A BRIEF INTRODUCTION TO COMPLEX DYNAMICS

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Abstract. This paper discusses the theory of dynamical systems of a single variable on the complex plane. We begin with a brief overview of the general theory of holomorphic dynamics on complex manifolds, defining the Julia set and exploring its elementary properties. We then examine the local dynamics of rational maps near a fixed point, demonstrating the existence and uniqueness of the Böttcher map, a normal form for the dynamics of any differentiable map near a superattracting fixed point. We conclude with Douady and Hubbard's proof of the connectedness of the Mandelbrot set.

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1. Introduction

The study of iterated holomorphic functions on the complex plane dates back to Ernst Schröder's work in 1870, but the field remains quite active and continues to expand, in part thanks to the development of computation. In 1870, Schröder first asked the question of whether or not an analytic function f fixing the point 0 and with derivative $a = f'(0)$ at the origin was conjugate to the linear map $z \mapsto (f'(0))z$ in a neighborhood of the origin.

In 1884, Gabriel Kœnigs answered this question in the affirmative for $0 < |a| < 1$. Indeed (see Theorem 4.2), the function $\varphi(z) = \lim_{n\to\infty} \frac{f^n(z)}{a^n}$ explicitly solves the equation $\varphi \circ f = a\varphi$. Note that the case $|a| > 1$ follows trivially by iterating the local inverse of f. Thus began the study of iterated complex functions.

The theory was further developed by the work of Auguste-Clémente Grévy, Lucyan Böttcher, and Leopold Leau, who studied the remaining cases $|a| = 0$ and $|a| = 1$. While the former case was answered completely by Böttcher, the latter was only answered by Leau in the *parabolic* case in which a is a root of unity, although the general case is now better understood due to the later work of Hubert Cremer and Carl Siegel, among others. Nevertheless, it is somewhat surprising that in all cases except the indefinite case $|a| = 1$, there is a single picture of local dynamics for all holomorphic functions. This is certainly not the case for real dynamical systems, and reflects just how strong a condition differentiability is for complex functions (indeed, this has deep connections to the Maximum Modulus Principle; see, for instance, Shwartz's Lemma 2.2).

Pierre Fatou and Gaston Julia studied the global theory of iterated complex maps, defining the Fatou and Julia sets and proving a great number of their properties despite the inability to see their shapes.

Finally, thanks to the development of the study of dynamical systems in the later half of the 20th century as well as to advances in computer graphics, the stage was set to study the quadratic family consisting of the maps $f_c(z) = z^2 + c$. Central to their study is the Mandelbrot set M of parameters c for which the Julia set of f_c is connected. Adrien Douady and John Hubbard studied the quadratic family extensively, proving in 1980 that M is connected. Still, a number of questions remain open, most notably the conjecture that M is *locally* connected.

This paper will lay out the foundations for the study of holomorphic dynamics. First, we explain a number of relevant results of complex analysis, such as the Uniformization Theorem (1.1) and the theory of covering spaces. We then begin our study of iterated functions, defining the Fatou and Julia sets and proving their basic properties, such as the density of iterated preimages in the Julia set J and the self-similarity of the geometry of J.

Third, we examine the local dynamics of functions near a fixed point, constructing Kœnigs' linearizing map near an attracting fixed point (Theorem 4.2) and the Böttcher map near a superattracting fixed point (Theorem 4.3). We conclude this section with the following theorem:

Theorem 4.6. *The Julia set $J(f)$ for a polynomial f of degree $d \geq 2$ is connected if and only if the filled Julia set $K(f)$ contains every critical point of f. In this case, the complement $\mathbb{C} \setminus K(f)$ is conformally isomorphic to the complement of the closed unit disk by the Böttcher map $\hat{\phi}$, conjugating f to the map $z \mapsto z^d$.*

If, instead, a critical point of f lies outside the filled Julia set, then $K(f)$, and hence $J(f)$, has uncountably many disconnected components.

Remark 5.1. Hence the Julia set for the quadratic map $f(z) = z^2 + c$ is connected if and only if the critical orbit $\{f^n(0) : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{C}$.

This theorem motivates the definition of the *Mandelbrot set*

$$\mathcal{M} := \{c \in \mathbb{C} : \exists C, \forall n \geq 0, |f_c^n(0)| < C\}.$$

Finally, we present the proof of Douady and Hubbard's famous theorem [DH82]:

Theorem 5.3. *The Mandelbrot set $\mathcal{M}$ is connected.*

2. Basic Theory

We begin with some basic definitions. If $U \subset \mathbb{C}$ is open, a function $f : U \to \mathbb{C}$ is called *holomorphic* if its derivative $f' : z \mapsto f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ is a continuous function from U to $\mathbb{C}$. A standard result in complex analysis states that a holomorphic function is in fact *analytic*, in the sense that it is infinitely differentiable and equal to its power series in the neighborhood of a point. A holomorphic function is called *conformal* if its derivative is nowhere zero. (In some works, the term *conformal* also requires that a function be one-to-one; here, we use the term *conformal isomorphism* to indicate this additional property.)

Henceforward, we will assume an understanding of the basic results of complex analysis (through [Con78], for example), though for convenience we will briefly summarize the major results used.

A *Riemann surface* S is a complex one-dimensional manifold. That is, a topological space S is a Riemann surface if for any point $p \in S$, there is a neighborhood U of p and a *local uniformizing parameter*

$$\Phi : U \to \mathbb{C}$$

mapping U homeomorphically into an open subset of the complex plane. Moreover, for any two such neighborhoods U and U' with nonempty intersection and local uniformizing parameters Φ and Ψ respectively, we require that $\Psi \circ \Phi^{-1}$ be a holomorphic function on $\Phi(U \cap U')$.

Two Riemann surfaces S and S' are said to be *conformally isomorphic* if there is a homeomorphism $f : S \to S'$ such that for neighborhoods $U \subset S$ and $U' = f(U) \subset S'$ with local uniformizing parameters Φ and Ψ, the map $\Psi \circ f \circ \Phi^{-1}$ is conformal from $\Phi(U)$ to $\Psi(U')$. One may check that this defines an equivalence relation on Riemann surfaces.

A connected space X is *simply connected* if, roughly speaking, every loop in X can be contracted. More formally, X is simply connected if every continuous function $f : S^1 \to X$ from the unit circle to X is homotopic to a constant function, i.e. there exists a continuous function $h : S^1 \times [0,1] \to X$ with $h(\theta, 0) = f(\theta)$ and $h(\theta, 1) = p$ for all $\theta \in S^1$ and some $p \in X$.

A standard result in complex analysis follows.

Theorem 2.1 (Uniformization Theorem). *Let S be a simply connected Riemann surface. Then S is conformally isomorphic to either*

1. *the complex plane $\mathbb{C}$,*
2. *the open disk $\mathbb{D}$ consisting of all $z \in \mathbb{C}$ with absolute value $|z| < 1$, or*
3. *the Riemann sphere $\hat{\mathbb{C}}$ consisting of $\mathbb{C}$ together with the point ∞ with local uniformizing parameter $\zeta(z) = \frac{1}{z}$ in a neighborhood of the point at infinity.*

The proof of Theorem **??** is rather deep, and is omitted in this paper. The following fact is also quite useful in our study.

Lemma 2.2 (Schwarz's Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map which fixes the origin. Then $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then $f : z \mapsto ze^{i\theta}$ is a rotation about the origin. On the other hand, if $|f'(0)| < 1$, then $|f(z)| < |z|$ for all $z \in \mathbb{D} \setminus \{0\}$.*

Proof. We apply the Maximum Modulus Principle, which asserts that any nonconstant holomorphic function cannot attain its maximum absolute value over a region in the region's interior. Notice that the quotient $q(z) := f(z)/z$ is holomorphic on the disk and attains the value $q(0) = f'(0)$. Suppose first that q is nonconstant (if $f(z) = cz$, then clearly $|c| \leq 1$). When $|z| = r < 1$, we have that $|q(z)| < 1/r$. Hence $|q(z)| < 1/r$ whenever $|z| \leq r$, since by the Maximum Modulus Principle, q cannot attain its maximum absolute value over the closed disk $\overline{\mathbb{D}_r}$ of radius r on any point of the interior of this region. Letting $r \to 1$, we obtain $|q(z)| \leq 1$ for all z in the disk. In particular, $|q(0)| = |f'(0)| \leq 1$.

Applying the Maximum Modulus Principle again, we see that if $|q(z)| = 1$ for any $z \in \mathbb{D}$, then $q(z) = c$ for all z, yielding $|f'(0)| = |q(0)| = 1$ and contradicting the assumption that q is nonconstant. Thus if $|f'(0)| < 1$ and q is nonconstant, then $|f(z)| < |z|$ for all $z \in \mathbb{D} \setminus \{0\}$, while if $|f'(0)| = 1$, then q is constant.

On the other hand, if $q(z) = c$ is constant, then clearly $|f'(0)| = |c| \leq 1$. If $|f'(0)| = 1$, then setting $c = e^{i\theta}$ we have $f(z) = ze^{i\theta}$, so that f is simply a rotation about the origin. If instead $|f'(0)| < 1$, then $|f(z)| = |cz| < |z|$ for all $z \in \mathbb{D} \setminus \{0\}$. □

Corollary 2.2.1. *If f is as above, then f is a conformal automorphism of the unit disk if and only if $|f'(0)| = 1$. Thus rotations are the only conformal automorphisms of the unit disk which fix the origin.*

Proof. In the case $|f'(0)| = 1$, then f is a rotation, and hence a conformal automorphism. Conversely, if f is a conformal automorphism of the disk, then it has a holomorphic inverse $g : \mathbb{D} \to \mathbb{D}$ also fixing the origin. In particular, as $f \circ g$ preserves absolute values, it follows that $|(f \circ g)'(0)| = |f'(0)||g'(0)| = 1$. Since $|f'(0)| \leq 1$ and $|g'(0) \leq 1$, we have that in fact $|f'(0)| = 1$. □

The conformal automorphisms of the three surfaces of Theorem 2.1 are as follows:

Theorem 2.3. *The group* $\mathrm{Aut}(\hat{\mathbb{C}})$ *of conformal automorphisms of the Riemann sphere consists of all* ***Möbius transformations*** $z \mapsto \frac{az+b}{cz+d}$ *with determinant* $ad - bc \neq 0$*. The group* $\mathrm{Aut}(\mathbb{C})$ *consists of all affine transformations* $z \mapsto az + b$*. The group* $\mathrm{Aut}(\mathbb{D})$ *consists of all maps of the form* $z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z}$*, with* $a \in \mathbb{D}$ *and* $\theta \in [0, 2\pi)$*.*

The inverse of a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ is given by $z \mapsto \frac{dz-b}{-cz+a}$, which is evidently also a Möbius transformation. The composition of two Möbius transformations $g : z \mapsto \frac{az+b}{cz+d}$ and $g' : z \mapsto \frac{a'z+b'}{c'z+d'}$ is given by

$$g \circ g' : z \mapsto \frac{(aa' + b'c)z + (a'b + b'd)}{(ac' + cd')z + (bc' + dd')}.$$

Thus the Möbius transformations form a group under composition. Moreover, noting that the coefficients of the composition of two functions correspond directly to those given by matrix multiplication, we see that this group is in fact isomorphic to the group $\mathbf{PSL}(2, \mathbb{C})$ of all 2×2 complex matrices with determinant 1, modulo $\{\pm I\}$ since multiplying the top and bottom of the fraction by a constant has no effect on the transformation.

Remark 2.4. The Möbius transformation

$$f(z) = \frac{(b-c)(z-a)}{(b-a)(z-c)}$$

takes three distinct complex numbers a,b, and c to 0, 1, and ∞ respectively. If $a = \infty$, this reduces to $f(z) = \frac{b-c}{z-c}$. If $b = \infty$, this becomes $f(z) = \frac{z-a}{z-c}$. Finally, if $c = \infty$, take $f(z) = \frac{z-a}{b-a} = \frac{z}{b} - 1$. Thus there is a Möbius transformation taking any three given points of the Riemann sphere to 0, 1, and ∞. By composing with the inverse of another such transformation, we may find a Möbius transformation mapping any three points to any other three points.

Proof of Theorem 2.3. It is a straightforward computation to check that any Möbius transformation $z \mapsto \frac{az+b}{cz+d}$, $ad - bc \neq 0$, is a conformal automorphism of the Riemann sphere. Suppose, conversely, that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a conformal automorphism. By the Remark, we may assume, up to composition with an appropriate Möbius transformation, that $f(0) = 0$ and $f(\infty) = \infty$. Then the quotient $q(z) = f(z)/z$

takes on the nonzero, finite value $q(0) = f'(0)$, and is clearly nonzero and finite for other $z \in \mathbb{C}$. Taking $\zeta = \zeta(z) = 1/z$ and $F(\zeta) = 1/f(1/\zeta)$, we have $q(z) = \zeta/F(\zeta) \to 1/F'(0)$ as $z \to \infty$, which is nonzero and finite. Thus q is bounded and holomorphic from $\mathbb{C} \setminus 0$ to itself, so the map $w \to q(e^w)$ is bounded and holomorphic on $\mathbb{C}$. Liouville's Theorem, a basic result in complex analysis, then gives that q must be constant. Thus since $f(1) = 1$, we have that $f(z) = z$, the identity transformation.

For the case of the complex plane, note that a conformal automorphism of $\mathbb{C}$ extends uniquely to a conformal automorphism of $\hat{\mathbb{C}}$, since $\lim_{z\to\infty} f(z) = \infty$. Thus $\mathbf{Aut}(\mathbb{C})$ is the subgroup of $\mathbf{Aut}(\hat{\mathbb{C}})$ consisting of all Möbius transformations which fix the point ∞. It is easy to see that this is simply the set of affine transformations of $\mathbb{C}$.

Finally, in the case of the unit disk, it is apparent that the map $f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$ is a Möbius transformation, and thus a conformal automorphism of $\hat{\mathbb{C}}$. Also, a short computation shows that for all $a \in \mathbb{D}$, $|z| < 1$ if and only if $|f(z)| < 1$, so that f is in fact a conformal automorphism of the disk.

Now suppose that g is an arbitrary conformal automorphism of $\mathbb{D}$. Let $a = g^{-1}(0)$, let $h(z) = \frac{z-a}{1-\bar{a}z}$, and let $f = g \circ h^{-1}$. Then $f(0) = 0$, and as f is a conformal automorphism, $f(z) = e^{i\theta}z$ is simply a rotation of the disk by Corollary 2.2.1. Thus $g(z) = e^{i\theta}h(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$ as desired. □

Note that the unit disk $\mathbb{D}$ is conformally isomorphic to the *upper half-plane* $\mathbb{H}$ consisting of all $z = a + bi \in \mathbb{C}$ with $b > 0$ under the map $f : \mathbb{D} \to \mathbb{H}$ given by

$$f(z) = \frac{i(1-z)}{1+z}$$

with inverse

$$f^{-1}(w) = \frac{i-w}{i+w}.$$

One may check by applying this map that the corresponding automorphisms of $\mathbb{H}$ are given by the Möbius transformations $w \mapsto \frac{aw+b}{cw+d}$ with $a, b, c, d \in \mathbb{R}$ and with determinant $ad - bc > 0$.

We now summarize a notion central to algebraic topology: the theory of covering spaces. A map $p : M' \to M$ between connected manifolds is a *covering map* if every point of M has a connected neighborhood U such that the restriction of p to each each component of $p^{-1}(U)$ is a homeomorphism onto U. M' is a *covering space* for M if there exists a covering map $p : M' \to M$. A *deck transformation* is a transformation $\gamma : M' \to M'$ such that $p \circ \gamma = p$.

Every manifold M has a unique (up to homeomorphism) simply connected covering space $\tilde{M}$, known as its *universal cover.* This covering is *normal*: if $p(x) = p(y)$, there exists a unique deck transformation γ taking x to y. The group $\Gamma = \pi_1(M)$ of deck transformations for the universal covering $p : \tilde{M} \to M$ is known as the *fundamental group* of M. Thus $N \cong \tilde{N}/\Gamma$.

Conversely, given a connected manifold M, a group Γ of homeomorphisms gives rise to a normal covering $p : M \to M/\Gamma$ if and only if Γ acts *freely* and *properly discontinuously* on M in the sense that firstly, every nonidentity element of Γ acts on M without fixed points, and secondly, any compact set intersects only finitely many elements of its orbit under the action of Γ. It follows that the action of this group must be a *discrete*; that is, the identity element is not a limit point of Γ in the topology of the Lie group $\mathbf{Aut}(\tilde{S})$.

Now let S be any Riemann surface, and let $\tilde{S}$ be its universal cover. Then $\tilde{S}$ plainly inherits a complex structure from S: let $p : \tilde{S} \to S$ be the universal covering of S, and let $x \in \tilde{S}$, V a neighborhood of $p(x)$ with local uniformizing parameter φ, and V' a neighborhood of $p(x)$ such that each component of $p^{-1}(V')$ maps to V' homeomorphically under p. Then the component U of $p^{-1}(V \cap V')$ has local uniformizing parameter $p \circ \varphi$. The deck transformations of $\tilde{S}$ correspond to conformal automorphisms of S.

By Theorem 2.1, then, every connected Riemann surface S has, up to conformal isomorphism, either $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$ as its universal cover. Thus there are three types of Riemann surfaces: *spherical* surfaces with universal cover isomorphic to $\hat{\mathbb{C}}$, *Euclidean* surfaces with universal cover isomorphic to $\mathbb{C}$, and *hyperbolic* surfaces with universal cover isomorphic to $\mathbb{D}$.

Theorem 2.5. *Any holomorphic map from a spherical surface to a Euclidean or hyperbolic surface must be constant, and any holomorphic map from a Euclidean surface to a hyperbolic surface must be constant.*

Proof. A holomorphic map $f : S \to S'$ lifts to a holomorphic map $\tilde{f} : \tilde{S} \to \tilde{S}'$ between universal covers. Since $\hat{\mathbb{C}}$ is open and compact, any holomorphic map $\tilde{f} : \hat{\mathbb{C}} \to U \subset \mathbb{C}$ must attain its maximum in the interior of $\hat{\mathbb{C}}$, and hence must be constant by the maximum modulus principle. Since $\hat{\mathbb{D}}$ is a bounded set, any map $\tilde{f} : \mathbb{C} \to \mathbb{D}$ must also be constant by Liouville's Theorem. $\square$

We now classify the spherical and Euclidean Riemann surfaces.

Theorem 2.6 (Spherical surfaces)**.** *Every spherical Riemann surface is conformally isomorphic to the Riemann sphere* $\hat{\mathbb{C}}$.

Proof. Let $S \cong \hat{\mathbb{C}}/\Gamma$ be a spherical surface. By Theorem 2.3, the automorphisms of $\hat{\mathbb{C}}$ have the form $z \mapsto \frac{az+b}{cz+d}$, $ad - bc \neq 0$, whose fixed points can be found by solving a quadratic equation, except in the special case $f(z) = az + b$, which either fixes the point $z = \frac{b}{1-a}$ in the extended complex plane, or, if $f(z) = z$, the entire Riemann sphere. Thus every nonidentical automorphism of $\hat{\mathbb{C}}$ fixes either one or two points. It follows that in order to act freely on $\hat{\mathbb{C}}$, the group Γ must be the trivial group, so that $S \cong \hat{\mathbb{C}}$. $\square$

Theorem 2.7 (Euclidean surfaces)**.** *Every Euclidean Riemann surface is conformally isomorphic to either*

(1) *the complex plane* $\mathbb{C}$,
(2) *the infinite cylinder* $\mathbb{C}/\mathbb{Z}$, *or*
(3) *the torus* $\mathbb{C}/\Lambda$, *where* $\Lambda = \mathbb{Z} \times \tau\mathbb{Z}$, $\tau \notin \mathbb{R}$ *is a lattice over* $\mathbb{C}$.

Proof. Let $S \cong \mathbb{C}/\Gamma$ be a Euclidean surface. Theorem 2.3 gives that every conformal automorphism of $\mathbb{C}$ has the form $f : z \mapsto \lambda z + c$. For $\lambda \neq 1$, f has a fixed point, and thus cannot be an element of the freely acting group Γ. If $\lambda = 1$, then $f : z \mapsto z + c$ is simply a translation of the plane. Hence Γ is isomorphic to a discrete additive subgroup of $\mathbb{C}$.

Such a group is clearly generated by at most two linearly independent elements (otherwise, there would be a linearly dependent set of vectors, whose integer multiples would approximate zero, violating discreteness). Thus we have three cases:

(1) If $\Gamma \cong 0$ is the trivial group, then $S \cong \mathbb{C}/\{0\} \cong \mathbb{C}$ is isomorphic to the complex plane.
(2) If Γ is generated by a single element, then $\Gamma \cong \alpha\mathbb{Z}$ for some nonzero complex number α, so that $S \cong \mathbb{C} \cong \{\alpha\mathbb{Z}\}$ is isomorphic (under multiplication by $1/\alpha$) to the infinite cylinder $\mathbb{C}/\mathbb{Z}$.
(3) If Γ is generated by two linearly independent elements α and β, then, multiplying by $1/\beta$ and setting $\tau = \alpha/\beta$, we have that $S \cong \mathbb{C}/\Gamma \cong \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} \times \tau\mathbb{Z}$ is a lattice on $\mathbb{C}$. □

Corollary 2.7.1. *If S is obtained by removing three or more points from the Riemann sphere, then S is hyperbolic.*

Proof. Note that the fundamental group of S is nonabelian, and thus S is not homeomorphic to any of the surfaces listed above. Hence S must be hyperbolic. □

Of importance is the fact that every Riemann surface is in fact a *metric space* (this is clear for spherical and Euclidean surfaces, since we may simply use the metric provided by angular distance on the sphere, or the Euclidean metric on the plane). In fact, every hyperbolic surface S has an important Riemannian metric known as the *Poincaré metric* on S. We will omit the details of this construction, but, briefly speaking, the Poincaré metric on the upper half plane $\mathbb{H}$ is given by

$$ds = |dw|/b, \text{ where } w = a + bi \in \mathbb{H}.$$

The great significance of this metric is that it is *invariant under conformal automorphisms of* $\mathbb{H}$. Moreover, such a metric is unique up to multiplication by a positive constant. This metric may then be consistently extended to any hyperbolic surface S, since the deck transformations between components of the preimages in $\mathbb{H}$ of a small neighborhood in S are conformal automorphisms and hence preserve the metric.

Thus we obtain a distance function on hyperbolic surfaces given by $\text{dist}_S(a, b) = \inf \int_P ds$, where the infimum is taken over all paths P from a to b. A fundamental property of the Poincaré metric is that every hyperbolic surface S is complete with respect to dist_S; that is, every Cauchy sequence in (S, dist_S) converges to a unique point of S. It follows that the infimum used in the definition of dist_S is in fact achieved.

Having developed the machinery to discuss notions such as uniform convergence, we now briefly summarize Montel's theory of normal families of functions, which will prove central to our study of dynamical systems.

Definition 2.8. A sequence $\{f_n\}$ of functions is said to converge *locally uniformly* on S if every point of S has a neighborhood in which the sequence converges uniformly. Equivalently in locally compact spaces such as Riemann surfaces, a sequence $\{f_n\}$ converges locally uniformly on S if it converges uniformly on every compact subset of S.

A family of holomorphic functions $\mathcal{F}$ from a Riemann surface S to a compact Riemann surface S' is *normal* if every infinite sequence of functions of $\mathcal{F}$ contains a subsequence which converges locally uniformly on S to some limit $g : S \to S'$.

Note that normality of a family $\mathcal{F}$ of functions on S is a local property, so that it makes sense to define the *domain of normality* of $\mathcal{F}$ to be the union of all points $p \in S$ which possess an open neighborhood U such that the set of restrictions

$\{f|_U : f \in \mathcal{F}\}$ is a normal family. The domain of normality is clearly an open subset of S.

For our purposes, the key result of this theory is the following.

Theorem 2.9 (Montel). *Let $\mathcal{F}$ be a family of maps from a Riemann surface S to the thrice-punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{a, b, c\}$. Then $\mathcal{F}$ is normal.*

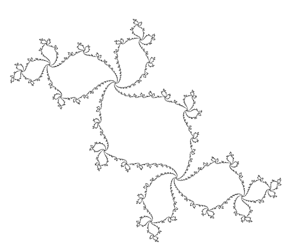
As the proof relies on a fair amount of theory which is largely unnecessary for the rest of this paper, it is omitted here. For a proof, see [Mil06], pp 30-36.

3. The Fatou and Julia sets

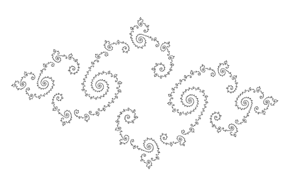
Consider now a holomorphic map f on the Riemann sphere (unless otherwise mentioned, we will always take this to be the case for f in this section). Such a map has a discrete (and hence finite by compactness of $\hat{\mathbb{C}}$) set of poles in $\mathbb{C}$, given by $\{z_1, \ldots, z_n\}$, with degrees $\{d_1, \ldots, d_n\}$. Then the function $p(z) = f(z) \prod_{j=1}^{n} (z - z_j)^{d_j}$ has no poles except perhaps at ∞, and is thus a polynomial. Hence f is a *rational map* of the form $f = p/q$, where p and q are polynomials which do not share any roots. The values of f where $q(z) = 0$ are taken to be infinity, and the value at infinity is given by the possibly infinite limit $f(\infty) = \lim_{z\to\infty} f(z)$. The *degree* of f is the maximum of the degrees of p and q. Intuitively, this value describes the number of preimages for a generic point z, so, for instance, it is homotopy invariant.

Let f^n denote the n-fold iterate of f. The domain of normality of $\{f^n\}$ is called the *Fatou set* for f, and its complement $J = J(f)$ is called the *Julia set* for f. Thus, the Fatou set $\hat{\mathbb{C}} \setminus J$ is open, while the Julia set J is closed.

Note that for polynomials of degree $d = 1$, the Julia set is simple to describe: the Julia set for an affine map $f(z) = \alpha z + c$ consists of the fixed point $-c/\alpha$ when $|\alpha| > 1$, while when $|\alpha| < 1$, the Julia set consists of the fixed point ∞. If f^m is the identity map for some m, then $J(f)$ is empty. For higher degree polynomials and rational maps, however, the Julia sets can form a wide variety of shapes, usually resembling complex fractals.



(A) Julia set for the map $z \mapsto z^2 + (-0.17 + 0.72i)$, a connected set known as the Douady Rabbit.



(B) Julia set for the map $z \mapsto z^2 + (-0.77 + 0.17i)$, a topological Cantor set.

We now prove some basic properties of the Julia set.

Theorem 3.1. *Let f be a rational map of degree 2 or higher. Then the Julia set $J(f)$ is nonempty.*

Proof. Suppose, instead, that $J(f)$ is empty. Then there is a sequence f^{n_j} of iterates which converges uniformly over the Riemann sphere (in the spherical metric σ given by angular distance) to some holomorphic limit g. For sufficiently large j, we have that $\sigma(f^{n_j}(z), g(z)) < \pi$ for all $z \in \hat{\mathbb{C}}$, so that $f^{n_j}(z)$ may be continuously deformed into $g(z)$ along the unique shortest geodesic. Thus f^{n_j} and g are homotopic and must then have the same degree. But $\deg(f^{n_j}) \geq 2^{n_j} \to \infty$ as $j \to \infty$, so g cannot be a rational map, which is a contradiction. □

Theorem 3.2 (Invariance)**.** *Let f be a holomorphic map on the Riemann sphere, and let $J = J(f)$ be its Julia set. Then $z \in J$ if and only if $f(z) \in J$.*

Proof. If f is constant, the results follow immediately, since the Julia set is empty. Assume then that f is nonconstant. Let $z \in \hat{\mathbb{C}} \setminus J$. Then there is a neighborhood U of z on which the iterates $\mathcal{F}_z = \{f^k|_U : k \geq 1\}$ form a normal family. Note that $f^{-1}(U)$ is a neighborhood of w, where $z = f(w)$, by continuity of f.

Let $K \subset f^{-1}(U)$ be compact, and let $\{f^{n_j}\}$ be some sequence of iterates of f defined on K. Then $f(K) \subset U$ is compact by continuity of f, so that the sequence $\{f^{n_j - 1}\}$ has a subsequence $\{f^{n_{j_k} - 1}\}$ which converges uniformly to a holomorphic limit on $f(K)$. Thus $\{f^{n_{j_k}}\}$ converges uniformly to a holomorphic limit on K, and hence $w \in \hat{\mathbb{C}} \setminus J$.

Now let $V \subset U$ be an open neighborhood of z such that $\overline{V} \subset U$. Note that $f(V)$ is a neighborhood of $f(z)$ by the Open Mapping Theorem (a standard result which asserts that every holomorphic map takes open sets to open sets, cf. [Con78], pg. 99). Suppose $K' \subset f(V)$ is compact, and let $\{f^{n'_j}\}$ be some sequence of iterates of f defined on K'. By the closed map lemma, which asserts that every continuous function from a compact space to a Hausdorff space is *closed* (i.e. closed sets are taken to closed sets) and *proper* (i.e. compact sets have compact preimages, cf. [Lee11], pg. 119), we have that $f^{-1}(K') \subset f^{-1}(f(V))$ is compact. Then $f^{-1}(K') \cap \overline{V} \subset U$ is also compact. Thus the sequence $\{f^{n'_j + 1}\}$ has a subsequence $\{f^{n'_{j_k} + 1}\}$ which converges uniformly to a holomorphic limit on $f^{-1}(K') \cap \overline{V}$. Then $\{f^{n'_{j_k}}\}$ converges uniformly to a holomorphic limit on $f\left(f^{-1}(K') \cap \overline{V}\right) = K'$, and hence $f(z) \in \hat{\mathbb{C}} \setminus J$.

Thus $z \in \hat{\mathbb{C}} \setminus J$ if and only if $f(z) \in \hat{\mathbb{C}} \setminus J$, or equivalently $z \in J$ if and only if $f(z) \in J$. □

Theorem 3.3 (Iteration)**.** *The Julia set $J(f^m)$ for the m-fold iterate of f is identically equal to $J(f)$.*

Proof. Note that if $z \in \hat{\mathbb{C}} \setminus J(f)$, then there is a neighborhood U of z such that every sequence of iterates of f on U has a subsequence which converges locally uniformly to a holomorphic limit. In particular, every sequence of the form $\{f^{mn_j}\}$ has a subsequence which converges locally uniformly to a holomorphic limit on U, so that $z \in \hat{\mathbb{C}} \setminus J(f^m)$, and hence $\hat{\mathbb{C}} \setminus J(f) \subseteq \hat{\mathbb{C}} \setminus J(f^m)$.

We now show that $\hat{\mathbb{C}} \setminus J(f^m) \subseteq \hat{\mathbb{C}} \setminus J(f)$. Suppose that $z \in \hat{\mathbb{C}} \setminus J(f^m)$. Then there is some neighborhood U of z such that every sequence $\{f^{mn_j}|_U\}$ has a subsequence which converges locally uniformly on U. Thus every sequence $\{f^{mn_j + k}|_U\}$ has a subsequence which converges locally uniformly on U.

Now let $\{f^{n_j}|_U\}$ be some sequence of iterates of f defined on U. Then there exists some $k \in \{0, 1, \ldots, m-1\}$ such that $n_j \equiv k \pmod m$ for infinitely many choices of j. Thus there is an increasing sequence $\{\tilde{n}_j\}$ of positive integers such that for each j, there exists j' with $m\tilde{n}_j + k = n_{j'}$. Then the sequence $\{f^{m\tilde{n}_j} + k\}$ is a subsequence of $\{f^{n_j}\}$, which by the above has a subsequence which converges locally uniformly on U. □

Definition 3.4. Let z and z' be two points of the Julia set $J = J(f)$. We say that (J, z) is *locally conformally isomorphic* to (J, z') if there are open neighborhoods U of z and U' of z' and a conformal isomorphism $\varphi : U \to U'$ such that $\varphi(z) = z'$ and $\varphi(J \cap U) = J \cap U'$.

A point z_0 is called *periodic* if there exists some $m \geq 1$ such that $f^m(z_0) = z_0$. The smallest positive m for which this holds is the *period* of z_0. The *multiplier* λ of a z is defined to be the first derivative of the m-fold iterate f^m, where m is the period of z. Thus we have

$$\lambda = \prod_{j=0}^{m-1} f'(f^j(z_0)) = f'(z_0)f'(z_1)\cdots f'(z_{m-1})$$

where $z_j = f^j(z_0)$ is the jth point of the *orbit* $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{m-1} \mapsto z_0$. Thus all points in a periodic orbit have the same multiplier. Note that in the case $z_j = \infty$, one may check with the local uniformizing parameter $\zeta = 1/z$ that we use $f'(\infty) = (\lim_{z\to\infty} f'(z))^{-1}$.

A periodic orbit is *geometrically attracting* if the multiplier λ has absolute value $0 < |\lambda| < 1$.If $\lambda = 0$, then the orbit contains a critical point, and the orbit is *superattracting*. On the other hand, an orbit is *repelling* if $|\lambda| > 1$.

An orbit is *indifferent* if $|\lambda| = 1$, in which case the dynamics are generally much harder to understand. If, however, the multiplier for an orbit is a root of unity, but no iterate of f is the identity map, then the points of the orbit are referred to as *parabolic* periodic points.

The *basin of attraction* for an attracting orbit of period m is the set U of all points z such that the sequence $f^m(z), f^{2m}(z), f^{3m}(z), \ldots$ converges to a point of the orbit.

Theorem 3.5. *Every attracting or superattracting orbit, along with its basin of attraction, belongs to the Fatou set. Every repelling orbit belongs to the Julia set.*

Proof. Let $\hat{z}$ be a fixed point with multiplier λ, so that the first derivative of f^n at $\hat{z}$ is λ^n.

If $|\lambda| > 1$, then this derivative tends to infinity, so f^n cannot converge uniformly to a holomorphic limit in a neighborhood of $\hat{z}$ by the Weierstrass Uniform Convergence Theorem, which asserts that is a sequence f_n of holomorphic functions converges uniformly to some limit g, then g is holomorphic and $f'_n \to g$ (cf. [Mil06], pg. 4). Thus $\hat{z} \in J(f)$.

If $|\lambda| < 1$, choose some r with $|\lambda| < r < 1$. Then there is a neighborhood U of $\hat{z}$ with $|f(z) - f(\hat{z})| \leq r|z - \hat{z}|$, so that $f^n|_U$ converges uniformly to the constant function $z \mapsto \hat{z}$. Hence $\hat{z}$ is a member of the Fatou set.

Now let z be an element of the basin of attraction $\mathcal{A}$ of an attracting fixed point $\hat{z}$. Choose sufficiently large k such that $f^k(z)$ is an element of the neighborhood U defined above, and choose a neighborhood $\tilde{U} \subset U$ of $f^k(z)$. Then $V := f^{-k}(\tilde{U})$ is a neighborhood of z, and the sequence $\{f^n|_V\}$ converges uniformly to the constant function $z \mapsto \hat{z}$. Thus $\mathcal{A} \subset \hat{\mathbb{C}} \setminus J(f)$.

If instead z_0 is periodic with period m, then f^m fixes z_0 with the same multiplier. By Theorem 3.3, $J(f^m) = J(f)$, so that $z_0 \in J(f)$ if $|\lambda| > 1$ and $z_0 \in \hat{\mathbb{C}} \setminus J(f)$ if $|\lambda| < 1$, and again if z_0 is attracting, then its basin of attraction is contained in the Fatou set. □

Theorem 3.6. *Every parabolic periodic point belongs to the Julia set.*

Proof. Up to composition with a Möbius transformation, let $\hat{z} = 0$ be the parabolic periodic point. Then for some $m > 1$, the local power series for f near $\hat{z}$ has the form

$$f^m(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots ,$$

with $n > 1$ and $a_n \neq 0$. Then $f^{mk}(z) = z + k a_n z^n +$ (higher terms), so that the n^{th} derivative of f^{mk} at 0 is $kn! a_n$, which tends to infinity as $k \to \infty$. But by the Weierstrass Uniform Convergence Theorem, no subsequence $\{f^{mk_n}\}$ can converge uniformly on a compact neighborhood of 0. Thus $\hat{z}$ cannot belong to the Fatou set, so it must be an element of the Julia set. □

Definition 3.7. The *grand orbit* of a point z_0 is the set

$$\mathrm{GO}(z_0, f) = \{z \in \hat{\mathbb{C}} : \exists m, n \in \mathbb{N},\ f^m(z) = f^n(z_0)\}.$$

consisting of all points whose orbit intersects the orbit of z_0. A point z_0 is *exceptional* if $\mathrm{GO}(z_0, f)$ is a finite set. The set of exceptional points will be denoted $\mathcal{E}(f)$.

The following theorem motivates the above terminology.

Theorem 3.8. *If f is a rational map of degree $d \geq 2$, then $\mathcal{E}(f)$ contains at most* 2 *points, each of which is superattracting and thus a member of the Fatou set $\hat{\mathbb{C}} \setminus J(f)$.*

Proof. The Fundamental Theorem of Algebra implies that any nonconstant rational map maps $\hat{\mathbb{C}}$ onto itself, so that in particular, $\mathrm{GO}(z, f)$ maps onto itself for any z. Thus if $z_0 \in \mathcal{E}(f)$, then $\mathrm{GO}(z_0, f)$ maps bijectively onto itself, since any map from a finite set onto itself must also be one-to-one. Thus $\mathrm{GO}(z_0, f)$ forms a single periodic orbit $z_0 \mapsto z_1 \mapsto \ldots \mapsto z_{k-1} \mapsto z_k = z_0$.

Also by the Fundamental Theorem of Algebra, every point $z \in \hat{\mathbb{C}}$ has precisely d preimages counted with multiplicity[1]. It is simple to check that any solution w with multiplicity greater than 1 must be a critical point of f, as

$$\begin{aligned} f'(w) &= \frac{p'(w)q(w) - p(w)q'(w)}{q(w)^2} = \frac{p'(w) - \hat{z}q'(w)}{q(w)} \\ &= \frac{n(w-w)^{n-1}\hat{p}(w) - (w-w)^n \hat{p}'(w)}{q(w)} = 0, \end{aligned}$$

where $\hat{p}$ is a polynomial and $q(w) \neq 0$. If $w = \infty$, we may apply the same argument after conjugating with the conformal automorphism $z \mapsto 1/z$.

Returning to the main argument, each point $z_j \in \mathrm{GO}(z_0, f)$ has exactly one preimage z_{j-1} which must then have multiplicity $d \geq 2$. Hence each element of

[1] A solution w to $f(w) = \hat{z}$ is said to have multiplicity n if $(z - w)^n$ divides the polynomial expression $p(z) - \hat{z}q(z)$, but $(z - w)^{n+1}$ does not, where $f(z) = p(z)/q(z)$ describes f as a ratio of polynomials in lowest terms.

$\mathrm{GO}(z_0, f)$, and indeed of $\mathcal{E}(f)$, is a critical point of f. Thus $\mathcal{E}(f)$ is finite, and every exceptional point is a superattracting periodic point and therefore a member of the Fatou set.

If $\mathcal{E}(f)$ contained three or more points, then f would map the region $\hat{\mathbb{C}} \setminus \mathcal{E}(f)$ to the thrice-punctured Riemann sphere, so that the set of iterates of f restricted to $\hat{\mathbb{C}} \setminus \mathcal{E}(f)$ would be normal by Montel's Theorem 2.9. Then $\mathcal{E}(f)$ and its complement would both be contained in the Fatou set, so that $J(f)$ is empty, contradicting Theorem 3.1. Hence $\mathcal{E}(f)$ can contain at most 2 elements. $\square$

We now prove a fundamental property of the Julia set as a dynamical system.

Theorem 3.9 (Transitivity)**.** *Let $z \in J = J(f)$ be an arbitrary point of the Julia set for a rational map f of degree $d \geq 2$. Let N be a neighborhood of z sufficiently small that $N \cap \mathcal{E}(f) = \emptyset$. Then*

$$U := \bigcup_{n \geq 0} f^n(N) = \hat{\mathbb{C}} \setminus \mathcal{E}(f).$$

Thus J is transitive *in the sense that f moves and expands neighborhoods of points of J to eventually cover the entire Riemann sphere with the exception of at most two points, neither of which is a member of the Julia set.*

Proof. Note that the iterates of f restricted to U cannot form a normal family, since U contains a neighborhood of a point of J. Since f maps U holomorphically into itself, we then have that the complement of U can contain at most two points by Montel's Theorem 2.9.

Since $f(U) \subset U$, for any $z \in \hat{\mathbb{C}} \setminus U$ we have that every iterated preimage of z is also an element of the complement $\hat{\mathbb{C}} \setminus U$, so that indeed f maps $\hat{\mathbb{C}} \setminus U$ into itself. Thus the grand orbit of any point of $\hat{\mathbb{C}} \setminus U$ is a subset of $\hat{\mathbb{C}} \setminus U$ and hence finite, or in other words $\hat{\mathbb{C}} \setminus U \subset \mathcal{E}(f)$.

Conversely, if $z \in \mathcal{E}(f)$, then $\mathrm{GO}(z) \subset \mathcal{E}$ cannot intersect N, so that $z \notin U$. Thus $U = \hat{\mathbb{C}} \setminus \mathcal{E}(f)$. $\square$

An immediate consequence of this theorem is the fact that J has an interior point if and only if $J = \hat{\mathbb{C}}$ is the entire Riemann sphere, as J is closed and the union of the images of a sufficiently small neighborhood of an interior point must be a subset of J which is dense in the Riemann sphere.

Corollary 3.9.1. *Let $z_0 \in \hat{\mathbb{C}} \setminus \mathcal{E}(f)$ be some point which is not exceptional (for instance, take $z_0 \in J$). Then the set $\{z : \exists n, f^n(z) = z_0\} \subset \mathrm{GO}(z_0)$ of iterated preimages of z_0 is dense in J.*

Proof. Since z_0 is not exceptional, given $z \in J$, any neighborhood U of z satisfies $f^n(U) \ni z_0$ for some n by Theorem 3.9. $\square$

Corollary 3.9.2 (Self-similarity)**.** *The Julia set $J = J(f)$ is* self-similar *in the sense that given a point $z_0 \in J$, the set*

$$\{z : (J, z) \text{ is locally conformally isomorphic to } (J, z')\}$$

is dense in J unless the following rare condition holds: For every sequence $\{z_j : j \geq 0\}$ with $f(z_j) = z_{j-1}$, there exists some $k > 0$ such that z_k is a critical point of f.

Proof. Suppose that the rare condition does not hold. By Theorem 3.9.1, the set of iterated preimages of z_0 is everywhere dense in J. Now suppose $z = z_m$ is contained in some backward orbit $z_0 \leftarrow\!\shortmid z_1 \leftarrow\!\shortmid \cdots$ under f such that no z_j, $0 \le j \le m$, is a critical point of f. Then f^m is conformal on a neighborhood N_1 of z, so that $N_2 := f^m(N_1) \ni z_0$ and $N_2 \cap J = f^m(N_1 \cap J)$. □

Corollary 3.9.3. *The Julia set $J = J(f)$ for a rational map f of degree 2 or higher contains no isolated points.*

Proof. By Theorem 3.1, there exists some $z_0 \in J$. In particular, z_0 is not an exceptional point, so $J \supset \mathrm{GO}(z_0)$ is an infinite set. Note also that J is compact as a closed subset of the compact space $\hat{\mathbb{C}}$. Thus J has some limit point $z \in J$. As the iterated preimages of z form a dense set of limit points, we have that J contains no isolated points. □

This concludes our discussion of the basic geometry of the Julia set.

4. Local Fixed Point Theory: The Kœnigs and Böttcher Maps

Consider now a fixed point $\hat{z}$ of a rational map f on the Riemann sphere. As above, a fixed point is *superattracting*, *geometrically attracting*, *indifferent*, or *repelling* according to the absolute value of its multiplier λ. We begin by motivating the use of the term *attracting* with a basic topological result.

Theorem 4.1. *Let $\hat{z}$ be a fixed point of f, with λ its multiplier. Then $|\lambda| < 1$ if and only if there exists a neighborhood U of $\hat{z}$ such that the iterates f^n are defined on U and such that the sequence $\{f^n|_U\}$ converges uniformly on U to the constant map $g(z) = \hat{z}$.*

Proof. Suppose first that $|\lambda| < 1$. We may assume up to composition with a Möbius transformation that $\hat{z} = 0$. Then the Taylor expansion of f near 0 is $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$. Choosing $C > |a_2|$, we have that there exists some $\varepsilon > 0$ such that $|f(z) - \lambda z| \le C|z^2|$ whenever $|z| < \varepsilon$.

Choose some c with $|\lambda| < c < 1$ and r with $0 < r < \min\left(\varepsilon, \frac{c-|\lambda|}{C}\right)$, and let $U := \{z : |z| < r\}$. Then for $z \in U$, we have

$$|f(z)| \le |\lambda z| + C|z^2| \le |\lambda||z| + (c - |\lambda|)|z| = c|z|.$$

Thus $|f^n(z)| \le c^n|z| < c^n r \to 0$ uniformly as $n \to \infty$ for $z \in U$.

Now suppose that a neighborhood U exists satisfying the properties stated in the claim. Then by the definition of uniform convergence, given $\epsilon > 0$ there exists N such that $|f^n(z)| < \epsilon/2$ for all $n \ge N$, for all $z \in U$. Choosing ϵ such that the disk $\mathbb{D}_\epsilon$ is a subset of U, we have that f^N maps $\mathbb{D}_\epsilon$ into the proper subset $\mathbb{D}_{\epsilon/2}$. Then by Schwarz's Lemma 2.2, the derivative λ^n of f^n at 0 satisfies $|\lambda^n| < 1$, so that $|\lambda| < 1$ as desired. □

We assert the following result, due to Kœnigs, which demonstrates a complete picture of the dynamics of rational maps near a geometrically attracting fixed point, showing that up to a change of coordinates, f is simply linear:

Theorem 4.2 (Kœnigs Normal Form). *If the multiplier λ satisfies $|\lambda| \notin \{0, 1\}$, then there exists a conformal change of coordinate $w = \varphi(z)$ on a neighborhood U of $\hat{z}$ such that $\phi(\hat{z}) = 0$ and $\varphi \circ f \circ \varphi^{-1}(w) = \lambda w$ for all $w \in U$. Moreover, φ is unique up to multiplication by a nonzero constant.*

As the geometrically attracting case is not relevant to our main result, we assert the above without proof, however the proof is quite similar to the proof of Theorem 4.3, taking $\varphi(z) = \lim_{k\to\infty} f^k(z)/\lambda^k$. There are also analogues to Corollary 4.3.1 (in fact, in the geometrically attracting case, φ itself extends holomorphically to all of $\mathcal{A}$) and Theorem 4.4, with similar proofs, though the behavior is far from identical to in the superattracting case (cf. [Mil06], pp 77-81).

We now focus our study on the *superattracting* case $\lambda = 0$, which is of particular importance to polynomial dynamics since these maps possess a superattracting point at ∞. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with a superattracting fixed point $\hat{z}$, and choose a local uniformizing parameter z which fixes 0. Then we have

$$f(z) = \sum_{j=n}^{\infty} a_j z^j,$$

with $n > 1$, $a_n \neq 0$. The power n of the lowest order term is called the *local degree* of the superattracting point.

Theorem 4.3 (Böttcher map)**.** *There exists a holomorphic change of coordinate $\hat{\phi} : U \to V$ between neighborhoods U of $\hat{z}$ and V of 0 such that $\hat{\phi}(0) = 0$ and $\hat{\phi} \circ f(z) = (\hat{\phi}(z))^n$. This map is unique up to multiplication by an $(n-1)$st root of unity.*

Proof. Choose some $(n-1)$st root c of a_n, so that $c^{n-1} = a_n$. Then conjugating f with multiplication by c, we have

$$cf(z/c) = z^n + b_1 z^{n+1} + b_2 z^{n+2} + \cdots = z^n(1 + b_1 z + b_2 z^2 + \cdots)$$

whose dynamics are analogous to those of f. Thus it suffices to assume f is itself monic, so that $f(z) = z^n(1 + b_1 z + b_2 z^2 + \cdots) = z^n(1 + g(z))$, where $g(z) = b_1 z + b_2 z^2 + \cdots$.

Choose some $0 < \varepsilon < 1/2$ such that $|g(z)| < 1/2$ when $|z| < \varepsilon$. Then for $|z| < 1/2$ we also have $|f(z)| \leq |z^n|(1 + 1/2) < 3/2^{n+1} \leq 3/8 < 1/2$, so that f maps the disk $\mathbb{D}_\varepsilon$ into itself. Also, $f(z) \neq 0$ for $0 < |z| < r$, since $|1 + g(z)| > 1/2$. Thus $f^m(z) = z^{n^m}(1 + n^{m-1}b_1 + \text{(higher terms)}$ takes $\mathbb{D}_\varepsilon$ into itself with the only zero occurring at $z = 0$.

Now define

$$\hat{\phi}_m(z) := (f^m(z))^{n^{-m}} = z(1 + n^{k-1}b_1 z + \cdots)^{n^{-m}} = z\left(1 + \frac{b_1}{n} z + \cdots\right),$$

where the rightmost term is obtained by taking the Taylor expansion of $\hat{\phi}_m$ near 0, implicitly choosing one of the n^m possible roots. Note that these $\hat{\phi}_m$ are chosen to satisfy the functional equation $\hat{\phi}_m(f(z)) = \left(\hat{\phi}_{m+1}(z)\right)^n$. The idea of the proof is then to show that the $\hat{\phi}_m$ converge uniformly on $\mathbb{D}_\varepsilon$ to some $\hat{\phi}$, which will then satisfy the equation $\hat{\phi}(f(z)) = \left(\hat{\phi}(z)\right)^n$.

To prove this convergence, we make the change of variables $z = e^\zeta$, where $\zeta \in \mathbb{H}_\varepsilon = \{x + iy : x < \log\varepsilon\}$. Under this substitution, f conjugates to

$$\begin{aligned}\tilde{f}(\zeta) &= \log(f(e^\zeta)) \\ &= \log(e^{n\zeta}(1 + g(e^\zeta))) \\ &= n\zeta + \log(1 + g(e^\zeta)) \\ &= n\zeta + \left(g(e^\zeta) - \frac{(g(e^\zeta))^2}{2} + \frac{(g(e^\zeta))^3}{3} - \cdots\right)\end{aligned}$$

where the final Taylor expansion provides an explicit choice of value for the logarithm function. As before, $\tilde{f}$ maps $\mathbb{H}_\varepsilon$ holomorphically into itself. Moreover, for $\zeta \in \mathbb{H}_\varepsilon$ we have that

$$|\tilde{f}(\zeta) - n\zeta| < 1 - 1/2 + 1/3 - \cdots = \log 2 < 1$$

by the Maximum Modulus Principle, as the map $z \mapsto \left(g(z) - \frac{(g(z))^2}{2} + \frac{(g(z))^3}{3} - \cdots\right)$ is holomorphic and defined in the region $|z-1| < 1/2$ with maximum modulus $\log 2$ on the boundary at $z = 1/2$.

Likewise, $\hat{\phi}_m$ conjugates to the map $\hat{\varphi}_m(\zeta) = \log(\hat{\phi}_m(e^\zeta)) = \tilde{f}^m(\zeta)/n^m$, which is again holomorphic on $\mathbb{H}_\varepsilon$. Then

$$\begin{aligned}|\hat{\varphi}_{m+1}(\zeta) - \hat{\varphi}_m(\zeta)| &= 1/n^{m+1}|\tilde{f}^{m+1}(\zeta) - n\tilde{f}^m(\zeta)| \\ &= n^{-(m+1)}|\tilde{f}(\tilde{f}^m(\zeta)) - n\tilde{f}^m(\zeta)| \\ &< n^{-(m+1)}.\end{aligned}$$

As the derivative of the exponential map has absolute value less than 1 throughout the left half plane, we have that the map reduces distances, so that indeed $|\hat{\phi}_{m+1}(z) - \hat{\phi}_m(z)| < n^{-(m+1)}$ for $z \in \mathbb{D}_\varepsilon$. Thus the sequence $\{\hat{\phi}_m\}$ satisfies the Cauchy condition for uniform convergence on $\mathbb{D}_\varepsilon$, so that as $m \to \infty$, $\hat{\phi}_m$ converges uniformly to a holomorphic limit $\hat{\phi}$. It follows immediately that $\hat{\phi}(f(z)) = \left(\hat{\phi}(z)\right)^n$.

To prove uniqueness, note that $\hat{\phi}$ conjugates f to the map $z \mapsto z^n$, so it suffices up to composition with ϕ to assume that $f(z) = z^n$. Suppose that some map $\tilde{\phi}(z) = a_1 z + a_k z^k + a_{k+1} z^{k+1} + \cdots$ conjugates f to itself. Then $\tilde{\phi}(z^n) = (\tilde{\phi}(z))^n$, so that

$$a_1 z^n + a_k z^{nk} + \cdots = a_1^n z^n + n a_1^{n-1} a_k z^{n+k-1} + \cdots$$

and hence, comparing terms, we see that $a_1^n = 1$ and $a_j = 0$ for $j > 1$. ☐

Thus in a neighborhood of a superattracting fixed point, the *Böttcher map* $\hat{\phi}$ conjugates f to the nth power map $z \mapsto z^n$. The following extension is quite relevant to the discussion of polynomial dynamics in the next section.

Corollary 4.3.1. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has a superattracting fixed point $\hat{z}$ with basin $\mathcal{A}$ and associated Böttcher map $\hat{\phi}$, then the map $z \mapsto |\hat{\phi}(z)|$ extends uniquely to a continuous map $|\hat{\phi}| : \mathcal{A} \to [0, 1)$ satisfying the functional equation $|\hat{\phi}(f(z))| = |\hat{\phi}(z)|^n$. Moreover, $|\hat{\phi}|$ is real analytic except at the iterated preimages of $\hat{z}$.*

Proof. Again we may assume up to a conformal automorphism of $\hat{\mathbb{C}}$ that $\hat{z} = 0$. For $z \in \mathcal{A}$, choose k sufficiently large that $f^k(z) \in \mathbb{D}_\varepsilon$ and simply define $|\hat{\phi}(z)| = |\hat{\phi}(f^k(z))|^{n^{-k}}$, noting that the value of $\hat{\phi}$ is independent of k for any z, k where it is defined. As the composition of analytic functions, we have that $|\hat{\phi}|$ is real analytic except when $f^k(z) = 0$, where the map $w \mapsto |w|^{n^{-k}}$ is not analytic. $\square$

Note that $\hat{\phi}$ is a conformal isomorphism between a neighborhood U of $\hat{z}$ and a sufficiently small disk $\mathbb{D}_\varepsilon$ centered at the origin. As before, we may then define a local inverse $\hat{\psi}_\varepsilon : \mathbb{D}_\varepsilon \to U$ such that $\hat{\psi}_\varepsilon(0) = \hat{z}$ and $\hat{\phi} \circ \hat{\psi}_\varepsilon(w) = w$ on $\mathbb{D}_\varepsilon$.

Theorem 4.4. *The local inverse $\hat{\psi}_\varepsilon$ extends uniquely to a function $\hat{\psi} : \mathbb{D}_r \to \mathcal{A}_0$, where $\mathbb{D}_r$ is an open disk of maximal radius $r \leq 1$, and $\mathcal{A}_0$ is the immediate basin of the superattracting point $\hat{z}$. If $r = 1$, then $\hat{\psi}$ is a conformal isomorphism from $\mathbb{D}$ to $\mathcal{A}_0$, and $\hat{z}$ is the only critical point in $\mathcal{A}_0$. Otherwise, if $r < 1$, then a critical point lies on the boundary $\partial\Big(\hat{\psi}(\mathbb{D}_r)\Big)$.*

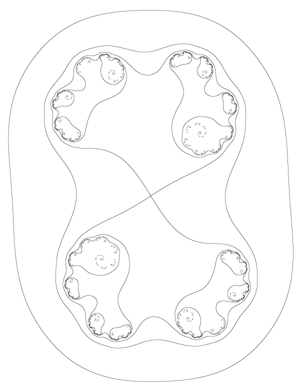


FIGURE 2. Level sets of $|\hat{\phi}|$, known as *equipotential lines*, for the Julia set of the map $z \mapsto z^2 + (0.5 + 0.1i)$. The third line from the outside is the equipotential line $|\hat{\phi}(z)| = r \approx 0.033742$, where $\hat{\psi}$ "pinches off" and becomes multivalued.

Proof. We may extend $\hat{\psi}_\varepsilon$ by analytic continuation to a holomorphic map $\hat{\psi}$ from a disk $\mathbb{D}_r$ of maximal radius $r \leq 1$ to its image $U = \psi(\mathbb{D}_r) \subseteq \mathcal{A}_0$. We claim that $\hat{\psi}$ is in fact a conformal isomorphism between $\mathbb{D}_r$ and U.

We first claim that $\hat{\psi}$ is conformal. To see this, suppose that if $\hat{\psi}'(w) = 0$ for some $w \in \mathbb{D}_r$. Then $\hat{\psi}'(w^n) = d/dw\, f(\hat{\psi}(w)) = f'(\hat{\psi}(w))\hat{\psi}'(w) = 0$, and thus we have a sequence $\{w^{n^k}\}_{k=0}^{\infty}$ of critical points. But this sequence converges to 0, contradicting that $\hat{\psi}_\varepsilon$ is a conformal isomorphism when $|z| < \varepsilon$.

It remains to show that $\hat{\psi}$ is one-to-one. Note that $|\hat{\phi}(\psi(w))| = |w|$ for $w \in \mathbb{D}_\varepsilon$, so by analytic continuation we have that this holds for all $w \in \mathbb{D}_r$. Thus if $\hat{\psi}(w_1) = \hat{\psi}(w_2)$, then $|w_1| = |w_2|$.

Let $s = \inf\{t : \exists w_1, w_2,\ w_1 \neq w_2,\ |w_1| = |w_2| = t,\ \hat{\psi}(w_2) = \hat{\psi}(w_2)\}$. Then there are sequences $w_1^{(j)}$, $w_2^{(j)}$ with $\hat{\psi}(w_1^{(j)}) = \hat{\psi}(w_2^{(j)})$ for all j and $|w_i^{(j)}| \to s$ as $j \to \infty$. But by compactness of $\hat{\mathbb{C}} \setminus \mathbb{D}_s$, these sequences must have accumulation points on the circle of radius s, two of which by continuity of $\hat{\psi}$ would also map to the same point under $\hat{\psi}$. Thus we may pick some w_1, w_2 of minimal absolute value such that

$\hat{\psi}(w_1) = \hat{\psi}(w_2)$. Since ψ is an open map, for $\tilde{w}_1$ sufficiently close to w_1, there exists $\tilde{w}_2$ near w_2 such that $\hat{\psi}(\tilde{w}_1) = \hat{\psi}(\tilde{w}_2)$. But we may take $|\tilde{w}_1| < |w_1|$, yielding a contradiction. Hence $\hat{\psi}$ is one-to-one on $\mathbb{D}_r$.

If $r = 1$, then the image U of $\hat{\psi}$ is certainly a subset of $\mathcal{A}_0$. If $U \subsetneq \mathcal{A}_0$, then the boundary ∂U must intersect $\mathcal{A}_0$. Let z be some point in this intersection. Then choosing a sequence $\{w_k\} \subset \mathbb{D}$ with $\hat{\psi}(w_k) \to z$, we see that $|w_k| = |\hat{\phi}(\hat{\psi}(w_k))| \to 1$, so that $|\hat{\phi}(z)| = 1$, which cannot occur. Thus in fact $U = \mathcal{A}_0$.

Now suppose that $r < 1$. Note that under the map $z \mapsto z^n$, the disk $\mathbb{D}_r$ is mapped to the proper subset $\mathbb{D}_{r^n}$, which is contained in the compact set $\overline{\mathbb{D}_{r^n}} \subset \mathbb{D}_r$. Thus $U = \hat{\psi}(\mathbb{D}_r)$ is mapped under f to some set $f(U)$ whose closure $\overline{f(U)}$ is a compact subset of U. By continuity of f, we have that $f(\overline{U}) \subset \overline{f(U)} \subset U \subset \mathcal{A}_0$. It follows that $\partial U \subset \mathcal{A}_0$.

It remains to show that ∂U contains a critical point of f. Otherwise, let $w_0 \in \partial\mathbb{D}_r$ be arbitrary, and let z_0 be some accumulation point for the curve $t \mapsto \hat{\psi}(tw_0), t \in [0, 1)$ as $t \to 1$. If z_0 is not a critical point of f, then by the Inverse Function Theorem, there is a neighborhood N of z_0 and a local inverse g which is holomorphic on N with $g(f(z)) = z$ for $z \in N$. But then we may simply extend $\hat{\psi}$ holomorphically on a neighborhood of w_0 by $\hat{\psi}(w) = g(\hat{\psi}(w^n))$. If there are no critical points on $\partial\mathbb{D}_r$, then these extensions allow $\hat{\psi}$ to be defined throughout some neighborhood of the closed disk $\overline{\mathbb{D}_r}$, which plainly contains some disk of slightly larger radius, contradicting maximality of r. Hence the boundary ∂U necessarily contains a critical point when $r < 1$. □

Consider now the dynamics of *polynomial* maps, which have a superattracting point at infinity. Define the *filled Julia set*

$$K = K(f) = \{z \in \mathbb{C} : \exists C > 0, \forall n, \ |f^n(z)| < C\}$$

to be the set of all z whose orbit under f remains bounded.

Lemma 4.5. *Let f be a polynomial of degree $d \geq 2$. The filled Julia set $K \subset \mathbb{C}$ is compact, with boundary $\partial K = J$ equal to the Julia set and with interior equal to the union of the bounded components of the Fatou set. The complement $\hat{\mathbb{C}} \setminus K$ is connected and equal to the basin $\mathcal{A}(\infty)$ of the superattracting point ∞.*

Proof. Write $f(z) = \sum_{k=0}^{d} a_k z^k$. Then clearly $f(z)/z^d \to a_d$ as $z \to \infty$. Choose some radius $R > \max(4/|a_d|, 1)$ such that $|f(z)/z^d - a_d| < |a_d|/2$ when $|z| > R$. Then for $|z| > R$, we have

$$|f(z)| > |a_d||z|^d/2 > R|a_d||z|^{d-1}/2 > 2|z|^{d-1} \geq 2|z|.$$

Thus any z with $|z| > R$ belongs to the attracting basin $\mathcal{A}$ of ∞. Clearly, then, if the orbit of some z is unbounded, then $|f^n(z)| > R$ for some n, so that $z \in \mathcal{A}$. Thus $K = \hat{\mathbb{C}} \setminus \mathcal{A}$, so that K is a compact subset of $\mathbb{C}$.

We now claim that $\partial\mathcal{A} = J$. If N is a neighborhood of some point $z \in J$, then by Theorem 3.9 we have that $N \cap \mathcal{A} \neq \emptyset$. As J is disjoint from $\mathcal{A}$, we have that $J \subset \partial\mathcal{A}$. Conversely, if $z' \in \partial\mathcal{A}$, then the iterates of f in a neighborhood N' of z' cannot form a normal family: since some elements of N' belong to $\mathcal{A}$ while others do not, the limit function g would necessarily have a jump discontinuity, so that in fact $z \in J$ and hence $J = \partial\mathcal{A} = \partial K$.

Now let U be some bounded component of the Fatou set $\hat{\mathbb{C}} \setminus J$. By the same argument as above, we have that $\partial U = J$. Also, if $z \in U$ and some iterate $f^m(z)$

satisfies $|f^m(z)| > R$, then by the Maximum Modulus Principle there exists some $z' \in \partial U$ with $|f^m(z')| > R$, so that $z' \in \mathcal{A}$. But $\partial U = J$ is disjoint from $\mathcal{A}$, so this is a contradiction. Thus $\mathcal{A}$ is connected, and any bounded component U of the Fatou set is contained in K (and hence in the interior of K). $\square$

The next result provides significant insight into polynomial dynamics.

Theorem 4.6. *The Julia set $J(f)$ for a polynomial f of degree $d \geq 2$ is connected if and only if the filled Julia set $K = K(f)$ contains every critical point of f. In this case, the complement $\mathbb{C} \setminus K$ is conformally isomorphic to the complement of the closed unit disk by the Böttcher map $\hat{\phi}$, conjugating f to the map $z \mapsto z^d$. If, instead, a critical point of f lies outside the filled Julia set, then $K(f)$, and hence $J(f)$, has uncountably many disconnected components.*

Proof. We may assume, up to conjugation $z \mapsto cf(z/c)$ with a linear change of variables, that f is monic. To understand the dynamics of f near ∞, we make the substitution $\zeta = 1/z$ and define $F(\zeta) = 1/f(1/\zeta)$, conjugating the superattracting point ∞ to the origin. Then

$$\lim_{\zeta \to 0} F(\zeta)/\zeta^d = \lim_{z \to \infty} z^d/f(z) = 1,$$

so that F has a superattracting fixed point at 0. Thus there is a Böttcher map $\hat{\phi} : \mathcal{A}(\infty) \to \mathbb{D}$ given by

$$\hat{\phi}(\zeta) = \lim_{k \to \infty} (F^k(\zeta))^{d^{-k}}$$

which satisfies $\hat{\phi} \circ F(\zeta) = (\hat{\phi}(\zeta))^d$. For $|\zeta|$ sufficiently small, this map is a conformal isomorphism with $\hat{\phi}'(0) = 1$.

Reversing the change of variables, we may define the reciprocal $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ by

$$\phi(z) = \frac{1}{\hat{\phi}\left(\frac{1}{z}\right)} = \lim_{k \to \infty} \left(f^k(z)\right)^{d^{-k}},$$

which is a conformal isomorphism from a neighborhood of ∞ to another neighborhood of ∞, with $\lim_{z \to \infty} \phi'(z) = 1$ and $\phi(f(z)) = (\phi(z))^d$ in a neighborhood on ∞.

Suppose first that the only critical point of f in the basin $\mathcal{A} = \mathcal{A}(\infty)$ of the superattracting fixed point ∞ is ∞ itself. By Theorem 4.4, we have that $\hat{\phi}$ extends to a conformal isomorphism from $\hat{\mathcal{A}}$, the basin of attraction of 0 under F, to the entire unit disk $\mathbb{D}$. Thus ϕ extends to a conformal isomorphism from $\mathcal{A} = \mathbb{C} \setminus K$ to $\mathbb{C} \setminus \overline{\mathbb{D}}$, so that the inverse $\psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ exists and is conformal.

It follows that the nested annuli $S_n = \{w : 1 < |w| < 1/n\}$ map under ψ to nested connected topological annuli, whose closures $\overline{S_n}$ are compact and contain the Julia set $J = \partial \mathcal{A}$. Thus the intersection $J = \bigcap_{n \geq 1} S_n$ is connected, and equivalently the filled Julia set K is connected.

Now suppose that there is some critical point of f in $\mathbb{C} \setminus K$. Then by Theorem 4.4, there exists some maximal $\hat{r} < 1$ such that the inverse $\hat{\psi}$ to $\hat{\phi}$ is a well-defined conformal isomorphism from $\mathbb{D}_{\hat{r}}$ onto some open subset $\hat{U} \subset \hat{\mathcal{A}}$ whose boundary $\partial \hat{U} \subset \hat{\mathcal{A}}$ is compact and contains at least one critical point. Inverting the change of variables, we have that there exists some minimal $r > 1$ such that the inverse to ϕ is a conformal isomorphism

$$\psi : \mathbb{C} \setminus \overline{\mathbb{D}}_r \xrightarrow{\cong} U \subset \mathbb{C} \setminus K,$$

where $\partial U \subset \mathbb{C} \setminus K$ is compact and contains some critical point $z_0 \in \partial U$ of f. Then the critical value $z_1 = f(z_0)$ satisfies $|\phi(z_1)| = r^d > r$, so that $z_1 \in U$. Now consider the ray $C = \{t\phi(z_1) : t \in [1, \infty)\} \subset \mathbb{C} \setminus \overline{\mathbb{D}}_r$ and define the *external ray to* z_1 *for the set* K to be the image $R = \psi(C) \subset U$.

Then the preimage $f^{-1}(R) \subset \overline{U}$ is

$$f^{-1}(\psi(C)) = \psi\left(\sqrt[d]{C}\right),$$

which evidently consists of d distinct components, corresponding to the d branches of $\sqrt[d]{C}$. Moreover, the representation above shows that these components are themselves external rays, corresponding to the d preimages of z_1. But as z_0 is a critical point, the expression $(z - z_0)^2$ divides $f(z) - z_1$, so that the solution $z = z_0$ to the equation $f(z) = z_1$ has multiplicity at least 2. Hence there are at least two external rays R_1, R_2 in the preimage of R which land at z_0. Since ψ is a conformal isomorphism when $|z| > r$, R_1 and R_2 intersect only at z_0 and thus the complement $\mathbb{C} \setminus (R_1 \cup R_2)$ consists of two disjoint open components V_1 and V_2.

Now note that each $f(V_j)$ is an open set by the Open Mapping Theorem. Suppose that some point p lies on the boundary of $\partial f(V_j)$. Then there is a sequence $\{p_n\}_{n=1}^{\infty} \subset V_j$ with $f(p_n) \to p$ as $n \to \infty$. As the sequence $\{p_n\}$ certainly does not have a subsequence tending to infinity, we may take it to be bounded, so that there is a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ converging to some $\hat{p}$. But $f(\hat{p}) = p \in \partial f(V_j)$ and $f(V_j)$ is open, so $\hat{p}$ cannot be an element of V_j and hence must lie on the boundary $\partial V_j = R_1 \cup R_2$. Thus $p \in f(R_1 \cup R_2) = R$, so that $\partial f(V_j) = R$.

As $\mathbb{C} \setminus R$ is connected, we have that $f(V_j) \supset \mathbb{C} \setminus R \supset K$. Thus we may set $J_1 = J \cap V_1$ and $J_2 = J \cap V_2$, with $f(J_1) = f(J_2) = J$ and $J_1 \cup J_2 = J$ (as certainly $J \cap R_j = \emptyset$). Note that J_1 and J_2 are compact. Furthermore, we may let $J_{\varepsilon_1, \varepsilon_2} = J_{\varepsilon_1} \cap f^{-1}(J_{\varepsilon_2})$, $\varepsilon_j \in \{1, 2\}$ to partition J into 4 disjoint compact sets, and we may continue this process inductively to split J into 2^{n+1} disjoint compact sets $J_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$ for any $n \geq 1$. Moreover, by definition, we have that

$$J_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \varepsilon_{n+1}} \subset J_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n},$$

so that for any infinite sequence $\varepsilon_1, \varepsilon_2, \ldots$ of 1s and 2s, the intersection

$$J_{\varepsilon_1, \varepsilon_2, \ldots} := \bigcap_{k=1}^{\infty} J_{\varepsilon_1, \ldots, \varepsilon_k}$$

is a compact, nonempty subset of J. Thus we see that in fact J has a distinct component for each sequence $\varepsilon_1, \varepsilon_2, \ldots$ of 1s and 2s, i.e. J consists of uncountably many disconnected components. $\square$

5. The Quadratic Family

In this final section we focus entirely on the *quadratic family* $\mathcal{Q}$ consisting of all maps of the form $f_c : z \mapsto z^2 + c$ with parameter $c \in \mathbb{C}$.

Remark 5.1. It follows from Theorem 4.6 that the Julia set for the quadratic map $f_c(z) = z^2 + c$ is connected if and only if the critical orbit $\{f_c^n(0) : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{C}$. Furthermore, if the Julia set if not connected, then it is totally disconnected, forming a topological Cantor set.

This motivates the following definition:

Definition 5.2. The *Mandelbrot set* is the set $\mathcal{M} \subset \mathbb{C}$ of all parameters c for which the Julia set of f_c is connected. Equivalently, $\mathcal{M} = \{c : \exists C, \forall n \geq 0, |f_c^n(0)| < C\}$ is the set of all c for which the orbit of 0 under f_c remains bounded.

It is immediate from the second definition that $\mathcal{M}$ is a compact subset of $\mathbb{C}$, since f_c^n depends continuously on c and since if $|c| > 2$, then $|f_c(0)| = 2 + \varepsilon$ for some $\varepsilon > 0$, so that $|f_c^n(z)| \geq 2 + n\varepsilon \to \infty$. The geometry of $\mathcal{M}$ is highly complex, though the following result provides significant insight.

Theorem 5.3 (Douady, Hubbard)**.** *The Mandelbrot set $\mathcal{M}$ is connected.*

Proof. Let $\phi_c : \mathbb{C} \setminus K_c \to \mathbb{C} \setminus \overline{\mathbb{D}}$ be the (inverted) Böttcher map for f_c in the basin $\mathcal{A}$ of ∞, as defined in the proof of Theorem 4.6. Define a function $\Phi : \hat{\mathbb{C}} \setminus \mathcal{M} \to \mathbb{C} \setminus \overline{\mathbb{D}}$, known as the Mandelbrot Böttcher map, by

$$\Phi(c) = \phi_c(c), \ \Phi(\infty) = \infty.$$

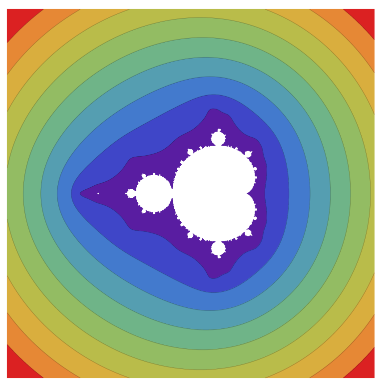


FIGURE 3. Contour lines for absolute value of the Mandelbrot Böttcher map Φ, which maps the complement of the Mandelbrot set conformally onto the complement of the closed unit disk.

Lemma 5.4. *Φ is holomorphic on $\hat{\mathbb{C}} \setminus \mathcal{M}$.*

Proof. Write $f_c(z) = z^2(1 + g_c(z))$, where $g_c(z) = c/z^2 \in O(z^{-2})$. Note that g_c depends holomorphically on c. Let $\phi_c^{(n)}(z) = (f_c^n(z))^{2^{-n}} = z + a_2 z^2 + \cdots$ (where the last expression chooses an explicit root), so that $\phi_c^{(n)} \to \phi_c$ pointwise. We claim that this convergence is uniform in c and z together. To see this, note that we may write ϕ_c as an infinite product

$$\phi_c(z) = z \cdot \prod_{k=0}^{\infty} \frac{\phi_c^{(k+1)}(z)}{\phi_c^{(k)}(z)} = z \cdot \frac{\phi_c^{(1)}(z)}{z} \cdot \frac{\phi_c^{(2)}(z)}{\phi_c^{(1)}(z)} \cdot \frac{\phi_c^{(3)}(z)}{\phi_c^{(2)}(z)} \cdots$$

whose convergence implies the uniform convergence $\phi_c^{(k)} \to \phi$. We have that

$$\begin{aligned}
\log|\phi_c(z)| &= \sum_{k=0}^{\infty} \left| \log \left(\frac{\sqrt{f_c(f_c^k(z))}}{f_c^k(z)} \right)^{1/2^k} \right| \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k} \left| \log \left(\frac{f_c^k(z)\sqrt{(1+g_c(f^k(z)))}}{f_c^k(z)} \right) \right| \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \left| \log \left(1 + g_c(f_c^k(z))\right) \right| \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \left| \log \left(1 + \frac{c}{(f_c^k(z))^2}\right) \right| \\
&\leq \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} |\log 2|
\end{aligned}$$

which converges and where the final inequality is satisfied for c in a neighborhood of 0 and z in some neighborhood of ∞.

As the maps $(z, c) \mapsto \phi_c^{(k)}(z)$ are clearly holomorphic, the Weierstrass uniform convergence theorem gives that the limit map $(z, c) \mapsto \phi_c(z)$ is also holomorphic for c in a neighborhood of 0 and z in a neighborhood of ∞. By analytic continuation, the extension of this map to $\{(z, c) : z \notin K_c = K(f_c)\}$ is also holomorphic, as we have already established convergence in Theorem 4.4. Hence the map $\Phi = [(z, c) \mapsto \phi_c(z)] \circ [c \mapsto (c, c)]$ is holomorphic on $\mathbb{C} \setminus \mathcal{M}$.

Note that $\lim_{c\to\infty} \phi_c^{(k)}(c) = \infty$ for all k, so that uniform convergence is immediate at ∞, and as each $\phi_c^{(k)}(c)$ is holomorphic on $\hat{\mathbb{C}}$, we have that Φ is holomorphic at ∞ and hence on all of $\hat{\mathbb{C}} \setminus \mathcal{M}$. $\square$

Lemma 5.5. *The function* $|\Phi|$ *extends continuously to the boundary* $\partial\mathcal{M}$*, with* $\lim_{c\to\mathcal{M}} |\Phi(c)| = 1$.

Proof. This will follow from the statement that the map $\mathrm{G} : (z, c) \mapsto \log|\phi_c(z)|$ is continuous. We already know that this is the case when $(z, c) \notin A := \{(z, c) : z \in K_c\}$. Furthermore, $\mathrm{G}(z, c) = 0$ when $(z, c) \in A$. Thus it is necessary only to prove continuity at the boundary ∂A.

To see this, let $r > 0$ be given. Then there exists r' such that K_c is contained in the ball $\mathcal{B}(0, r') = \{z : |z| < r'\}$ for all $|c| \leq r$. For instance, take $r' = r + 2$, so that for $|c| \leq r$, $|z| > r'$,

$$|f(z)| = |z^2 + c| \geq |z|^2 - r > |z|^2 - r|z| \geq |z|(r' - r) = 2|z|.$$

Now let $R := \mathrm{G}(r', r)$, so that if $|c| \leq r$ and $\mathrm{G}(z, c) \geq R$, then $|z| \geq r'$ by the maximum modulus principle. Thus the set $B = \{(z, c) : \mathrm{G}(z, c) \geq R\} \cap \{(z, c) : |c| \leq r\}$ is bounded away from A. Since G is continuous except possibly at ∂A, we have that in fact B is closed, since preimages of closed sets are closed.

Now let $\varepsilon > 0$ be given and define

$$A_n := \{(z, c) : \mathrm{G}(z, c) \geq \varepsilon\} = \{(z, c) : \mathrm{G}(f_c^n(z), c) \geq 2^n\varepsilon\}.$$

Choosing n sufficiently large that $2^n\varepsilon \geq R$, we have that the set

$$\{\, (z, c) : \mathrm{G}(f_c^n(z), c) \geq 2^n\varepsilon \,\} \cap \{\, (z, c) : |c| \leq r \,\}$$

is closed, so by continuity of the map $(z,c) \mapsto (f^n(z), c)$, we have that $A_n \cap \{(z,c) : |c| \leq r\}$ is closed. But certainly $A_n \cap \{(z,c) : |c| \geq r\}$ is closed for r sufficiently large, so that in face A_n is closed. Hence $\mathrm{G}^{-1}([0,\varepsilon)) \subset \mathbb{C}^2$ is open for all $\varepsilon > 0$, so that G is continuous. □

Recall from the proof of Theorem 3.2 that a map is *proper* if the preimage of any compact set is compact.

Lemma 5.6. *Φ is a proper map.*

Proof. Let $K \subset \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be compact. By the closed map lemma, Φ is proper on $\hat{\mathbb{C}} \setminus U$ for any open neighborhood U of $\mathcal{M}$. Thus if $\overline{\Phi^{-1}(K))} \cap \mathcal{M} = \emptyset$, then $\Phi^{-1}(K)$ is compact. As K is bounded away from $\mathbb{D}$, we have that $\Phi^{-1}(K)$ is bounded away from M and hence compact. □

Lemma 5.7. *Under Φ, the image of any open set is open, and the image of any closed set is closed.*

Since Φ is holomorphic, the first statement follows from the open mapping theorem. To see that Φ is also a closed map, let $F \subset \hat{\mathbb{C}} \setminus \mathcal{M}$ be closed. Let N be some neighborhood of F whose closure does not intersect $\mathcal{M}$. As Φ is an open map, $\Phi(N)$ is a neighborhood of $\Phi(F)$, and as $\hat{\mathbb{C}} \setminus \mathcal{M}$ is precompact, $\overline{N}$ is in fact compact. Then $\Phi|_{\overline{N}}$ is a continuous mapping from a compact space to a Hausdorff space, so it is a closed map by the closed map lemma. Thus $\Phi(F)$ is closed in the neighborhood $\Phi(N)$, or equivalently, in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Hence Φ is also a closed map.

Lemma 5.8. *Φ maps $\hat{\mathbb{C}} \setminus \mathcal{M}$ onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.*

Proof. As Φ is an open map, the image $\Phi(\hat{\mathbb{C}} \setminus \mathcal{M})$ is open in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Also, it follows from Lemma 5.5 that $\Phi(\partial(\hat{\mathbb{C}} \setminus \mathcal{M})) = \partial(\Phi(\hat{\mathbb{C}} \setminus \mathcal{M})) = \partial\mathbb{D}$, since Φ takes topological circles around $\mathcal{M}$ to topological circles around $\mathcal{M}$. Thus in fact $\Phi(\hat{\mathbb{C}} \setminus \mathcal{M}) = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. □

Lemma 5.9. *Φ is one-to-one.*

Proof. Let $\zeta = 1/z$ be a local uniformizing parameter in a neighborhood of ∞, and let $\hat{\Phi}(z\zeta) = 1/\Phi(1/\zeta)$ be the map which conjugates a neighborhood of ∞ to one of the origin. By Lemma 5.6, the preimage of any point $\zeta_0 \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ under Φ is finite. Thus there is some closed loop C which encloses every point of $\hat{\Phi}^{-1}(0)$. By the argument principle (cf. [Con78], pg. 123), we have that

$$\#(\Phi^{-1}(\infty)) = \#(\hat{\Phi}^{-1}(0)) = \frac{1}{2\pi i} \int_C \frac{\hat{\Phi}'(z)}{\Phi(z) - z_0} dz.$$

But $\Phi^{-1}(\infty) = \{\infty\}$, so that the integral on the right must equal 1. But this integral is constant for any ζ enclosed by C, so that $\#(\Phi^{-1}(z)) = 1$ in a neighborhood of ∞, i.e. whenever $|z| \geq R$ for some $R > 0$. In fact, we may let C tend to the boundary $\partial\mathbb{D}$, so that since $\hat{\Phi}$ is surjective, the preimage $\hat{\Phi}^{-1}(\zeta)$ contains one element for any $\zeta \in \mathbb{D}$, and hence $\Phi^{-1}(z)$ contains one element for any $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$. □

Thus Φ is a bijection which is an open map, so it is a homeomorphism. Hence $\hat{\mathbb{C}} \setminus \mathcal{M}$ is homeomorphic to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, so that the complement of $\mathcal{M}$ is simply connected. Equivalently, $\mathcal{M}$ is connected. □

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