COCOMPLETENESS OF CATEGORIES OF ALGEBRAS - AN OVERVIEW

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Abstract. It is well-known that if a category $\mathcal{C}$ is complete, then the category $\mathcal{C}^{\mathbb{T}}$ of algebras over a monad $\mathbb{T}$ on $\mathcal{C}$ is also complete. The situation for colimits is more complicated: It is not generally true that if $\mathcal{C}$ is cocomplete, then so is $\mathcal{C}^{\mathbb{T}}$. The aim of this paper is to give an overview of sufficient conditions on $\mathcal{C}$ and $\mathbb{T}$ that ensure cocompleteness of $\mathcal{C}^{\mathbb{T}}$.

Contents

1. Introduction

Consider a monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$. It is a well-known fact that completeness of $\mathcal{C}$ guarantees completeness of $\mathcal{C}^{\mathbb{T}}$, which is possibly the most desirable completeness theorem for $\mathcal{C}^{\mathbb{T}}$. Cocompleteness of $\mathcal{C}^{\mathbb{T}}$ is more difficult. Unfortunately, cocompleteness of $\mathcal{C}$ does not imply cocompleteness of $\mathcal{C}^{\mathbb{T}}$, as was shown by Adámek: In [1], he constructs a monad $\mathbb{S}$ on the category of graphs **Graph**, which is cocomplete, such that $\mathbf{Graph}^{\mathbb{S}}$ is not cocomplete. Nevertheless, it is possible to find viable sufficient conditions for the existence of colimits in $\mathcal{C}^{\mathbb{T}}$.

An important tool for the generation of cocompleteness theorems for $\mathcal{C}^{\mathbb{T}}$ was developed by Linton, who showed in [9] that $\mathcal{C}^{\mathbb{T}}$ has small colimits whenever it has reflexive coequalizers. This means that instead of finding sufficient conditions for the existence of general colimits in $\mathcal{C}^{\mathbb{T}}$, we can restrict our attention to finding conditions that ensure that $\mathcal{C}^{\mathbb{T}}$ will have reflexive coequalizers. There are different approaches to finding such conditions on $\mathcal{C}$ and $\mathbb{T}$, the most fruitful of which involve some preservation property of $\mathbb{T}$ (for example, T preserves reflexive coequalizers or

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epimorphisms), combined with some hypotheses on $\mathcal{C}$. The aim of this paper is to give an overview of cocompleteness theorems for categories of algebras.

2. Preliminaries

2.1. **Monads, Algebras, and the Eilenberg-Moore Category.**

We briefly review the notion of a monad and an algebra over a monad. For an excellent introduction to monads, please see [13, Chapter 5].

Definition 2.1. Let $\mathcal{C}$ be a category. A **monad** $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{C}$ consists of

1. an endofunctor $T : \mathcal{C} \to \mathcal{C}$,
2. a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow T$ called unit,
3. a natural transformation $\mu : T^2 \Rightarrow T$ called multiplication

such that the following diagrams commute:

$$\begin{tikzcd} T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "\eta T"'] \arrow[dl, Rightarrow, "1_T"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd} \tag{2.2}$$

More generally, we can define a monad in any 2-category, not just **Cat**. However, we shall not consider monads in such generality here.

Definition 2.3. An **algebra** over a monad $\mathbb{T}$ on a category $\mathcal{C}$ is an object X of $\mathcal{C}$ together with a $\mathcal{C}$-morphism $\xi : TX \to X$ such that the diagrams

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "T\xi"] \arrow[d, "\mu_X"] & TX \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"] & X \end{tikzcd} \tag{2.4}$$

commute. Let (A, α) and (B, β) be $\mathbb{T}$-algebras. A **$\mathbb{T}$-algebra morphism** from (A, α) to (B, β) is a $\mathcal{C}$-morphism $f : A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{2.5}$$

The **category of $\mathbb{T}$-algebras** (or **Eilenberg-Moore category**), denoted $\mathcal{C}^{\mathbb{T}}$, is the category whose objects are $\mathbb{T}$-algebras and whose morphisms are $\mathbb{T}$-algebra morphisms. Importantly, we can form a free-forgetful adjunction between a category $\mathcal{C}$ and the category $\mathcal{C}^{\mathbb{T}}$ of algebras over a monad $\mathbb{T}$ on $\mathcal{C}$:

Proposition 2.6. *There is an adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F^{\mathbb{T}}"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathbb{T}} \arrow[l, bend left, "U^{\mathbb{T}}"] \end{tikzcd} \tag{2.7}$$

where

- *$U^{\mathbb{T}}$ is the evident forgetful functor;*
- *$F^{\mathbb{T}}$ is the functor that carries an object A of $\mathcal{C}$ to the free $\mathbb{T}$-algebra (TA, μ_A) and carries a $\mathcal{C}$-morphism $f : A \to B$ to the morphism between free $\mathbb{T}$-algebras $Tf : (TA, \mu_A) \to (TB, \mu_B)$.*

Proof. The proof of this can be found in any textbook covering monads, for example in [13, Lemma 5.2.8]. ☐

Monads are intimately related to adjunctions. This relationship is described by the following proposition, whose proof can be found in [13, Lemmas 5.1.3, 5.2.8]:

Proposition 2.8. *Given an adjunction between categories $\mathcal{C}$ and $\mathcal{D}$*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd} \tag{2.9}$$

with unit $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and counit $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$, the triple

$$(GF : \mathcal{C} \to \mathcal{C}, \eta : 1_{\mathcal{C}} \Rightarrow GF, G\epsilon F : (GF)^2 \Rightarrow GF) \tag{2.10}$$

forms a monad on $\mathcal{C}$. Given a monad $\mathbb{T}$ on $\mathcal{C}$, the monad $(U^{\mathbb{T}}F^{\mathbb{T}}, \eta, U^{\mathbb{T}}\epsilon F^{\mathbb{T}})$ constructed from the adjunction between $\mathcal{C}$ and $\mathcal{C}^{\mathbb{T}}$ described in Proposition 2.6 is equal to $\mathbb{T}$.

We complete this section by introducing briefly a monad which will appear in Section 2, Corollary 3.29.

Example 2.11. Let $\mathcal{C}$ be a closed symmetric monoidal category with countable coproducts. The category $\mathbf{Mon}\mathcal{C}$ consists of

- objects: monoids in $\mathcal{C}$, i.e. objects M of $\mathcal{C}$ together with a multiplication $\mu : M \otimes M \to M$ and a unit $\eta : I \to M$ satisfying associativity and left and right unit laws;
- morphisms: morphisms in $\mathcal{C}$ of underlying objects that respect the monoid structure.

Given an object C of $\mathcal{C}$ we can form the free monoid on M as the monoid with underlying object

$$\textstyle\sum_{n \geq 0} C^{\otimes_n} = 1 + C + (C \otimes C) + (C \otimes C \otimes C) + \dots, \tag{2.12}$$

with multiplication given by juxtaposition and unit given by the coprojection map of 1 into the coproduct (2.12). A detailed construction of the free monoid can be found in [10, Theorem 7.3.2]. We can form a free-forgetful adjunction

$$\begin{tikzcd} F : \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Mon}\mathcal{C} : U \arrow[l, shift left] \end{tikzcd} \tag{2.13}$$

This adjunction induces a monad $\mathbb{T} = (UF, \eta, U\epsilon F)$, where $\eta : 1_{\mathcal{C}} \Rightarrow UF$ and $\epsilon : FU \Rightarrow 1_{\mathbf{Mon}\mathcal{C}}$ are the unit and counit of the adjunction, respectively. UF sends an object C of $\mathcal{C}$ to (2.12). The component of an object C of the unit η is given by the coprojection map $C \to \sum_{n \geq 0} C^{\otimes_n}$. The multiplication $U\epsilon F$ of the monad is given by concatenation.

This monad is called the **free monoid monad** on $\mathcal{C}$. In fact, $F \dashv U$ is a **monadic adjunction**, meaning that the category of algebras over $\mathbb{T}$, $\mathcal{C}^{\mathbb{T}}$, is equivalent to $\mathbf{Mon}\mathcal{C}$. The free monoid monad on **Set** is also known as the **list monad** and is important in functional programming.

2.2. **Properties of Adjunctions and Colimits.**

Definition 2.14. Let $\mathcal{C}$ be a category and I be a small category. Let $\Delta : \mathcal{C} \to \mathcal{C}^I$ denote the constant diagram functor, which carries an object C of $\mathcal{C}$ to the constant functor $I \to \mathcal{C}$ valued at C. Let $d : I \to \mathcal{C}$ be a diagram of shape I in $\mathcal{C}$. Recall that a colimit (X, η) of d consists of an object X and a universal cone $\eta : d \Rightarrow \Delta X$ under d. For any class of diagrams $d : I \to \mathcal{C}$, we say that a functor $F : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to a category $\mathcal{D}$

- **preserves** those colimits if whenever (X, η) is a colimit of any diagram $d : I \to \mathcal{C}$, then $(FX, F\eta)$ is a colimit of the composite diagram Fd;
- **reflects** those colimits if any cone under a diagram $d : I \to \mathcal{C}$, whose image upon applying F is a colimit cone under the composite diagram Fd, is a colimit cone under d;
- **creates** those colimits if for any diagram $d : I \to \mathcal{C}$, whenever the composite diagram Fd has a colimit, then there is some colimit cone under Fd that can be lifted to a colimit cone under d, and in addition F reflects colimits in this class of diagrams.

Proposition 2.15. *If a category $\mathcal{C}$ has coequalizers and coproducts, then $\mathcal{C}$ is cocomplete.*

Proof. See [10, Theorem 5.2.1]. □

We now state two results about adjunctions, of which we will make repeated use later.

Theorem 2.16. *Left adjoints preserve colimits, right adjoints preserve limits.*

Proof. See [13, Theorems 4.5.2, 4.5.3]. □

Theorem 2.17. *Let $\mathcal{C}$ be a category and let I be a small category. Then the functor Δ defined in Definition 2.14 admits a left adjoint if and only if $\mathcal{C}^I$ admits all colimits of shape I.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "\Delta"'] \arrow[r, phantom, "\perp"] & \mathcal{C}^I \arrow[l, shift right, "colim"'] \end{tikzcd} \tag{2.18}$$

Proof. See [13, Proposition 4.5.1]. □

2.3. **Split Coequalizers and Reflexive Coequalizers.** We introduce the notions of reflexive coequalizers and split coequalizers, which will play important roles later. Reflexive coequalizers will be essential in the formulation of cocompleteness theorems for categories of algebras (see Theorem 3.3). Split coequalizers will appear in the proof that the category of algebras of any monad on **Set** is cocomplete (see Proposition 3.19).

Definition 2.19. A **fork** is a diagram of the form

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C \tag{2.20}$$

with $h \circ f = h \circ g$.

Definition 2.21. A **reflexive pair** is a parallel pair $f, g : A \rightrightarrows B$ having a common section, i.e. a morphism $s : B \to A$ such that $f \circ s = g \circ s = 1_B$. A **reflexive coequalizer** is a coequalizer of a reflexive pair.

Example 2.22. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. Let (X, ξ) be a $\mathbb{T}$-algebra. Using the axioms of a monad and an algebra, we may check that the following diagram is a reflexive coequalizer diagram in $\mathcal{C}^{\mathbb{T}}$:

$$\begin{tikzcd} (T^2X, \mu_{TX}) \arrow[r, shift left, "\mu_X"] \arrow[r, shift right, "T\xi"'] & (TX, \mu_X) \arrow[l, bend left=50, "T\eta_X"] \arrow[r, "\xi"] & (X, \xi) \end{tikzcd} \tag{2.23}$$

and that $U^{\mathbb{T}}$ preserves this reflexive coequalizer, i.e. that

$$\begin{tikzcd} T^2X \arrow[r, shift left, "\mu_X"] \arrow[r, shift right, "T\xi"'] & TX \arrow[l, bend left=50, "T\eta_X"] \arrow[r, "\xi"] & X \end{tikzcd} \tag{2.24}$$

is a reflexive coequalizer diagram in $\mathcal{C}$.

Definition 2.25. A **split coequalizer** is a diagram of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left=50, "s"] \arrow[r, "p"] & C \arrow[l, bend left=50, "h"] \end{tikzcd} \tag{2.26}$$

where $g \circ s = 1_B, p \circ h = 1_C$ and $f \circ s = h \circ p$.

It is not difficult to see that in any split coequalizer as in (2.26), the morphism $p : B \to C$ is a coequalizer of the pair f, g. Moreover, p is an **absolute coequalizer**, meaning it is preserved by any functor.

Example 2.27. Let B be a set and $E \subset B \times B$ be an equivalence relation on B. Let $p : B \to B/E$ be the map that sends an element of B to its image modulo E. Let $\pi_1, \pi_2 : E \to B$ be the evident projection maps. Assuming the axiom of choice, we can construct a map $r : B/E \to B$ that picks a representative for each equivalence class in B/E. Then the following diagram is a split coequalizer:

$$\begin{tikzcd} E \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & B \arrow[l, bend left=50, "\langle rp, 1\rangle"] \arrow[r, "p"] & B/E \arrow[l, bend left=50, "r"] \end{tikzcd} \tag{2.28}$$

The following result about split coequalizers will be used in the next section, together with Example 2.27, to show that any category of algebras over a monad on **Set** is cocomplete.

Lemma 2.29. *Given a pair $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ in $\mathcal{C}^{\mathbb{T}}$ such that $f, g : A \rightrightarrows B$ has a split coequalizer $B \to C$ in $\mathcal{C}$, then the pair f, g has a coequalizer in $\mathcal{C}^{\mathbb{T}}$ that is created by $U^{\mathbb{T}}$.*

Proof. This is implied by Beck's Monadicity Theorem. See e.g. [13, Theorem 5.5.1]. □

3. Reflexive Coequalizers and Cocompleteness

We begin this section by describing a basic tool that allows us to relate limits and colimits of $\mathcal{C}$ to limits and colimits of $\mathcal{C}^{\mathbb{T}}$.

Proposition 3.1. *Let $\mathbb{T}$ be a monad on a category $\mathcal{C}$. The forgetful functor $U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$*

(1) creates all limits that $\mathcal{C}$ has;

(2) creates colimits that $\mathcal{C}$ has and $\mathbb{T}$ preserves.

Proof. See [6, Propositions 4.3.1, 4.3.2]. □

Proposition 3.1 fully answers the question in the case of completeness:

Corollary 3.2. *Let $\mathcal{C}$ be a complete category. Then $\mathcal{C}^{\mathbb{T}}$ is complete and limits of diagrams in $\mathcal{C}^{\mathbb{T}}$ can be computed in $\mathcal{C}$.*

As discussed in the introduction, the situation in the case of cocompleteness is more complicated. The following theorem due to Linton (see [9, Corollary 2]) reduces the problem of finding sufficient conditions for the cocompleteness of $\mathcal{C}^{\mathbb{T}}$ to finding conditions that ensure that $\mathcal{C}^{\mathbb{T}}$ has (reflexive) coequalizers. We shall make use of this throughout the paper.

Theorem 3.3. *(Linton). Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a cocomplete category $\mathcal{C}$. Then the following are equivalent:*

(1) $\mathcal{C}^{\mathbb{T}}$ is cocomplete;
(2) $\mathcal{C}^{\mathbb{T}}$ has reflexive coequalizers.

Our proof is a simplified version of Linton's original proof, using elements of the proof sketch in [14, Theorem 2.2].

Proof. Suppose that $\mathcal{C}^{\mathbb{T}}$ has reflexive coequalizers. We will show how to construct small coproducts and general coequalizers from reflexive coequalizers and coproducts. By Proposition 2.15, this will imply that $\mathcal{C}^{\mathbb{T}}$ is cocomplete.

We first show that $\mathcal{C}^{\mathbb{T}}$ has coproducts. Let $(X_i, \xi_i)_{i \in I}$ be a collection of $\mathbb{T}$-algebras indexed by the set I. Note that the coproduct $\sum_i U^{\mathbb{T}}(X_i, \xi_i) = \sum_i X_i$ of their underlying objects in $\mathcal{C}$ must exist. Since $F^{\mathbb{T}}$ preserves coproducts (see Proposition 2.16), we can form coproducts of free $\mathbb{T}$-algebras as follows:

$$(T\sum_i X_i, \mu_{\sum_i X_i}) = F^{\mathbb{T}} \sum_i X_i \cong \sum_i F^{\mathbb{T}} X_i = \sum_i (TX_i, \mu_{X_i}). \tag{3.4}$$

Precomposition of the coprojection maps into $\sum_i (TX_i, \mu_{X_i})$ with the maps $(\mu_{X_i})_{i \in I}$ yields a cone with nadir $\sum_i (TX_i, \mu_{X_i})$ under the $(T^2 X_i, \mu_{TX_i})$'s, so by the universal property of the coproduct $\sum_i (T^2 X_i, \mu_{TX_i})$, this cone uniquely factors through a map

$$\sum_i \mu_{X_i} : \sum_i (T^2 X_i, \mu_{TX_i}) \to \sum_i (TX_i, \mu_{X_i}). \tag{3.5}$$

We define the maps $\sum_i T\xi_i$ and $\sum_i T\eta_{X_i}$ in a similar fashion. By the universal property of $\sum_i (TX_i, \mu_{X_i})$, these maps define a reflexive pair:

$$\begin{tikzcd} \sum_i (T^2 X_i, \mu_{TX_i}) \arrow[r, shift left, "\sum_i \mu_{X_i}"] \arrow[r, shift right, "\sum_i T\xi_i"'] & \sum_i (TX_i, \mu_{X_i}) \arrow[l, bend left=60, "\sum_i T\eta_{X_i}"] \end{tikzcd} \tag{3.6}$$

Let

$$p : (T\sum_i X_i, \mu_{\sum_i X_i}) \to (X, \xi) \tag{3.7}$$

denote the coequalizer of (3.12) in $\mathcal{C}^{\mathbb{T}}$. Let

$$j_i : X_i \to \sum_i X_i \tag{3.8}$$

denote the ith coprojection map into the coproduct $\sum_i X_i$ in $\mathcal{C}$. Set

$$h_i := p \circ \eta_{\sum_i X_i} \circ j_i : X_i \to X. \tag{3.9}$$

See [9, Proposition 2] for a detailed justification that each h_i defines a $\mathbb{T}$-algebra morphism. The h_i's assemble to form a cone with nadir (X, ξ) under the (X_i, ξ_i)'s. We will now show that this cone is in fact a coproduct cone.

Suppose there are maps

$$c_i : (X_i, \xi_i) \to (C, \gamma) \tag{3.10}$$

for all $i \in I$. Precomposing each c_i with ξ_i gives us a cone with nadir (C, γ) under the (TX_i, μ_{X_i})'s. By the universal property of the coproduct $(T\sum_i X_i, \mu_{\sum_i X_i})$, there is a unique $\mathbb{T}$-algebra morphism

$$g : (T\sum_i X_i, \mu_{\sum_i X_i}) \to (C, \gamma) \tag{3.11}$$

such that $g \circ Tj_i = c_i \circ \xi_i$ for all $i \in I$. It is straightforward to verify that

$$\begin{tikzcd} \sum_i(T^2X_i, \mu_{TX_i}) \arrow[r, shift left, "\sum_i \mu_{X_i}"] \arrow[r, shift right, "\sum_i T\xi_i"'] & \sum_i(TX_i, \mu_{X_i}) \arrow[l, bend left=60, "\sum_i T\eta_{X_i}"] \arrow[r, "g"] & (C, \gamma) \end{tikzcd} \tag{3.12}$$

defines a fork. Thus, g factors through p via a unique map $c : (X, \xi) \to (C, \gamma)$. A diagram chase shows that each c_i factors through h_i via c:

$$\begin{aligned} c \circ h_i &= c \circ p \circ \eta_{\sum_i X_i} \circ j_i \\ &= g \circ \eta_{\sum X_i} \circ j_i \\ &= g \circ Tj_i \circ \eta_{X_i} \\ &= c_i \circ \xi_i \circ \eta_{X_i} \\ &= c_i \end{aligned} \tag{3.13}$$

It is not difficult to see that the factorization of each c_i through h_i is unique, which means that (X, ξ) is indeed the coproduct of the (X_i, ξ)'s.

Now we construct general coequalizers in $\mathcal{C}^{\mathbb{T}}$ from coproducts and reflexive coequalizers. Consider a parallel pair $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ in $\mathcal{C}^{\mathbb{T}}$. Define a $\mathcal{C}$-morphism f' to be the unique map making the following diagram in $\mathcal{C}^{\mathbb{T}}$ commute:

$$\begin{tikzcd} (A, \alpha) \arrow[r, "p_A"] \arrow[dr, "f"'] & (A, \alpha) + (B, \beta) \arrow[d, dashed, "\exists!\, f'"] & (B, \beta) \arrow[l, "p_B"'] \arrow[dl, equal] \\ & (B, \beta) & \end{tikzcd} \tag{3.14}$$

where p_A and p_B denote the coprojection maps into the product $(A, \alpha) + (B, \beta)$. Define a morphism g' analogously. These maps allow us to construct the following reflexive pair in $\mathcal{C}^{\mathbb{T}}$:

$$
\begin{tikzcd}
(A,\alpha)+(B,\beta) \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & (A,\alpha) \arrow[l, bend left=50, "p"]
\end{tikzcd}
\tag{3.15}
$$

We can form an adjunction between the diagram categories $(\mathcal{C}^{\mathbb{T}})^{\bullet \overset{\rightrightarrows}{\leftarrow} \bullet}$ and $(\mathcal{C}^{\mathbb{T}})^{\bullet \rightrightarrows \bullet}$

$$
\begin{tikzcd}
(\mathcal{C}^{\mathbb{T}})^{\bullet \rightrightarrows \bullet} \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & (\mathcal{C}^{\mathbb{T}})^{\bullet \overset{\rightrightarrows}{\leftarrow} \bullet} \arrow[l, shift left, "U"]
\end{tikzcd}\ ,
\tag{3.16}
$$

the right adjoint U being the evident forgetful functor and the left adjoint R being the functor that sends a parallel pair $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ to the reflexive coequalizer (3.15). Since $\mathcal{C}^{\mathbb{T}}$ has reflexive coequalizers, by Theorem 2.17 the constant diagram functor

$$
\Delta_1 : \mathcal{C}^{\mathbb{T}} \to (\mathcal{C}^{\mathbb{T}})^{\bullet \overset{\rightrightarrows}{\leftarrow} \bullet}
\tag{3.17}
$$

has a left adjoint. By composing this left adjoint with the left adjoint R from (3.16), we can form a left adjoint to the constant diagram functor

$$
\Delta_2 : \mathcal{C}^{\mathbb{T}} \to (\mathcal{C}^{\mathbb{T}})^{\bullet \rightrightarrows \bullet},
\tag{3.18}
$$

so by Theorem 2.17, $\mathcal{C}^{\mathbb{T}}$ has general coequalizers. □

Note that the construction of general coequalizers from binary coproducts and reflexive coequalizers in the proof of Theorem 3.3 works in any category, whereas the construction of arbitrary small coproducts from reflexive coequalizers was specific to the category $\mathcal{C}^{\mathbb{T}}$.

We can directly apply Theorem 3.3 to show that any category of algebras of a monad on **Set** is cocomplete. In particular, this means that all varieties of algebras - such as the categories of rings, groups, and Lie algebras - are cocomplete.

Proposition 3.19. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on **Set**. Then, assuming the axiom of choice, **Set**$^{\mathbb{T}}$ is cocomplete.*

Proof. Let (A, α) and (B, β) be $\mathbb{T}$-algebras, and let $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ be a pair of $\mathbb{T}$-algebra morphisms. Since $U^{\mathbb{T}}$ creates limits (see Lemma 3.1 (1)) and **Set** has them, there exists a unique map $\xi : T(B \times B) \to B \times B$ such that $(B \times B, \xi)$ is a $\mathbb{T}$-algebra.

Let $E \subset B \times B$ be the smallest equivalence relation on B which contains $f(A) \times g(A)$ and such that $TE \subset E$. Then $(E, \xi|_E)$ is a subalgebra of $(B \times B, \xi)$. We may check that the projection maps $\pi_1, \pi_2 : (E, \xi|_E) \to (B, \beta)$ are $\mathbb{T}$-algebra morphisms.

The parallel pair π_1, π_2 extends to the following split coequalizer diagram in **Set**, with the maps as described in Example 2.27:

$$
\begin{tikzcd}
E \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & B \arrow[l, bend left=50, "\langle rp,1\rangle"] \arrow[r, "p"] & B/E \arrow[l, bend left=50, "r"]
\end{tikzcd}
\tag{3.20}
$$

Since E and B have $\mathbb{T}$-algebra structures, by Lemma 2.29 the coequalizer $p : B \to B/E$ from diagram 3.20 lifts to a coequalizer $p : (B, \beta) \to (B/E, \gamma)$ of the pair $\pi_1, \pi_2 : (E, \xi|_E) \rightrightarrows (B, \beta)$ in **Set**$^{\mathbb{T}}$.

Notice that E is the smallest equivalence relation that the $\mathbb{T}$-algebra morphism $R = \langle f, g\rangle : (A, \alpha) \to (B \times B, \xi)$ factors through:

$$\begin{tikzcd} (A,\alpha) \arrow[r, "R"] \arrow[d] & (B \times B, \xi) \\ (E, \xi|_E) \arrow[ur, hook, "\text{inclusion}"'] & \end{tikzcd} \tag{3.21}$$

Thus, we see that the coequalizer $p : (B, \beta) \to (B/E, \gamma)$ of the pair π_1, π_2 in $\mathcal{C}^{\mathbb{T}}$ must also be a coequalizer of the pair $f, g : (A, \alpha) \rightrightarrows (B, \beta)$. So by Theorem 3.3, $\mathbf{Set}^{\mathbb{T}}$ is cocomplete. □

Proposition 3.1 (2) tells us that the forgetful functor $U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates reflexive coequalizers if $\mathcal{C}$ has them and $\mathbb{T}$ preserves them. An immediate corollary of Theorem 3.3 and Proposition 3.1 (2) is the following:

Corollary 3.22. *Let $\mathbb{T}$ be a monad on a category $\mathcal{C}$ and suppose $\mathbb{T}$ preserves reflexive coequalizers. Then if $\mathcal{C}$ has reflexive coequalizers, $\mathcal{C}^{\mathbb{T}}$ is cocomplete. In particular, if $\mathcal{C}$ is cocomplete and $\mathbb{T}$ preserves reflexive coequalizers, then $\mathcal{C}^{\mathbb{T}}$ is cocomplete.*

This poses the following question: In which situations does $\mathbb{T}$ preserve reflexive coequalizers? We will explore this question for the remainder of this section. The following result tells us that the category of algebras of a finitary monad on **Set** is always cocomplete. We have seen that if we assume the axiom of choice, this is true for *any* monad on **Set** (see Proposition 3.19). For a definition and further discussion of finitary monads see Section 5.

Proposition 3.23. *Let $\mathbb{T}$ be a finitary monad on **Set**. Then $\mathbb{T}$ preserves reflexive coequalizers.*

Proof. See [15, Theorem 2.5]. □

Note that the preservation of reflexive coequalizers by $\mathbb{T}$ is not a necessary condition for the cocompleteness of $\mathcal{C}^{\mathbb{T}}$. Consider for instance the covariant **power set monad** on **Set**, which is induced by the free-forgetful adjunction

$$F : \mathbf{Set} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{SupLat} : U \tag{3.24}$$

where **SupLat** is the category of suplattices (a suplattice is a poset which has all joins). This adjunction is monadic, and **SupLat** is cocomplete. Note also that the power set monad UF is infinitary. It is shown in [2, Corollary 3.5, Example 3.9] that UF does not preserve reflexive coequalizers, in an argument that is too long to be presented here.

The following proposition will allow us to formulate a cocompleteness theorem for monoidal categories.

Proposition 3.25. *A bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserves reflexive coequalizers if and only if the functors $F(C, -) : \mathcal{D} \to \mathcal{E}$ and $F(-, D) : \mathcal{C} \to \mathcal{E}$ preserve reflexive coequalizers for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

Proof (sketch). Let $C \in \mathcal{C}$ and $D \in \mathcal{D}$. If h and g are reflexive coequalizers in $\mathcal{C}$ and $\mathcal{D}$, respectively, then $1_C \times g$ and $h \times 1_D$ are reflexive coequalizers in $\mathcal{C} \times \mathcal{D}$. Since $F(C, g) = F(1_C, g)$ and $F(h, D) = F(h, 1_D)$, if the bifunctor F preserves reflexive coequalizers, so do the functors $F(C, -)$ and $F(-, D)$.

Now suppose that $F(C,-)$ and $F(-,D)$ preserve reflexive coequalizers for all $C \in \mathcal{C}$ and all $D \in \mathcal{D}$. Suppose

$$(C_1, D_1) \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; (C_2, D_2) \longrightarrow (C_3, D_3) \tag{3.26}$$

is a reflexive coequalizer in $\mathcal{C} \times \mathcal{D}$. This induces reflexive coequalizers

$$C_1 \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} C_2 \to C_3 \qquad \text{and} \qquad D_1 \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} D_2 \to D_3$$

in $\mathcal{C}$ and $\mathcal{D}$, respectively. Then in the following diagram each row and each column is a reflexive coequalizer:

$$\begin{tikzcd}
F(C_1, D_1) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] & F(C_1, D_2) \arrow[l] \arrow[r] \arrow[d, shift left=2] \arrow[d, shift right=2] & F(C_1, D_3) \arrow[d, shift left=2] \arrow[d, shift right=2] \\
F(C_2, D_1) \arrow[u] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d] & F(C_2, D_2) \arrow[u] \arrow[l] \arrow[r] \arrow[d] & F(C_2, D_3) \arrow[u] \arrow[d] \\
F(C_3, D_1) \arrow[r, shift left=2] \arrow[r, shift right=2] & F(C_3, D_2) \arrow[l] \arrow[r] & F(C_3, D_3)
\end{tikzcd} \tag{3.27}$$

Since (3.27) is serially commutative, [5, Lemma 4.4.2] implies that the induced diagram

$$F(C_1, D_1) \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; F(C_2, D_2) \longrightarrow F(C_3, D_3), \tag{3.28}$$

which is the image of the coequalizer (3.26) under F, is a reflexive coequalizer. For a more detailed proof of the dual statement see [8, Corollary 1.2.12]. $\square$

Proposition 3.25 has the following nice consequence:

Corollary 3.29. *Let $\mathcal{C}$ be a cocomplete, closed symmetric monoidal category, and let $\mathbb{T}$ be the free monoid monad on $\mathcal{C}$ introduced in Example 2.11. Then $\mathbb{T}$ preserves reflexive coequalizers and* ***Mon***$\mathcal{C}$ *is cocomplete.*

Proof. Since $\mathcal{C}$ is closed symmetric monoidal, the functors $- \otimes C : \mathcal{C} \to \mathcal{C}$ and $C \otimes - : \mathcal{C} \to \mathcal{C}$ have right adjoints and thus preserve reflexive coequalizers (see Proposition 2.16). It follows by Proposition 3.25 and the construction of $\mathbb{T}$ in Example 2.11 that $\mathbb{T}$ preserves reflexive coequalizers. By Theorem 3.3, the category of algebras over $\mathbb{T}$, which is equivalent to **Mon**$\mathcal{C}$, is cocomplete. $\square$

4. Regularity, Factorization Systems, and Cocompleteness

In this section we introduce the notions of regularity and a factorization system to obtain an alternative set of sufficient conditions on $\mathcal{C}$ that guarantee cocompleteness of $\mathcal{C}^{\mathbb{T}}$. We will show that if $\mathbb{T}$ is a monad on a regular category $\mathcal{C}$, or more generally on a category with a proper factorization system, then mild additional conditions on $\mathbb{T}$ or $\mathcal{C}$ suffice to guarantee cocompleteness of $\mathcal{C}^{\mathbb{T}}$.

A factorization system consists of two classes of maps E and M such that every map factors into an E-map followed by an M-map, and where E-maps and M-maps satisfy some properties.

Definition 4.1. A factorization system (E, M) in a category $\mathcal{C}$ consists of classes E, M of morphisms of $\mathcal{C}$ such that

(1) E and M both contain all isomorphisms of $\mathcal{C}$ and are closed under composition;

(2) every morphism f of $\mathcal{C}$ can be factored as $f = m \circ e$ for some morphisms $e \in E$ and $m \in M$;
(3) for every commutative square

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dashed, "d", "\exists!"'] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd} \tag{4.2}$$

where $e \in E, m \in M$, there exists a unique morphism d making both triangles commute.

An (E, M)-factorization system is called **proper** if all E-morphisms are epimorphisms and all M-morphisms are monomorphisms.

Factorization systems as defined in 4.1 are often called an **orthogonal factorization system**. Our definition of a factorization system is equivalent to the perhaps more common one found in [12, Definition 1.1]. We refer the reader to [12] for a proof of this. We present one other definition of a factorization system that is equivalent to ours:

Definition 4.3. A factorization system (E, M) in a category $\mathcal{C}$ consists of classes E, M of morphisms of $\mathcal{C}$ such that
(1) E and M both contain all isomorphisms of $\mathcal{C}$ and are closed under composition;
(2) every morphism f of $\mathcal{C}$ can be factored *uniquely up to unique isomorphism* as $f = m \circ e$ for some morphisms $e \in E$ and $m \in M$.

Remark 4.4. Proper factorization systems have many convenient properties that we will make use of to formulate a cocompleteness theorem (see [1]):
(1) E contains all coequalizers;
(2) in a pushout

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d, "p_1"] \\ \bullet \arrow[r, "p_2"] & \bullet \end{tikzcd} \tag{4.5}$$

if g is an E-morphism then so is p_1, and if f is an E-morphism then so is p_2;
(3) if in a set-indexed family of morphisms $f_i : A \to B_i$ with common source each f_i is in E, then in its multiple pushout (colimit) C, if it exists, all coprojection maps are E-morphisms.

$$\begin{tikzcd} & & A \arrow[dll, "f_{i_1}"'] \arrow[dl, "f_{i_2}"] \arrow[dd] \arrow[dr, "f_{i_3}"] & & \\ B_{i_1} \arrow[drr] & B_{i_2} \arrow[dr] & & B_{i_3} \arrow[dl] & \cdots \\ & & C & & \end{tikzcd} \tag{4.6}$$

Below we will show that we can form a proper factorization system in each regular category. We first introduce the notion of regularity.

Definition 4.7. An epimorphism is called **regular** if it can be written as the coequalizer of some pair of morphisms. A category $\mathcal{C}$ is called regular if

(1) it is finitely complete;
(2) the pullback of a regular epimorphism along any morphism is a regular epimorphism;
(3) if $X \times_Y X$ is the kernel pair of any morphism $f : X \to Y$

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{4.8}$$

then the parallel pair $p_1, p_2 : X \times_Y X \to X$ has a coequalizer.

A **regular monad** is a monad on a regular category which preserves finite limits and coequalizers of kernel pairs.

Example 4.9. A regular category has an (E, M)-factorization system, where E is the class of all regular epimorphisms and M is the class of all monomorphisms.

To see why this is true, let $\mathcal{C}$ be a regular category, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Let $e : X \to Z$ be the coequalizer of the kernel pair of f. By the universal property of e, there exists a unique morphism $m : Z \to Y$ such that $f = m \circ e$. Using the axioms of a regular category, it can be shown that m is a monomorphism and that the factorization of f into a regular epimorphism followed by a monomorphism is unique up to unique isomorphism (see [11, Proposition 4.2]). It is straightforward to verify that the class of regular epimorphisms contains all isomorphisms and is closed under compositions. Thus by Definition 4.3, the classes of regular epimorphisms and monomorphisms form a factorization system of $\mathcal{C}$. This factorization system is easily seen to be proper.

We now present two theorems that justify our earlier claim that if $\mathbb{T}$ is a monad on a regular category $\mathcal{C}$, $\mathcal{C}^{\mathbb{T}}$ is cocomplete under mild additional conditions. The proofs of these theorems both rely on the fact that a regular category has a factorization system of regular epimorphisms and monomorphisms, which we introduced in Example 4.9. Due to limited space, we will not present these proofs here but instead refer the reader to [7, Theorem 20.33] for a proof of Theorem 4.10 and to [6, Proposition 4.5.3] for a proof of Theorem 4.11.

Theorem 4.10. *If $\mathbb{T}$ is a regular monad on a regular category $\mathcal{C}$, then $\mathcal{C}^{\mathbb{T}}$ has all colimits that exist in $\mathcal{C}$. In particular, if $\mathcal{C}$ is cocomplete then so is $\mathcal{C}^{\mathbb{T}}$.*

Theorem 4.11. *Let $\mathcal{C}$ be a complete and cocomplete, regular category in which every regular epimorphism has a section. Then for any monad $\mathbb{T}$ on $\mathcal{C}$, $\mathcal{C}^{\mathbb{T}}$ is cocomplete.*

Next, we want to formulate a cocompleteness theorem for categories with more general proper factorization systems. To this end, we first introduce the notion of a category being well-powered or cowell-powered.

Definition 4.12. Let $\mathcal{C}$ be a category. We say that the $\mathcal{C}$-epimorphisms $C \twoheadrightarrow A$ and $C \twoheadrightarrow B$ with common source are **equivalent** if there are (necessarily unique) maps for which the diagram

$$\begin{tikzcd} C \arrow[r, two heads] \arrow[d, two heads] & A \arrow[dl, shift left] \\ B \arrow[ur, shift left] & \end{tikzcd} \tag{4.13}$$

commutes. We say that $\mathcal{C}$ is **cowell-powered** if for each object C of $\mathcal{C}$, the equivalence classes of epimorphisms with source C form a set.

If $\mathcal{C}$ has a proper factorization system (E, M), then we say that $\mathcal{C}$ is ***E*-cowell-powered** if for each object C of $\mathcal{C}$ there is a set Ω of E-epimorphisms with the property that every E-epimorphism is equivalent to one in Ω.

Definition 4.14. Let $\mathbb{T}$ be a monad on a category $\mathcal{C}$. Let $\mathcal{C}(\mathbb{T})$ denote the category whose objects are pairs $(X, \xi : TX \to X)$ (with no restrictions on ξ) and whose morphisms $f : (A, \alpha) \to (B, \beta)$ are $\mathcal{C}$-morphisms $f : A \to B$ such that $f \circ \alpha = \beta \circ Tf$.

Note that $\mathcal{C}^{\mathbb{T}}$ is a full subcategory of $\mathcal{C}(\mathbb{T})$. The following lemma tells us that often if $\mathcal{C}$ has a factorization system when aiming at a cocompleteness theorem for $\mathcal{C}^{\mathbb{T}}$, we can work in the larger category $\mathcal{C}(\mathbb{T})$ instead.

Lemma 4.15. *Suppose $\mathcal{C}$ has a factorization system (E, M), and suppose that T preserves E (i.e. $Te \in E$ for all $e \in E$). Then for every coequalizer in $\mathcal{C}(\mathbb{T})$*

$$(A, \alpha) \overset{f}{\underset{g}{\rightrightarrows}} (B, \beta) \xrightarrow{h} (C, \gamma) \tag{4.16}$$

such that (B, β) is a $\mathbb{T}$-algebra, (C, γ) is also a $\mathbb{T}$-algebra. In particular, if $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ is a parallel pair in $\mathcal{C}^{\mathbb{T}}$ with a coequalizer $h : (B, \beta) \to (C, \gamma)$ in $\mathcal{C}(\mathbb{T})$ then $h : (B, \beta) \to (C, \gamma)$ will also be its coequalizer in $\mathcal{C}^{\mathbb{T}}$.

Proof. See [1, Lemma 2.3]. □

The following cocompleteness theorem was first discovered by Linton (see [9, Proposition 4]) and independently by Barr (see [3, Theorem 3.3]), and later improved upon by Adámek, who proved that we do not need to include completeness of $\mathcal{C}$ as one of the sufficient conditions of the theorem (see [1, Theorem 2.4]). The proof we present here is due to Adámek.

Theorem 4.17. *Let $\mathcal{C}$ be a cocomplete, E-cowell-powered category with a proper factorization system (E, M). Let $\mathbb{T}$ be a monad preserving E. Then $\mathcal{C}^{\mathbb{T}}$ is cocomplete.*

Proof. Let $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ be a parallel pair of morphisms in $\mathcal{C}^{\mathbb{T}}$. Let Ω be the class of all E-epimorphisms $k : B \to K$ such that for every $\mathbb{T}$-algebra morphism $l : (B, \beta) \to (D, \delta)$ with $l \circ f = l \circ g$ there exists a $\mathcal{C}$-morphism $l_k : K \to D$ with $l = l_k \circ k$. Since $\mathcal{C}$ is E-cowell-powered, Ω is a set. Thus the diagram Ω has a colimit. Let K_0 denote the colimit, let r_k denote the coprojection map from the object K, the target of the map $k : B \to K$, and let r_0 denote the coprojection map $B \to K_0$. Note that commutativity of the colimit cone implies that $r_0 = r_k \circ k$ for all $k \in \Omega$.

Suppose $h : (B, \beta) \to (C, \gamma)$ is a morphism in $\mathcal{C}(\mathbb{T})$ with $h \circ f = h \circ g$. Then by definition, h can be factored through each $k \in \Omega$ via a map $h_k : K \to C$. This means that there is a cone with nadir C under the diagram Ω. By the universal property of K_0, there exists a unique map $h_0 : K_0 \to C$ with $h_0 \circ r_k = h_k$ for each

$k \in \Omega$ and $h_0 \circ r_0 = h$ (h_0 is the unique map with this property since r_0 is an epimorphism).

$$
\begin{tikzcd}
 & & B, \arrow[dll, "k"'] \arrow[dl, "k'"] \arrow[dd, "r_0"] \arrow[dr, "k''"] & & \\
K \arrow[drr, "r_k"] \arrow[ddrr, "h_k"'] & K' \arrow[dr, "r_{k'}"] \arrow[ddr, "h_{k'}"] & & K'' \arrow[dl, "r_{k''}"'] \arrow[ddl, "h_{k''}"] & \cdots \\
 & & K_0 \arrow[d, dashed, "h_0"] & & \\
 & & C & &
\end{tikzcd}
\tag{4.18}
$$

Notice that for each $k \in \Omega$, $k \in E$ and $r_k \in E$ (K_0 is a multiple pushout of E-morphisms - see Remark 4.4 (3)). Thus, $r_0 \in E$ and $Tr_0 \in E$.

Define the maps p and q to be the pushout

$$
\begin{tikzcd}
TB \arrow[r, "r_0 \circ \beta"] \arrow[d, "Tr_0"'] \arrow[dr, phantom, "\lrcorner"] & K_0 \arrow[d, "q"] \\
TK_0 \arrow[r, "p"] & R
\end{tikzcd}
\tag{4.19}
$$

We will now show that q is an isomorphism. It suffices to show that $q \circ r_0 \in \Omega$ since then r_0 can be factored as $r_0 = r_{q \circ r_0} \circ q \circ r_0$, which implies that $1 = r_{q \circ r_0} \circ q$, and so q is both an epimorphism (since q is opposite an E-morphism in a pushout - see Remark 4.4 (2)) and a split monomorphism; and thus an isomorphism.

To see that $q \circ r_0 \in \Omega$, first notice that $q \circ r_0 \in E$. Since $h_0 \circ r_0 = h$ is a morphism between $\mathbb{T}$-algebras, the square

$$
\begin{tikzcd}
TB \arrow[r, "r_0 \circ \beta"] \arrow[d, "Tr_0"'] & K_0 \arrow[d, "h_0"] \\
TK_0 \arrow[r, "\gamma \circ Th_0"] & C
\end{tikzcd}
\tag{4.20}
$$

commutes, which by the universal property of the pushout (4.19) implies that there exists a unique morphism $R \to C$, which we denote by $h_{q \circ r_0}$, such that $h_0 = h_{q \circ r_0} \circ q$ and $\gamma \circ Th_0 = h_{q \circ r_0} \circ p$. Then $h = h_{q \circ r_0} \circ q \circ r_0$ and so $q \circ r_0 \in \Omega$, as desired.

Finally, we show that the $\mathcal{C}(\mathbb{T})$-morphism

$$
r_0 : (B, \beta) \to (K_0, q^{-1} \circ p)
\tag{4.21}
$$

is a coequalizer of the parallel pair f, g in $\mathcal{C}(\mathbb{T})$. First note that by the pushout (4.19), r_0 is indeed a morphism in $\mathcal{C}(\mathbb{T})$. Let $c : B \to D$ denote the coequalizer of f, g in $\mathcal{C}$. Then $c \in E$ (see Remark 4.4 (1)) and $c \in \Omega$, and hence we can factor r_0 as $r_0 = r_c \circ c$. This shows that $r_0 \circ f = r_0 \circ g$. We have already shown above that the $\mathcal{C}(\mathbb{T})$-morphism $h : (B, \beta) \to (C, \gamma)$ factors uniquely through r_0 as $h = h_0 \circ r_0$.

It remains to check that $h_0 : K_0 \to C$ is a morphism in $\mathcal{C}(\mathbb{T})$:

$$
\begin{aligned}
h_0 \circ q^{-1} \circ p \circ Tr_0 &= h_0 \circ q^{-1} \circ q \circ r_0 \circ \beta \\
&= h \circ \beta \\
&= \gamma \circ Th \\
&= \gamma \circ Th_0 \circ Tr_0,
\end{aligned}
\tag{4.22}
$$

so since Tr_0 is an epimorphism, $h_0 \circ q^{-1} \circ p = \gamma \circ Th_0$. By Lemma 4.15, r_0 is the coequalizer of f, g in $\mathcal{C}^{\mathbb{T}}$. This shows that $\mathcal{C}^{\mathbb{T}}$ has coequalizers, and thus by Theorem 3.3 that $\mathcal{C}^{\mathbb{T}}$ is cocomplete. □

The statement of Theorem 4.17 appears quite different in [9] and [3]. However, these results turn out to be equivalent to Theorem 4.17, with the exception that completeness of $\mathcal{C}$ is included as one of the sufficient conditions in both [9] and [3]. See [1, Section 4] for a detailed discussion of this.

We have seen in Example 4.9 that every regular category has a proper factorization system. Many other types of categories come with factorization systems, too. For example, each cowell-powered category has a proper factorization system (E, M), where E is given by the class of epimorphisms, and M is given by the class of extremal monomorphisms (see [7, Corollary 15.17] for details). This gives us the following corollary of Theorem 4.17:

Corollary 4.23. *Suppose $\mathcal{C}$ is cowell-powered and cocomplete. Then if $\mathbb{T}$ preserves epimorphisms, $\mathcal{C}^{\mathbb{T}}$ is cocomplete.*

We state an analogous result for regular categories, using the factorization system described in Example 4.9:

Corollary 4.24. *Suppose $\mathcal{C}$ is regular, regularly cowell-powered, and cocomplete. Then if $\mathbb{T}$ preserves regular epimorphisms, $\mathcal{C}^{\mathbb{T}}$ is cocomplete.*

5. Finitary Monads and Cocompleteness

In this section, we present yet another approach to determining cocompleteness of $\mathcal{C}^{\mathbb{T}}$. We will see that the category of algebras over a finitary monad is cocomplete under mild conditions.

Definition 5.1. A **filtered category** is a category J in which there is a cone under every finite diagram. A **filtered colimit** is a colimit of a diagram with filtered indexing category.

Note in particular that a sequential colimit - a colimit of a diagram whose indexing category is an ordinal or its opposite - is a filtered colimit.

Definition 5.2. A functor is called **finitary** if it preserves filtered colimits. In particular, a monad $T : \mathcal{C} \to \mathcal{C}$ is finitary if it preserves filtered colimits in $\mathcal{C}$.

Note that given an adjunction $F \dashv G$, if a G is finitary then so is the monad GF because F, being a left adjoint, preserves all colimits (see Proposition 2.16). We now state our main result in this section:

Theorem 5.3. *Let $\mathbb{T}$ be a monad on a category $\mathcal{C}$ which has equalizers and is finitely cocomplete. If $\mathbb{T}$ preserves sequential colimits where the indexing category is a countable ordinal, then $\mathcal{C}^{\mathbb{T}}$ is cocomplete.*

Proof. See [5, Theorem 9.3.9]. □

The following corollary of Theorem 5.3 is proved directly in [6, Proposition 4.3.6]. A more modern write-up of this proof is available in [13, Theorem 5.6.12].

Theorem 5.4. *If $T : \mathcal{C} \to \mathcal{C}$ is a finitary monad on a complete and cocomplete, locally small category $\mathcal{C}$, then the category $\mathcal{C}^{\mathbb{T}}$ is also complete and cocomplete.*

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