Complex cobordism and algebraic topology

...it was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry ...

S. Lefschetz, A page of mathematical autobiography, BAMS 74 (1968) 854 - 879

Vers les années 1962 m'arriva de rencontrer Lefschetz aux Etats-Unis. Il me fit alors cette confidence: "Que lá Topologie était belle avant 1935! Après, elle est devenue beaucoup trop algébrique."

R. Thom, Problèmes recontrés dans mon parcours mathématique: un bilan, Publ. Math. IHES **70** (1989)

§ Introduction

This note is an attempt to supplement recent [A2,Su2] accounts of René Thom's work on cobordism theory with a description of that subject's later evolution, under the influence of Novikov, Quillen, Sullivan, and others.

Thom thought of himself above all as a geometer, but his thinking was grounded in formidable algebraic insight. Much of mathematics might be described as a dialogue between geometry and algebra (cf. e.g. Descartes), and although I will focus here on developments in **algebraic** topology, I will tell this story in a lingua franca based on algebraic **geometry**, not so far from the dialect used by Thom himself in his work [TL] on singularities of differentiable maps. I will try to justify the utility of that language in the course of a historical sketch.

What follows is at best a tissue of generalities; it is a 'soft' account, intended to be as accessible as possible. It therefore contains nothing about explicit calculations, which already have an extensive and accessible literature, cf. eg. [R1,2; WSW].

§1 Preliminaries

One of the most intimidating aspects of classical cobordism theory is the sheer size of the rings involved. Thom showed that the unoriented cobordism ring is graded polynomial, with one generator of each degree not of the form $2^k-1,\ k \geq 1$, and he showed that over the rationals, the oriented cobordism ring is generated by the complex projective spaces. Wall and others completed the calculation of the latter ring, which has a considerable amount of two-torsion; but interest eventually centered on the **complex** cobordism ring Ω^{U}_*, which was shown by Milnor and Novikov to be torsion-free, with one generator in each even degree. We are therefore faced from the outset with rings which are not finitely generated; but Ω^{U}_* has the technical advantage of being in some sense locally finitely presented (it is **coherent** [S]) which makes its homological algebra reasonably tractable. Thus, although complex cobordism is in some ways less geometrically natural than oriented cobordism, its algebraic accessibility makes it a fundamental technical tool.

It took a while for attention to shift from the cobordism rings to their associated (co)homology theories; indeed the multiplicity of such objects played an important role in crystallizing our present notion of a spectrum. That what we now call Thom spectra define cohomology theories was used by Atiyah [A1] as a device to study relations between cobordism rings associated to various families of classical groups, but before the work of Novikov it was the cobordism rings *per se*, not their associated cohomology theories, which were regarded as the significant subjects for investigation. Modern thinking tries to understand a cohomology theory and its operations together, but with the technology available in the 60's, this was an intimidating task [La]. Just as all cats look alike in the dark, over the rationals all cohomology theories are equivalent: tensored with $\mathbb{Q}$, the ring of cobordism operations must act (in some sense) freely and transitively on the coefficients of the theory. The algebras of cobordism operations are therefore approximately as enormous as the coefficient rings themselves, and had Novikov not shown the remarkable efficiency of an Adams spectral sequence based on complex cobordism as a tool for calculating stable homotopy groups of spheres (and in particular for clarifying the role played there by of the J-homomorphism [No]), our understanding of these operations would probably have stayed limited for a considerable time.

§2 The geometric picture

It was Quillen who gave us an economical language for these topics; but he was very much concerned with making his paper as elementary and self-contained as possible, so that language was backgrounded and largely implicit. His formalism was strongly influenced by Grothendieck's contemporary thinking about motives, and one way to paraphrase his approach is to describe complex cobordism as the universal multiplicative cohomology theory carrying good Chern classes for complex vector bundles [Q; see also LM]. The work of Borel and Hirzebruch on characteristic classes and symmetric functions reduces this to a question about Chern classes for **line** bundles, and the key issue becomes the existence of a canonical (Euler, or first Chern) class $\mathbf{e}$ in the universal example

$$\Omega^*_{\mathrm{U}}(\mathbb{C}P^\infty) \cong \Omega^*_{\mathrm{U}}[[\mathbf{e}]] \;;$$

thus any multiplicative cohomology theory E^* with a Chern class for line bundles is the recipient of a unique natural multiplicative transformation $\Omega^*_{\mathrm{U}} \to E^*$ of cohomology theories.

2.1 Now the classifying space $\mathbb{C}P^\infty$ for complex line bundles is a commutative H-space, so $\Omega^*_{\mathrm{U}}(\mathbb{C}P^\infty)$ acquires the structure of a completed Hopf Ω^*_{U}-algebra on a single formal generator $\mathbf{e}$. Conveniently enough, such 'formal group laws' had been extensively studied by Dieudonné, Lazard, and others, as a theory of Abelian arithmetic Lie groups. It is trivial to show the existence of a universal one-dimensional formal group law (take the ring generated by the coefficients of the Hopf diagonal, modulo the ideal of relations imposed by associativity and commutativity), but Lazard demonstrated the highly **non**-trivial fact that the resulting ring L is (implicitly graded) polynomial over $\mathbb{Z}$, with one generator in each even positive degree. Miščenko had previously identified the formal logarithm for the complex cobordism ring as

$$\log_\Omega(\mathbf{e}) = \sum_{n \geq 1} \mathbb{C}P^{n-1} \ \frac{\mathbf{e}^n}{n} \ ,$$

and Quillen went on to show that the homomorphism

$$L^* \to \Omega^*_{\mathrm{U}}$$

classifying the formal group law defined by $\mathbb{C}P^\infty$ is in fact an isomorphism: the formal group law of complex cobordism is **universal**. If only for psychological reasons, this is quite a remarkable result; it provides the (enormous, amorphous) complex cobordism ring with something very much like a preferred system of coordinates.

[In fact explicit coordinates can be defined, at least locally with respect to a prime p, by writing

$$[p]_\Omega(\mathbf{e}) = p\mathbf{e} +_\Omega \cdots +_\Omega v_k \mathbf{e}^{p^k} +_\Omega \cdots \ ;$$

where $[p]_\Omega$ denotes the effect on the formal group of multiplication by p in the H-space structure on $\mathbb{C}P^\infty \sim H(\mathbb{Z}, 2)$, $+_\Omega$ denotes addition with respect to the formal group law, and $v_k \in \Omega^{\mathrm{U}}_{2(p^k-1)} \otimes \mathbb{Z}_{(p)}$ is a certain recursively defined polynomial generator. These (Araki-Milnor) generators are very useful for computations, but I will not pursue this question further: the purpose of this note is really to sketch a **coordinate-free** description of the cobordism ring.]

2.2.1 It is here that the language of algebraic geometry becomes relevant. In commutative algebra one associates to a (commutative) ring A, the space spec A of its prime ideals: such ideals $\mathfrak{p}$ have integral domain quotients $A/\mathfrak{p}$, which imbed in their quotient fields, so we can equally well think of an element of spec A as the kernel of some ring homomorphisms from A to a field k. In classical algebraic geometry one restricted attention to **maximal** prime ideals, which correspond to points closed in the natural (Zariski) topology on spec A; in particular if k is algebraically closed, and $A = \mathbb{Z}[x_1, \ldots, x_n]$ is polynomial, then such closed points correspond to assignments of values in k for the coordinate functions x_k, thus recovering classical n-dimensional affine space over k. The language of schemes [AM] provides a dictionary equating commutative rings to a category of topological spaces endowed with a structure sheaf whose stalks are **local** rings; modules over these rings can thus be described as sheaves of modules over this ringed space.

In particular, a (graded-commutative) multiplicative cohomology functor

$$X \mapsto E^*(X)$$

whose coefficient ring $E^*(\mathrm{pt}) = E^*$ is concentrated in even degrees can without loss of generality be interpreted as taking values in a category of (even-odd graded) sheaves over the ringed space spec E obtained by ignoring the grading (for the moment; we'll return to this issue below).

2.2.2 A more modern point of view, however, sees the prime ideal spectrum as a special instance of a more general functor of points

$$\mathrm{spec}\ A(-) := \mathrm{Hom}_{\mathrm{rings}}(A, -) : (\text{Commutative rings}) \to (\text{Sets}) ,$$

the idea being that the classical prime ideal spectrum yields inconveniently few points. Yoneda's lemma guarantees that the enriched functor $A \mapsto \mathrm{spec}\ A(-)$ is faithful, ensuring the existence of an abundance of points.

This interpretation developed out of the study of moduli problems, and it fits perfectly with Lazard's result: if k is a general commutative algebra, then the elements of

$$\Lambda(k) := \mathrm{spec}\ L(k) := \mathrm{Hom}_{\mathrm{rings}}(L, k)$$

are precisely the formal group laws

$$\{F \in k[[X,Y]] \mid F(X, F(Y,Z) = F(F(X,Y),Z),\ F(X,Y) = F(Y,X) = X+Y+\cdots\}$$

with coefficients in k. This allows us to describe spec $L(k)$ as the set of 'all' (one-dimensional) formal group laws (i.e. over **any** convenient commutative ring). Lazard's theorem then describes this set of group laws as an infinite-dimensional affine space Λ: specifying the particular ring we're working over is irrelevant, because this formulation of the theorem behaves naturally under 'change of rings'.

2.3.1 Quillen's theorem thus tells us that we can think of complex cobordism as a cohomology theory taking values in the category of sheaves of modules over the ringed space of 'all' one-dimensional formal group laws; moreover, when the functor is applied to a finite complex, the resulting sheaf will be coherent. But picturesque as this interpretation may be, it is perhaps not particularly compelling. It becomes more interesting, however, when we consider cohomology operations.

In the language of §2.2.2 above, a group object in the category of schemes is precisely a representable functor from commutative rings to groups; the additive group

$$\mathbb{G}_a(A) = A = \mathrm{Hom}_{\mathrm{rings}}(\mathbb{Z}[t], A)$$

and the multiplicative group

$$\mathbb{G}_m(A) = A^\times = \mathrm{Hom}_{\mathrm{rings}}(\mathbb{Z}[t,t^{-1}], A)$$

are typical (and important) examples. The operations in complex cobordism can be described most easily in terms of the functor

$$A \mapsto \Gamma(A) := \{t(T) = \sum_{k \geq 0} t_k T^{k+1} \mid g_0 \in A^\times\}$$

which sends a commutative ring to the set of invertible formal power series with coefficients in A. This set is in fact a group, with composition $t, t' \mapsto t \circ t' := t(t'(T))$ as group operation; the functor is representable (it is the spectrum, in the functorial sense, of the polynomial ring

$$S = \mathbb{Z}[t_0, t_0^{-1}][t_k \mid k \geq 1] = S_*[t_0, t_0^{-1}] \ ,$$

and the diagonal

$$\Delta t(T) = (t \otimes 1)((1 \otimes t)(T)) \in (S \otimes S)[[T]]$$

makes S into a commutative (but not cocommutative) Hopf algebra. Thus Γ is a groupscheme; it is easily seen to be an extension

$$1 \to \Gamma_0 \to \Gamma \to \mathbb{G}_m \to 1$$

of the multiplicative group by iterated copies of the additive group, split by a canonical imbeddding of the multiplicative torus $\mathbb{G}_m \to \Gamma$. It acts naturally on the space Λ of formal group **laws:** if $F \in \text{spec}\, L(A)$ is a group law, and $t \in \Gamma(A)$ is an invertible power series, then

$$F^t(X, Y) := t^{-1}(F(t(X), t(Y))) \in \text{spec}\, L(A)$$

is another formal group law. This action, called 'change of coordinates', is represented by a ring homomorphism

$$L \mapsto L \otimes S$$

satisfying the axioms in [Adams]; the resulting equivalence classes, or orbits of the action of Γ on Λ, are called (one-dimensional) 'formal groups'. The multiplicative subgroup of Γ plays a special role; it is sometimes useful to distinguish between isomorphisms of formal group laws (represented by elements of Γ) and **strict** isomorphisms (represented by elements of Γ_0).

2.3.2 The usual analysis of the operations on the (co)homology theory associated to a spectrum $\mathbf{E}$ requires the existence of a Künneth theorem strong enough to guarantee that $E^*(\mathbf{E})$ and $E_*(\mathbf{E})$ are in some sense dual Hopf E^*-algebras. This holds for the complex cobordism spectrum $\mathbf{MU}$; in particular,

$$\Omega^{\mathrm{U}}_*(\mathbf{MU}) \cong L_* \otimes S_*$$

as (bilateral) Hopf algebras, from which it follows [Adams] that

$$\Omega^{\mathrm{U}}_*(X) = \pi_*(X \wedge \mathbf{MU})$$

has a natural Hopf coaction

$$\Omega^{\mathrm{U}}_*(X) \mapsto \Omega^{\mathrm{U}}_*(X) \otimes S_* \ .$$

The role played by the grading in this brief summary has been suppressed, and needs to be clarified. The category of comodules over the Hopf algebra

$S_0 := \mathbb{Z}[t_0, t_0^{-1}]$ representing the multiplicative group is equivalent to the usual category of $2\mathbb{Z}$-graded modules, via the functor which assigns to the evenly graded module M_* its associated ungraded module $M = \oplus_k M_{2k}$, with coaction

$$\sum m_{2k} \mapsto \sum m_{2k} t_0^k : M \to M \otimes S_0 \; .$$

The algebra S_0 acts on a formal group

$$F(X, Y) = \sum F_{i,j} X^i Y^j$$

sending the coefficient $F_{i,j}$ to $t_0^{i+j-1} F_{i,j}$, thus endowing L with its intrinsic (even) grading; similarly, $t_k \in S_* \mapsto t_0^k t_k$. But the cobordism $\Omega_*^{\mathrm{U}}(X)$ of a space will in general not be concentrated in even degree, so it is natural to regard it as an even-odd graded comodule over $L \otimes S$, and to endow the tensor category of such comodules with the Koszul associativity constraint usual in algebraic topology [CK].

2.4.1 This discussion can now be summarized by saying that the bordism module of a space X can be interpreted, without loss of generality, as an element $\Omega_\bullet^{\mathrm{U}}(X)$ of the tensor category of even-odd graded sheaves of modules over the **stack** Λ/Γ of one-dimensional formal groups: in other words, it is a Γ-equivariant sheaf on the moduli space Λ of formal group laws.

We can of course do homological algebra this category. The bordism modules $\Omega_\bullet^{\mathrm{U}}(S^n)$ of the spheres define analogues of the Tate twists in algebraic geometry, and Novikov's Adams spectral sequence for stable maps between spaces X and Y has the groups

$$\mathrm{Ext}^i_{\Lambda/\Gamma}(\Omega_\bullet^{\mathrm{U}}(X), \Omega_\bullet^{\mathrm{U}}(S^k Y))$$

of extensions in this tensor category as its E_2 term. In particular, the sheaf cohomology

$$H^i_\Gamma(\Lambda, \Omega_\bullet^{\mathrm{U}}(S^k)) \Rightarrow \pi^S_{i-k}(\mathrm{pt}) \; .$$

of the moduli stack is the E_2 term of Novikov's spectral sequence for the stable homotopy groups of the spheres.

2.4.2 The stable homotopy category is the original, motivating example of what is now called a tensor triangulated category, and the derived category of sheaves of modules over the stack Λ/Γ is another. Complex cobordism defines a homological functor on the first category, and it is tempting to suspect that the Adams-Novikov spectral sequence is a consequence of the existence of a lift of this homology theory to a functor between these triangulated categories. It is significant that **this is not the case:** a homology theory which lifts to the derived category, in the sense that it takes distinguished triangles in the stable homotopy category to distinguished triangles in some derived category of modules, is necessarily ordinary homology [[CF]; but see, however, [Fr] and [Ne]].

The derived category of sheaves of modules over the stack of one-dimensional formal groups is a remarkably good model for the stable homotopy category, but their precise relation is deep and is very far from being well-understood. A great deal of research in recent years has been concerned with this question; the next section will describe some of that work.

§3 The orbit stratification

It is a consequence of Lie theory that over a field of characteristic zero, all (one-dimensional) formal groups are isomorphic to the additive group $F_{\mathbb{G}_a}(X,Y) = X+Y$; this translates as the assertion that over such a field, the stack Λ/Γ is equivalent to the category with one object, and the group $\mathbb{G}_m$ as morphisms. The category of even-odd graded sheaves over this quotient is thus equivalent (by H. Miller's categorical version of Shapiro's lemma [M2]) to the category of $\mathbb{Z}$-graded rational vector spaces. This is another reflection of the fact that as far as pure homotopy theory is concerned, all cohomology theories in characteristic zero are equivalent.

In positive characteristic things are much more interesting. If F is a formal group law over a field k of characteristic $p > 0$, it is easy to see (e.g. by symmetric functions) that the operation of multiplication by p in the group structure is represented by a power series of the form

$$[p]_F(T) = g(T^q)$$

for some power $q = p^n$ of p and some series g with $g'(0) \neq 0$ (unless $[p]_F = 0$, when we take $n = \infty$ and say that F is of additive type). The integer n is an invariant of the group **law**, called its height; for example, the multiplicative formal group $F_{\mathbb{G}_m}(X,Y) = X+Y+XY$ has

$$[p]_{\mathbb{G}_m}(T) = (1+T)^p - 1 \equiv T^p \bmod p\ ,$$

and thus has height one. All positive integers occur: for example Honda's logarithm

$$\log_{F_n}(T) = \sum_{k \geq 0} p^{-k} T^{q^k}$$

defines a group law over $\mathbb{Z}_p$ whose reduction modulo p has $[p]_{F_n}(T) = T^q$.

Height is a complete invariant for (one-dimensional) formal groups over a separably closed field; in other words, in characteristic p the set of geometric points of Λ stratifies into orbits indexed by the set $\mathbb{N} \cup \infty$. The orbit of a grouplaw F is a homogeneous space $\Gamma/\mathbb{S}_F$, and an equivariant sheaf of modules over such a quotient is **locally free**; indeed, it can be recovered from the action of the isotropy group $\mathbb{S}_F$ of the orbit on the fiber of the sheaf above the point F. This suggests the hope of understanding sheaves over the moduli stack in terms of representations of the isotropy groups of the orbits. These isotropy groups are very beautiful; they were the focus of studies in the early sixties by several arithmeticians, and in this section I will present some of their results.

3.1.0 The Chern class for line bundles in classical cohomology is additive, which makes $n = \infty$ a reasonable example to begin with; but, as we will see, formal groups of infinite height are in many ways anomalous. The foregrounding of classical cohomology in most presentations of algebraic topology may therefore to have distorted our expectations of the nature of the stable homotopy category. Solving the Hopf invariant one problem using classical cohomology requires all the machinery of the Adams spectral sequence; this is in striking contrast to the elegance and simplicity of Adams and Atiyah's proof using K-theory. Ordinary cohomology lies on a stratum of infinite codimension in the moduli space of complex-oriented theories: it may be simple in some ways, but in others it is extremely degenerate.

3.1.1 The isotropy group for the additive group law over a field k of characteristic p assigns to a commutative k-algebra A, the group

$$\mathbb{S}_{\mathbb{G}_a}(A) = \{t \in \Gamma(A) \mid t(X+Y) = t(X) + t(Y)\} \;;$$

but in characteristic p the function defined by $t(T) = \sum t_k T^{k+1}$ is linear iff the coefficients vanish unless $k = p^s$ is a power of p. A typical element of this group thus has the form

$$a(T) = \sum a_k T^{p^k} \;;$$

the composition

$$(a \circ b)(T) = \sum a_k (\sum b_j T^{p^j})^{p^k} = \sum_i (\sum_{i=j+k} a_k b_j^{p^k}) T^{p^i} \;;$$

of two such elements is represented by Milnor's formula

$$\Delta \xi_i = \sum_{i=j+k} \xi_k \otimes \xi_j^{p^k}$$

for the diagonal of the Hopf algebra $\mathbb{F}_p[\xi_k \mid k \geq 1]$ dual to Steenrod's reduced pth powers.

When $p = 2$ this exhausts the operations in ordinary cohomology, but there are more operations when the prime is odd. We will recover the remaining operations below, but even without them it is no exaggeration to paraphrase this little calculation by saying that the automorphisms of the fiber of the cobordism sheaf, above the stratum of formal group laws of additive type, are essentially the operations of ordinary cohomology.

3.1.2 The isotropy group $\mathbb{S}_{\mathbb{G}_m}$ for the multiplicative group law is much simpler. Multiplication by an integer in the formal group structure defines an embedding

$$n \mapsto [n]_{\mathbb{G}_m}(T) = (1+T)^n - 1 : \mathbb{Z} \to \mathrm{End}_k(\mathbb{G}_m) \,,$$

which extends to an embedding

$$\alpha \mapsto \sum_{k \geq 1} \binom{\alpha}{k} T^k \in k[[T]]$$

of the ring $\mathbb{Z}_p$ of p-adic integers in the endomorphisms. The units of this ring exhaust the automorphisms of the theory; they correspond to the Adams operations in p-adically completed K-theory.

It is important, and typical, that this groupscheme is essentially constant; when F has finite height over a field, the Lie algebra of $\mathbb{S}_F$ is trivial. For if t is an automorphism of a grouplaw with $[p]_F(T) = T^q$ (which, when k is separably closed, we can assume without loss of generality), then $t \circ [p]_F = [p]_F \circ t$ implies that if $t(T) = t_0 T + \epsilon T^i$ mod higher order terms is an infinitesimal automorphism (with $\epsilon^2 = 0$), then

$$t_0 T^q + \epsilon T^{iq} \equiv t_0^q T^q + \epsilon^q T^{iq} \equiv t_0^q T^q$$

mod higher order terms; i.e. $\epsilon = 0$. In consequence, the groupschemes $\mathbb{S}_F$ are étale when the height of F is finite; in particular, if $F = F_n$ is the group law defined by the Honda logarithm, its isotropy group $\mathbb{S}_{F_n} = \mathbb{S}_n$ can be identified with the group of units of the noncommutative integral domain

$$\mathbf{o}_n := W(\mathbb{F}_q)\langle F \rangle / (F^n - p) \; ;$$

here $W(k)$ denotes the ring of Witt vectors of k (which can be identified when $k = \mathbb{F}_q$ with the ring obtained by adjoining a primitive $(q-1)$th root of unity to $\mathbb{Z}_p$), and the Frobenius element F (representing the endomorphism $F(T) = T^p$ of F_n) satisfies the identity

$$a^\sigma F = Fa$$

for $a \in W(\mathbb{F}_q)$ (with $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ the usual pth power automorphism of the field with q elements).

The isotropy groups of group laws of finite height thus become constant, once the field of definition is large enough. Reduction modulo p defines a ring homomorphism

$$W(\mathbb{F}_q)\langle F \rangle / (F^n - p) \to \mathbb{F}_q \langle F \rangle / (F^n)$$

and thus a homomorphism on groups of units. The target is a truncation of the group

$$(\mathbb{F}_q \langle\langle F \rangle\rangle)^\times = \mathbb{S}_{\mathbb{G}_a}(\mathbb{F}_q)$$

defined by the reduced power Hopf algebra; this leads to a consistency condition for the fibers of an equivariant sheaf over the moduli stack. A finite complex defines a system of mod p representations of the groups $\mathbf{o}_n^\times$ of strict units in a family of p-adic division algebras, and these representations converge in a certain sense to a representation of the Steenrod algebra. In intuitive terms: a finite complex has limited ramification.

3.2.0 Quillen's theorem thus allows us to associate to a space, a family of invariants associated to the stratification by height of the moduli stack of formal groups. This stratification is in some ways quite simple: the nth layer Λ_n is the locus where the polynomial generators $p = v_0, v_1, \cdots, v_{n-1}$ vanish, and v_n is invertible. There is a big open stratum, defined by formal groups of height one, and the fiber of the cobordism sheaf above this generic orbit is (the mod p reduction of) classical K-theory; while at the bottom of the stratification we have a closed orbit, carrying a sheaf with properties reminiscent of ordinary (mod p) cohomology.

At this point we are in the position of topologists in the days of Emmy Noether, faced with the analog of torsion coefficients; we would like a better understanding of the significance of these invariants. Ideas of Sullivan permit us to interpret them in terms of closely related extraordinary cohomology theories.

3.2.1 To return for a moment to the wishful thinking in §2.4.2: if we **did** have a lifting

$$(\text{Stable Homotopy}) \to D(\Lambda/\Gamma - \text{Modules})$$

of complex cobordism to a functor between tensor-triangulated categories, then composition with the left-derived functor Li_n^* defined by the inclusion morphism

$$i_n : \Lambda_n/\Gamma \to \Lambda/\Gamma$$

would define a functor taking values in equivariant sheaves over a homogeneous space (and thus, after specializing at a typical fiber F, a cohomology theory taking values in modular representations of $\mathbb{S}_F$). Such a derived functor would of course be more complicated than simple restriction; in particular, it would imply the existence of a kind of universal coefficient spectral sequence

$$E_2^{*,*} = \mathrm{Tor}^{\Omega_{\mathrm{U}}}(\Omega_*^{\mathrm{U}}(X), v_n^{-1}\Omega_*^{\mathrm{U}}/I_n) \Rightarrow Li_n^*\Omega_*^{\mathrm{U}}(X) \ ,$$

I_n being the ideal $(p, v_1, \cdots, v_{n-1})$ defining the locus Λ_n.

It is quite remarkable that Sullivan's theory of cobordism with singularities provides us with such a cohomology theory, taking values in equivariant sheaves over Λ_n, calculable by just such a spectral sequence. We cannot construct such a theory by applying homological algebra naively to the coefficients, but we can mimic a standard construction in the homotopy category. The sequence $v_0, \cdots, v_{n-1}$ of elements is regular in the sense of commutative algebra, so the tensor product of the complexes

$$0 \longrightarrow \Omega_*^{\mathrm{U}} \xrightarrow{v_k} \Omega_*^{\mathrm{U}} \longrightarrow 0$$

(for $0 \leq k \leq n-1$) is a (Koszul) resolution of $\Omega_*^{\mathrm{U}}/I_n$. In the homotopy category we can emulate this by taking the cofiber of a suitable cubical diagram of spectra, constructed as a product of elementary cofibrations realizing the elementary complexes above [M1]. The fiber of the resulting sheaf above the generic orbit is the mod p reduction of complex K-theory, while the fiber above (a typical point of) the closed orbit is ordinary (mod p) cohomology. More generally, the fiber above a point of Λ_n defines (the homological version of) the 'extraordinary K-theory' $K(n)^\bullet(X)$.

Nowadays we have very elegant accounts [EKMM,HSS] of the stable category which make these constructions relatively straightforward. I believe it is fair historically to say that the need to put these, and related, examples on a solid foundation provided much of the impetus which led to our current highly polished understanding of the category of spectra.

3.2.2 It is not immediate from the description above that the cohomology theories $K(n)$ are naturally $\mathbb{S}_n$-representations; to prove this requires us to exhibit a Hopf algebra structure on the operations, and for that we need a rudimentary multiplicative structure [Ba,Ro]; there is a detailed account in [Wu]. The representing objects are definitely not E_∞ ringspectra; very few mod p theories are. It is however possible to identify their operation algebras completely (Boardman has completed the $p = 2$ case only recently), and there is more to the story than the isotropy groups.

Sullivan's constructions use stratified sets with carefully controlled singularities modelled on cones over a list of special example manifolds [Su1]; in our case these models represent the classes v_k. Each such model defines a generalized Bockstein operation on the resulting theory, and in our case the operation defined by v_k corresponds (in the sense of §3.1.2) to Milnor's operation Q_k. The full Hopf algebra of operations is thus an extension of some version of the group algebra of $\mathbb{S}_n$ by an exterior algebra $E(Q_0, \cdots, Q_{n-1})$; to complete the description, we need to understand the action of the isotropy groups on the algebra of Bocksteins.

For this, a coordinate-free viewpoint is again useful. Lubin and Tate constructed a cohomology theory $H^*(F, k)$ for formal groups over a field k, and in particular showed that the second cohomology group classified deformations. They calculate the rank of $H^2(F, k)$ to be the height of F minus one; an alternate formulation of their result identifies this cohomology group (under its natural $\mathbb{S}_F$-action) with the normal bundle at F of its orbit in Λ. We can thus think of the Bocksteins as differentiation operators parametrized by deformations of the group law. This accounts for all the Bocksteins except Q_0, which, in this language, measures deformations in a direction away from p, toward characteristic zero.

§4 Some more recent developments

Stable homotopy theory allows us to understand spaces, which are fundamental and perhaps irreducible objects of our imagination, in intrinsically **algebraic** terms. On the other hand, the deep formal properties of complex cobordism described above provide us with a **geometric** language for describing these objects (as sheaves of modules over the structure space Λ/Γ). This intriguing double reversal embeds algebraic topology into algebraic geometry in a way which, it seems fair to say, was probably not anticipated by the creators of the subject.

This concluding section collects some very brief remarks about more recent work which takes for granted this circle of ideas, and pushes them forward. It is idiosyncratic and incomplete, and in particular I will say little about elliptic cohomology, which is really the most fertile and striking new development in the subject. In the language presented above, the starting point of that theory is the idea that the formal completion of an elliptic curve at its origin defines a one-dimensional formal group. This defines a functor

$$(\text{Elliptic curves}) \to (\text{Formal groups}) \sim \Lambda/\Gamma \ ,$$

or, more precisely, a morphism of **stacks**. Roughly speaking, elliptic cohomology is the pullback of complex cobordism, regarded as a sheaf over the range of this morphism, to its domain; but this description misrepresents the facts. For example, elliptic cohomology is **not** complex-oriented, and its construction (which is a tour-de-force [HGMR,Re]) requires a host of new ideas and techniques which I am not competent to summarize.

Elliptic cohomology is moreover a **global** object: taking it apart one prime at a time can be technically useful, but loses a great deal of perspective. Its relations with conformal field theory on one hand, and the equivariant topology of free loopspaces on the other, provide it with the kind of rich connections to other fields which signal a major conceptual advance [H2,AHS]. It is too early to attempt a summary; this story is just beginning.

4.1 The theory developed above derives about equally from algebra (the theory of formal groups) and geometry (in particular, cobordism with singularities). This cross-fertilization brings into focus the geometrically unexpected existence of a **hierarchy** of theories of singularity, parametrized by the systems $p, v_1, \cdots, v_{n-1}$. There are in some sense too many possible theories of cobordism with singularities, and it is remarkable that algebra singles out this particular sequence of elements as especially interesting.

Sullivan's constructions work with manifolds, not cobordism classes, and the classes v_n are represented (modulo their predecessors) by the p- dric hypersurfaces

$$\{[z_0 : \cdots : z_q] \in \mathbb{C}P^q \mid \sum z_k^p = 0\}$$

(where $q = p^n$). This suggests that the geometric significance of such singularities should be sought in algebraic geometry in characteristic p; but the question seems to have attracted little attention.

These models are extremely symmetric: they possess stationary-point-free actions of $(\mathbb{Z}/p\mathbb{Z})^n$ [F]. There is reason to think that the height filtration of Λ/Γ and the Atiyah-Swan filtration of the spectrum of the cohomology of a finite group (by rank of supporting elementary abelian groups) are two extreme aspects of some unified phenomenon, whose understanding awaits progress in the study of equivariant cobordism [HKR,GS,TCB]

4.2 The deeper (arithmetic) aspects of the theory of formal groups have been backgrounded in this account, through its use of the language of algebraic varieties over fields. It is now clear that the moduli spaces of **liftings** of formal groups (from characteristic p to characteristic zero) [LT] provide natural sites for the localizations of stable homotopy theory, and work of Bousfield [B,BN] provides us with a systematic way to restrict to subcategories of spectra supported on suitably closed substacks of Λ/Γ. In particular, the notion that the classical Euclidean primes have chromatic overtones [H1] has become standard imagery. This is reflected by the deep, purely homotopy-theoretic fact [HS] that a the ring of stable endomorphisms of a finite complex has a center of Krull dimension one, generated either by some power of a prime p, or by a self-map which can be interpreted as a power of some v_n. This is reminiscent of the existence of a nontrivial element of the center of a finite p-group: it provides a natural, 'internal' way for decomposing spaces inductively.

Our understanding of the relations between neighboring chromatic primes is very primitive; the subject is really at the forefront of current research. Waldhausen's chromatic red-shift principle [AR] suggests that free loopspaces raise chromatic level; besides its connection with elliptic cohomology, this seems to have arithmetic echoes [AMS,TT]. Hopkins has proposed a chromatic splitting conjecture [Hv], which seems to require some further tinkering. There is now a very elegant analysis [GHMR] of the sphere at height two (and $p = 3$) which may serve as a model for future research.

Underlying these developments looms the enormous question of the homotopy-theoretic underpinnings of many of the constructions of classical algebra. Quillen taught us that the K-theory of a ring, which before him was purely a matter of generators-and-relations algebra, was in fact part of homotopy theory; and that the K-groups themselves were really just the homotopy groups of a much richer invariant defined by the K-theory spectrum. Now we know that the moduli stack of formal groups is the tip of another such homotopy-theoretic iceberg, and the list of such objects has grown, to include the Lubin-Tate moduli spaces, the elliptic moduli stack, and (in unpublished work of Hopkins) moduli spaces of $K3$ surfaces and more general Calabi-Yau manifolds. Waldhausen has embraced this brave new world enthusiastically, and his point of view has revolutionized algebraic K-theory. One is reminded of Hilbert's remark about Cantor, that no one will expel us from the garden that he has shown to us.

4.3 There is good reason to suspect that certain Shimura varieties [C2 §6.4], which play a central role in the proof of the local Langlands conjecture, are ripe for such homotopification. These objects parametrize Abelian varieties decorated in various ways, e.g. with level structures as generalized by Drinfel'd, and they lead to constructions with enormous symmetry groups: roughly the product $\mathrm{Gl_n}(\mathbb{Q}_\mathrm{p}) \times D_n^\times \times \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (with D_n the division algebra defined by the field of quotients of the ring $\mathbf{o}_n$ of §3.1.2). Such objects define a kind of Morita transformation relating automorphic representations of $\mathrm{Gl}_n(\mathbb{Q}_p)$ to representations of the unit groups of division algebras and to representations of Galois groups.

Work on the Langlands program has focussed on applications of this construction to representation theory, but it is quite remarkable that **precisely** the same objects arise in algebraic topology in the context sketched here [GH,HG,RZ]; though now it is the cohomology of these objects (in the sense of §2.4.1), rather than their representations, which is of interest. The discovery of these objects was one of the major accomplishments of mathematics in the 20th century; understanding their significance is a comparable challenge for the coming one.

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