ultrafilter on κ and $2^\kappa = \kappa^+$ or E is a uniform ultrafilter on κ and $\kappa = 2^{<\kappa}$.*

That is, to prove Observation 5.5, we want to show that we can find a sequence $\langle g_\alpha : \alpha < 2^\kappa \rangle$ of members of ${}^\kappa \kappa$ so that $\alpha < \beta \implies g_\alpha \neq g_\beta \mod E$. When E is regular, this follows from Fact 5.1. We give two proofs for the complete case, using inaccessibility of κ.

First proof of Obs. 5.5 – counting functions. Suppose then that κ is measurable, thus inaccessible. For each $\alpha < \kappa$, let $\Gamma_\alpha = \langle \gamma_\eta : \eta \in {}^\alpha 2 \rangle$ be a sequence of pairwise distinct ordinals $< \kappa$. For each $\eta \in {}^\kappa 2$ let $g_\eta : \kappa \to \kappa$ be given by $g_\eta(\alpha) = \gamma_{\eta|_\alpha}$. So $\{g_\eta : \eta \in {}^\kappa 2\} \subseteq {}^\kappa \kappa$. By construction, all we need is one point of difference to know the functions diverge: $\eta \neq \nu \in {}^\kappa 2, \eta(\beta) \neq \nu(\beta) \implies \{\alpha < \kappa : g_\eta(\alpha) = g_\nu(\alpha)\} \subseteq \{\alpha : \alpha < \beta\} = \emptyset \mod E$ as E is uniform. $\qquad\square$

Second proof of Obs. 5.5 – realizing types. Suppose then that κ is measurable, thus inaccessible. So we may choose M, $|M| = \kappa$ to be a κ-saturated model of the theory of the random graph. To show that $|M^\kappa/E| \geq 2^\kappa$, it will suffice to show that 2^κ-many distinct types over the diagonal embedding of M in the ultrapower N are realized. Let $p(x) = \{xRf_\alpha^0 \land \neg xRf_\alpha^1 : \alpha < \kappa\}$ be such a type, with each $f_\alpha^i = {}^\kappa\{m\}$ for some $m \in M$ and of course $\alpha, \beta < \kappa \implies f_\alpha^0 \neq f_\beta^1$. For each $t \in \kappa$, let $p_t(x) = \{xRf_\alpha^0(t) \land \neg xRf_\alpha^1(t) : \alpha < t\}$. Note that since the elements f_α^i are constant, for each $t \in \kappa$ we have that $p_t(x)$ is a consistent partial type in M. Choose a new element $h \in {}^\kappa M$ so that $t \in \kappa$

implies $h(t)$ satisfies $p_t(x)$ in M. By the saturation of M, some such h exists. By uniformity of E h realizes the type $p(x)$, that is, for $\alpha < \kappa$,

$$|\kappa \setminus \{t \in \kappa : M \models h(t)Rf_\alpha^0(t) \wedge \neg h(t)Rf_\alpha^1(t)\}| \leq \alpha < \kappa$$

As no such h can realize two distinct types over M in N, we finish. $\qquad\square$

Discussion 5.7. *We may wonder: for $\mathcal{D}$ an ultrafilter on λ, M saturated enough, $T = Th(M)$, how saturated is $M^\lambda/\mathcal{D}$? How locally saturated? Let $\kappa = \kappa(\mathcal{D})$ be the completeness of $\mathcal{D}$. In questions of saturation, we naturally assume $|T| < \kappa(\mathcal{D})$, or that $\mathcal{D}$ is $(|T|, \kappa)$-good.*

Claim 5.8 is a result addressing 5.7. Further work in this direction appears in a paper in preparation of the authors.

Claim 5.8. *Suppose that:*

(1) *$\mathcal{D}$ is an ultrafilter on λ*
(2) *$\mathcal{D}$ is κ-complete not κ^+-complete (so κ is measurable),*
(3) *M is a κ-universal model*
   (which means that every type with $\leq \kappa$ formulas and $\leq \kappa$ variables, but no parameters, is realized in M; if $\kappa \geq |\tau(M)|$, this means every $N \equiv M$ of cardinality $\leq \kappa$ is elementarily embeddable in M.)
(4) *$Th(M)$ is independent, i.e. some formula has the independence property.*

Then $M^\lambda/\mathcal{D}$ is not $(\mu^*)^+$-saturated, where

$$\mu^* = \mu^*(\mathcal{D}) = \min\{\prod_{i<\lambda} \kappa_i/\mathcal{D} \; : \; \bigwedge_i \kappa_i < \kappa \text{ and } \prod_i \kappa_i/\mathcal{D} \geq \kappa \text{ and } \prod_i 2^{\kappa_i}/\mathcal{D} < 2^{\prod_i \kappa_i/\mathcal{D}}\}$$

Remark 5.9. *(a) If $\mathcal{D}$ is (λ, κ)-regular, then there is such $\mu^*(\mathcal{D})$ and it is $\leq 2^\lambda$, as $2^\lambda = \prod_i \kappa_i/\mathcal{D}$ for some $\kappa_i < \kappa$. Otherwise, it is not immediately clear whether μ^* is well defined. (b) In 5.8, could just use $< \kappa$-universal.*

Proof. (of 5.8) Let $\bar\kappa = \langle \kappa_i : i < \lambda \rangle$ be a sequence witnessing $\mu^*(\mathcal{D})$, i.e. $\prod_i \kappa_i/\mathcal{D} = \mu^*(\mathcal{D})$. Without loss of generality, M is a model of the random graph, so $\tau(M) = \{R\}$ and xRy has the independence property. [1] For $i < \alpha < \kappa$, let $a_{\alpha,i} \in M$ be such that $\langle a_{\alpha,i}Rx : i < \alpha \rangle$ is an independent sequence of formulas, using the $(< \kappa)$- universality of M. For each i, choose $A_i \subseteq M$, $A_i = \{a_{\kappa_i,\epsilon} : \epsilon < \kappa_i\}$. We then choose $B_i \subseteq M$, $|B| = 2^{\kappa_i}$ such that for every $A \subseteq A_i$ there is $b = b_A^i \in B_i$ such that $(\forall a \in A)(aR^M b \iff a \in A)$. Let $\tau^* = \{R, P, Q\}$ where P, Q are unary predicates. Let M_i be the τ_*-model expanding M with $P^{M_i} = A_i$, $Q^{M_i} = B_i$. Let $N = M^\lambda/\mathcal{D}$, $N_* = \prod_{i<\lambda} M_i/\mathcal{D}$, so $N = N_* \upharpoonright \tau$. Let $A = P^{N_*}$, so $|A| = \mu^*$.

Now we claim there is $A' \subseteq A$ such that no $b \in N$ realizes $p(x) = \{aRx : a \in A'\} \cup \{\neg aRx : a \in A \setminus A'\}$, which is a consistent partial type in N (e.g. by Łoś' theorem, since $\{xRa : a \in P^{M_i}\}$ is independent in each M_i). First, notice that it suffices to show there is no realization $b \in Q^{N_*}$, since for every $b_1 \in N$ there is $b_2 \in Q^{N_*}$ such that $a \in P^N \implies aR^N b_1 \equiv aR^N b_2$ [by Łoś' theorem and the choice of M_i]. Second, $|\{A' : A' \subseteq A\}| = 2^{|P^{N_*}|} = \prod_i \kappa_i/\mathcal{D}$, whereas the number of $b \in Q^{N_*}$ is $|Q^{N_*}| = \prod_i 2^{\kappa_i}/\mathcal{D}$. Moreover, each such b determines an A' uniquely. But by the choice of $\langle \kappa_i : i < \lambda \rangle$, $|Q^{N_*}| < 2^{|P^{N_*}|}$. This completes the proof. $\qquad\square$

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[1] It is straightforward to translate omission of a type in the random graph to omission of a type in a formula with the independence property, see e.g. [13] §5.

6. Flexible and ok ultrafilters

In this section we verify that "OK" may be substituted for "flexible" in our theorems below. We repeat here thanks to Kunen for suggesting the connection to the first author. The direction 6.1 $(2) \to (1)$ is in Malliaris' paper [13], Claim 8.4.

Claim 6.1. *Suppose that $\mathcal{D}$ is an $\aleph_1$-incomplete ultrafilter on I. Then the following are equivalent.*

(1) $\mathcal{D}$ *is λ-O.K.*
(2) $\mathcal{D}$ *is λ-flexible.*

Proof. $(1) \to (2)$ Let M be given with $(\mathbb{N}, <) \preceq M$ and let $h_0 \in {}^I M$ be any $\mathcal{D}$-nonstandard integer. Let $\{Z_n : n < \omega\} \subseteq \mathcal{D}$ witness that $\mathcal{D}$ is $\aleph_1$-incomplete. Without loss of generality, $n < n' \implies Z_n \supseteq Z_{n'}$. Let $h_1 \in {}^I \mathbb{N}$ be given by $h_1(t) = \max\{n : t \in Z_n\}$. Define $h \in {}^I \mathbb{N}$ by

$$h(t) = \min\{h_0(t), h_1(t)\}$$

Then for each $n \in \mathbb{N}$, $X_n := \{t : h(t) \geq n\} \in \mathcal{D}$ and $X_n \subseteq Z_n$, thus $\bigcap\{X_n : n \in \mathbb{N}\} = \emptyset$.

Define a function $f : \mathcal{P}_{\aleph_0}(\lambda) \to \mathcal{D}$ by $f(u) = X_{|u|}$. As $\mathcal{D}$ is λ-OK, we may choose g to be a multiplicative refinement of f, and consider $\mathbf{Y} = \{Y_i : i < \lambda\}$ given by $Y_i = g(\{i\})$.

First, we verify that $\mathbf{Y}$ is a regularizing family, by showing that each $t \in I$ can only belong to finitely many elements of $\mathbf{Y}$. Given $t \in I$, let $m = h(t) + 1 < \omega$, so $t \notin X_m$. Suppose there were $i_1 < \cdots < i_m < \lambda$ such that $t \in g(\{i_1\}) \cap \cdots \cap g(\{i_m\})$. As g is multiplicative and refines f, this would imply $t \in g(\{i_1, \ldots i_m\}) \subseteq f(\{i_1, \ldots i_m\}) = X_m$, a contradiction. Thus $\mathbf{Y}$ is a regularizing family. Moreover, as t was arbitrary, we have shown that

$$|\{i < \lambda : t \in g(\{i\})\}| \leq h(t) \leq h_0(t)$$

and thus that $\mathbf{Y}$ is a regularizing family below h_0. As h_0 was an arbitrary nonstandard integer, this completes the proof.

$(2) \to (1)$ Let $f : \mathcal{P}_{\aleph_0}(\lambda) \to \mathcal{D}$ be such that $|u| = |v| \implies f(u) = f(v)$, and we will construct a multiplicative refinement for f. Let $\langle Z_n : n < \omega \rangle$ witness the $\aleph_1$-incompleteness of $\mathcal{D}$, and as before, we may assume $n < n' \implies Z_n \supseteq Z_{n'}$. For each $t \in I$, let $\rho \in {}^I \mathbb{N}$ be given by $\rho(t) = \max\{n \in \mathbb{N} : t \in f(n) \cap Z_n\}$, which is well defined by the choice of the Z_n. Now for each $m \in \mathbb{N}$, recalling that m is an element of $[\lambda]^m$ via the convention $m = \{0, \ldots, m-1\}$,

$$\{t \in I : \rho(t) > m\} \supseteq \bigcap\{f(n) \cap Z_n : n \leq m\} \in \mathcal{D}$$

so ρ is $\mathcal{D}$-nonstandard. Applying the hypothesis of flexibility, let $\{Y_i : i < \lambda\}$ be a λ-regularizing family below ρ. Let $g : \mathcal{P}_{\aleph_0}(\lambda) \to \mathcal{D}$ be given by $g(\{i\}) = f(\{i\}) \cap Y_i$ and for $|u| > 1$, $g(u) = \bigcap\{g(\{i\}) : i \in u\}$. Thus g is multiplicative, by construction. Let us show that it refines f. Given any $n < \omega$ and $i_1 < \cdots < i_n < \lambda$, observe that by definition of "below ρ" we have $t \in Y_{i_1} \cap \cdots \cap Y_{i_n} \implies \rho(t) \geq n$. Applying this fact and the definitions of g and f,

$$g(\{i_1, \ldots i_n\}) \subseteq \bigcap\{Y_{i_j} : 1 \leq j \leq n\} \subseteq \{t \in I : \rho(t) \geq n\} \subseteq f(n) = f(\{i_1, \ldots i_n\})$$

thus g refines f, which completes the proof.   □

7. For κ measurable, $\lambda \geq 2^\kappa$ there is $\mathcal{D}$ on λ flexible but not good for T_{rg}

We begin by characterizing when flexibility is preserved under products of ultrafilters, Definition 2.11. The first observation says that λ-flexibility of the first ultrafilter ensures there are λ-regularizing families in $\mathcal{D}$ below certain nonstandard integers, namely those which are a.e. $\mathcal{D}_1$-nonstandard.

Observation 7.1. *Let* $\lambda, \kappa \geq \aleph_0$ *and let* $\mathcal{D}_1, E$ *be ultrafilters on* λ, κ *respectively. Let* $\mathcal{D} = \mathcal{D}_1 \times E$ *be the product ultrafilter on* $\lambda \times \kappa$. *Suppose that we are given* $n_* \in {}^{\lambda \times \kappa}\mathbb{N}$ *such that:*

(1) $n \in \mathbb{N} \implies \{(s,t) \in \lambda \times \kappa \ : \ n_*(s,t) > n\} \in \mathcal{D}$, *i.e.* n_* *is* $\mathcal{D}$-*nonstandard*
(2) $N := \{t \in \kappa : n \in \mathbb{N} \implies \{s \in \lambda : n_*(s,t) > n\} \in \mathcal{D}_1\} \in E$
    i.e. E-*almost all of its projections are* $\mathcal{D}_1$-*nonstandard*

Then (a) $\implies$ (b), *where:*

(a) $\mathcal{D}_1$ *is* λ-*flexible*
(b) *there is a regularizing set* $\langle X_i : i < \lambda \rangle \subseteq \mathcal{D}$ *below* n_*,
    i.e. such that for all $(s,t) \in \lambda \times \kappa$, $|\{i < \lambda : (s,t) \in X_i\}| \leq n_*(s,t)$

Proof. For each $t \in N$, let $\langle X_i^t : i < \lambda \rangle \subseteq \mathcal{D}_1$ be a regularizing family below $n_*(-,t)$, that is, such that for each $s \in \lambda$, $|\{i < \lambda : s \in X_i^t\}| \leq n_*(s,t)$. Such a family is guaranteed by the λ-flexibility of $\mathcal{D}_1$ along with the definition of N, since the latter ensures that $n_*(-,t) \in {}^{\lambda}\lambda$ is $\mathcal{D}_1$-nonstandard. Now define $\langle X_i : i < \lambda \rangle$ by $X_i = \{(s,t) : s \in X_i^t\}$. We verify that:

- $\langle X_i : i < \lambda \rangle \subseteq \mathcal{D}$, as $\{t \in \kappa : \{s \in \lambda : (s,t) \in X_i\} \in \mathcal{D}_1\} \supseteq N$ and $N \in E$ by hypothesis.
- $\langle X_i : i < \lambda \rangle$ is below n_*, since for each $(s,t) \in \lambda \times \kappa$,
  $|\{i : (s,t) \in X_i\}| = |\{i : (s,t) \in X_i^t\}| \leq n_*(s,t)$ by construction.

This completes the proof. $\qquad\square$

The next claim shows that $\mathcal{D}$-nonstandard integers project E-a.e. to $\mathcal{D}_1$-nonstandard integers precisely when the second ultrafilter E is at least $\aleph_1$-complete.

Claim 7.2. *Let* $\lambda, \kappa \geq \aleph_0$ *and let* $\mathcal{D}_1, E$ *be uniform ultrafilters on* λ, κ *respectively. Let* $\mathcal{D} = \mathcal{D}_1 \times E$ *be the product ultrafilter on* $\lambda \times \kappa$. *Then the following are equivalent.*

(1) *If* $n_* \in {}^{\lambda \times \kappa}\mathbb{N}$ *is such that* $n \in \mathbb{N} \implies \{(s,t) \in \lambda \times \kappa \ : \ n_*(s,t) > n\} \in \mathcal{D}$,
    then $N \in E$ *where* $N := \{t \in \kappa : n \in \mathbb{N} \implies \{s \in \lambda : n_*(s,t) > n\} \in \mathcal{D}_1\}$.
(2) E *is* $\aleph_1$-*complete.*

Proof. (1) $\to$ (2) Suppose E is not $\aleph_1$-complete, so it is countably incomplete and we can find $\langle X_n : n < \omega \rangle \subseteq E$ such that $\bigcap\{X_n : n \in \omega\} = \emptyset \mod \mathcal{D}$. Without loss of generality, $n < \omega \to X_{n+1} \subsetneq X_n$. Let $n_* \in {}^{\lambda \times \kappa}\mathbb{N}$ be given by:

$$ t \in \kappa \wedge t \in X_n \setminus X_{n+1} \implies n_*(-,t) = {}^{\lambda}\{n\} $$

Then n_* is $\mathcal{D}$-nonstandard but its associated set N is empty (as a subset of κ, so a fortiori empty modulo $\mathcal{D}$).

(2) $\to$ (1) Suppose on the other hand that E is $\aleph_1$-complete, and let some $\mathcal{D}$-nonstandard n_* be given. For each $n \in \mathbb{N}$, define $X_n = \{t \in \kappa : \{s \in \lambda : n_*(s,t) > n\} \in \mathcal{D}_1\}$. Then by completeness, $N \supseteq \bigcap\{X_n : n \in \mathbb{N}\} \in E$. $\qquad\square$

Corollary 7.3. *Let* $\lambda, \kappa \geq \aleph_0$ *and let* $\mathcal{D}_1, E$ *be ultrafilters on* λ, κ *respectively where* $\kappa > \aleph_0$ *is measurable. Let* $\mathcal{D} = \mathcal{D}_1 \times E$ *be the product ultrafilter on* $\lambda \times \kappa$. *Then:*

(1) *If* $\mathcal{D}_1$ *is* λ-*flexible and* E *is* $\aleph_1$-*complete, then* $\mathcal{D}$ *is* λ-*flexible.*
(2) *If* $\lambda \geq \kappa$ *and* $\mathrm{lcf}(\aleph_0, \mathcal{D}_1) \geq \lambda^+$, *then* $\mathrm{lcf}(\aleph_0, \mathcal{D}) \geq \lambda^+$, *so in particular,* $\mathcal{D} = D \times E$ *will* λ^+-*saturate any countable stable theory.*

Proof. (1) By Claim 7.2 and Observation 7.1.

(2) Let us show that the $\mathcal{D}_1$-nonstandard integers are cofinal in the $\mathcal{D}$-nonstandard integers. Let $M = (\mathbb{N}, <)^{\lambda}/\mathcal{D}_1$, $N = M^{\kappa}/E$. Let $n_* \in N$ be $\mathcal{D}$-nonstandard. By Claim 7.2, the set

$X = \{t \in \kappa : n_*(t)$ is a $\mathcal{D}_1$-nonstandard element of $M\} \in E$. Since $\operatorname{lcf}(\aleph_0, \mathcal{D}_1) \geq \lambda^+ > \kappa$, there is $m_* \in M$ which is $\mathcal{D}_1$-nonstandard and such that $t \in X \implies M \models n_*(t) > m_*$. Then the diagonal embedding of m_* in N will be $\mathcal{D}$-nonstandard but below n_*, as desired. The statement about stable theories follows by §4, Theorem G and Theorem 4.2(2)→(1).   □

Theorem 7.4. *Assume $\aleph_0 < \kappa < \lambda$, $2^\kappa \leq \lambda$, κ measurable. Then there exists a regular ultrafilter $\mathcal{D}$ on λ such that $\mathcal{D}$ is λ-flexible, yet for any model M of the theory of the random graph, $M^\lambda/\mathcal{D}$ is not $(2^\kappa)^+$-saturated. However, $\mathcal{D}$ will λ^+-saturate any countable stable theory.*

A fortiori, $\mathcal{D}$ is neither good nor good for equality, and it will fail to $(2^\kappa)^+$-saturate any unstable theory.

Proof. Let $\mathcal{E}$ be a uniform $\aleph_1$-complete ultrafilter on κ. Let $\mathcal{D}_1$ be any λ-flexible (thus, λ-regular) ultrafilter on λ, e.g. a regular λ^+-good ultrafilter on λ. Let $\mathcal{D} = \mathcal{D}_1 \times \mathcal{E}$ be the product ultrafilter on $I = \lambda \times \kappa$. Then $|I| = \lambda$, and we have that $\mathcal{D}$ is λ-flexible, thus regular, by Corollary 7.3(1). $\mathcal{D}$ saturates countable stable theories by 7.3(2).

Since for any model M,

$$\prod_{\lambda \times \kappa} M/(\mathcal{D}_1 \times \mathcal{E}) \cong \prod_\kappa \left(\prod_\lambda M/\mathcal{D}_1 \right) / \mathcal{E}$$

the right hand side shows, by Claim 5.2, that the resulting ultrafilter will not $(2^\kappa)^+$-saturate the random graph.

Finally, the "a fortiori" clause holds by Theorem 4.2.   □

Corollary 7.5. *In the construction just given, by Claim 7.2, $\operatorname{lcf}(\aleph_0, \mathcal{D}) = \operatorname{lcf}(\aleph_0, \mathcal{D}_1) \geq \lambda^+$ since $\mathcal{D}_1$ is λ^+-good and the nonstandard integers of $\mathcal{D}_1$ are cofinal in the nonstandard integers of $\mathcal{D}$. Thus consistently, a regular ultrafilter on $\lambda > \kappa$ may have large lower cofinality of $\aleph_0$ while failing to $(2^\kappa)^+$-saturate the random graph.*

By [20].VI.4, a necessary condition for a regular ultrafilter $\mathcal{D}$ on λ to saturate some unstable theory is that $\operatorname{lcf}(\aleph_0, \mathcal{D}) \geq \lambda^+$; Corollary 7.5 shows it is not sufficient.

8. $M^\lambda/\mathcal{D}$ IS NOT λ^{++}-SATURATED FOR λ REGULAR AND $Th(M)$ NON-SIMPLE

In this section we prove that there is a loss of saturation in ultrapowers of non-simple theories. As mentioned above, this is a new proof of a result from [20].VI.4.7, which reflects an interest (visible elsewhere in this paper e.g. 10.1) in controlling the distribution of sets of indices.

Definition 8.1. *A first-order theory has the* tree property (*more precisely, the* 2-tree property) *if there is a formula $\varphi(x; \overline{y})$ which does, where this means that in any $\aleph_1$-saturated model $M \models T$ there exist $\langle \overline{a}_\eta : \eta \in {}^{\aleph_0 >}2 \rangle$ such that:*

(1) *for $\eta \in {}^{\aleph_0 >}2$ and $i, j < \omega$, $\{\varphi(x; \overline{a}_{\eta^\frown i}), \varphi(x; \overline{a}_{\eta^\frown j})\}$ is inconsistent.*
(2) *for $\eta \in {}^{\aleph_0}2$, $\{\varphi(x; \overline{a}_{\eta|_i}) : i < \aleph_0\}$ is a consistent partial φ-type.*

If T has the tree property, we say it is not simple, *otherwise it is* simple.

We will use $\trianglelefteq$ to indicate comparability in the tree, i.e. $\eta \trianglelefteq \rho$ means η is before ρ in the partial tree order.

Remark 8.2. *Definition 8.1 remains agnostic about whether $\{\varphi(x; \overline{a}_\eta), \varphi(x; \overline{a}_\rho)\}$ is consistent when η, ρ are incomparable but not necessarily successors of the same node. Theorem H says, in some*

sense, that we may additionally assume (perhaps after changing the formula) the answer to this question is either always yes or always no.

Theorem H. ([20] III.7.11, in our language, or [21]) *If T is not simple then T contains either a formula with TP_1 (equivalently SOP_2) or a formula with TP_2, where this means:*

(1) *T has TP_1, or equivalently SOP_2, if there is a formula $\varphi(x; \overline{y})$ which does, where this means that in any $\aleph_1$-saturated model $M \models T$ there exist $\langle \overline{a}_\eta : \eta \in {}^{\aleph_0>}2 \rangle$ such that:*
   (a) *for $\eta, \rho \in {}^{\aleph_0>}2$ incomparable, i.e. $\neg(\eta \trianglelefteq \rho) \wedge \neg(\rho \trianglelefteq \eta)$, we have that $\{\varphi(x; \overline{a}_\eta), \varphi(x; \overline{a}_\rho)\}$ is inconsistent.*
   (b) *for $\eta \in {}^{\aleph_0}2$, $\{\varphi(x; \overline{a}_{\eta|_i}) : i < \aleph_0\}$ is a consistent partial φ-type.*
(2) *T has TP_2 if there is a formula $\varphi(x; \overline{y})$ which does, where this means that in any $\aleph_1$-saturated model $M \models T$ there exists an array $A = \{\overline{a}^i_j : i < \omega, j < \omega\}$ of tuples, $\ell(\overline{a}^i_j) = \ell(\overline{y})$ such that: for any finite $X \subseteq \omega \times \omega$, the partial type*

$$\{\varphi(x; \overline{a}^i_j) : (i, j) \in X\}$$

is consistent if and only if

$$(i, j) \in X \wedge (i', j') \in X \wedge i = i' \implies j = j'$$

Observe that the tree property is strong enough to ensure that all instances at a given *level* are inconsistent:

Observation 8.3. *Let T have the tree property and M be a λ^{++}-saturated model of T. Then there is a formula φ in T and a sequence $\langle \overline{a}_\eta : \eta \in {}^{\lambda^+>}\lambda \rangle$ of elements of M such that:*

(1) *for $\gamma < \lambda$ and $\eta, \rho \in {}^\gamma\lambda$, i.e. $\lg(\eta) = \lg(\rho)$, $\{\varphi(x; \overline{a}_\eta), \varphi(x; \overline{a}_\rho)\}$ is inconsistent.*
(2) *for $\eta \in {}^\omega 2$, $\{\varphi(x; \overline{a}_{\eta|_i}) : i < \omega\}$ is a consistent partial φ-type.*

Proof. Apply Theorem H and in either case, we argue by compactness that we can find such a tree for $\{\eta : \eta \in {}^n m\}$ for n, m finite. If some φ has TP_1, this is immediate from the definition. Otherwise, some φ has TP_2, so let A be an array $\{\overline{a}_{i,j} : i < \omega, j < \omega\}$ as in Theorem H(2). For each $k < n$, let $\{X^k_\ell : \ell \in {}^k m\}$ be a partition of $\{\overline{a}_{k,j} : j < \omega\}$ into disjoint infinite sets. Then choose $\langle \overline{a}_\eta : \eta \in {}^n m \rangle$ so that $\overline{a}_\emptyset = \overline{a}_{0,0}$ and for each $\eta \in {}^{n-1}m$, $\{\overline{a}_{\eta^\frown i} : i < m\}$ are distinct elements of $X^{\lg(\eta)}_\eta$. □

Fact 8.4. *Suppose that $\mathcal{D}$ is a regular ultrafilter on λ, $\lambda = \lambda^{<\lambda}$ or just $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ (see [6]). Let $\kappa = \aleph_0$. Then for each $\epsilon < \lambda$ we may choose a sequence of sets $\overline{u}_\epsilon = \langle u_{\epsilon,\alpha} : \alpha < \lambda^+ \rangle$ such that:*

(1) *$u_{\epsilon,\alpha} \subseteq \alpha$*
(2) *$|u_{\epsilon,\alpha}| < \lambda$*
(3) *$\alpha \in u_{\epsilon,\beta} \implies u_{\epsilon,\alpha} = u_{\epsilon,\beta} \cap \alpha$*
(4) *if $u \subseteq \lambda^+$, $|u| < \kappa$ then*

$$\{\epsilon < \lambda : \exists\alpha(u \subseteq u_{\epsilon,\alpha})\} \in \mathcal{D}$$

Proof. By Kennedy-Shelah-Vaananen [7] p. 3 this is true when λ satisfies the stated hypothesis and $\mathcal{D}$ is regular. Note that as briefly mentioned there, in the case of singular λ, the claim may be true; but it is also consistent that it may fail. □

Claim 8.5. *Suppose $\mathcal{D}$ and λ satisfy the hypotheses of Fact 8.4. Suppose T is not simple and let $M \models T$ be λ^{++}-saturated. Then $M^\lambda/\mathcal{D}$ is not λ^{++}-saturated, and in particular is not λ^{++}-saturated for φ-types for some formula φ with the tree property.*

Proof. Let φ and $\langle \bar{a}_\eta : \eta \in {}^{\lambda^+ >}\lambda \rangle$ be given by Observation 8.3, so this is a tree in the index model M. Let $\bar{u}$ be given by Fact 8.4. For each $\epsilon < \lambda$, $\alpha < \lambda^+$ let $\eta_{\epsilon,\alpha}$ list $u_{\epsilon,\alpha} \cup \{\alpha\}$ in increasing order. So $\eta_{\epsilon,\alpha}$ is an element of ${}^{(\lambda>)}\lambda^+$, and by our choice of tree,

(1) $\qquad\qquad \mathrm{otp}(u_{\epsilon,\alpha}) = \mathrm{otp}(u_{\epsilon,\beta}) \implies \{\varphi(x, \bar{a}_{\eta_{\epsilon,\alpha}}), \varphi(x, \bar{a}_{\eta_{\epsilon,\beta}})\}$ is inconsistent.

From this tree in the index model, we now build an omitted type in the ultrapower. For each $\alpha < \lambda^+$, define a function f_α from I to ${}^{\mathrm{lg}(y)}\mathfrak{C}$ by $f_\alpha(\epsilon) = \bar{a}_{\eta_{\epsilon,\alpha}}$. Now, for each $\alpha_0 < \alpha_1 < \lambda^+$, the set

$$\{\epsilon < \lambda : (\exists \alpha)(\{\alpha_0, \alpha_1\} \subseteq u_{\epsilon,\alpha}\} \in \mathcal{D}$$

which implies that for each such α, $u_{\epsilon,\alpha} \cap \alpha_1 = u_{\epsilon,\alpha_1}$. Thus, $\alpha_0 \in u_{\epsilon,\alpha_1}$ on this large set. More generally, for $n < \omega$ and $\alpha_0 < \cdots < \alpha_n < \lambda^+$, the set

$$\{\epsilon < \lambda : \ell < n \implies \alpha_\ell \in u_{\epsilon,\alpha_{\ell+1}}\} \in \mathcal{D}$$

Thus for any $n < \omega$ and any $\alpha_0 < \cdots < \alpha_n < \lambda^+$, $\{\epsilon < \lambda : \eta_{\epsilon,\alpha_0} \triangleleft \eta_{\epsilon,\alpha_1} \triangleleft \cdots \triangleleft \eta_{\epsilon,\alpha_n}\} \in \mathcal{D}$ and therefore

$$\{\epsilon < \lambda : M \models \exists x \bigwedge_\ell \varphi(x; a_{\epsilon,\alpha_\ell})\} \in \mathcal{D}$$

Since $n, \alpha_0, \ldots \alpha_n$ were arbitrary, this shows

$$p = \{\varphi(x; f_\alpha/\mathcal{D}) : \alpha < \lambda^+\}$$

is a consistent partial type.

Assume for a contradiction that p is realized, say by $f_* \in {}^\lambda M$. For each $\alpha < \lambda^+$, define $J_\alpha = \{\epsilon < \lambda : M \models \varphi(f_*(\epsilon), f_\alpha(\epsilon))\}$ to be the set of indices on which the supposed realization satisfies the α-th formula of the type. We assumed $f_* \models p$, so $\alpha < \lambda^+$ implies $J_\alpha \in \mathcal{D}$ thus $J_\alpha \neq \emptyset$ and we may choose some $\epsilon_\alpha \in J_\alpha$. Since $\lambda^+ > \lambda$ is regular, there is some $\epsilon_* < \lambda$ such that $|S_0| = \lambda^+$, where $S_0 = \{\alpha < \lambda^+ : \epsilon_\alpha = \epsilon_*\} \subseteq \lambda^+$. As the set $\{\mathrm{otp}(u_{\epsilon,\alpha}) : \alpha \in S_0\}$ has cardinality λ, there is $\gamma < \lambda$ such that $|S_1| = \lambda^+$, where

$$S_1 = \{\alpha \in S_0 : \mathrm{otp}(u_{\epsilon,\alpha}) = \gamma\}$$

In particular, $|S_1| \geq 2$, so let $\alpha \neq \beta$ be distinct elements of S_1. Then by choice of ϵ_*, $f(\epsilon_*)$ realizes

(2) $\qquad\qquad \{\varphi(x, f_\alpha(\epsilon_*)), \varphi(x, f_\alpha(\epsilon_*))\}$ i.e. $\{\varphi(x, \bar{a}_{\eta_{\epsilon_*,\alpha}}), \varphi(x, \bar{a}_{\eta_{\epsilon_*,\beta}})\}$

But (1) contradicts the consistency of (2). So p is not realized, which completes the proof. $\qquad\square$

Conclusion 8.6. *Let $\lambda = \lambda^{<\lambda}$ and let $\mathcal{D}$ be a regular ultrafilter on λ. If T is not simple and $M \models T$, then there is φ such that $M^\lambda/\mathcal{D}$ is not λ^{++}-saturated for φ-types.*

Remark 8.7. *Let D_1, D_2 be ultrafilters on λ, κ respectively and suppose that $\kappa = \kappa^{<\kappa}$. If $\lambda \geq \kappa^+$ and D_2 is regular, then by Theorem F(5) and Conclusion 8.6, $D_1 \times D_2$ cannot be good for equality.*

9. κ-COMPLETE NOT κ^+-COMPLETE IMPLIES NO (κ, κ)-CUTS

Claim 9.1. *Suppose that $\mathcal{E}$ is a κ-complete but not κ^+-complete ultrafilter on I and M_1 is a κ^+-saturated model in which a linear order L and tree T are interpreted. Then in $M_2 = M_1^I/\mathcal{E}$:*

  (a) *the linear order L^{M_2} has no (κ, κ)-cut, and moreover no (θ, σ)-cut for $\theta, \sigma < \kappa$ both regular.*
  (b) *the tree T^{M_2} has no branch (i.e. maximal linearly ordered set) of cofinality $\leq \kappa$.*

Remark 9.2. *In the statement of Claim 9.1:*

  (1) *in (a), the κ-saturation of M_1 is necessary in the following sense: if there is a sequence $\bar{\theta} = \langle \theta_t : t \in I \rangle$, which certainly need not be distinct, such that M has a (θ_t, θ_t)-cut for each $t \in I$ and $(\prod_{t \in I} \theta_t, <_\mathcal{E})$ has cofinality κ, then the conclusion of Claim 9.1(a) is false.*

(2) *By this Claim, we may add to the conclusion of Theorem 7.4 that $(\kappa, \kappa) \notin \mathcal{C}(\mathcal{E})$, Definition 2.13, since in that Theorem the ultrafilter E is a κ-complete uniform ultrafilter on κ and thus not κ^+-complete.*

Proof. (of Claim 9.1)

(a) The "moreover" clause in (a) follows from the fact that M_1 and M_2 are $L_{\kappa,\kappa}$-equivalent, by the completeness of $\mathcal{E}$, and the hypothesis on saturation of M_1.

So we consider a potential (κ, κ)-cut in M_2, i.e. a (κ, κ)-pre-cut given by $\langle f_\alpha : \alpha < \kappa \rangle$, $\langle g_\alpha : \alpha < \kappa \rangle$ where if $\alpha < \beta < \kappa$ then

$$M_2 \models (f_\alpha/\mathcal{E}) <_L (f_\beta/\mathcal{E}) <_L < (g_\beta/\mathcal{E}) <_L (g_\alpha/\mathcal{E})$$

For $0 < \gamma < \kappa$ let

$$A_\gamma = \{t : \text{ if } \alpha < \beta < \gamma \text{ then } M_1 \models f_\alpha(t) <_L f_\beta(t) <_L g_\beta(t) <_L g_\alpha(t)\}$$

Let $A_0 = A_1 = I$. Then $\overline{A} = \langle A_\gamma : \gamma < \kappa \rangle$ is a continuously decreasing sequence of elements of $\mathcal{E}$, i.e.:

- $\gamma_1 < \gamma_2 \implies A_{\gamma_1} \supseteq A_{\gamma_2}$
- for limit $\delta < \kappa$, $A_\delta = \bigcap\{A_\gamma : \gamma < \delta\}$
- each $A_\gamma \in \mathcal{E}$, by choice of the functions and κ-completeness

As we assumed $\mathcal{E}$ is κ-complete but not κ^+-complete, there is a sequence $\overline{B} = \langle B_\gamma : \gamma < \kappa \rangle$ of elements of $\mathcal{E}$ such that $\bigcap\{B_\gamma : \gamma < \kappa\} = \emptyset$. We may furthermore assume that $\overline{B}$ is a continuously decreasing sequence (if necessary, inductively replace B_δ by $\bigcap\{B_\gamma : \gamma < \delta\}$ using κ-completeness).

Thus given $\overline{A}, \overline{B}$, for each $t \in I$ we may define

$$\gamma(t) = \min\{\alpha : t \notin A_{\alpha+1} \cap B_{\alpha+1}\}$$

By choice of $\overline{B}$, $t \mapsto \gamma(t)$ is a well-defined function from I to κ, and $t \in A_{\gamma(t)} \cap B_{\gamma(t)}$. Recall that we want to show that our given (κ, κ)-sequence is not a cut. Choose $f_\kappa, g_\kappa \in {}^I M$ so that first, for each $t \in I$, $f_\kappa(t), g_\kappa(t) \in L^{M_1}$, and second, for each $t \in I$ and all $\alpha < \gamma(t)$,

$$M_1 \models f_\alpha(t) \leq_L f_\kappa(t) <_L g_\kappa(t) \leq_L g_\alpha(t)$$

This we can do by the choice of $\overline{A}$ as a continuously decreasing sequence (so the function values f_α, g_β below $\gamma(t)$ in each index model are correctly ordered) and the saturation hypothesis on M_1. Thus for each $\alpha < \kappa$, we have that

$$\{t : f_\alpha(t) \leq_L f_\kappa(t) <_L g_\kappa(t) \leq_L g_\alpha(t)\} \supseteq A_{\alpha+1} \cap B_{\alpha+1} \in \mathcal{E}$$

which completes the proof.

(b) Similar proof, but we only need to use one sequence $\langle f_\alpha : \alpha < \kappa \rangle$ which we choose to potentially witness that the cofinality of the branch is at most κ. $\square$

We prove a related fact for normal filters, Definition 2.6.

Claim 9.3. *Assume $\mathcal{E}$ is a normal filter on λ and M is a λ^+-saturated dense linear order. Then $M^I/\mathcal{E}$ is λ^+-saturated.*

Proof. Suppose that $\langle f_\alpha/\mathcal{E} : \alpha < \kappa_1 \rangle$ is increasing in $M^I/\mathcal{E}$, and $\langle g_\beta/\mathcal{E} : \beta < \kappa_2 \rangle$ is decreasing in $M^I/\mathcal{E}$, with $\kappa_1, \kappa_2 \leq \lambda$ and $f_\alpha/\mathcal{E} < g_\beta/\mathcal{E}$ for $\alpha < \kappa_1, \beta < \kappa_2$. Let

$$X_{\alpha,\beta} = \{t \in \lambda : f_\alpha(t) < g_\beta(t)\} \in \mathcal{E}$$

for $\alpha < \beta < \lambda$. Without loss of generality, suppose $\kappa_1 \le \kappa_2$. For each $\beta < \kappa_2$, let

$$Y_\beta = \{\alpha \in \lambda : (\forall j < (1 + \alpha) \cap \beta)(j \in X_{\alpha,\beta})\} \in \mathcal{E}$$

by normality. For $\beta \ge \kappa_2$, let $Y_\beta = I$. Finally, define

$$Z = \{\beta \in \lambda : (\forall k < (1 + \beta) \cap \kappa_2)(j \in Y_\beta)\} \in \mathcal{E}$$

Now if $t \in Z$ (so t plays the role of β) we have that

$$p_t = \{f_\alpha(t) < x < g_t(t) : \alpha < t\}$$

is a consistent partial type, realized in M by the saturation hypothesis. Choose $h \in {}^\lambda M$ such that for each $t \in Z$, $h \models p_t$. Then h realizes the type. □

Note that Claim 9.3 implies by Fact 11.7 of the Appendix that the relevant $\mathcal{E}$ is good.

10. $\mathcal{E}$ NORMAL AND κ-COMPLETE ON κ IMPLIES A (κ^+, κ^+)-CUT

The use of the additional hypothesis "normal" in this section comes in Step 3 of Claim 10.1, and consequently in later results which rely on it. Recall Definition 2.6 and Fact 2.7.

Claim 10.1. *Assume κ measurable, $\mathcal{E}$ a normal κ-complete ultrafilter on κ, $\lambda \ge \kappa$, M_1 a λ-saturated model with $(L_M, <_M)$ a dense linear order. Let $M_2 = M_1^\kappa/\mathcal{E}$. Then L_{M_2} has a (κ^+, κ^+)-cut.*

Proof. The proof has several steps.

Step 1: Fixing sequences of indices. For each $\alpha < \kappa^+$ choose $\overline{\mathcal{U}}_\alpha = \langle u_{\alpha,\epsilon} : \epsilon < \kappa \rangle$ so that:

  (a) this sequence is $\subseteq$-increasing and continuous, and for each $\alpha < \kappa^+$, $u_{\alpha,0} = \emptyset$
  (b) for each $\epsilon < \kappa$, $|u_{\alpha,\epsilon}| < \kappa$
  (c) $\bigcup\{u_{\alpha,\epsilon} : \epsilon < \kappa\} = \alpha$
  (d) (for coherence) for $\beta < \alpha < \kappa^+$,

$$\beta \in u_{\alpha,\epsilon} \implies u_{\beta,\epsilon} \subseteq u_{\alpha,\epsilon}$$

Such a sequence will always exist as $|\alpha| \le \kappa$. [Details: Clearly such a sequence exists for $\alpha \le \kappa$: let $u_{\kappa,\epsilon} = \epsilon \cap \alpha$, so for arbitrary $\kappa \le \alpha < \kappa^+$, fixing a bijection to κ let $\overline{V}_\alpha = \langle v_{\alpha,\epsilon} : \epsilon < \kappa \rangle$ be the preimage of the sequence for κ. Having thus fixed, for each $\alpha < \kappa^+$, a sequence satisfying (a)-(c) we may then inductively pad these sequences to ensure coherence. For $\beta = 0$ and each $\epsilon < \kappa$, let $u_{\beta,\epsilon} = v_{\beta,\epsilon}$. For $0 < \beta < \kappa^+$, for each $\epsilon < \kappa$ let $u_{\beta,\epsilon} = \bigcup\{u_{\alpha,\epsilon} : \alpha \in v_{\beta,\epsilon}\}$, and note that this will preserve (b), (a), (c) and ensure (d).]

Step 2: The inductive construction of the (pre-)cut. We now construct a cut. We will first describe the construction, and then show that it is in fact a cut (i.e. we will show that we have indeed constructed a pre-cut, and that this pre-cut is not realized).

By induction on $\alpha < \kappa^+$ we will choose $f_\alpha, g_\alpha \in {}^\kappa(L_{M_1})$. The intention is that each $u_{\alpha,\epsilon}$ represents a small set of prior functions which we take into account when choosing the values for f_α, g_α at the index $\epsilon \in \kappa$.

At stage α, for each index $\epsilon < \kappa$ define

$$
\begin{aligned}
w_{\alpha,\epsilon} = \{\beta \in u_{\alpha,\epsilon} : \quad &\langle f_\gamma(\epsilon) : \gamma \in u_{\alpha,\epsilon} \cap (\beta + 1)\rangle \text{ is } <_{L(M_1)}\text{-increasing,} \\
&\langle g_\gamma(\epsilon) : \gamma \in u_{\alpha,\epsilon} \cap (\beta + 1)\rangle \text{ is } <_{L(M_1)}\text{-decreasing,} \\
&\text{and } f_\beta(\epsilon) <_{L(M_1)} g_\beta(\epsilon) \}
\end{aligned}
$$

Our aims in defining f_α, g_α are, on the one hand, to continue describing a pre-cut, and on the other, to stay as close to the linearly ordered $w_{\alpha,\epsilon}$ as possible, as we now describe. That is, for fixed α for each ϵ, we will choose f_α, g_α so that:

(e) For all $\beta \in w_{\alpha,\epsilon}$, $M_1 \models f_\beta(\epsilon) < f_\alpha(\epsilon) < g_\alpha(\epsilon) < g_\beta(\epsilon)$
   i.e. *locally we continue the pre-cut described by $w_{\alpha,\epsilon}$.*

(f) For all $\beta < \alpha$, neither $M_1 \models f_\beta(\epsilon) < f_\alpha(\epsilon) < g_\beta(\epsilon) < g_\alpha(\epsilon)$ nor $M_1 \models f_\alpha(\epsilon) < f_\beta(\epsilon) < g_\alpha(\epsilon) < g_\beta(\epsilon)$
   i.e. *the intervals are either nested or disjoint.*

(g) If $\gamma \in \alpha$ and $f_\gamma(\epsilon), g_\gamma(\epsilon)$ satisfy:
$$\beta \in w_{\alpha,\epsilon} \implies f_\beta(\epsilon) < f_\gamma(\epsilon) < g_\gamma(\epsilon) < g_\beta(\epsilon)$$
   then the intervals $[f_\alpha(\epsilon), g_\alpha(\epsilon)]_{L(M_1)}$, $[f_\beta(\epsilon), g_\beta(\epsilon)]_{L(M_1)}$ are disjoint
   i.e. *inside the pre-cut given by $w_{\alpha,\epsilon}$ we avoid any further refinements: we realize* exactly *the intitial segment given by $w_{\alpha,\epsilon}$.*

We will show in step 3 that by the hypothesis on κ, (a)-(g) imply the further condition that for each fixed $\alpha < \kappa^+$,

(h) For all $\beta < \alpha$, $\{\epsilon : f_\beta(\epsilon) < f_\alpha(\epsilon) < g_\alpha(\epsilon) < g_\beta(\epsilon)\} \in \mathcal{E}$, i.e. *the functions chosen will ultimately describe a pre-cut.*

At each index $\epsilon < \kappa$, we may choose $f_\alpha(\epsilon), g_\alpha(\epsilon)$ to satisfy (e),(f),(g) simply because L_{M_1} is dense and λ^+-saturated; the definition of $w_{\alpha,\epsilon}$ ensures (e) describes a pre-cut; and since (f) is inductively satisfied, (g) is possible.

Step 3: For $\beta < \alpha$, $f_\beta < f_\alpha < g_\alpha < g_\beta$. In this step we verify that for the objects constructed in the previous step, for each $\alpha < \kappa^+$ and all $\beta < \alpha$,
$$X_{\alpha,\beta} = \{\epsilon < \kappa : \beta \in u_{\alpha,\epsilon}, f_\beta(\epsilon) < f_\alpha(\epsilon) < g_\alpha(\epsilon) < g_\beta(\epsilon)\} \in \mathcal{E}$$

(By conditions (a)-(d) requiring $\beta \in u_{\alpha,\epsilon}$ does not affect membership in $\mathcal{E}$.) Suppose this is not the case, so let $\alpha < \kappa^+$ be minimal for which there is $\beta < \alpha$ with $X_{\alpha,\beta} \notin \mathcal{E}$, and having fixed α, let $\beta < \alpha$ be minimal such that $X_{\alpha,\beta} \notin \mathcal{E}$. For the remainder of this step we fix this choice of α, β.

Since $\beta < \alpha$, by construction (that is, by (e),(f),(g) of Step 2)
$$X_{\alpha,\beta} \subseteq \{\epsilon < \kappa : \beta \in u_{\alpha,\epsilon} \setminus w_{\alpha,\epsilon}\}$$

Define a function $\mathbf{x} : \kappa \to \kappa$ by
$$t \mapsto \max\{\epsilon \le t : \quad \langle f_\gamma(t) : \gamma \in u_{\alpha,\epsilon}\rangle \text{ is } <_{L(M_1)}\text{-increasing,}$$
$$\langle g_\gamma(t) : \gamma \in u_{\alpha,\epsilon}\rangle \text{ is } <_{L(M_1)}\text{-decreasing,}$$
$$\text{and } \gamma \in u_{\alpha,\epsilon} \implies f_\gamma(t) <_{L(M_1)} g_\gamma(t) \}$$

This is well defined by Step 1, condition (a): 0 belongs to the set on the righthand side, and by continuity, there are no new conditions at limits.

For each $\epsilon < \kappa$, the set $\{t < \kappa : \mathbf{x}(t) > \epsilon\} \in \mathcal{E}$. This is because:

(1) by (c) $|u_{\alpha,\epsilon}| < \kappa$
(2) by minimality of α, for any $\gamma < \gamma' < \alpha$ (e.g. any two elements of $u_{\alpha,\epsilon}$) we have that $X_{\gamma,\gamma'} \in \mathcal{E}$
(3) $\mathcal{E}$ is κ-complete
(4) by (a) $\epsilon' < \epsilon \implies u_{\alpha,\epsilon'} \subseteq u_{\alpha,\epsilon}$

Notice that for any $t < \kappa$, $\mathbf{x}(t) = t$ implies $u_{\alpha,t} = w_{\alpha,t}$. So if $\mathbf{x}(t) = t$ on an $\mathcal{E}$-large set, $X_{\alpha,\beta} \in \mathcal{E}$, which would finish the proof. Suppose, then, that $Y = \{t < \kappa : \mathbf{x}(t) < t\} \in \mathcal{E}$. By normality (Fact 2.7), there is $Z \subseteq Y$, $Z \in \mathcal{E}$ on which $\mathbf{x}(t) = \epsilon_*$ for some fixed $\epsilon_* < \kappa$. But this contradicts the first sentence of the previous paragraph.

These contradictions prove that for no α, β can it happen that $X_{\alpha,\beta} \notin \mathcal{E}$, which finishes the proof of Step 3.

Step 4: The pre-cut is not realized, i.e. it is indeed a cut. In this step we assume, for a contradiction, that there is $h \in {}^{\kappa}M_1$ such that for each $\alpha < \kappa^+$

$$f_\alpha/\mathcal{E} <_L h/\mathcal{E} <_L g_\alpha/\mathcal{E}$$

i.e. h realizes the type. Fixing such an h, let

$$A_\alpha = \{\epsilon < \kappa : f_\alpha(\epsilon) <_L h(\epsilon) <_L g_\alpha(\epsilon)\} \in \mathcal{E}$$

Since to each α we may associate a choice of index in A_α, by Fodor's lemma for some $\epsilon_* < \kappa$,

$$S_1 = \{\delta : \delta < \kappa^+, \ \mathrm{cf}(\delta) = \kappa, \ \epsilon_* \in A_\delta\}$$

is stationary. Furthermore, since $|u_{\epsilon_*,\alpha}| < \kappa$, there is some $w_* \subseteq u_{\epsilon,\alpha}$ for which

$$S_2 = \{\delta \in S_1 : w_{\epsilon_*,\delta} = w_*\} \subseteq \kappa^+$$

is stationary. Let $\delta_* \in S_2$ be such that $|\delta_* \cap S_2| = \kappa$. As $|w_{\epsilon_*,\delta_*}| \leq |u_{\epsilon_*,\delta_*}| < \kappa$, we may choose $\gamma_* \in S_2 \cap \{\delta_* \setminus w_{\epsilon_*,\delta_*}\}$.

Now $w_{\epsilon_*,\delta_*} = w_{\epsilon_*,\gamma_*} = w_*$ since $\delta_*, \gamma_* \in S_2$, and note $\gamma_* < \delta_*$. The definition of the sets w (here, w_*) and Step 3, condition (e) means that when choosing $f_{\delta_*}(\epsilon_*), g_{\delta_*}(\epsilon_*)$ we would have ensured that

$$\beta \in w_* \implies f_\beta(\epsilon_*) < f_{\delta_*}(\epsilon_*) < g_{\delta_*}(\epsilon_*) < g_\beta(\epsilon_*)$$

and likewise that

$$\beta \in w_* \implies f_\beta(\epsilon_*) < f_{\gamma_*}(\epsilon_*) < g_{\gamma_*}(\epsilon_*) < g_\beta(\epsilon_*)$$

On the other hand, $\gamma_* < \delta_*$, and $\gamma_* \notin w_*$. So when choosing $f_{\gamma_*}(\epsilon), g_{\gamma_*}(\epsilon)$, Step 3, condition (g) would have meant we chose the intervals $[f_{\delta_*}(\epsilon_*), g_{\delta_*}(\epsilon_*)]_{L(M_1)}$, $[f_{\gamma_*}(\epsilon_*), g_{\gamma_*}(\epsilon_*)]_{L(M_1)}$ to be disjoint.

But we also know that $\gamma_*, \delta_* \in S_1$, so $h(\epsilon_*)$ must belong to both intervals. This contradiction completes Step 4 and the proof. $\square$

Remark 10.2. *We know that if D is any ultrafilter on κ and M is a model whose theory is not simple, then M^κ/D is not κ^{++}-saturated. Still, Claim 10.1 gives more precise information about the size of the cut: we are guaranteed a cut of type (κ^+, κ^+) as opposed to e.g. (κ^+, κ). On the importance of symmetric cuts, see [17].*

Claim 10.3. *Assume κ measurable, $\mathcal{E}$ a κ-complete filter on κ, $\lambda \geq \kappa$, M_1 a λ-saturated model with $(L_M, <_M)$ a dense linear order. Let $M_2 = M_1^\kappa/\mathcal{E}$. Then L_{M_2} has no (θ, σ)-cut with $\theta < \kappa$ and $\sigma < \lambda$.*

Proof. Suppose for a contradiction that there were such a cut given by $\langle f_\alpha : \alpha < \theta \rangle$, $\langle g_\beta : \beta < \sigma \rangle$ with $\alpha_1, \alpha_2 < \theta, \beta_1, \beta_2 < \sigma \implies \{t : M_1 \models f_{\alpha_1}(t) <_L f_{\alpha_2}(t) <_L g_{\alpha_2}(t) <_L g_{\alpha_1}(t)\} \in \mathcal{E}$. Expand the language to add constants $\{c_\alpha : \alpha < \theta\}$ where in the tth copy of the index model M_1, denoted $M_1[t]$, c_α is interpreted as $f_\alpha(t)$. Then in the ultrapower (which in the expanded language is an ultraproduct), $\langle c_\alpha : \alpha < \theta \rangle$ forms the lower half of the supposed cut. For each $\beta < \sigma$, the set

$$A_\beta := \bigcap \{\{t \ : \ M_1[t] \models c_\alpha <_L g_\beta[t]\} : \alpha < \theta\} \in \mathcal{E}$$

by κ-completeness.

But recall that M_1 is a λ-saturated model, and $\sigma < \lambda$. Since for each t, we have

$$|\{\beta < \sigma : t \in A_\beta\}| \leq \sigma < \lambda$$

we may choose a new element $h \in {}^I M_1$ so that for each t, $h(t)$ satisfies $\alpha < \theta \implies M_1[t] \models c_\alpha <_L h(t)$ and $t \in A_\beta \implies M_1[t] \models h(t) <_L g_\beta(t)$. By Łos' theorem h realizes our cut, which is the desired contradiction. □

In a forthcoming paper the authors have shown that:

Theorem I. (Malliaris and Shelah [17]) *If $\mathcal{D}$ is a regular ultrafilter on λ which saturates some theory with SOP_2, and M is a model of linear order, then among other things:*

(1) *for all $\mu \leq \lambda$ $M^\lambda/\mathcal{D}$ has no (μ, μ)-cut,*
(2) *for all $\mu \leq \lambda$ there is at most one $\rho \leq \lambda$ such that $M^\lambda/\mathcal{D}$ has a (μ, ρ)-cut*

Conclusion 10.4. *Let $\kappa < \lambda$ and suppose κ is measurable. Then there exists a regular ultrafilter $\mathcal{D}$ on I, $|I| = \lambda$ which is flexible but not good, specifically not good for any theory with SOP_2.*

Proof. Let D be a λ^+-good, λ-regular ultrafilter on λ. Let $\mathcal{E}$ be a normal κ-complete, not κ^+-complete ultrafilter on κ. Let $\mathcal{D} = D \times \mathcal{E}$ be the product ultrafilter. Then $\mathcal{D}$ is flexible by Claim 7.3. On the other hand, by Claim 10.1, any $\mathcal{D}$-ultrapower of linear order will omit a (κ^+, κ^+)-cut. By Theorem I, $\mathcal{D}$ cannot saturate any theory with SOP_2. □

Remark 10.5. *On one hand, the advantage of Conclusion 10.4 over Theorem 7.4 is in the greater range of cardinals: we ask only that $\kappa < \lambda$, not $2^\kappa \leq \lambda$. On the other hand, Theorem 7.4 gives an a priori stronger failure of goodness, since the random graph is minimum among unstable theories in Keisler's order.*

11. FINITE ALTERNATIONS OF SYMMETRIC CUTS

In this section we iterate the results of §9, §10 to produce regular ultrafilters $\mathcal{D}$ whose library of cuts, $\mathcal{C}(\mathcal{D})$, contains any fixed finite number of alternations (or gaps). The following definition is stated for regular ultrafilters only so that the choice of index model will not matter.

Definition 11.1. *Let κ be a cardinal. Say that the regular ultrafilter $\mathcal{D}$ on $\lambda \geq \aleph_0$ has κ alternations of cuts if there exist cardinals $\langle \mu_\ell : \ell < \kappa \rangle$, $\langle \rho_\ell : \ell < \kappa \rangle$ such that:*

- *$\ell_1 < \ell_2 < \kappa \implies \aleph_0 < \rho_{\ell_1} < \mu_{\ell_1} < \rho_{\ell_2} < \mu_{\ell_2} < \lambda$*
- *for each $0 \leq \ell < \kappa$, $(\rho_\ell, \rho_\ell) \in \mathcal{C}(\mathcal{D})$, i.e. $(\mathbb{N}, <)^\lambda/\mathcal{D}$ has some (ρ_ℓ, ρ_ℓ)-cut*
- *for each $0 \leq \ell < \kappa$, $(\mu_\ell, \mu_\ell) \notin \mathcal{C}(\mathcal{D})$, i.e. $(\mathbb{N}, <)^\lambda/\mathcal{D}$ has no (μ_ℓ, μ_ℓ)-cut*

We will start by proving a theorem for products of complete ultrafilters, Theorem 11.3, and then extend it to regular ones in Theorem 11.4 by adding one more iteration of the ultrapower.

First we observe that taking ultrapowers will not fill symmetric cuts whose cofinality is larger than the size of the index set.

Observation 11.2. *Suppose M is a λ-saturated model of linear order, $\kappa < \lambda$, D an ultrafilter on κ. If M contains a (κ_*, κ_*)-cut, where $\kappa_* = \mathrm{cf}(\kappa_*) > \kappa$, then M^κ/D will also contain a (κ_*, κ_*)-cut. More precisely, the image of the cut from M under the diagonal embedding will remain unrealized in M^κ/D.*

Proof. Let the cut in M be given by $(\langle f_\alpha : \alpha < \kappa_* \rangle, \langle g_\beta : \beta < \kappa_* \rangle)$, and we consider the pre-cut given by $(\langle f_\alpha/D : \alpha < \kappa_* \rangle, \langle g_\beta/D : \beta < \kappa_* \rangle)$ in the ultrapower M^κ/D. Suppose for a contradiction that there were a realization $h \in {}^\kappa M$. Let $\mathbf{x} : \kappa_* \to \kappa$ be a function which to each $\alpha < \kappa_*$ associates some

index $\epsilon < \kappa$ for which $M \models f_\alpha(\epsilon) < h(\epsilon) < g_\alpha(\epsilon)$. By Fodor's lemma, there is a stationary subset $X \subseteq \kappa_*$ on which $\mathbf{x}$ is constant and equal to, say, ϵ_*. Then in M, $(\langle f_\alpha(\epsilon_*) : \alpha \in X \rangle, \langle g_\beta(\epsilon_*) : \beta \in X \rangle)$ will be cofinal in the original cut, but by choice of X it will be realized by $h(\epsilon_*)$, contradiction. $\square$

Since the proof of Theorem 11.3 involves an inductive construction, it will be convenient to index the cardinals κ_ℓ in reverse order of size.

Theorem 11.3. *Suppose that we are given:*

(a) $n < \omega$ *and* $\kappa_n < \cdots < \kappa_0 < \kappa_{-1} = \lambda$
(b) $\mathcal{E}_\ell$ *a normal κ_ℓ-complete ultrafilter on κ_ℓ, for $\ell \leq n$.*
(c) M_0 *a λ-saturated model which is, or contains, a dense linear order* $<$
(d) $M_{\ell+1} = (M_\ell)^{\kappa_\ell}/\mathcal{E}_\ell$ *for $\ell \leq n$*

Then:

(α) *for $\ell \leq n$, $M_{\ell+1}$ is κ_ℓ-saturated*
(β) *if $\ell < k \leq n+1$ then M_k has a $(\kappa_\ell{}^+, \kappa_\ell{}^+)$-cut*
(γ) *for $i, \ell \leq n+1$, M_ℓ has no (κ_i, κ_i)-cut*
(δ) *for $\ell \leq n+1$, M_ℓ has no (θ, θ)-cut for $\theta < \lambda$ weakly compact*

Thus, for each $\ell \leq n$, $(\kappa_\ell{}^+, \kappa_\ell{}^+) \in \mathcal{C}(M_{n+1})$, and for any weakly compact $\theta < \lambda$, in particular $\theta = \kappa_\ell$, $(\theta, \theta) \notin \mathcal{C}(M_{n+1})$.

Proof. The "thus" clause summarizes (α)-(δ). Recall that for transparency all languages are countable. Note that condition (b) implies the cardinals κ_ℓ are measurable cardinals, thus limit cardinals, so condition (γ) can never contradict condition (β).

(α) By induction on $-1 \leq \ell \leq n$ we verify that $M_{\ell+1}$ is κ_ℓ-saturated. For $\ell = -1$, M_0 is λ-saturated and $\lambda > \kappa_0$. For $\ell > -1$, use the fact that $M_{\ell+1} = (M_\ell)^{\kappa_\ell}/\mathcal{E}_\ell$ thus $M_{\ell+1} \equiv_{L_{\infty,\kappa_\ell}} M_\ell$ by Łoś' theorem for L_{κ_i,κ_i}.

(β) By Claim 10.1 and Observation 11.2.

(γ) Follows from (δ) as measurable implies weakly compact.

(δ) We prove this by induction on $\ell \leq n$. For $\ell = -1$, M_0 is λ-saturated. For $\ell > -1$, let $\theta < \lambda$ be given and suppose we have a pre-cut in $M_{\ell+1}$ given by $(\langle f_\alpha : \alpha < \theta \rangle, \langle g_\beta : \beta < \theta \rangle)$. There are three cases. If $\theta < \kappa_\ell$, then use (α). If $\theta = \kappa_\ell$, use Claim 9.1. So we may assume $\kappa_\ell < \theta$. Since θ is weakly compact, therefore inaccessible, $2^{\kappa_\ell} < \theta$. For $\alpha < \beta < \theta$ let

$$A_{\alpha,\beta} = \{\epsilon < \kappa_\ell : f_\alpha(\epsilon) < f_\beta(\epsilon) < g_\beta(\epsilon) < g_\alpha(\epsilon)\} \in \mathcal{E}_\ell$$

As θ is weakly compact, by Fact 2.5 the function $\mathbf{x} : \theta \times \theta \to 2^{\kappa_\ell} < \theta$ is constant on some $\mathcal{U} \in [\theta]^\theta$. Call this constant value A_*. Now for $\epsilon \in A_*$, the sequence $(\langle f_\alpha(\epsilon) : \alpha \in \mathcal{U} \rangle, \langle g_\beta(\epsilon) : \beta \in \mathcal{U} \rangle)$ is a pre-cut in M_ℓ, meaning that $\alpha < \beta \in \mathcal{U} \implies f_\alpha(\epsilon) < f_\beta(\epsilon) < g_\beta(\epsilon) < g_\alpha(\epsilon)$.

Let $B_* = \{\epsilon \in A_* : $ in M_ℓ there is c such that $\alpha \in \mathcal{U} \implies f_\alpha(\epsilon) <_{M_\ell} c <_{M_\ell} g_\alpha(\epsilon)\}$. Now if $A_* \setminus B_* \neq \emptyset$, for any $\epsilon \in A_* \setminus B_*$ we have that $(\langle f_\alpha(\epsilon) : \alpha \in \mathcal{U} \rangle, \langle g_\beta(\epsilon) : \beta \in \mathcal{U} \rangle)$ is not just a pre-cut but also a cut in M_ℓ, contradicting the inductive hypothesis. Thus for every $\epsilon \in A_*$ we may choose a realization $c(\epsilon)$ of the relevant pre-cut. For $\epsilon \in \kappa \setminus A_*$, let $c(\epsilon)$ be arbitrary. Then $\langle c(\epsilon) : \epsilon < \kappa_\ell \rangle/\mathcal{E}_\ell \in M_{\ell+1}$ realizes $(\langle f_\alpha : \alpha < \theta \rangle, \langle g_\beta : \beta < \theta \rangle)$, as desired. $\square$

By appending the construction of Theorem 11.3 to a suitable regular ultrafilter, we may produce regular ultrapowers with n alternations of cuts for any finite n.

Theorem 11.4. *Let λ be an infinite cardinal, $n < \omega$ and suppose that there exist measurable cardinals $\kappa_n < \cdots < \kappa_0 < \lambda$. Then there is a regular ultrafilter $\mathcal{D}$ on I, $|I| = \lambda$ such that:*

(1) $(\kappa_\ell{}^+, \kappa_\ell{}^+) \in \mathcal{C}(\mathcal{D})$ *for* $\ell \leq n$
(2) $(\theta, \theta) \notin \mathcal{C}(\mathcal{D})$ *for* $\theta < \lambda$ *weakly compact, in particular* $\ell \leq n$, $\theta = \kappa_\ell$
(3) $\mathcal{D}$ *is* $\kappa_n{}^+$*-good*
(4) $\mathcal{D}$ *is* λ*-flexible*
(5) $\mathcal{D}$ *is* λ^+*-good for countable stable theories*
(6) $\mathcal{D}$ *is not* $(2^{\kappa_n})^+$*-good for unstable theories*

Proof. Let $\mathcal{D}_1$ be a λ-regular, λ^+-good ultrafilter on λ. Let $\mathcal{E}$ be the ultrafilter on κ_n given by $\mathcal{E}_0 \times \cdots \times \mathcal{E}_n$, where the $\mathcal{E}_\ell$ are as in the statement of Theorem 11.3. Let $\mathcal{D} = \mathcal{D}_1 \times \mathcal{E}$. We will show that $\mathcal{D}$ has the desired properties.

(1)-(2) This follows from having chosen M_0 in Theorem 11.3 to be a $\mathcal{D}_1$-ultrapower, thus λ^+-saturated by the λ^+-goodness (and regularity) of $\mathcal{D}_1$.
   (3) By Observation 11.6.
   (4) By induction on $\ell \leq n$, using Claim 7.3(1) and the completeness of $\mathcal{E}_\ell$, $\ell \leq n$.
   (5) By induction on $\ell \leq n$, using Claim 7.3(2). Since $\mathcal{D}_1$ is regular and λ^+-good, $\mathrm{lcf}(\aleph_0, \mathcal{D}) \geq \lambda^+$.
   (6) By Claim 5.2.

$\square$

Question 11.5. *Can Theorem 11.4 be generalized to any number of alternations, not necessarily finite?*

Appendix

In this appendix, we collect several known facts, easy proofs or extensions of proofs relevant to the material in the paper.

Observation 11.6. (1) *Any regular ultrafilter is uniform.*
   (2) *In the definition of good ultrafilter, 1.4, we may assume the functions are monotonic.*
   (3) *If $\mathcal{E}$ is a κ-complete ultrafilter on κ then $\mathcal{E}$ is κ^+-good.*

Proof. (1) Keisler [5] Theorem 1.1.

(2) By induction on $|u|$, replace a given function f, not necessarily monotonic, by its monotonic refinement f' given by $f'(u) = f(u) \cap \bigcap_{v \subsetneq u} f'(v)$.

(3) Adapt the proof of [20] Claim 3.1 p. 334 that any ultrafilter is $\aleph_1$-good. $\square$

Note, however, that while a good regular ultrafilter produces saturated ultrapowers, this need not be the case when the ultrafilter is complete, *unless* the index models are also saturated. That is, the goodness of the ultrafilter D on I is equivalent to saturation of the ultrapower M^I/D when (a) we have regularity of the ultrafilter, *or* (b) we assume the model M is saturated.

Fact 11.7. *Let D be an ultrafilter on I and λ a cardinal. Then the following are equivalent:*

   (1) *D is λ-good*
   (2) *for any model M in a countable signature which is λ-saturated, M^I/D is λ-saturated*

and if D is regular, (1) *is equivalent to:*

   (3) *for any model M in a countable signature, M^I/D is λ-saturated*

Proof. See [20] VI.2 in particular Theorems 2.2-2.3, Claim 2.4 and Lemma 2.11. $\square$

By re-presenting the proof of [20] Theorem VI.4.8 p. 379 to emphasize the role of incompleteness, we obtain a more general result.

Theorem 11.8. *Let M be a λ^+-saturated model of an unstable theory T, φ an unstable formula and E a κ-complete, κ^+-incomplete ultrafilter on λ. Let $\delta = \mathrm{lcf}(\kappa, \mathcal{E})$. Then $M^I/\mathcal{E}$ is not $(\kappa + \mathrm{lcf}(\kappa, \mathcal{E}))^+$-saturated for φ-types.*

Proof. First consider the countably incomplete case. We build a correspondence between a φ-type and a $<$-type in an expanded language.

Let $\varphi = \varphi(x; y)$, where without loss of generality $\ell(x) = 1$ but $\ell(y)$ need not be 1. Choose, for each $m < n < \omega$, sequences $\bar{a}_m^n$ where $\ell(\bar{a}_m^n) = \ell(y)$, so that φ has the order property over each $\langle \bar{a}_m^n : m < n \rangle$: i.e., $k < n < \omega$ implies $\{\varphi(x; \bar{a}_m^n)^{if(m>k)} : m < n\}$ is consistent.

Let $\langle b_n : n < \omega \rangle$ be a sequence of distinct elements. Let P be a new unary relation symbol, $<$ a new binary relation symbol, and for each $\ell < \ell(y)$ let F_ℓ be a new binary function symbol. Let M_1 denote the expansion of M by these new symbols, as follows. $P^{M_1} = \{b_n : n < \omega\}$ and $<^{M_1} = \{\langle b_k, b_n \rangle : k < n < \omega\}$. Finally, interpret the functions F_ℓ so that for each $k < n < \omega$, $\bar{a}_k^n = \langle F_0(b_k, b_n), \dots F_{\ell(y)-1}(b_k, b_n) \rangle$.

Now we take the ultrapower $N = M^\lambda/E$ and let N_1 denote the corresponding ultrapower in the expanded language. In N_1, P is a linear order and so by the hypothesis on E we have a δ-cut over the diagonal embedding of the sequence $\langle b_n : n < \omega \rangle$. Choose a sequence $\langle c_i : i < \delta \rangle$ witnessing this, so (1) each $c_i \in P^{N_1}$, (2) $n < \omega$ and $i < j < \delta$ implies $N_1 \models b_n < c_i < c_j$, (3) for no $c \in P^{N_1}$ is it the case that for each $n < \omega, i < \delta$, $N_1 \models b_n < c < c_i$.

We now translate back to a φ-type. Consider:

$$q(x) = \{\neg\varphi(x, F_0(b_n, c_0), \dots F_{\ell(y)-1}(b_n, c_0)) : n < \omega\} \cup \{\varphi(x, F_0(c_i, c_0), \dots F_{\ell(y)-1}(c_i, c_0)) : 0 < i < \delta\}$$

This is a consistent partial φ-type by Łoś' theorem, since for $n < \omega, i < \delta$ we have that $b_n < c_i$ mod E. We will show that q is omitted. Suppose it were realized, say by a. Let $\langle X_n : n < \omega \rangle$ be a sequence of elements of E witnessing that E is $\aleph_1$-incomplete. Let $\langle Y_n : n < \omega \rangle$ be a sequence of elements of E given by

$$Y_n = \{t \in \lambda : M \models \bigwedge\{\neg\varphi(a[t], F_0(b_k, c_0[t]), \dots F_{\ell(y)-1}(b_k, c_0[t])) : k \le n\}$$

which exists by Łoś' theorem. (We write b_k rather than $b_k[t]$ since these are essentially constant elements. Note that in each index model, $c_0[t]$ is simply one of the b_ns.)

For each $t \in I$ define $\rho(t) = \min\{n : t \notin X_{n+1} \cap Y_{n+1}\}$. Then ρ is well defined by $\aleph_1$-incompleteness. Define $b \in {}^\lambda M$ by: $b[t] = b_{\rho(t)}$. By Łoś' theorem $P^{N_1}(b)$. Where does it fit under $<^{N_1}$? First, for each $n < \omega$, we have $b > b_n$ mod E, as E is ω-complete (i.e. a filter) thus:

$$\{t : c[t] > b_n[t]\} \supseteq \bigcap_{j \le n}(X_{j+1} \cap Y_{j+1}) \in \mathcal{D}$$

On the other hand, suppose that for some $i < \delta$ we had $c_i \le b$ mod E. Then by Łoś' theorem and definition of ρ, it would have to be the case that

$$N_1 \models \neg\varphi(a[t], F_0(c_i[t], c_0[t]), \dots F_{\ell(y)-1}(c_i[t], c_0[t]))$$

contradicting the definition of q. So for each $i < \delta$, we have that $b < c_i$ mod E.

Thus from a realization $a \models q$ we could construct a realization b of the $(\aleph_0, \delta)$-cut in P^N. Since the latter is omitted, q must be as well, which completes the proof for countably incomplete filters.

Now for the general case: If $\lambda = \kappa$, modify the argument of Theorem 11.8 by replacing ω with κ everywhere in the proof just given, and Łoś' theorem by Łoś' theorem for $L_{\kappa,\kappa}$. If $\lambda > \kappa$, begin by choosing a surjective map $\mathbf{h} : \lambda \to \kappa$ so that $\mathcal{E} = h(E)$ is a nonprincipal ultrafilter on κ, thus κ-complete not κ^+-complete. $\qquad\square$

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