HILBERT'S THEOREM ON IMMERSION OF THE HYPERBOLIC PLANE

JOHN M. DEWHURST

Abstract. Hilbert's theorem is a key result in early differential and hyperbolic geometries, proving that no complete model of hyperbolic geometry can be isometrically immersed in $\mathbb{R}^3$. This expository paper presents a proof of Hilbert's theorem, much of which follows the same logic used in Hilbert's original proof.

Contents

1. Introduction

For hundreds, if not thousands, of years, geometers tried and failed to prove Euclid's parallel postulate, but in the early 19th century several mathematicians discovered what is now known as hyperbolic geometry. Lobachevsky, Bolyai, and a few others independently formulated self-consistent geometries in which Euclid's parallel postulate did not hold. Their discovery set off a great deal of exploration in geometry, which began to converge with another blossoming field at the time.

A few years earlier, Gauss had published his *Theorema Egregium* on the curvature of surfaces, the crowning result in the early differential geometry of surfaces. Gauss had investigated surfaces in three dimensions, but the *Theorema Egregium* showed that the curvature of a surface was intrinsic to the surface itself. That

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distinction allowed for the treatment of geometry on surfaces and the construction of an abstract surface that modeled the geometry discovered by Lobachevsky and others, called the hyperbolic plane.

It was not yet known, however, whether such a surface could exist in three dimensional space. The hyperbolic plane was shown to have constant negative Gaussian curvature, but no such 'complete' surface in $\mathbb{R}^3$ was known. There exist surfaces with constant negative curvature, but they were found to have holes or edges, unlike the models of the hyperbolic plane. Some surfaces have negative-everywhere curvature and no edges, but have regions that are almost flat, which also contradicted the hyperbolic plane. In 1901, David Hilbert proved that such a surface cannot exist in $\mathbb{R}^3$.

In this paper, we first lay out the necessary tools of elementary differential geometry for Hilbert's theorem. Section 3 presents a few particular properties that we then use in the proof of Hilbert's theorem, after which some generalizations are discussed.

2. Properties of Surfaces

We begin by laying out the tools to describe the geometry of surfaces.

Definition 2.1. A *surface* is a set S with a family of injective maps $\mathbf{x}_a : U_a \to S$ from open sets $U_a \subset \mathbb{R}^2$ with the following properties:

(a) The images $\bigcup_a \mathbf{x}_a(U_a) = S$ cover S
(b) For any two U_a, U_b with non-disjoint images $V := \mathbf{x}_a(U_a) \cap \mathbf{x}_b(U_b) \neq \emptyset$, the preimages $\mathbf{x}_a^{-1}(V), \mathbf{x}_b^{-1}(V)$ are open, and the maps $\mathbf{x}_a^{-1} \circ \mathbf{x}_b, \mathbf{x}_b^{-1} \circ \mathbf{x}_a$ (restricted to the appropriate sets) are differentiable.

For $p \in U_a$, we call $\mathbf{x}_a$ a *parametrization of* S *at* p, and $V := \mathbf{x}_a(U_a)$ a *coordinate neighborhood of* $\mathbf{x}(p)$. For a point $q = \mathbf{x}(u,v) \in V$, we call $(u,v) \in \mathbb{R}^2$ the *coordinates of* q *in the parametrization* $\mathbf{x}_a$.

This definition characterizes a surface as any topological space that is locally diffeomorphic to $\mathbb{R}^2$.

Example 2.2 (One-Sheet Hyperboloid)**.** The one-sheet hyperboloid H_1 (Fig. 2.3) given by

$$x^2 + y^2 - z^2 = 1$$

can be parametrized except for one meridian by

$$\mathbf{x}_1 : (-\pi, \pi) \times \mathbb{R} \to H_1 \subset \mathbb{R}^3$$
$$\mathbf{x}_1(\theta, v) = (\cosh v \cos\theta,\ \cosh v \sin\theta,\ \sinh v),$$

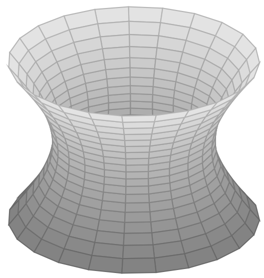
and the same map can be rotated around the z axis to cover H_1:

$$\mathbf{x}_2(\theta, v) = (-\cosh v \cos\theta,\ -\cosh v \sin\theta,\ \sinh v).$$

Each map is bijective onto its image and differentiable with non-singular differential, so the compositions $\mathbf{x}_1^{-1} \circ \mathbf{x}_2$, $\mathbf{x}_2^{-1} \circ \mathbf{x}_1$ are differentiable. Hence H_1 equipped with $\mathbf{x}_1, \mathbf{x}_2$, and $U_1 = U_2 = (-\pi, \pi) \times \mathbb{R}$ is a surface.

Definition 2.4. A function $\varphi : S_1 \to S_2$ from surface S_1 to surface S_2 is *differentiable* at $p \in S_1$ if for parametrizations $\mathbf{x}_p$ of S_1 at p and $\mathbf{y}_{\varphi(p)}$ of S_2 at $\varphi(p)$ the map $\mathbf{y}_{\varphi(p)}^{-1} \circ \varphi \circ \mathbf{x}_p$ is differentiable. A differentiable function $\alpha : I \to S_1$ from an interval I to S_1 is a *curve* in S_1.

Figure 2.3. One-Sheet Hyperboloid



Definition 2.5. Let $\alpha : I \to S$ be a curve in a surface S, and let D be the set of differentiable functions $f : S \to \mathbb{R}$. We define the *tangent vector of* α *at* t_0 as the function $\alpha'(t_0) : D \to \mathbb{R}$ given by

$$[\alpha'(t_0)](f) = \frac{d(f \circ \alpha)}{dt}\bigg|_{t_0}$$

A *tangent vector to* S *at* $p \in S$ is the tangent vector at 0 of some curve α with $\alpha(0) = p$.

Note in particular that in $\mathbb{R}^2$ as a surface, this definition coincides with the usual notion of tangent vectors in $\mathbb{R}^2$.

In a neighborhood of $p \coloneqq \alpha(0)$ we can express the curve $\alpha(t) = \mathbf{x}(u(t), v(t))$ and the function $f = f(u, v)$ and write

$$\begin{aligned}[\alpha'(0)](f) &= \frac{d}{dt}\Big(f[\mathbf{x}(u(t), v(t))]\Big)_0 \\ &= u'(0)\frac{\partial(f \circ \mathbf{x})}{\partial u}\bigg|_0 + v'(0)\frac{\partial(f \circ \mathbf{x})}{\partial v}\bigg|_0,\end{aligned}$$

hence we distinguish the function $\hat{\mathbf{u}}(f) \coloneqq \frac{\partial(f \circ \mathbf{x})}{\partial u}\Big|_0$ and likewise $\hat{\mathbf{v}}$ analogously. We can consider $\hat{\mathbf{u}}$ to be the tangent vector to the coordinate curve $\alpha(t) = \mathbf{x}(t, 0)$ at 0 (and similarly for $\hat{\mathbf{v}}$). One can directly check that the set of tangent vectors at p is a vector space under usual function operations, with $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\}$ as a basis. We will denote this vector space $T_p(S)$.

Definition 2.6. Let S_1 and S_2 be surfaces and $\varphi : S_1 \to S_2$ be differentiable at $p \in S_1$. For each $w \in T_p(S_1)$, choose a curve α_w in S_1 with $\alpha(0) = p$, $\alpha'(0) = w$. Then the *differential of* φ *at* p is the linear map $d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ given by

$$d\varphi_p(w) = (\varphi \circ \alpha_w)'(0).$$

Definition 2.7. Let S be a surface. A *Riemannian metric on* S is an inner product $\langle \cdot, \cdot \rangle_p$ on the tangent plane $T_p(S)$ at each point p such that for every p in S there is some parametrization $\mathbf{x} : U \to S$ of p on which the functions

$$E(q) \coloneqq \langle \hat{\mathbf{u}}, \hat{\mathbf{u}} \rangle_q \qquad F(q) \coloneqq \langle \hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle_q \qquad G(q) \coloneqq \langle \hat{\mathbf{v}}, \hat{\mathbf{v}} \rangle_q$$

are differentiable at every $q \in U$.

From now on we will assume unless stated otherwise that every surface comes equipped with a Riemannian metric.

2.1. **The First Fundamental Form.** A Riemannian metric on a surface S allows us to define a quadratic form $I_p : T_p(S) \to \mathbb{R}$ by

$$I_p(w) = \langle w, w \rangle_p,$$

called the *first fundamental form of* S. Notice that $\sqrt{I_p}$ defines a norm on $T_p(S)$, so for a vector $w \in T_p(S)$ we write $|w| = \sqrt{I_p(w)}$.

Using the first fundamental form, we can define various metric properties on abstract surfaces.

Definition 2.8. The *arc length* of a curve α in S is the value

$$L = \int_I \sqrt{I_p(\alpha'(t))}\, dt.$$

Recalling that in a parametrization $\mathbf{x}$, α' can be expressed $\alpha'(t) = u'(t)\hat{\mathbf{u}} + v'(t)\hat{\mathbf{v}}$, we can rewrite the arc length as

$$L = \int_I \sqrt{(u')^2 E + 2u'v'F + (v')^2 G}\, dt$$

where E, F, and G are evaluated at $\alpha(t)$. We also say that a curve $\alpha : I \to S$ is *parametrized by arc length* if $|\alpha'(t)| = 1$ for all $t \in I$.

For the standard inner product of vectors $v, w \in \mathbb{R}^3$, we have that $\langle v, w \rangle = |v||w| \cos\theta$, where θ is the angle between v and w, which leads us to the following definition in an abstract surface:

Definition 2.9. The angle θ between the coordinate directions $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ at a point $p \in S$ is given by $\cos\theta = \left(\frac{F}{\sqrt{EG}}\right)$.

We can also define area in a surface, as follows.

Definition 2.10. A *simple region* of S is a subset $R \subset S$ that is the image by a differentiable homeomorphism of a closed disc in $\mathbb{R}^2$.

To compute the area, we note that in $\mathbb{R}^2$, the area of the parallelogram formed by vectors v and w is given by $\sqrt{|v|^2|w|^2 - \langle v, w \rangle^2}$, where $\langle v, w \rangle$ is the usual inner product of $\mathbb{R}^2$. We adapt this to the following definition of the area.

Definition 2.11. The *area* of a simple region $R \subset S$ contained in a parametrization $\mathbf{x}$ is the value

$$\iint_{\mathbf{x}^{-1}(R)} \sqrt{EG - F^2}\, du\, dv$$

We claim that this value is independent of the parametrization chosen, and various proofs of this fact can be found (e.g. [1, p.100]).

2.2. **Isometry and Immersion.** To proceed further, we need to introduce the tools that describe maps between surfaces, as well as notions of a surface as a subset of $\mathbb{R}^3$.

Definition 2.12. A bijection $\varphi : S_1 \to S_2$ is a *diffeomorphism* if φ and φ^{-1} are both differentiable. A map $\psi : S_1 \to S_2$ is a *local diffeomorphism at* $p \in S_1$ if there is some neighborhood V of p such that the restriction $\bar{\psi} : V \to \psi(V)$ of ψ to V is a diffeomorphism.

We say ψ is a *local diffeomorphism* if it is a local diffeomorphism at every point of S_1. Surfaces S_1 and S_2 are *diffeomorphic* if there exists a diffeomorphism between them.

Diffeomorphism captures the concept of a smooth map between surfaces, and we similarly define the stronger condition that a map preserves a surface's metric.

Definition 2.13. A diffeomorphism $\varphi : S_1 \to S_2$ is an *isometry* if for all $p \in S_1$ and $v, w \in T_p(S_1)$, we have

$$\langle v, w\rangle_p = \langle d\varphi_p(v), d\varphi_p(w)\rangle_{\varphi(p)}.$$

A local diffeomorphism $\psi : S_1 \to S_2$ is a *local isometry* if for all $p \in S_1$ there exists a neighborhood V of p such that the restriction of ψ to V is an isometry onto $\psi(V)$. Surfaces S_1 and S_2 are said to be *isometric* if there exists an isometry between them.

Remark 2.14. It follows immediately that isometry preserves the first fundamental form; if φ is a local isometry at p, then $I_p(w) = I_{\varphi(p)}(d\varphi_p(w))$. Additionally, a Riemannian metric is symmetric and the first fundamental form captures its diagonal values, so a map that preserves the first fundamental form must be a (local) isometry. Hence all the metric properties discussed so far are preserved under local isometry, and in general the properties that are preserved by isometry are called the *intrinsic geometry* of a surface.

Example 2.15 (Hypberbolic plane)**.** Let $\mathbb{H}^2 \coloneqq \mathbb{R}^2$ be a surface, and define a Riemannian metric by

$$E(u,v) = 1 \qquad\qquad F(u,v) = 0 \qquad\qquad G(u,v) = e^{2u}$$

The surface $\mathbb{H}^2$ is called the hyperbolic plane.

We can see immediately that the hyperbolic plane is diffeomorphic to the Euclidean plane ($\mathbb{R}^2$ equipped with $E = G = 1, F = 0$), but not isometric. Also note that the map $\varphi : \mathbb{H}^2 \to (0,\infty) \times \mathbb{R}$, $\varphi(u,v) = (v, e^{-u})$ is an isometry to the Poincaré half-plane model of hyperbolic geometry.

In Example 2.2, we defined H_1 as a subset of $\mathbb{R}^3$, whereas we define the hyerbolic plane abstractly, without reference to an ambient space, which leads us to formalize the notion of a surface 'sitting in' a space.

Definition 2.16. A differentiable map $\varphi : S \to \mathbb{R}^3$ is an *immersion (into* $\mathbb{R}^3$*)* if the differential $d\varphi_p$ is non-singular (i.e. injective) for all p. If also for all $p \in S$,

$$\langle v, w\rangle_p = \langle d\varphi_p(v), d\varphi_p(w)\rangle_{\varphi(p)},$$

where the second inner product is the usual inner product of $\mathbb{R}^3$, then φ is an *isometric immersion.*

Note that an immersion φ need not be injective, only its differential; an immersion requires only that the tangent space not collapse to a line (or point) anywhere. An immersion into any differentiable manifold can also be naturally defined, but only $\mathbb{R}^3$ is used in this paper. In the example of the hyperbolic plane, the map

$\varphi(u, v) = (u, v, 0)$ is an immersion but is not isometric, and Hilbert's theorem proves the non-existence of an isometric immersion.

Definition 2.17. An (isometric) immersion $\varphi : S \to \mathbb{R}^3$ is an *(isometric) embedding* if it is a homeomorphism onto the image $\varphi(S)$.

Unlike an immersion, an embedding must be injective. A *regular* or *embedded surface* is the image by an isometric embedding of a surface into $\mathbb{R}^3$.

2.3. **The Gauss Map and Second Fundamental Form.** At a point p of a regular surface S, we can treat $T_p(S)$ as a vector subspace of $\mathbb{R}^3$, in which the basis $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\}$ of $T_p(S)$ allows us to define a unit normal vector $N \in \mathbb{R}^3$ by $N_p = \frac{\hat{\mathbf{u}} \times \hat{\mathbf{v}}}{|\hat{\mathbf{u}} \times \hat{\mathbf{v}}|}$.

Definition 2.18. The map $N : S \to S^2,\ N(p) = \frac{\hat{\mathbf{u}} \times \hat{\mathbf{v}}}{|\hat{\mathbf{u}} \times \hat{\mathbf{v}}|}$, where S^2 denotes the unit sphere in $\mathbb{R}^3$, is called the *Gauss map*.

Remark 2.19. The Gauss map is defined by reference to $\mathbb{R}^3$; hence the properties developed from the Gauss map are referred to as the *extrinsic geometry* of an embedded surface.

Remark 2.20. The Gauss map is defined independently of the parametrization $\mathbf{x}$ up to orientation. For any two parametrizations $\mathbf{x}(u, v)$, $\mathbf{y}(s, t)$ of a coordinate neighborhood of a point $p \in S$, either $\frac{\hat{\mathbf{s}} \times \hat{\mathbf{t}}}{|\hat{\mathbf{s}} \times \hat{\mathbf{t}}|} = \frac{\hat{\mathbf{u}} \times \hat{\mathbf{v}}}{|\hat{\mathbf{u}} \times \hat{\mathbf{v}}|}$ or $\frac{\hat{\mathbf{t}} \times \hat{\mathbf{s}}}{|\hat{\mathbf{t}} \times \hat{\mathbf{s}}|} = \frac{\hat{\mathbf{u}} \times \hat{\mathbf{v}}}{|\hat{\mathbf{u}} \times \hat{\mathbf{v}}|}$. Reversing the coordinates of a parametrization is equivalent to a sign change in N, so the choice of parametrization determines an *orientation* on V.

Definition 2.21. The *Gaussian curvature* K at a point $p \in S$ is the determinant $\det(dN_p)$ of the differential of the Gauss map at p.

It follows from Remark 2.20 that Gaussian curvature is independent of parametrization.

As with the first fundamental form, we can define a quadratic form on embedded surfaces:

Definition 2.22. The quadratic form $II_p : T_p(S) \to \mathbb{R},\ II_p(w) = -\langle dN_p(w), w\rangle$ is called the *second fundamental form of S at p.*

It can be shown that the differential of the Gauss map is self-adjoint (see [1, Proposition 1, p.142]), which allows us to express the second fundamental form of a vector $w = \alpha\hat{\mathbf{u}} + \beta\hat{\mathbf{v}}$ in some parametrization as

$$II_p(w) = \alpha^2 e + 2\alpha\beta f + \beta^2 g$$

with the values

$$\begin{aligned} e &= -\langle dN_p(\hat{\mathbf{u}}), \hat{\mathbf{u}}\rangle \\ f &= -\langle dN_p(\hat{\mathbf{u}}), \hat{\mathbf{v}}\rangle = -\langle dN_p(\hat{\mathbf{v}}), \hat{\mathbf{u}}\rangle \\ g &= -\langle dN_p(\hat{\mathbf{v}}), \hat{\mathbf{v}}\rangle. \end{aligned}$$

We call e, f, and g the *coefficients of the second fundamental form.*

Lemma 2.23. *Gaussian curvature can be written as* $K = \frac{eg - f^2}{EG - F^2}$.

This property follows from the definition of curvature K as the determinant of dN.

Example 2.24 (Pseudosphere)**.** The tractricoid (Fig. 2.25) is a surface of revolution of a curve called the tractix. The tractricoid, minus one meridian, can be given by the following parametrization:

$$\mathbf{x}(\theta, v) = (\operatorname{sech} v \cos\theta, \operatorname{sech} v \sin\theta, v - \tanh v) \qquad -\pi < \theta < \pi,\ 0 < v$$

We can compute directly the coefficients of the first fundamental form:

$$E = \operatorname{sech}^2 v \qquad\qquad F = 0$$
$$G = (\tanh v \operatorname{sech} v)^2 + \tanh^4 v = \tanh^2 v$$

From here, the computation of the coefficients of the second fundamental form is straightforward, and we find

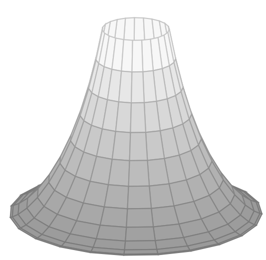
$$\begin{aligned} e &= \langle N, \partial_\theta \hat{\theta} \rangle = \frac{\langle \hat{\theta} \times \hat{\mathbf{v}},\, \partial_\theta \hat{\theta} \rangle}{\sqrt{EG - F^2}} = \ \cdots\ = \operatorname{sech} v \tanh v \\ f &= \langle N, \partial_v \hat{\theta} \rangle = 0 \\ g &= \langle N, \partial_v \hat{\mathbf{v}} \rangle = \ \cdots\ = -\operatorname{sech} v \tanh v. \end{aligned}$$

Applying Lemma 2.23 shows that at each point of the tractricoid

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-\operatorname{sech}^2 v \tanh^2 v}{\operatorname{sech}^2 v \tanh^2 v} = -1.$$

Because the tractricoid has constant negative curvature, it is more commonly known as the *pseudosphere*, in analogy with the usual sphere which has constant positive curvature. The pseudosphere is not, however, a solution to Hilbert's problem, because it has a boundary at the unit circle in the xy plane, and it is thus not a *complete* surface (cf. Sec. 2.5).

FIGURE 2.25. The Pseudosphere



2.4. **Covariant Derivative.** To continue building the necessary structures of intrinsic geometry for our purposes, we introduce the covariant derivative, which generalizes differentiation to tangent vector fields on a surface. For vector fields X, Y, Z the covariant derivative $D_X Y$ is uniquely defined by the properties

(a) Compatibility with the metric; $\partial_X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$
(b) Symmetry; $D_{\hat{\mathbf{u}}_i} \hat{\mathbf{u}}_j - D_{\hat{\mathbf{u}}_j} \hat{\mathbf{u}}_i = 0$

where $i, j \in \{1, 2\}$ and $u_1 = u$, $u_2 = v$. A formal treatment is outside the scope of this paper, but can be found in [2, p.53–55] and [3]. The above properties imply that the covariant derivative is fully determined by the expressions

$$D_{\hat{\mathbf{u}}_i}(\hat{\mathbf{u}}_j) = \Gamma^1_{ij}\hat{\mathbf{u}} + \Gamma^2_{ij}\hat{\mathbf{v}}.$$

The values Γ^k_{ij} are called *Christoffel symbols*, and are determined by the derivatives of the Riemannian metric. Note that for an embedded surface, the covariant derivative is the orthogonal projection of the derivative in $\mathbb{R}^3$ onto the tangent plane.

We then take the notational convention of writing $g_{ij} = \langle \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j \rangle$, and if we consider g_{ij} as a matrix, then g^{ij} is the inverse matrix. We can then use the following formula for the Christoffel symbols.

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\Big(\partial_{u_j} g_{il} + \partial_{u_i} g_{jl} - \partial_{u_l} g_{ij}\Big) \tag{2.26}$$

Definition 2.27. Let $\alpha : I \to S$ be a parametrized curve with $\alpha(t) = \mathbf{x}(u(t), v(t))$, and let w be a differentiable tangent vector field along α given by $w(t) = a(\alpha(t))\hat{\mathbf{u}} + b(\alpha(t))\hat{\mathbf{v}}$. The value

$$\begin{aligned}\frac{Dw}{dt} &= \left(\frac{da}{dt} + \Gamma^1_{11}au' + \Gamma^1_{12}av' + \Gamma^1_{21}bu' + \Gamma^1_{22}bv'\right)\hat{\mathbf{u}} \\ &\quad + \left(\frac{db}{dt} + \Gamma^2_{11}au' + \Gamma^2_{12}av' + \Gamma^2_{21}bu' + \Gamma^2_{22}bv'\right)\hat{\mathbf{v}}, \qquad t \in I\end{aligned}$$

with terms evaluated at t and $\alpha(t)$, is the *covariant derivative of w at t*.

Theorem 2.28 (Theorema Egregium — Gauss)**.** *Gaussian curvature of an embedded surface is invariant under local isometry.*

Proof Sketch. By differentiating $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ as vectors in $\mathbb{R}^3$ and expressing them in terms of the Christoffel symbols, a calculation (see [1, p.235–37]) shows that

$$K = \frac{-1}{E}\Big(\partial_u\Gamma^2_{12} - \partial_v\Gamma^2_{11} + \Gamma^2_{11}\Gamma^1_{12} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\Big), \tag{2.29}$$

hence locally the Riemannian metric totally determines curvature. □

This result lets us conclude that Gaussian curvature is in fact part of a surface's intrinsic geometry, and we can use the above formula for the curvature of an arbitrary surface, not just an embedded surface.

A similar result, the Bonnet Theorem, states that a neighborhood U of $\mathbb{R}^2$ and differentiable functions E, F, G, e, f, and g, satisfying $E > 0$, $G > 0$, $EG - F^2 > 0$, and the Gauss and Mainardi-Codazzi compatibility equations, on U characterize a neighborhood of a regular surface up to rigid motion. See [1, Theorem (Bonnet), p.239].

Example 2.30 (Curvature of the Hyperbolic Plane)**.** Recalling the hyperbolic plane $\mathbb{H}^2$ (Ex. 2.15), we can use (2.29) to find its curvature. From (2.26), we find all the Christoffel symbols are zero except

$$\Gamma^2_{21} = \Gamma^2_{12} = 1 \qquad\qquad \Gamma^1_{22} = -e^{2u}$$

which we can then substitute into (2.29) and find

$$K = -1.$$

Remark 2.31. The hyperbolic plane can be embedded in three dimensional Minkowski space M^3, which is $\mathbb{R}^3$ equipped with a metric of signature (2,1). In M^3, one sheet of the two-sheet hyperboloid is isometric to the hyperbolic plane. For a history and more complete treatment, see [5].

2.5. **Geodesics and Completeness.** Using the covariant derivative, we can define a 'straight line' on a surface as a parametrized curve that does not turn in the surface:

Definition 2.32. A non-constant parametrized curve $\gamma : I \to S$ is a *geodesic* if for all $t \in I$ its tangent vector has zero covariant derivative; $\frac{D\gamma'}{dt} = 0$.

One important tool that arises from geodesic curves is the exponential map, a map from the tangent plane to a surface that preserves distance from some center point in the surface. The following property of geodesics is necessary for this construction, but for conciseness is presented without proof. For proof, see [1, Proposition 5, p.257].

Lemma 2.33. *In a neighborhood $V \subset S$ of a point p, for every non-zero vector $w \in T_p(S)$ there is $\varepsilon > 0$ such that there exists exactly one parametrized geodesic $\gamma : (-\varepsilon, \varepsilon) \to V$ with $\gamma(0) = p$ and $\gamma'(0) = w$.*

Definition 2.34. For a point $p \in V \subset S$ of a neighborhood of a regular surface and a vector $w \in T_p(S)$, let $\gamma_w : [0,1] \to S$ be the unique geodesic of S with $\gamma(0) = p$ and $\gamma'(0) = w$. The *exponential map*

$$\exp_p : U \subset T_p(S) \to S, \quad \exp_p(0) = p, \ \exp_p(w) = \gamma_w(1)$$

sends w to the point obtained by traveling a length $|w|$ along a geodesic from p in the direction of w.

Note that $\exp_p$ is not always defined on the whole tangent plane. For instance, on the upper hemisphere of the unit sphere, denoting by $\mathcal{N}$ the north pole, $\exp_{\mathcal{N}}$ is defined only on the open disc of radius $\frac{\pi}{2}$ centered at the origin of $T_{\mathcal{N}}(S)$.

It also follows from the definition of a geodesic that $|\gamma_w'(t)| = |w|$ is constant, and hence that the arc length $L(\gamma_w)$ is $|w|$, as noted above.

We now can set another constraint on surfaces, which is necessary for Hilbert's theorem.

Definition 2.35. A surface S is *geodesically complete* if every parametrized geodesic $\gamma : I \to S$ can be extended to a parametrized geodesic $\bar{\gamma} : \mathbb{R} \to S$.

We use this particular definition because of the Hopf–Rinow theorem, which further characterizes geodesic completeness using the following tool: for points $p, q \in S$, let α_p^q denote a parametrized curve with $\alpha_p^q(0) = p, \alpha_p^q(1) = q$. The *intrinsic metric* d_l on S is then defined as $d_l(p,q) \coloneqq \inf\{L(\alpha_p^q)\}$.

Theorem 2.36 (Hopf–Rinow)**.** *For any surface S, the following are equivalent:*

(a) S is geodesically complete.
(b) Any closed and bounded subset of S is compact.
(c) S is a complete metric space under the intrinsic metric.

Proof Sketch. That (a) implies (b) follows from the fact that a closed and bounded subset of S is contained in the image by the exponential map of a closed disc. It follows from (b) that the closure of a Cauchy sequence is compact, and therefore contains a convergent subsequence, which implies (c). By considering a Cauchy sequence along a geodesic, (c) implies that the maximal domain a geodesic can be extended to is closed, open, and non-empty, and must therefore be the real line, hence (a).

For a full proof, see [4, Theorem 1.48, p.27–30]. □

3. Prerequisites to Hilbert's Theorem

In this section we discuss several particular concepts used in our proof of Hilbert's theorem.

Definition 3.1. The coordinate curves $\mathbf{x}(u, \text{const.}),\ \mathbf{x}(\text{const.}, v)$ form a *Chebychev net* if every quardilateral formed by those curves has opposite sides of equal length.

Lemma 3.2. *If the coordinate curves of a neighborhood $V \subset S$ satisfy the condition $\partial_v E = \partial_u G = 0$, then the coordinate curves form a Chebychev net.*

Proof. In a Chebychev net, any curve given by $\alpha : [u_1, u_2] \to S,\ \alpha(u) = \mathbf{x}(u, \text{const.})$ has the same arc length for all v. By construction α varies only in u, so we have for v constant that

$$\int_{u_1}^{u_2} |\alpha'(u)|\, du = \int_{u_1}^{u_2} \sqrt{E(u)}\, du = \text{ const.}$$

Differentiating by v yields that

$$\int_{u_1}^{u_2} \partial_v \sqrt{E(u)}\, du = 0,$$

which holds for $\partial_u E = 0$. The same logic can be applied to curves given by $\mathbf{x}(\text{const.}, v)$ to prove the lemma. □

Intuitively, if the 'length element' along one coordinate curve is independent of position along the other, then any quadrilateral formed by the coordinate curves must have opposing sides of the same length.

Definition 3.3. A vector $w \in T_p(S)$ lies in an *asymptotic direction at p* if $II_p(w) = 0$.

Note that all vectors linearly dependent with w lie in the same asymptotic direction.

Remark 3.4. If the Gaussian curvature at a point p is positive, then II_p has no zeroes in $T_p(S)$, and thus no asymptotic directions. If $K = 0$ at p, then either there is exactly one asymptotic direction or all directions are asymptotic. If $K < 0$, then there are exactly two asymptotic directions at p.

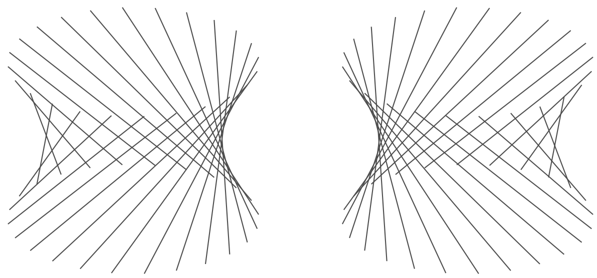
Example 3.5 (Asymptotic Directions of the Hyperboloid)**.** To find the asymptotic directions at a point $p \in H_1$ (Ex. 2.2) we note that the lines $\alpha_\phi, \beta_\phi : \mathbb{R} \to \mathbb{R}^3$ given by

$$\alpha_\phi(t) = (\cos\phi - t\sin\phi,\ \sin\phi + t\cos\phi,\ t)$$
$$\beta_\phi(t) = (\cos\phi - t\sin\phi,\ \sin\phi + t\cos\phi,\ -t)$$

lie in H_1 (see Fig. 3.6). Since α_ϕ is a straight line, $\alpha'_\phi(t)$ must be an asymptotic direction at $\alpha(t)$, and likewise for β_ϕ. We also have by Remark 3.4 that the curvature of H_1 is non-positive. In fact, a computation similar to that for the pseudosphere (Ex. 2.24) shows that the curvature at (θ, v) is

$$K = -\Big(\sinh^2(v) + \cosh^2(v)\Big)^{-2}.$$

FIGURE 3.6. Asymptotic Curves in One-Sheet Hyperboloid



The following two tools are necessary for the proof of Hilbert's theorem.

Lemma 3.7. *In a parametrization with $E = G = 1$ and $F = \cos\theta$, the curvature is given by $K = -\frac{\partial_{uv}\theta}{\sin\theta}$.*

Proof. As in the Example 2.30, the Christoffel symbols may be calculated directly and substituted into (2.29). □

Lemma 3.8. *In a complete surface S with non-positive curvature, the exponential map $\exp_p$ is a local diffeomorphism for any $p \in S$. That is, for any $p, q \in S$, there exists a neighborhood V of q such that $\exp_p$ restricted to V is a diffeomorphism.*

Proof Sketch. This follows from the fact that in Euclidean or hyperbolic spaces geodesics that start together tend to 'spread out' as they travel, and thus the differential of the exponential map is non-singular.

For a full proof, see [1, p.363–72]. □

4. Hilbert's Theorem

Theorem 4.1 (Hilbert)**.** *There exists no isometric immersion of a complete surface with constant negative Gaussian curvature in $\mathbb{R}^3$.*

Hilbert's theorem describes how the intrinsic geometry of the hyperbolic plane cannot be reconciled with the external structure of $\mathbb{R}^3$, so a reasonable place to look for this incompatibility is in the *extrinsic* structure an immersion into $\mathbb{R}^3$ would induce on a surface. In particular, one would look at the extrinsic structure unique to points of negative curvature. At any hyperbolic point in an immersed surface, there are exactly two asymptotic directions.

For the remainder of this section, we will assume that there exists an isometric immersion of the hyperbolic plane $\mathbb{H}^2$. Let S be an immersed surface isometric to $\mathbb{H}^2$. We will develop a parametrization of S by the asymptotic curves, and then show a contradiction with the area of rectangles formed by these parametric curves, and finally generalize the result to any immersion of a complete surface of constant negative curvature.

Proposition 4.2. *S can be locally parametrized by asymptotic curves that form a Chebychev net.*

Proof. $K < 0$ in $\mathbb{H}^2$, so at any point $p \in S$, there is a neighborhood $V \subset S$ that can be parametrized by its asymptotic curves.

To prove this, we refer to the Gauss map and second fundamental form of S. The defining characteristic of our choice of parametrization is that at a point p, we have $II_p(\hat{\mathbf{u}}) = e = 0$. Because N is defined to have norm 1, $\langle \partial_u N, N\rangle = 0$ as well, so $\partial_u N$ is perpendicular to both N and $\hat{\mathbf{u}}$. Likewise, $\partial_v N$ is perpendicular to both N and $\hat{\mathbf{v}}$. This property proves the proposition with the following manipulation.

Since $\partial_u N$ is perpendicular to $\hat{\mathbf{u}}$ and likewise for $\partial_v N$, $\hat{\mathbf{v}}$, the angle between $\partial_u N$ and $\partial_v N$ is the same as the angle between $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, so we find

$$\begin{aligned} \partial_u N \times \partial_v N &= dN(\hat{\mathbf{u}}) \times dN(\hat{\mathbf{v}}) = \left(\frac{|\partial_u N||\partial_v N|}{|\hat{\mathbf{u}}||\hat{\mathbf{v}}|}\right)\left(\hat{\mathbf{u}} \times \hat{\mathbf{v}}\right) \\ &= \left(\det dN\right)\left(N\sqrt{EG-F^2}\right) \end{aligned}$$

$$\partial_u N \times \partial_v N = -N\sqrt{EG-F^2}. \tag{4.3}$$

We then differentiate and find

$$\begin{aligned} \frac{\partial}{\partial v}(\partial_u N \times N) &= (\partial_{vu} N \times N) + (\partial_u N \times \partial_v N) \\ \frac{\partial}{\partial u}(\partial_v N \times N) &= (\partial_{uv} N \times N) + (\partial_v N \times \partial_u N) \end{aligned}$$

$$2(\partial_u N \times \partial_v N) = \partial_v(\partial_u N \times N) - \partial_u(\partial_v N \times N). \tag{4.4}$$

We can then further expand this to

$$\begin{aligned} \partial_v(\partial_u N \times N) &= \partial_v\left(\partial_u N \times \frac{(\hat{\mathbf{u}} \times \hat{\mathbf{v}})}{|\hat{\mathbf{u}} \times \hat{\mathbf{v}}|}\right) \\ &= \partial_v\left(\frac{1}{|\hat{\mathbf{u}} \times \hat{\mathbf{v}}|}\hat{\mathbf{u}}\langle \partial_u N, \hat{\mathbf{v}}\rangle - \hat{\mathbf{v}}\langle \partial_u N, \hat{\mathbf{u}}\rangle\right) \\ &= \partial_v\left(\frac{f\hat{\mathbf{u}}}{\sqrt{EG-F^2}}\right) \end{aligned}$$

since $\langle \partial_u N, \hat{\mathbf{u}}\rangle = e = 0$. However, we have $K = \frac{eg-f^2}{EG-F^2}$, and since K is assumed to be constant, this simplifies to

$$\partial_v(\partial_u N \times N) = \pm\sqrt{-K}(\partial_{vu}\mathbf{x}) = \pm\partial_{vu}\mathbf{x}. \tag{4.5}$$

Applying the same manipulations gives also

$$\partial_u(\partial_v N \times N) = \pm\partial_{uv}\mathbf{x}. \tag{4.6}$$

Now, we can combine (4.5), (4.6), (4.4), and (4.3) and we find

$$N\sqrt{EG-F^2} = \pm\partial_{uv}\mathbf{x},$$

and therefore $\partial_{uv}\mathbf{x}$ is proportional to N. It follows that $\langle \partial_{uv}\mathbf{x}, \hat{\mathbf{u}}\rangle = 0$, and hence $\partial_v E = 2\langle \partial_{vu}\mathbf{x}, \hat{\mathbf{u}}\rangle = 0$. Likewise, $\partial_u G = 0$, proving the proposition. ☐

Remark 4.7. The manipulation here boils down to showing that $\partial_{uv}\mathbf{x}$ is parallel to N, which means intuitively that as you travel along the curve $\mathbf{x}(u, \text{const.})$, the $\hat{\mathbf{v}}$ direction *twists* around the $\hat{\mathbf{u}}$ direction, but is not stretched at all (and the same for $\mathbf{x}(\text{const.}, v)$).

Remark 4.8. Since E depends only on u and G only on v, the coordinate curves $\bar{\mathbf{x}}(u, v_1)$ and $\bar{\mathbf{x}}(u_1, v)$ can be reparametrized by arc length:

$$s(u) = \int_{u_0}^{u} \sqrt{E(\bar{u})}d\bar{u} \qquad\qquad t(v) = \int_{v_0}^{v} \sqrt{G(\bar{v})}d\bar{v}$$

Thus for the parametrization $\mathbf{x} = \bar{\mathbf{x}}(s(u), t(v))$ we have $E = G = 1$, and therefore $F = \cos\theta(q)$, where $\theta(q)$ is the angle between the coordinate curves at a point q.

By Lemma 3.7, we then have in a parametrization of S by the asymptotic curves that $\partial_{uv}\theta = \sin\theta$.

4.1. **Parametrizing S by Asymptotic Curves.**

Construction 4.9. We now fix a point $p_0 \in S$ and define a map $\mathbf{x} : \mathbb{R}^2 \to S$ as follows. Fix an asymptotic curve α, parametrized by arc length, with $\alpha(0) = p_0$. Since S is complete, the domain of α can be extended to the whole real line. We define $\mathbf{x}(u, 0) \coloneqq \alpha(u)$. To define $\mathbf{x}(u, v)$ in general, we take the asymptotic curve β_u parametrized by arc length with $\beta_u(0) = \mathbf{x}(u, 0)$ and $\alpha'(u) \times \beta_u'(v) = \lambda(N(u, v))$ for $\lambda > 0$ (so $\beta_u'(0)$ is always on the 'same side' of α) and let $\mathbf{x}(u, v) \coloneqq \beta_u(v)$.

To prove that $\mathbf{x}(u, v)$ is well defined for all $(u, v) \in \mathbb{R}^2$, note that by Proposition 4.2, α is well-defined on some open interval around 0. Thus if α cannot be extended to the whole real line, then there is minimal $u_0 > 0$ such that $\alpha(u_0)$ is not defined. However, for a sequence $u_n < u_0$ that converges to u_0, $\lim_{n\to\infty} \alpha(u_n) \in S$, by completeness. Again by Proposition 4.2, $\lim_{n\to\infty} \alpha(u_n)$ can be parametrized locally by asymptotic curves one of which must agree with α, so α can be extended to u_0. Thus α can be extended to all of $\mathbb{R}$, and the same argument can be applied to β_u to show that $\mathbf{x}(u, v)$ can be extended to all of $\mathbb{R}^2$.

Proposition 4.10. *The curves $\alpha_v(u) \coloneqq \mathbf{x}(u, v = \text{const.})$ are asymptotic curves.*

Proof. For any u_0, v_0 with $v_0 \neq 0$, the segment of curve β_{u_0} between $\mathbf{x}(u_0, 0)$ and $\mathbf{x}(u_0, v_0)$ is compact, and can thus be covered by a finite family U_i of coordinate neighborhoods such that, by Proposition 4.2, each U_i can be parametrized by asymptotic curves which form a Chebychev net. Let U_1 be a neighboorhood of $\mathbf{x}(u_0, 0)$ and α_{v_1} intersect U_1. Then, by the construction of α_{v_1}, the curves β_u all trace out the same arc length between $\alpha(u)$ and $\alpha_{v_1}(u)$. Since the asymptotic curves form a Chebychev net in U_1 and α is an asymptotic curve, α_{v_1} must also be an asymptotic curve. In particular, if α_{v_1} were not an asymptotic curve, then there would exist an asymptotic curve γ such that the curves β_u did not trace out equal arc lengths between α and γ, which contradicts Proposition 4.2.

Suppose U_2 intersects U_1. Then there is an asymptotic curve α_{v_1} in the intersection, by the above paragraph. We can extend the argument to all α_{v_2} in U_2 by noting that the point $\alpha_{v_2}(u)$ is found by tracing a length $(v_2 - v_1)$ along β_u, starting at $\alpha_{v_1}(u)$. By the same argument as above, since asymptotic curves form a Chebychev net in U_2 and α_{v_1} is an asymptotic curve, α_{v_2} must be an asymptotic curve as well.

The same argument can be repeated for all U_i; hence α_{v_0} is an asymptotic curve in a the neighborhood of (u_0, v_0), which, since u_0, v_0 were arbitrary, proves the proposition. $\square$

Remark 4.11. The curves α_v are parametrized by arc length, as each traces out the same length in S as α on a given interval.

Proposition 4.12. *The map* $\mathbf{x}$ *is a local diffeomorphism.*

Proof. Every point $\mathbf{x}(u_0, v_0)$ has a neighborhood V that can be parametrized by asymptotic curves by Proposition 4.2. By Proposition 4.10, the coordinate curves of $\mathbf{x}$ are asymptotic curves, so since each point has only two asymptotic curves, $\mathbf{x}$ restricted to some neighborhood U of (u_0, v_0) is a parametrization of S as $\mathbf{x}(u_0, v_0)$. Hence $\mathbf{x}$ restricted to U is a diffeomorphism. $\square$

Proposition 4.13. *The map* $\mathbf{x}$ *is surjective.*

Proof. Suppose $\mathbf{x}$ is not surjective. Because S is homeomorphic to a plane, it is simply connected, and thus the boundary $\partial(\mathbf{x}(\mathbb{R}^2))$ is non-empty. Choose a point p in $\partial(\mathbf{x}(\mathbb{R}^2))$. Since $\mathbf{x}$ is a local diffeomorphism, $\mathbf{x}(\mathbb{R}^2)$ is open and thus p is not in $\mathbf{x}(\mathbb{R}^2)$. We choose some neighborhood V of p such that, by Proposition 4.2, V is parametrized by asymptotic curves that form a Chebychev net. One of the asymptotic curves γ at p must then pass through $\mathbf{x}(\mathbb{R}^2)$ (or else the asymptotic curves would be tangent at p, and $\mathbf{x}$ would not be a parametrization), so we pick a point $q \in \mathbf{x}(\mathbb{R}^2)$ along γ. Because q has exactly two asymptotic curves, γ must then be a coordinate curve of $\mathbf{x}$, and hence p must be in $\mathbf{x}(\mathbb{R}^2)$. $\square$

Injectivity follows from topological considerations that can be found in Appendix A. We then have that $\mathbf{x}$ is a diffeomorphism.

4.2. **Area in S.**

Proposition 4.14. *Any rectangle formed by coordinate curves of* $\mathbf{x}$ *has finite area.*

Proof. Let $R \subset S$ be a rectangular region with vertices $\mathbf{x}(u_0, v_0), \mathbf{x}(u_1, v_0), \mathbf{x}(u_1, v_1)$, and $\mathbf{x}(u_0, v_1)$, whose interior angles are $\phi(u_0, v_0)$, etc. By Remark 4.8 we have $\partial_{uv}\theta = \sin\theta$, so the area of R is given by

$$\begin{aligned} A(R) &= \iint_{\mathbf{x}^{-1}(R)} \sqrt{EG - F^2}\, du\, dv = \iint_{\mathbf{x}^{-1}(R)} \sin\theta\, du\, dv \\ &= \iint_{\mathbf{x}^{-1}(R)} \partial_{uv}\theta \ du\, dv \\ &= \phi(u_1, v_1) - (\pi - \phi(u_1, v_0)) - (\pi - \phi(u_0, v_0)) + \phi(u_0, v_1) \\ &< 2\pi \end{aligned}$$

and is therefore finite. $\square$

Lemma 4.15. *The area of* S *is infinite.*

Proof. Since S is isometric to $\mathbb{H}^2$, it suffices to find $A(\mathbb{H}^2)$ in the parametrization of Example 2.15

$$A(\mathbb{H}^2) = \iint_{\mathbb{R}^2} \sqrt{EG - F^2}\, du\, dv = \iint_{\mathbb{R}^2} e^u\, du\, dv = \infty.$$

$\square$

With this discrepancy in hand, we can now prove the theorem. By Proposition 4.14, we can cover S with coordinate rectangles $R_a := \mathbf{x}\big([-a,a]\times[-a,a]\big)$ such that for $a < b$, R_a is a proper subset of R_b. Every R_a has area at most 2π, though, which contradicts that the area of S is infinite. Hence there exists no isometric immersion in $\mathbb{R}^3$ of a surface homeomorphic to a plane with curvature $K = -1$.

To generalize this result to any complete surface of constant negative curvature, note we can transform any surface of constant negative curvature into one with $K = -1$ by multiplying its metric by a constant factor.

For a complete surface S' with $K = -1$ not homeomorphic to a plane, we have that the exponential map $\exp_p : T_p(S') \to S'$ is a local diffeomorphism (Lemma 3.8). We can then treat $T_p(S')$ as a surface, and give it a metric by *pulling back* the metric of S' via the exponential map. That is, for $v, w \in T_q(T_p(S'))$, we define

$$\langle v, w\rangle_q := \langle d(\exp_p)_q(v),\ d(\exp_p)_q(w)\rangle_p.$$

Under this metric on $T_p(S')$, we have that the exponential map is an isometry, and therefore $T_p(S')$ has constant negative curvature. Then if there existed an isometric immersion φ of S', we would also have $\varphi\circ\exp_p$ being an isometric immersion of $T_p(S')$, which we have just proven does not exist.

5. Further Results

In 1963, N. V. Efimov [6] published a generalization of Hilbert's theorem, proving that any any complete surface with curvature $K \le \delta < 0$ cannot be isometrically immersed in $\mathbb{R}^3$ (The one-sheet hyperboloid (Ex. 2.2) is not a counterexample, as the curvature approaches zero for large z).

The Nash embedding theorem states that for a sufficiently differentiable manifold M, there exists a finite n such that M can be isometrically embedded in $\mathbb{R}^n$. In the case of the hyperbolic plane, it guarantees the existence of an embedding in $\mathbb{R}^{51}$. In fact, D. Blanusa [7] found in 1955 an explicit smooth isometric embedding of H^2 into $\mathbb{R}^6$.

Appendix A. Arc Lifting

The proof that $\mathbf{x}$, as defined in Construction 4.9, is a diffeomorphism relies on a topological argument which is laid out in this appendix.

Definition A.1. An *arc from p to q* is a continuous map $a : [0,1] \to X$ with $a(0) = p,\ a(1) = q$. An arc a is *closed* if $a(0) = a(1)$.

Definition A.2. Let a and b be arcs from p to q. A *homotopy between a and b* is a continuous map $H : [0,1]\times[0,1] \to S$ with $H(0,t) = p,\ H(1,t) = q,\ H(s,0) = a(s),\ H(s,1) = b(s)$. Arcs a and b are *homotopic* if there exists a homotopy between them.

Lemma A.3. *Every closed arc in $\mathbb{R}^2$ is homotopic to a constant arc; $\mathbb{R}^2$ is simply connected.*

Proof. Let $a : [0,1] \to \mathbb{R}^2$ be a closed arc and $r = a(0) = a(1)$. Consider the map

$$\bar{H}(s,t) = tr + (1-t)a(s).$$

It is immediate that $\bar{H}$ is a homotopy between a and the constant arc $r(s)$. □

Definition A.4. Let $\varphi : X \to Y$ be a continuous map. If for every arc a in Y and point p in X with $\varphi(p) = a(0)$ there exists a unique arc $\bar{a}$ in X such that $\bar{a}(0) = p$ and $\varphi \circ \bar{a} = a$, then φ has the *unique arc lifting property.*

Lemma A.5. *Let $\varphi : X \to Y$ be a local homeomorphism with the unique arc lifting property. Then for every homotopy H in Y and point r in X such that $\varphi(r) = H(0,t)$, there is a unique homotopy $\bar{H}$ in X with $\varphi \circ \bar{H} = H$ and $\bar{H}(0,t) = r$.*

Proof. To prove uniqueness, suppose there exist homotopies $\bar{H}, \bar{H}'$ in X that satisfy the above conditions. Then the set $P \subset [0,1] \times [0,1]$ of points (s,t) such that $\bar{H}(s,t) = \bar{H}'(s,t)$ is non-empty. Since, however, $\bar{H}$ and $\bar{H}'$ are continuous, P must be closed in $[0,1] \times [0,1]$, and since φ is a local homeomorphism, P must also be open. The set $[0,1] \times [0,1]$ is connected, so $P = [0,1] \times [0,1]$ and thus $\bar{H} = \bar{H}'$.

We can construct $\bar{H}$ as follows: let a_t be the arc $t =$ const. of H. Then we define $\bar{H}(s,t) \coloneqq \bar{a}_t(s)$, since φ has the arc lifting property. All the desired properties except continuity follow immediately.

To prove that $\bar{H}$ is continuous, we fix (s_0, t_0) and define ψ as the restriction of φ to some neighborhood U of (s_0, t_0) such that ψ is a homeomorphism from $\bar{H}(U)$ to $\psi(\bar{H}(U))$. We have that $\psi \circ \bar{H} = H$ and ψ is invertible, so in some neighborhood of (s_0, t_0), we can write $\bar{H} = \psi^{-1} \circ H$. Since (s_0, t_0) was arbitrary, $\bar{H}$ is continuous, proving the lemma. □

Proposition A.6. *The map $\mathbf{x}$ as defined in Construction 4.9 is closed; the image of a closed set by $\mathbf{x}$ is closed.*

Proof. Suppose $Y \subset \mathbb{R}^2$ is closed but $\mathbf{x}(Y)$ is not closed. Then there is a convergent sequence z_n whose limit $z_0 \coloneqq \lim_{n\to\infty} z_n$ is in Y, but $\lim_{n\to\infty} \mathbf{x}(z_n)$ is not in $\mathbf{x}(Y)$. Consider a compact set $C \subset \mathbb{R}^2$ that contains the sequence z_n. The closed map lemma states that a continuous function from a compact set to a Hausdorff space is closed, so $\mathbf{x}$ restricted to C is a closed map. Hence $\mathbf{x}(C \cap Y)$ is closed. In particular, $z_n \in C \cap Y$, so we have that $\mathbf{x}(z_n) \in \mathbf{x}(C \cap Y)$ and therefore $\lim_{n\to\infty} \mathbf{x}(z_n) \in \mathbf{x}(Y)$. □

Proposition A.7. *The map $\mathbf{x}$ as defined in Construction 4.9 has the unique arc lifting property.*

Proof. Let a be an arc in S with $\mathbf{x}(p) = a(0)$, and suppose there does not exist a unique arc $\bar{a}$ in $\mathbb{R}^2$ with $\bar{a}(0) = p$ and $\mathbf{x} \circ \bar{a} = a$. Since $\mathbf{x}$ is a local diffeomorphism, there is a neighborhood U of p such that $\mathbf{x}$ restricted to U is a homeomorphism. Let $\mathbf{y}$ be the restriction of $\mathbf{x}$ to U. Then in U, the arc $\mathbf{y}^{-1} \circ a$ must agree with $\bar{a}$, and since U is open, there exists minimal $s_0 \in (0,1]$ such that the arc a_{s_0} given by the restriction of a to $[0, s_0]$ has no unique lifting.

In particular, for any strictly increasing sequence t_n with $\lim_{n\to\infty} t_n = s_0$, the sequence $\bar{a}_s(t_n)$ has no convergent subsequence; if such a subsequence converged to q, then by continuity of $\mathbf{x}$, we would have $\lim_{s\to s_0}(a_s(s)) = a(s_0) = \mathbf{x}(q)$ and the restriction of $\mathbf{x}$ onto a neighborhood of q would produce a unique lifting of a_{s_0}. Since no such sequence exists, we have that the set $\{\bar{a}_s(s) \mid 0 < s < s_0\} \cup \{p\}$ is closed in $\mathbb{R}^2$. However, the set $\{a(s) \mid 0 \leq s < s_0\}$ is not closed in S, which contradicts that $\mathbf{x}$ is closed. □

Remark A.8. This proposition is adapted from the slightly more general case in [8, Proposition 6.14, p.144].

Proposition A.9. *The map $\mathbf{x}$ as defined in Construction 4.9 is a diffeomorphism.*

Proof. Since we have that $\mathbf{x}$ is a surjective local diffeomorphism, it suffices to show that $\mathbf{x}$ is injective. Supposing the contrary, there are $p, q \in \mathbb{R}^2$ such that $\mathbf{x}(p) = \mathbf{x}(q)$. Let a be an arc from p to q. The composition $b := \mathbf{x} \circ a$ is then a closed arc. Because S is homeomorphic to a plane, there exists a homotopy H between b and a constant arc c. Since $\mathbf{x}$ has the path lifting property, by Lemma A.5 there is a unique homotopy $\bar{H}$ in $\mathbb{R}^2$ with $\bar{H}(0,t) = p$. By uniqueness, we have that $\bar{H}(s,0) = a(s)$, and also that $\bar{H}(s,1) = p$; hence $a(1) = p = q$. □

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