Beilinson's 'office hours' on Bloch's lecture of 11.3.03.

The γ-filtration; an unorthodox approach.

We use some basic facts about Karoubi-Villamayor K-groups. For a ring R and $i \geq 1$ one has $K_n^{KV}(R) := \pi_{n-1}GL(R.) = \pi_n BGL(R.)$. Here $R.$ is the usual simplicial ring, $R_n := R[t_0, .., t_n]/\Sigma t_i = 1$, $GL := \cup GL_n$, so $GL(R.)$ is a simplicial group, and we consider the homotopy groups of its realization.

Since elementary matrices lie in the connected component of 1 in $GL(R.)$, we see that $K_1^{KV}(R)$ is commutative. Therefore the obvious embedding $BGL(R) \hookrightarrow BGL(R.)$ extends (canonically) to a morphism $BGL(R)^+ \to BGL(R.)$. Passing to the homotopy groups, we get a canonical morphism $K_n(R) \to K_n^{KV}(R)$.

From now on R is a commutative regular ring. Then $K_n(R) \xrightarrow{\sim} K_n^{KV}(R)$. The key fact needed for the proof is that the assumption on R implies that $K_n(R) \xrightarrow{\sim} K_n(R[t])$. (Drinfeld's question: is it true if R is not supposed to be commutative?)

Below $\Delta_R^n := \mathrm{Spec} R_n$, and $\Sigma_R^{n-1} \subset \Delta_R^n$ is the union of hyperplanes $t_i = 0$; let $I_n \subset R_n$ be its ideal. We also denote by Δ^n the standard simplex (which is a simplicial set), and $\Sigma^{n-1} \subset \Delta^n$ is its boundary (a simplicial sphere).

Proposition. For $n \geq 1$ one has $K_n(R) \xrightarrow{\sim} \mathrm{Coker}(K_1(R) \to K_1(\Sigma_R^{n-1}))$.

Proof. Use the fact that $GL(R.)$ (being a simplicial group) is a Kahn's simplicial set: $K_n(R) = \pi_{n-1}(R.) = \mathrm{Coker}(Maps(\Delta^n, GL(R.)), Maps(\Sigma^{n-1}, GL(R.))) = \mathrm{Coker}(GL(\Delta_R^n) \to GL(\Sigma_R^{n-1}))$. Recall that $K_1(R) = K_1(R_n) = GL(R_n)/E(R_n)$ and $K_1(\Sigma_R^{n-1}) = GL(\Sigma_R^{n-1})/E(\Sigma_R^{n-1})$ where E is the subgroup generated by the elementary matrices. Taking quotients mod E, we get a morphism $\mathrm{Coker}(GL(\Delta_R^n) \to GL(\Sigma_R^{n-1})) \to \mathrm{Coker}(K_1(R) \to K_1(\Sigma_R^{n-1}))$. This is an isomorphism since the map $E(R_n) \to E(\Sigma_R^{n-1})$ is surjective (this immediately follows from the surjectivity of the map $R_n \to \Sigma_R^{n-1}$). $\square$

Now let $Z_R^{n-1} \subset \Delta_R^n$ be the union of all closed subschemes that do not intersect Σ_R^{n-1}; equivalently, this is the union of hypersurfaces $(f = 0)$ where f is a function such that $f(\Sigma_R^{n-1}) = 1$. Our Z_R^{n-1} is an ind-scheme; we define the Chow groups $CH_a(Z^{n-1})$ as the inductive limit of $CH_a(Y)$ with respect to all subschemes $Y \subset Z^{n-1}$. Define $K_0'(Z^{n-1})$ in a similar way; it carries a filtration $K_0'(Z^{n-1}).$ by the dimension of support, and one has the usual identifications $\mathrm{gr}_a K_0'(Z^{n-1})_{\mathbb{Q}} \xrightarrow{\sim} CH_a(Z^{n-1})_{\mathbb{Q}}$.

Let $\tilde{R}_n$ be the localization of R_n with respect to all functions that equal 1 on Σ_R^{n-1}, $\tilde{\Delta}_R^n := \mathrm{Spec}\tilde{R}_n$. So Σ_R^{n-1} is a closed subscheme of $\tilde{\Delta}_n^R$; let $\tilde{I}_n \subset \tilde{R}_n$ be its ideal, so $1 + \tilde{I}_n \subset \tilde{R}_n^\times \subset K_1(\tilde{R}_n)$. We have a usual "boundary" map $\partial : K_1(\tilde{R}_n) \to K_0'(Z_R^{n-1})$, so for $f \in R_n$ which is invertible in $\tilde{R}_n$, i.e., on Σ_R^{n-1}, one has $\partial(f) = [R_n/fR_n] \in K_0'(Z_R^{n-1})$.

Theorem. One has $K_n(R) = K_0'(Z_R^{n-1})/\partial(1 + \tilde{I}_n)$.

Proof. (a) We have $K_n(R) = \mathrm{Coker}(K_1(R_n) \to K_1(\Sigma_R^{n-1})) \xrightarrow{\sim} K_0(\Delta_R^n, \Sigma_R^{n-1})$. The latter arrow is the boundary map in the exact sequence for relative K-groups; it is an isomorphism since the restriction map $i : K_0(R_n) \to K_0(\Sigma_R^{n-1})$ is injective (any R-point x on the smooth part of Σ_R^{n-1} yields the pull-back map $K_0(\Sigma_R^{n-1}) \to K_0(R) \xrightarrow{\sim} K_0(R_n)$, which is a left inverse to i).

(b) Consider the exact category of finitely generated R_n-modules P such that $\tilde{R}_n \otimes P$ is a projective $\tilde{R}_n$-module. It contains the exact category of R_n-modules supported on Z_R^{n-1}, the quotient category equals the category of finitely generated projective $\tilde{R}_n$-modules. We get the corresponding exact sequence of K-groups. We also have the exact sequence of relative K-groups with respect to the closed subscheme Σ_R^{n-1} which looks as $K_1(\tilde{\Delta}_R^n, \Sigma_R^{n-1}) \to K_0'(Z_R^{n-1}) \to K_0(\Delta_R^n, \Sigma_R^{n-1}) \to K_0(\tilde{\Delta}_R^n, \Sigma_R^{n-1})$. Since $\tilde{I}_n$ lies in the Jakobson ideal of $\tilde{R}$, the last K-group vanishes, and the first equals $1+\tilde{I}_n$, hence $K_0(\Delta_R^n, \Sigma_R^{n-1}) = \mathrm{Coker}(\partial : 1+\tilde{I}_n \to K_0'(Z_R^{n-1}))$, and we are done by (a). $\square$

Spencer also claims that the γ-filtration on $K_n(R)$ coincides with the decreasing filtration coming from the filtration by the codimension of support on $K_0'(Z^{n-1})$. I will consider the increasing filtration by the dimension of support instead denoting it by $K_0'(Z^{n-1})_a$.

Suppose $n \geq 2$ and $R = k$ is a field. Then Σ_k^{n-1} is connected, and $1+\tilde{I}_n$ is the group of all codimension 1 cycles on Δ_k^n that do not intersect Σ_k^{n-1}. So the composition $1+\tilde{I}_n \to K_0'(Z^{n-1}) \to K_0'(Z^{n-1})/K_0'(Z^{n-1})_{n-2}$ is an isomorphism, and we can rewrite the theorem as an identification

$$K_n(k) \xrightarrow{\sim} K_0'(Z^{n-1})_{n-2}. \tag{1}$$

Passing to gr, and using the motivic cohomology notation, we get for $a \leq n-2$ a canonical identification

$$H_{\mathcal{M}}^{n-2a}(k, \mathbb{Q}(n-a)) \xrightarrow{\sim} CH_a(Z^{n-1})_{\mathbb{Q}}. \tag{2}$$

For $a > n-2$ the l.h.s. in (2) vanishes.

Therefore the vanishing conjecture about motivic cohomology groups amounts to the vanishing of the Chow groups $CH_a(Z^{n-1})_{\mathbb{Q}}$ for $a \in [n-2, n/2]$. (Notice that the non-emptiness of the latter interval means that $n \geq 4$). Can one prove this?