INFINITE-DIMENSIONAL VECTOR BUNDLES IN ALGEBRAIC GEOMETRY

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To Izrail Moiseevich Gelfand on his 90th birthday

Contents

Partially supported by NSF grant DMS-0100108.

I have to write the introduction! Details will appear elsewhere...

Let us agree that "S-family of vector spaces" means the same as "vector bundle on S".

1. Families of discrete infinite-dimensional vector spaces

Is there a reasonable notion of not necessarily finite-dimensional vector bundle on a scheme? We know due to Serre [S] that a finite-dimensional vector bundle on an affine scheme $\operatorname{Spec} R$ is the same as a finitely generated projective R-module. So it is natural to give the following definition.

Definition. A vector bundle on a scheme X is a quasicoherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that for every open affine subset $\operatorname{Spec} R \subset X$ the R-module $H^0(\operatorname{Spec} R, \mathcal{F})$ is projective.

Key question: is this a local notion? More precisely, the question is as follows. Let $\operatorname{Spec} R = \bigcup_i U_i$, $U_i = \operatorname{Spec} R_i$. Let M be a (not necessarily finitely generated) R-module such that $M \otimes_R R_i$ is projective for all i. Does it follow that M is projective?

The question is difficult: the arguments used in the case that M is finitely generated fail for modules of infinite type. Nevertheless Grothendieck [Gr2] conjectured that the answer is positive. This was proved by Raynaud and Gruson [RG] (in Ch. 1 for countably generated modules and in Ch. 2 for arbitrary ones). Moreover, they proved that projectivity is a local property for the fpqc topology, not only for Zariski (in other words, if R' is flat over R and the morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ is surjective then projectivity of $M \otimes_R R'$ implies projectivity of M). In fact, they derived it as an easy corollary of the following remarkable and nontrivial theorem due to Kaplansky [Ka] and Raynaud-Gruson [RG], which explains what projectivity really is. In this theorem the ring R is not assumed to be commutative.

Theorem 1.1. *An R-module M is projective if and only if the following properties hold:*
(a) M is flat;
(b) M is a direct sum of countably generated modules;
(c) M is a Mittag–Leffler module.

The fact that a projective module can be represented as a direct sum of countably generated ones was proved by Kaplansky [Ka]. The remaining part of Theorem 1.1 is due to Raynaud and Gruson [RG]. The key notion of Mittag–Leffler module was introduced in [RG]. In §5 we recall the definition and basic properties of Mittag–Leffler modules. Here I prefer only to explain what a flat Mittag–Leffler module is. By the Govorov–Lazard lemma, a flat R-module M can be represented as the inductive limit of a directed family of finitely generated projective modules P_i. According to [RG], in this situation M is Mittag–Leffler if and only if the projective system formed by the dual (right) R-modules $P_i^* := \operatorname{Hom}_R(P_i, R)$ satisfies the Mittag–Leffler condition: for every i there exists $j \geq i$ such that $\operatorname{Im}(P_j^* \to P_i^*) = \operatorname{Im}(P_k^* \to P_i^*)$ for all $k \geq j$.

Remarks. (i) One gets a sightly different definition of not necessarily finite-dimensional vector bundle on a scheme if one replaces projectivity by the property of being a flat Mittag–Leffler module. The product of infinitely many copies of $\mathbb{Z}$ is an example of a flat Mittag–Leffler $\mathbb{Z}$-module which is not a projective $\mathbb{Z}$-module (see 5.6(?,?) for more details). Informally the property of being a flat Mittag–Leffler module can be regarded as "projectivity with human face". **Should I ask Hirschfeldt to help me say something more precise??** E.g., one does not need the axiom of choice to prove that a vector space over a field is a flat Mittag–Leffler module, but it is not clear how to prove without this axiom that $\mathbb{R}$ is a direct summand of a free

$\mathbb{Q}$-module (by the way, without the axiom of choice it is not true that if F is a direct summand of a free module then F is projective, i.e., every epimorphism $M \to F$ has a section).

(ii) Instead of property (c) from Theorem 1.1 the authors of [RG] used a slightly different one, which is harder to formulate. Probably their property has some technical advantages.

(iii) Here are some more comments regarding the work [RG]. First, there is no evidence that the authors of [RG] knew that the local nature of projectivity had been conjectured by Grothendieck. Second, their notion of Mittag–Leffler module and their results on infinitely generated projective modules were probably largely forgotten (even though they deserve being mentioned in algebra textbooks). Probably they were "lost" among many other powerful and important results of [RG] (mostly in the spirit of EGA IV).

2. Families of Tate vector spaces, almost projective modules, and the K_{-1}-functor

2.1. **A class of topological vector spaces.** We consider topological vector spaces over a discrete field k.

Definition. A topological vector space is *linearly compact* if it is the topological dual of a discrete vector space.

Example: $k[[t]] \simeq k \times k \times \ldots = (k \oplus k \oplus \ldots)^*$.

A topological vector space V is linearly compact if and only if it has the following 3 properties:
1) V is complete and Hausdorff,
2) V has a base of neighborhoods of 0 consisting of vector subspaces,
3) each open subspace of V has finite codimension.

Definition. A *Tate space* is a topological vector space isomorphic to $P \oplus Q^*$, where P and Q are discrete.

A topological vector space T is a Tate space if and only if it has an open linearly compact subspace.

Example: $k((t))$ equipped with its usual topology (the subspaces $t^n k[[t]]$ form a base of neighborhoods of 0). This is a Tate space because it is a direct sum of the linearly compact sapce $k[[t]]$ and the discrete space $t^{-1}k[t^{-1}]$, or because $k[[t]] \subset k((t))$ is an open linearly compact subspace.

Tate spaces play an important role in the algebraic geometry of curves (e.g., the ring of adeles corresponding to an algebraic curve is a Tate space) and also in the theory of ∞-dimensional Lie algebras and Conformal Field Theory. In fact, they were introduced by Lefschetz ([L], p.78–79) under the name of locally linearly compact spaces. The name "Tate space" was introduced by Beilinson because these spaces are implicit in Tate's remarkable work [T]. In fact, the approach to residues on curves developed in [T] can be most naturally interpreted in terms of the canonical central extension of the endomorphism algebra of a Tate space, which is also implicit in [T].

2.2. **What is a family of Tate spaces?** Probably this question has not been considered. We suggest the following answer. We introduce the notion of *Tate module* over a (not necessarily commutative) ring R. If R is commutative then we suggest to consider Tate R-modules as "families of Tate spaces".

Definition. An *elementary Tate R-module* is a topological R-module isomorphic to $P \oplus Q^*$, where P, Q are discrete projective R-modules (P is a left module, Q is a right one). A *Tate R-module* is a direct summand of an elementary Tate R-module. A Tate R-module M is *quasi-elementary* if $M \oplus R^n$ is elementary for some $n \in \mathbb{N}$.

Remark. It is easy to see that if every projective R-module is a direct sum of finitely generated ones then every quasi-elementary Tate R-module is elementary. E.g., according to the following theorem due to Bass (Corollary 4.5 from [Ba2]) this is the case if R is a commutative Noetherian ring.

Theorem 2.1. *If R is a commutative Noetherian ring whose spectrum is connected then every nonfinitely generated projective R-module is free.*

Examples. 1) $R((t))^n$ is an elementary Tate R-module.

2) A finitely generated projective $R((t))$-module is a Tate R-module. In general, it is not quasi-elementary. E.g., let k be a field, $R := \{f \in k[x] | f(0) = f(1)\}$ and

$$M := \{u = u(x,t) \in k[x]((t)) \mid u(1,t) = tu(0,t)\}. \tag{2.1}$$

Then M is a finitely generated projective $R((t))$-module which is not quasi-elementary as a Tate R-module (see 2.6).

Theorem 2.2. *The notion of Tate module over a commutative ring R is local for the flat topology, i.e., for every faithfully flat commutative R-algebra R' the category of Tate R-modules is canonically equivalent to that of Tate R'-modules equipped with a descent datum.*

The proof is based on the Raynaud–Gruson technique. See 2.11 for more details.

Remark. For Theorem 2.2 to be true it is essential that we consider not only elementary Tate modules (see Theorem 2.3).

2.3. **The class of a Tate R-module in $K_{-1}(R)$.** How to prove that a Tate R-module is not quasi-elementary? Let $\mathcal{T}_R$ denote the additive category of Tate R-modules. It is easy to see that a quasi-elementary Tate R-module has zero class in $K_0(\mathcal{T}_R)$ (one can show that $K_0(\mathcal{T}_R)$ makes sense even though $\mathcal{T}_R$ is not equivalent to a small category). The converse is also true. One also shows that $K_0(\mathcal{T}_R)$ is canonically isomorphic to $K_{-1}(R)$ and that every element of $K_0(\mathcal{T}_R) = K_{-1}(R)$ can be represented as a class of a finitely generated projective $R((t))$-module.

The definition of the morphism $K_0(\mathcal{T}_R) \to K_{-1}(R)$ is easy if one uses the following definition of $K_{-1}(R)$, which is slightly nonstandard but equivalent to the standard ones (C. Weibel, private communication).

Definition. $K_{-1}(R) := K_0(C^{\mathrm{Kar}})$, where C^{Kar} is the Karoubi envelope[1] of the category C whose objects are projective R-modules and whose morphisms are defined by $\mathrm{Hom}_C(P,P') := \mathrm{Hom}(P,P')/\mathrm{Hom}_f(M,M')$. Here $\mathrm{Hom}_f(P,P') \subset \mathrm{Hom}(P,P')$ is the group of finite rank operators (an R-linear operator $P \to P'$ is said to have finite rank if it can be decomposed as $P \to R^n \to P'$, $n \in \mathbb{N}$).

Remark. One easily shows that $K_0(C)$ makes sense (even though C is not equivalent to a small category) and that the morphism $K_0(C^{\aleph_0}) \to K_0(C)$ is an isomorphism. Here $C^{\aleph_0} \subset C$ is the full subcategory of countably generated projective R-modules. The category $C^{\aleph_0}$ is equivalent to that of finitely generated projective modules over the "algebraic Calkin ring" $\mathrm{Calk}(R) := \mathrm{End}_C R^{(\infty)}$, $R^{(\infty)} := R \oplus R \oplus \ldots$; the equivalence is defined by $P \mapsto \mathrm{Hom}_C(P, R^{(\infty)})$, $P \in C^{\aleph_0}$. The ring $\mathrm{Calk}(R)$ is an algebraic version of the analysts' Calkin algebra, which is defined to be the quotient of the ring of continuous endomorphisms of a Banach space by the ideal of compact operators.

Now the morphism $K_0(\mathcal{T}_R) \to K_{-1}(R) = K_0(C^{\mathrm{Kar}})$ is defined to be the one induced by the following functor $\Phi : \mathcal{T}_R \to C^{\mathrm{Kar}}$. Let $E_R \subset \mathcal{T}_R$ be the full subcategory of elementary Tate modules. One gets a functor $\Psi : E_R \to C$ by setting $\Psi(P \oplus Q^*) := P$ (here P, Q are discrete projective modules) and defining $\Psi(f) \in \mathrm{Hom}_C(P, P_1)$, $f : P \oplus Q^* \to P_1 \oplus Q_1^*$, to be the image of the composition $P \hookrightarrow P \oplus Q^* \xrightarrow{f} P \oplus Q^* \twoheadrightarrow Q^*$ in $\mathrm{Hom}_C(P, P_1)$ (one easily checks that $\Psi(f'f) = \Psi(f')\Psi(f)$). The functor Ψ extends to $\Psi^{\mathrm{Kar}} : \mathcal{T}_R = E_R^{\mathrm{Kar}} \to C^{\mathrm{Kar}}$ and therefore

[1] Recall that the Karoubi envelope of a category $\mathcal{K}$ is the category one gets by formally adding the images of idempotent endomorphisms.

induces a morphism $K_0(\mathcal{T}_R) \to K_0(C^{\mathrm{Kar}}) = K_{-1}(R)$. One can show that this morphism is an isomorphism.

2.4. **Comparison between the definitions of K_{-1}.** Should I rename this subsection into "Various definitions of K_{-1}"??

Does the Karoubi-Villamayor functor C commute with filtering inductive limits??

Recall that our $K_{-1}(R)$ is equal (more or less by definition) to $K_0(\mathrm{Calk}(R))$, $\mathrm{Calk}(R) := \operatorname{End} R^{(\infty)} / \operatorname{End}_f R^{(\infty)}$, where $\operatorname{End}_f R^{(\infty)} \subset \operatorname{End} R^{(\infty)}$ is the ideal of finite rank operators.

On the other hand, J. B. Wagoner [Wa] defined the negative K-theory of R by $K_0^W := K_0$ and $K_{i-1}^W(R) := K_i^W(lR/mR)$, where $lR \subset \operatorname{End} R^{(\infty)}$ is the subring of matrices with finitely many non-zero entries in each row (not only in each column) and $mR := lR \cap \operatorname{End}_f R^{(\infty)}$. A similar definition had been introduced by M. Karoubi and O. Villamayor [KV1, KV2]: $K_0^{KV} := K_0$ and $K_{i-1}^{KV}(R) := K_i^{KV}(CR/mR)$, where CR is a certain subring of lR. (This definition was developed by S. M. Gersten [Ge] at the level of spectra.) It is known that the morphisms $K_i^{KV} \to K_i^W$ are isomorphisms for all $i \leq 0$. C. Weibel informed me (private communication) that the morphism $K_{-1}^W(R) \to K_{-1}(R) = K_0(\mathrm{Calk}(R))$ is an isomorphism.

What about [P, PW]??

H. Bass [Ba] defined the negative K-theory of R by

$$K_{i-1}(R) := \operatorname{Coker}(K_i(R[t]) \oplus K_i(R[t^{-1}]) \to K_i(R[t,t^{-1}])), \quad i \leq 0. \tag{2.2}$$

Let us denote by K_{-1}^B the K_{-1}-functor defined by (2.2). As before, we put $K_{-1}(R) := K_0(C^{\mathrm{Kar}})$. The isomorphism between $K_{-1}^B \xrightarrow{\sim} K_{-1}$ is defined as follows. We have the morphism

$$K_0(R[t,t^{-1}]) \to K_0(\mathcal{T}_R) = K_0(C^{\mathrm{Kar}}) = K_{-1}(R) \tag{2.3}$$

that sends the class of a finitely generated projective $R[t,t^{-1}]$-module P to the class of the Tate R-module $R((t)) \otimes_{R[t,t^{-1}]} P$. This class is opposite to that of the Tate R-module $R((t^{-1})) \otimes_{R[t,t^{-1}]} P$ (to see this notice that P embeds into $R((t)) \otimes_{R[t,t^{-1}]} P \oplus R((t^{-1})) \otimes_{R[t,t^{-1}]} P$ as a discrete submodule and the quotient is dual to a discrete projective R-module). This implies that the images of both $K_0(R[t])$ and $K_0(R[t^{-1}])$ in $K_0(\mathcal{T}_R)$ are zero, so we get a morphism $K_{-1}^B(R) \to K_{-1}(R)$. The fact that it is an isomorphism was essentially proved[2] in [**?**] (reference??). This implies that (2.3) is surjective.

Remarks. (i) It follows from (2.2) that all the K_i-functors, $i \leq 0$, commute with filtering inductive limits. On the other hand, proving this for the K_{-1}-functor defined by $K_{-1}(R) := K_0(C^{\mathrm{Kar}})$ requires some efforts.

(ii) Bass proved in [Ba] that if R is left regular (i.e., it is left noetherian and every finitely generated R-module has a finite projective resolution) or right regular then $K_i(R) = 0$ for all $i < 0$. This follows from (2.2) because according to [Ba], ch. XII, §3 the morphisms $K_0(R) \to K_0(R[t]) \to K_0(R[t,t^{-1}])$ are isomorphisms if R is left or right regular. One can also prove the vanishing of K_{-1} for regular rings using the definition $K_{-1}(R) := K_0(C^{\mathrm{Kar}})$.

2.5. **The dimension torsor.** Let R be commutative. Then it follows from Theorem 8.5 of [We2] that there is a canonical epimorphism $K_{-1}(R) \to H^1_{\mathrm{et}}(\operatorname{Spec} R, \mathbb{Z})$, so a Tate R-module M should define $\alpha_M \in H^1_{\mathrm{et}}(\operatorname{Spec} R, \mathbb{Z})$. We will define α_M explicitly as a class of a certain $\mathbb{Z}$-torsor Dim_M on $\operatorname{Spec} R$ canonically associated to M. Dim_M is called "the torsor of dimension theories" or "dimension torsor".

Let us recall the well known definition of Dim_M in the case that R is a field (so M is a Tate space). Notice that if $L \subset M$ is open and linearly compact then usually $\dim L = \infty$ and $\dim(M/L) = \infty$. But for any open linearly compact L, $L' \subset M$ one has the *relative dimension* $d_L^{L'} := \dim(L'/L' \cap L) - \dim(L/L' \cap L) \in \mathbb{Z}$.

[2] Why "essentially"??

Definition. A *dimension theory* on a Tate space M is a function

$$d : \{\text{open linearly compact subspaces } L \subset M\} \to \mathbb{Z}$$

such that $d(L') - d(L) = d_L^{L'}$.

A dimension theory exists and is unique up to adding $n \in \mathbb{Z}$. So dimension theories on a Tate space form a $\mathbb{Z}$-torsor. This is Dim_M.

Example. Let T be a $\mathbb{Z}$-torsor, let $k^{(T)}$ be the vector space over a field k freely generated by T. Then $\mathbb{Z}$ acts on $k^{(T)}$, so $k^{(T)}$ becomes a $k[z, z^{-1}]$-module (multiplication by z coincides with the action of $1 \in \mathbb{Z}$). Put $M := k((z)) \otimes_{k[z,z^{-1}]} k^{(T)}$. Then one has a canonical isomorphism

$$\mathrm{Dim}_M \xrightarrow{\sim} T : \tag{2.4}$$

to $t \in T$ one assoicates the dimension theory d_t such that $d_t(L_t) = 0$, where $L_t \subset M$ is the $k[[z]]$-submodule generated by t.

Definitions. Let M be a Tate module over a (not necessarily commutative) ring R. A submodule $L \subset M$ is a *lattice* if it is open and L/U is finitely generated for every open submodule $U \subset L$. A lattice $L \subset M$ is *coflat* if M/L is projective.

Remarks. (i) One can show that a lattice $L \subset M$ is coflat if and only if M/L is flat.

(ii) It is easy to see that every Tate R-module M has a lattice. On the other hand, M has a coflat lattice if and only if M is elementary.

If M is a Tate module and $L \subset L' \subset M$ are coflat lattices then L'/L is a finitely generated projective R-module, so if R is commutative then $d_L^{L'} := \mathrm{rank}(L'/L) \in H^0(\mathrm{Spec}\, R, \mathbb{Z})$ is well-defined.

Definition. Let M be a Tate module over a commutative ring R. A *dimension theory* on M is a rule that associates to each R-algebra R' and each coflat lattice $L \subset R'\hat{\otimes}_R M$ a locally constant function $d_L : \mathrm{Spec}\, R' \to \mathbb{Z}$ in a way compatible with base change and so that $d_{L_2} - d_{L_1} = \mathrm{rank}(L_2/L_1)$ for any pair of coflat lattices $L_1 \subset L_2 \subset R'\hat{\otimes}_R M$. Here $R'\hat{\otimes}_R M$ denotes the completed tensor product.

One defines Dim_M to be the $\mathbb{Z}$-torsor of dimension theories. To show that this is indeed a $\mathbb{Z}$-torsor, one uses the following fact. The following theorem implies that this is indeed a $\mathbb{Z}$-torsor and that if the functions d_L with the above properties are defined for all etale R-algebras then there exists a unique way to extend the definition to all R-algebras.

Theorem 2.3. *Let R be a commutative ring. Then every Tate R-module M is Nisnevich-locally elementary; in other words, there exists a Nisnevich covering* $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ *such that $R'\hat{\otimes}_R M$ has a coflat lattice L'. Moreover, for every lattice $L \subset M$ one can choose R' and L' so that $L' \supset R'\hat{\otimes}_R L$.*

Recall that a morphism $\pi : \mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ is said to be a Nisnevich covering if it is etale and there exist closed subschemes $\mathrm{Spec}\, R = F_0 \supset F_1 \supset \ldots \supset F_n = \emptyset$ such that each F_i is defined by finitely many equations and π admits a section over $F_{i-1} \setminus F_i$, $i = 1, \ldots, n$. (If R is Noetherian then an etale morphism $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ is a Nisnevich covering if and only if it admits a section over each point of $\mathrm{Spec}\, R$). So the Nisnevich topology is weaker than etale but stronger than Zariski. The following table may be helpful:

Topology	Stalks of $\mathcal{O}_X$, $X = \mathrm{Spec} R$
Zariski	Localizations of R
Nisnevich	Henselizations of R
Etale	Strict henselizations of R

Now let us return to dimension torsors. One has a canonical isomorphism

$$\mathrm{Dim}_{M_1 \oplus M_2} \xrightarrow{\sim} \mathrm{Dim}_{M_1} + \mathrm{Dim}_{M_2}\,. \tag{2.5}$$

So one gets a morphism $K_0(\mathcal{T}_R) = K_{-1}(R) \to H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z})$. It is surjective. Indeed, let T be a $\mathbb{Z}$-torsor on $S := \operatorname{Spec} R$. Then the free $\mathcal{O}_S$-module $\mathcal{O}_S^{(T)}$ generated by the sheaf of sets T is equipped with an action of $\mathbb{Z}$, so it is a module over $\mathcal{O}_S[z, z^{-1}]$ (multiplication by z coincides with the action of $1 \in \mathbb{Z}$). This module is locally free of rank one, so its global sections form a projective $R[z, z^{-1}]$-module $R^{(T)}$ of rank 1. Therefore $R((z)) \otimes_{R[z,z^{-1}]} R^{(T)}$ is a Tate R-module. Its dimension torsor is canonically isomorphic to T (cf. (2.4)).

2.6. **Example.** Let M be the Tate module (2.1) over $R := \{f \in k[x] | f(0) = f(1)\}$. Then the $\mathbb{Z}$-torsor Dim_M is nontrivial (its pullback to $S := \operatorname{Spec}(R \otimes_k \bar{k})$ corresponds to the universal covering of S). So the class of M in $K_0(\mathcal{T}_R) = K_{-1}(R)$ is nontrivial and therefore M is not quasi-elementary. This example also shows that *in Theorem 2.3 one cannot replace "Nisnevich" by "Zariski"*.

2.7. **Remark.** The kernel of the morphism $K_{-1}(R) \to H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z})$ may be nonzero, even if R is local. Examples can be found in [We3]. More precisely, §6 of [We3] contains examples of algebras R over a field k such that $H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z}) = 0$ but $K_{-1}(R) \neq 0$. In each of these examples $\operatorname{Spec} R$ is a normal surface with one singular point x. Let R_x denote the local ring of x. According to [We1], the map $K_{-1}(R) \to K_{-1}(R_x)$ is an isomorphism, so $K_{-1}(R_x) \neq 0$.

2.8. **The dimension torsor of a projective $R((t))$-module.** Let R be a commutative ring. Let M be a finitely generated projective $R((t))$-module equipped with an isomorphism $\varphi : \det M \xrightarrow{\sim} R((t))$. If R is a field then M has an $R[[t]]$-stable lattice; moreover, there is a lattice $L \subset M$ such that

$$R[[t]]L \subset L, \quad \varphi(\det L) = R[[t]]. \tag{2.6}$$

So it is easy to see that if R is a field then there is a unique dimension theory d_φ on M such that $d_\varphi(L) = 0$ for all lattices $L \subset M$ satisfying (2.6). Therefore if R is any commutative ring then the $\mathbb{Z}$-torsor Dim_M is trivialized over each point of $\operatorname{Spec} R$.

Proposition 2.4. *These trivializations come from a (unique) trivialization d_φ of the $\mathbb{Z}$-torsor* Dim_M.

Remarks. (i) By Proposition 2.4 the morphism $K_0(R((t))) \to H^1(\operatorname{Spec} R, \mathbb{Z})$ which sends th class of a projective $R((t))$-module M to the class of Dim_M annihilates the kernel of the epimorphism $\det : K_0(R((t))) \twoheadrightarrow \operatorname{Pic} R((t))$, so we get a morphism $\operatorname{Pic} R((t)) \to H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z})$ such that the diagram

$$\begin{array}{ccc} K_0(R((t))) & \xrightarrow{\det} & \operatorname{Pic} R((t)) \\ \downarrow & \swarrow & \\ H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z}) & & \end{array} \tag{2.7}$$

commutes. The composition $\operatorname{Pic} R[t, t^{-1}]) \to \operatorname{Pic} R((t)) \to H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z})$ was studied in [We2].

(ii) The interested reader can easily lift the diagram (2.7) of abelian groups to a commutative diagram of appropriate Picard groupoids (in the sense of 2.12.1).

2.9. **The determinant gerbe.** Given a Tate space M over a field Kapranov [Ka3] defines its *the groupoid of determinant theories*. The definition is based on the notion of relative determinant of two lattices in a Tate space and goes back to J.-L. Brylinski [Br] (and further back to the Japanese school and [ACK]). If M is a Tate module over a commutative ring R then rephrasing the definition from [Ka3] in the obvious way one gets a sheaf of groupoids on the Nisnevich topology of $S := \operatorname{Spec} R$ (details will be explained in 2.12). This sheaf of groupoids is, in fact, an $\mathcal{O}_S^*$-gerbe. We call it *the determinant gerbe of M*. Associating the class of this gerbe to a Tate R-module M one gets a morphism

$$K_0(\mathcal{T}_R) = K_{-1}(R) \to H^2_{\text{Nis}}(S, \mathcal{O}_S^*). \tag{2.8}$$

Probably it is well known to K-theorists. One can get the restriction of (2.8) to $\mathrm{Ker}(K_{-1}(R) \to H^1_{\mathrm{et}}(\mathrm{Spec}\, R, \mathbb{Z}))$ (and possibly the morphism (2.8) itself) from the Brown–Gersten–Thomason spectral sequence ([TT], §10.8).

2.10. **Almost projective modules.** Recall that every Tate R-module has a lattice but not necessarily a coflat one. If M is a Tate R-module and $L \subset M$ is a lattice then M/L is almost projective in the following sense.

Definition. An *elementary almost projective R-module* is a module isomorphic to a direct sum of a projective R-module and a module of finite presentation. An *almost projective R-module* is a direct summand of an elementary almost projective module. An almost projective R-module M is *quasi-elementary* if $M \oplus R^n$ is an elementary almost projective R-module for some $n \in \mathbb{N}$.

Remark. It is easy to see that an R-module M is a quasi-elementary almost projective module if and only if it is isomorphic to P/N with P projective and $N \subset P$ finitely generated.

Theorem 2.5. *(i) The notion of almost projective module over a commutative ring R is local for the flat topology, i.e., for every faithfully flat commutative R-algebra R' the category of almost projective R-modules is canonically equivalent to that of almost projective R'-modules equipped with a descent datum.*

(ii) For every almost projective module M over a commutative ring R there exists a Nisnevich covering $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ *such that $R' \otimes_R M$ is elementary.*

Remarks. 1) A quasi-elementary almost projective module M over a commutative ring R becomes elementary already *Zariski*-locally. This can be easily deduced from the following theorem due to Kaplansky [Ka]: a projective module over a local ring is free (even if it is not finitely generated).

2) My impression is that statement (ii) is more important than (i) even though it is much easier to prove. Statement (i) gives you a peace of mind (without it one would have two candidates for the notion of almost projectivity), but in the examples of almost projective modules that I know one can prove almost projectivity directly rather than showing that the property holds locally.

A submodule L of an almost projective R-module M is said to be a *coflat lattice* if L is finitely generated and M/L is projective. It is easy to show that in this situation L has finite presentation, so coflat lattices exist if and only if M is elementary.

Now let R be commutative. We define a *dimension theory* (resp. *upper semicontinuous dimension theory*) on an almost projective R-module M to be a rule that associates to each R-algebra R' and each coflat lattice $L \subset R' \otimes_R M$ a locally constant (resp. an upper semicontinuous) function $d_L : \mathrm{Spec}\, R' \to \mathbb{Z}$ in a way compatible with base change and so that $d_{L_2} - d_{L_1} = \mathrm{rank}(L_2/L_1)$ for any pair of coflat lattices $L_1 \subset L_2 \subset R' \otimes_R M$. The notion of dimension theory (or upper semicontinuous dimension theory) does not change if one considers only etale R-algebras instead of arbitrary ones. Dimension theories on an almost projective R-module M form a $\mathbb{Z}$-torsor for the Nisnevich topology of $\mathrm{Spec}\, R$, which is denoted by Dim_M. One defines the *canonical upper semicontinuous dimension theory* d^{can} on M by $d^{\mathrm{can}}_L(x) := \dim_{K_x}(K_x \otimes_{R'} L)$, where R' is an R-algebra, $L \subset R' \otimes_R M$ is a coflat lattice, $x \in \mathrm{Spec}\, R'$, and K_x is the residue field of x. An upper semicontinuous dimension theory on M is the same as an *upper semicontinuous section* of Dim_M, by which we mean a $\mathbb{Z}$-antiequivariant morphism from the $\mathbb{Z}$-torsor Dim_M to the sheaf of upper semicontinuous $\mathbb{Z}$-valued functions on $\mathrm{Spec}\, R$. Clearly d^{can} is a true (i.e., locally constant) section of Dim_M if and only if the quotient of M modulo the nilradical $I \subset M$ is projective over R/I. In this case d^{can} defines a trivialization of Dim_M.

If N is a Tate R-module and $L \subset N$ is a lattice then the dimension torsor of the almost projective module N/L canonically identifies with that of N.

2.11. **On the proofs.** Theorems 2.2 and 2.3 are reduced to Theorem 2.5. To prove Theorem 2.5(i) we introduce the following definition: an R-module M is *almost flat* if there is a morphism $N \to M$ such that N is an R-module of finite presentation and the corresponding morphism $\mathrm{Tor}_1(L,N) \to \mathrm{Tor}_1(L,M)$ is surjective for every right R-module L. Then we show that in Theorem 1.1 one can replace "projective" and "flat" by "almost projective" and "almost flat". Theorem 2.5(ii) is easy and more or less equivalent to the following one.

Theorem 2.6. *Every element of $K_{-1}(R)$ vanishes Nisnevich-locally.*

Remark. According to Example 8.5 of [We2] (which goes back to L. Reid's work [Re]), this is not true for K_i, $i < -1$.

Proof. According to our definition of K_{-1}, it suffices to prove that if P is a projective R-module and $\pi \in \operatorname{End} P$ is such that $\operatorname{rank}(\pi^2 - \pi) < \infty$ then after Nisnevich localization there exists $\widetilde{\pi} \in \operatorname{End} P$ such that $\widetilde{\pi}^2 = \widetilde{\pi}$ and $\operatorname{rank}(\widetilde{\pi} - \pi) < \infty$. The idea is to consider the spectrum of π. There exists monic $f \in R[\lambda]$ such that $f(\pi^2 - \pi)|_{\mathrm{Im}(\pi^2-\pi)} = 0$. So $f(\pi^2 - \pi)(\pi^2 - \pi) = 0$. Put $g(\lambda) := (\lambda^2 - \lambda)f(\lambda^2 - \lambda)$. We have $R[\lambda]/(g) \to \operatorname{End} P$, $\lambda \mapsto \pi$. Now consider $S := R[\lambda]/(g) \subset \operatorname{Spec} R \times \mathbb{A}^1$. Clearly $S \supset \underline{0} \cup \underline{1}$, where $\underline{0} = \operatorname{Spec} R \times \{0\}$, $\underline{1} = \operatorname{Spec} R \times \{1\}$. Suppose we have $S = S_0 \coprod S_1$, where S_0 and S_1 are open subsets such that $S_0 \supset \underline{0}$ and $S_1 \supset \underline{1}$. Define $e \in R[\lambda]/(g) = H^0(S, \mathcal{O}_S)$ by $e|_{S_0} = 0$, $e|_{S_1} = 1$. Then we can put $\widetilde{\pi} :=$ image of e in $\operatorname{End} P$. We claim that the decomposition $S = S_0 \coprod S_1$ exists *locally* with respect to the Nisnevich topology of $\operatorname{Spec} R$. This is clear because such a decomposition exists if R is Henselian. □

2.12. **Determinants and dimensions combined together.** Following §2 of [BBE], we combine the dimension torsor and the determinant gerbe into a single object, which is a Torsor over a certain Picard algebroid (these notions are defined below). The reason why it is convenient and maybe necessary to do this is explained in 2.12.3. The reader may prefer to skip this subsection and go directly to §3. Our terminology is slightly different from that of [BBE], and our determinant Torsor is inverse to that of [BBE].

2.12.1. *Terminology.* According to §1.4 of [Del], a *Picard groupoid* is a symmetric monoidal category $\mathcal{A}$ such that all the morphisms of $\mathcal{A}$ are invertible and the semigroup of isomorphism classes of the objects of $\mathcal{A}$ is a group. A Picard groupoid is said to be *strictly commutative* if for every $a \in \mathrm{Ob}\ \mathcal{A}$ the commutativity isomorphism $a \otimes a \xrightarrow{\sim} a \otimes a$ equals id_a. As explained in §1.4 of [Del], there is also a notion of *sheaf of Picard groupoids* (champ de catégories de Picard) on a site.

We will work with the following simple example.

Example. For a commutative ring R we have the Picard groupoid $\mathbb{Z}\text{-}\mathrm{Pic}_R$ of $\mathbb{Z}$-graded invertible R-modules (it is not stricly commutative because we use the "super" commutativity constraint $a \otimes b \mapsto (-1)^{p(a)p(b)} b \otimes a$). For a scheme S denote by $\mathbb{Z}\text{-}\mathrm{Pic}_S$ the sheaf of Picard groupoids on the Nisnevich site of S formed by $\mathbb{Z}$-graded invertible $\mathcal{O}_S$-modules.

Should I write $\mathbb{Z}\text{-}\mathrm{Inv}_R$ instead of $\mathbb{Z}\text{-}\mathrm{Pic}_R$??

We need more terminology. An *Action* of a monoidal category $\mathcal{A}$ on a category $\mathcal{C}$ is a monoidal functor from $\mathcal{A}$ to the monoidal category $\mathrm{Funct}(\mathcal{C},\mathcal{C})$ of functors $\mathcal{C} \to \mathcal{C}$. Suppose $\mathcal{A}$ acts on $\mathcal{C}$ and $\mathcal{C}'$, i.e., one has monoidal functors $\Phi : \mathcal{A} \to \mathrm{Funct}(\mathcal{C},\mathcal{C})$ and $\Phi' : \mathcal{A} \to \mathrm{Funct}(\mathcal{C}',\mathcal{C}')$. Then an *$\mathcal{A}$-functor* $\mathcal{C} \to \mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ equipped with isomorphisms $F\Phi(a) \xrightarrow{\sim} \Phi'(a)F$ satisfying the natural compatibility condition (the two ways of constructing an isomorphism $F\Phi(a_1 \otimes a_2) \xrightarrow{\sim} \Phi'(a_1 \otimes a_2)F$ must give the same result). An *$\mathcal{A}$-equivalence* $C \to C'$ is an $\mathcal{A}$-functor $C \to C'$ which is an equivalence.

There is also an obvious notion of Action of a sheaf of monoidal categories $\mathcal{A}$ on a sheaf of categories $\mathcal{C}$, and given an Action of $\mathcal{A}$ on $\mathcal{C}$ and $\mathcal{C}'$ there is an obvious notion of $\mathcal{A}$-functor $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{A}$-equivalence $\mathcal{C} \to \mathcal{C}'$.

Definition. Let $\mathcal{A}$ be a sheaf of Picard groupoids on a site. A sheaf of categories $\mathcal{C}$ equipped with an Action of $\mathcal{A}$ is an $\mathcal{A}$-*Torsor* if it is locally $\mathcal{A}$-equivalent to $\mathcal{A}$.

Remark. The notion of Torsor makes sense even if $\mathcal{A}$ is non-symmetric. But $\mathcal{A}$ has to be symmetric if we want to have a notion of product of $\mathcal{A}$-Torsors.

2.12.2. *The determinant Torsor.* Let R be a commutative ring, $S := \operatorname{Spec} R$. Slightly modifying the construction of [Ka3], we will associate a Torsor over $\mathbb{Z}$-Pic_S to an almost projective R-module M. Recall that a coflat lattice $L \subset M$ is a finitely generated (or presented??) submodule such that M/L is projective. The set of coflat lattices $L \subset M$ will be denoted by $G(M)$. In general, $G(M)$ may be empty, and it is not clear if every $L_1, L_2 \in G(M)$ are contained in some $L \in G(M)$. But it follows from Theorem 2.5(ii) that these properties hold after Nisnevich localization (to show that every $L_1, L_2 \in G(M)$ are Nisnevich-locally contained in some coflat lattice apply Theorem 2.5(ii) to $M/(L_1 + L_2)$ or use Remark 1 after the theorem). In other words, for every $x \in \operatorname{Spec} R$ the inductive limit of $G(R' \otimes_R M)$ over the filtering category of all etale R-algebras R' equipped with an R-morphism $x \to \operatorname{Spec} R'$ is a non-empty directed set.

For each pair $L_1 \subset L_2$ in $G(M)$ one has the invertible R-module $\det(L_2/L_1)$. It is equipped with a $\mathbb{Z}$-grading (the determinant of an n-dmensional vector space has grading n).

Definition. A *determinant theory* on M (resp. a *weak determinant theory* on M) is a rule Δ which associates to each R-algebra R' and each $L \in G(R' \otimes_R M)$ an invertible graded R'-module $\Delta(L)$ (resp. an invertible R'-module $\Delta(L)$), to each pair $L_1 \subset L_2$ in $G(R' \otimes_R M)$ an isomorphism

$$\Delta_{L_1 L_2} : \Delta(L_1) \otimes \det(L_2/L_1) \xrightarrow{\sim} \Delta(L_2), \tag{2.9}$$

and to each morphsism $f : R' \to R''$ of R-algebras a collection of base change morphisms $\Delta_f = \Delta_{f,L'} : \Delta(L') \to \Delta(R''L')$, $L' \in G(R' \otimes_R M)$. These data should satisfy the following conditions:

(i) each $\Delta_{f,L'}$ induces an isomorphism $R'' \otimes_{R'} \Delta(L') \xrightarrow{\sim} \Delta(R''L')$;
(ii) $\Delta_{f_2 f_1} = \Delta_{f_2} \Delta_{f_1}$;
(iii) the isomorphisms (2.9) commute with base change;
(iv) for any triple $L_1 \subset L_2 \subset L_3$ in $G(R' \otimes_R M)$ the obvious diagram

$$\Delta(L_1) \otimes \det(L_2/L_1) \otimes \det(L_3/L_3) \xrightarrow{\sim} \Delta(L_1) \otimes \det(L_3/L_1) \downarrow \downarrow \Delta(L_2) \otimes \det(L_3/L_2) \xrightarrow{\sim} \Delta(L_3)$$

commutes.

Remark. It follows from Theorem 2.5(ii) that the notion of (weak) determinant theory does not change if one considers only etale R-algebras instead of arbitrary ones.

The groupoid of all determinant theories on M is equipped with an obvious Action of the Picard groupoid $\mathbb{Z}$-Pic_R of invertible $\mathbb{Z}$-graded R-modules: $P \in \mathbb{Z}$-Pic_R sends Δ to $P\Delta$, where $(P\Delta)(L) := P \otimes_R \Delta(L)$.

Determinant theories on $R' \otimes_R M$ for all etale R algebras R' form a sheaf of groupoids Det_M on the Nisnevich site of $S := \operatorname{Spec} R$, which is equipped with an Action of the sheaf of Picard groupoids $\mathbb{Z}$-Pic_S. It follows from Theorem 2.5(ii) that Det_M is a Torsor over $\mathbb{Z}$-Pic_S. We call it the *determinant Torsor* of M.

Remark. Consider the category whose set of objects is $\mathbb{Z}$ and whose only morphisms are the identities. We will denote it simply by $\mathbb{Z}$. Addition of integers defines a functor $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, so $\mathbb{Z}$ becomes a Picard groupoid. We have a canonical Picard functor from $\mathbb{Z}$-Pic_S to the constant sheaf $\mathbb{Z}$ of Picard groupoids: an invertible $\mathcal{O}_S$-module placed in degree n goes to n. The $\mathbb{Z}$-torsor corresponding to the ($\mathbb{Z}$-Pic_S)-Torsor Det_M is the dimension torsor Dim_M from 2.5.

2.12.3. *On the notion of determinant gerbe.* We also have the forgetful functor from the category of $\mathbb{Z}$-graded invertible R-modules to that of plain invertible R-modules. It defines a monoidal functor F (but not a Picard functor!) from $\mathbb{Z}\text{-Pic}_S$ to the sheaf of Picard categories $\mathcal{O}_S^*[1]$ whose category of sections over an etale S-scheme S' is the Picard category of $\mathcal{O}_S^*$-torsors. Applying F to the $(\mathbb{Z}\text{-Pic}_S)$-Torsor Det_M one gets an $\mathcal{O}_S^*[1]$-Torsor, i.e., an $\mathcal{O}_S^*$-gerbe. This is the *determinant gerbe* considered by Kapranov [Ka3] and mentioned in 2.9. Its sections are weak determinant theories. As F does not commute with the commutativity constraint, there is no canonical equivalence between the $\mathcal{O}_S^*$-gerbe corresponding to a direct sum of almost projective modules M_i, $i \in I$, $\mathrm{Card}\, I < \infty$, and the product of the $\mathcal{O}_S^*$-gerbes corresponding to M_i, $i \in I$ (but there is an equivalence which depends on the choice of an ordering of I). This is the source of the numerous signs in [ACK] and the reason why we prefer to consider Torsors over $\mathbb{Z}\text{-Pic}_S$ rather than pairs consisting of a $\mathcal{O}_S^*$-gerbe and a $\mathbb{Z}$-torsor (as Kapranov does in [Ka3]).

I should mention (where??) that the $(\mathbb{Z}\text{-Pic}_S)$-Torsor Det_M depends only on M as an object of the "up to isogeny" category. Is it true that (at least at the level of groupoids) this is a full subcategory of the category of ind-objects of the groupoid of R-modules (to an almost projective module M one associates the formal inductive limit of the quotients M/F with F finitely generated)?? In particular, one can define the determinant Torsor of a Tate R-module (this is also clear directly).

2.12.4. *The Fermion module.* Suppose we are given a Tate R-module M equipped with a weak determinant theory τ. of its determinant gerbe. If R is a field it is well known that these data define a graded T-module $\bigwedge M$ over the Clifford algebra of $M \oplus M^*$, where T is the dimension torsor of M (the graded components of M are the semi-infinite exterior powers of M). In fact, one can define the "Fermion module" $\bigwedge_\tau M$ for any commutative ring R (locally for the Nisnevich topology of $\mathrm{Spec}\, R$ one can construct $\bigwedge_\tau M$ just as in the case that R is a field). $\bigwedge_\tau M$ is projective: Nisnevich-locally this is clear, then use the Raynaud-Gruson theorem on the local nature of projectivity (see §1).

2.12.5. *A vague picture.* Let S be a spectrum in the sense of algebraic topology. We put $\pi^i(S) := \pi_{-i}(S)$ and define $\tau^{\le k}S$ to be the spectrum equipped with a morphism $\tau^{\le k}S \to S$ such that $\pi^i(\tau^{\le k}S) = 0$ for $i > k$ and the morphism $\pi^i(\tau^{\le k}S) \to \pi^i(S)$ is an isomorphism for $i \le k$. There is a notion of torsor over a spectrum S, which depends only on $\tau^{\le 1}S$. Namely, an S-torsor is a point of the infinite loop space L corresponding to $(\tau^{\le 1}S)[1]$ (or, equivalently, a morphism from the spherical spectrum to $S[1]$). A homotopy equivalence between torsors is a path connecting the corresponding points of L, so equivalence classes are parametrized by $\pi^1(S) := \pi_{-1}(S)$.

According to Beilinson, to an almost projective R-module M there should correspond a torsor over the K-theory spectrum $K(R)$ whose class in $\pi_{-1}(K(R)) = K_{-1}(R)$ should be the class $[M]$ defined in 2.3 (here and in what follows the status of "should" is not clear to the author, so the picture becomes vague). If R is commutative then by Thomason's localization theorem ([TT], §10.8) $K(R) = R\Gamma(S, \mathcal{K})$, where $\mathcal{K}$ is the sheaf of K-theories of $\mathcal{O}_S$ (this is a sheaf of spectra on the Nisnevich site of S). So the notion of $K(R)$-torsor should coincide with that of $\mathcal{K}$-torsor. Both of them should coincide with that of $\tau^{\le 1}\mathcal{K}$-torsor. By Theorem 2.6, $\mathcal{K}^1 := \mathcal{K}_{-1} = 0$, so $\tau^{\le 1}\mathcal{K} = \tau^{\le 0}\mathcal{K}$ and therefore we get a morphism $\tau^{\le 1}\mathcal{K} = \tau^{\le 0}\mathcal{K} \to \mathcal{K}_{[0,1]} := \mathcal{K}^{[-1,0]} := \tau^{\ge -1}\tau^{\le 0}\mathcal{K}$. So to an almost projective R-module M there should correspond a $\mathcal{K}_{[0,1]}$-torsor Δ_M. According to Beilinson, $\mathcal{K}_{[0,1]}$ and Δ_M should identify with $\mathbb{Z}\text{-Pic}_S$ and the Torsor Det_M from 2.12.2 via the following dictionary, which goes back to A. Grothendieck.

According to Corollary 1.4.17 of [Del], a sheaf of strictly commutative Picard groupoids is essentially the same as a complex of sheaves of abelian groups with cohomology concentrated in degrees 0 and -1. As far as I understood Lawrence Breen (24 June 2003, private communication), this remains true if one removes the strict commutativity condition, replaces "complex" by "spectrum" (in the sense of algebraic topology) and the cohomology groups H^i by the homotopy groups π_{-i}, but a precise reference is unfortunately not available. Hopefully, a torsor over the spectrum corresponding to a Picard groupoid $\mathcal{A}$ is the same as an $\mathcal{A}$-Torsor in the sense of 2.12.1. One also hopes that this is true for sheaves of spectra and sheaves of Picard groupoids.

Problem: *make the above vague picture precise.* The notion of determinant Torsor is very useful, and its rigorous interpretation in the standard homotopy-theoretic language of algebraic K-theory would be helpful.

3. Application: "refined" motivic integration

In this section we fix a field k and assume all k-algebras to be commutative.

3.1. **A class of schemes.** We will use the following notation for affine spaces: $\mathbb{A}^I := \operatorname{Spec} k[x_i]_{i\in I}$, $\mathbb{A}^\infty := \mathbb{A}^{\mathbb{N}}$.

We say that a k-scheme is *nice* if it is isomorphic to $X \times \mathbb{A}^I$, where X is of finite type (the set I may be infinite). $\mathbb{A}^I$ is an affine space, i.e., $\mathbb{A}^I := \operatorname{Spec} k[x_i]_{i\in I}$ (I may be infinite). An affine scheme is nice if and only if it can be defined by finitely many equations in a (not necessarily finite-dimensional) affine space.

Definition. A k-scheme is *locally nice* (resp. *Zariski-locally nice, etale-locally nice*) if it becomes nice after Nisnevich localization (resp. Zariski or etale localization).

I do not know if etale-local niceness implies local niceness. Local niceness does not imply Zariski-local niceness (see 3.2 below).

If $X = \operatorname{Spec} R$ is locally nice then Ω^1_R is almost projective by Theorem 2.5(i). So we have the *dimension torsor* $\operatorname{Dim}_X := \operatorname{Dim}_{\Omega^1_X}$. In the next subsection we will see that Dim_X may be nontrivial, and on the other hand, there exists a locally nice k-scheme with trivial dimension torsor which is not Zariski-locally nice. The dimension torsor of any locally nice k-scheme is defined by gluing together the torsors Dim_U for all open affine $U \subset X$.

3.2. **Examples.** (i) Define $i : \mathbb{A}^\infty \to \mathbb{A}^\infty$ by $i(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$. Take $\mathbb{A}^1 \times \mathbb{A}^\infty$ and then glue $(0, x) \in \mathbb{A}^1 \times \mathbb{A}^\infty$ with $(1, i(x)) \in \mathbb{A}^1 \times \mathbb{A}^\infty$. Thus one gets a scheme X whose dimension torsor is nontrivial and even not Zariski-locally trivial.

(ii) Let M be an almost projective module over a finitely generated k-algebra R. Let X denote the spectrum of the symmetric algebra of M. Then X is locally nice by Theorems 2.5(ii) and 2.1. It is easy to deduce from Theorem 2.1 that X is Zariski-locally nice if and only if the class of M in $K_{-1}(R)$ vanishes locally for the Zariski topology (to prove the "only if" statement consider the restriction of Ω^1_X to the zero section $\operatorname{Spec} R \hookrightarrow X$). If R and M are as in (2.1) then we get the above Example (i).

(iii) There exists a locally nice k-scheme X with trivial dimension torsor which is not Zariski-locally nice.[3] According to (ii), to get such an example it suffices to find a finitely generated k-algebra R and an almost projective R-module M such that $H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z}) = 0$ but the class of M in $K_{-1}(R)$ is not Zariski-locally trivial. §6 of [We3] contains examples of finitely generated normal k-algebras R with $K_{-1}(R) \neq 0$. In each of them $\operatorname{Spec} R$ has a unique singular point x, and according to [We1], the map $K_{-1}(R) \to K_{-1}(R_x)$ is an isomorphism. Now take any nonzero element of $K_{-1}(R)$ and represent it as a class of an almost projective R-module M.

3.3. **A class of ind-schemes.** Functors from the category of k-algebras to that of sets will be called "spaces". E.g., a k-scheme can be considered as a space. An *ind-scheme* is a space which can be represented as the union of a directed family of closed subschemes. An ind-scheme is *ind-affine* if its closed subschemes are affine.

Main example. Let Y be an affine scheme over $F := k((t))$. Define a functor Y_F from the category of k-algebras to that of sets by $Y_F(R) := Y(R\hat{\otimes}F)$, $R\hat{\otimes}F := R((t))$. It is well known and easy to see that Y_F is an ind-affine ind-subscheme. If Y is an affine scheme of *finite type* over F then Y_F is "reasonable" in the sense of the following definition (which is due to A. Beilinson).

[3] In 3.12.3 we will see that one can get such X from the loop space of a smooth affine manifold.

Definition. Let X be an ind-scheme. A closed quasi-compact subscheme $Y \subset X$ is called *reasonable* if for any closed subscheme $Z \subset X$ such that $Y \subset Z$ the ideal of Y in $\mathcal{O}_Z$ is finitely generated locally on Z. **Do I really need quasicompactness??** We say that X is *reasonable* if X is a union of its reasonable subschemes, i.e., it may be represented as the direct limit of X_α where all X_α are reasonable.

(A closed ind-subscheme of a reasonable ind-scheme is a reasonable ind-scheme; a product of two reasonable ind-schemes is reasonable. **Do I need this??**)

Definition. A reasonable ind-scheme X is *ind-smooth* if

(i) every reasonable closed subscheme of X is locally nice;

(ii) X is formally smooth, i.e., for every k-algebra R and any nilpotent ideal $I \subset R$ the map $X(R) \to X(R/I)$ is surjective.

Remark. In the above definition we do not require every closed subscheme of X to be contained in a formally smooth subscheme. It is not clear if this property holds for $SL(n)_F$ or for the affine Grassmannian $SL(n)_F/SL(n)_O$, $O := k[[(t]]$, even though these ind-schemes are ind-smooth. See also Remark (ii) from the next subsection.

3.4. **Loops of an affine manifold.**

Theorem 3.1. *If an affine F-scheme Y is smooth then Y_F is ind-smooth.*

Remarks. (i) In fact, the formal smoothness of Y_F immediately follows from the smoothness of Y and the definition of Y_F. It is property (i) from the definition of ind-smoothness that requires some efforts.

(ii) If Y is a smooth affine F-scheme then by Theorem 3.1 every reasonable closed subscheme $X \subset Y_F$ is locally nice. But there exist Y and $X \subset Y_F$ as above such that X is not Zariski-locally nice. One can choose Y and X so that Dim_X is not Zariski-locally trivial. But one can also choose Y and X so that Dim_X is trivial but X is not Zariski-locally nice. See 3.12 for examples of these situations. As K_{-1} of a regular ring is zero (see 2.4), in these examples Y_F cannot be represented (even Zariski-locally) as the union of an increasing sequence of smooth closed subschemes.

Before sketching the proof of Theorem 3.1 I will formulate a slightly more general theorem. The point is that if Y is embedded into an affine space then the intersection of Y with a polydisk is a very special kind of an affinoid analytic space over F (it is defined by polynomial equations rather than holomorphic ones). In the next subsection we formulate an analog of Theorem 3.1 for any affinoid analytic space over F.

3.5. **Loops of an affinoid space.** We will use the terminology from [BGR] (which goes back to Tate) rather than the one from [Be]. Let $F\langle z_1, \ldots, z_n\rangle \subset F[[z_1, \ldots z_n]]$ be the algebra of power series which converge in the polydsik $|z_i| \leq 1$. As $F = k((t))$ one has $F\langle z_1, \ldots, z_n\rangle = k[z_1, \ldots, z_n]((t))$. For every k-algebra R the F-algebra $R\hat{\otimes}F = R((t))$ is equipped with the norm whose unit ball is $R[[t]]$. In particular, $F\langle z_1, \ldots, z_n\rangle$ is a Banach algebra. An *affinoid F-algebra* is a topological F-algebra isomorphic to a quotient of $F\langle z_1, \ldots, z_n\rangle$ for some n. All morphisms between affinoid F-algebras are automatically continuous (see, e.g., §6.1.3 of [BGR]). The category of *affinoid analytic spaces* is defined to be dual to that of affinoid F-algebras; the affinoid space corresponding to an affinoid F-algebra A will be denoted by $\mathcal{M}(A)$.

For an affinoid analytic space $Z = \mathcal{M}(A)$ and a k-algebra R denote by $Z_F(R)$ the set of continuous F-homomorphisms from A to the Banach F-algebra $R\hat{\otimes}F = R((t))$. It is easy to see that the functor Z_F is a reasonable ind-affine ind-scheme which has a closed subscheme X such that the ideal of X in every bigger closed subscheme of Z_F is nilpotent. X can be constructed as follows: choose[4] a representation of A as $A_0[t^{-1}]$ with A_0 isomorphic to a quotient

[4] If Z is reduced there is no choice: A_0 is the set of $f \in A$ such that $|f(z)| \leq 1$ for all $z \in Z$

of $k[[t, z_1, \dots z_n]]$ which is flat over $O = k[[t]]$ and define $X(R) \subset Z_F(R)$ to be the set of F-morphsims $f : A \to R\hat{\otimes}F$ such that $f(A_0) \subset R\hat{\otimes}O$.

Theorem 3.2. *If an affinoid space Z is smooth then Z_F is ind-smooth.*

3.6. **On the proof of Theorems 3.1 and 3.2.** Theorem 3.1 follows from Theorem 3.2. The formal smoothness of the ind-scheme Z_F from Theorem 3.2 immediately follows from the definitions. So it remains to show that every reasonable closed subscheme $X = \operatorname{Spec} R \subset Z_F$ is locally nice.

The first step is to study Ω^1_R. For every closed subscheme $Y \subset Z_F$ containing X let M_Y be the R-module corresponding to the pullback of Ω^1_Y to $X = \operatorname{Spec} R$. The projective limit M of the R-modules M_Y is a topological R-module. The following lemma is easy.

Lemma 3.3. *(i) Let A be the affinoid F-algebra corresponding to Z. Consider the homomorphism of topological F-algebras $A \to R((t))$ corresponding to the morphism $\operatorname{Spec} R = X \hookrightarrow Z_F$. Then the topological R-module M identifies with $R((t)) \otimes_A \Omega^1_A$ (here Ω^1_A is understood in the sense of affinoid F-algebras rather than abstract algebras, so Ω^1_A is a finitely generated projective A-module). Therefore M is a finitely generated projective $R((t))$-module and therefore a Tate R-module.*

(ii) $\Omega^1_R = M/L$ for some lattice $L \subset M$.

By Lemma 3.3(i) and Theorem 2.3 M becomes quasi-elementary after Nisnevich localization. One easily shows that if $\operatorname{Spec} R$ is connected then a quasi-elementary Tate R-module which comes from a finitely generated projective $R((t))$-module is isomorphic to $P \oplus Q^*$, where P and Q are *free* discrete R-modules. Applying Lemma 3.3(ii) we see that after Nisnevich localization Ω^1_R becomes a direct sum of a free module and a module of finite presentation. This is a linearized version of Theorem 3.2.

To deduce the theorem from its linearized version one works with the implicit function theorem.

3.7. **The renormalized dualizing complex.** Fix a prime $l \neq \operatorname{char} k$. Let $D^b_c(X, \mathbb{Z}_l)$ denote the appropriately defined bounded constructible l-adic derived category on a scheme X (see [E, Ja]). For a general locally nice k-scheme X there is no natural way to define the dualizing complex $K_X \in D^b_c(X, \mathbb{Z}_l)$. Indeed, if X is the product of $\mathbb{A}^\infty$ and a k-scheme Y of finite type and if $\pi : X \to Y$ is the projection then K_X should equal $\pi^* K_Y \otimes (\mathbb{Z}_l[2](1))^{\otimes\infty}$, which makes no sense. But suppose that the dimension $\mathbb{Z}$-torsor Dim_X is trivial and that we have chosen its trivialization η. Then one can define the *renormalized dualizing complex* $K^\eta_X \in D^b_c(X, \mathbb{Z}_l)$. The definition (which is straightfroward) is given below. The reader can skip it and go directly to 3.8.

First assume that X is nice, i.e., there exists a morphism $\pi : X \to Y$ such that Y is a k-scheme of finite type and X is Y-isomorphic to $Y \times \mathbb{A}^I$ for some set I. Let C_X be the category of all such pairs (Y, π). A morphism $f : (Y, \pi) \to (Y', \pi')$ is defined to be a morphism $f : Y \to Y'$ such that $\pi' = f\pi$. Such f is unique if it exists. The category C_X is equivalent to a directed set. So to define K^η_X it suffices to define a functor

$$C_X \to D^b_c(X, \mathbb{Z}_l), \quad (Y, \pi) \mapsto K^{\eta,\pi}_X \tag{3.1}$$

which sends all morphisms to isomorphisms.

If $(Y, \pi) \in C_X$ then $\pi^*\Omega^1_Y \subset \Omega^1_X$ is locally of finite presentation and $\Omega^1_X/\pi^*\Omega^1_Y$ is locally free. So for every open affine $U \subset X$ one has the coflat lattice $\Gamma(U, \pi^*\Omega^1_Y) \subset \Gamma(U, \Omega^1_X)$ and therefore a section of the torsor Dim_X over U. These sections agree with each other, so we get a global section η_π of Dim_X. Put

$$m := \eta_\pi - \eta \in H^0(X, \mathbb{Z}), \tag{3.2}$$

$$K^{\eta,\pi}_X := \pi^* K_Y \otimes (\mathbb{Z}_l[2](1))^{\otimes m}, \tag{3.3}$$

Now let $f : (Y, \pi) \to (Y', \pi')$ be a morphism. One easily shows that $f : Y \to Y'$ is smooth[5], so one has a canonical isomorphism

$$K_Y \xrightarrow{\sim} f^* K_{Y'} \otimes (\mathbb{Z}_l[2](1))^{\otimes d}, \tag{3.4}$$

where d is the relative dimension of Y over Y'. It is easy to see that $\pi^* d = \eta_{\pi'} - \eta_\pi$, so (3.4) induces an isomorphism $\alpha_f : K_X^{\eta,\pi} \to K_X^{\eta,\pi'}$. We define (3.1) on morphisms by $f \mapsto \alpha_f$.

So we have defined K_X^η if X is nice. The formation of K_X^η commutes with etale localization of X. It is easy to see that $\mathrm{Ext}^i(K_X^\eta, K_X^\eta) = 0$ for $i < 0$. So by Theorem 3.2.4 of [BBD] there is a unique way to extend the definition of K_X^η to all etale-locally nice k-schemes X so that the formation of K_X^η still commutes with etale localization.

3.8. **$R\Gamma_c$ of a locally nice scheme.** Suppose we are in the situation of 3.7, i.e., we have a locally nice k-scheme X, a trivialization η of its dimension torsor, and a prime $l \neq \mathrm{char}\, k$. Assume that X is quasicompact and quasiseparated. Then we put

$$R\Gamma_c^\eta(X \otimes \bar{k}, \mathbb{Z}_l) := R\Gamma(X \otimes_k \bar{k}, K_X^\eta)^*, \tag{3.5}$$

where K_X^η is the renormalized dualizing complex defined in 3.7. $R\Gamma_c^\eta(X \otimes \bar{k}, \mathbb{Z}_l)$ is an object of $D_c^b(\mathrm{Spec}\, k, \mathbb{Z}_l)$, i.e., of the appropriately defined bounded constructible derived category of l-adic representations of $\mathrm{Gal}(k^s/k)$, where k^s is a separable closure of k.

Problem. Define an object of the triangulated category of k-motives whose l-adic realization equals $R\Gamma_c^\eta(X \otimes \bar{k}, \mathbb{Z}_l)$ for each $l \neq \mathrm{char}\, k$. Voevodsky says this can probably be done. **What about Voevodsky's stable homotopy category??**

3.9. **"Refined" motivic integration.** Suppose that in the situation of Theorem 3.2 the canonical bundle $\det \Omega_Z^1$ is trivial. Choose a trivialization of $\det \Omega_Z^1$, i.e., a differential form $\omega \in H^0(Z, \det \Omega_Z^1)$ with no zeros. As explained in 3.5, the ind-scheme Z_F has a (reasonable) closed subscheme X such that the ideal of X in every bigger closed subscheme of Z_F is nilpotent. Choose such X. By 2.8 and Lemma 3.3 our trivialization of $\det \Omega_Z^1$ induces a trivialization η of the dimension torsor $\mathrm{Dim}\, X$. We put

$$\int_Z |\omega| := R\Gamma_c^\eta(X, \mathbb{Z}_l) \in D_c^b(\mathrm{Spec}\, k, \mathbb{Z}_l), \tag{3.6}$$

where $R\Gamma_c^\eta(X, \mathbb{Z}_l)$ is defined by (3.5). Clearly $\int_Z |\omega|$ does not depend on the choice of X.

3.10. **Comparison with usual motivic integration.** In the situation of 3.9 (i.e., integrating a holomorphic form with no zeros over an affinoid domain) the usual motivic integral [**?**] belongs to $M_k := M_k'[L^{-1}]$, where M_k' is the Grothendieck ring of k-varieties[6] and $L \in M_k'$ is the class of the affine line. Its definition can be reformulated as follows.

Given a connected nice k-scheme X and a trivialization η of its dimension torsor one chooses $\pi : X \to Y$ as in 3.7, defines $m \in H^0(X, \mathbb{Z}) = \mathbb{Z}$ by (3.2) and puts $[X]^\eta := [Y]L^m \in M_k$. If X is any quasicompact quasiseparated locally nice k-scheme choose closed subschemes $X = F_0 \supset F_1 \supset \ldots \supset F_n = \emptyset$ so that each F_i is defined by finitely many equations and $F_i \setminus F_{i+1}$ is nice and connected; then put $[X]^\eta := \sum_i [F_i \setminus F_{i+1}]^\eta$. Finally, in the situation of 3.9 one puts

$$(\int_Z |\omega|)_{usual} := [X]^\eta \in M_k, \tag{3.7}$$

[5]Choosing a section $Y \to X$ one sees that Y is Y'-isomorphic to a retract of $Y' \times \mathbb{A}^J$ for some J. So f is formally smooth and therefore smooth.

[6]M_k' is generated by elements $[X]$ corresponding to isomorphism classes of k-schemes of finite type, and the defining relations are $[X] = [Y] + [X \setminus Y]$ for any k-scheme X of finite type and any closed subscheme $Y \subset X$. In particular, these relations imply that $[X]$ depends only on the reduced subscheme corresponding to X.

Clearly (3.7) is well-defined, and the images of (3.7) and (3.6) in $K_0(D^b_c(\operatorname{Spec} k, \mathbb{Z}_l))$ are equal. So (3.7) and (3.6) can be considered as different refinements of the same object of $K_0(D^b_c(\operatorname{Spec} k, \mathbb{Z}_l))$. Unless the map $M_k \to K_0(D^b_c(\operatorname{Spec} k, \mathbb{Z}_l))$ is injective (which seems unlikely), the "refined" motivic integral (3.6) cannot be considered as the refinement of the usual motivic integral (3.7). This is why I am using quotation marks.

3.11. **Remark.** Our definition of "refined" motivic integration works only in the case of integrating a holomorphic form with no zeros over an affinoid domain (which is probably too special for serious applications).

On the other hand, in an unpublished manuscript V. Vologodsky defined a different kind of "refined motivic integration" in the case of K3 surfaces. More precisely, let $\omega \neq 0$ be a regular differential form on a K3 surface X over $F = k((t))$, $\operatorname{char} k = 0$. Let A denote the Grothendieck ring of the category of Grothedieck motives over k, and let I_n denote the motivic integral of ω over $X \otimes_F k((t^{1/n}))$ viewed as an object of $A \otimes \mathbb{Q}$. Vologodsky defined objects M_1, M_2, M_3 of the category of Grothedieck motives so that I_n is a certain linear combination of the classes of M_1, M_2, M_3. The objects M_1, M_2, M_3 depend functorially on (X, ω). His definition of M_1, M_2, M_3 is mysterious.

3.12. **Counterexamples.** Here are the examples promised in Remark (ii) of 3.4.

3.12.1. *Nontrivial dimension torsor.* Put $Y := (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Gamma_f$, where $\mathbb{P}^1$ is the projective line over $F := k((t))$ and Γ_f is the graph of a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $n > 0$. Clearly Y is affine, and the canonical bundle of Y is isomorphic to $p_1^*\mathcal{O}(-2) \otimes p_2^*\mathcal{O}(-2) = p_1^*\mathcal{O}(2n-2)$, where $p_1, p_2 : Y \to \mathbb{P}^1$ are the projections. We claim that if $n > 1$ then the dimension torsor of Y_F **(what is this??)** is nontrivial. Moreover, there exists a morphism $\phi : \operatorname{Spec} R \to Y_F$, $R := \{f \in k[x] | f(0) = f(1)\}$, such that $\phi^* \operatorname{Dim}_{Y_F}$ is nontrivial. One constructs ϕ as follows. Consider the $R((t))$-module M defined by (2.1). One can represent M as a direct summand of $R((t))^2$. Indeed, the $R((t))$-module

$$\{u = u(x,t) \in k[x]((t))^2 \,|\, u(1,t) = A(t)u(0,t)\}, \quad A := \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$$

is isomorphic to $R((t))^2$ because there exists $A(x,t) \in SL(2, k[x,t,t^{-1}])$ such that $A(0,t)$ is the unit matrix and $A(1,t) = A(t)$ (to find $A(x,t)$ represent $A(t)$ as a product of elementary matrices). Representing M as a direct summand of $R((t))^2$ one gets a morphism $g : \operatorname{Spec} R((t)) \to \mathbb{P}^1$. As $p_1 : Y \to \mathbb{P}^1$ is a locally trivial fibration with fiber $\mathbb{A}^1$, one can represent g as $p_1\varphi$ for some $\varphi : \operatorname{Spec} R((t)) \to Y$. The morphism $\phi : \operatorname{Spec} R \to Y_F$ corresponding to φ has the desired property, i.e., $\phi^* \operatorname{Dim}_{Y_F}$ is nontrivial.

3.12.2. *Not Zariski-locally trivial dimension torsor.* Let Y be the space of triples (v, l, l'), where l, l' are transversal 1-dimensional subspaces in F^2 and $v \in l$. The dimension torsor of Y_F is not Zariski-locally trivial. Moreover, slightly modifying the construction of 3.12.1 one gets a morphism $\phi : \operatorname{Spec} R \to Y_F$, $R := \{f \in k[x] | f(0) = f(1)\}$, such that $\phi^* \operatorname{Dim}_{Y_F}$ is not Zariski-locally trivial.

3.12.3. *Any "unpleasant thing" can happen.* This is what the following theorem essentially says. E.g., combining statement (ii) of the theorem with Weibel's examples mentioned in 2.7 one sees that for some smooth scheme Y over $F = k((t))$ with trivial canonical bundle there exists a reasonable closed subscheme of Y_F which is not Zariski-locally nice (even though its dimension torsor is trivial).

Theorem 3.4. *Let R be a k-algebra and $u \in K_{-1}(R)$.*

(i) There exists a smooth scheme Y over $F = k((t))$ and a morphism $f : \operatorname{Spec} R \to Y_F$ such that the pullback of the cotangent sheaf of Y_F to $\operatorname{Spec} R$ has class u.

(ii) If the image of u in $H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z})$ equals 0 then one can choose Y to have trivial canonical bundle (in this case the dimension torsor of Y_F is trivial).

To prove the theorem we generalize the construction of 3.12.2.

Proof. As the morphism (2.3) is surjective one can represent $u \in K_{-1}(R)$ as the image of some element $u' \in K_0(R[t,t^{-1}])$. One can choose u' to be representable as the class of a projective $R[t,t^{-1}]$-module P. Without loss of generality we can assume that P has constant rank m. Represent P as a direct summand of $R[t,t^{-1}]^{m+n}$. Such a representation defines a morphism $\operatorname{Spec} R[t,t^{-1}] \to G/H$, where $G = \operatorname{Aut}(k^m \oplus k^n)$, $H = \operatorname{Aut} k^m \times \operatorname{Aut} k^n$ (G and H are viewed as algebraic groups over k). The composition $\operatorname{Spec} R((t)) \to \operatorname{Spec} R[t,t^{-1}] \to G/H$ defines a morphism $\operatorname{Spec} R \to (G/H)_F$.

Now we define the desired manifold Y by $Y = (G \times V)/H$, where V is a suitable representation of H. The morphism $f : \operatorname{Spec} R \to Y_F$ is defined to be the composition of the above morphism $\operatorname{Spec} R \to (G/H)_F$ and the zero section $(G/H)_F \to Y_F$. It is easy to see that to prove (i) it suffices to take $V = ((\operatorname{Lie} G)/(\operatorname{Lie} H))^* \oplus W^*$, where W is the representation of H in k^m. We claim that to prove (ii) it suffices to take $V = ((\operatorname{Lie} G)/(\operatorname{Lie} H))^* \oplus W^* \oplus \det W$.

To justify this claim we have to show that if u has zero image in $H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z})$ then the class of $\det P$ in $K_0(R[t,t^{-1}])$ has zero image in $K_{-1}(R)$. The commutative diagram (2.7) shows that the class of $\det P$ has zero image in $H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z})$. So it remains to prove that if $v \in \operatorname{Pic} R[t,t^{-1}]$ has zero image in $H^1_{\text{et}}(\operatorname{Spec} R, \mathbb{Z})$ then v has zero image in $K_{-1}(R)$.

According to the main result of [We2] (see Lemma 1.5.1 and Theorem 5.5 of loc.cit.), v can be represented as

$$v = v' + v'', \quad v' \in \operatorname{Im}(\operatorname{Pic} R[t] \to \operatorname{Pic} R[t,t^{-1}]),\ v'' \in \operatorname{Im}(\operatorname{Pic} R[t^{-1}] \to \operatorname{Pic} R[t,t^{-1}]).$$

As $\operatorname{Pic} R[[t]] = \operatorname{Pic} R$ the image of v in $\operatorname{Pic} R((t))$ belongs to $\operatorname{Im}(\operatorname{Pic} R[t^{-1}] \to \operatorname{Pic} R((t)))$. So the image of v in $K_0(R((t)))$ belongs to the image of $K_0(R[t^{-1}])$ in $K_0(R((t)))$. Finally, the composition $K_0(R[t^{-1}]) \to K_0(R((t))) \to K_{-1}(R)$ equals 0 (see 2.4). □

4. Families of vector bundles on a punctured polydisk

R commutative, $S := (\operatorname{Spec} R[[t_1, \ldots, t_n]]) - \underline{0}$, where $\underline{0} \subset \operatorname{Spec} R[[t_1, \ldots, t_n]]$ is defined by $t_1 = \ldots = t_n = 0$. Let C be the category of vector bundles on S (of finite rank).

If $n > 1$ then for every $\mathcal{L} \in C$ the top cohomology $H^{n-1}(S, \mathcal{L})$ is an almost projective R-module.

Related construction (it was Beilinson who suggested me that it should exist). Consider $D^{\text{perf}}(S) = K^b(C) :=$\{homotopy category of bounded complexes in C\}. Then $R\Gamma : K^b(C) \to D(R)$ can be decomposed as

$$K^b(C) \xrightarrow{(R\Gamma)_{\text{topol}}} K^b(\mathcal{T}_R) \xrightarrow{\text{Forget}} D(R),$$

where $\mathcal{T}_R$ is the category of Tate R-modules.

Remark. Commutativity is unnecessary in the theorem and in the construction (it is easy to define C, $R\Gamma$, and $(R\Gamma)_{\text{topol}}$ without the commutativity assumption).

5. Appendix: Mittag-Leffler modules (after Raynaud and Gruson)

This is a brief overview of the results by Raynaud and Gruson [RG] on Mittag-Leffler modules. All the results and proofs in this section are taken from [RG].

5.1. **Set-theoretical Mittag-Leffler condition.** Suppose we have a projective system of sets $(E_i, u_{ij} : E_j \to E_i)$, $i \in I$, where I is a directed set. Recall that according to EGA 0_{III} 13.1.2 such a system satisfies the *Mittag-Leffler condition*[7] if for every $i \in I$ there exists $j \geq i$ such that $u_{ij}(E_j) = u_{ik}(E_k)$ for all $k \geq j$. This condition is satisfied if and only if the projective system (E_i, u_{ij}) is equivalent[8] to a projective system $(\widetilde{E}_\alpha, \widetilde{u}_{\alpha\beta})$ in which the maps $\widetilde{u}_{\alpha\beta}$ are surjective.

[7]Its relation to Mittag-Leffler's theorem from complex analysis is explained at the end of §3.5 of Ch. II of [Bo1].

[8]That is, isomorphic as a pro-object of the category of sets. See §8 of [GV] for the basic notions concerning ind-objects and pro-objects

To prove the "only if" statement it suffices to put $\widetilde{E}_i := u_{ij}(E_j)$ for j big enough. Let us recall the following basic lemma.

Lemma 5.1. *If a projective system of non-empty sets $(Y_i)_{i\in I}$ parametrized by a countable directed set I satisfies the Mittag-Leffler condition then its projective limit is non-empty.*

Proof. Replacing our projective system by an equivalent one we can assume that all the maps $E_j \to E_i$ are surjective. Then the statement becomes obvious because as I is countable one can replace it by $\mathbb{N}$ equipped with the usual order. $\square$

The following example shows that the countability assumption is essential. Let I be the set of finite subsets of an uncountable set A (I is ordered by inclusion). For every finite $F \subset A$ the set E_F of injections $F \to \mathbb{N}$ is nonempty, and the restriction maps $E_{F'} \to E_F$, $F' \supset F$, are surjective, but the projective limit of the sets E_F is empty.

Remark. Of course, the statement of the lemma still holds if I is any directed set with a cofinal countable or finite subset. According to [Bo2], Ch. III, §7, Exercise 4d, there is a converse theorem: if a directed set I is such that every projective system of non-empty sets labeled by I with surjective transition maps has non-empty projective limit then I has a cofinal finite or countable subset. To show this, consider the set E_i of finite sequences $i_1, \ldots, i_n \in I$ ($n \in \mathbb{N}$ is not fixed) such that $i_n \geq i$ and there is no $m \in \{1, \ldots, n-1\}$ with $i_m \geq i$. For $i \geq j$ define a map $E_i \to E_j$ by $(i_1, \ldots, i_n) \mapsto (i_1, \ldots, i_m)$, where $m \in \{1, \ldots, n\}$ is the smallest number such that $i_m \geq j$. Thus we get a projective system of nonempty sets such that all the transition maps $E_i \to E_j$, $i \geq j$, are surjective. Finally, a point of the projective limit of the sets E_i is the same as a sequence[9] $i_1, i_2, \ldots$ of different elements of I such that the subset $\{i_1, i_2, \ldots\} \subset I$ is cofinal (i.e., for every $i \in I$ there exists k with $i_k = i$).

What was Deligne's example about?

5.2. Definition of Mittag-Leffler module.

5.2.1. Let A be a ring. Denote by $\mathcal{C}$ the category of A-modules of finite presentation. According to [RG], p.69 an A-module M is said to be a *Mittag-Leffler module* if every morphism $f : F \to M$, $F \in \mathcal{C}$, can be decomposed as $F \xrightarrow{u} G \to M$, $G \in \mathcal{C}$, so that for every decomposition of f as $F \xrightarrow{u'} G' \to M$, $G' \in \mathcal{C}$, there is a morphism $\varphi : G' \to G$ such that $u = \varphi u'$.

5.2.2. Suppose that $M = \varinjlim M_i$, $i \in I$, where I is a directed ordered set and $M_i \in \mathcal{C}$. For $i \leq j$ let u_{ij} denote the morphism $M_i \to M_j$. According to loc.cit, M is a Mittag-Leffler module if and only if

$$\forall i \in I\, \exists j \geq i\, \forall k \geq i \ \ u_{ij} = \varphi_{ijk} u_{ik} \text{ for some } \varphi_{ijk} : M_k \to M_j \tag{5.1}$$

A similar statement holds if I is a filtering category. If I is the category of all morphisms from objects of $\mathcal{C}$ to M and $M_i \in \mathcal{C}$ is the source of the morphism i then the statement is tautological.

Remark. Property (5.1) of inductive systems (M_i), $M_i \in \mathcal{C}$, makes sense if $\mathcal{C}$ is replaced by any category $\mathcal{C}'$. It is easy to see that (5.1) depends only on the equivalence class of the inductive system, so there is a notion of Mittag-Leffler ind-object. If $\mathcal{C}'$ is dual to the category of sets (i.e., if one has a projective system of sets $(E_i, u_{ij} : E_j \to E_i)$) then (5.1) is nothing but the Mittag-Leffler condition from 5.1.

[9] This sequence may be infinite or finite (and even empty if $I = \emptyset$).

5.2.3. Suppose that $M = \varinjlim M_i$, $M_i \in \mathcal{C}$. According to [RG] M is a Mittag-Leffler module if and only if for any contravariant functor Φ from $\mathcal{C}$ to the category of sets the projective system $(\Phi(M_i))$ satisfies the Mittag-Leffler condition (to prove the "if" statement consider the functor $\Phi(N) = \operatorname{Hom}(N, \prod_i M_i)$ or $\widetilde{\Phi}(N) = \bigsqcup_i \operatorname{Hom}(N, M_i)$).

Assume that M is flat. Put $M_i^* = \operatorname{Hom}(M_i, A)$. According to [RG], M is a Mittag-Leffler module if and only if the projective system (M_i^*) satisfies the Mittag-Leffler condition. This is clear if the modules M_i are projective. The general case follows by the Govorov–Lazard lemma (there is an inductive system equivalent to (M_i) which consists of finitely generated projective modules).

5.3. **Definition of strictly Mittag-Leffler module.** According to [RG], p.74 a module M is *strictly Mittag-Leffler* if for every $f : F \to M$, $F \in \mathcal{C}$, there exists $u : F \to G$, $G \in \mathcal{C}$, such that $f = gu$ and $u = hf$ for some $g : G \to M$, $h : M \to G$ (recall that $\mathcal{C}$ is the category of modules of finite presentation). If $M = \varinjlim M_i$, $M_i \in \mathcal{C}$, and $u_{ij} : M_i \to M_j$, $u_i : M_i \to M$ are the canonical maps then M is strictly Mittag-Leffler if and only if for every i there exists $j \geq i$ such that $u_{ij} = \varphi_{ij} u_j$ for some $\varphi_{ij} : M \to M_j$.

5.4. **Relations between various classes of modules.** Clearly a stritly Mittag-Leffler is Mittag-Leffler. It is also easy to see (?) that a projective module is strictly Mittag-Leffler and flat. The converse statements are not true in general (see 5.6). The following theorem is due to Raynaud and Gruson [RG].

Theorem 5.2. *The following conditions are equivalent:*

(i) M is a flat Mittag-Leffler module;

(ii) every finite or countable subset of M is contained in a countably generated projective submodule $P \subset M$ such that M/P is flat;

(iii) every finite subset of M is contained in a projective submodule $P \subset M$ such that M/P is flat.

In particular, a projective module is Mittag-Leffler and a countably generated[10] *flat Mittag-Leffler module is projective.*

The implication (iii)$\Rightarrow$(i) is easy. (It suffices to show that if F and F' are modules of finite presentation and $\varphi : F \to F'$, $\psi : F' \to M$ are morphisms such that $\psi\varphi(F) \subset P$ then there exists $\widetilde{\psi} : F' \to M$ such that $\widetilde{\psi}(F') \subset P$ and $\widetilde{\psi}\varphi = \psi\varphi$; use the fact that $\operatorname{Hom}(L, M) \to \operatorname{Hom}(L, M/P)$ is surjective for every L of finite presentation, in particular for $L = \operatorname{Coker} \varphi$).

The implication (i)$\Rightarrow$(ii) is proved in [RG], p.73–74. The key argument is as follows. Suppose we have a sequence $P_1 \to P_2 \to \dots$ where $P_1, P_2, \dots$ are finitely generated projective modules and the projective system (P_i^*) satisfies the Mittag-Leffler property. To prove that $P := \varinjlim P_i$ is projective one has to show that for every exact sequence $0 \to N' \to N \to N'' \to 0$ the map $\operatorname{Hom}(P, N) \to \operatorname{Hom}(P, N'')$ is surjective. For each i the sequence

$$0 \to \operatorname{Hom}(P_i, N') \to \operatorname{Hom}(P_i, N) \xrightarrow{\pi_i} \operatorname{Hom}(P_i, N'') \to 0$$

is exact and the problem is to show that the projective limit of these sequences is exact. According to EGA 0_{III} 13.2.2 this follows from the Mittag-Leffler property of the projective system $(?_i)$, $?_i := \operatorname{Hom}(P_i, N')$ and the countability of the set of indices i. In other words, if $f \in \operatorname{Hom}(P, N'')$ and f_i is the image of f in $\operatorname{Hom}(P_i, N'')$ then to show that the projective limit of the sets $f \in \operatorname{Hom}(P, N'')$ belongs .

Here is another proof of the projectivity of P (in fact, another version of the same proof). Denote by f_i the map $P_i \to P_{i+1}$. The Mittag-Leffler property means that after replacing the sequence $\{P_i\}$ by its subsequence there exist $g_i : P_{i+1} \to P_i$ such that $g_{i+1} f_{i+1} f_i = f_i$. Put

[10]The countable generatedness assumption is essential; see 5.6.

$\mathcal{P} := \bigoplus_i P_i$. Denote by $f : \mathcal{P} \to \mathcal{P}$ and $g : \mathcal{P} \to \mathcal{P}$ the operators induced by the f_i and g_i. Then $gf^2 = f$. We have the exact sequence

$$0 \to \mathcal{P} \xrightarrow{1-f} \mathcal{P} \to P \to 0$$

Since $\mathcal{P}$ is projective it suffices to show that this sequence splits, i.e., there is an $h : \mathcal{P} \to \mathcal{P}$ such that $h(1-f) = 1$. Indeed, set $h = 1 - (1-g)^{-1}gf$ and use the equality $gf^2 = f$.[11]

5.5. **Mittag-Leffler $\mathcal{O}$-modules on schemes.**

Proposition 5.3. *Let $A \to B$ be a morphism of commutative rings. If M is a Mittag-Leffler A-module then $B \otimes_A M$ is a Mittag-Leffler B-module. If B is faithfully flat over A then the converse is true.*

This is proved in [RG]. The proof is easy: represent M as an inductive limit of modules of finite presentation and use 5.2.2.

So the notion of a Mittag-Leffler $\mathcal{O}$-module on a scheme is clear **as well as the notion of Mittag-Leffler $\mathcal{O}^p$-module on an ind-scheme, but where should I mention this??**.

Proposition 5.4. *A flat Mittag-Leffler $\mathcal{O}$-module $\mathcal{F}$ of countable type on a noetherian scheme S is locally free. If S is affine and connected, and $\mathcal{F}$ is of infinite type then $\mathcal{F}$ is free.*

This is an immediate consequence of Theorems 5.2 and 2.1.

5.6. **Examples.**

5.6.1. *Mittag-Leffler modules over Dedekind rings.* Let A be a Dedekind ring and K be its field of fractions. A flat (i.e., torsion-free) A-module M is Mittag-Leffler if and only if for every finite-dimensional subspace $V \subset M \otimes_A K$ the A-submodule $V \cap M$ is finitely generated.

5.6.2. *The module A^I.* (i) According to [RG], p.77, 2.4.1 for every noetherian A and projective A-module P the A-module $P^* := \mathrm{Hom}_A(P, A)$ is strictly Mittag-Leffler and flat. To prove that P^* is strictly Mittag-Leffler one can argue as follows: for any $f : F \to P^*$ with F of finite type the image of $f^* : P \to F^*$ is generated by some $l_1, \ldots, l_n \in F^*$; the l_i define $u : F \to A^n$ such that $f = gu$ and $u = hf$ for some $g : A^n \to P^*$, $h : P^* \to A^n$.

In particular, if A is noetherian then for every set I the A-module A^I is strictly Mittag-Leffler and flat.

(ii) According to Baer (see p.48 and p.82 of [Ka2]), if A is a Dedekind ring and not a field then A^I is not projective for infinite I. Indeed, we can assume that I is countable. Fix a non-zero prime ideal $\mathfrak{p} \subset A$ and consider the submodule M of elements $a = (a_i) \in A^I$ such that $a_i \to 0$ in the $\mathfrak{p}$-adic topology. If A^I were projective the localization $M_{\mathfrak{p}}$ would be free. Since $M/\mathfrak{p}M$ has countable dimension $M_{\mathfrak{p}}$ would have countable rank. But M contains a submodule isomorphic to A^I, so $(A^I)_{\mathfrak{p}}$ would have countable rank. This is impossible because the dimension of $(A^I)_{\mathfrak{p}}/\mathfrak{p} \cdot (A^I)_{\mathfrak{p}} = (A/\mathfrak{p})^I$ is uncountable.

(iii) Suppose that A is finitely generated over $\mathbb{Z}$ or over a field[12]. If A is not Artinian and I is infinite then A^I is not projective: use 5.6.2(ii) and the existence of a Dedekind ring B finite over A.

What is the correct place for the following paragraph?

(b) If L is a non-projective flat Mittag-Leffler module then there exists a non-split exact sequence $0 \to N' \to N \to L \to 0$ where N and N' are flat Mittag-Leffler modules. Indeed, if N is a projective module and $N \to L$ is an epimorphism then it does not split and $\mathrm{Ker}(N \to L)$ is Mittag-Leffler ([RG], p.71, 2.1.6).

[11] D.Arinkin noticed that it is clear a priori that if f and g are elements of a (non-commutative) ring R such that $gf^2 = f$ and $1-g$ has a left inverse then $1-f$ has a left inverse. Indeed, denote by $\mathbf{1}$ the image of 1 in $R/R(1-f)$. Then $f\mathbf{1} = \mathbf{1}$, $gf^2\mathbf{1} = g\mathbf{1}$, so $g\mathbf{1} = \mathbf{1}$ and therefore $\mathbf{1} = 0$.

[12] We do not know whether it suffices to assume A noetherian.

5.6.3. *Non-strictly Mittag-Leffler modules.* (i) It is noticed in [RG] that if

$$0 \to A \xrightarrow{f} M' \to M \to 0 \tag{5.2}$$

is a non-split exact sequence of A-modules and M is flat and Mittag-Leffler then M' is Mittag-Leffler but not strictly Mittag-Leffler (an example of this situation will be given in (ii) below). Indeed, if M' were strictly Mittag-Leffler then there would exist a module G of finite presentation and a morphism $u : A \to G$ such that $f = gu$ and $u = hf$ for some $g : G \to M'$, $h : M' \to G$. Since M is a direct limit of finitely generated projective modules one can assume that $\operatorname{Im} g \subset \operatorname{Im} f$. Then gh would define a splitting of (5.2), i.e., one gets a contradiction.

Here is another argument (**which uses material from the next section??**). The fiber of $F_{M'}$ over $0 \in F_M$ is a closed subscheme of $F_{M'}$ canonically isomorphic to $\operatorname{Spec} A \times \mathbb{A}^1$; if (5.2) is non-split then the projection $\operatorname{Spec} A \times \mathbb{A}^1 \to \mathbb{A}^1$ cannot be extended to a function $F_{M'} \to \mathbb{A}^1$, so by 6.2.4 M' is not strictly Mittag-Leffler.

(ii) Let A be a Dedekind ring which is neither a field nor a complete local ring. Then according to [RG], p.76 there is a non-split exact sequence (5.2) such that M is a flat strictly Mittag-Leffler A-module. Here is a construction. Let K denote the field of fractions of A. Fix a non-zero prime ideal $\mathfrak{p} \subset A$ and consider the completions $\widehat{A}_\mathfrak{p}$, $\widehat{K}_\mathfrak{p}$; then $\widehat{A}_\mathfrak{p} \neq A$, $\widehat{K}_\mathfrak{p} \neq K$. Denote by M the module of sequences (a_n) such that $a_n \in \mathfrak{p}^{-n}$ and (a_n) converges in $\widehat{K}_\mathfrak{p}$; we have the morphism $\lim : M \to \widehat{K}_\mathfrak{p}$. Notice that M is a strictly Mittag-Leffler module[13]. Indeed, according to 5.6.2(i) above $\prod_{n=1}^\infty \mathfrak{p}^{-n}$ is strictly Mittag-Leffler and $(\prod_{n=1}^\infty \mathfrak{p}^{-n})/M$ is flat, so M is strictly Mittag-Leffler. We claim that $\operatorname{Ext}(M, A) \neq 0$, i.e., the morphism $\varphi : \operatorname{Hom}(M, K) \to \operatorname{Hom}(M, K/A)$ is not surjective. More precisely, let $l : M \to K/A$ be the composition of $\lim : M \to \widehat{K}_\mathfrak{p}$ and the morphisms $\widehat{K}_\mathfrak{p} \to \widehat{K}_\mathfrak{p}/\widehat{A}_\mathfrak{p} \hookrightarrow K/A$. We will show that $l \notin \operatorname{Im} \varphi$.

Suppose that l comes from $\widetilde{l} : M \to K$. The restriction of $\widetilde{l}$ to $\mathfrak{p}^{-n} \subset M$ defines $c_n \in \operatorname{Hom}(\mathfrak{p}^{-n}, A) = \mathfrak{p}^n$. Then $\widetilde{l} = \widetilde{l}'$ where $\widetilde{l}' : M \to K_\mathfrak{p}$ maps $(a_n) \in M$ to

$$\sum_{n=1}^\infty c_n a_n + \lim_{n\to\infty} a_n \,. \tag{5.3}$$

Indeed, $\widetilde{l}' - \widetilde{l}$ is a morphism $M/M_0 \to \widehat{A}_\mathfrak{p}$ where M_0 is the set of $(a_n) \in M$ such that $a_n = 0$ for n big enough; on the other hand, $\operatorname{Hom}(M/M_0, \widehat{A}_\mathfrak{p}) = 0$ because M/M_0 is $\mathfrak{p}$-divisible (i.e., $\mathfrak{p}M + M_0 = M$). Since $\widetilde{l}' = \widetilde{l}$ the expression (5.3) belongs to $K \subset \widehat{K}_\mathfrak{p}$ for every sequence $(a_n) \in M$. This is impossible (consider separately the case where the number of nonzero c_n's is finite and the case where it is infinite).

Remark. In (ii) we had to exclude the case where A is a complete local ring. The true reason for this is explained by the following results.

(a) According to [Je] if A is a complete local noetherian ring, M is a flat A-module, and N is a finitely generated A-module then $\operatorname{Ext}(M, N) = 0$.

(b) If A is a projective limit of Artinian rings then every flat Mittag-Leffler A-module is strictly Mittag-Leffler (see [RG], p.76, Remark 4 from 2.3.3).

6. Appendix: Mittag-Leffler modules and ind-schemes

Where should I write that A is a commutative ring?

In this section (whose results are not used in the rest of the work) we show that the notion of flat Mittag-Leffler module is, in some sense, a linearized version of the notion of formally smooth ind-scheme of ind-finite type (see 6.1, 6.2.2, 6.2.3). Using the fact that countably generated flat Mittag-Leffler modules are projective we describe formally smooth affine $\aleph_0$-formal schemes of ind-finite type (see 6.2.10, 6.2.11).

[13]The fact that M is a Mittag-Leffler module is clear: A is a Dedekind ring, M is flat, and for every finite-dimensional subspace $V \subset M \otimes K$ the module $V \cap M$ is finitely generated

In **??** we assumed that "ind-scheme" meant "ind-scheme over a field". In this section we drop this assumption.

6.1. **Where should I put this?** Consider the following two classes of functors from the category of A-modules to the category of abelian groups:

1) For an A-module M one has the functor

$$L \mapsto L \otimes_A M; \tag{6.1}$$

2) For a projective system of A-modules N_i (where i belong to a directed ordered set) one has the functor

$$L \mapsto \varinjlim_i \operatorname{Hom}(N_i, L) \tag{6.2}$$

6.1.1. *Proposition.* (i) The functor (6.1) is isomorphic to a functor of the form (6.2) if and only if M is flat.

(ii) The functor (6.1) is isomorphic to the functor (6.2) corresponding to a projective system (N_i) with surjective transition maps $N_j \to N_i$, $i \leq j$, if and only if M is a flat Mittag-Leffler module.

(iii) The functor (6.2) corresponding to a projective system (N_i) with surjective transition maps $N_j \to N_i$, $i \leq j$, is isomorphic to a functor of the form (6.1) if and only if the functor (6.2) is exact and the modules N_i are finitely generated.

Proof. If (6.1) and (6.2) are isomorphic then (6.1) is left exact, so M is flat. If M is flat then by the Govorov–Lazard lemma $M = \varinjlim P_i$ where the modules P_i are projective and finitely generated, so the functor (6.2) corresponding to $N_i = P_i^*$ is isomorphic to (6.1).

We have proved (i). To deduce (ii) from (i) notice that for P_i as above the projective system (P_i^*) is equivalent to a projective system (N_i) with surjective transition maps $N_j \to N_i$ if and only if (P_i^*) satisfies the Mittag-Leffler condition (see 5.1).

To prove (iii) notice that functors of the form (6.1) are those additive functors which are right exact and commute with infinite direct sums (then they commute with inductive limits). A functor of the form (6.2) is right exact if and only if it is exact. If the modules N_i are finitely generated then (6.2) commutes with infinite direct sums. If the transition maps $N_j \to N_i$ are surjective and (6.2) commutes with inductive limits then the modules N_i are finitely generated. □

6.1.2. According to 6.1.1 a flat Mittag-Leffler module is "the same as" an equivalence class of projective systems (N_i) of finitely generated modules with surjective transition maps $N_j \to N_i$, $i \leq j$, such that the functor (6.2) is exact. More precisely, $M = \varinjlim_i \operatorname{Hom}(N_i, A)$ (then the functors (6.1) and (6.2) are isomorphic).

6.2. **How should I call this subsection??**

Proposition 6.1. *Let X be a formally smooth ind-scheme of ind-finite type over a field. Then the $\mathcal{O}^p$-modules Θ_X, $\mathcal{D}_X$, $\mathcal{D}_{iX}$ (see* **??***) are flat Mittag-Leffler modules.*

Proof. Let us prove that the restriction of $\mathcal{D}_X$ to a closed subscheme $Y \subset X$ is a flat Mittag-Leffler $\mathcal{O}_Y$-module (the same argument works for Θ_X and $\mathcal{D}_{iX}$). We can assume that Y is affine (otherwise replace X by $X \setminus F$ for a suitable closed $F \subset Y$). According to 6.1.1 it suffices to prove that

(1) The functor $L \mapsto L \otimes \mathcal{D}_X$ defined on the category of $\mathcal{O}_Y$-modules is exact,
(2) it has the form (6.2) where the $\mathcal{O}_Y$-modules N_i are coherent.

By definition, $L \otimes \mathcal{D}_X$ is the sheaf $\mathcal{D}(L)$ defined by (**??**). So (ii) is clear. We have proved (i) in **??**. □

6.2.1. *Proposition.* Let X be a formally smooth $\aleph_0$-ind-scheme of ind-finite type over a field, $Y \subset X$ a locally closed subscheme. Then the restriction of Θ_X to Y is locally free. If Y is affine and connected, and the restriction of Θ_X to Y is of infinite type then it is free.

This follows from Propositions 6.1 and 5.4.

6.2.2. *Proposition.* Let A be a ring, M an A-module. Define an "A-space" F_M (i.e., a functor from the category of A-algebras to that of sets) by $F_M(R) = M \otimes R$. Then F_M is an ind-scheme if and only if M is a flat Mittag-Leffler module. In this case F_M is formally smooth over A and of ind-finite type over A.

Proof. If M is a flat Mittag-Leffler module then by 6.1.1(ii) F_M is an ind-scheme and by 6.1.1(iii) it is of ind-finite type over A. Formal smoothness follows from the definition. Now suppose that F_M is an ind-scheme. Represent F_M as $\varinjlim S_i$ where the S_i are closed subshemes of F_M containing the zero section $0 \in F_M(A)$. Denote by N_i the restriction of the cotangent sheaf of S_i to $0 : \operatorname{Spec} A \hookrightarrow S_i$. Then the functor (6.2) is isomorphic to (6.1), so by 6.1.1(ii) M is a flat Mittag-Leffler module. □

Remark. If M is an arbitrary flat A-module then M is an inductive limit of a directed family of finitely generated projective A-modules M_i, so $F_M = \varinjlim F_{M_i}$ is an ind-scheme in the broad sense (the morphisms $F_{M_i} \to F_{M_j}$ are not necessarily closed embeddings). It is easy to see that if F_M is an ind-scheme in the broad sense then M is flat.

6.2.3. *Proposition.* Let $(N_i)_{i \in I}$ be a projective system of finitely generated A-modules parametrized by a directed set I such that all the transition maps $N_j \to N_i$, $j \geq i$, are surjective. Put $\mathbb{A}(N_i) := \operatorname{Spec} \operatorname{Sym}(N_i)$, $S := \varinjlim_i \mathbb{A}(N_i)$. The ind-scheme S is formally smooth over A if and only if S is isomorphic to the ind-scheme F_M from 6.2.2 corresponding to a flat Mittag-Leffler module M.

Proof. S is formally smooth if and only if the functor (6.2) is exact (apply the definition of formal smoothness to A-algebras of the form $A \oplus J$, $A \cdot J \subset J$, $J^2 = 0$). Now use 6.1.1(iii). □

6.2.4. *Proposition.* Let M be a flat Mittag-Leffler module, F_M the ind-scheme from 6.2.2. The following conditions are equivalent:

(1) the pro-algebra corresponding to F_M (see **??**(i)) is a topological algebra;
(2) M is a strictly Mittag-Leffler module in the sense of **??**.

Proof. Represent M as $\varinjlim P_i$ where the modules P_i are finitely generated and projective. Put $N_i := \operatorname{Im}(P_j^* \to P_i^*)$ where j is big enough. Consider the following conditions:

(a) the maps $\varphi_i : \varprojlim_r \operatorname{Sym}(N_r) \to \operatorname{Sym}(N_i)$ are surjective;
(b) $\operatorname{Im} \varphi_i \supset N_i$ for every i;
(c) the map $\varprojlim_r N_r \to N_i$ is surjective for every i;
(d) for every i there exists $j \geq i$ such that the images of $\operatorname{Hom}(M, A)$ and $\operatorname{Hom}(P_j, A)$ in $\operatorname{Hom}(P_i, A)$ are equal.

Clearly (i)⇔(a)⇔(b)⇔(c)⇔(d). For $i \leq j$ consider the maps $u_{ij} : P_i \to P_j$ and $u_i : P_i \to M$. To show that (d)⇔(ii) it suffices to prove that the images of $\operatorname{Hom}(M, A)$ and $\operatorname{Hom}(P_j, A)$ in $\operatorname{Hom}(P_i, A)$ are equal if and only if $u_{ij} = \varphi u_j$ for some $\varphi : M \to P_j$. To prove the "only if" statement notice that the images of $\operatorname{Hom}(M, P_j)$ and $\operatorname{Hom}(P_j, P_j)$ in $\operatorname{Hom}(P_i, P_j)$ are equal and therefore the image of $\mathrm{id} \in \operatorname{Hom}(P_j, P_j)$ in $\operatorname{Hom}(P_i, P_j)$ is the image of some $\varphi \in \operatorname{Hom}(M, P_j)$. □

6.2.5. Before passing to the structure of formally smooth affine $\aleph_0$-ind-schemes let us discuss the relation between the definition of formal scheme from **??** and Grothendieck's definition (see EGA I). They are not equivalent even in the affine case. A formal affine scheme in our sense is an ind-scheme X that can be represented as $\varinjlim \operatorname{Spec} R_\alpha$ where (R_α) is a projective system of rings such that the maps $u_{\alpha\beta} : R_\beta \to R_\alpha$, $\beta \geq \alpha$, are surjective and the elements of $\operatorname{Ker} u_{\alpha\beta}$ are nilpotent. Grothendieck requires the possibility to represent X as $\varinjlim \operatorname{Spec} R_\alpha$ so that the maps

$$\varprojlim_{\beta} R_\beta \to R_\alpha \tag{6.3}$$

are surjective[14] and the ideals $\operatorname{Ker} u_{\alpha\beta}$ are nilpotent. A reasonable $\aleph_0$-formal scheme in our sense is a formal scheme in the sense of EGA I. A quasi-compact formal scheme in Grothendieck's sense having a fundamental system of "defining ideals (English?)" ("Idéaux de définition"; see EGA I 10.5.1) is a formal scheme in our sense; in particular, this is true for noetherian formal schemes in the sense of EGA I.

Since we are mostly interested in affine $\aleph_0$-formal schemes of ind-finite type over a field the difference between our definition and that of EGA I is not essential.

6.2.6. *Proposition.* Let X be a formally smooth $\aleph_0$-ind-scheme of ind-finite type over A, $S \subset X$ a closed subscheme such that $S \to \operatorname{Spec} A$ is an isomorphism. Suppose that $X_{\mathrm{red}} = S_{\mathrm{red}}$ (in particular, X is a formal scheme). Let M denote the A-module of global sections of the restriction of the relative tangent sheaf $\Theta_{X/A}$ to S. Then M is a countably generated projective module and (X, S) is isomorphic to the completion $\widehat{F}_M$ of the ind-scheme F_M (see 6.2.2) along the zero section.

Remark. The $\mathcal{O}^p$-module $\Theta_{X/A}$ on a formally smooth ind-scheme X of ind-finite type over A is defined just as in the case $A = \mathbb{C}$ (see **??**, **??**).

Proof. Just as in 6.1 one shows that M is a flat Mittag-Leffler module. The $\aleph_0$ assumption implies that M is countably generated. By Theorem 5.2 M is projective.

Represent X as $\varinjlim X_n$, $n \in \mathbb{N}$, where the X_n are closed subschemes of X containing S such that $X_n \subset X_{n+1}$. Let $X^{(1)}$ be the first infinitesimal neighbourhood of S in X, i.e., $X^{(1)}$ is the union of the first infinitesimal neighbourhoods of S in X_n, $n \in \mathbb{N}$. Clearly $X^{(1)} = F_M^{(1)}$:=the first infinitesimal neighbourhood of $0 \in F_M$. The embedding $X^{(1)} \to \widehat{F}_M$ can be extended to a morphism $\varphi : X \to \widehat{F}_M$ (to construct φ define $\varphi_n : X_n \to \widehat{F}_M$ so that $\varphi_n|_{X_{n-1}} = \varphi_{n-1}$ and the restriction of φ_n to $X_n \cap X^{(1)}$ is the canonical embedding $X_n \cap X^{(1)} \hookrightarrow F_M^{(1)}$; this is possible because $\widehat{F}_M$ is formally smooth over A). Quite similarly one extends the embedding $F_M^{(1)} = X^{(1)} \hookrightarrow X$ to a morphism $\psi : \widehat{F}_M \to X$. Since φ and ψ induce isomorphisms between $F_M^{(1)}$ and $X^{(1)}$ we see that φ and ψ are ind-closed embeddings and $\varphi\psi$ is an isomorphism. So φ and ψ are isomorphisms. □

6.2.7. *Example.* We will construct a pair (X, S) satisfying the conditions of 6.2.6 except the $\aleph_0$ assumption such that (X, S) is not A-isomorphic to a formal scheme of the form $\widehat{F}_M$.

Suppose we have a nontrivial extension of flat Mittag-Leffler modules

$$0 \to N' \to N \to L \to 0. \tag{6.4}$$

Such extensions do exist for "most" rings A; see **??**[15], 5.6.2(iii), 5.6.3(ii). After tensoring (6.4) by $A[t]$ we get the extension $0 \to N'[t] \to N[t] \to L[t] \to 0$. Multiplying this extension by t we get

[14] This is stronger than surjectivity of $u_{\alpha\beta}$; e.g., if M is a flat Mittag-Leffler A-module that is not strictly Mittag-Leffler then the arguments from 6.1.1 show that the completion of F_M along the zero section cannot be represented as $\varinjlim \operatorname{Spec} R_\alpha$ so that the maps (6.3) are surjective.

[15] This was a reference to the following: If L is a non-projective flat Mittag-Leffler module then there exists a non-split exact sequence $0 \to N' \to N \to L \to 0$ where N and N' are flat Mittag-Leffler modules. Indeed, if N

$0 \to N'[t] \to Q \to L[t] \to 0$. The ind-scheme F_Q is formally smooth over $A[t]$ and therefore over A. Let $S \subset F_Q$ be the image of the composition of the zero sections $\operatorname{Spec} A \to \operatorname{Spec} A[t] \to F_Q$. Denote by X the completion of F_Q along S.

Before proving the desired property of (X, S) let us describe X more explicitly. For an A-algebra R an R-point of F_Q is a pair consisting of an A-morphism $A[t] \to R$ and an element of $Q \otimes_{A[t]} R$. In other words, an R-point of F_Q is defined by a triple (n, l, t), $n \in N \otimes_A R$, $l \in L \otimes_A R$, $t \in R$, such that

$$\pi(n) = tl \tag{6.5}$$

where π is the projection $N \otimes_A R \to L \otimes_A R$.

So F_Q is a closed ind-subscheme of $F_N \times F_L \times \mathbb{A}^1$ defined by the equation (6.5). Therefore $X \subset \widehat{F}_N \times \widehat{F}_L \times \widehat{\mathbb{A}}^1$ is defined by the same equation (6.5) (here $\widehat{\mathbb{A}}^1$ is the completion of $\mathbb{A}^1$ at $0 \in \mathbb{A}^1$).

Now suppose that (X, S) is A-isomorphic to $\widehat{F}_M$. Then M is the module of global sections of the restriction of $\Theta_{X/A}$ to S. Linearizing (6.5) we see that

$$M = N' \oplus L \oplus A \subset N \oplus L \oplus A. \tag{6.6}$$

The composition

$$\widehat{F}_M \xrightarrow{\sim} X \hookrightarrow \widehat{F}_N \times \widehat{F}_L \times \mathbb{A}^1 \tag{6.7}$$

is defined by a "Taylor series" $\sum_{n=1}^{\infty} \varphi_n$ where φ_n is a homogeneous polynomial map $M \to N \oplus L \oplus A$ of degree n; clearly φ_1 is the embedding (6.6). Put $f = \mathrm{pr}_N \circ \varphi_2$ where pr_N is the projection $N \oplus L \oplus A \to N$. Since $M = N' \oplus L \oplus A$ the module of quadratic maps $M \to N$ contains as a direct summand the module of bilinear maps $L \times A \to N$, i.e., $\operatorname{Hom}(L, N)$. The image of f in $\operatorname{Hom}(L, N)$ defines a splitting of (6.4) (use the fact that the morphism (6.7) factors through the ind-subscheme $X \subset \widehat{F}_N \times \widehat{F}_L \times \mathbb{A}^1$ defined by the equation (6.5)). So we get a contradiction.

6.2.8. *Proposition.* Let X be a formally smooth ind-scheme over a ring A. Suppose that one of the following two assumptions holds:

(i) X is ind-affine;
(ii) A is noetherian and X is of ind-finite type over A.

Then X is the union of a directed family of ind-closed $\aleph_0$-ind-schemes formally smooth over A.

Proof. It suffices to show that for every increasing sequence of closed subschemes $Y_n \subset X$ there is an ind-closed $\aleph_0$-ind-scheme $Y \subset X$ formally smooth over A such that $Y \supset Y_n$ for all n.

Suppose that X is ind-affine. Then each Y_n is affine. Represent Y_n as a closed subscheme of a formally smooth scheme V_n over A (e.g., represent the coordinate ring of Y_n as a quotient of a polynomial algebra over A). Let $Y_n' \subset V_n$ be the first infinitesimal neighbourhood of Y_n in V_n. Since X is formally smooth the morphism $Y_n \hookrightarrow X$ extends to a morphism $Y_n' \to Z_n \subset X$ for some closed subscheme $Z_n \subset X$. Put $Y_n^{(2)} := Z_1 \cup \ldots \cup Z_n$. Now apply the above construction to $(Y_n^{(2)})$ and get a new sequence $(Y_n^{(3)})$, etc. The union of all $Y_n^{(k)}$ is formally smooth over A.

If X is ind-quasicompact but not ind-affine an obvious modification of the above construction yields an ind-closed $\aleph_0$-ind-scheme $Y \subset X$ containing all the Y_n such that for any affine scheme S over A and any closed subscheme $S_0 \subset S$ defined by an Ideal $\mathcal{I} \subset \mathcal{O}_S$ with $\mathcal{I}^2 = 0$ every A-morphism $S_0 \to Y$ extends *locally* to a morphism $S \to Y$. If assumption (ii) holds then this implies the existence of a global extension. □

is a projective module and $N \to L$ is an epimorphism then it does not split and $\operatorname{Ker}(N \to L)$ is Mittag-Leffler ([RG], p.71, 2.1.6).

6.2.9. **Should I rather denote the ground field by k?? In this case I should avoid using k for other purposes!**

We are going to describe formally smooth affine $\aleph_0$-formal schemes of ind-finite type over a field C (according to 6.2.8 the general case can, in some sense, be reduced to the $\aleph_0$ case). First of all we have the following examples.

(0) Put $R_{mn} := C[x_1, \ldots, x_m][[x_{m+n}, \ldots, x_{m+n}]]$. Then $\operatorname{Spf} R_{mn}$ is a formally smooth affine $\aleph_0$-formal scheme over C.

(i) Let $I \subset R_{mn}$ be an ideal, $A := R_{mn}/I$. Denote by $\mathcal{I}$ the sheaf of ideals on $\operatorname{Spf} R_{mn}$ corresponding to I. Of course, $\operatorname{Spf} A$ is an affine $\aleph_0$-formal scheme of ind-finite type over C. It is formally smooth if and only if for every $u \in \operatorname{Spf} A$ the stalk of $\mathcal{I}$ at u is generated by some $f_1, \ldots, f_r \in I$ such that the Jacobi matrix $(\frac{\partial f_i}{\partial x_j}(u))$ has rank r.

(ii) Suppose that A is as in (i) and $\operatorname{Spf} A$ is formally smooth. Then $\operatorname{Spf} A[[y_1, y_2, \ldots]]$ is a formally smooth affine $\aleph_0$-formal scheme of ind-finite type over C.

In 6.2.10 and 6.2.11 we will show that every connected formally smooth affine $\aleph_0$-formal scheme of ind-finite type over a field is isomorphic to a formal scheme from Example (i) or (ii).

6.2.10. *Proposition.* Let X be a formally smooth affine formal scheme of ind-finite type over a field C such that Θ_X is coherent (i.e., the restriction of Θ_X to every closed subscheme of X is finitely generated). Then X is isomorphic to a formal scheme from Example 6.2.9(i).

Proof. Represent X as $\varinjlim \operatorname{Spec} A_i$ so that for $i \leq j$ the morphism $A_j \to A_i$ is surjective with nilpotent kernel. The algebras A_i are of finite type. We can assume that the set of indices i has a smallest element 0. Put $I_i := \operatorname{Ker}(A_i \to A_0)$.

Lemma 6.2. *For every $k \in \mathbb{N}$ there exists i_1 such that the morphisms $A_i/I_i^k \to A_{i_1}/I_{i_1}^k$ are bijective for all $i \geq i_1$.*

Assuming the lemma set $A_{(k)} := A_i/I_i^k$ for i big enough, $I_{(k)} := \operatorname{Ker}(A_{(k)} \to A_0)$. Clearly $A_{(1)} = A_0$, $A_{(k)} = A_{(k+1)}/I_{(k+1)}^k$, $I_{(k)} = I_{(k+1)}/I_{(k+1)}^k$. One has $X = \operatorname{Spf} A$, $A := \varprojlim A_{(k)}$. Choose generators $\bar{x}_1, \ldots, \bar{x}_m$ of the algebra $A_{(1)} = A_0$ and generators $\bar{x}_{m+1}, \ldots, \bar{x}_{m+n}$ of the A_0-module $I_{(2)}$. Lift $\bar{x}_1, \ldots, \bar{x}_{m+n}$ to $\widetilde{x}_1, \ldots, \widetilde{x}_{m+n} \in A$. Put $R_{mn} := C[x_1, \ldots, x_m][[x_{m+1}, \ldots, x_{m+n}]]$. There is a unique continuous homomorphism $f : R_{mn} \to A$ such that $x_i \mapsto \widetilde{x}_i$. Clearly f is surjective. Moreover, f induces surjections $\mathfrak{a}^k \to \operatorname{Ker}(A \to A_{(k)})$, where $\mathfrak{a} \subset R_{mn}$ is the ideal generated by $x_{m+1}, \ldots, x_{m+n}$. So f is an open map. Therefore f induces a topological isomorphism between A and a quotient of R_{mn}. The proposition follows.

It remains to prove the lemma. There exists i_0 such that for every $i \geq i_0$ the morphism $\operatorname{Spec} A_{i_0} \to \operatorname{Spec} A_i$ induces isomorphisms between tangent spaces (indeed, since the restriction of Θ_X to $\operatorname{Spec} A_0$ is finitely generated the functor (6.2) corresponding to the A_0-modules $N_i := \Omega_i \otimes_{A_i} A_0$ is isomorphic to the functor $L \mapsto \operatorname{Hom}(Q, L)$ for some A_0-module Q, so there exists i_0 such that $N_i = N_{i_0}$ for $i \geq i_0$). We can assume that $i_0 = 0$. Put $Y_i := \operatorname{Spec} A_i/I_i^k$ (in particular, $Y_0 = \operatorname{Spec} A_0$). The morphisms $Y_0 \to Y_i$ induce isomorphisms between tangent spaces.

Represent A_0 as $C[x_1, \ldots, x_n]/J$ and set $\widetilde{Y}_0 := \operatorname{Spec} C[x_1, \ldots, x_n]/J^k$. Since X is formally smooth the morphism $Y_0 \hookrightarrow X$ extends to a morphism $\widetilde{Y}_0 \to X$. Its image is contained in Y_{i_1} for some i_1. Let us show that for $i \geq i_1$ the embedding $\nu : Y_{i_1} \hookrightarrow Y_i$ is an isomorphism. We have the morphism $f : \widetilde{Y}_0 \to Y_{i_1}$. On the other hand, the morphism $Y_0 \hookrightarrow \widetilde{Y}_0$ extends to $g : Y_i \to \widetilde{Y}_0$. The composition $\nu f g : Y_i \to Y_i$ induces the identity on Y_0. So $\nu f g$ is finite and induces isomorphisms between tangent spaces. Therefore $\nu f g$ is a closed embedding. Since Y_i is noetherian a closed embedding $Y_i \to Y_i$ is an isomorphism. So $\nu f g$ is an isomorphism and therefore ν is an isomorphism. $\square$

6.2.11. *Proposition.* Let X be a connected formally smooth affine $\aleph_0$-formal scheme of ind-finite type over a field C such that Θ_X is not coherent (i.e., the restriction of Θ_X to X_{red} is of infinite type). Then X is isomorphic to a formal scheme from Example 6.2.9(ii).

Proof. We will construct a formally smooth morphism

$$X \to \operatorname{Spf} C[[y_1, y_2, \ldots]]$$

whose fiber over $0 \in \operatorname{Spf} C[[y_1, y_2, \ldots]]$ is a formal scheme from 6.2.9(i). Represent X as $\varinjlim \operatorname{Spec} A_n$, $n \in \mathbb{N}$, so that for every n the morphism $A_{n+1} \to A_n$ is surjective with nilpotent kernel. The algebras A_n are of finite type. By 6.2.1 the restriction of Θ_X to $\operatorname{Spec} A_n$ is free; it has countable rank. This means that for every n the projective system $(\Omega_{A_i} \otimes_{A_i} A_n)$, $i \geq n$, is equivalent to the projective system

$$\ldots \to A_n^3 \to A_n^2 \to A_n$$

(here the map $A_n^{k+1} \to A_n^k$ is the projection to the first k coordinates). So after replacing the sequence (A_n) by its subsequence one gets the diagram

$$\ldots \twoheadrightarrow \Omega_{A_3} \twoheadrightarrow F_2 \twoheadrightarrow \Omega_{A_2} \twoheadrightarrow F_1 \twoheadrightarrow \Omega_{A_1}$$

where the F_n are finitely generated free A_n-modules and the A_n-modules $G_n := \operatorname{Ker}(F_{n+1} \otimes_{A_{n+1}} A_n \to F_n)$ are also free. For each n choose a base $e_{n1}, \ldots, e_{nk_n} \in G_n$. Lift e_{ni} to $\widetilde{e}_{ni} \in \operatorname{Ker}(\Omega_{A_{n+2}} \otimes_{A_{n+2}} A_n \to F_n) \subset \operatorname{Ker}(\Omega_{A_{n+2}} \otimes_{A_{n+2}} A_n \to \Omega_{A_n})$ and represent $\widetilde{e}_{ni}$ as df_{ni}, $f_{ni} \in \operatorname{Ker}(A_{n+2} \to A_2)$. Finally lift f_{ni} to $\widetilde{f}_{ni} \in A := \varprojlim_m A_m$ and organize the f_{ni}, $n \in \mathbb{N}$, $i \leq k_n$, into a sequence $\varphi_1, \varphi_2, \ldots$. This sequence converges to 0, so one has a continuous morphism $C[[y_1, y_2, \ldots]] \to A$ such that $y_i \mapsto \varphi_i$. It induces a morphism

$$f : X \to Y := \operatorname{Spf} C[[y_1, y_2, \ldots]] \tag{6.8}$$

It follows from the construction that the differential

$$df : \Theta_X \to f^*\Theta_Y \tag{6.9}$$

is surjective and its kernel is coherent (indeed, it is clear that these properties hold for the restriction of (6.9) to $\operatorname{Spec} A_1 \subset X$, so they hold for the restriction to $\operatorname{Spec} A_n$, $n \in \mathbb{N}$).

Lemma 6.3. *A morphism $f : X \to Y$ of formally smooth ind-schemes of ind-finite type is formally smooth if and only if its differential* (6.9) *is surjective. In this case $\Theta_{X/Y}$ is the kernel of* (6.9).

Assuming the lemma we see that (6.8) is formally smooth and $\Theta_{X/Y}$ is coherent. So the fiber X_0 of (6.8) over $0 \in Y$ satisfies the conditions of Proposition 6.2.10. Therefore X_0 is isomorphic to a formal scheme from Example 6.2.9(i). Let us show that X is isomorphic to $\widetilde{X} := X_0 \times Y$. Indeed, since X is formally smooth over Y the embedding $X_0 \hookrightarrow X$ extends to a Y-morphism $\alpha : \widetilde{X} \to X$. Since $\widetilde{X}$ is formally smooth over Y the embedding $X_0 \hookrightarrow \widetilde{X}$ extends to a Y-morphism $\beta : X \to \widetilde{X}$. Both α and β are ind-closed embeddings (if a morphism $\nu : Y \to Z$ of schemes of finite type induces an isomorphism $Y_{\mathrm{red}} \to Z_{\mathrm{red}}$ and each geometric fiber of ν is reduced then ν is a closed embedding). The Y-morphism $\beta\alpha : X_0 \times Y \to X_0 \times Y$ induces the identity over $0 \in Y$, so $\beta\alpha$ is an isomorphism. Therefore α and β are isomorphisms, so we have proved the proposition.

The proof of the lemma is standard. The statement concerning $\Theta_{X/Y}$ follows from the definitions. To prove the first statement take an affine scheme S with an Ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I}^2 = 0$ and let $S_0 \subset S$ be the subscheme corresponding to $\mathcal{I}$. For a morphism $\psi : S_0 \to X$ denote by $E_X(S, \mathcal{I}, \psi)$ (resp. $E_Y(S, \mathcal{I}, \psi)$) the set of extensions of ψ (resp. of $f\psi$) to a morphism $S \to X$ (resp. $S \to Y$). Formal smoothness of f means that $f_* : E_X(S, \mathcal{I}, \psi) \to E_Y(S, \mathcal{I}, \psi)$ is surjective for all S, $\mathcal{I}$, ψ as above. Since X and Y are formally smooth $E_X(S, \mathcal{I}, \psi)$ and $E_Y(S, \mathcal{I}, \psi)$ are non-empty. According to 16.5.14 from [Gr] they are torsors (i.e., non-empty affine spaces) over $V_X(S, \mathcal{I}, \psi) := \operatorname{Hom}(\psi^*\Omega_X, \mathcal{I}) = \Gamma(S_0, \psi^*\Theta_X \otimes \mathcal{I})$ and $V_Y(S, \mathcal{I}, \psi) = \Gamma(S_0, \psi^* f^*\Theta_Y \otimes \mathcal{I})$. The map f_* is affine and the corresponding linear map $\Gamma(S_0, \psi^*\Theta_X \otimes \mathcal{I}) \to \Gamma(S_0, \psi^* f^*\Theta_Y \otimes \mathcal{I})$ is induced by (6.9). So the first statement of the lemma is clear. □

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