ULTRAFILTER AND HINDMAN'S THEOREM

GUANYU ZHOU

Abstract. In this paper, we present various results of Ramsey Theory, including Schur's Theorem and Hindman's Theorem. With the focus on the proof of Hindman's Theorem, we introduce ultrafilter and equip topology and operation to the space of ultrafilters. In the end, we generalize Hindman's Theorem by proving its analogous statements such as Hindman's Theorem on the power set and Hindman's Theorem with respect to finite product set.

Contents

1. Introduction

The main theme of this paper is the connection between ultrafilters and combinatorics. In particular, we explore ultrafilters' application in Ramsey Theory, a branch of combinatorics that is concerned with the phenomenon of preservation of highly organized structures under finite partition. With the main goal to prove Hindman's Theorem, we will analyze the space constructed by ultrafilters by endowing with a topology and algebraic structure and discussing its various interesting characterisitics, such as Stone-Čech compactification and existence of idempotent elements. To get a feeling of Ramsey Theory, to which Hindman's Theorem belongs, we first consider Fermat's Theorem over finite fields, which requires a weaker statement than Hindman's Theorem to prove.

Date: AUGUST 14, 2017.

2. Application of Ramsey Theory: Schur's Theorem and its Corollary

Schur's Theorem, an application of Ramsey's Theorem, was proved by Issai Schur in 1916 with the goal to solve the problem of Fermat's Equation modulo a prime number. Schur's Theorem can be considered as describing "the local version" of Fermat's Equation.

Definition 2.1. The *Ramsey number* $R(n_1, \ldots n_r)$ is the smallest positive integer such that for any $m \geqslant R(n_1, \ldots, n_r)$ and any r coloring of the edges of the complete graph K_m, there is some $i \in \{1, \ldots, r\}$ such that there exists a monochromatic complete subgraph of size n_i whose edges all have the i^{th} color. In particular, if $n_1 = n_2 = \cdots = n_r = 3$, we denote $R(n_1, \ldots n_r)$ by $S(r)$.

From Ramsey's Theorem we know that $R(n_1, \ldots n_r)$ always exists. In fact, to prove Schur's Theorem, we only need the following special case of Ramsey's Theorem.

Lemma 2.2. *Let $r \in \mathbb{N}$, then for all sufficiently large m, for any r coloring of the edges of the complete graph K_m, there is always a monochromatic triangle, i.e. $S(r) = R(\underbrace{3, 3, \ldots, 3}_{r})$ exists.*

Proof. We prove this by induction on r.
Base case: For $r = 1$, we have $S(1) = R(1) = 3$. Let $r \in \mathbb{N}$ and assume $S(r)$ exists. Now we show $S(r+1)$ exists.

Let $m = (r+1)S(r) + 1$. Consider any $r+1$ coloring of K_m; we pick any vertex $v \in K_m$.
Claim: Let $\mathcal{E} = \{$the $m-1$ edges which are connected to $v\}$. Then there exists a monochromatic subcollection $\mathcal{E}_0 \subset \mathcal{E}$ of at least $\lceil \frac{m-1}{r+1} \rceil$ edges.
Proof: For any color $i \in \{1, 2, \ldots, r+1\}$, let $\mathcal{E}_i = \{e \in \mathcal{E} \mid \text{e has the color } i\}$, and let $n_i = |\mathcal{E}_i|$. Assume that $n_i < \lceil \frac{m-1}{r+1} \rceil$ for all i. Then $n_i \leqslant \lceil \frac{m-1}{r+1} \rceil - 1 < \lfloor \frac{m-1}{r+1} \rfloor \leqslant \frac{m-1}{r+1}$. Then $|\mathcal{E}| = \sum_{i=1}^{r+1} |\mathcal{E}_i| = \sum_{i=1}^{r+1} n_i < (r+1)\frac{m-1}{r+1} = m-1$, contradiction. ■

Now we can finish the proof of the lemma. Let

$$V = \{\text{vertices in } K_m \text{ that are connected to } v \text{ via an edge in } \mathcal{E}_0\}.$$

Then $|V| = |\mathcal{E}_0| \geqslant \lceil \frac{m-1}{r+1} \rceil \geqslant \frac{m-1}{r+1} = S(r)$. Let K_0 be the complete subgraph generated by V. If one of the edges in K_0 has color i, say edge e_0 connecting $v_1, v_2, \in V$, then the triangle with vertices $\{v, v_1, v_2\}$ has all edges in the same color i. If none of the edges in K_0 has color i, then we have a complete graph K_0 with at least $S(r)$ vertices whose edges are colored by r colors, so by the inductive hypothesis, K_0 contains a monochromatic triangle.

Therefore, we know some $S(r+1) \leqslant m$ exists. Hence by induction, we have proved the lemma. □

Theorem 2.3 (Schur's Theorem)**.** *For any $N \geqslant S(r)$, for any r-coloring of $\{1, 2, \ldots, N\}$, there exist distinct $x, y, z \in \{1, 2, \ldots, N\}$ of the same color such that $x + y = z$.*

Proof. Let $N \geqslant S(r)$. Given any r-coloring $C : \{1, \ldots, N\} \to \{1, \ldots, r\}$, we can define an r-coloring of the complete graph K_N with vertices labelled $1, 2, \ldots, N$ by coloring the edge (i, j) with $C(|i-j|)$.

Since $N \geqslant S(r)$, there exists a monochromatic triangle. In other words, there exist $a, b, c \in \{1, 2, \ldots, N\}$ such that $C(|a-b|) = C(|b-c|) = C(|c-a|)$. Without loss of generality, assume $a < b < c$ and let $x = b-a$, $y = c-b$, $z = c-a$. Then $x, y, z \in \{1, 2, \ldots, N\}$ satisfy $x + y = z$ and $C(x) = C(y) = C(z)$.

$\square$

Theorem 2.4 (Fermat's Last Theorem mod p)**.** *Let $n \in \mathbb{N}$. Then $x^n + y^n \equiv z^n$ (mod p) has a nontrivial solution for all sufficiently large prime numbers p.*

Proof. Let p be a prime number. Denote $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_p$ and let $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$. Then $(\mathbb{Z}_p, +, \cdot)$ is a field and $(\mathbb{Z}_p^*, \cdot)$ is an abelian group. Now it suffices to show that for large enough p, there exist $x, y, z \in \mathbb{Z}_p^*$ such that $x^n + y^n = z^n$ in $\mathbb{Z}_p^*$.

Let $G_n = \{x^n \colon x \in \mathbb{Z}_p^*\}$. Then G_n is a subgroup of $(\mathbb{Z}_p^*, \cdot)$. Consider the homomorphism $\phi\colon \mathbb{Z}_p^* \to G_n, x \mapsto x^n$. By the first isomorphism theorem $G_n \cong \mathbb{Z}_p^*/\ker\phi$. Thus, $|G_n| = |\mathbb{Z}_p^*|/|\ker\phi|$. Since $(\mathbb{Z}_p, +, \cdot)$ is a field, the polynomial $x^n - 1$ has at most n roots, so the number of cosets $[\mathbb{Z}_p^* \colon G_n] = |\mathbb{Z}_p^*|/|G_n| = |\ker\phi| = |\{x \in \mathbb{Z}_p^* \colon x^n = 1\}| \leqslant n$.

Now we can partition $\mathbb{Z}_p^*$ into disjoint left cosets, i.e. $\mathbb{Z}_p^* = a_1 G_n \cup a_2 G_n \cup \ldots \cup a_r G_n$, where $r \leqslant n$ and $a_1, \ldots, a_r \in \mathbb{Z}_p^*$. Define an r-coloring $C\colon \mathbb{Z}_p^* \to \{1, \ldots, r\}$ by $C(x) = i$ iff $x \in a_i G_n$. Suppose $p - 1 \geqslant S(n) \geqslant S(r)$. Then by Schur's theorem, there exist $x_0, y_0, z_0 \in \mathbb{Z}_p^*$ such that $x_0 + y_0 = z_0$ and $C(x_0) = C(y_0) = C(z_0)$. So there exists $i \in \{1, \ldots, r\}$, such that $x_0, y_0, z_0 \in a_i G_n$, i.e. there exist $x, y, z \in \mathbb{Z}_p^*$ such that $x_0 = a_i x^n, y_0 = a_i y^n, z_0 = a_i z^n$. So $a_i x^n + a_i y^n = a_i z^n$; multiplying both sides by a_i^{-1}, we get $x^n + y^n = z^n$ in $\mathbb{Z}_p^*$. $\square$

3. Introducing Hindman's Theorem and Ultrafilters

Similar to Schur's Theorem, Hindman's Theorem is also about the existence of monochromatic subsets in positive integers with a certain arithmetic structure. In fact Hindman's Theorem is a much stronger generalization of Schur's Theorem.

Definition 3.1. Let $A \subseteq \mathbb{N}$; we define finite sum set of A by

$$FS(A) := \left\{ \sum_{a \in A'} a \ : \ A' \subseteq A, \ |A'| < \infty \right\}.$$

Now let $(n_j)_{j \in \mathbb{N}}$ be any increasing sequence in $\mathbb{N}$, we define

$$FS(n_j)_{j=1}^{\infty} := FS(\{n_j |\, j \in \mathbb{N}\}) = \{n_{j_1} + n_{j_2} + \cdots + n_{j_k} : j_1 < j_2 < \cdots < j_k, \ k \in \mathbb{N}\}.$$

Now we state the main theorem in this paper.

Theorem 3.2 (Hindman's Theorem)**.** *Given any finite coloring $\mathbb{N} = \dot{\bigcup}_{i=1}^{r} C_i$, one of the C_i contains an infinite sequence $(n_j)_{j \in \mathbb{N}}$ together with its $FS(n_j)_{j=1}^{\infty}$.*

In order to prove Hindman's Theorem, we need to introduce a concept called ultrafilter.

Definition 3.3. A *filter* on a set S is a collection of subsets $\mathcal{F} \subset \mathcal{P}(S)$, such that for all $A, B \subseteq S$,

(1) $S \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.
(2) If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
(3) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Example 3.4. $\mathcal{F}_0 = \{A \subseteq S \mid S \smallsetminus A \text{ is finite}\}$ is an example of filter and is called a *cofinite filter*.

It is easy to check a cofinite filter satisfies the definition of a filter.

Definition 3.5. A filter $\mathcal{F}$ on S is an *ultrafilter* if for all $A \subseteq S$, either $A \in \mathcal{F}$ or $A^{\mathsf{C}} \in \mathcal{F}$.

Definition 3.6. A filter $\mathcal{F}$ on S is a maximal filter if for any $A \subseteq S$ and $A \notin \mathcal{F}$, $\mathcal{F} \cup \{A\}$ is not a filter. (i.e. $\mathcal{F}$ cannot be extended)

Proposition 3.7. *A filter $\mathcal{F}$ is an ultrafilter if and only if it is a maximal filter.*

Proof. First we show every ultrafilter is a maximal filter. For an ultrafilter $\mathcal{F}$, suppose we extend it by adding $A \subset S$ to this ultrafilter $\mathcal{F}$. Since $A \notin \mathcal{F}$ before, $A^{\mathsf{C}} \in \mathcal{F}$. However, if $A^{\mathsf{C}}, A \in \mathcal{F}$, $A^{\mathsf{C}} \cap A = \emptyset \in \mathcal{F}$, a contradiction.

Then we show every maximal filter is an ultrafilter. Let $\mathcal{F}$ be maximal. If $\mathcal{F}$ is not an ultrafilter, there exists $A \subset S$ such that $A \notin \mathcal{F}$ and $A^{\mathsf{C}} \notin \mathcal{F}$. Either A or A^{C} must intersect with all sets in $\mathcal{F}$, since if not, there exists set $B \in \mathcal{F}$, $B \subseteq A$ or $B \subseteq A^{\mathsf{C}}$, from which it follows that A or A^{C} is in $\mathcal{F}$, which contradicts our assumption. Without loss of generality, we assume $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Define $\mathcal{F}' = \{C \in \mathcal{P}(S)\colon A \cap F \subseteq C \text{ for some } F \in \mathcal{F}\}$. We claim $\mathcal{F}'$ is a filter containing $\mathcal{F}$. For all $F \in \mathcal{F}$, $A \cap F \subseteq F$, so $\mathcal{F} \subset \mathcal{F}'$. It is clear that $\emptyset \notin \mathcal{F}'$ and $S \in \mathcal{F}'$. For $C_1 \in \mathcal{F}'$ and $C_1 \subset C_2$, since $A \cap F \subseteq C_1 \subset C_2$, $C_2 \in \mathcal{F}'$. For $C, D \in \mathcal{F}'$, $A \cap F \subseteq C$ and $A \cap F \subset D$, so $A \cap F \subset C \cap D$ and $C \cap D \in \mathcal{F}'$. Our claim is proved but such claim contradicts the maximality; hence, every maximal filter is an ultrafilter.

□

Example 3.8. The *principal ultrafilter* generated by $x \in S$ is a collection of subsets: $\mathcal{F}_x = \{A \subseteq S\colon x \in A\}$. We denote such a principal ultrafilter with $\langle x \rangle$.

Lemma 3.9. *Given an ultrafilter $\mathcal{F}$, if $A \cup B \in \mathcal{F}$, then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

Proof. We will prove the contrapositive of this claim. Suppose $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$, then $A^{\mathsf{C}}, B^{\mathsf{C}} \in \mathcal{F}$. It follows $A^{\mathsf{C}} \cap B^{\mathsf{C}} = (A \cup B)^{\mathsf{C}} \in \mathcal{F}$. Therefore $A \cup B \notin \mathcal{F}$.

□

Lemma 3.10. *If an ultrafilter $\mathcal{F}$ on $\mathbb{N}$ contains a finite set, then $\mathcal{F} = \langle n \rangle$ for some $n \in \mathbb{N}$.*

Proof. We first show that some principal ultrafilter $\langle n \rangle \subseteq \mathcal{F}$. When ultrafilter $\mathcal{F}$ contains a finite set A with cardinality k, by Lemma 3.9, for some $a \in A$, either $\{a\} \in \mathcal{F}$ or $A \setminus \{a\} \in \mathcal{F}$. When $\{a\} \in \mathcal{F}$, principal ultrafilter $\langle a \rangle \subset \mathcal{F}$. When $A \setminus \{a\} \in \mathcal{F}$ we can proceed this process until we find a singleton set in $\mathcal{F}$, which is feasible since $|A \setminus \{a\}| = k - 1$. Thus some principal ultrafilter $\langle n \rangle$ is contained in $\mathcal{F}$. We claim that $\mathcal{F} = \langle n \rangle$. If not, there exists B such that $n \notin B$ but $B \in \mathcal{F}$. Then $n \in B^{\mathsf{C}}$ and $B^{\mathsf{C}} \in \langle n \rangle \subseteq \mathcal{F}$, which contradicts the fact that $\mathcal{F}$ is an ultrafilter.

□

We can also relate principal ultrafilters with cofinite filters, but we first need to establish the existence of non-principal ultrafiilters.

Definition 3.11. For a collection of sets $\mathcal{A}$, a *chain* in $\mathcal{A}$ is a collection $\{A_\alpha\colon \alpha \in I\}$ of elements of $\mathcal{A}$ such that for any $\alpha, \beta \in I$, either $A_\alpha \subseteq A_\beta$ or $A_\beta \subseteq A_\alpha$.

Lemma 3.12 (Zorn's Lemma)**.** *Let $\mathcal{A}$ be a nonempty collection of sets such that for every chain $\{A_\alpha : \alpha \in I\}$ in $\mathcal{A}$, $\bigcup_{\alpha\in I} A_\alpha \in \mathcal{A}$. Then there is a maximal element in $\mathcal{A}$, i.e. there is no $B \in \mathcal{A}$ with $A \subsetneq B$.*

Remark 3.13. Note that the essence of *Zorn's Lemma* is the statement that if every totally ordered subset of a partially ordered set has an upper bound, then the partially ordered set has a maximal element. Therefore, the relation on $\mathcal{A}$ doesn't necessarily need to be $\subseteq$, the relation $\supseteq$ also makes $\mathcal{A}$ a partially ordered set. In this case, *Zorn's Lemma* can be restated as: Let $\mathcal{A}$ be a nonempty collection of sets such that for every chain $\{A_\alpha : \alpha \in I\}$ in $\mathcal{A}$, we have $\bigcap_{\alpha\in I} A_\alpha \in \mathcal{A}$. Then there is a minimal element in $\mathcal{A}$.

Theorem 3.14 (Ultrafilter Theorem)**.** *Every filter $\mathcal{F}$ extends to an ultrafilter.*

Proof. For some filter $\mathcal{F}$, let $\mathcal{A}$ be the set of all filters $\mathcal{F}'$ on X such that $\mathcal{F} \subseteq \mathcal{F}'$. $\mathcal{A}$ is nonempty since $\mathcal{F} \in \mathcal{A}$. We claim that for each chain in $\mathcal{A}$, $\{\mathcal{F}_\alpha : \alpha \in I\}$, their union $\bigcup_{\alpha\in I} \mathcal{F}_\alpha$is still a filter in $\mathcal{A}$. It is clear that $\emptyset \notin \bigcup_{\alpha\in I} \mathcal{F}_\alpha$ and $X \in \bigcup_{\alpha\in I} \mathcal{F}_\alpha$. For an element $A \in \bigcup_{\alpha\in I} \mathcal{F}_\alpha$, $A \in \mathcal{F}_\alpha$ for some $\alpha \in I$. Then for all B such that $A \subseteq B$, $B \in \mathcal{F}_\alpha \subset \bigcup_{\alpha\in I} \mathcal{F}_\alpha$. Similarly, for $C, D \in \bigcup_{\alpha\in I} \mathcal{F}_\alpha$, $C \in \mathcal{F}_\alpha$ and $B \in \mathcal{F}_\beta$. Without loss of generality, we assume $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$; thus, $B \in \mathcal{F}_\alpha$ and $A \cap B \in \mathcal{F}_\alpha \subseteq \bigcup_{\alpha\in I} \mathcal{F}_\alpha$. By *Zorn's Lemma*, there exists a maximal element in $\mathcal{A}$ and by Proposition 3.7, it is an ultrafilter.

□

Corollary 3.15. *Let S be an infinite set. Then there exist non-principal ultrafilters on S.*

Proof. By Ultrafilter Theorem, we can extend a cofinite filter $\mathcal{F}_0$ to an ultrafilter $\mathcal{F}$. $\mathcal{F} \setminus \mathcal{F}_0$ can only contain infinite sets. Suppose $\mathcal{F} \setminus \mathcal{F}_0$ contains a finite set A. Since $\mathcal{F}_0$ is cofinite, $S \setminus A \in \mathcal{F}_0$ and $S \setminus A$ and A are both in ultrafilter $\mathcal{F}$, which is a contradiction. Therefore ultrafilter $\mathcal{F}$ contains no finite set, so it is non-principal. □

In fact, only by containing a cofinite filter can an ultrafilter be non-principal .

Corollary 3.16. *Every non-principal ultrafilter $\mathcal{F}$ contains a cofinite filter.*

Proof. Let $x \in S$. Since $\mathcal{F}$ is an ultrafilter, then either $\{x\}$ or $S \setminus \{x\} \in \mathcal{F}$. Since $\mathcal{F}$ is non-principal, $S \setminus \{x\} \in \mathcal{F}$. Now for all $A \subseteq S$ and A being finite, $S \setminus A = \bigcap_{x\in A} S \setminus \{x\} \in \mathcal{F}$.

□

4. Topological Space of Ultrafilters βS

We would like to consider ultrafilters on an infinite set S, which means $\mathbb{N}$ is an example of S. Let βS denote the collection of ultrafilters on S and in this section we will equip βS with a topology and discuss properties of βS. Since we consider the set βS as a topological space, it's natural to denote its elements (ultrafilters) by lowercase letter p. And for $A \in p$, we say A is p-large.

First, let us review some basic definitions and theorems of topology.

Definition 4.1. A *topology* on X is a collection $\mathscr{T}$ of subsets of X that satisfy the following properties:

(1) X and $\emptyset$ are elements of $\mathscr{T}$.

(2) The union of an arbitrary collection of sets in $\mathscr{T}$ is also in $\mathscr{T}$.
(3) The intersection of a finite number of sets in $\mathscr{T}$ is also in $\mathscr{T}$.

The elements of $\mathscr{T}$ are called the *open sets* of X. The set X with the structure of the topology $\mathscr{T}$ is called a *topological space*.

Definition 4.2. An *open neighborhood* U of a point x in a topological space X is an open set U such that $x \in U$.

Definition 4.3. Basic open sets are open sets such that any open set is the union of some basic open sets. In other words, for any point in an open set, we can find a basic open set containing such point which is a subset of the original open set.

Definition 4.4. A set $A \subset X$ is *closed* if its complement is open.

Definition 4.5. The *closure* $\overline{A}$ of $A \subseteq X$ in a topological space X is the set of all points x such that every open neighborhood of x intersects A.

Lemma 4.6. *A set $A \subseteq X$ is closed if and only if $\overline{A} = A$.*

Lemma 4.7. *For $A, B \subseteq X$, if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

Definition 4.8. Let X and Y be topological spaces, $f\colon X \to Y$. f is *continuous* if at any $a \in X$, for each open set V containing $f(a)$, there exists an open neighborhood U of a such that $f(U) \subseteq V$.

Definition 4.9. A *Hausdorff space* is a topological space such that for all distinct $x, y \in X$, there are disjoint sets $U, V \in \mathscr{T}$ such that $x \in U$, $y \in V$.

Definition 4.10. For a set S, a *basis* for a topology of S is a collection $\mathcal{B}$ of subsets of S (called *basis elements* such that:

(1) For all $x \in S$, there exists $B \in \mathcal{B}$ such that $x \in B$.
(2) If $A, B \in \mathcal{B}$ and $x \in A \cap B$, then there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

Similar with the basis of a vector space, a basis for a topology of S generates a topology $\mathscr{T}$ on S.

Lemma 4.11. *Let $\mathcal{B}$ be a basis for a topology satisfying the properties above. Define $\mathscr{T}_{\mathcal{B}} \colon = \{A \subseteq S \colon \forall x \in A, \exists B \in \mathcal{B}, x \in B \subseteq A\}$. Then $\mathscr{T}_{\mathcal{B}}$ defines a topology on S.*

Note that with our definition the basis $\mathcal{B}$ automatically becomes the collection of basic open sets of the topology $\mathscr{T}$ it generates.

Definition 4.12. A *compact* space is a topological space X such that every open cover has a finite subcover.

Lemma 4.13. *For a compact set K, a closed subset $D \subseteq K$ is compact.*

Lemma 4.14. *Let K be a compact Hausdorff space and $x \in K$. For each open neighborhood U of x, there is an open neighborhood V of x such that $\overline{V} \subseteq U$.*

Proof. As K is a compact Hausdorff space, its closed subset $U^{\complement}$ is compact. For each $y \in U^{\complement}$, choose disjoint open neighborhoods V_y and U_y of x and y, respectively. Then $U^{\complement} \subseteq \bigcup_{y \in U^{\complement}} U_y$. Since $U^{\complement}$ is compact, there exists $n \in \mathbb{N}$ such that $U^{\complement} \subseteq U_{y_1} \cup \cdots \cup U_{y_n}$. Let $C = (U_{y_1} \cup \cdots \cup U_{y_n})^{\complement} = U_{y_1}^{\complement} \cap \cdots \cap U_{y_n}^{\complement}$. Thus, $C \subseteq U$. Since $U_{y_i}^{\complement}$ is closed for all $1 \leqslant i \leqslant n$, C is closed. Let $V = V_{y_1} \cap \cdots \cap V_{y_n}$ and V is an open neighborhood of x. Since $V_{y_i} \cap U_{y_i} = \emptyset$ for all $1 \leqslant i \leqslant n$, $V_{y_i} \subseteq U_{y_i}^{\complement}$, so $V \subseteq C$. Therefore, by Lemma 4.7, $\overline{V} \subseteq \overline{C}$ and since C is closed, by Lemma 4.6 $\overline{V} \subseteq \overline{C} = C \subseteq U$.

□

Lemma 4.15. *For a Hausdorff space K, a compact set $D \subset K$ is closed.*

Proof. When $D \subset K$, take any $x \in D^{\complement}$. For every $y \in D$, since the space is Hausdorff, there are open neighborhoods U_y of y and V_y of x such that $U_y \cap V_y = \emptyset$. $\{U_y \mid y \in D\}$ is an open cover of D and since D is compact, there exists $y_1, \ldots, y_m$ such that $D \subseteq U_{y_1} \cup \cdots \cup U_{y_m}$. Let $U = U_{y_1} \cup \cdots \cup U_{y_m}$ and $V = V_{y_1} \cap \cdots \cap V_{y_m}$. Thus, $U \cap V = \emptyset$ and U, V are open. Moreover, $V \subset D^{\complement}$. And since x is arbitrary, for every point in $D^{\complement}$, there exists an open neighborhood containing such point in $D^{\complement}$, so $D^{\complement}$ is open and D is closed. □

Now we try to construct a basis for a topology on the space of ultrafilters βS by finding a collection of subsets of βS that satisfies the properties listed in Definition 4.10.

Definition 4.16. For a set $A \subseteq S$, define $[A]\colon = \{p \in \beta S\colon A \in p\}$.

We can show that for $A, B \subseteq S$, $[A]$ and $[B]$ have the following properties:

(1) $[\emptyset] = \emptyset$; $[S] = \beta S$.
(2) $[A] \subseteq [B]$ if and only if $A \subseteq B$, and it follows $[A] = [B]$ if and only if $A = B$.
(3) $[A] \cap [B] = [A \cap B]$.
(4) $[A] \cup [B] = [A \cup B]$
(5) $[A^{\complement}] = [A]^{\complement}$.

The reasons follow:

(1) Since there exists no $p \in \beta S$ such that $\emptyset \in p$, $[\emptyset] = \emptyset$. For all $p \in \beta S$, $S \in p$; thus $[S] = \beta S$.
(2) Suppose $A \subseteq B$, then for $p \in [A]$, we have $A \in p$, so $B \in p$, and $p \in [B]$. Thus $[A] \subseteq [B]$. Suppose $[A] \subseteq [B]$ and $A \nsubseteq B$, then $A' = A \setminus B \neq \emptyset$. Therefore, we can choose ultrafilter p such that A' is p-large. Since $A' \subset A$, $A \in p$ and $p \in [A]$. It follows that $B \in p$. However, $\emptyset = B \cap A' \in p$, a contradiction.
(3) The $[A] \cap [B] \subseteq [A \cap B]$ follows from the definition. For $p \in [A \cap B]$, $A \cap B \in p$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $A \in p$ and $B \in p$. Thus $p \in [A] \cap [B]$ and $[A \cap B] \subseteq [A] \cap [B]$.
(4) For $p \in [A] \cup [B]$, either $A \in p$ or $B \in p$. Thus $A \cup B \in p$ and $p \in [A \cup B]$. For $p \in [A \cup B]$, $A \cup B \in p$. Assume $p \notin [A]$ and $p \notin [B]$. Then $A \notin p$ and $B \notin p$. So $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement} \in p$, which is a contradiction. Therefore, $[A \cup B] = [A] \cup [B]$.
(5) For $p \in [A^{\complement}]$, $A^{\complement} \in p$ so $A \notin p$. Thus $p \in [A]^{\complement}$ and $[A^{\complement}] \subseteq [A]^{\complement}$. For $p \in [A]^{\complement}$, $p \notin [A]$. Thus $A \notin p$ and $A^{\complement} \in p$. So $p \in [A^{\complement}]$ and $[A]^{\complement} \subseteq [A^{\complement}]$.

The definition of $[A]$ and Property 3 allow $\mathcal{B} = \{[A]\colon A \subseteq S\}$ to satisfy the conditions of Definition 4.10 for being a basis of a topology on βS. We deine the topology on βS to be the one generated by the basis $\mathcal{B} = \{[A]\colon A \subseteq S\}$.

Theorem 4.17. *βS is a compact Hausdorff space.*

Proof. Assume βS is not compact. Then there exists an open cover of βS with no finite subcover. We can assume cover is the form $\{[A_\alpha]\colon \alpha \in I\}$. Then for any

$\alpha_1, \ldots, \alpha_n \in I$, $[A_{\alpha_1} \cup A_{\alpha_2} \cup \cdots \cup A_{\alpha_n}] = [A_{\alpha_1}] \cup [A_{\alpha_2}] \cup \cdots \cup [A_{\alpha_n}] \neq \beta S = [S]$. It follows that $A_{\alpha_1} \cup A_{\alpha_2} \cup \cdots \cup A_{\alpha_n} \neq S$, or in other words, $A_{\alpha_1}^{\complement} \cap A_{\alpha_2}^{\complement} \cap \cdots \cup A_{\alpha_n}^{\complement} \neq \emptyset$. Consider the collection of all sets containing at least one of $A_{\alpha_i}^{\complement}$; $\{A_{\alpha}^{\complement} \colon \alpha \in I\}$ extends to a filter and this filter is contained in some ultrafilter $p \in \beta S$. Since $\{[A_\alpha] \colon \alpha \in I\}$ covers βX, we can choose the α such that $p \in [A_\alpha]$. Then $A_\alpha^{\complement}$ and A_α are both in p, which is a contradiction.

For $p, q \in \beta S$ and p and q being distinct ultrafilters, there exists $A \in p$ such that $A \notin q$, which implies $A^{\complement} \in q$. $[A]$ and $[A^{\complement}]$ are two open sets containing p and q respectively. Since $[A^{\complement}] = [A]^{\complement}$, $[A]$ and $[A^{\complement}]$ are also disjoint. Thus βS is a compact Hausdorff space.

□

The nature of this topology on βS is interesting and worth noticing:

Definition 4.18. A clopen set in a topological space X is a set which is both open and closed.

Since $[A^{\complement}] = [A]^{\complement}$ for all $A \subseteq S$, the sets $[A]$ are clopen.

We can also identify every principal ultrafilter $\langle x \rangle$ with the point x, since the function from S to βS defined by $x \mapsto \langle x \rangle$ is injective on its image. With such indentification, S becomes a topological subspace of βS.

Definition 4.19. The discrete topology on X is the topology defined by $\mathscr{T} = \mathcal{P}(X)$

Definition 4.20. A set U is dense in set V, if $\overline{U} = V$.

Theorem 4.21. *S is a discrete topological subspace of βS, and S is dense in βS.*

Proof. From Definition 4.19, it follows easily that a topological space is discrete if and only if every singleton set $\{x\}$ is open. In other words, if for every $x \in S$, there exists an open set containing only x, then S is discrete. By definition, for any $p \in [\{x\}]$, $\{x\} \in p \in \beta S$; thus $p = \langle x \rangle$. Therefore, $[\{x\}] = \{\langle x \rangle\}$, which is a basic open set, so S is a discrete topological space.

For any $q \in \beta S$ and every basic open set $[A]$ such that $q \in [A]$, we have $A \in q$. Thus A is nonempty. Fix $x \in A$, $A \in \langle x \rangle$, i.e. $\langle x \rangle \in [A]$. Therefore, $[A] \cap S \neq \emptyset$ and S is dense in βS. □

5. The Stone-Čech Compactification

Definition 5.1. The Stone-Čech compactification βX is a compact Hausdorff space together with a continuous map $\beta \colon X \to \beta X$, such that every continuous map from topological space X to a compact Hausdorff space K, $f \colon X \to K$, extends uniquely to a continuous map $\beta f \colon \beta X \to K$.

$$\begin{tikzcd}
\beta X \arrow[dr, dotted, "\beta f"] & \\
X \arrow[u, hook, "\beta"] \arrow[r, "f"] & K
\end{tikzcd}$$

We will show that βS, the space of ultrafilters on S, is a Stone-Čech compactification of S.

Definition 5.2. Recall the definition of a filter from Definition 3.3. Let $\mathcal{F}$ be a filter on a topological space X. A point $x \in X$ is a limit point of $\mathcal{F}$ if for each open neighborhood U of x, we have $U \in \mathcal{F}$. We say $\mathcal{F}$ converges to x if x is a limit point of $\mathcal{F}$, and we write $\lim \mathcal{F} = x$.

Lemma 5.3. *For a Hausdorff topological space X, a filter $\mathcal{F}$ on X can have at most one limit point.*

Proof. Assume $\lim \mathcal{F} = x$ and $\lim \mathcal{F} = y$ but $x \neq y$. By the Hausdorff property, there exist disjoint open neighborhoods U and V of x and y, respectively. Since x, y are limit points of $\mathcal{F}$, there are sets $A, B \in \mathcal{F}$ such that $A \subseteq U$ and $B \subseteq V$. Then $\emptyset = A \cap B \in \mathcal{F}$, which is a contradiction.

□

Theorem 5.4. *A Hausdorff topological space X is compact if and only if every ultrafilter on X is convergent.*

Proof. Only if: Let p be an ultrafilter on X without a limit point. Then for all $x \in X$, there exists open neighborhood $U_x \notin p$. Since X is compact, the open cover $\{U_x \colon x \in X\}$ has a finite subcover, i.e. $X = U_{x_1} \cup \cdots \cup U_{x_n}$. Pick a set $A \in p$. As $A \subseteq X$, $A = A \cap X = A \cap (U_{x_1} \cup \cdots \cup U_{x_n}) = (A \cap U_{x_1}) \cup \cdots \cup (A \cap U_{x_n}) \in p$. Thus by Lemma 3.9, there exists $1 \leqslant i \leqslant n$ such that $A \cap U_{x_i} \in p$. Then $U_{x_i} \in p$, a contradiction.

If: Suppose X is not compact. Let $\{U_\alpha \colon \alpha \in I\}$ be an open cover of X with no finite subcover. Then for all $\alpha_1, \dots, \alpha_n \in I$, since $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \neq X$, $(U_{\alpha_1} \cup \cdots \cup U_{\alpha_n})^{\complement} = U_{\alpha_1}^{\complement} \cap \cdots \cap U_{\alpha_n}^{\complement} \neq \emptyset$. In other words, any finite intersection of elements of $\mathcal{A} = \{U_\alpha^{\complement} \colon \alpha \in I\}$ is nonempty. Then $\mathcal{F} = \{F \subseteq X \colon A \subseteq F \text{ for some } A \in \mathcal{A}\}$ is a filter on X and by Theorem 3.14 $\mathcal{F}$ can be extended to an ultrafilter p on X. Let $\lim p = x$. Pick $\alpha \in I$ such that $x \in U_\alpha$. By definition of limit point, $U_\alpha \in p$. Therefore we have $U_\alpha \in p$ and $U_\alpha^{\complement} \in p$, which is a contradiction. □

Lemma 5.5. *Let $f \colon X \to Y$. For each ultrafilter p on X, the family of sets $f(p) = \{A \colon f^{-1}(A) \in p\}$ is an ultrafilter on Y.*

Proof. It follows from the definition of $f(p)$ that $Y \in f(p)$ and $\emptyset \notin f(p)$.

For $A, B \in f(p)$, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \in p$. Thus, $A \cap B \in f(p)$.

For $A \in f(p)$ and $A \subseteq A'$, $f^{-1}(A) \subseteq f^{-1}(A')$. Since $f^{-1}(A) \subseteq f^{-1}(A') \in p$, $A' \in f(p)$.

Suppose $A \notin f(p)$; then $f^{-1}(A) \notin p$ and $X \setminus f^{-1}(A) \in p$. Since $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A) \in p$, $Y \setminus A \in f(p)$. Thus we have proved $f(p)$ is an ultrafilter on Y.

□

Theorem 5.6. *Let K be a compact Hausdorff space and let βS be the space of ultrafilters on S. Every function $f \colon S \to K$ extends, in a unique manner, to a continuous function $\beta f \colon \beta S \to K$.*

$$\begin{tikzcd}
\beta S \arrow[dr, dotted, "\beta f"] & \\
S \arrow[u, hook, "id"] \arrow[r, "f"] & K
\end{tikzcd}$$

Proof. We first prove the existence of βf. For each $p \in \beta S$, since p is an ultrafilter on S and $f : S \to K$, by Lemma 5.5, $f(p)$ is an ultrafilter on K. Since K is a compact Hausdorff space, by Theorem 5.4 $f(p)$ converges to a unique point in K. Define $\beta f(p) :=\lim f(p)$.

The function βf extends f: For each $x \in S$,

$$\begin{aligned} f(\langle x\rangle) &= \{A \subseteq K : f^{-1}(A) \in \langle x\rangle\} \\ &= \{A \subseteq K : x \in f^{-1}(A)\} \\ &= \{A \subseteq K : f(x) \in A\} \\ &= \langle f(x)\rangle. \end{aligned}$$

For any open neighborhood U of $f(x)$, $\{f(x)\} \subseteq U$, so $U \in \langle f(x)\rangle = f(\langle x\rangle)$ and $\lim f(\langle x\rangle) = f(x)$. Thus, with $\langle x\rangle$ identified with x, $\beta f(x) = \lim f(\langle x\rangle) = f(x)$.

The function βf is continuous: Let $p \in \beta S$ and W be an open neighborhood of $\beta f(p)$ in K. By Lemma 4.14, there exists V as an open neighborhood of $\beta f(p)$ such that $\overline{V} \subseteq W$. As $\beta f(p) = \lim f(p) \in V$, $f^{-1}(V) \in p$. Recall that $[f^{-1}(V)] = \{p \in \beta S : f^{-1}(V) \in p\}$ and $[f^{-1}(V)]$ is a basic open set, so $[f^{-1}(V)]$ is an open neighborhood of p. For any $q \in [f^{-1}(V)]$, $f^{-1}(V) \in q$, so $V \in f(q)$. Take any open neighborhood U of $\beta f(q) = \lim f(q) \in K$, $U \in f(q)$. Thus, $U \cap V \neq \emptyset \in f(q)$. Therefore $\beta f(q) = \lim f(q) \in \overline{V} \subseteq W$. By definition of continuity, we have proved βf is continuous.

Now we prove the uniqueness of βf. Suppose there exists a distinct extension $\phi f : \beta S \to K$. Since βf and ϕf are extensions of $f : S \to K$, their restrictions to S are the same, i.e. $f = \beta f\mid_S = \phi f\mid_S$. Thus, there exists $x \in \beta S \setminus S$ such that $\phi f(x) \neq \beta f(x)$. Since K is Hausdorff, there exist open sets $\beta f(x) \in V_1$ and $\phi f(x) \in V_2$ such that $V_1 \cap V_2 = \emptyset$. Let $U_1 = \beta f^{-1}(V_1)$ and $U_2 = \phi f^{-1}(V_2)$. Then $x \in U_1 \cap U_2$ and since βf and ϕf are continuous, $U_1 \cap U_2$ is open in βS. Thus $U_1 \cap U_2$ is an open neighborhood of x. Recall that S is dense in βS, so $(U_1 \cap U_2) \cap S \neq \emptyset$. It follows that $\beta f(U_1) \cap \phi f(U_2) = V_1 \cap V_2 \neq \emptyset$, which is a contradiction.

□

Recall that S is a discrete topological space, so the identification function from S to βS and every function from S to K are continuous. Therefore, Theorem 5.6 verifies the universal property in Definition 5.1, showing that the space of ultrafilters on S is a Stone-Čech compactification of S.

6. Algebraic Structure on $\beta\mathbb{N}$

Now we try to construct an algebraic structure on $\beta\mathbb{N}$.

Definition 6.1. A semigroup is a nonempty set S with an associative product $*$.

Definition 6.2. A subsemigroup of $(S, *)$ is a nonempty subset T of S that is closed under multiplication, i.e. $T * T \subseteq T$.

Schur's theorem can be applied to prove the existence of idempotent element in a finite semigroup, and the existence of idempotent element is a critical requirement for us to prove Hindman's Theorem.

Theorem 6.3. *Every finite semigroup has an idempotent element.*

Proof. Since $(S,*)$ is a finite semigroup (suppose $|S|=k$), for some $a\in S$, by Pigeonhole principle, there exist $n_1,n_2\in\mathbb{N}$ such that $a^{n_1}=a^{n_2}$. Fix such $a\in S$. For all $s\in S$, let $C_s=\{n\in\mathbb{N}\mid a^n=s\}$. Thus $\mathbb{N}$ are colored with k numbers, i.e. $\mathbb{N}=\dot\bigcup_{s\in S}C_s$.

Coloring natural numbers is a special case of Schur's theorem, which can be restated as: for each finite coloring of $\mathbb{N}$, there are numbers $n,m,n+m\in C_s$. In other words, $a^n=a^m=a^{n+m}=s$. Therefore $s^2=(a^n)^2=a^n\cdot a^m=a^{m+n}=a_n=s$, so s is idempotent.

□

In order to be more closely connected to the proof of Hindman's Theorem, from now on we consider the example of $S=\mathbb{N}$.

Definition 6.4. We define operation $+$ on $\beta\mathbb{N}$ in the following way:

$$p+q=\{A\subseteq\mathbb{N}\colon\{n\in\mathbb{N}\colon(A-n)\in p\}\in q\}$$

where $A-n=\{m\in\mathbb{N}\mid m+n\in A\}$.

We will show that the operation defined above renders $\beta\mathbb{N}$ a semigroup.

We first show that $p+q\in\beta\mathbb{N}$. It is obvious that $\emptyset\notin p+q$ and $\mathbb{N}\in p+q$. Let $A,B\in p+q$. Then $(A\cap B-n)\in p$ if and only if $(A-n)\in p$ and $(B-n)\in p$. So $\{n\in\mathbb{N}\colon(A\cap B-n)\in p\}=\{n\in\mathbb{N}\colon A-n\in p\}\cap\{n\in\mathbb{N}\colon B-n\in p\}$. Since $\{n\in\mathbb{N}\colon A-n\in p\}\in q$ and $\{n\in\mathbb{N}\colon B-n\in p\}\in q$, we have $\{n\in\mathbb{N}\colon(A\cap B-n)\in p\}\in q$ and $A\cap B\in p+q$.

Let $C\in p+q$ and $C\subseteq D$. For all $n\in\mathbb{N}$, $C-n\subseteq D-n$. Thus $(C-n)\in p$ implies $(D-n)\in p$. Thus $\{n\in\mathbb{N}\colon(C-n)\in p\}\subseteq\{n\in\mathbb{N}\colon(D-n)\in p\}$ and $\{n\in\mathbb{N}\colon(D-n)\in p\}\in q$. Therefore, $D\in p+q$.

Let $A\subseteq N$ with $A\notin p+q$. Then $\{n\in\mathbb{N}\colon(A-n)\in p\}\notin q$. So we have:

$$\begin{aligned}&\{n\in\mathbb{N}\colon(A^{\complement}-n)\in p\}=\{n\in\mathbb{N}\colon(A-n)^{\complement}\in p\}\\&=\{n\in\mathbb{N}\colon(A-n)\notin p\}=\{n\in\mathbb{N}\colon(A-n)\in p\}^{\complement}\in q\end{aligned}$$

Therefore, $A^{\complement}\in p+q$.

Now we show the associativity of this operation.

Let $A\subseteq N$. Then $A\in p+(q+r)$ iff $\{n\in\mathbb{N}\colon(A-n)\in p\}\in q+r$ and more explicitly iff $\{m\in\mathbb{N}\colon(\{n\in\mathbb{N}\colon(A-n)\in p\}-m)\in q\}\in r$.

$$\begin{aligned}&\{n\in\mathbb{N}\colon(A-n)\in p\}-m\\&=\{n-m\colon(A-n)\in p\}\\&=\{n\in\mathbb{N}\colon[(A-m)-n]\in p\}\end{aligned}$$

So now $A\in p+(q+r)$ iff $\{m\in\mathbb{N}\colon\{n\in\mathbb{N}\colon[(A-m)-n]\in p\}\in q\}\in r$
iff $\{m\in\mathbb{N}\colon(A-m)\in p+q\}\in r$
iff $A\in(p+q)+r$.

Lemma 6.5. $\langle x\rangle+\langle y\rangle=\langle x+y\rangle$

Proof. For any $A\in\langle x+y\rangle$, since $(A-y)=\{m\in\mathbb{N}\mid m+y\in A\}$ and $x+y\in A$, $x\in(A-y)$. Thus $(A-y)\in\langle x\rangle$ and $y\in\{n\in\mathbb{N}\colon(A-n)\in\langle x\rangle\}\in\langle y\rangle$. Therefore, $\langle x+y\rangle\subseteq\langle x\rangle+\langle y\rangle$.

For any A such that $\{n\in\mathbb{N}\colon(A-n)\in\langle x\rangle\}\in\langle y\rangle$, there exists $y\in\{n\in\mathbb{N}\colon(A-n)\in\langle x\rangle\}$, i.e. $(A-y)\in\langle x\rangle$. Therefore $x\in A-y=\{m\in\mathbb{N}\mid m+y\in A\}$

and it follows $x+y \in A$ and $A \in \langle x+y \rangle$. Therefore $\langle x \rangle + \langle y \rangle \subseteq \langle x+y \rangle$, so we have $\langle x \rangle + \langle y \rangle = \langle x+y \rangle$

□

This lemma indicates that principal ultrafilter is not idempotent, a result which we would use in the proof of Hindman's Theorem.

Theorem 6.6. *For any $p \in \beta\mathbb{N}$, the function $\lambda\colon q \mapsto p+q$, with $\lambda_p(q) = p+q$, is continuous.*

Proof. Let $q \in \beta\mathbb{N}$ and V be an open set containing $\lambda_p(q)$. Then there exists $A \subset \mathbb{N}$ such that $\lambda_p(q) = p+q \in [A] \subseteq V$. Then $A \in p+q$. Let $B\colon = \{n \in \mathbb{N}\colon (A-n) \in p\}$, so $B \in q$ and $q \in [B]$. Let $r \in [B]$ then $B \in r$, i.e. $\{n \in \mathbb{N}\colon (A-n) \in p\} \in r$. Thus $A \in p+r$ and $p+r \in [A]$. Hence for all $r \in [B]$, $\lambda_p(r) \in [A] \subseteq V$. Therefore, $\lambda_p(q)$ is continuous. □

7. Existence of Idempotent Elements in $(\beta\mathbb{N}, +)$

Definition 7.1. A left topological semigroup $(S, *)$ is a semigroup with a topology and operation $*$ such that for every $s \in S$ operation with s on the left, i.e.$\lambda_s\colon x \mapsto s * x$ with $\lambda_s(x) = s * x$, is continuous.

Thus, by Theorem 6.6, $\beta\mathbb{N}$ is a left topological semigroup. Moreover, since $\beta\mathbb{N}$ is a compact Hausdorff space, $\beta\mathbb{N}$ is a compact left topological semigroup. We will use this property to show $\beta\mathbb{N}$ has idempotent elements.

Lemma 7.2 (Finite Intersection Property)**.** *For a compact space K and a family of closed sets $\{D_\alpha\colon \alpha \in I\}$ in K, if every intersection of finitely many members of this family is nonempty, then the entire intersection $\bigcap_{\alpha \in I} D_\alpha$ is nonempty.*

Proof. Suppose $\bigcap_{\alpha \in I} D_\alpha = \emptyset$. Then its complement, $\bigcup_{\alpha \in I} D_\alpha^{\complement} = K$ and is an open cover of K since D_α is closed. Since K is compact, open cover $\bigcup_{\alpha \in I} D_\alpha^{\complement}$ has a finite subcover, i.e. there exist $D_{\alpha_1}^{\complement}, D_{\alpha_2}^{\complement}, \ldots, D_{\alpha_n}^{\complement}$ such that $D_{\alpha_1}^{\complement} \cup D_{\alpha_2}^{\complement} \cup \cdots \cup D_{\alpha_n}^{\complement} = K$. The complement of this union is $D_{\alpha_1} \cap D_{\alpha_2} \cap \ldots, \cap D_{\alpha_n} = \emptyset$, which is a contradiction. Therefore, $\bigcap_{\alpha \in I} D_\alpha$ is nonempty.

□

Theorem 7.3. *If $(S, *)$ is a compact left topological semigroup, then S has an idempotent element.*

Proof. We first show that every compact left topological group has a minimal compact subsemigroup. Define $\mathcal{T} = \{$ compact subsemigroups of $S\}$. $\mathcal{T}$ is nonempty since $S \in \mathcal{T}$ and $\mathcal{T}$ is partially ordered by containment $\subseteq$. For any chain $\mathcal{C} = \{C_i\colon i \in I\}$ of $\mathcal{T}$ and any $k, j \in I$, either $C_k \subseteq C_j$ or $C_j \subseteq C_k$, so every intersection of finitely many members in $\mathcal{C}$ is nonempty. Since every subsemigroup C is compact and thus closed, $\mathcal{C}$ has the finite intersection property, i.e. $\bigcap_{i \in I} C_i \neq \emptyset$. Moreover, since an arbitrary intersection of compact sets is compact, $\bigcap_{i \in I} C_i$ is compact; thus, $\bigcap_{i \in I} C_i$ is a compact subsemigroup, or in other words $\bigcap_{i \in I} C_i \in \mathcal{T}$. Therefore, we can apply Zorn's Lemma: there exists a minimal compact subsemigroup $T \subseteq S$.

Then we show that for a minimal compact subsemigroup $T \subseteq S$, if $e \in T$, then e is idempotent.

Claim: For $eT = \{et\colon t \in T\}$, $eT = T$.

Proof: To prove the claim, we show eT is a compact subsemigroup of T. $eT \subseteq T$ as a subset and $eT \neq \emptyset$ are clear from definition. Now we check eT is also a subsemigroup of T. For $et_1, et_2 \in Te$, $et_1et_2 = e(t_1et_2)$ and $t_1et_2 \in T$, so $et_1et_2 \in eT$. eT is also compact as image of compact set T under continuous function λ_e. Therefore, eT is a compact subsemigroup of T. By the minimality of T, $eT = T$. ■

Define $B = \{t \in T : et = e\}$.
Claim: B=T.
Proof: $B \subseteq T$ as a subset is clear from definition. Since $eT = T$, $e \in eT$; in other words, there exists $x \in T$ such that $ex = e$, so B is nonempty. We will show B is a subsemigroup of T. For $t_1, t_2 \in B$, $e(t_1t_2) = (et_1)t_2 = et_2 = e$, so B is a subsemigroup of T. Note that by definition $B = \lambda_e^{-1}(\{e\}) \cap T$. Since the topological space is Hausdorff, the one-point set $\{e\}$ is closed. And because the left multiplication λ_e is continuous, its preimage $\lambda_e^{-1}(\{e\})$ is closed hence comapct, and it follows $B = \lambda_e^{-1}(\{e\}) \cap T$ is compact. Therefore B is a compact subsemigroup of T and by minimality of T, $B = T$. ■

In particular, $e \in B$; thus $e * e = e$.

□

Therefore, there exists idempotent element in $(\beta\mathbb{N}, +)$.

8. Proof of Hindman's Theorem

Recall that for an increasing sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$, we define

$$FS(n_j)_{j=1}^{\infty} := \{n_{j_1} + n_{j_2} + \cdots + n_{j_k} : j_1 < j_2 < \cdots , j_k,\ k \in \mathbb{N}\}.$$

Theorem 8.1 (Hindman's Theorem)**.** *Given any finite coloring $\mathbb{N} = \dot{\bigcup}_{i=1}^{r} C_i$, one of C_i contains an infinite sequence $(n_j)_{j\in\mathbb{N}}$ together with its $FS(n_j)_{j=1}^{\infty}$.*

Proof. Define

$$\mathcal{A} := \left\{A \subseteq \mathbb{N} : FS(n_j)_{j=1}^{\infty} \subseteq A \text{ for some } (n_j)_{j\in\mathbb{N}} \text{ increasing in } \mathbb{N}\right\}.$$

Let $p \in (\beta\mathbb{N}, +)$ be an idempotent ultrafilter (we know its existence from Section 5). First we will show that $p \subseteq \mathcal{A}$. Let $A \in p$, we define

$$A^* := \{n \in \mathbb{N} : A - n \in p\}.$$

Since $p + p = p$, we have $A \in p + p$, so $\{n \in \mathbb{N} : A - n \in p\} \in p$, i.e. $A^* \in p$. So $A \cap A^* \in p$, hence nonempty. Moreover, $A \cap A^*$ is an infinite set, because only principal ultrafilters can contain finite sets (Lemma 3.10), but idempotent ultrafilters cannot be principal (Lemma 6.5).

Now we define our increasing sequence $(n_j)_{j\in\mathbb{N}}$ contained in A recursively. Let $A_1 = A$. Since we have just shown that $A_1 \cap A_1^*$ is an infinite set, let's pick some $n_1 \in A_1 \cap A_1^*$. Then $n_1 \in A_1^*$, so $A_1 - n_1 \in p$. Let $A_2 = (A_1 - n_1) \cap A_1$. Then $A_2 \in p$, and as proved above, $A_2 \cap A_2^*$ is infinite. Thus we can choose $n_2 \in A_2 \cap A_2^*$ such that $n_2 > n_1$. Set $A_3 = (A_2 - n_2) \cap A_2$, and then A_3 belongs to p as well. We can continue this process: at each stage, choose $n_j \in A_j \cap A_j^*$ such that $n_j > n_{j-1}$ and define $A_{j+1} = (A_j - n_j) \cap A_j$.

Now we show $FS(n_j)_{j=1}^{\infty} \subseteq A$, and it suffices to show the following:
Claim: For all $k \in \mathbb{N}$ and $j_1 < j_2 < \cdots < j_k \in \mathbb{N}$, we have $n_{j_1} + n_{j_2} + \cdots + n_{j_k} \in A_{j_1}$.
Proof: We prove this by induction on k.

The base case is clear because for all $j_1 \in \mathbb{N}$, we have $n_{j_1} \in A_{j_1}$.

Now let $k \in \mathbb{N}$, and assume that the statement holds for k. Then we show it holds for $k+1$ as well. Let $j_1 < j_2 < \cdots < j_{k+1} \in \mathbb{N}$, and let $m = n_{j_2} + n_{j_3} + \cdots + n_{j_{k+1}}$. Then by induction hypothesis we know $m \in A_{j_2}$. Since $j_2 > j_1$, then $j_2 \geqslant j_1 + 1$, so $A_{j_2} \subseteq A_{j_1+1}$. Also we know $A_{j_1+1} \subseteq A_{j_1} - n_{j_1}$; hence $m \in A_{j_1} - n_{j_1}$. Thus $m + n_{j_1} \in A_{j_1}$, i.e.

$$n_{j_1} + n_{j_2} + \cdots + n_{j_{k+1}} \in A_{j_1}.$$

Now by the principle of induction we have proved our claim. ■

Therefore, $FS(n_j)_{j=1}^{\infty} \subseteq A$, so $A \in \mathcal{A}$. This holds for all $A \in p$, so $p \subseteq \mathcal{A}$.

Since $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, by Lemma 3.9, we have one of $C_i \in p \subseteq \mathcal{A}$. Thus we have a set C_i that contains an $FS(n_j)_{j=1}^{\infty}$ for some increasing sequence $(n_j)_{j\in\mathbb{N}}$. □

9. Hindman's Theorem on Power Set and Partition Regularity

In this section, we would try to generalize Hindman's Theorem into a stronger statement, but first we consider the analogous statement of Hindman's Theorem on the power set on $\mathbb{N}$.

Definition 9.1. Let $(A_n)_{n\in\mathbb{N}}$ be an infinite sequence of disjoint sets; we define finite union set of $(A_n)_{n\in\mathbb{N}}$

$$FU(A_n)_{n=1}^{\infty} := \{A_{n_1} \cup \ldots \cup A_{n_k} : n_1 < \cdots < n_k, k \in \mathbb{N}\}.$$

Theorem 9.2 (Hindman's Theorem on $\mathcal{P}(\mathbb{N})$)**.** *For $\mathcal{P}(\mathbb{N}) = \bigcup_{i=1}^{r} \widetilde{C_i}$, there exists i such that $\widetilde{C_i}$ contains an infinite sequence of disjoint subsets of $\mathbb{N}(A_n)_{n\in\mathbb{N}}$ together with its $FU(A_n)_{n=1}^{\infty}$.*

Proof. For $m \in \mathbb{N}$, m can be uniquely represented as $m = \sum_{i=1}^{\infty} m_i 2^{i-1} = m_1 + 2m_2 + 2^2 m_3 + \cdots + 2^{k-1} m^k$ with $m_i \in \{0, 1\}$. Given $\mathcal{P}(\mathbb{N}) = \bigcup_{i=1}^{r} \widetilde{C_i}$, define function $f \colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ as $f(m) = \{i \in \mathbb{N} \mid m_i = 1\}$.

Now we can easily check some properties of our map f.

Let $m, n, k \in \mathbb{N}$. Then

(1)$f(m) \cap [1, k] = \emptyset$ iff $2^k | m$

(2)$f(m) \cap [1, k] = f(n) \cap [1, k]$ iff $m \equiv n \mod 2^k$

(3)If $f(m) \cap f(n) = \emptyset$, then $f(m + n) = f(m) \cup f(n)$.

Thus, $\mathcal{P}(\mathbb{N}) = \bigcup_{i=1}^{r} \widetilde{C_i}$ induces a coloring on $\mathbb{N}$ by f, explicitly, $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ where $C_i = f^{-1}(\widetilde{C_i})$. Therefore, by Theorem 8.1, there exists i such that C_i contains finite sum set $FS(x_n)_{n=1}^{\infty}$ of an infinite increasing sequence $(x_n)_{n\in\mathbb{N}}$.

Claim: There exists an increasing sequence $(y_n)_{n\in\mathbb{N}}$ in finite sum set $FS(x_n)_{n=1}^{\infty}$ such that all $A_n = f(y_n)$ are disjoint.

Proof:

Let's construct a sequence $(y_n)_{n\in\mathbb{N}}$ of the form:

$$y_n = \sum_{k=i_n}^{j_n} x_k = x_{i_n} + x_{i_n+1} + \cdots + x_{j_n}, \text{for some} \quad i_n \leqslant j_n.$$

Let $A_n = f(y_n)$, and $r_n = \max A_n$. We define i_n, j_n, y_n, r_n recursively. For $n = 1$, take $i_1 = j_1 = 1$. Then $y_1 = x_1$. Now suppose for $n \in \mathbb{N}$, we have already defined $y_n = \sum_{k=i_n}^{j_n} x_k$. Then we define y_{n+1} as following. Consider $(z_l) = \sum_{k=1}^{l} x_{j_n+k}$, for $l \in \{1, \cdots, 2^{r_n} + 1\}$. Since $|\mathcal{P}([1, r_n])| = 2^{r_n} < 2^{r_n} + 1$, by Pigeonhole principle, there exist $l_1 > l_2$ such that $f(z_{l_1}) \cap [1, r_n] = f(z_{l_2}) \cap [1, r_n]$. So $z_{l_1} \equiv z_{l_2} \mod 2^{r_n}$.

Let $y_{n+1} = z_{l_1} - z_{l_2}$. Then $2^{r_n}|y_{n+1}$, so $A_{n+1} = f(y_{n+1}) \cap [1, r_n] = \emptyset$. Hence $\max A_n < \min A_{n+1}$. Now it is clear that all $A_n = f(y_n)$ are disjoint.

■

Moreover, observe that by definition of $(y_n)_{n\in\mathbb{N}}$, for any $y_a, y_b \in (y_n)_{n\in\mathbb{N}}$ with $y_a = x_{i_a} + \cdots + x_{j_a}$ and $y_b = x_{i_b} + \cdots + x_{j_b}$, either $j_a < i_b$ or $i_b < j_a$. In other words, y_a and y_b are sums of two disjoint subsequences of $(x_n)_{n\in\mathbb{N}}$. Therefore, $y_a + y_b \in FS(x_n)_{n=1}^{\infty}$ and $FS(y_n)_{n=1}^{\infty} \subset FS(x_n)_{n=1}^{\infty} \subset C_i$. So $f(FS(y_n)_{n=1}^{\infty}) \subset \widetilde{C_i}$. Since for any $y_a, y_b \in (y_n)_{n\in\mathbb{N}}$, $f(y_a) \cap f(y_b) = \emptyset$, $f(y_a + y_b) = f(y_a) \cup f(y_b)$. Therefore, for any $A_{n_1}, \ldots, A_{n_k} \in (A_n)$ with $f(y_{n_i}) = A_{n_i}$, $A_{n_1} \cup \ldots \cup A_{n_k} = f(y_{n_1}) \cup \ldots \cup f(y_{n_k}) = f(y_{n_1} + \cdots + y_{n_k}) \in \widetilde{C_i}$. Thus we have proved $FU(A_n)_{n=1}^{\infty}$ of an infinite sequence of disjoint subsets $(A_n)_{n\in\mathbb{N}}$ is contained in some $\widetilde{C_i}$.

□

With Theorem 9.2, we can prove a stronger version of Hindman's Theorem, from which Hindman's Theorem follows easily.

Theorem 9.3 (Partition Regularity of Finite Sum Set)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be an infinite increasing sequence in $\mathbb{N}$. Given $FS(x_n)_{n=1}^{\infty} = \dot\bigcup_{i=1}^{r} C_i$, one of C_i contains finite sum set $FS(y_n)_{n=1}^{\infty}$ of an increasing sequence $(y_n)_{n\in\mathbb{N}}$ in C_i.*

Proof. Define function $g\colon \mathcal{P}(\mathbb{N}) \to FS(x_n)$ with $A \mapsto \sum_{j\in A} x_j$. Note that by definition, if $A_1 \cap A_2 = \emptyset$, $g(A_1) + g(A_2) = g(A_1 \cup A_2)$.

Given $FS(x_n)_{n=1}^{\infty} = \dot\bigcup_{i=1}^{r} C_i$, g induces a coloring on $\mathcal{P}(\mathbb{N})$, explicitly, $\mathcal{P}(\mathbb{N}) = \dot\bigcup_{i=1}^{r} \widetilde{C_i}$ where $\widetilde{C_i} = g^{-1}(C_i)$. So there exists $\widetilde{C_i}$ that contains $FU(A_n)_{n=1}^{\infty}$ with $(A_n)_{n\in\mathbb{N}}$ as an infinite sequence of disjoint subsets of $\mathbb{N}$.

Let $B = \{g(A) \mid A \in (A_n)_{n\in\mathbb{N}}\}$. Then B is an infinite set because for all $b \in \mathbb{N}$, there only exist finitely many $A \subset \mathbb{N}$ such that $\sum_{j\in A} x_j = b$. Therefore we can choose an increasing sequence $(y_n)_{n\in\mathbb{N}}$ in B. For any $y_{n_i} \in (y_n)_{n\in\mathbb{N}}$, there exists some $A_{n_i} \in (A_n)_{n\in\mathbb{N}}$ such that $g(A_{n_i}) = y_{n_i}$. Therefore, for any $y_{n_1}, \ldots, y_{n_k} \in (y_n)$, $y_{n_1} + \cdots + y_{n_k} = g(A_{n_1}) + \cdots + g(A_{n_k})$. Since $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets, $g(A_{n_1}) + \cdots + g(A_{n_k}) = g(A_{n_1} \cup \ldots \cup A_{n_k})$. Because $A_{n_1} \cup \ldots \cup A_{n_k} \in \widetilde{C_i} = g^{-1}(C_i)$, $y_{n_1} + \cdots + y_{n_k} = g(A_{n_1} \cup \ldots \cup A_{n_k}) \in C_i$. Therefore, $FS(y_n)_{n=1}^{\infty} \subset C_i$.

□

10. Generalization of Hindman's Theorem

In this section, we'd like to consider further generalizations of Hindman's Theorem. First, we generalize the relation between idempotents and finite sum set we discovered in the proof of Hindman's Theorem. We have shown that if $p \in (\beta\mathbb{N}, +)$ is idempotent, then every p-large set contains a finite sum set $FS(n_j)_{j=1}^{\infty}$. However, this statement does not guarantee that $FS(n_j)_{j=1}^{\infty}$ itself is a p-large set. Nonetheless, the following lemma shows that for any finite sum set $FS(n_j)_{j=1}^{\infty}$, there exists an idempotent $q \in (\beta\mathbb{N}, +)$ such that $FS(n_j)_{j=1}^{\infty} \in q$.

Lemma 10.1. *Given any sequence $(n_j)_{j\in\mathbb{N}}$, there exists an idempotent $p \in (\beta\mathbb{N}, +)$ such that for any $m \in \mathbb{N}$, $FS(n_j)_{j=m}^{\infty} \in p$.*

Proof. Let $S = \bigcap_{m=1}^{\infty} [FS(n_j)_{j=m}^{\infty}]$. We will show that S is a compact left topological semigroup. Recall that for any $B \subseteq \mathbb{N}$, $[B]$ is clopen in $\beta\mathbb{N}$. Thus, S is an arbitrary intersection of decreasing sequence of closed sets. Since $\beta\mathbb{N}$ is Hausdorff,

S is compact and nonempty. We now show that S is a semigroup. For $p, q \in S$, to show $p+q \in S$ is to show for any $m \in \mathbb{N}$, $A = FS(n_j)_{j=m}^{\infty} \in p+q$, which is equivalent to showing $\{x \in \mathbb{N} \colon (A-x) \in p\} \in q$. Let $a \in A$. Then $a = n_{j_1} + \cdots + n_{j_l}$, where $m \leqslant j_1 < j_2 < \cdots < j_l$. Let $k = l+1$. Then $FS(n_j)_{j=k}^{\infty} \subseteq A - a$. However, $FS(n_j)_{j=k}^{\infty} \in p$ and it follows that $A - a \in p$. Therefore, $A \subseteq \{x \in \mathbb{N} \colon (A-x) \in p\}$ and since $A \in q$, $\{x \in \mathbb{N} \colon (A-x) \in p\} \in q$. Moreover, note that S inherits the same operation $+$ on $\beta\mathbb{N}$; thus S is a compact left topological semigroup. By Theorem 7.3, there exists idempotent $p \in S$. In other words, there exists p such that $\bigcap_{m=1}^{\infty} FS(n_j)_{j=m}^{\infty} \in p$ and it follows that for any $m \in \mathbb{N}$, $FS(n_j)_{j=m}^{\infty} \in p$. $\square$

Lemma 10.1 gives answer to another natural question to raise: which ultrafilters besides idempotents have all their members containing a finite sum set $FS(n_j)_{j=1}^{\infty}$?

Let Γ be the closure in $(\beta\mathbb{N}, +)$ of the set of idempotent ultrafilters:

$$\Gamma \colon = \overline{\{p \in (\beta\mathbb{N}, +) \colon p + p = p\}}.$$

Theorem 10.2. *An ultrafilter p belongs to Γ if and only if every p-large set contains some finite sum set $FS(n_j)_{j=1}^{\infty}$.*

Proof. Only if: For $p \in \Gamma$, let $A \in p$. Then $[A]$ is an open neighborhood of p in $\beta\mathbb{N}$, so there is $q \in \beta\mathbb{N}$ such that $q = q + q$ and $A \in q$. Then by the proof of Hindman's Theorem, A has to contain a finite sum set $FS(n_j)_{j=1}^{\infty}$.

If: Given $p \in \beta\mathbb{N}$, assume for all $A \in p$ there exists $FS(n_j)_{j=1}^{\infty} \subseteq A$. We need to show $p \in \Gamma$. Fix $A \in p$ and let E denote the finite sum set $FS(n_j)_{j=1}^{\infty}$ contained in A. By Lemma 10.1, there exists idempotent q such that $E \in q$. Therefore $q \in [E]$ and $q \in [A]$. Hence, we have shown that for any $A \in p$, $[A] \cap \{q \in (\beta\mathbb{N}, +) \colon q + q = q\} \neq \emptyset$, which means every neighborhood of p intersects the set of idempotents, i.e. $p \in \Gamma$. $\square$

Another generalization we'd like to consider is the multiplicative analog of Hindman's Theorem.

Definition 10.3. Let $(n_j)_{j \in \mathbb{N}}$ be an increasing sequence in $\mathbb{N}$. Define finite product set

$$FP(n_j)_{j=1}^{\infty} \colon = \{n_{j_1} \cdot n_{j_2} \cdot \ldots \cdot n_{j_k} \colon j_1 < j_2 < \ldots < j_k; k \in \mathbb{N}\}.$$

Theorem 10.4. *Given $\mathbb{N} = \dot{\bigcup}_{i=1}^{r} \widetilde{C_i}$, there exists i such that $\widetilde{C_i}$ contains an infinite sequence $(n_j)_{j \in \mathbb{N}}$ together with its $FP(n_j)_{j=1}^{\infty}$.*

Proof. Define r-coloring on $\mathbb{N}$ by letting $C_i = \{n \in \mathbb{N} \colon 2^n \in \widetilde{C_i}\}$. By Theorem 8.1, there exists C_i such that $FS(m_j)_{j=1}^{\infty} \subseteq C_i$. In other words, there exists $(m_j)_{j \in \mathbb{N}}$ such that for any $j_1 < j_2 < \cdots < j_k$ with $k \in \mathbb{N}$, $m_{j_1} + m_{j_2} + \cdots + m_{j_k} \in C_i$, which implies $2^{m_{j_1} + m_{j_2} + \cdots + m_{j_k}} = 2^{m_{j_1}} \cdot \ldots \cdot 2^{m_{j_k}} \in \widetilde{C_i}$. Hence we define $(n_j)_{j \in \mathbb{N}} = (2^{m_j})_{j \in \mathbb{N}}$ and we have $FP(n_j)_{j=1}^{\infty} \subseteq \widetilde{C_i}$. $\square$

Remark 10.5. Replicating the approach to Theorem 8.1 one can also prove Theorem 10.4. In such a proof, we need to define multiplication on $\beta\mathbb{N}$ and establish the existence of idempotents in $(\beta\mathbb{N}, \cdot)$.

Definition 10.6. We define multiplication on $\beta\mathbb{N}$ in the following way:
$$p \cdot q = \{A \subseteq \mathbb{N} \colon \{n \in \mathbb{N} \colon An^{-1}) \in p\} \in q\}$$
where $An^{-1} = \{m \in \mathbb{N} \mid mn \in A\}$.

We can show that multiplication defined above also renders $\beta\mathbb{N}$ a semigroup, and function $\phi \colon q \mapsto p \cdot q$ with $\phi_p(q) = p \cdot q$ is continuous. Therefore, $(\beta\mathbb{N}, \cdot)$ is a compact left topological semigroup.

With Definition 10.6, in a similar way to the proof of Hindman's Theorem, we are able to get the following analogous theorem.

Theorem 10.7. *For any multiplicative idempotent $p \in (\beta\mathbb{N}, \cdot)$, every p-large set contains a finite product set $FP(n_j)_{j=1}^{\infty}$.*

Now we try to combine the existence of finite sum set and finite product set into one theorem.

Theorem 10.8. *Given $\mathbb{N} = \dot{\bigcup}_{i=1}^{r} C_i$, there exist C_i and two increasing sequences $(m_j)_{j\in\mathbb{N}}$, $(n_j)_{j\in\mathbb{N}}$ such that $FS(m_j)_{j=1}^{\infty} \subseteq C_i$ and $FP(n_j)_{j=1}^{\infty} \subseteq C_i$.*

Proof. Claim: $\Gamma = \overline{\{p \in (\beta\mathbb{N}, +) \colon p + p = p\}}$ contains a multiplicative idempotent $q = q \cdot q$.
Proof: We first show that for any $p \in \Gamma$, $p \cdot \beta\mathbb{N} \subseteq \Gamma$. Let $p \in \Gamma$ and $q \in \beta\mathbb{N}$. For any $A \in p \cdot q$, by definition of operation $\cdot$, $\{n \in \mathbb{N} \colon An^{-1}) \in p\} \in q$. Take any $x \in \mathbb{N}$ such that $Ax^{-1} \in p$. Since $Ax^{-1} \in p \in \Gamma$, there exists $FS(m_j)_{j=1}^{\infty} \subseteq Ax^{-1}$. Therefore, $FS(x \cdot m_j)_{j=1}^{\infty} \subseteq A$ and it follows $p \cdot q \in \Gamma$.

Therefore, Γ is closed under multiplication. And since $(\Gamma, \cdot) \subseteq (\beta\mathbb{N}, \cdot)$, Γ is a left topological semigroup. Moreover, since Γ is defined as the closure of a nonempty set, Γ is closed and thus compact. Therefore, by Theorem 7.3, $(\Gamma, \cdot)$ has a multiplicative idempotent q.

■

Since $\mathbb{N} = \dot{\bigcup}_{i=1}^{r} C_i$, by Lemma 3.9, we have one of $C_i \in q$. Since $q \in \Gamma$, C_i contains a finite sum set $FS(m_j)_{j=1}^{\infty}$. On the other hand, since q is a multiplicative idempotent, by Theorem 10.7, C_i contains a finite product set $FP(n_j)_{j=1}^{\infty}$.

□

Acknowledgments. It is a pleasure to thank my mentor, Boming Jia, whose guidance and help reach far beyond the content of this paper but illuminate the path of my further pursuit of mathematical research.

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